FREE LATTICES GENERATED BY PARTIALLY ORDERED SETS

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ABSTRACT

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Let P be a partially ordered set. FL(P) is the free lattice generated by P preserving existing binary join and meet in P . A quasi-order is constructed on the lattice polynomials over P in terms of certain ideals and dual ideals of P ; under the induced partial order the quotient set is the lattice FL(P) . A characterization of FL(P) is derived and a generalization of a theorem of Sorkin on the extension of isotone maps is proved.

$\mathcal{M}$ and $\mathcal{N}$ are families of finite subsets of P such that every element of $\mathcal{M}$ has a least upper bound in P and every element of $\mathcal{N}$ has a greatest lower bound in P . FL(P ; $\mathcal{M}$, $\mathcal{N}$) , a generalization of FL(P) , is the free lattice generated by P preserving joins of elements of $\mathcal{M}$ and meets of elements of $\mathcal{N}$. The results on FL(P) are extended to FL(P ; $\mathcal{M}$, $\mathcal{N}$) . As an application certain results of R.A. Dean on completely free lattices are derived.

The results concerning FL(P) and FL(P ; $\mathcal{M}$, $\mathcal{N}$) are applied to solve the word problem for free products of lattices, partially ordered free products of lattices, and

amalgamated free products where the amalgamated sublattice is of finite length.

A lattice L generated by a partially ordered set P is said to admit canonical representations if every element of L can be represented by a polynomial over P of shortest length which is unique up to commutativity and associativity. Those free products and partially ordered free products of lattices that admit canonical representations are characterized.

PREFACE

The construction of free universal algebras was first accomplished by G. Birkhoff (see [7], Chapter 4). Birkhoff's construction essentially defines, in a highly non-effective manner, a congruence relation on the algebra of polynomials over the generating set. Whitman [10] analysed the structure of the free lattice generated by a totally unordered set; he defined a quasi-order on the lattice polynomials in an effective manner and thus was able to give an effective construction of the congruence relation yielding the free lattice. Among his conclusions was the result that every element of the free lattice can be represented by a polynomial of shortest length which is unique up to commutativity and associativity, that is, that the free lattice admits canonical representations.

In his analysis of the problem of embedding lattices in complemented lattices, Dilworth [4] had occasion to discuss the lattices FL(P) and CF(P) generated by a partially ordered set P . To construct FL(P) he defined a quasi-order on the lattice polynomials in an inductive manner, which entailed knowing the quasi-order on a lower level on the polynomials and not just on P ; thus effectiveness was lost. Of more import to our work, he had occasion to introduce certain elements of the generating set, the lower and upper covers, in a very special situation. These covers were further exploited by Chen and Grätzer [2] in their extension of Dilworth's results on embedding lattices in complemented lattices.

In [3] Dean completely analysed CF(P) and, generalizing Whitman's method, he showed that CF(P) admits canonical representations.

In [3a] Dean constructed FL(P) and a generalization $FL(P ; \mathcal{M}, \mathcal{N})$, the free lattice preserving more general sup and inf in P , in terms of certain ideals and dual ideals of P . Our construction of these lattices is essentially that of Dean; however, we use the results of Theorems 7 and 9 of Dean's paper to define the quasi-order that yields these lattices. This approach adheres more closely to the "cover" approach mentioned above; covers are now ideals and dual ideals of P , rather than elements.

For didactic reasons it was thought best to divide this work into two parts--one treating FL(P) and the other treating $FL(P ; \mathcal{M}, \mathcal{N})$. Consequently all sections relating to $(\mathcal{M}, \mathcal{N})$ concepts are marked with an asterisk (*); it is recommended that the reader omit these sections at first reading.

A special case of FL(P) is the free product of lattices. Sorkin [9] discussed free products of lattices and solved the problem of which are finite sets. One of the tools he used was the fact that isotone maps--and not necessarily lattice homomorphisms--from the factors to a lattice can be extended to an isotone map from the free product to the lattice. He also presented an example showing that the free product does not always admit canonical representations.

In Chapter I of this work we present the basic material needed in the analysis of FL(P) and FL(P ; $\mathcal{M}, \mathcal{N}$) ; most of these results are well-known, although assignment of specific references can be rather difficult.

In Chapter II we present the construction of FL(P) essentially due to Dean [3a]. We characterize FL(P) and, as an application of these methods, we present a generalization of Sorkin's theorem on the extension of isotone maps. In Chapter III these results are extended to FL(P ; $\mathcal{M}, \mathcal{N}$) . We apply these results to derive the results of Dean [3] concerning CF(P) .

In Chapter IV the results of Chapters II and III are specialized to free products, amalgamated free products, and the concept we chose to call partially ordered free products. In these cases the upper and lower covers reduce to <u>elements</u> of the factors.

Chapter VI summarizes the results of this work in the context of the "word problem".

And now a word on notation: The lattice-theoretic notation is explained in the text. Set-theoretic notation is standard; we need only mention that set union and intersection are denoted by $\bigcup, \bigcap$ while $\smile, \frown$ are used for "join" and "meet" of lattice polynomials. Set difference is denoted $A - B$. The symbol $\subseteq$ is used both for the concept "subset" and for the quasi-order defined on the lattice polynomials; there will be no danger of confusion. Maps are written on the left; thus fg is g followed by f.

The theorems and definitions are numbered consecutively in each chapter. In referring to a theorem, definition, or section the chapter number is given only if that theorem, definition, or section is in a chapter other than the one in which the reference is made. Thus, for example, "Lemma 5" refers to Lemma 5 of that chapter while "Definition 3.2" refers to Definition 2 of Chapter III.

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CHAPTER I

INTRODUCTORY CONCEPTS

1. Posets and lattices.

A relational system $\langle P ; \leqslant \rangle$ with a binary relation $\leqslant$ is said to be a partially ordered set (poset) if the following three properties hold:

1) Reflexive, for all $x \in P$, $x \leqslant x$;

2) Antisymmetric, for all $x, y \in P$, if $x \leqslant y$ and $y \leqslant x$ then $x = y$;

3) Transitive, for all $x, y, z \in P$, if $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$.

The relation $\leqslant$ is said to be a partial order. The statement $x \leqslant y$ is often also written $y \geqslant x$. If $x \leqslant y$ and $x \neq y$ we write $x < y$.

If, in addition, the property

4) for all $x, y \in P$ either $x \leqslant y$ or $y \leqslant x$;

holds then the order $\leqslant$ is said to be a total order and P is said to be a chain.

A binary relation $\subseteq$ that is reflexive and transitive, but not necessarily antisymmetric, is said to be a quasi-order. Lemma 1 on p. 21 of [1] states:

1. Lemma. In any quasi-ordered set $\langle S ; \subseteq \rangle$ define $x \sim y$ when $x \subseteq y$ and $y \subseteq x$. Then:

(i) $\sim$ is an equivalence relation on S ;

(ii) if E and F are two equivalence classes for $\sim$, then $x \subseteq y$ either for no $x \in E$, $y \in F$ or for all $x \in E$, $y \in F$;

(iii) the quotient set $S/\sim$ is a poset if $E \leqslant F$ is defined to mean that $x \subseteq y$ for some $x \in E$, $y \in F$.

If $\langle P ; \leqslant \rangle$ is a poset, A a subset of P, and $x \in P$, then x is said to be an <u>upper bound</u> of A if $x \geqslant a$ for all $a \in A$. x is said to be the <u>least upper bound</u> of A, denoted by $\sup A$ (or $\sup_P A$ if the poset P is to be stressed) if

(i) x is an upper bound of A ;

(ii) if y is an upper bound of A then $x \leqslant y$.

The concepts of <u>lower bound</u> and <u>greatest lower bound</u>, denoted $\inf A$, are dual to the above. It is clear from the definition that if $\sup A$ (and dually $\inf A$) exists it is unique.

A <u>lattice</u> L is a poset in which every set consisting of a pair of elements has a least upper bound and a greatest lower bound. We denote $\sup \{x, y\}$ by $x \vee y$, "join", and $\inf \{x, y\}$ by $x \wedge y$, "meet". It is to be stressed that in this work $\vee$ and $\wedge$ are used only in a lattice, i.e. when all pairs have a sup and an inf.

A lattice L can also be thought of as an algebra with two binary operations, $\vee$ and $\wedge$, satisfying:

1) for all $x \in L$, $x \vee x = x \wedge x = x$;

2) for all $x, y \in L$, $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$;

3) for all $x, y, z \in L$, $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$;

4) for all $x, y \in L$, $x \vee (x \wedge y) = x \wedge (x \vee y) = x$.

The connection between these two approaches to a lattice is provided by:

$x \leqslant y$ is equivalent to $x \vee y = y$ which is equivalent to $x \wedge y = x$ ([1] p. 8).

A lattice L may or may not have a greatest element and a least element; if L has both a greatest and a least element it is said to be <u>bounded</u>. Any lattice L may be embedded in a bounded lattice L^b. To construct L^b from L we adjoin two symbols 0 and 1 to L ; $L^b = L \cup \{0, 1\}$. The partial order on L^b is that defined on L along with the requirement that $1 \geqslant x$ for all $x \in L^b$ and $0 \leqslant x$ for all $x \in L^b$. Thus, for all $x \in L^b$,
$1 \vee x = 1$, $1 \wedge x = x$, $0 \vee x = x$, $0 \wedge x = 0$.
We should like to point out that even if L is bounded L^b consists of two more elements than L ; the reason for this approach is to preserve a degree of effectiveness in certain constructions, because for an arbitrary lattice L there is no algorithm to decide whether or

not L is bounded.

If L is a poset such that all subsets have a sup and an inf then L is said to be a complete lattice. By the method of "completion by cuts", [1] p. 126:

2. Lemma. Any lattice L can be embedded as a sublattice of a complete lattice L*.

Finally we mention a metatheorem, the principle of duality:

Principle of duality. Any theorem about a poset remains true if $\leqslant$ is replaced by $\geqslant$ and sup and inf are interchanged.

The principle of duality will be referred to rather frequently in the sequel in order to reduce the length of proofs.

2. Ideals, homomorphisms, hereditary sets, and isotone maps.

Let $\langle P ; \leqslant \rangle$ be a poset.

3. Definition. a) A subset I of P is said to be an ideal of P if:

(i) $x \in I$ and $y \leqslant x$ imply $y \in I$;

(ii) $x, y \in I$ and $\sup \{x, y\}$ exists imply $\sup \{x, y\} \in I$.

b) A subset D of P is said to be a <u>dual ideal</u> of P if:

(i) $x \in D$ and $y \geqslant x$ imply $y \in D$;

(ii) $x, y \in D$ and $\inf\{x, y\}$ exists imply $\inf\{x, y\} \in D$.

We observe that P itself is always an ideal and a dual ideal. Also, since the empty set $\emptyset$ satisfies the conditions vacuously, it will also be considered to be an ideal and a dual ideal even if P is a lattice. For lattices this differs from the usual convention.

4. <u>Lemma</u>. If $(I_\lambda \mid \lambda \in \Lambda)$ is a family of ideals (resp. dual ideals) of P then $\bigcap(I_\lambda \mid \lambda \in \Lambda)$ is an ideal (resp. dual ideal) of P.

<u>Proof</u>: We prove the lemma for the case of ideals and invoke the principle of duality to establish the result for dual ideals.

(i) Let $x \in \bigcap(I_\lambda \mid \lambda \in \Lambda)$ and let $y \leqslant x$. For each $\lambda \in \Lambda$ $x \in I_\lambda$ and thus $y \in I_\lambda$. Consequently $y \in \bigcap(I_\lambda \mid \lambda \in \Lambda)$.

(ii) Let $x, y \in \bigcap(I_\lambda \mid \lambda \in \Lambda)$ and let $\sup\{x, y\}$ exist. Then, for each $\lambda \in \Lambda$, $x, y \in I_\lambda$ and so $\sup\{x, y\} \in I_\lambda$. Thus $\sup\{x, y\} \in \bigcap(I_\lambda \mid \lambda \in \Lambda)$.

Thus $\bigcap(I_\lambda \mid \lambda \in \Lambda)$ is an ideal and the lemma is established.

Lemma 4 implies that the ideals and dual ideals of P form closure systems (Moore families, in the terminology of [1]). Thus ([1] p. 112):

5. Lemma. The ideals (resp. dual ideals) of a poset P form a complete lattice under set inclusion . If $(I_\lambda \mid \lambda \in \Lambda)$ is a family of ideals (resp. dual ideals) then $\inf (I_\lambda \mid \lambda \in \Lambda) = \bigcap (I_\lambda \mid \lambda \in \Lambda)$ and $\sup (I_\lambda \mid \lambda \in \Lambda) = \bigcap (J \mid J$ is an ideal (dual ideal) and $\bigcup_\lambda I_\lambda \subseteq J)$.

6. Definition. a) For each $x \in P$ the set $(x] = \{y \mid y \leqslant x\}$ is said to be the principal ideal generated by x .

b) For each $x \in P$ the set $[x) = \{y \mid y \geqslant x\}$ is said to be the principal dual ideal generated by x .

To justify this definition it should be noted that:

7. Lemma. $(x]$ is an ideal of P and is the set intersection of all ideals containing x ; $[x)$ is a dual ideal of P and is the set intersection of all dual ideals containing x .

Proof: By condition a)(i) of Definition 3 every ideal of P containing x includes $(x]$.

(i) Let $y \in (x]$, i.e. $y \leqslant x$, and let $z \leqslant y$. Then $z \leqslant x$, and so $z \in (x]$.

(ii) Let $y, z \in (x]$ and let $\sup\{y, z\}$ exist. Since $y, z \leqslant x$ then $\sup\{y, z\} \leqslant x$. Thus $\sup\{y, z\} \in (x]$.

Consequently the lemma is established for $(x]$ and, by duality, the lemma also holds for $[x)$.

8. Definition. An ideal (resp. dual ideal) I is said to be a pseudo-principal ideal (resp. dual ideal) of P if it can be obtained by taking a finite sequence of binary joins and meets of principal ideals (resp. principal dual ideals).

The pseudo-principal ideals (resp. pseudo-principal dual ideals) are a sublattice of the lattice of all ideals (resp. dual ideals) and could be described as the sublattice generated by the principal ideals (resp. principal dual ideals).

The concepts of ideal and dual ideal can be weakened by requiring only that order be preserved. This leads to the concept of hereditary sets.

9. Definition. A subset I of P is said to be a hereditary subset (resp. dual hereditary subset) of P if $x \in I$, $y \leqslant x$ (resp. $y \geqslant x$) imply $y \in I$ for all $x, y \in P$.

Every ideal of P is clearly a hereditary subset, and dually. The set $(x]$ is the smallest hereditary subset containing x and $[x)$ is the smallest dual hereditary subset containing x. The families of hereditary subsets and dual hereditary subsets of P are again lattices, but in this case

$$\sup (I_\lambda | \lambda \in \Lambda) = \bigcup (I_\lambda | \lambda \in \Lambda)$$

whenever $(I_\lambda | \lambda \in \Lambda)$ is a family of hereditary (resp. dual hereditary) subsets of P. Thus the sets of hereditary and dual hereditary subsets of P are distributive lattices, and, indeed, sublattices of the lattice of all subsets of P. We recall that a distributive lattice is one where one, and hence both, of the following properties hold:

1) for all $x, y, z \in L$ $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;

2) for all $x, y, z \in L$ $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

As in the case of ideals, one can define <u>pseudo-principal</u> hereditary and dual hereditary sets. A case of interest in the sequel is when $\sup \{x, y\}$ and $\inf \{x, y\}$ exist only when x and y are <u>comparable</u>, i.e. when either $x \leqslant y$ or $y \leqslant x$. In this case the concepts of ideal and hereditary set coincide, and dually.

10. <u>Definition</u>. A mapping $f : P \to L$, where L is a lattice, is said to be a <u>homomorphism</u> if

(i) $x, y \in P$ and $\sup \{x, y\}$ exists imply

$$f(\sup \{x, y\}) = f(x) \vee f(y) ;$$

and

(ii) $x, y \in P$ and $\inf \{x, y\}$ exists imply

$$f(\inf \{x, y\}) = f(x) \wedge f(y) .$$

If P is also a lattice this concept agrees with that of "lattice homomorphism".

A weaker situation is:

11. *Definition*. A mapping $f : P \to L$, L a lattice, is said to be an *isotone map* if $x, y \in P$, $x \leqslant y$ imply $f(x) \leqslant f(y)$.

Of course, the concept of "isotone map" is meaningful even if L is a poset, not necessarily a lattice.

3. *Free lattices*.

Lattices of various degrees of freeness were discussed by Whitman [10] and [11], Dilworth [4], and Dean [3], [3a].

12. *Definition*. (Whitman [10]). The *free lattice on $\mathfrak{m}$ generators* consists of a set X of cardinality $\mathfrak{m}$, a lattice denoted $FL(\mathfrak{m})$, and a set injection $\varphi : X \to FL(\mathfrak{m})$ such that

(i) $\varphi(X)$ generates $FL(\mathfrak{m})$;

(ii) if L is a lattice and f_o is a set mapping $f_o : X \to L$ then there is a lattice homomorphism

$f : FL(\mathcal{W}) \longrightarrow L$ such that $f\varphi = f_o$.

13. *Definition* (Dilworth [4]). The *free lattice generated by a poset* P consists of a lattice denoted $FL(P)$ and an injective homomorphism $\varphi : P \to FL(P)$ such that

(i) $\varphi(P)$ generates $FL(P)$;

(ii) given any lattice L and homomorphism $f_o : P \to L$ there is a lattice homomorphism $f : FL(P) \longrightarrow L$ such that $f\varphi = f_o$.

14. *Definition* (Dilworth [4]). The *completely free lattice generated by a poset* P consists of a lattice denoted $CF(P)$ and an isotone injection $\varphi : P \to CF(P)$ such that

(i) $\varphi(P)$ generates $CF(P)$;

(ii) given a lattice L and an isotone map $f_o : P \to L$ there is a lattice homomorphism $f : CF(P) \longrightarrow L$ such that $f\varphi = f_o$.

Certain special cases of $FL(P)$ wil be discussed in the sequel. Let $(L_\lambda \mid \lambda \in \Lambda)$ be a family of mutually disjoint lattices indexed by a set Λ. Then $\bigcup(L_\lambda \mid \lambda \in \Lambda)$ can be regarded as a poset P where $\sup\{x, y\}$ and $\inf\{x, y\}$ exist if and only if $x, y \in L_\lambda$ for some $\lambda \in \Lambda$; thus $x \leqslant y$ if and only if $x, y \in L_\lambda$ for some $\lambda \in \Lambda$ and $x \leqslant y$ in that L_λ. In this case $FL(P)$ is said to be the free product of the lattices $(L_\lambda \mid \lambda \in \Lambda)$. An alternative definition is:

15. Definition. If $(L_\lambda \mid \lambda \in \Lambda)$ is an indexed family of lattices then the free product of the lattices $(L_\lambda \mid \lambda \in \Lambda)$ consists of a lattice L and an indexed family $(\varphi_\lambda \mid \lambda \in \Lambda)$ of lattice injections $\varphi_\lambda : L_\lambda \to L$ such that

(i) $\bigcup(\varphi_\lambda(L_\lambda) \mid \lambda \in \Lambda)$ generates L ;

(ii) given any lattice L' and lattice homomorphisms $f_\lambda : L_\lambda \to L'$, $\lambda \in \Lambda$, there is a lattice homomorphism $f : L \to L'$ such for each $\lambda \in \Lambda$ $f\varphi_\lambda = f_\lambda$.

The concept of free product of lattices can be extended in two directions. The first is:

16. Definition. Let $(L_\lambda \mid \lambda \in \Lambda)$ be an indexed family of lattices, let M be a lattice, and for each $\lambda \in \Lambda$ let $\psi_\lambda : M \to L_\lambda$ be a lattice injection. The amalgamated free product of the $(L_\lambda \mid \lambda \in \Lambda)$ over M consists of a lattice L and lattice injections $\varphi_\lambda : L_\lambda \to L$ such that for each $\lambda, \mu \in \Lambda$ $\varphi_\lambda \psi_\lambda = \varphi_\mu \psi_\mu$, satisfying:

(i) $\bigcup(\varphi_\lambda(L_\lambda) \mid \lambda \in \Lambda)$ generates L ;

(ii) given any lattice L' and lattice homomorphisms $f_\lambda : L_\lambda \to L'$ such that for $\lambda, \mu \in \Lambda$ $f_\lambda \psi_\lambda = f_\mu \psi_\mu$ then there is a lattice homomorphism $f : L \to L'$ such that $f\varphi_\lambda = f_\lambda$.

The concept of amalgamated free product, as well as that of free product, is quite general and properly belongs to the field of universal algebra. An alternative generalization of the concept of free product is peculiar to lattice theory. Let the indexing set Λ be a <u>poset</u>. Let $P = \bigcup(L_\lambda \mid \lambda \in \Lambda)$. A partial order $\leqslant$ is defined on P by:

(i) if $x, y \in L_\lambda$, $\lambda \in \Lambda$, then $x \leqslant y$ if and only if $x \leqslant y$ in L_λ ;

(ii) if $\lambda \neq \mu$, $x \in L_\lambda$ and $y \in L_\mu$ then $x \leqslant y$ if and only if $\lambda < \mu$.

However, if x, y are incomparable $\sup \{x, y\}$ and $\inf \{x, y\}$ wil be considered only if x and y are in the same lattice. With this restriction on sup and inf $FL(P)$ is called the <u>partially ordered free product</u> of the $(L_\lambda \mid \lambda \in \Lambda)$.

<u>Example 1.</u> Let Λ be the poset depicted in Fig. 1. Let L_c have a smallest element w. Then if $x \in L_a$, $y \in L_b$ w is the least upper bound of x and y. However, by the above convention, $\sup \{x, y\}$ will not exist.

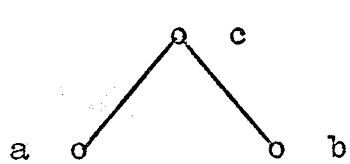

Fig. 1

An alternative approach to partially ordered free products is provided by:

17. Definition. Let Λ be a poset and $(L_\lambda \mid \lambda \in \Lambda)$ be a family of lattices indexed by Λ. The partially ordered free product of the $(L_\lambda \mid \lambda \in \Lambda)$ consists of a lattice L and lattice injections $\varphi_\lambda : L_\lambda \to L$ such that if $\lambda \leqslant \mu$ $\varphi_\lambda(x) \leqslant \varphi_\mu(y)$ for all $x \in L_\lambda$, $y \in L_\mu$, satisfying:

(i) $\bigcup(L_\lambda \mid \lambda \in \Lambda)$ generates L ;

(ii) given a lattice L' and lattice homomorphisms $f_\lambda : L_\lambda \to L'$ such that if $\lambda < \mu$ $f_\lambda(x) \leqslant f_\mu(y)$ for all $x \in L_\lambda$, $y \in L_\mu$, then there is a lattice homomorphism $f : L \to L'$ such that $f\varphi_\lambda = f_\lambda$ for all $\lambda \in \Lambda$.

*4. $(\mathcal{M},\mathcal{N})$-structures.

The concepts of $\mathcal{M}$-ideals, $\mathcal{N}$-dual ideals, $(\mathcal{M},\mathcal{N})$-morphisms, and $FL(P; \mathcal{M},\mathcal{N})$ serve both as unifying concepts and generalizations of the ideas outlined in Sections 2 and 3. (See Dean [3a].)

18. Definition. Let $\langle P ; \leqslant \rangle$ be a poset. An $(\mathcal{M},\mathcal{N})$-structure on P consists of two families $\mathcal{M}$, $\mathcal{N}$ of finite non-empty subsets of P such that

(i) $A \in \mathcal{M}$ implies that $\sup A$ exists ;

and

(ii) $A \in \mathcal{N}$ implies that $\inf A$ exists.