PHOTOELASTIC AND ANALYTICAL INVESTIGATION OF STRESSES IN DEEP BEAMS WITH OPENINGS

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SYNOPSIS

Simply supported deep beams with circular openings were investigated, both analytically and photoelastically.

The six beams studied were divided into two series. Series A consisted of three beams, loaded in pure bending, each with depth/span ratio of 2/3, and containing a circular hole with diameter/depth ratio of 1/3. However, the holes were located along the central vertical axis at different depths.

The three beams of series B also had depth/span ratios of 2/3 but were loaded in bending and shear with a point load at mid-span. Each beam had a pair of circular holes, diameter/depth ratio 1/3, centred on the vertical axes at quarter-span from either support, and along the same horizontal line. The varying parameter in this case was the distance of the line of centres of holes from the top edge.

All six beams were experimentally studied, using two-dimensional photoelasticity. The method used for separating principal stresses was the direct numerical solution of the Laplace equation for the sum of principal stresses. For an analytical solution, an original numerical method, philosophically similar to the compatibility method of structural analysis, but using finite element techniques, was developed. A relatively coarse discretization was then applied to one beam of the B series and stresses obtained, which were then compared to the experimental results. They were seen to compare favorably.

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LIST OF SYMBOLS

a = length of rectangular element

b = width of rectangular element

L = span of beam

H = depth of beam

d = flexibility coefficient

D = displacement

m,n = interface numbers

N = normal stress resultant

S = shearing stress resultant

q = intensity of distributed load

X = redundant

σ = normal stress

τ = shearing stress

ϵ = linear strain

γ = shearing strain

σ_1 = major principal stress

σ_2 = minor principal stress

f_σ = material fringe value

f'_σ = Model fringe value

σ_s = extreme fibre stress as calculated with simple beam theory

E = modulus of elasticity

G = shear modulus

1. INTRODUCTION

A deep beam may be defined as one whose depth is comparable to its span. Beams of this type, both in steel and in reinforced concrete, often arise in the construction of bins, hoppers or similar structures, as well as in more ordinary construction of foundation walls or in cases in which walls are supported on individual columns or footings. The vertical or horizontal diaphragms used to transmit wind forces in tall buildings (walls or floors) are frequently of such dimensions as to be classified as deep beams.

The problem of holes in beams, especially beams with large depth/span ratios, is not uncommon. In almost all buildings and related structures, openings of some sort are required either for access or for passing piping, ventilation or electrical systems to different compartments. Such material discontinuities cause stress concentrations in structural members and, if neglected, may endanger the otherwise well designed structure. It is, therefore, the duty of the structural engineer to carefully evaluate these stress concentrations and properly allow for them in the overall design.

As is well known, the ordinary theory of bending, the Bernoulli-Euler theory, cannot be applied to a beam whose depth is greater than about half its span. The stress distribution in deep beams has received the attention of mathematicians and research workers since the turn of the century. One common approach is by solving the bi-harmonic equation for the Airy stress function, and then obtaining the normal and shear stresses as second partial derivatives of the stress function. Unfortunately, in the case of the deep beam, there are six boundary conditions to be sarisfied (15) in the general loading case and only four constants in the bi-harmonic so that no general solution is possible. Approximate solutions, however, have been obtained for certain simple loading cases by choosing a particular solution for the bi-harmonic and then finding a polynomial which, when super-imposed on the particular solution, will satisfy boundary conditions.

Neglecting body forces, and assuming all boundary forces to be given, the stress distribution in two-dimensional problems involving simply-connected bodies is independent of the material. However, the presence of openings greatly complicates the mathematical solution, since in multiply-connected bodies, the possibility exists that strains may be multi-valued even when the stresses are single-valued [10]. In such problems, not only must the mathematical solution satisfy the conditions at the boundaries, but in addition the expressions for displacement must be examined to see that they are single-valued. The mathematical solution of deep beams with openings thus becomes a very tedious and involved, and very often impossible, task.

Classical mathematics, despite its ever-increasing sophistication, is basically capable of solving only highly idealized situations while demanding at the same time a great deal of skilled time, which could otherwise be more usefully employed in the process of design. Fortunately, with the advent of digital computers which perform arithmetical operations at an amazing speed, methods of numerical analysis have been developed which are easily adaptable to electronic computations. The finite difference and the up-coming finite element methods are fine examples of such techniques. Unlike solutions employing classical mathematics, numerical solutions encounter no difficulty in satisfying boundary conditions. Numerical methods give approximate solutions, but so do mathematical analyses which make use of series convergence. In the latter case, only a series summed to infinite number of terms will yield an exact solution. In the finite difference method, the approximation lies in the solution of the bi-harmonic, whereas in the finite element method, the approximation is in the assumption of behaviour of individual elements.

Before numerical solutions of real problems dealing with complex continua can be performed, it is necessary to limit the number of unknowns involved --- a process known as discretization. Naturally, the finer the discretization, the closer will be the solution obtained to its exact value. The digital computer, with its tremendous capacity and speed, is ideally suited to solve the large number of linear algebraic equations resulting from the numerical analysis.

Alternatively, the complicated problems of stress distribution in multiply-connected continua may be solved experimentally with the relatively simple and yet very powerful photoelastic methods. Photoelastic methods excel in yielding a whole-field representation of stress distribution and concentration.

The science of photoelasticity first appeared in the engineering world at the turn of the century, and has since become a powerful tool for stress analysts. It is applicable to any state of stress, but it can be most conveniently applied in studies of two-dimensional state of stress. The present problem may be studied as a plane-stress case if it is assumed that either the thickness of the beam is small compared with its other dimensions, or the load is applied uniformly throughout its thickness. Thus only two-dimensional photoelastic equipment and technique are required.

Photoelastic data include principal stress directions and differences in principal stresses. Any standard method of separation of principal stresses --- be it the interferometer, the lateral extensometer, the oblique incidence or the shear difference method, or the semi-analytical method of numerical solution of the Laplace equation for sums of principal stresses --- completes the analysis for stresses in a given continuum.

2. REVIEW OF LITERATURE

The departure of stresses in deep beams from straight-line distribution is well known. In 1932 Dischinger[6] used trigonometric series to determine the stresses in continuous deep beams. The Portland Cement Association[5] prepared an expanded version of Dischinger's paper and added solution for simply supported spans. Conway, Chow and Morgan[3], in 1951, proposed a method of solving problems of simply supported deep beams by satisfying lateral boundary conditions by super-imposing on the primary Airy stress function, a second stress function obtained with a strain energy method. Later in 1952, Chow, Conway and Winter[1] used finite difference equations to deal with the same problems. Many cases of loading and depth/span ratios were studied for the simply supported deep beam, but, unfortunately, it was later brought out in a discussion of the paper that a considerable amount of error existed because of the coarseness of the net and inherent rounding-off of peak values.

An exhaustive compilation of solution to problems of plates, with or without openings, loaded transversely or in their planes, with various support conditions, was prepared by Weinberg[22]. Timoshenko[20] also gave solutions to beams with circular openings, but only limited to openings that are small compared with dimensions of the beam, and a further simplification of uniform stress field was made. Muskhelishvili[13], in his book on problems in elasticity, indicated the use of conformal mapping in solving problems dealing with multiply-connected bodies. Boundary conditions, however, were again idealized. Timoshenko[20] demonstrated the application of finite difference techniques to problems of multiply-connected bodies, with special considerations to how conditions at secondary boundaries could be satisfied. The work of Savin on stress concentration around holes has been translated into English and published in the form of a monograph [17]. It is an extensive theoretical investigation of the influence of various holes in the state of stress in a non-uniform stress field in a plane, based on one important assumption --- that this influence is localized.

The difficulty of satisfying boundary conditions no longer exists with the recent development of finite element techniques, which consider the discretized continuum as an assemblage of elements whose behaviour is studied with the basic methods of structural analysis, namely the force and displacement methods. The large number of resulting linear algebraic equations are expediently solved with high speed electronic computers. Clough and Zienkiewicz [24] and [25] have made important contributions to this field, while Gaonkar [11] explores the feasibility of various numerical methods in solving problems of inclusions.

Literature on photoelastic analysis of deep beams or beams with openings are relatively scarce. Saad and Hendry [14] investigated simply supported deep beams, depth/span ratios of 2/3, 1 and 1.59, each carrying a central concentrated load. The conclusion was that the simple beam theory was adequate in the case of beams whose spans exceeded 1.5 times their depths. Gibson and Jenkins [12] photoelastically analysed a simply supported shallow beam with a circular opening at the centre of the beam and carrying a concentrated load at mid-span. Holes of different sizes were studied and it was found that the pattern of local stress concentration around the holes could not be predicted by mathematical theory.

No literature on deep beams with openings have been located, even with the help of the exhaustive Geodex Structural Information Index.

It is therefore hoped that this contribution to the study of the problem will provide useful information to designers and serve to encourage further investigation of the problem.

3. A FLEXIBILITY COEFFICIENT METHOD IN PLANE STRESS ANALYSIS

3.1 Introduction

Any method of analysis of indeterminate structures must satisfy the following conditions:

(1) Equilibrium of the structure is maintained.

(2) Stress-strain relationship of the elastic material is obeyed.

(3) Compatibility exists between different parts of the structure before fracture.

Consequently, all methods are basically similar except for the order in which the above conditions are satisfied. In particular, the flexibility coefficient method, also known as the force or compatibility method, initially assumes equilibrium of the structure. Further equations are then obtained by stipulating compatibility requirement between parts of the structure. The stress-strain relationship is implicitly obeyed in the process of determining flexibility coefficients.

The same concepts may be used in problems of stress analysis, and most conveniently in plane stress problems. When the body is discretized into a finite number of elements, it can be considered as an ordinary indeterminate structure. In this particular method, the unknowns are the normal and shear stress resultants, which are assumed to be acting at the centres of the interfaces between elements. Thus the discretized structure is statically equivalent to an assemblage of elements connected by fictitious hinges at the centres of interfaces (Fig. 3.1).

Basically, the relation between the method presented here and the usual finite element method can be compared to the relation between the force and displacement methods used in analysis of indeterminate structures.

As will be seen later, the procedure used in solution of any plane stress problem by flexibility coefficients closely parallels the steps used in setting up and solving the elastic equations of the force method.

As compared to the usual finite element technique, the flexibility coefficient method offers several advantages. Geometrical and stress conditions at the boundaries are automatically satisfied, and in most cases the number of unknowns is substantially less than in the finite element approach. One of the outstanding features of the method is its remarkable simplicity and the ease with which it can be adapted for the computer by making use of a few simple standard sub-routines.

3.2 The Primary Structure and the Calculation of Flexibility Coefficients

In the flexibility coefficient approach, any elastic body in a state of plane stress can be considered as an assemblage of mostly rectangular elements with trapezoidal or truncated-rectangular elements at curvilinear boundaries (Fig. 3.2). In the subsequent development, it is simpler to concentrate one's attention on a typical rectangular element as is shown in Fig. 3.3a.

Assuming the body to be of unit thickness, the well-known equilibrium equations for such an element can be written in a finite difference form [(21) and (7)] as

$$(\sigma_{x,m+1} - \sigma_{x,m})b + (\tau_{yx,n+1} - \tau_{yx,n})a + abX = 0 \qquad (3.1)$$

$$(\sigma_{y,n+1} - \sigma_{y,n})a + (\tau_{xy,m+1} - \tau_{xy,m})b + abY = 0 \qquad (3.2)$$

where X and Y are body stresses. As can be seen from Fig. 3.3b, these equations assume that the variations of the stresses σ_x, τ_{xy}, σ_y, τ_{yx} across the lengths of the element are linear. And by considering the

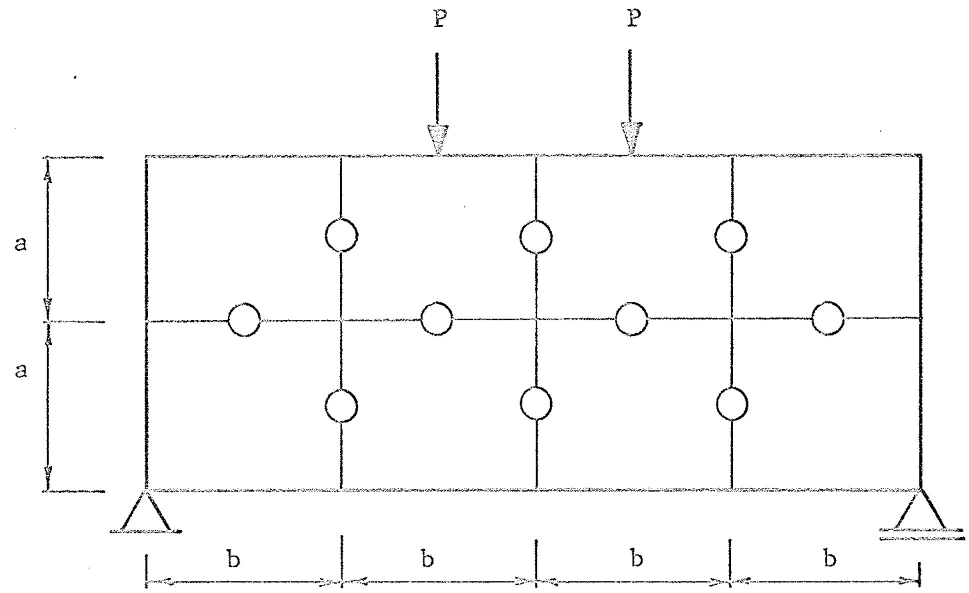

Fig. 3.1 - Discretized Structure as Assemblage of Finite Elements Connected with Fictitious Hinges at Centres of Interfaces

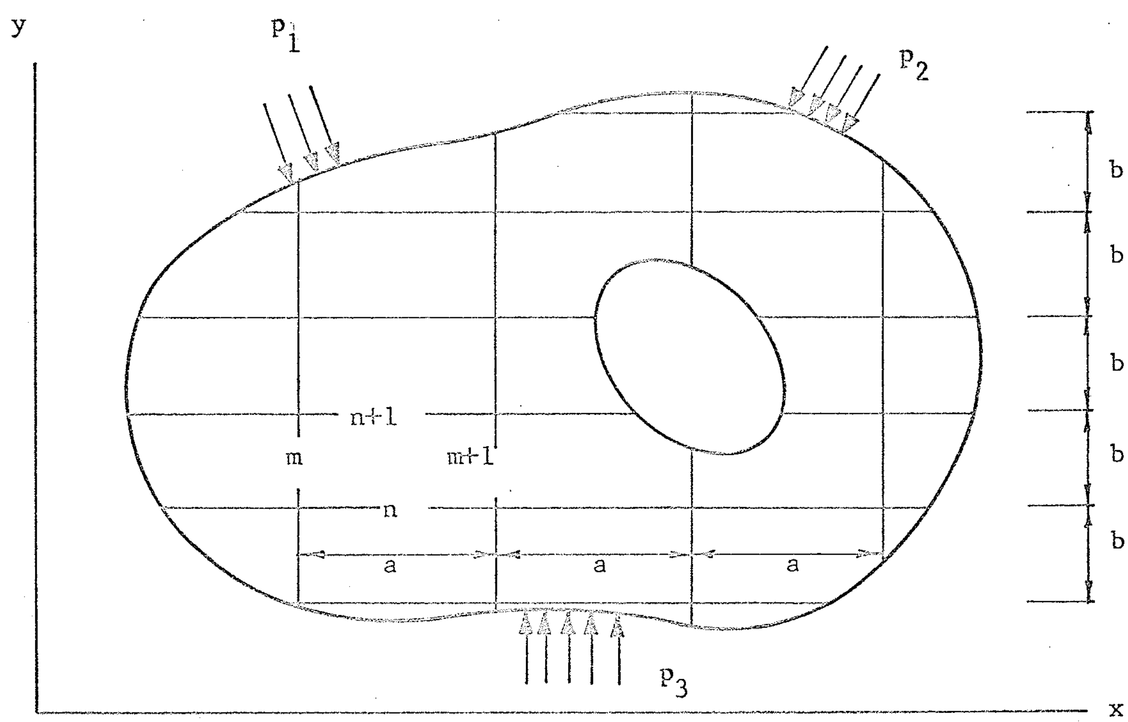

Fig. 3.2 - Discretized Doubly-Connected Continuum

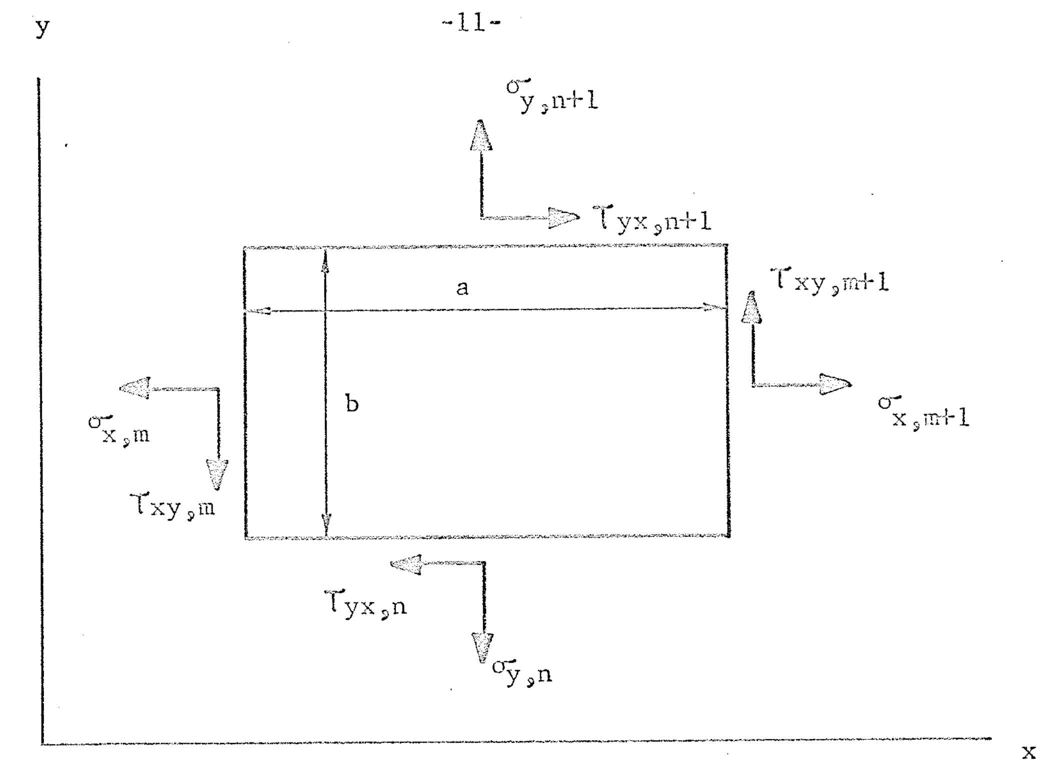

Fig. 3.3a - Stresses on Typical Rectangular Element

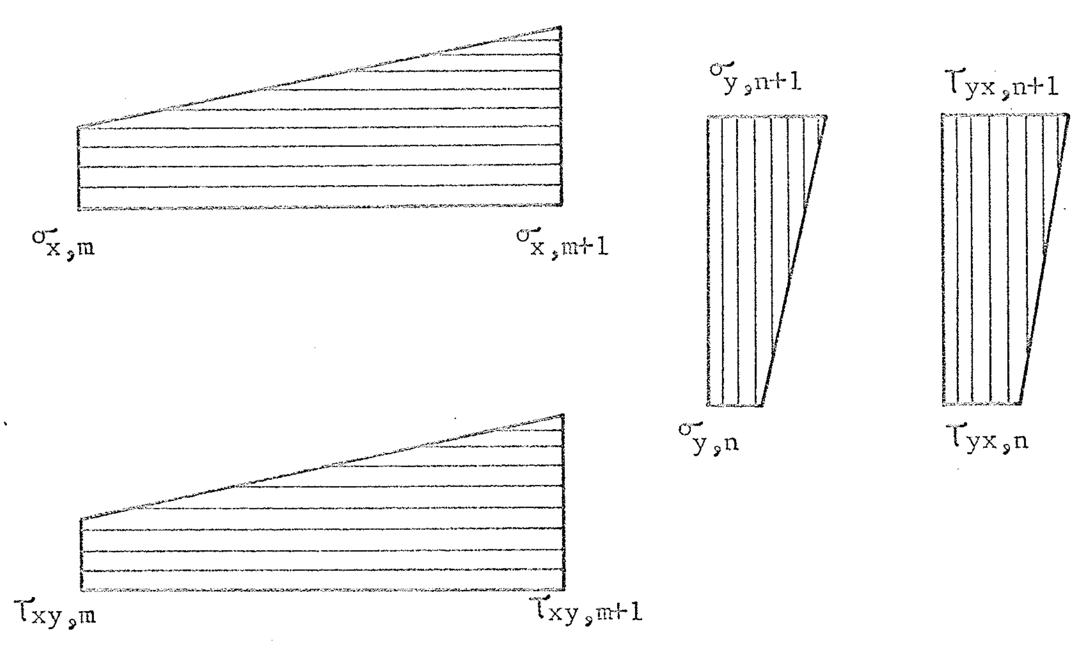

Fig. 3.3b - Typical Stress Variations

stress resultants to act at the centres of interfaces, one has inherently assumed uniform stress distribution over the interfaces between the finite elements. Further, it may be noted that since elements of finite lengths are being considered, $\tau_{xy} \neq \tau_{yx}$, since equality applies only to infinitesimally small elements.

Two such equilibrium equations can be written out for each element, so that if the body of Fig. 3.2 were divided into E elements, a total of 2E equilibrium equations would be available. At each interface such as m in Fig. 3.2, normal and shearing stresses σ and τ are acting. If F is the number of interfaces, then the number of stresses to be determined is 2F. The assemblage of E elements constituting the body is therefore indeterminate I times where

$$I = 2(F - E) \qquad (3.3)$$

Of the 2F stresses to be found, I stresses can be treated as redundants and on setting these as zero, a primary structure will be obtained which should be determinate and geometrically stable. Bearing this in mind, it is not difficult to decide on which of the 2F stresses can be treated as redundants.

Consider now the primary structure. Under the action of the applied loads, primary stresses which can be denoted by σ^o, τ^o are produced at the interfaces as shown in Fig. 3.4a. These can be evaluated by writing out equation (3.1) and (3.2) for each element and by considering the equilibrium of groups of such elements taken as rigid bodies.

Due to the deformations of the individual elements, discontinuities will be produced in the direction of the redundants at the interfaces. These deformations which can be denoted in general by D^o, are evaluated by