

UNIVERSITY OF MANITOBA

THE SIMULATION
OF
TIME-VARIABLE DELAY

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ABSTRACT

Delays often occur in dynamic systems when it is necessary to transmit information over a distance; if the properties of the transmitting medium vary, then the delay will vary and this might affect the system's behavior. A literature survey is made. This indicates that the theory of systems with a time-variable delay has not yet been developed. The best method of solving these systems is by analog computer. Several methods of simulating a time-variable delay are discussed in detail and one of these is used in constructing a variable delay unit. This device is used to study a sinusoidally varying delay. It is recognized that an important subclass of systems with a time-variable delay is composed of those systems in which the variable delay is due to the flow, at a variable rate, of an incompressible fluid through a pipe of constant volume. Problems in this subclass must be solved by quite different methods.

PREFACE

When the work for this thesis was started, the intention was to study systems which contained a time-variable delay and to see how the behavior of these systems was influenced by the variations in the delay. Analog simulation was to be used. It soon became apparent that the art of analog simulation had not yet been developed to the point where rapidly changing delays could be simulated. This realization precipitated the redefinition of the thesis topic.

Thus, this thesis is concerned primarily with the problems of simulating a variable delay. A number of techniques were investigated; their advantages and disadvantages are discussed.

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CHAPTER I

INTRODUCTION

It often happens in the study and analysis of physical systems that one must take into consideration the fact that there is at least one state of the system whose reaction to a stimulus is not instantaneous: rather, the reaction begins a finite amount of time after the stimulus has been applied. This time delay phenomenon occurs when information or energy must be transmitted over a distance (for example, a pressure wave in a pneumatic control line), in some economic problems, or in control systems involving human operators. The delays encountered are often constant, or very nearly so. In some cases, however, the delays are variable and the effect of this is noticeable in the behavior of the system. The desire to study such systems provided the motivation for this thesis.

A great many such systems may be described by an equation of the form,

$$\sum_{i=0}^n \sum_{k=0}^m a_{ik}(t) x^{(i)}(t - \tau_k(t)) = f(t), \quad (1-1)$$

which is a linear differential-difference equation with variable coefficients. In the systems usually encountered,

many of the coefficients, $a_{ik}(t)$, are constant and most of the delays, $\tau_k(t)$, are zero. Because negative delays do not occur in physical systems, the delays, $\tau_k(t)$, are restricted to have non-negative values.

The solution of differential-difference equations is difficult when the coefficients and the delays are constant. When this constancy is denied, a solution is impossible to achieve except by numerical or analog means.

It is clear that the simplest method of solving this type of problem is to use an analog technique. To do this, it is necessary to have a variable delay unit (VDU) for the analog computer to simulate the varying delay. Because such devices are not readily available, most of this thesis is devoted to a review of the work which has been done in the study and simulation of systems with a time-variable delay, and to some detailed discussions of several approaches to the simulation problem.

Chapter II contains some basic observations and distinctions which enable the reader to appreciate the significance of some of the comments in later chapters. A review of the history of the study of systems with time-variable delay is presented in Chapter III. Chapter IV presents a detailed discussion of several different approaches to the construction of a VDU.

One of these approaches to the construction of a VDU was actually used to construct a device. This particular

VDU is described in Chapter V. With the existence of an operating VDU, it was possible to give consideration to the effect of time-variable delays on linear systems.

The case of a sinusoidally varying delay with a sinusoidal input is considered in Chapter VI. Some theoretical predictions about which frequencies will appear in the output are made and verified experimentally.

The final chapter, Chapter VII, contains a review of the conclusions and suggestions for further work, of which there is much.

CHAPTER II

SOME BASIC OBSERVATIONS

Before entering the main part of the study it will be found useful to make some fundamental observations about systems with a variable delay. In the light of these observations, the full significance of some of the later remarks will be more readily apparent.

2-1 A Closely Related Problem

Although the stated purpose of this thesis is to study systems with time-variable delay, it should be borne in mind that a closely related class is the class of systems with a state-variable delay. A state-variable delay is one which does not vary in a predetermined fashion but which varies with some state (position, for example) of the system. The fundamental difference is that systems with a state-variable delay are automatically nonlinear.

In a later discussion of various methods of constructing variable delay simulators it will be useful to ascertain which simulators could be used in simulating a state-variable delay. Such a simulator would be more useful than one which could simulate only a time-variable delay.

2-2 Analysis of Time-Variable Systems Requires Caution.

Whenever a physical system is analysed mathematically, certain simplifying assumptions are usually made. The usefulness of the analysis is then limited by these assumptions. As the demands on the analysis increase, the assumptions must be changed or dropped completely. All too often this results in a very much more difficult analysis. So it is in the case of systems in which there is a delay.

In any analysis, it is possible to neglect any delays which are much less than the significant time constants of the system. When this condition does not hold, the delay must be considered in all accurate analyses.

Numerous techniques have been developed for studying systems with a constant delay. An entire book (1)* has been written about differential-difference equations, the equations which describe systems with delay. This book contains discussions of many types of differential-difference equations and many of their properties but almost nothing about the case of variable differences.

When one is dealing with any time-variable system, there is a great temptation to apply those ideas which may be used in time-invariant systems. Such carelessness will often have disastrous results.

*Numbers in brackets refer to the List of References, pg.54.

A well-known concept of linear time-invariant system theory is that of the "poles" of the system and how they can be used to describe its behavior. In the case of time-variable systems, it would seem reasonable to postulate that the poles are variable. However, this possibility has been eliminated by a theorem proved by Cruz (2) which states that if the input and output variables of a stable system are related by a linear differential equation, then the system function cannot have time-varying singularities. This theorem notwithstanding, Zadeh (3) has pointed out that if the system is varying slowly, the concept of a moving pole is useful in approximate analyses.

The above comments, which apply to all time-variable systems, certainly apply to systems with a time-variable delay. The solution of these problems requires specialized techniques but, because there has been no great need to study this type of problem, there has been no concerted effort to develop these techniques.

In many cases where there is a variable delay, it is quite possible to ignore it completely or at least partially. If the overall variation in the delay is small compared with its mean value, then the variation will likely have no noticeable effect on the behavior of the system and can be ignored completely. If the variation in the delay is slow, then Zadeh's comment about moving poles prevails and it is possible to obtain the solution

to the problem in a piecewise manner by choosing a different constant value for the delay in each interval of time.

The duration of the variation in the delay is also an important factor in determining whether the variation needs to be considered in the analysis. If the variation is short-lived, then the delay will be constant for most of the time and, except when the delay is varying, conventional methods of analysis may be used. If the variation continues for some time, then it must be taken into consideration in the analysis of the system.

2-3 The Pipe Problem

During the course of the study of the present problem, it became quite clear that in the class of systems with a variable delay there is an important subclass which in many respects is easier to solve.

Representative of this subclass is the case where a variable delay is caused by the flow of an incompressible fluid through a pipe of constant volume. When the rate of fluid flow varies, the delay varies. This subclass includes all analogous situations in which there is a constant amount of "fluid" (or information) in transit at all times but does not include such cases as that in which the delay varies because of variations in the distance through which the information must travel. This distinction is very important.

Because of the importance of this particular subclass of problems, it has been designated the pipe problem.

CHAPTER III

HISTORY

3-1 Early Studies

The earliest work appears to have been done by Polosukhina in 1910 (4). She considered equations of the form

$$y'(t) + M(t) y(\alpha(t)) = f(t) \quad (3-1)$$
$$a \leq t \leq b; \quad a \leq \alpha(t) \leq b$$

and also analogous equations of higher order by reducing them to integral equations and applying the theory of Fredholm. She succeeded in proving the existence and uniqueness of a solution under the assumption that $\alpha(t)$ is less than or equal to t which ensures that the system is not a predicting system.

Tinbergen (5) did some work in 1935 on the quantitative analysis of business cycles and recognized the fact that the lags in his mathematical model of the economy were not necessarily constant. However, he did not offer any solution to the problem in the case where the delay varied.

Between 1949 and 1952, the Russian mathematician Myskis published a number of papers dealing with differential-difference equations with varying differences (6-11). These papers deal with such questions as the existence of solu-

tions, and such properties of those solutions as whether they are damped or oscillatory. He has considered both first and second order equations. Unfortunately, only one of these papers (6) has been translated from Russian into English. English abstracts (see List of References) of all of the papers are available but they are not sufficient to give a clear indication of the value, and relevance, of Myskis' work to the present investigation.

3-2 A Graphical Solution

Yeh and Kim (12) have published details of a graphical method of solving equations of the form

$$\dot{x}(t) = f[x(t), x(t-\tau(t))]. \quad (3-2)$$

It is a modification of a method first used by Cunningham (13) who treated the case of constant delay. The method employs a construction on the $x(t) - \dot{x}(t)$ plane and leads to a solution similar to the phase-plane solution used for second-order ordinary differential equations. A severe limitation of the method is that the equations must be of the first order.

3-3 Systems With Discrete Values of Delay

Most of the work on systems with variable delays has considered only continuous variations in the delay. There are cases, however, in which the delay varies discontinuously and assumes only certain discrete values.

Litsyn (14) and Teverovskii (15) have both done work on systems where the delay has two discrete values. This situation can arise in systems using relays. There is a delay in the operation of the relay but it is usually not the same on closing and releasing. Because of the relay in the system, it is nonlinear. Furthermore, because the delay depends upon the input to the relay, it is state-variable rather than time-variable. Because of the nonlinearity of the system, oscillations might be expected. These oscillations are the subject of the papers by Litsyn and Teverovskii.

3-4 Some Mathematical Techniques

There is a great need for a mathematical technique capable of providing a solution with reasonable accuracy. None of the useful methods for solving time-invariant systems can be applied to the problem of a variable delay. Either a new approach or a novel use of existing techniques will probably have to be used. Mitropol'skiy* has suggested that an extension of the integral manifold method to systems of differential equations with a slowly varying delay is possible but he has not developed the idea.

Although a number of people have given consideration to the problem, the only product seems to have been the

*See Ref. (16), page 14.

development of a number of algorithms to facilitate the numerical solution of problems involving a variable delay (17, 18). These algorithms are necessary to reduce the errors inherent in other less accurate methods involving, for instance, storage of previous values and interpolation.

An accurate algorithm (17), which requires that the delay be such that $t-\tau(t)$ is monotonic increasing, determines the value of the system variable on successive intervals in time: $t_N \leq t \leq t_{N+1}$. The interstitial numbers t_1, t_2, \dots are determined from the recursion relation:

$$t_{N+1} - \tau(t_{N+1}) = t_N \quad (3-3)$$

where t_0 is defined to be zero.

While this method is accurate, it is very laborious and is to be avoided if possible. If a solution is desired on the interval $(0, t_{k+1})$, $\frac{(k+1)(k+2)}{2}$ equations must be integrated. It is easily seen that if k is at all large, the solution will require a great amount of computer time.

A paper by Ball (19) describes a variable time-delay subroutine which has been developed for some of the digital simulation codes used for nuclear reactor analysis and control problems. The subroutine is written in FORTRAN. It is designed for use only in the solution of a pipe problem. The technique used is to sample the input quantity and the flow rate and then to keep track of each sample of fluid as it flows through the pipe.

It is sometimes possible to perform a transformation on a differential equation with a variable delay to convert it to a differential equation with varying coefficients but a constant delay. This approach to solving such equations was first suggested by Bellman and Cooke (18). This transformation is discussed in some detail in Appendix A where it is pointed out that this method is only useful when the problem being solved is a pipe problem.

3-5 Approximate Simulators

Many of the variable delay simulators which are described in the literature (20-24) are only approximate devices which may be realized readily using the operational amplifiers and multipliers of an analog computer. In all of these cases, the parametric transfer function of the variable delay unit, $e^{-s\tau(t)}$, is approximated by a function which may be realized on an analog computer.

Constant delays are often simulated in a similar manner but as has been mentioned in the literature (25, 26), serious errors may result if the techniques for reproducing a constant delay are applied mechanically to the simulation of a variable delay.

The parametric transfer function, as in the case of constant delay, can be approximated by the first few terms of the Padé series. For example, the second order approximation is

$$e^{-s\tau(t)} \approx \frac{s^2\tau^2(t) - 6s\tau(t) + 12}{s^2\tau^2(t) + 6s\tau(t) + 12} \quad (3-4)$$

The transfer function obtained in this manner will have two variable poles. It is very difficult to reproduce such a transfer function on an analog computer since, according to Cruz (2), equations whose transfer functions have the form of that in Equ. (3-4) do not exist among the class of linear differential equations with variable parameters.

One method of realizing this transfer function approximately is to simulate a time-variable parameter equation in which the operators on the right- and left-hand sides correspond, respectively, to the numerator and denominator of the Padé approximand. Zadeh (3) has developed a method of synthesizing this transfer function. The error between the desired and the realized transfer functions increases with an increase in the rate of change of delay so this type of approximation is suitable only when the delay varies slowly.

It would be possible to obtain an approximation to $e^{-s\tau(t)}$ by expanding it in a Taylor series. Although this type of approximation would have constant poles, it has the serious restriction that, because of its slow convergence, it would require considerable circuitry for its realization.

Chernyshev (20) has proposed the construction of a series which combines some of the properties of the Padé series with some of those of the Taylor series. He suggests that the parametric transfer function be represented in the form

$$e^{-s\tau(t)} = [e^{-s\tau_0}]^{a(t)} \quad (3-5)$$

where τ_0 is a constant and $a(t)$ is defined by the equation

$$s\tau(t) = s\tau_0 a(t). \quad (3-6)$$

$e^{-s\tau_0}$ is the transfer function of a constant delay unit and may be represented by a Padé approximation. Equation (3-5) may then be expanded as a binomial series which converges under appropriate conditions. The resulting series can be realized on an analog computer.

There are two sources of error in this method: the Padé expansion of $e^{-s\tau_0}$ is only approximate, and the binomial series must be truncated when it is being realized. One advantage, however, is that, unlike the previously discussed approximation, there is no restriction upon the rate of change of the delay.

The fundamental difficulty encountered in all of these approximate delay simulators is that they are all low-pass devices. In the case of constant delay simulators, it is possible to ensure a certain level of accuracy as long as the product, $w\tau$, is restricted sufficiently. The same

idea will hold for variable delays although the condition will not be as simply defined because of the variation in τ and the resulting variation in w . This variation in w is discussed in Appendix B.

In many cases the restriction on frequency would not cause any undue difficulty. It is necessary that the analyst decide whether one of these approximate variable delay simulators will be suitable for the particular problem he is studying. If he is not sure how his system is going to operate, or if he expects that it will be necessary that the variable delay unit transmit high frequency signals, he would be well advised to try a more exact approach to the simulation problem.

Margolis and O'Donnell (27) have constructed a variable delay simulator for use in studies of the pipe problem. Their approach was to simulate a transfer function which was approximately equal to that of an ideal variable delay unit. Before this was done, the equation was transformed so that the independent variable was no longer time, t , but rather total flow, q . In the transformed system, the "delay" is constant and equal to the volume of the pipe, Q . A delay simulator with a constant delay, Q , may be simulated using an adaptation of a standard Padé approximation.

However, in this synthesis of a Padé approximation, the integrators must integrate with respect to q rather than t . Because the pipe problem is being considered, this

synthesis is easily accomplished. It is shown in Appendix A that the rate of fluid flow, $V(t)$, is actually equal to dq/dt . Thus by preceding each standard integrator by a multiplier which multiplies the input by $V(t)$, it is possible to obtain an integrator which integrates with respect to q . This is illustrated in Fig. 3-1.

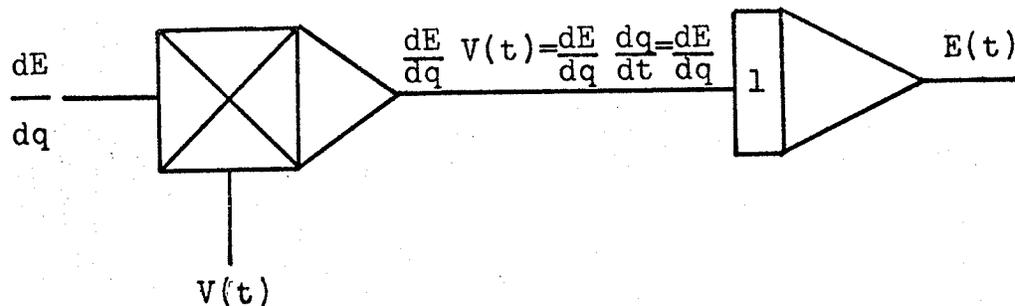


Figure 3-1. Integration With Respect to q

Because this simulator includes a Padé approximation, its accuracy is limited. However, there is no restriction on the rate of change of the delay.

3-6 Special Purpose Simulators

Although most of the published work on variable delay simulators has employed elements readily available on analog computers, some special purpose devices have been constructed.

Doganovskii and Ivanov (28) have written a book which they claim "can . . . be used as a manual on controlled-delay devices." * They discuss the operation and construc-

* See Ref. (28), page ix.

tion of a number of delay simulators which are based upon several different principles. Although not mentioned, it is a fact that in all of these simulators, unless special precautions are taken, the delay must change only slowly if they are to operate accurately. When the rate of change of the delay is appreciable, it is incorrect to control the delay in the obvious manner. Special approaches must be used but no mention of them is made in this book. A more thorough discussion of the problems of controlling the delay will be given in Chapter IV.

Robb (29), in the simulation of a radar tracking system, used a magnetic tape recorder. The recorder had two capstan drives: the first, at the recording head, had a constant speed while the second, at the reading head, had a variable speed. There was provision for tape storage between the two capstans. The author stated that the operation was quite satisfactory for the system being simulated. It should be pointed out, however, that the rate of change in the delay was quite small.

Walker (30) has constructed a digital variable delay unit which contains a 256-word magnetic core memory. The input is sampled, converted to digital form, and stored in the memory. The storage takes place in a circulating manner--that is, after word 255 is filled, the next point of data is stored in word 000. Before a new data point is stored, the existing data point is read from the memory,

converted to analog form, and appears at the output. The sampling rate is determined by an oscillator the frequency of which is controlled by a control voltage which can vary between 0 and 10 volts.

Although not mentioned in the previously cited paper, it is apparent (31) that Walker's unit is intended for the simulation of one particular class of problem--the pipe problem--because the operation of this type of simulator is completely analogous to the physical situation being simulated. It would, in principle, be possible to use this simulator for problems which do not fall into the class of the pipe problem, but the determination of the necessary control signal would be very difficult. Walker does not even mention this problem but it is explored in some detail in Chapter IV of this thesis.

Von Perry (32) has built a device which is similar in principle to Walker's but in this case, the memory consists of a dual-gun cathode ray storage tube in which the data is stored in a pulse-width modulated form. Von Perry claims that his system "offers a capability of providing a time delay which varies as the computer runs" * but gives no idea about how he would achieve this accurately. As is pointed out in Chapter IV of this thesis, conventional modulation techniques such as pulse width modulation are

*See Ref. (32), page 4.

not accurate when used in variable delay simulators. Apart from this, this system is similar to Walker's and the same comments apply.

3-7 Optimal Control

Once the behavior of a system is known, a question which often arises is: Can it be controlled and, if so, how well can it be controlled?

Ragg has published a paper (33) in which he states necessary conditions for the optimal control of systems with varying delays. The systems which he considers are very general: they may be nonlinear, have a multivariable control and contain a number of different delays.

The conditions for optimal control which he derives are the Euler-Lagrange equations, the first corner condition, and the transversality condition of the calculus of variations.

CHAPTER IV

CONSIDERATIONS OF SEVERAL SCHEMES FOR CONSTRUCTING VDU'S

Several methods are currently being used for the simulation of constant delays. It would seem very logical to make appropriate modifications to these existing systems to permit them to operate as variable delay simulators. This is, in fact, done but it will be shown below that this is not, in most cases, as easy as it would first seem.

4-1 Requirements

A major drawback of most of the schemes used to construct variable delay simulators is that they are approximations. They may well give accurate results but it might be difficult for the experimenter to decide how accurate they are.

The behavior of systems with time-variable delay has not been widely studied, so in most cases one does not know what to expect. Not knowing what to expect, it is hard to select an approximation which is "good enough." The solution to this dilemma is to make the simulator as good as possible.

It is important that the VDU be able to simulate a rapidly changing delay because it is this case which is really the most significant. If the delay is changing

slowly, then the behavior of the system will differ only slightly from that of a system with constant delay.

Furthermore, it is important that the VDU have a sufficiently wide bandwidth to transmit all of the significant frequencies. Because of these requirements, those VDU's which are based upon some approximation of the parametric transfer function $e^{-s\tau(t)}$ are deemed insufficiently accurate and will not be considered further.

4-2 Moving Paper Chart

One method of producing a constant delay is to use a strip chart recorder and a line follower. The input function is recorded by a pen on the paper chart as the chart moves past the pen at a constant rate. After the paper has passed the pen, it passes a line follower which converts the position of the line on the paper to an analog voltage. By adjusting the distance, L , between the pen and the line follower or by adjusting the speed, v , of the paper, it is possible to get any delay, τ , desired.

These quantities are related, in the case of a constant delay, by the equation

$$\tau = \frac{L}{v} . \quad (4-1)$$

It is clear that a variable delay simulator could be realized by any one (or combination) of the following actions:

- i) the position of the pen could be varied.
- ii) the position of the line follower could be varied.
- iii) the velocity of the chart could be varied.

What may not be immediately obvious is that Equ. (4-1) does not necessarily describe the situation even when the time-dependence of τ and either L or v is included. That is, the equation

$$\tau(t) = \frac{L(t)}{v} \quad (4-2)$$

does not apply in all cases. In fact, it describes only case ii above. The correct equations which describe cases i and iii are more complex and, consequently, it is a major problem to control the delay in the desired manner.

While all three cases are not of equal importance for the moving paper delay simulator, the problems encountered are interesting. Furthermore, the analyses of some other variable delay simulators are quite analogous to one or another of the above cases so that when these three cases have been analysed, the operation of other simulators may be described by referring to the appropriate case here.

The reason that Equ. (4-2) cannot be used to describe the case where the pen position is varied is that the delay at time t is determined by the distance the paper had to move from the pen to the line follower. Although

L is a function of time, t , this distance is not $L(t)$, but rather $L(t-\tau(t))$. Thus, rather than Equ. (4-2) the correct relationship is

$$\tau(t) = \frac{L(t-\tau(t))}{v} . \quad (4-3)$$

It can be seen that, although L is a function of time, its value at a particular time, t , does not determine the value of the delay at that time.

$\tau(t)$ is the desired time-variable delay. The problem now is to determine $L(t)$ such that Equ. (4-3) holds.

It will be helpful later if the independent variable, t , in Equ. (4-3) is replaced by the symbol z . Now, Equ. (4-3) becomes

$$\tau(z) = \frac{L(z-\tau(z))}{v} . \quad (4-4)$$

It is convenient to introduce the variable, t , which is now defined by the equation

$$t = z - \tau(z) . \quad (4-5)$$

If t is monotonic increasing with z , then there exists a single-valued inverse function which gives z in terms of t :

$$z = f(t) . \quad (4-6)$$

The definitions of Eqs. (4-5) and (4-6) may be used in Equ. (4-2) which, when it is multiplied by v , becomes

$$L(t) = v\tau(f(t)) . \quad (4-7)$$

The variable, t , above must represent time because the function L was defined to be a function of time.

Equation (4-7) gives the separation, $L(t)$, between the pen and the pick-up head as an explicit function of time.

It has been seen that $t-\tau(t)$ must be monotonic. If $t-\tau(t)$ were monotonic nonincreasing, the delay would be increasing faster than, or at least as fast as, time and nothing that entered the delay simulator would ever come out. This is obviously not an interesting case and so may be eliminated. It is concluded that $t-\tau(t)$ must be monotonic increasing. This places the restriction on the rate of change of $\tau(t)$:

$$\tau'(t) < 1 . \quad (4-8)$$

The physical reason for this restriction is that each bit of history may be read only once with a simulator of this particular type. The delayed function can be read only while it is under the line follower. Each point on the paper passes the line follower once.

In practical situations, the limit on $\tau'(t)$ might be somewhat less than that expressed in (4-8). In such cases, the limit would be imposed by limitation on the rate of

change in the distance between the pen and the line follower. This, of course, is a property of the equipment being used.

While this is a restriction, it is not nearly as restrictive as that encountered in the usual method of approximating $e^{-s\tau(t)}$. In one example in their article, Kogan and Chernyshev (22) restrict the magnitude of $\tau'(t)$ to be less than 0.05 in order to achieve acceptable operation.

The major drawback of this method is the difficulty in determining $L(t)$ from Equ. (4-7). The solution for $L(t)$ requires a priori knowledge of $\tau(t)$ and, consequently, this method could not be used for the simulation of a state-variable delay.

If the delay were to be varied by controlling the position of the line follower, then Equ. (4-2) is correct as it is written. The delay at any particular time, t , is dependent only on the position of the line follower at that time. It does not depend upon the line follower position at past or future times.

There are no fundamental restrictions in this method. Any errors would arise through inaccuracies in the equipment. This approach, through its straight-forward operation, has much to recommend it. The fact that the delay is determined only by the instantaneous position of the line follower means that this approach could be used in a state-variable delay simulator.

The third possibility, that of controlling the delay by controlling the velocity with which the chart moves from the pen to the line follower, must be considered for two different cases: the pipe problem, and all other problems.

A moment's reflection will show that the simulator with a variable speed chart is exactly analogous to the pipe problem. The distance, L , between the pen and the line follower corresponds to the volume, Q , of the pipe; the velocity of the paper chart, $v(t)$, corresponds to the rate of flow, $V(t)$, through the pipe. Once this analogy is recognized, it is very easy to control the variable delay simulator--the chart velocity is made proportional to the rate of flow.

This type of simulator can be used readily in the simulation of problems with a state-variable delay as long as they are of the pipe problem type.

If the problem being simulated is not a pipe problem, it must be converted to one by determining the chart velocity, $v(t)$. It is the delay, $\tau(t)$, which is known, rather than the flow rate. The delay and the chart velocity are related by the equation

$$L = \int_{t-\tau(t)}^t v(x) dx \quad (4-9)$$

in which x is the variable of integration. The difficulties

which would be encountered in the solution of Equ. (4-9) for $v(t)$ are such as to be prohibitive.

It is clear that a simulator with a variable speed chart is very easily used in simulating a pipe problem but cannot readily be used to simulate a general time-variable delay.

4-3 Magnetic Tape Systems

Constant delay simulators are often built using a magnetic tape on which is recorded the input signal for subsequent playback after the desired delay. In order to construct a variable delay, it would seem like a very logical adaptation of this if the tape speed or the position of the reading or writing heads were variable. The same effect as moving the heads could be achieved by using a tape device with two independently controlled capstans and provision for some tape storage between them.

The three cases possible here are identical with those discussed earlier in the discussion of variable delay simulators using a moving paper chart. The same advantages and disadvantages discussed there apply in this case.

In addition to the difficulties already mentioned, magnetic tape systems are beset with problems which arise due to the very nature of the recording process. Analog signals cannot usually be recorded directly on magnetic tape; instead, they are used to modulate a high frequency

carrier in some manner and the resultant modulated carrier is recorded on the tape.

All of the usual modulating schemes, such as amplitude, frequency, pulse code, pulse width, or pulse position modulation, have an unfortunate characteristic. For freedom from amplitude distortion, the relative velocity between the tape and the read head must equal the relative velocity between the tape and the recording head when the tape was recorded. It is clear that this requirement cannot be met with a variable delay simulator.

In some cases, it might, in principle, be possible to correct for this distortion. Every combination of modulation method and driving technique will have its own unique problems. There is no point in investigating all possible combinations, but one of the simpler cases will be examined to illustrate a possible approach to the problem.

Consider, for example, amplitude modulation of the signal when the delay is controlled by varying the position of the reading head.

Let v_w , a constant, denote the relative velocity between the tape and the writing head and $v_r(t)$, a variable, denote that between the tape and the read head. Equation (4-2) correctly describes the situation and can be used to express the delay, $\tau(t)$, in terms of the two velocities:

$$\tau(t) = \frac{v_w t - \int_0^t v_r(x) dx + L_0}{v_w} \quad (4-10)$$

where L_0 , a constant, is equal to the length of tape between the two heads at $t = 0$. Differentiation of Equ. (4-10) with respect to time yields

$$\tau'(t) = 1 - \frac{v_r(t)}{v_w} \quad (4-11)$$

When the tape with the signal recorded on it moves past the read head, an e.m.f. is induced in this head. The magnitude of the e.m.f. depends upon the rate of change of the magnetization on the section of the tape immediately beneath the head. If the recording is made at a velocity v_w and is read at a velocity $v_r(t)$, then the induced e.m.f. will be in error by the factor $\frac{v_r(t)}{v_w}$.

A rearrangement of Equ. (4-11) yields

$$\frac{v_r(t)}{v_w} = 1 - \tau'(t) \quad (4-12)$$

This means that the signal is in error by the factor $1 - \tau'(t)$. It is clear that the error is small when the rate of change in delay is small. However, a requirement of a variable delay simulator is that it be able to handle the simulation of rapidly changing delays. It is obvious that the correct signal could be obtained by taking the

output of the demodulator and dividing it by $1 - \mathcal{V}(t)$. This could be done by analog techniques.

The error brought about by the varying relative velocity between the tape and the heads is probably easiest to correct in this case. However, amplitude modulation is inherently less accurate than other methods of modulation. This would mean that for an accurate simulator, some other modulating technique would have to be used.

Other conventional modulation techniques would have similar problems but the solution would probably not be so easy to find. It would be possible to design a recording technique which was not subject to this difficulty, but the effort is not warranted. It is unlikely that many variable delay simulators will be built using magnetic tape as the storage medium. Other approaches would be more easily realized.

4-4 Sample-And-Store Systems

Another technique of delay simulation is to sample the input voltage at discrete points in time, store the sampled signal in some sort of memory and subsequently read out these discrete values. The memory elements are set up so that all of them contain information. As each new sample is read into the memory, it takes the place of the sample which was stored at a time \mathcal{T} previously and which has just been read out at the output.

The storage may be one of two types. The signal may be stored in analog form such as on a charged capacitor or it may be stored in a coded form such as used by Walker and Von Perry* in their delay simulators. It is possible to construct a simulator with digital storage on a hybrid computer.

The delay is controlled by the sampling rate of the system--as the sampling rate increases, the delay decreases. For the case of a constant delay, the delay, τ , is related to the number of memory elements, N , and the sampling frequency, f , by the equation

$$\tau = \frac{N}{f} \quad (4-13)$$

Unfortunately, Equ. (4-13) does not hold when the delay is varying. The reason for the failure of Equ. (4-13) is immediately apparent when it is realized that the operation of this type of simulator is quite analogous to the operation of the simulator which employs a paper chart with a variable velocity. It was pointed out earlier in this chapter that this type of simulator is ideal for the simulation of a pipe problem but poorly suited for any other problem.

*See the description of these devices in Chapter III.

CHAPTER V

THE CONSTRUCTION OF A PARTICULAR VDU

In this study of systems with a time-variable delay, the objective was to study such systems in general and to make whatever observations were possible. Because of the scarcity of information about these systems and the lack of theoretical techniques for their analysis, the method of analog simulation had to be used in the study.

5-1 The Requirements

It was desired that the VDU be as accurate as possible because it was not really known what specific properties were necessary for the achievement of meaningful results. This requirement eliminated the possibility of using a Padé approximation or some similar approximate technique. Furthermore, because this was to be a preliminary study and it was not known whether the project would develop into a lengthy study, it was desired that the simulator be as flexible as possible so as not to require the expenditure of a large sum of money for equipment which might be used only briefly.

5-2 The Simulator

The simulation method used met these requirements well. It was a moving paper chart device with a movable pen.

Although the control of a movable pen system is more difficult than that of other systems, it was used because it could be assembled entirely from standard equipment.

All components of the system are standard devices manufactured by the Hewlett-Packard Company. They are a 2D-2A X-Y recorder, a 17007A chart drive and a 7500A line follower. Figure 5-1 shows the assembled VDU.

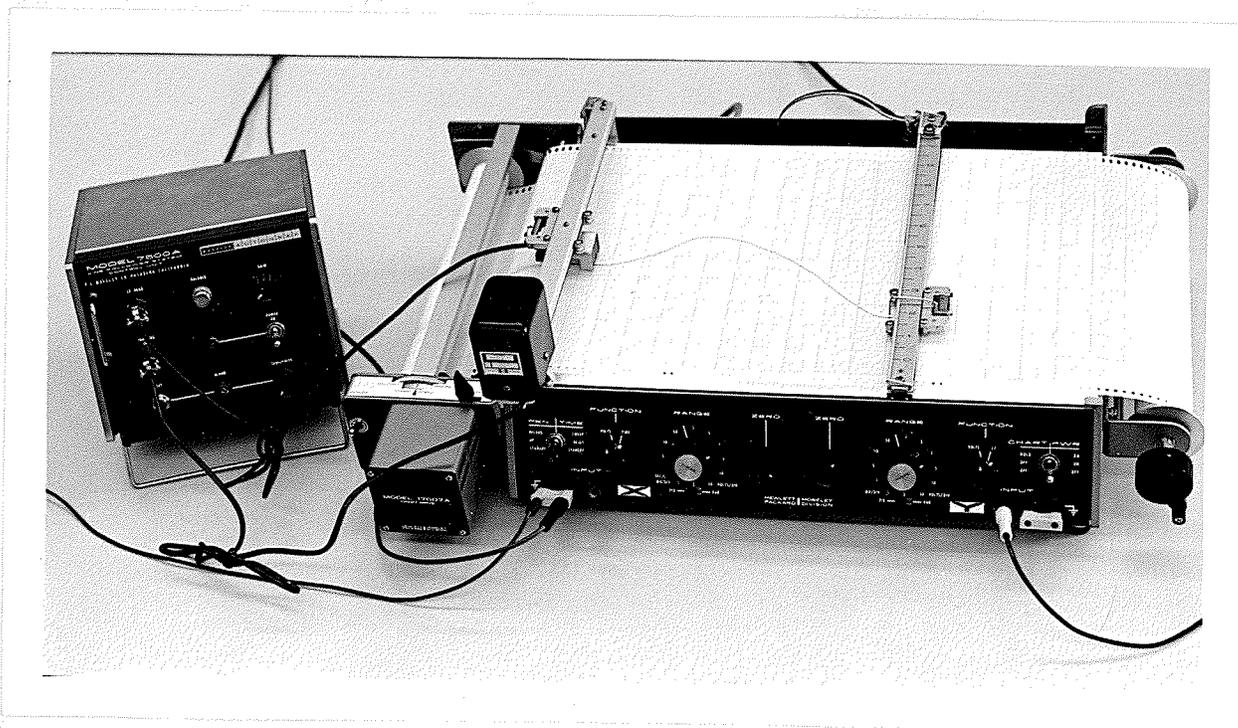


Figure 5-1. A Moving Paper Chart VDU With Variable Pen Position. (The line follower control is at the left.)

The chart drive is mounted on the left end of the X-Y recorder with the paper supply at the right end so that the paper chart is thus driven from right to left. The line follower is mounted at the extreme left of the recorder table.

The minimum delay is determined by the minimum possible separation between the pen and the line follower head. It was found that by reversing the line follower (i.e. installing it with the motor at the lower edge of the table rather than at the upper edge) it was possible to reduce the minimum delay obtainable. This change was readily accomplished by removing the stop-arm from the line follower assembly and two trim strips from the recorder.

The input signal, $e_i(t)$, was used to drive the pen in the Y-direction in the usual manner.

The delay was controlled by moving the pen in the X-direction. The manner in which the pen moved was determined by the solution of Equ. (4-7) which solution could be obtained in either of two ways. Equation (4-7) could be solved numerically on a digital computer, using the program in Appendix E, and then a function generator could be set to provide the correct voltage for the control of the delay. This method is inconvenient in that, whenever $\mathcal{Z}(t)$ is changed, then a new digital solution must be obtained and the function generator must be reset. A more satisfactory alternative is to solve Equ. (4-7) on an analog computer and thus generate the control voltage directly. This approach is described in Appendix C.

If the Y-direction attenuator on the recorder was set to k inches/volt, then it was found that the output from the line follower was a voltage equal to $K-0.55 ke_i(t-\mathcal{Z}(t))$

where K is a constant depending upon the choice of origin. This constant voltage had to be subtracted from the output before the system operated correctly as a VDU. This subtraction was performed by amplifier 9 in Fig. 5-2. Potentiometer 9 could be adjusted so that the output of the VDU had the correct value at the outset of each simulation.

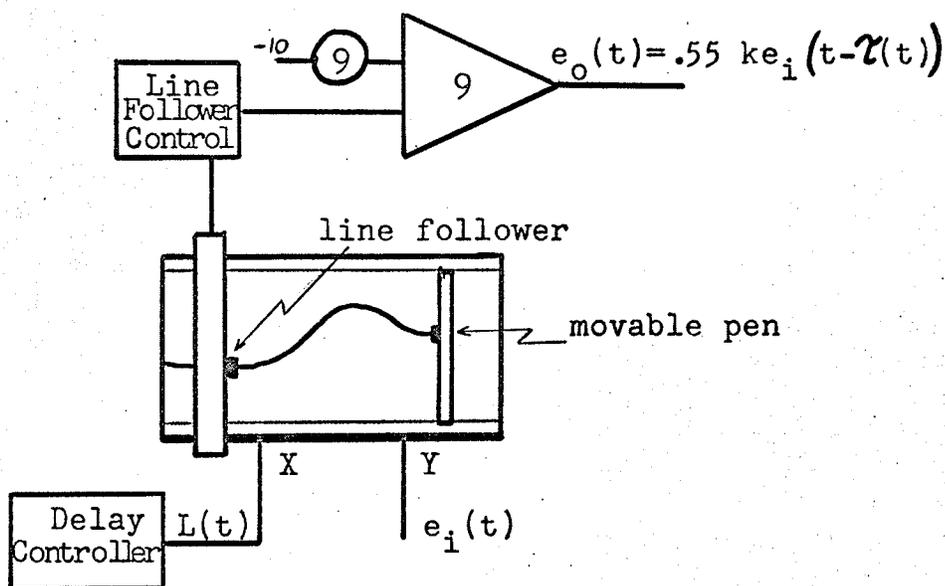


Figure 5-2. The Complete VDU

5-3 Test Of The VDU

In order to check the theory, as developed in Chapter IV, and also the operation of this particular type of VDU, it was tested by determining how well it simulated the delay,

$$\tau(t) = 10(1 + e^{-.1t}) . \quad (5-1)$$

For this test, the first method of controlling the delay was used. Equation (4-7) was solved for this particular delay and the results were used to set a function generator, the output of which drove the pen in the X-direction.

Measurement of the delay of the VDU was accomplished by using a ramp input signal. The output of the VDU, amplified to remove the factor $0.55k$, was then subtracted from the input signal. The difference was

$$t - (t - \tau(t)) = \tau(t) . \quad (5-2)$$

The results of this test are shown in Fig. 5-3. Because the initial function stored in the VDU was zero, rather than a ramp, the difference in Equ. (5-2) will only

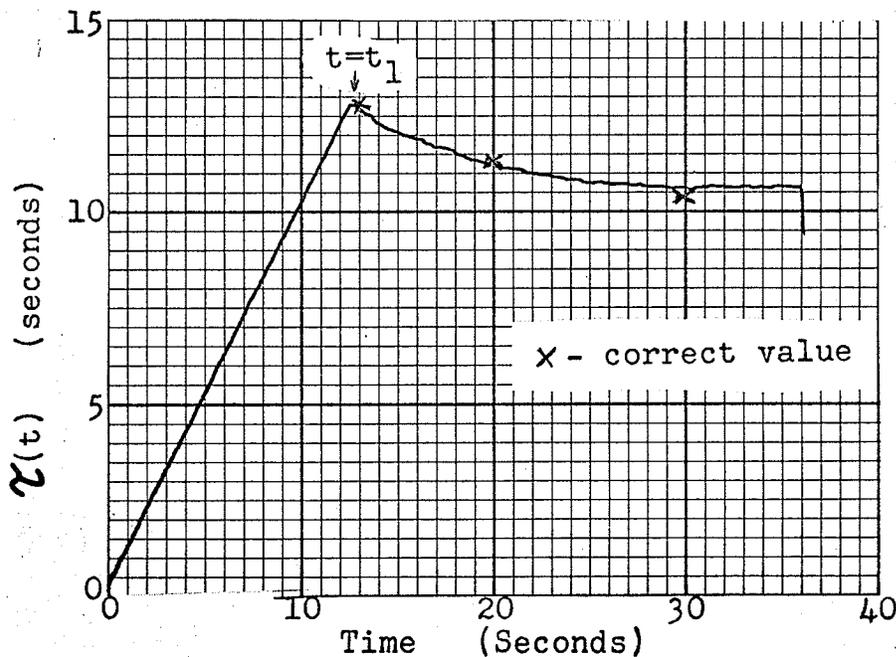


Figure 5-3. A Test of the VDU

be equal to $\tau(t)$ for $t > t_1$ where t_1 is defined by

$$t_1 - \tau(t_1) = 0 . \quad (5-3)$$

In this case, t_1 is equal to 12.8 seconds. After this point, the curve in Fig. (5-3) represents $\tau(t)$.

It can be seen that the experimental results were close to the expected values. The deviation between the expected and actual values of delay at higher values of t is the result of practical difficulties: the slide-wire on the line follower was found to be slightly nonlinear. This nonlinearity will have no effect on the actual delay-- it is only the method of measuring the delay which makes it appear as though the delay is in error. In actual operation, the effect of this nonlinearity would be to cause a slight distortion in the output.

The agreement was close enough to justify the use of this VDU in the solution of other problems.

CHAPTER VI

SYSTEMS WITH A SINUSOIDALLY VARYING DELAY

With the development of the successful VDU as described in Chapter V it became possible to give consideration to the study of actual systems with a time-variable delay. The systems considered were those with a sinusoidally varying delay given by

$$\tau(t) = a + b \sin \omega t \quad (6-1)$$

In particular, the response to a sinusoidal input was investigated. An equation which predicts which frequencies will occur in the output was developed theoretically and verified experimentally.

6-1 Some Basic Remarks

When a linear system with a sinusoidally varying delay is subjected to a sinusoidal input, it is clear that the system output will be periodic if the input and delay frequencies are commensurable. The period of the output will be such that, at the end of a period, the input and the delay are exactly the same as they were at the beginning of the period. The output period is the lowest common multiple of the periods of the input and the delay.

Consequently, the output frequency is the greatest common divisor of the input and delay frequencies.

Let w_d and w_i be the frequencies of the delay and the input, respectively. If these frequencies are commensurable then there exist positive integers, p and q , such that it is possible to define the output frequency w_o such that the following two equations must hold:

$$w_d = pw_o \quad (6-2a)$$

$$w_i = qw_o \quad (6-2b)$$

If w_d and w_i are not commensurable, or if they are such that p and q would have to be very large numbers, it is possible to consider the output to be "almost periodic" and then to choose p and q so that they give approximate results.

Because the output is periodic, it is possible to represent it as a Fourier series with a fundamental frequency w_o . At this time, it would be reasonable to expect the output to contain all frequencies which are integral multiples of w_o although it is realized that some of the components may be reduced by the filtering action of the fixed portion of the system. It will be seen later that, in fact, only some of these frequencies are present.

Recognizing that this sinusoidally varying delay has the effect of a modulator, one wonders whether the output would contain the familiar sum and difference frequencies: $w_d \pm w_i$. Equations (6-2a) and (6-2b) can be used to write

$$\begin{aligned} w_d \pm w_i &= p w_o \pm q w_o \\ &= (p \pm q) w_o \end{aligned} \quad (6-3)$$

Because p and q are both integers, $p \pm q$ is an integer and it is clear that the sum and difference frequencies, being integral multiples of w_o , may be expected in the output.

Bellman, Buell, and Kalaba (34) have done a small amount of theoretical work on the particular equation

$$u'(t) = -u(t-l-b \sin w_d t) + \sin at \quad (6-4)$$

They found that the solution of this equation had a non-zero mean for the particular case $w_d = a$ and said that this does not occur when $w_d \neq a$.

6-2 The Frequencies In The Output Of A Sinusoidally Varying Delay Unit

It is possible to deduce considerably more information about which frequencies are present in the output. In Appendix D it is shown that the output of a delay unit with sinusoidally varying delay, with frequency $q w_o$, and

a sinusoidal input, with frequency ω_0 , has components with frequencies given by

$$\omega = (np \pm q)\omega_0 \quad (6-5)$$

where n is an integer. Equation (6-5) gives the frequencies of all expected components of the output. These frequencies are a subset of the set which consists of all frequencies which are integral multiples of ω_0 .

If the output is to have a nonzero mean, then the Fourier series which expresses the output must contain a constant term. This constant term has a frequency of zero radians per second. Thus, the output will have a nonzero mean only if there exists an integral value of n such that the equation

$$np - q = 0 \quad (6-6)$$

holds. The output will have a nonzero mean only if the input frequency is an integral submultiple of the delay frequency. This is a more general statement than that offered by Bellman et al. (34).

6-3 The Frequencies In The Output Of A System With A Sinusoidally Varying Delay

In a typical linear closed loop control system which contains a sinusoidally varying delay element, the steady-state response to a sinusoidal input will consist of many

frequencies. Thus the input to the variable delay element will, in general, be composed of components of a number of frequencies. The question arises whether these frequencies give rise to a new set of frequencies in the output of the VDU or whether the same set of frequencies is regenerated. This question is readily resolved.

For any linear system with time-variable delay, it is possible to draw the block diagram of the system in the form shown in Fig. 6-1.

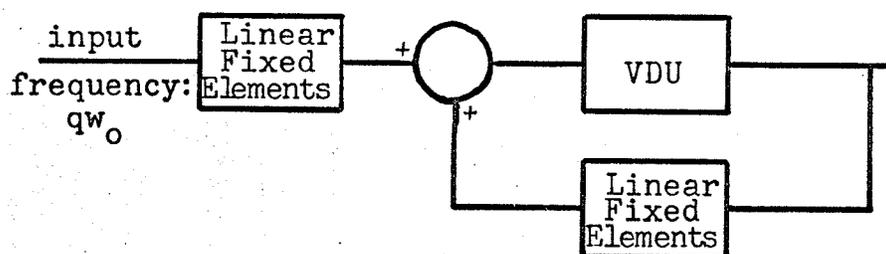


Figure 6-1. A Linear System With A VDU

The output from the VDU is assumed to contain the set of frequencies:

$$\Omega = \left\{ \omega / \omega = (np \pm q)\omega_0, n = 0, 1, 2, \dots \right\}. \quad (6-7)$$

The linear portions of the system will affect the amplitude and phase of the various components but they will not introduce any new frequencies. Thus the output of the linear portions, and consequently the input to the VDU, contain those frequencies which belong to the set Ω .

It is shown in Appendix D that when a sinusoidal signal enters the VDU, the output consists of a combination of sinusoids whose frequencies are equal to the input frequency shifted by integral multiples of the delay frequency.

Consider the particular input frequency $(mp + q)w_0$ which is an element of Ω . It will produce the set of frequencies Ω' where the definition of Ω' is

$$\Omega' = \left\{ w/w = (mp + q)w_0 \pm npw_0, n = 0, 1, 2, \dots \right\}. \quad (6-8)$$

It is obvious that the typical element of Ω' ,

$$w = ((m \pm n)p + q)w_0 \quad (6-9)$$

is a member of the set Ω . This same argument can be applied to each of the input frequencies. It may then be concluded that the two sets, Ω , and Ω' , are identical.

It is therefore possible to conclude that in any linear system with a single sinusoidally varying delay, the steady-state response to a sinusoidal input is given by a sum of sinusoids whose frequencies are given by Equ. (6-5).

6-4 Experimental Study

In order to verify Equ. (6-5) and to try to gain insight into the behavior of systems with a sinusoidally varying delay, experimental work was required. A sinusoidally varying delay simulator, as described in Appendix C, was used.

The tests were designed to elucidate the behavior of the VDU alone. The delay unit was set in operation and various sinusoidal inputs were fed into it. The paper chart which emerged from the simulator had the output of the simulator recorded on it. One period of this output signal was sampled manually and a harmonic analysis was performed on the data by digital computer. Appendix F contains the computer program.

It is necessary to have data for one complete period in order to do a harmonic analysis. As was mentioned before, the period of the output of the delay unit is given by the lowest common multiple of the period of the input signal and the period of the delay. Thus, to ensure that the period of the output was short enough to be handled easily, it was necessary that the ratio between the input frequency and the delay frequency be in the form of a fraction whose numerator and denominator were both small integers. That is, p and q were necessarily small positive integers.

The frequency of the delay variation was the most difficult to adjust so it was left fixed while the input frequency was adjusted to be in the desired ratio. Even the fine frequency control on the frequency generator proved to be quite coarse so it was only possible to get the input frequency to within one or two percent of the desired value.

It was observed early in the study that the phase of the input signal compared with that of the delay was an important factor in determining the output. A sine wave of arbitrary phase may be represented as a linear combination of a sine wave and a cosine wave. Thus, by knowing the output of the VDU due to both a sine wave and a cosine wave, it would be possible to determine the output due to an input of any given phase. For this reason, both sine wave and cosine wave inputs were considered.

Tests were made with a number of different combinations of p and q . It was clear from the results of these tests that Equ. (6-5) is valid. It was not, however, possible to verify Equ. (6-6) directly from these tests but this does not mean that it has been disproved. This equation predicts the existence of the zeroth harmonic in some cases but it gives no indication of its magnitude. Presumably these constant terms are small and thus are easily hidden by the "noise" of sampling and by errors arising from the failure to set the frequencies accurately enough. Because Equ. (6-6) follows immediately from Equ. (6-5) verification of the latter is sufficient to verify Equ. (6-6).

Table 6-1 on page 49 contains the results of two tests. The magnitude and phase (assuming a cosine series) of the first ten harmonics in the output are entered in the table. Those harmonics which are predicted by Equ. (6-5) to be in the output are also indicated.

Although the phases of the harmonics are included in the table, their actual values are of little real importance because they depend upon the choice of time origin. Changing the time origin will cause a change in all of the phase angles but the change will be different for each harmonic.

In these tests, the origin used in writing the Fourier series was the time t_1 , where t_1 is defined by

$$t_1 - \mathcal{Z}(t_1) = 0. \quad (6-10)$$

The reason for this was that initially it took t_1 seconds for the applied signal to emerge from the VDU.

Not all of the magnitudes which were expected to be zero were, in fact, zero. However, those which were not zero were very close to it. These errors could be attributed to the fact that the input and the delay frequencies could not be set exactly to the correct ratio. This would mean that the portion of the output which was sampled was not, in fact, a whole period.

It appears that in all cases the magnitudes of each frequency component are the same for both sine and cosine inputs--only the phases are different.

It was observed that in many cases the phase difference between those harmonics produced by a sine wave input and those produced by a cosine wave input was surprisingly close to $\frac{\pi}{2}$ radians. This was not so for

all harmonics but it happened often enough to suggest that the phases may follow some simple pattern. This pattern, if it exists, has not yet been discovered. Apart from this, there seems to be no particular pattern to either the magnitudes or the phases.

TABLE 6-1

THE HARMONICS IN THE OUTPUT OF A SINUSOIDALLY VARYING VDU

Harmonic Number	Predicted Magnitude	Magnitude of N-th Harmonic For Input:		Angle (radians) of N-th Harmonic For Input:	
		$-\sin q\omega_0 t$	$-\cos q\omega_0 t$	$-\sin q\omega_0 t$	$-\cos q\omega_0 t$
CASE 1: $p = 3$ $q = 2$					
0	0	-.01	-.01	-	-
1	-	.18	.19	-.90	.39
2	-	.90	.90	.91	-.58
3	0	.02	.01	.52	-.37
4	-	.02	.02	-3.32	-3.04
5	-	.18	.18	.82	-.71
6	0	.00	.00	-	-
7	-	.01	.00	.73	-
8	-	.02	.02	.74	-.85
9	0	.00	.00	-	-
10	-	.00	.00	-	-
CASE 2: $p = 2$ $q = 3$					
0	0	.00	.00	-	-
1	-	.36	.40	-2.31	-2.00
2	0	.01	.01	.61	-.58
3	-	.76	.76	.72	-.82
4	0	.01	.01	-2.37	-3.60
5	-	.37	.36	.79	-.77
6	0	.00	.00	-	-
7	-	.09	.09	.86	-.71
8	0	.01	.00	.32	-
9	-	.02	.02	.94	-.75
10	0	.00	.00	-	-

CHAPTER VII

CONCLUSION

There are a great many problems associated with the solution of equations with a time-variable delay. This thesis has dealt primarily with a survey of the work which has been done heretofore and a discussion of several possible methods of constructing VDU's. There are many aspects of the problem which have not yet been investigated.

7-1 Summary Of Findings

Several questions should be answered by the analyst before the study progresses very far. The most important question is, "Does the delay variation really have to be considered?" Because of the difficulty in analysing systems with a time-variable delay, one should satisfy himself that the effort is actually necessary. One should also decide whether the delay is state-variable or time-variable; the possible methods of solution depend on the type of variation. Finally, it is absolutely necessary that the distinction be made between the pipe problem and all other problems. In the pipe problem, it is not the delay, but rather the flow rate which is known. Although these two quantities are related, it is not easy to convert from one to the other so it is best to solve the problem in the form given.

It is clear from the survey of the history of the study of systems with a variable delay that there has not been any unifying theory developed. In this thesis, it was pointed out that the pipe problem can be transformed first to convert the problem to one with a constant delay and variable coefficients. Even with this transformation, the problem is difficult to solve.

The only generally applicable method of solving problems with a variable delay is the method of analog simulation. The problem of simulating a time-variable delay has been discussed in considerable detail. Several different approaches to the problem were considered. A knowledge of this work would be of considerable value to one who wishes to construct a VDU.

There is one rule which is of utmost importance-- the best method of simulation is the one which is exactly analogous to the system being simulated. Thus, if a pipe problem is to be simulated, the best simulator is one which operates in a similar fashion, namely, one like Walker's or one which has a moving chart with a variable velocity. If another type of problem is being considered, then a moving chart with a movable line follower might be the best simulator.

The approach which was taken in the construction of the VDU was chosen more for short-term economy than for ease of operation. The device was assembled entirely from

standard components which could be used elsewhere when they were no longer needed in the VDU. Although the device was readily built, it was so difficult to operate that a minimum of tests were run with it.

A large portion of the work in this thesis is devoted to the control of the delay in the particular VDU employed. If a more straightforward approach were employed in the VDU, then the simulation of the variable delay would have presented much less difficulty. The experimental work did not include a study of differential equations with time-variable delay; it was only concerned with the behavior of the VDU. Most of the tests served only to verify that the VDU actually operated correctly. A few tests were also made on a VDU with a sinusoidal delay. These tests verified a theoretical prediction of which frequencies would appear in the output of such a VDU when the input was a sinusoid.

7-2 Suggestion For Further Work

There is still a need to study whole systems which contain one, or more, variable delays. The most difficult problems will probably be encountered in the study of nonlinear systems. The nonlinearity may be due to separate nonlinear elements or to the fact that the delay is state-variable. The existence and stability of oscillations should be investigated.

More work must be done to develop a unified mathematical approach to these problems. Certainly better algorithms for numerical solutions can be found. The work of Myskis should be studied carefully to see to what extent it pertains to the present problem. The suggestion of Mitropol'skiy that the method of integral manifolds be used in the study of systems with variable delays should be investigated.

Consideration should also be given to the possibility that for some particular cases, special techniques might be developed. It was found that the pipe problem is most easily simulated with a VDU which does not work well in the general case. This suggests the question of whether there are some mathematical techniques which are useful only for the pipe problem. One of these, the transformation to a constant delay, has been found but there may be others. It is intuitively appealing that an approximate solution could be obtained by substituting a constant delay for the variable delay. How this constant delay should be chosen, and what errors would be expected, are other problems which should be considered.

If further simulation of time-variable delays is considered, a different type of VDU should definitely be employed. Which type of VDU would be best depends upon whether or not the pipe problem is being considered.

LIST OF REFERENCES

1. Bellman, R.E., and K.L. Cooke, Differential-Difference Equations. Academic Press Inc., New York, 1963. 465 pp.
2. Cruz, J.B., "On the Realizability of Linear Differential Systems," IRE Trans on Circuit Theory, vol. CT-7, No. 3, pp. 347-348, Sept. 1960.
3. Zadeh, L.A., "Frequency Analysis of Variable Networks," Proc. IRE, vol. 38, No. 3, pp. 291-298, March 1950.
4. Polosukhina, O., Über eine besondere Klasse von differentialen Funktional gleichungen. Inaugural-Dissertation, Zurich, 1910.
5. Tinbergen, J., "Annual Survey: Suggestions on Quantitative Business Cycle Theory," Econometrica, vol. 3, p. 241, 1935.
6. Myskis, A.D., "General Theory of Differential Equations With a Retarded Argument," American Mathematical Society Translations, Series 1, vol. 4, American Mathematical Society, Providence, R.I., 1962. pp. 207-267.
7. _____, "Hystero-differential Equations," Uspekhi Matemat. Nauk, vol. 4, No. 1, pp. 190-193, 1949. (In Russian. An English review appears in Mathematical Reviews, vol. 11, p. 365, 1950.)
8. _____, "Linear Homogeneous Differential Equations of the First Order With Retarded Argument," Uspekhi Matemat. Nauk, vol. 5, No. 2, pp. 160-162, 1950. (In Russian. An English review appears in Mathematical Reviews, vol. 12, p. 106, 1951.)
9. _____, "On Solutions of Linear Homogeneous Differential Equations of the Second Order of Periodic Type With a Retarded Argument," Matemat. Sbornik, vol. 70 (N.S. 28), pp. 15-54, 1951. (In Russian. An English review appears in Mathematical Reviews, vol. 13, p. 43, 1952.)
10. _____, "On Solutions of Linear Homogeneous Differential Equations of the First Order of Stable Type With a Retarded Argument," Matemat. Sbornik, vol. 70 (N.S. 28), pp. 641-658, 1951. (In Russian. An English review appears in Mathematical Reviews, vol. 13, p. 246, 1952.)

11. _____, Naumovic, "A Refinement of the Method of Reciprocal Sequences for the Investigation of Differential Equations With Lagging Argument." Dokl. Akad. Nauk SSSR, vol. 124, pp. 976-979, 1959. (In Russian. An English review appears in Mathematical Reviews, vol. 22, #9695, 1961.)
12. Yeh, C.H., and M. Kim, "On the Differential-Difference Equations With Time-Variable Lag," Proc. IEEE, vol. 53, No. 5, pp. 504-505, May 1965.
13. Cunningham, W.J., "Graphical Solution of Certain Nonlinear Differential-Difference Equations," Jour. Franklin Inst., vol. 261, pp. 621-629, June 1956.
14. Litsyn, N.M., "Relay Control of Linearly-Asymmetrical Objects," Automat. Remote Control, vol. 22, No. 11, pp. 1354-1358, April 1962.
15. Teverovskii, V.I., "Periodic Operating Mode of Relay Systems With Variable 'Pure' Lags," Automat. Remote Control, vol. 27, No. 1, pp. 73-79, August 1966.
16. Mitropol'skiy, Y.A., "The Method of Integral Manifolds," pp. 1-14, International Symposium on Nonlinear Differential Equations and Nonlinear Mechanics, edited by J.P. LaSalle and S. Lefschetz, Academic Press, New York, 1963. 1086 pp.
17. Bellman, R.E., J.D. Buell, and R.E. Kalaba, Numerical Integration of a Differential-Difference Equation With a Decreasing Time-Lag. The RAND Corporation, RM-4375-NIH, December 1964.
18. Bellman, R.E., and K.L. Cooke, On the Computational Solution of a Class of Functional Differential Equations. The RAND Corporation, RM-4287-PR, October 1964.
19. Ball, S.J., "A Variable Time-Delay Subroutine For Digital Simulation Program," Simulation, vol. 7, No. 1, pp. 23-24, July 1966.
20. Chernyshev, M.K., "Analysis of the Limiting Possibilities of Devices for the Reproduction of Variable Delay by Operational Amplifiers," Automat. Remote Control, vol. 26, No. 12, pp. 2172-2181, July 1966.
21. Grinya, Ya.I., and M.K. Chernyshev, "Rational Selection of Circuits for Reproduction of Variable Delays With Operational Amplifiers," Automat. Remote Control, vol. 27, No. 1, pp. 122-127, August 1966.

22. Kogan, B. Ya., and M.K. Chernyshev, "Delay Simulation With the Aid of Operational Amplifiers," Automat. Remote Control, vol. 27, No. 3, pp. 505-519, November 1966.
23. Novosel'tseva, Zh.A., "Analog Model of Variable Delay Using Standard Analog Computer Elements," Automat. Remote Control, vol. 25, No. 1, pp. 85-88, July 1964.
24. Single, C.H., Review of Transport Delay Representations for an Analog Computer. Applied Dynamics, Inc., August 1963.
25. Fifer, S., Analog Computation Theory, Techniques and Applications, vol. IV, McGraw-Hill Book Co., Inc., New York, 1961, pp. 1254-1257.
26. Korn, G.A., "Network Approximations of Variable Time Delay Require Caution," IRE Trans. on Electronic Computers, vol. EC-11, No. 1, p. 82, February 1962.
27. Margolis, S.G., and J.J. O'Donnell, "Rigorous Treatments of Variable Time Delays," IEEE Trans. on Electronic Computers, vol. EC-12, No. 3, pp. 307-309, June 1963.
28. Doganovskii, S.A., and V.A. Ivanov, Controlled-Delay Devices. The MacMillan Co., New York, 1963. 63 pp.
29. Robb, R.E., "Tracking System Simulation With Variable Delay," Simulation, vol. 5, No. 5, pp. 290-297, November 1965.
30. Walker, B.A., Time-Variable Time Delay for Analog Computer Simulations. General Electric Co., HW-83092.
31. ———, personal communication.
32. Von Perry, J., An Analog Time Delay Unit. Ph.D. dissertation, University of North Carolina at Raleigh, 1964.
33. Ragg, B.C., "Necessary Conditions for the Optimal Control of a System With Time-Varying Transport Lags," IEEE Trans. on Automatic Control, vol. AC-11, No. 4, pp. 738-740, October 1966.
34. Bellman, R.E., J.D. Buell, and R.E. Kalaba, Mathematical Experimentation In Time-Lag Modulation. The RAND Corporation, RM-4432-NIH, February 1965.
35. Levine, L., and A. Levine, Methods for Solving Engineering Problems Using Analog Computers. McGraw-Hill Book Co., New York, 1964. p. 160.

36. Gradshteyn, I.S. and I.M. Ryzhik, Table of Integrals, Series, and Products: Academic Press Inc., New York, 1965. Section 1.320, #1.
37. Nielsen, K.L., Methods In Numerical Analysis. The MacMillan Company, New York, 1964. pp. 313-315.

APPENDIX A

A CHANGE OF VARIABLE WHICH SIMPLIFIES THE PIPE PROBLEM

It is possible, in some cases, to perform a transformation on a differential equation with a variable delay to convert it to a differential equation with a constant delay. This approach to the problem was suggested in a paper by Bellman and Cooke (18).

This transformation will involve a change of the independent variable from time, t , to something else, say q .

Assume the existence of a function T such that t and q are related by the equation

$$t = T(q) . \quad (A-1)$$

Because neither q nor T has yet been specified (although once one has been specified, the other is determined) it is possible to define an arbitrary constant, Q , and write the equation

$$t - \tau(t) = T(q - Q) . \quad (A-2)$$

The variable, t , may be eliminated between Eqs. (A-1) and (A-2) to yield the equation

$$T(q) - \tau(T(q)) = T(q - Q) . \quad (A-3)$$

Equation (A-3) could, in principle, be solved for $T(q)$ but this would be very difficult and in most cases would require a numerical solution. The reason that the solution of Equ. (A-3) is required will become apparent later.

Suppose that the time-variable delay system which is being studied has a differential equation

$$\dot{u}(t) = f\{u(t), u(t - \tau(t))\} . \quad (A-4)$$

When the substitutions of Eqs. (A-1) and (A-2) are made in Equ. (A-4), it becomes

$$\frac{d}{dt} u(T(q)) = f\{u(T(q)), u(T(q - Q))\} . \quad (A-5)$$

With this substitution, it is seen that the system variable, u , is now a function of q but, because of the different functional form, it is helpful to define the function U by the equation

$$u(T(q)) = U(q) . \quad (A-6)$$

The left side of Equ. (A-5) can be simplified by using the chain rule. Equation (A-1) can be used to determine dq/dt and Equ. (A-5) becomes

$$\frac{1}{T'(q)} \frac{d}{dq} U(q) = f\{U(q), U(q - Q)\} . \quad (A-7)$$

In order that this be correct, it is necessary that $T'(q)$ never be zero. This condition is satisfied if t is monotonic increasing with q .

When Equ. (A-7) is multiplied by $T'(q)$, it becomes

$$\frac{d}{dq} U(q) = T'(q) f\{U(q), U(q - Q)\} \quad (A-8)$$

which is a differential-difference equation with a constant delay and variable parameters.

In general, even though the delay is now constant, Equ. (A-8) is not really easier to solve. It is very difficult to determine $T'(q)$ which would be obtained by differentiating the solution of Equ. (A-3).

It happens that there is one very important and often-occurring class of systems with time-variable delay for which the determination of $T'(q)$ presents no problem. This is that class of problems which has been designated the pipe problem.

For the sake of definiteness, suppose that the delay is caused by the flow of a fluid at $V(t)$ cubic feet per second through a pipe of constant volume, Q cubic feet.

It is convenient to define the integrated flow, $q(t)$, by the equation

$$q(t) = \int_{-\infty}^t V(x) dx \quad (A-9)$$

This is the total flow out of the pipe up to time t . If $V(t)$ is restricted to positive values, then $q(t)$ is a

monotonic increasing function of time and there is a one-to-one correspondence between t and $q(t)$. Equation (A-9) is the inverse of Equ. (A-1).

Because of the incompressibility of the fluid, exactly Q cubic feet of fluid must leave the pipe during the passage of any fluid particle from the inlet to the outlet.

There is now a strong suggestion that the independent variable of the transformed problem should be chosen to be the integrated flow. In this case, the "delay" would be Q , the volume of the pipe.

The validity of this particular transformation may be further justified by seeing that it satisfies Equ. (A-3). In terms of the present interpretation of the symbols, Equ. (A-3) indicates that the time at which the total flow is q minus the delay at the time at which the total flow is q is equal to the time at which the total flow is $q - Q$. This is correct.

The problem of determining $T'(q)$ still remains but in this case is readily solved by differentiating Eqs. (A-1) and (A-9) with respect to q and then equating the two expressions for dt/dq . This results in the relationship

$$T'(q) = \frac{1}{V(t)} . \quad (A-10)$$

But in problems where delay arises because of flow in a pipe, it is the flow rate $V(t)$ which is known. Thus in

such problems, Equ. (A-3) need not be solved in order to obtain $T'(q)$. Equation (A-10) may be used instead.

Although $V(t)$ is known, it must be expressed as a function of q in Equ. (A-8) where the independent variable is q . This is easy in principle; it requires only the solution of Equ. (A-9) for t in terms of q so that $V(t)$ can be written as $V(T(q))$.

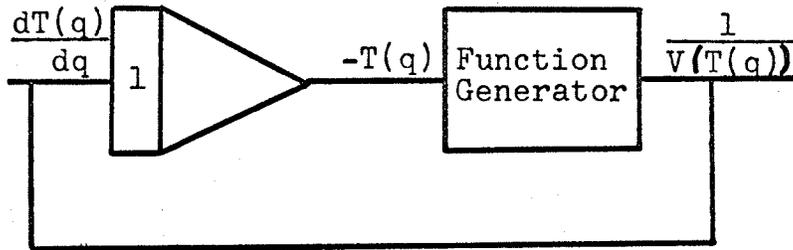
Equation (A-8) can now be written

$$\frac{d}{dq} U(q) = \frac{1}{V(T(q))} f\{U(q), U(q - Q)\} . \quad (A-11)$$

The easiest method of solving either the original differential equation (A-4) or the transformed equation (A-11) is probably by means of an analog computer. But because of the many problems encountered in constructing a variable delay simulator, the solution could most easily be obtained by solving Equ. (A-11) on an analog computer equipped for simulating a constant delay.

When Equ. (A-11) is solved on the analog computer, the independent variable, q , will be represented on the computer by computer time. The only difficulty then is the determination of the variable coefficient, $\frac{1}{V(T(q))}$. However, if $T(q)$ is substituted for t in Equ. (A-10), then it is clear that this equation is just a first order non-linear differential equation. It can be solved readily on the analog computer, the required variable coefficient

being produced as a by-product. Figure A-1 shows how this could be done.



Note that q here represents computer time.

Figure A-1. The Determination Of $\frac{1}{V(T(q))}$

Because $V(t)$ must be known in advance in order to set the function generator, this method could not be used to simulate a state-variable delay.

APPENDIX B

RELATIONSHIP BETWEEN FREQUENCIES AT THE INPUT AND THE OUTPUT OF A VDU

Consider a typical VDU in which the output, $e_o(t)$, is related to the input, $e_i(t)$, by the equation,

$$e_o(t) = e_i(t - \tau(t)). \quad (B-1)$$

Let the input to the VDU be a signal of a single frequency, for example, $\cos wt$. Now the output is given by

$$e_o(t) = \cos w(t - \tau(t)). \quad (B-2)$$

It is understood that the "frequency" of the output is determined by differentiating the argument of the cosine function with respect to t . Thus the output frequency is given by

$$\frac{d}{dt} w(t - \tau(t)) = w(1 - \tau'(t)). \quad (B-3)$$

This frequency shift occurs for all frequencies. Thus the bandwidth of the output signal is modified accordingly.

APPENDIX C

THE SOLUTION OF EQUATION (4-7) ON THE ANALOG COMPUTER

C-1 The General Solution

The equation which is to be solved is

$$L(t) = v\mathcal{L}(f(t)) \quad (C-1)$$

in which $f(t)$ is defined to be the inverse of the equation

$$t = z - \mathcal{L}(z) . \quad (C-2)$$

Equation (C-2) gives z as an implicit function of t . The analog computer circuit (35) shown in Fig. C-1 will solve this equation for z .

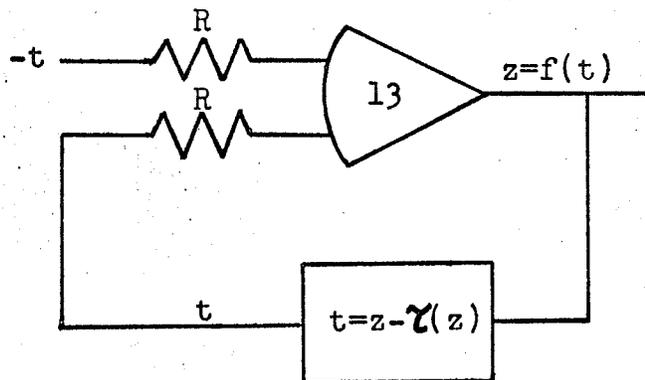


Figure C-1. The Solution Of Equation (C-2)

To ensure the stability of this circuit in solving for z implicitly, it is necessary that $z - \mathcal{L}(z)$ be monotonic

increasing. This is the same condition which is required by the particular method of simulation being used and consequently is already satisfied.

Both z and t represent time: z is the time at which the pen must be in such a position that the moving paper will carry the record to the line follower at time t . At the time of reading, the record will have been delayed by a time $\tau(t)$.

Because both z and t represent real time, they will both be monotonic increasing during the solution on the analog computer. Both of these variables are represented by voltages in the analog computer and, as such, are subject to saturation of the amplifiers. Therefore, the simulation can only proceed for a limited length of time until saturation occurs. In order to increase the permissible duration of operation, it is helpful to multiply Equ. (C-2) by the factor β which is less than unity. This will permit the simulator to operate $1/\beta$ times as long without saturating.

By using a function generator and perhaps other components on the analog computer, it is possible to generate $\tau(z)$ when the input is z . But z is equal to $f(t)$ so that Equ. (C-2) is readily simulated.

When these modifications and additions are incorporated into the circuit of Fig. C-1, the circuit of Fig. C-2 results.

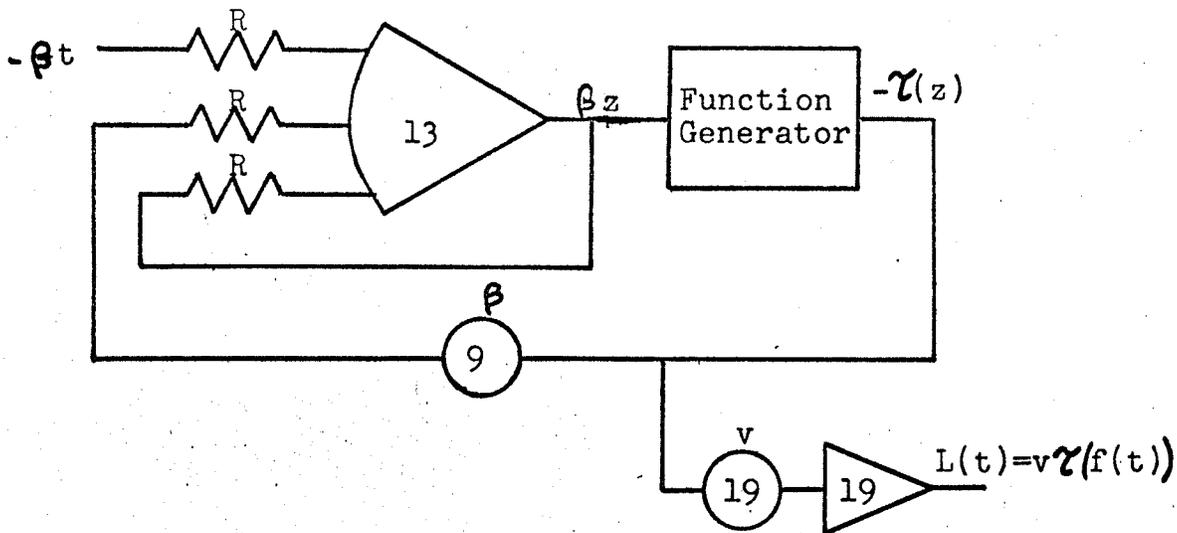


Figure C-2. Circuit For Determining $L(t)$

The output of amplifier 19 is used to drive the X-axis of the X-Y recorder. The X-axis gain and zero control of the recorder must be adjusted so that the pen moves through the correct range.

C-2 Periodically Varying Delay

There is one large class of problems in which the delay continues to vary significantly but which can be studied with this simulator. This is the class of problems with a periodically varying delay.

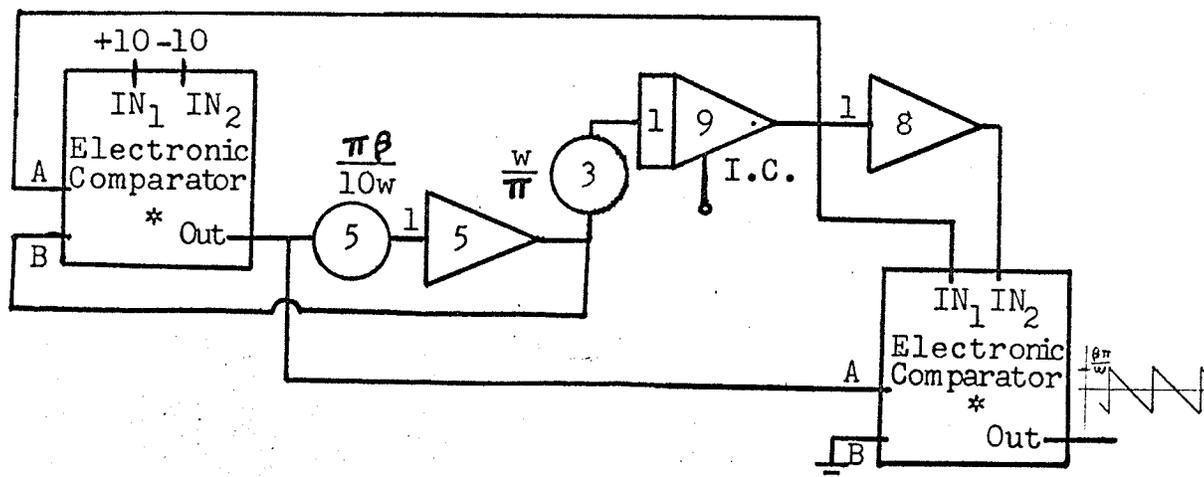
The operation of the simulator in this case is made possible by the readily-verified fact that when the delay varies periodically, $L(t)$ also varies periodically. The stratagem employed is to operate the delay simulator re-

peatedly for exactly one period of the delay. At the end of each period, the simulator reverts to the state in which it was at the start of the period. (This reversion has no effect on L which is periodic.) In each successive period, both t and z have the same range as they did in the original cycle so that if they do not saturate in the first cycle, they will never saturate.

When the simulator is operated in this manner, it is driven, not by a ramp, but rather by a sawtooth wave. The period of the sawtooth is just equal to the period of the delay. In this case, it will be necessary to adjust β so that the amplitude of the sawtooth wave is less than 10 volts.

A requirement for the sawtooth generator is that it must be able to operate for a considerable length of time without drifting. Although not essential, it is preferable that the sawtooth wave be symmetrical about zero volts. It is necessary that the sawtooth should switch very quickly at the discontinuity.

The sawtooth generator shown in Fig. C-3 meets these requirements. It generates a sawtooth wave with a slope of $-\beta$. The frequency of the wave is w . If necessary, the phase can be controlled at the outset of an experiment by adjusting the initial condition on integrator 9.



*Operation of Electronic Comparator:

$$\begin{aligned} \text{Out} &= -\text{IN}_1 & ; & \quad A + B > 0 \\ \text{Out} &= -\text{IN}_2 & ; & \quad A + B < 0 \end{aligned}$$

Figure C-3. Sawtooth Wave Generator

There is one aspect of the simulation set-up which requires particular care by the operator. This care is necessary because the factors β and w each occur in the setting of more than one potentiometer. β occurs in the setting of potentiometers 5 and 9 and once in the function generator which produces $\mathcal{U}(z)$; w occurs in 3 and 5 and once in the function generator.

In order that the simulator operate correctly, it is necessary that the settings on the above-mentioned potentiometers be consistent. If this is not so, then in one period of the sawtooth generator the output of the function generator would go through somewhat more, or less,

than one period of $\mathcal{V}(t)$. This will result in a discontinuity in the output of the function generator.

If the frequency of operation is critical, the sawtooth generator should be set first (potentiometers 3 and 5) and checked for the correct frequency of operation. When this is achieved, the rest of the simulator (potentiometer 9 and the function generator) should be set. The operation of the simulator should then be observed.

It often happens that $L(t)$ is slightly discontinuous due to the inconsistent potentiometer settings. This discontinuity can be eliminated by a slight adjustment in the setting of the potentiometer in the function generator.

C-3 A Particular Example: $\mathcal{V}(t) = a + b \sin wt$

Because the delay is periodic, it is possible to use the sawtooth generator to drive the simulator.

The rest of the circuitry required is shown in Fig. C-4. This figure shows the circuitry which is peculiar to this particular problem and is not shown in Fig. C-2.

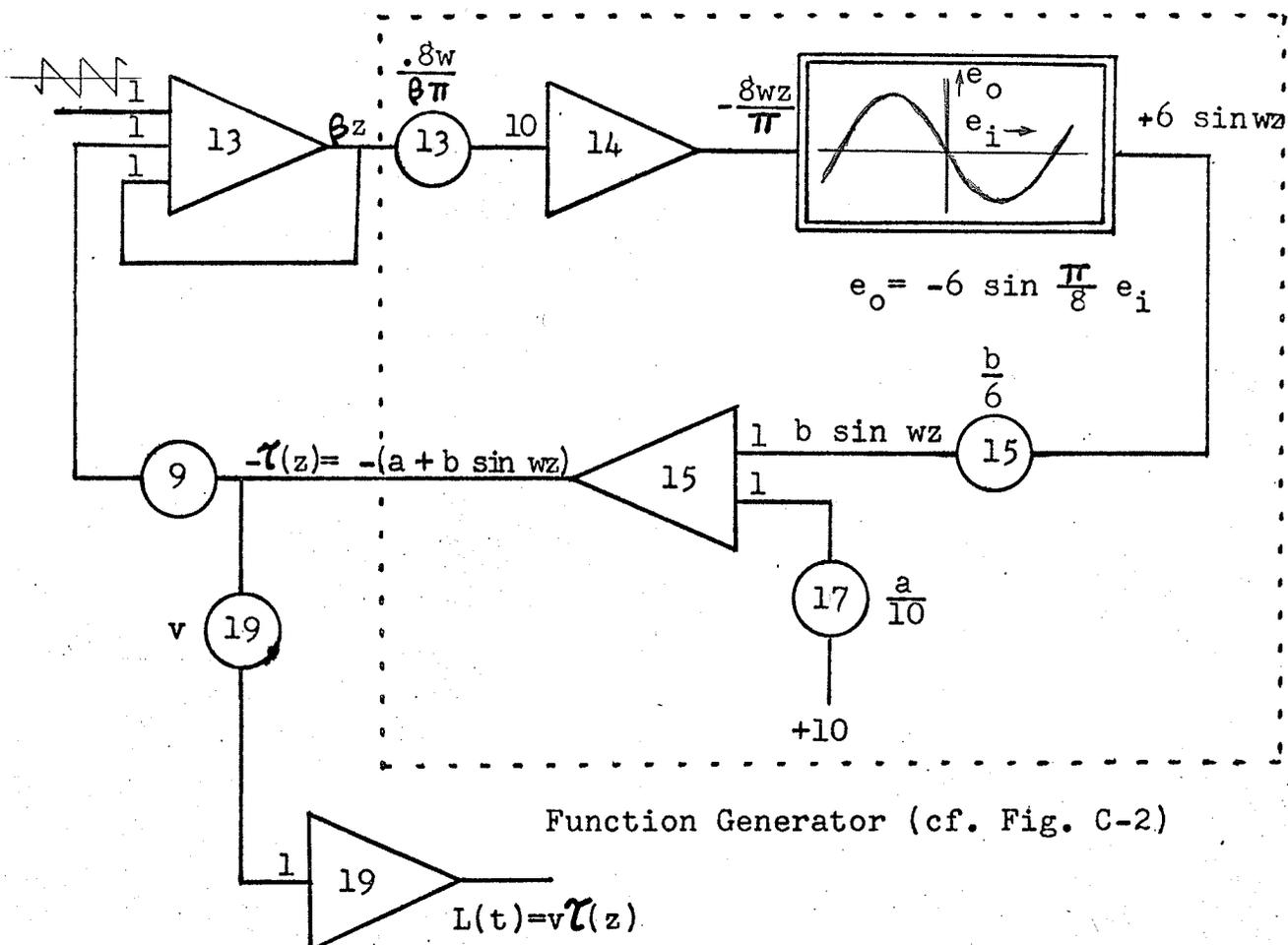


Figure C-4. Determination of $L(t)$ For The Case: $\mathcal{L}(t) = a + b \sin wt$

Because of certain constraints which are imposed by the limitations of the equipment used in the simulator, it is necessary to restrict the range of permissible values for some of the parameters. These restrictions are summarized below.

The necessity that $t - \tau(t)$ be monotonic increasing requires the condition

$$1 - \tau'(t) > 0 . \quad (C-3)$$

For a sinusoidally varying delay, this condition becomes

$$1 - bw > 0 \quad (C-4)$$

or

$$w < \frac{1}{b} . \quad (C-5)$$

In addition, w is also restricted by the frequency response of the X-Y recorder.

The maximum separation between the pen and the line follower is 15 inches; the minimum separation is 0.4 inches. These conditions impose the following restrictions:

$$\frac{(a + b)}{v} < 15 \quad (C-6)$$

$$\frac{(a - b)}{v} > 0.4 . \quad (C-7)$$

The amplitude of the sawtooth wave is $\frac{\beta\pi}{w}$. This amplitude must be less than 10 volts, which condition is written:

$$\frac{\beta\pi}{w} < 10 . \quad (C-8)$$

It could easily happen that, due to the above conditions, it would be impossible to simulate certain problems in their original form. In this event, it will be necessary to time-scale the problem. This involves replacing t by αt in the original differential equations which describe the system. When this is done, the derivative must be multiplied by α and $\mathcal{L}(t)$ must be replaced by $\frac{1}{\alpha} \mathcal{L}(\alpha t)$. The time-scaling also necessitates changes in the foregoing conditions: w must be replaced by αw and a and b must be replaced by $\frac{a}{\alpha}$ and $\frac{b}{\alpha}$ respectively.

Figure C-5 shows the solution of Equ. (4-7) for a particular case of sinusoidally varying delay. The fact that the solution differs considerably from a sine wave illustrates the fact that the VDU with a moving pen is fundamentally different from one with a moving read head. The rapid rate of change of $L(t)$ suggests another problem: if the maximum velocity of the X-Y recorder in the X-direction is less than that required to follow $L(t)$, then the VDU will not operate accurately. This imposes another restriction on $\mathcal{L}'(t)$.

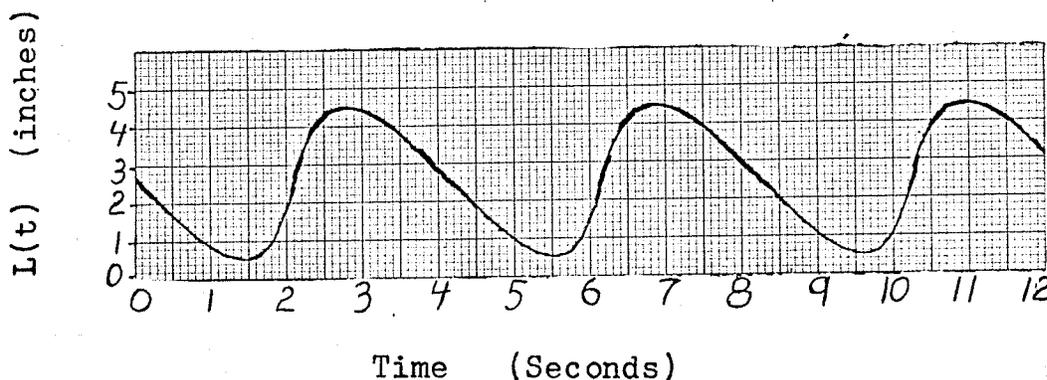


Figure C-5. The Solution Of Equ. (4-7) For The Case Of A Sinusoidal Delay ($a=5$, $b=4$, $w=1.53$, $v=.5$)

APPENDIX D

DERIVATION OF EQUATION (6-5)

In an ideal VDU, the output, $e_o(t)$, is related to the input, $e_i(t)$ by the equation

$$e_o(t) = e_i(t - \tau(t)) \quad (D-1)$$

where $\tau(t)$ is the delay.

Consider the case where the delay has the form

$$\tau(t) = a + b \sin pw_0 t \quad (D-2)$$

and the input to the system is given by

$$e_i(t) = \sin qw_0 t \quad (D-3)$$

The frequencies, pw_0 and qw_0 , are as defined in Eqs.(6-2a) and (6-2b).

The right-hand side of Equ. (D-1) may be written as a Taylor expansion about $(t - a)$. This, combined with Equ. (D-2), results in the equation

$$e_o(t) = e_i(t-a) - b \sin pw_0 t e_i'(t-a) + \frac{b^2 \sin^2 pw_0 t e_i''(t-a)}{2!} \dots$$

$$+ \frac{(-b \sin pw_0 t)^n e_i^{(n)}(t-a)}{n!} + \dots \quad (D-4)$$

It is possible* to expand $\sin^n p\omega_0 t$:

$$\sin^n p\omega_0 t = \sum_{j=0}^n k_j \sin j p\omega_0 t \quad (D-5)$$

where the k_j 's are constants, half of which are zero. With this modification, Equ. (D-4) can be written

$$e_o(t) = \sum_{n=0}^{\infty} \sum_{j=0}^n k_j \sin j p\omega_0 t \frac{e_i^{(n)}(t-a)}{n!} \quad (D-6)$$

Because the input, $e_i(t)$, is just a simple sine wave, its derivatives of all orders will be sinusoids with exactly the same frequency, $q\omega_0$. Using this fact and the familiar trigonometric identities for the product of two sinusoids, one can readily see that the term, $\sin j p\omega_0 t e_i^{(n)}(t-a)$, will consist of two frequencies: $j p\omega_0 + q\omega_0$ and $j p\omega_0 - q\omega_0$. But j takes on all integral values from 0 to n and n runs from 0 to ∞ so that it is possible to say that j takes on all of the same values that n does, namely, the positive integers.

It has been shown that when the input to a sinusoidally varying delay unit is a sine wave whose frequency is commensurable with that of the delay, then the output contains only those frequencies given by

$$w = (np \pm q)\omega_0 \quad (D-7)$$

where n is a positive integer and ω_0 is the fundamental frequency.

*See, for example, Ref. (36).

APPENDIX E
DIGITAL SOLUTION OF EQUATION (4-7)

THIS PROGRAM IS WRITTEN IN FORTRAN II.

THE FIRST TWO CARDS CONTAIN FUNCTION STATEMENTS DEFINING
TAUF(T), THE DELAY, AND TAUDAF(T), THE DERIVATIVE OF THE
DELAY.

```
TAUF(T)= 10. + 10.*EXP(-.1*T)
TAUDAF(T) = - EXP(-.1*T)
ZLAST=0.
DEL=.001
V=.5
T=0.
DT=60./20.
DO 52 J=1,21
C...EQUATION (4-5) WILL NOW BE SOLVED FOR Z=F(T) BY AN ITERATIVE
C...TECHNIQUE.
Z=ZLAST
49 Z=Z-(Z-T-TAUF(Z))/(1.-TAUDAF(Z))
IF (DEL-ABS(Z-ZLAST)) 50, 51, 51
50 ZLAST=Z
GO TO 49
51 ZLAST=Z
C...THE VALUE OF Z=F(T) IS USED TO EVALUATE L(T).
ELT=V*TAUF(ZLAST)
J0=J-11
PUNCH 100, J0, T, ZLAST, ELT
52 T=T+DT
100 FORMAT (I7,3F12.3)
CALL EXIT
END
```

APPENDIX F
HARMONIC ANALYSIS PROGRAM

THIS PROGRAM IS WRITTEN IN FORTRAN IV

USING THE METHOD OF NIELSEN(37), THIS PROGRAM FINDS THE COEFFICIENTS A(K) AND B(K) OF THE FOURIER SERIES WHEN 2N EQUIDISTANT VALUES WITHIN A SINGLE PERIOD OF THE FUNCTION ARE GIVEN. THE MAGNITUDE AND PHASE ARE ALSO CALCULATED. THIS IS DONE FOR M HARMONICS.

THE FIRST DATA CARD CONTAINS NSET--THE NUMBER OF SETS OF DATA TO BE PROCESSED-- IN FORMAT I6.

EACH SET OF DATA MUST APPEAR AS FOLLOWS-

DESCRIPTION CARD--THE CONTENTS OF THIS CARD, WHICH APPEAR IN THE TYPED OUTPUT, CAN BE USED TO IDENTIFY THE DATA.

N,M CARD-- CONTAINS N AND M IN FORMAT 2I6.

DATA CARDS-- CONTAIN Y(J) IN FORMAT 8F10.3.

```

DIMENSION Y(100), A(50), B(50), D(50)
READ (1, 100) NSET
PI = 3.14159
DO 4 NUM=1, NSET
  READ (1,104)
  WRITE (3, 104)
  READ (1, 100) N,M
  N2=2*N
  EN=N
  DX=3.14159/EN
  READ (1, 101)(Y(J), J=1,N2)
  SUM=0.
  DO 1 I=1,N2
1    SUM=SUM+Y(I)
  A0=SUM/2./EN
  WRITE (3 , 105)
  WRITE (3, 102) A0
C...A0 HAS BEEN CALCULATED AND TYPED.
  R=0.
  DO 3 K=1, M
  R=R+1.
  SUMA=0.
  SUMB=0.
  X=0.
  DO 2 I=1,N2
  X=X+DX
  SUMA=SUMA +Y(I)*COS(R*X)
2    SUMB = SUMB + Y(I)*SIN(R*X)
  A(K) = SUMA/EN
3    B(K) =SUMB/EN
C...A(K) AND B(K) HAVE BEEN CALCULATED FOR M VALUES OF K.
  DO 4 K=1, M
  D(K)=SQRT(A(K)*A(K)+B(K)*B(K))
  IF (B(K)) 10, 10, 11
11   PHI = - ATAN (B(K)/A(K)) - PI
  GO TO 12
10   PHI = - ATAN (B(K)/A(K))
12   CONTINUE

```

```
C
C...THE MAGNITUDE AND PHASE HAVE BEEN CALCULATED.
4   WRITE (3,103) K, A(K), B(K), D(K) , PHI
C...THE OUTPUT HAS BEEN TYPED.
100  FORMAT (2I6)
101  FORMAT (8F10.3)
102  FORMAT (3H 0, F12.4)
103  FORMAT (I3, 4F12.4)
104  FORMAT (54H
105  FORMAT (12X,1HA, 11X, 1HB, 7X, 9HMAGNITUDE, 9H   ANGLE)
      CALL EXIT
      END
```