# APPLICATION OF THE REACTION CONCEPT TO THE PROBLEM OF DETERMINING MUTUAL IMPEDANCE BETWEEN A PAIR OF COUPLED DIPOLE ANTENNAS

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ABSTRACT

The problem of defining impedance parameters for cylindrical dipole antennas is discussed. A review of the one-dimensional formulation of the antenna model is outlined. The elements of the reaction concept are presented and then specialized to the case of coupled cylindrical dipoles. A method of improving the trial approximations by the reaction method is discussed. Application of the reaction method then leads to an approximate expression for mutual impedance. A computer program is then used to calculate numerical results from this formula. Finally, graphs of computed numerical impedances are presented.

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# CHAPTER I

## INTRODUCTION

The concept of reaction was introduced by Rumsey[1*] as a fundamental observable to simplify the formulation of boundary value problems in electromagnetic theory. Beginning with the idea that classical analyses of electromagnetic problems are based on the theory of fields which satisfy Maxwell's equations, Rumsey suggests that, from the point of view of an experimenter, the postulate of fields may be questioned on the grounds that any experiment designed to measure these fields necessarily consists of measuring the effects of the fields over a small but finite region. The postulate is therefore incompatible with the process of performing the observation. Rumsey then introduced a physical observable which he termed "reaction" and gave it the symbol $\langle a,b \rangle$.

The reaction $\langle a,b \rangle$ is a scalar, and gives a measure of the coupling between two sources "a" and "b". While the field quantities are implicitly included in the formulation, the reaction method does not attempt to measure (or compute) these fields at a point, but rather includes the information carried by the field quantities as an integrated effect over the measuring or observing device.

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* The numeral denotes reference number as listed in bibliography

Approximate solutions to many problems in electromagnetic theory may be obtained by means of the reaction technique. Consider for example, the problem of determining the mutual impedance between a pair of coupled dipole antennas. This cannot be solved directly, since the current distributions on the antennas are not known. However, if assumed current distributions are used, application of the reaction concept leads directly to an approximate solution which is stationary (in the sense of the Calculus of Variations) with respect to small variations of the assumed current distributions about the true current distributions. That is, instead of attempting to solve the actual, but more difficult problem, the reaction method seeks to replace the correct (but unknown) current distributions with approximate distributions which are then adjusted so that their reactions with certain "test" sources are correct. In essence, the procedure is to make the approximate sources "look" the same as the correct sources according to the physical tests which are inherent in the problem.

The purpose of this investigation was to formulate by means of the reaction method an approximation to the mutual impedance between a pair of coupled dipole antennas. A computer program for the calculation of numerical results is derived from this formulation, and graphs of computed mutual impedances are presented.

In Chapter II, a brief summary of the circuit aspects of a dipole antenna is presented. The end effects and the gap problem are discussed, and the one-dimensional formulation reviewed.

Chapter III outlines the elements of the reaction concept method.

In Chapter IV, the reaction concept is applied to the problem of a pair of coupled dipole antennas, and the mutual impedance approximation derived.

Curves of mutual impedance are presented in Chapter V, along with a discussion of the results obtained.

# CHAPTER II

## DEVELOPMENT OF THE CIRCUIT MODEL

The problem of determining the impedance parameters of an antenna system is essentially a problem of attempting to find a solution to a set of three dimensional vector wave equations that satisfies the specified boundary conditions. No general method is available to handle this. Instead, the usual approach (developed by E. Hallén[7]) is to replace the three dimensional problem by a quasi-one dimensional problem and attempt to solve the latter.

The situation is further complicated by the fact that any pratical antenna is fed from a transmission line. This aspect of the problem must be carefully examined in order to gain an understanding of the operational significance of the defined impedances. This is the so called gap problem in antenna theory.

A discussion of this gap problem and an outline of the one dimensional formulation is presented in this chapter. The one dimensional model discussed is that to which the reaction method is to be applied in Chapter IV.

### I THE GAP PROBLEM AND THE END EFFECT

Before considering the problem of coupled antennas, it is well to examine the single, cylindrical dipole antenna shown in figure 2 - 1. The antenna is center driven from a transmission line with a conductor separation $b = 2\delta$. If an attempt is made to define an input impedance $Z_0$ for this antenna, it is found that transmission line effects cannot be ignored when b has a nonzero value. That is, with nonzero separation, an impedance cannot be defined that is a property of the dipole antenna alone. However, King[7] shows that if the separation is made sufficiently small (i.e., the following inequality is satisfied, namely, $\beta b \ll 1$, where $\beta$ is the phase constant), the coupling between charges on the antenna and those in adjacent parts of the line is reduced sufficiently that an impedance $Z_\delta$ can be defined as

$$Z_\delta = \frac{V_\delta}{I_\delta}$$

As $\delta$ is made to approach zero, $Z_\delta$ approaches $Z_0$, and the impedance so defined is a property of the antenna structure alone, independent of the circuit to which it is connected.

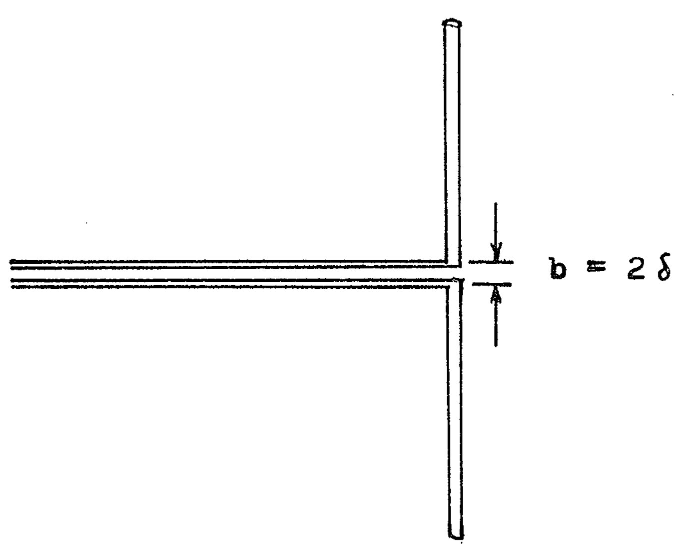


Figure 2-1. Center driven dipole antenna.

The result is a hypothetical antenna which extends unbroken from $z = -\ell$ to $z = +\ell$. In effect, the condition is equivalent to replacing the scalar potential difference $V_\delta$ across terminals that are separated by a finite distance by a discontinuity in scalar potential across terminals that are separated by a vanishingly small distance. This hypothetical driving source is termed a slice or belt generator.

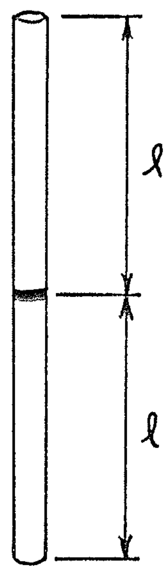


Figure 2-2. Cylindrical Antenna Driven by a Slice Generator.

Correlation of Theory and Experiment

Experiments[7] have shown that if the apparent impedance of an antenna terminating a transmission line is measured repeatedly as the spacing of the conductors is decreased progressively, and the values so obtained are extrapolated to zero line spacing, the values at zero spacing may be identified with those calculated from the configuration of figure 2-2 with $\delta = 0$. In this manner, an operational significance is given to the properties of antennas driven by slice generators.

<u>Current Source Representation</u>

In a preceeding section, the antenna driving mechanism was pictured as a hypothetical slice generator feeding current to the antenna conductors. However, as far as application of the reaction method is concerned, it is more convenient to use the current source representation of Harrington[8]. In this representation, the feeding mechanism is viewed as a short column of impressed current $I_o$ existing across the gap as shown in figure 2-3. As Tai[4] points out, this representation of the gap problem leads to the same solution as the slice generator representation.

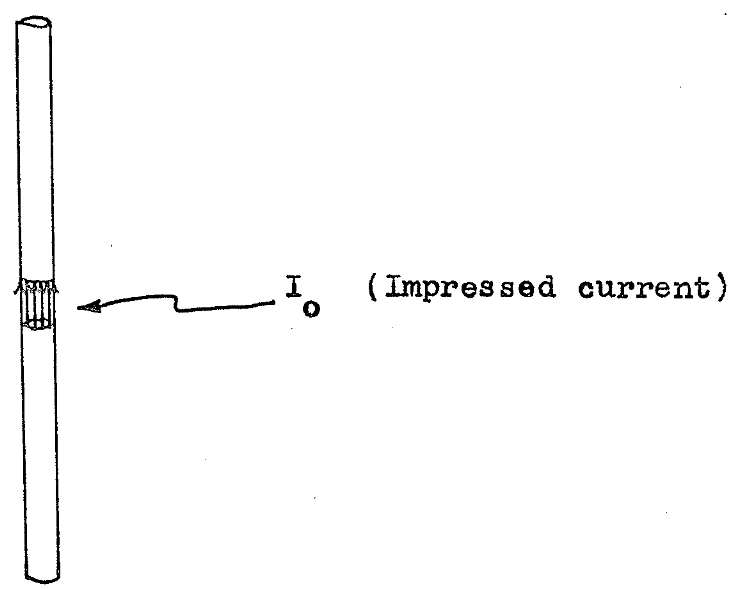


Figure 2-3. Antenna driven from a current source.

Coupled Antennas

If a pair of dipole antennas is considered, the transmission line effects noted previously are further complicated by the fact that coupling occurs not only between an antenna and its transmission line, but also between it and the transmission line of the adjacent antenna. In order to arrive at impedance parameters that are properties of the antenna configuration alone (independent of the external circuit), the same technique of using slice generators may be employed. The fundamental circuit of figure 2-4 then results. Impedance

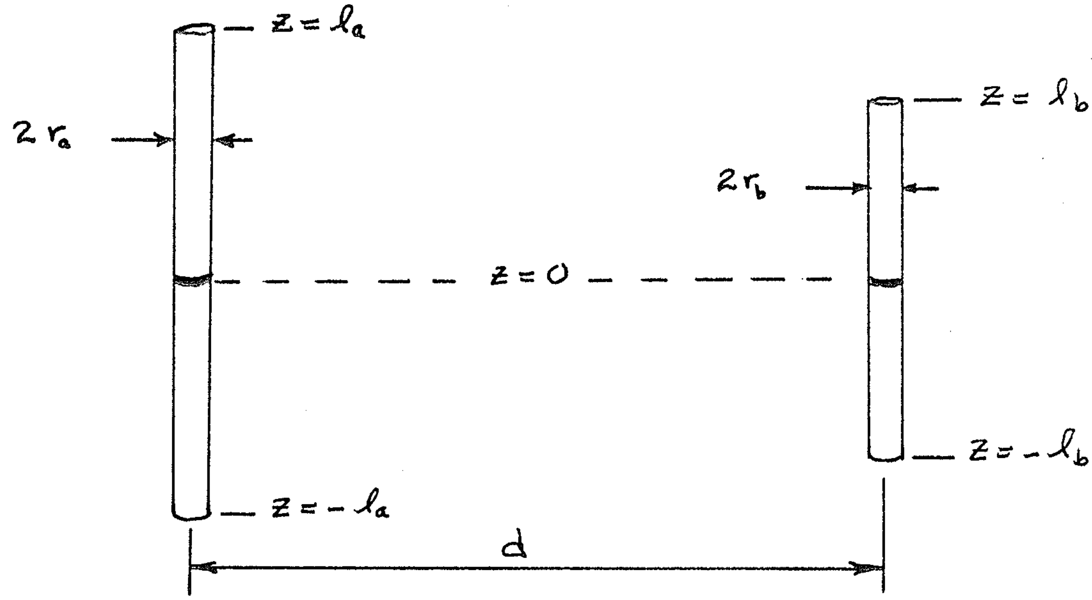


Figure 2-4. Coupled Dipole Antennas.

parameters for this case are governed only by the physical geometry of the system and the frequency. If all dimensions are expressed in wavelengths, the impedance parameters are determined only by the quantities $r_a, \ell_a, r_b, \ell_b$ and d. Operational significance is again obtained by comparing the computed impedances with measured values which have been determined by extrapolating the measured results to zero gap spacing.

End Effect

In the one dimensional formulation to follow, the condition that the current is zero at the ends of the dipoles is imposed. While this is true for an antenna with hemispherical ends, it is not true when the antenna is composed of a solid cylindrical conductor with flat ends or a tube with open ends. However, for the latter two cases, King states that the effective half length of the antenna exceeds the physical half length by an amount that is difficult to determine accurately, but that is of the order of magnitude of the radius.

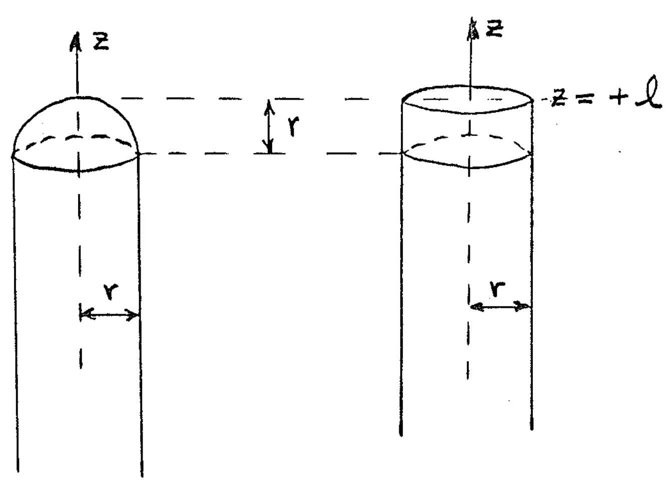


Figure 2-5. Antennas with Hemispherical and Plane Ends.

The model illustrated in figure 2-5 is a physically unavailable cylinder of radius r and half-length $\ell$ that has no chargeable surfaces beyond the edges at $z = \pm \ell$.

## II ONE DIMENSIONAL FORMULATION

It was mentioned in a previous section that the three dimensional coupled antenna problem was not amenable to analysis and therefore, a one dimensional formulation would be presented instead. In essence, this involves replacing the volume distribution of current density in the antenna conductors by an axially distributed filamentary current along the center of the dipole. Rigorous justification of this procedure may be found in the literature[6].

### Vector Potential Formulation

The electric field intensity $\mathbb{E}$ may be related to the charges and currents in a system through the potential functions $\Phi$ and $\mathbb{A}$. The defining relationships are

$$\mathbb{E} = -\nabla\Phi - \frac{\partial \mathbb{A}}{\partial t}$$

and

$$\nabla \cdot \mathbb{A} = -\mu\epsilon \frac{\partial \Phi}{\partial t}$$

When the time variations are harmonic, these reduce to

$$\mathbb{E} = -\nabla\Phi - j\omega\mathbb{A} \quad \text{.......... } 2 - 1$$

$$\nabla\cdot\mathbb{A} = -j\omega\mu\epsilon\Phi \quad \text{.......... } 2 - 2$$

Solving 2 - 2 for $\Phi$ and substituting into 2 - 1 yields an expression for $\mathbb{E}$ entirely in terms of $\mathbb{A}$. It is

$$\mathbb{E} = -j\frac{\omega}{\beta^2}\nabla(\nabla\cdot\mathbb{A}) - j\omega\mathbb{A} \quad \text{.......... } 2 - 3$$

where $\beta = \omega\sqrt{\mu\epsilon}$ .......... 2 - 4

The vector potential $\mathbb{A}$ is related to the currents in the system through the integral[8]

$$\mathbb{A} = \mu\iiint_{Volume}\frac{\mathbb{J}e^{-j\beta r}}{4\pi r}\,d\tau \quad \text{.......... } 2 - 5$$

where r is the distance from the point at which $\mathbb{A}$ is being determined to the element of integration. $\mathbb{J}$ is the volume distribution of current density.

It may be seen from equation 2 - 5 that $\mathbb{A}$ is a vector in the same direction as $\mathbb{J}$. Thus, if the current distribution is entirely z-directed, $\mathbb{A}$ will have only a z-component as well. In the one dimensional formulation, only the z-component of $\mathbb{E}$ is of interest. Solving equation 2 - 3 for the z-component of $\mathbb{E}$ yields

$$E_z = - j\omega \left(A_z + \frac{1}{\beta^2} \frac{\partial^2 A_z}{\partial z^2}\right) \qquad \text{.......... } 2 - 6$$

Consider the single dipole antenna shown in figure 2-6.

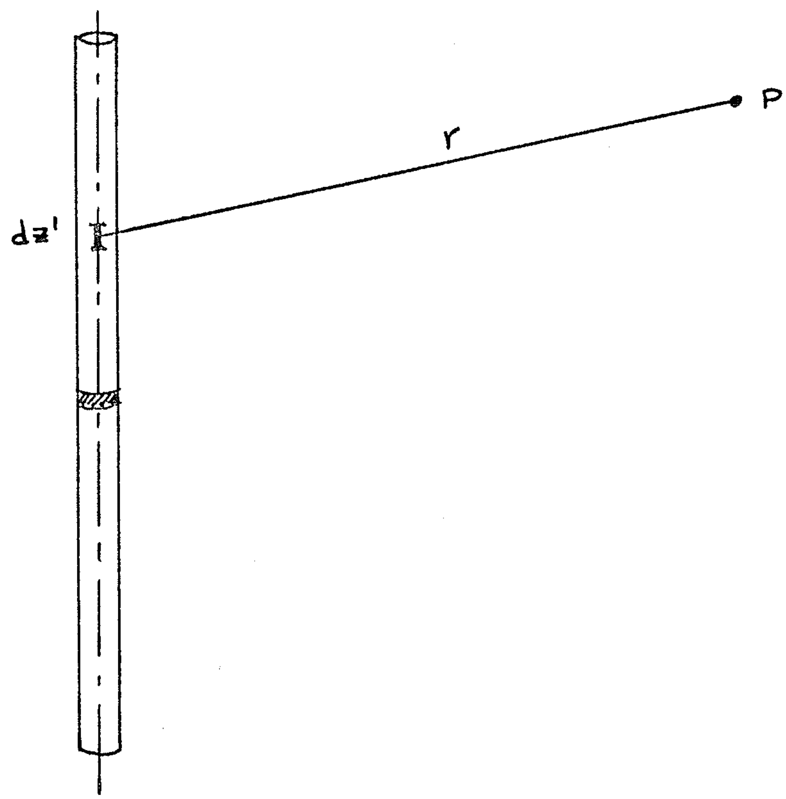


Figure 2-6. Single Dipole Antenna.

In the quasi-one dimensional formulation, the volume distribution of current density $\mathbb{J}$ is replaced by a filamentary z-directed current I(z'), and $A_z$ calculated according to the integral

$$A_z = \frac{\mu}{4\pi}\int_{-\ell}^{\ell} I(z')\frac{e^{-j\beta r}}{r}\,dz' \qquad \text{.......... } 2 - 7$$

Note that the primed variable refers to the axis of the antenna.

Combining equations 2 - 6 and 2 - 7 yields an expression for $E_z$ in terms of the current $I(z')$. It is

$$E_z = -j30\beta\int_{-\ell}^{\ell} I(z')\left(1 + \frac{1}{\beta^2}\frac{\partial^2}{\partial z^2}\right)\frac{e^{-j\beta r}}{r}\,dz' \qquad \text{.......... } 2 - 8$$

Consider the two antenna system of figure 2-7.

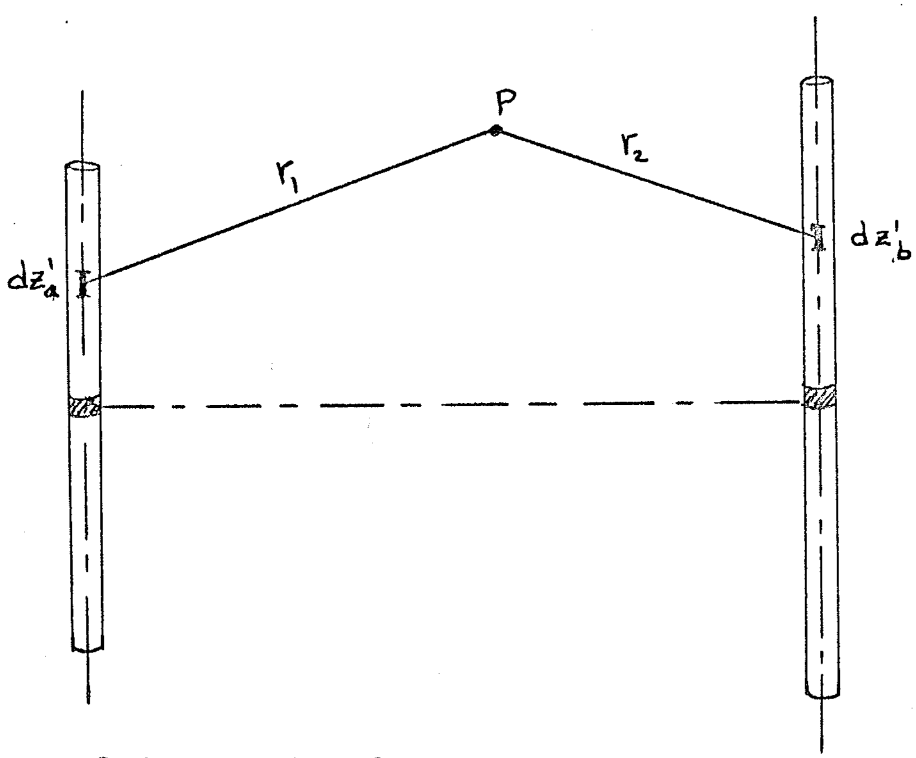


Figure 2-7. Geometry for two antennas.

The vector potential at any point P outside the antenna surfaces is obtained by summing the contributions due to each antenna. Thus

$$A_z = \frac{\mu}{4\pi}\left\{ \int_{-l_a}^{l_a} I_a(z_a') \frac{e^{-j\beta r_1}}{r_1} dz_a' + \int_{-l_b}^{l_b} I_b(z_b') \frac{e^{-j\beta r_2}}{r_2} dz_b' \right\} \qquad \text{.......... } 2 - 9$$

If the true current distributions $I_a(z_a')$ and $I_b(z_b')$ were known, it would theoretically be possible to perform the integration and obtain $A_z$. Application of equation 2 - 6 would then yield $E_z$. In principle, it would thus be possible to solve the problem directly.

In view of the fact that the current distributions $I_a(z_a')$ and $I_b(z_b')$ are not known, and in fact can not be solved for directly, some sort of approximating procedure must be used. Iterative type solutions for the current distribution and the impedance parameters have been presented by King and Harrison[2] and by Tai[3]. Both these analyses were limited to the case of identical antennas. The results obtained were quite good for thin antennas.

## CHAPTER III

## THE REACTION CONCEPT

An outline of the reaction concept is presented in this chapter. The definition and properties of the reaction are considered and the impedance properties of a two port network are expressed in terms of the various reactions involved. A procedure for improving the trial approximations used in calculating the reactions is discussed.

### Reciprocity Theorems

Consider two sets of AC sources $\mathbb{J}_a$ and $\mathbb{J}_b$ of the same frequency and existing in the same linear medium. Denote the fields produced by the "a" source acting alone as $\mathbb{E}_a$ and $\mathbb{H}_a$ and those produced by the "b" source acting alone as $\mathbb{E}_b$ and $\mathbb{H}_b$. These two sets of quantities may be related in a single equation known as a reciprocity theorem. Two forms of pure field reciprocity theorems are considered below.

Carson[9] has presented a pure field reciprocity theorem in the form of volume integrals involving electric current density and electric field intensity. It is

$$\iiint_{V_a} (\mathbb{E}_b \cdot \mathbb{J}_a)\, d\tau = \iiint_{V_b} (\mathbb{E}_a \cdot \mathbb{J}_b)\, d\tau \qquad \text{.......... 3 - 1}$$

where volume $V_a$ includes antenna "a" and $V_b$ includes antenna "b".

A second pure field reciprocity theorem involving electric- and magnetic-field intensities was derived by Lorentz[9] in the form of the surface integral expression below.

$$\iint_{s_a} (\mathbb{E}_a \times \mathbb{H}_b - \mathbb{E}_b \times \mathbb{H}_a) \cdot d\mathbb{S} = \iint_{s_b} (\mathbb{E}_b \times \mathbb{H}_a - \mathbb{E}_a \times \mathbb{H}_b) \cdot d\mathbb{S} \quad \text{.......... } 3 - 2$$

Surface $s_a$ encloses antenna "a" and surface $s_b$ encloses antenna "b".

The steps leading to equations 3 - 1 and 3 - 2 are outlined in Appendix A.

<u>Definition of Reaction</u>

Rumsey has given the name "reaction" to the integrals appearing in equations 3 - 1 and 3 - 2. By definition, the reaction of field "a" on source "b" is

$$\langle a,b \rangle = \iiint_{V_b} (\mathbb{E}_a \cdot \mathbb{J}_b)\, d\tau \quad \text{.......... } 3 - 3$$

In this notation, the reciprocity theorem becomes

$$\langle b,a\rangle = \langle a,b\rangle \qquad \cdots\cdots\cdots\cdots \; 3-4$$

In view of the equivalence of the two reciprocity equations, an alternate statement of reaction is

$$\langle a,b\rangle = \iint_{S_b} (\mathbb{E}_b \times \mathbb{H}_a - \mathbb{E}_a \times \mathbb{H}_b) \cdot d\mathbb{S} \qquad \cdots\cdots\cdots\cdots \; 3-5$$

Useful Identities

Let "c" represent a third source of the same frequency as "a" and "b", and existing in the same linear medium. Making use of the linearity of the field equations, the following useful identity is obtained

$$\langle a,(b+c)\rangle = \langle a,b\rangle + \langle a,c\rangle \qquad \cdots\cdots\cdots\cdots \; 3-6$$

Another useful identity is

$$\langle Aa,b\rangle = A\langle a,b\rangle = \langle a,Ab\rangle \qquad \cdots\cdots\cdots\cdots \; 3-7$$

where A is a scalar quantity.

<u>A Reciprocity Theorem of the Mixed Type</u>

Kouyoumjian[10] has developed an expression for the voltage induced in one antenna by another in terms of the reaction. The physical situation is depicted in figures 3-1 and 3-2.

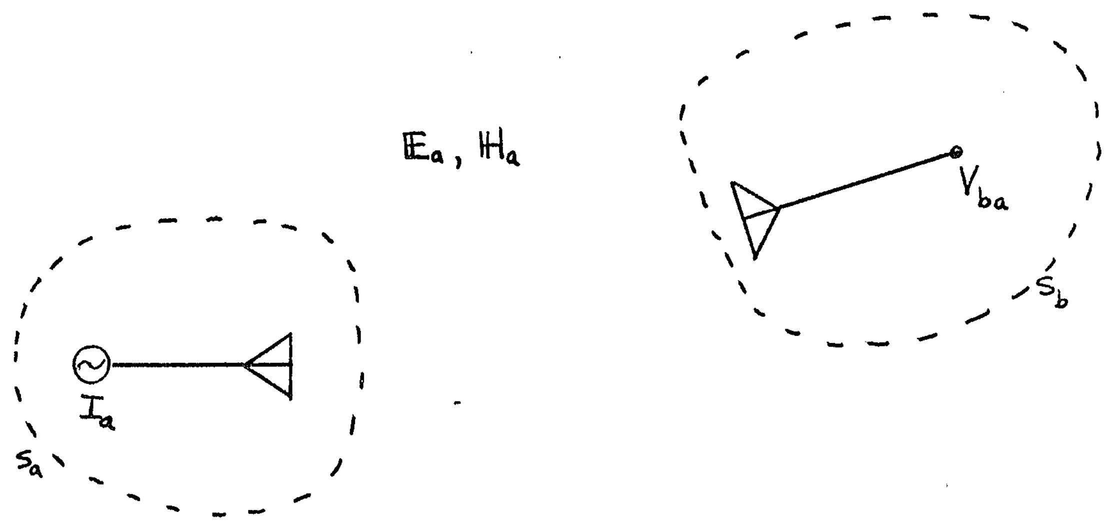


Figure 3-1. First situation: Antenna "a" transmits and antenna "b" receives.

As shown in figure 3-1, antenna "a" is driven by a current source $I_a$ at its terminals, and antenna "b" is open circuited. Voltage $V_{ba}$ is the open circuit voltage at the terminals of antenna "b".

In the second situation shown in figure 3-2, antenna "b" is driven by a current source $I_b$, and voltage $V_{ab}$ is the open circuit voltage induced at the terminals of antenna "a".

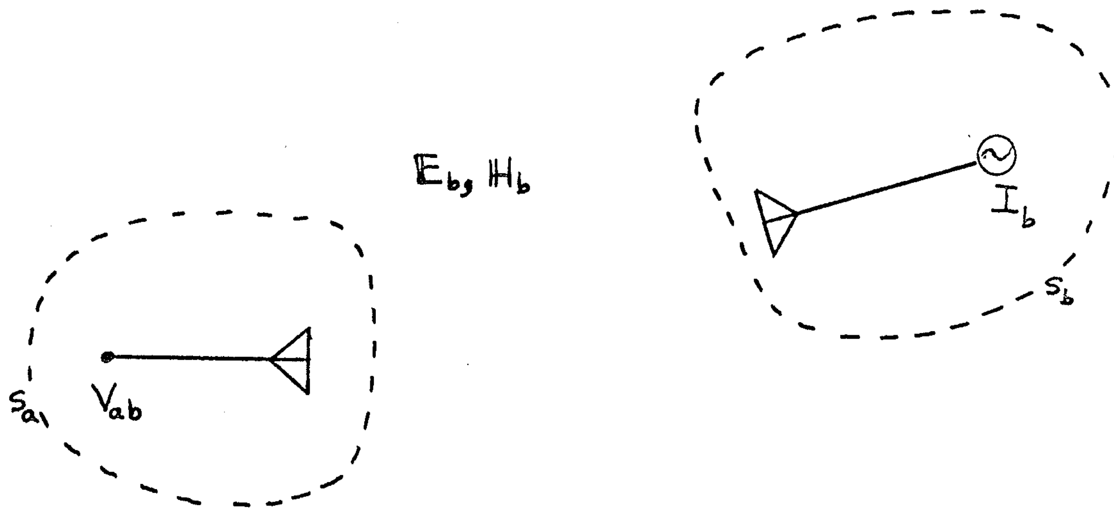


Figure 3-2. Second situation: Antenna "b" transmits and antenna "a" receives.

In terms of the above defined quantities, Kouyoumjian derived the following expressions

$$\langle a,b\rangle = - V_{ba} I_b \qquad \text{.......... } 3 - 8$$

$$\langle b,a\rangle = - V_{ab} I_a \qquad \text{.......... } 3 - 9$$

Equations 3 - 8 and 3 - 9 relate field quantities to terminal (circuit) quantities, and thus, they are reciprocity theorems of the mixed (field-circuit) type.

Impedance in Terms of Reactions

To relate the above reciprocity theorems to the usual circuit theory representation of a two port network, let the two antennas be represented (insofar as their terminal behaviour is concerned) by the following matrix equation

$$\begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} Z_{aa} & Z_{ab} \\ Z_{ba} & Z_{bb} \end{bmatrix} \begin{bmatrix} I_a \\ I_b \end{bmatrix} \qquad \text{..........} \; 3 - 10$$

Let the partial response $V_{ij}$ be the voltage at port "i" due to source $I_j$ at port "j". Each current source sees the other port open-circuited; hence

$$Z_{ij} = \frac{V_{ij}}{I_j} \qquad \text{..........} \; 3 - 11$$

In terms of the circuit reactions, $\langle j,i \rangle = - V_{ij}\, I_j$; thus

$$Z_{ij} = - \frac{\langle j,i \rangle}{I_i\, I_j} \qquad \text{..........} \; 3 - 12$$

Equation 3 - 10 may now be expanded as

$$\begin{aligned} I_a\, V_a &= -\langle a,a \rangle \;-\; \langle a,b \rangle \\ I_b\, V_b &= -\langle b,a \rangle \;-\; \langle b,b \rangle \end{aligned} \qquad \text{..........} \; 3 - 13$$

These equations are to be applied to the analysis of a pair of coupled dipole antennas in Chapter IV.

## A Modified Reaction Integral

Neither equation 3 - 3 nor equation 3 - 5 is suitable for evaluating the reaction $\langle a,b \rangle$ , since the total fields $\mathbb{E}_a$ and $\mathbb{E}_b$ vanish on the perfectly conducting surface of the antenna. Richmond[11] however, presents an alternate equation which is useful for computing $\langle a,b \rangle$ . By resolving the fields into incident and scattered components, he obtains the following expression

$$\langle a,b \rangle = -\iint_{S_b} (\mathbb{E}_a^i \times \mathbb{H}_b) \cdot d\mathbb{S} \qquad \text{.......... 3 - 14}$$

where $\mathbb{E}_a^i$ is the incident electric field intensity. Note that this no longer vanishes on the antenna surface, since the incident, rather than the total field is used.

The surface current $\mathbb{J}_{s_b}$ on the metal can be introduced in place of $\hat{n} \times \mathbb{H}_b$ to obtain the following result

$$\langle a,b \rangle = \iint_{S_b} (\mathbb{E}_a^i \cdot \mathbb{J}_{s_b}) \, dS \qquad \text{.......... 3 - 15}$$

Combining equations 3 - 12 and 3 - 15 gives

$$Z_{ab} = -\frac{1}{I_a I_b} \iint_{S_b} (\mathbb{E}_a^i \cdot \mathbb{J}_{S_b})\, dS \qquad \text{.......... } 3 - 16$$

If the true expressions for $\mathbb{E}_a^i$ and $\mathbb{J}_{S_b}$ were known, they could be substituted into equation 3 - 16, and (at least in principle), the integration performed to obtain the mutual impedance directly. However, such is not the case, and some sort of approximating procedure must be employed.

<u>Constraints</u>

In many cases, evaluation of $\langle a,b \rangle$ by application of any of the defining integrals is impossible because the true fields and sources are unknown. However, it is often possible to determine approximations to the desired reactions by assuming trial fields (or sources) to approximate the true fields (or sources). To be specific, suppose an approximation to the reaction $\langle a,b \rangle$ is desired. Let the correct value of $\langle a,b \rangle$ be denoted by $\langle c_a, c_b \rangle$. (The "c" stands for correct). If it were possible to adjust the approximation $\langle a,b \rangle$ such that

$$\langle a,b \rangle = \langle c_a, c_b \rangle \qquad \text{.......... } 3 - 17$$