A NEW APPROACH TO THE REALIZATION OF NONMINIMUM BIQUADRATIC FUNCTIONS

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ABSTRACT

In this study, a new approach to the synthesis of driving point impedance functions is considered.

The proposed new technique applies the real part padding method to relocate the even part zeros of a nonminimum biquadratic function so that one of a number of specified favourable conditions will be satisfied.

It has been illustrated that the new approach yields a simpler network as compared with other methods in the classes of techniques that do not use either transformers or gyrators. Various combinations of coefficients of the given biquadratic nonminimum impedance function determine the realizability in one of the specified networks. Proof of the method, as well as illustrative examples are provided. in the text.

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CHAPTER I

INTRODUCTION

In general, all existing techniques of realizing positive real functions can be classified into two groups. This classification is based on whether or not the manipulation which disturbs the location of the even part zeros is applied. For a positive real function, the zeros of it's even part exhibit quadrantal symmetry.

The synthesis techniques belonging to the first group subtract a certain amount of positive real number from the given function so as to cause a set of the quadrantal zeros to be relocated on the imaginary axis in the complex s-plane, prior to the removal of the set. These realizations will, in general, require the use of ideal transformers, as was demonstrated by the Brune (1) realization. However, Bott and Duffin (2) showed that by employing the use of an excessive number of elements, ideal transformers may be avoided in the realization.

The second group does not disturb the even part zeros in the complex s-plane. The methods in this group begin immediately to remove whole sets of zeros either at once, or set by set. This group has an advantage over the first group in that the tedious procedure for finding the maximum amount of permissible real part that may be subtracted from the given function is eliminated. However, in general, the second group of techniques requires ideal transformers or gyrators for realizations in cascade structures as was demonstrated by Hazony (4). If the constraint on the network structure is eased, a network may be realized with a smaller number of elements than its counterpart in group one which avoids the use of transformers and gyrators. This realization was developed by Kim (5). Synthesis procedures, belonging to this group, advocated by Miyata (6) and Kuh (7) results in simple realizations, requiring no transformers or gyrators and except for the simplest functions, considerably fewer elements are necessary than are required for the techniques discussed previously. Unfortunately, the applicability of the two procedures depends upon the locations of the even part zeros in the s-plane, consequently, only restricted classes of positive real functions may be realized.

The new approach suggested in this investigation applies the concept of real part padding. This manipulation shifts the sets of even part zeros from their original positions, relocating them somewhere between the locations of group one and group two, such that they now occupy favourable regions in the s-plane. When the even part zeros have been placed in these favourable regions, simple realizations are obtained containing reasonable numbers of elements. The conditions corresponding to these favourable regions in which the even part zeros are to be relocated, have been found (8), (9).

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CHAPTER II

DISCUSSION OF THE PROPOSED TECHNIQUE

In this chapter, the function class, upon which the new method will be applied, is defined. With this definition taken into consideration, a discussion of the theoretical basis of the technique will be undertaken.

I. DEFINITION OF THE FUNCTION CLASS

The function class is the general, biquadratic, driving point impedance function Z(s), and may be written as

$$Z(s) = \frac{K(s^2 + a_1 s + a_o)}{s^2 + b_1 s + b_o}. \tag{2-1}$$

The function Z(s) is the driving point impedance of a lumped, linear, passive, bilateral, and time-invarient network. The assumption is made that the quantities a_1, a_o, b_1, b_o, and K are real, positive, and non-zero numbers. As a result of the above definition and assumptions, Z(s) is a positive real function of the complex variable s. Since the coefficients are positive real numbers, the necessary and sufficient condition that the function Z(s) be a positive real function of the complex variable s is

$$(\sqrt{a_o} - \sqrt{b_o})^2 \leq a_1 b_1 \, . \tag{2-2}$$

Because the coefficients of Z(s) are non-zero, it is automatically minimum reative and minimum susceptive, since it has no poles or zeros, respectively, on the imaginary axis in the complex plane. No generality is lost with the assumption of non-zero coefficients, since the case that does occur with zero coefficients may be handled by reactance or susceptance reduction, yielding a simple and economical realization. This case being solved, it is of no further interest.

II. THE PROPOSED TECHNIQUE

The principles underlying the proposed technique are quite simple. It is based on the fact that the zeros of the even part of a positive real function exhibit quadrantal symmetry. For the general biquadratic function Z(s), as defined earlier, a typical set of even part zeros would appear as shown in Figure 1.

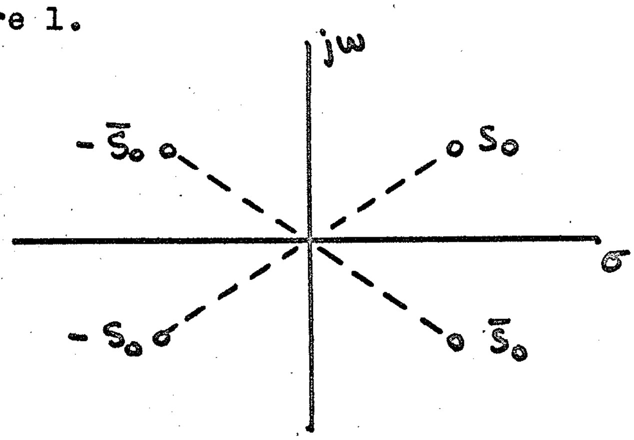

Figure 1. Zeros of the even part of Z(s) exhibiting quadrantal symmetry.

From Figure 1, the zero of the even part of Z(s) located in the first quadrant is defined as

$$s_o = \sigma_o + jw_o. \qquad (2\text{-}3)$$

The real part, o_o, and the imaginary part, w_o, of the even part zero s_o are positive quantities.

In general, the even part zeros shown will not be located such that the desired realization is possible. However, since the function being dealt with is nonminimum, the even part zeros may be relocated by extracting a portion of the minimum resistance from the function. A remainder function, $Z_1(s)$, may be written as

$$Z_1(s) = Z(s) - R_o,$$

or

$$Z_1(s) = \frac{(K - R_o)s^2 + (a_1K - b_1R_o)s + a_oK - b_oR_o}{s^2 + b_1s + b_o}. \quad (2\text{-}4)$$

The remainder function must be positive real, consequently, the value of the extracted resistance, R_o, must not exceed that of the minimum resistance, R_m, of the original function $Z(s)$. The subtraction of R_o from $Z(s)$ tends to shift the even part zeros towards the jw-axis. If, as a consequence of shifiting even part zeros, one or more of the following relations (8), (9) are satisfied;

$$\sigma_1 = w_1, \quad (2\text{-}5)$$

$$\sigma_1U_1 + w_1V_1 = 0, \quad V_1 < 0, \quad (2\text{-}6)$$

$$\sigma_1U_1 - w_1V_1 = 0, \quad V_1 > 0, \quad (2\text{-}7)$$

$$V_1 = 0, \quad (2\text{-}8)$$

$$\sigma_1 = 0, \quad Z_1(0) = 4Z_1(\infty), \quad (2\text{-}9)$$

$$\sigma_1 = 0, \quad Z_1(\infty) = 4Z_1(0), \quad (2\text{-}10)$$

then the synthesis of the given function, $Z(s)$, is complete. The quantities U_1 and V_1 are defined by the equation

$$Z_1(s_1) = U_1 + jV_1, \quad (2\text{-}11)$$

where U_1 and V_1 are the real and imaginary parts, respectively, of $Z_1(s)$ evaluated at the even part zero s_1. From Eq. (2-4), it may be seen that the location of the even part zeros in the complex plane, of the remainder function, is a function of R_o. A proper selection of R_o, where possible values of R_o range from zero to the value of the minimum resistance of Z(s), causes the even part zeros to shift, so that one or more of the conditions in Eqs. (2-5) to (2-10) may be satisfied, resulting in the network shown in Figure 2.

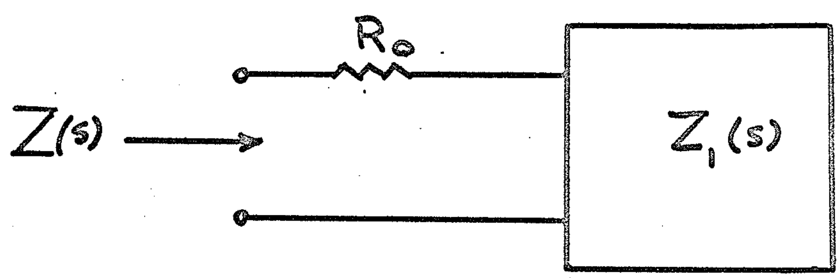

Figure 2. Realization of the given function Z(s).

The networks that realize $Z_1(s)$ are given in Appendix A, and were developed from networks produced by Kim (8). The use of these networks and their associated element values realize $Z_1(s)$ with six elements, consisting of two resistors, two inductors and two capacitors. Consequently, Z(s) is realized with a total of seven elements. If conditions in Eq. (2-9) or Eq. (2-10) are satisfied, then a six element realization, in general, is possible, consisting of three reactive and three resistive elements (9).

The remainder of this study will be concerned with finding expressions for the amount of resistance R_o, that must be extracted from Z(s) in order to satisfy one of the conditions. Also, an attempt will be made to define the subclass of the general class of functions as defined by $Z(s)$, to which the proposed technique will be amenable. This will be carried out by determining the necessary and sufficient conditions on the coefficients of the given function so that the proposed realization technique may be applied successfully. A flow diagram outlining these conditions will be presented.

CHAPTER III

THE REALIZATION OF Z(s)

This chapter will be devoted to determining the amount of resistance that must be removed from Z(s) in order to force the remainder function, $Z_1(s)$, into the class of functions that will satisfy at least one of the desired relations contained in Eqs. (2-5) to (2-10). Consideration of the positive real property of $Z_1(s)$ yields the necessary and sufficient con ditions on the coefficients of the given function, Z(s), required for the specific relation to be satisfied.

I. DETERMINATION OF THE RESISTANCE R_o

Before developing specific expressions for R_o, it will be convenient to determine the quantities o_1, w_1, U_1, and V_1, in terms of the coefficients and the gain constant of Z(s) and as functions of R_o.

Equation (2-4) may be rewritten in an alternate form as the following,

$$Z_1(s) = \frac{(K - R_o)\left[s^2 + \dfrac{(a_1K - b_1R_o)s}{(K - R_o)} + \dfrac{(a_oK - b_oR_o)}{(K - R_o)}\right]}{s^2 + b_1 s + b_o}, \quad (3\text{-}1)$$

or

$$Z_1(s) = \frac{K'(s^2 + a_1' s + a_o')}{s \quad + b_1 s + b_o}. \quad (3\text{-}2)$$

The coefficients in Eq. (3-2), a', a', and K' are defined as

$$a_1' = \frac{a_1K - b_1R_o}{K - R_o}, \tag{3-3}$$

$$a_o' = \frac{a_oK - b_oR_o}{K - R_o}, \tag{3-4}$$

and $$K' = K - R_o. \tag{3-5}$$

The function $Z_1(s)$ must be positive real, consequently, the quantities a_1' and a_o', and the gain conatant K', must be positive numbers. Furthermore, the coefficients of $Z_1(s)$ must satisfy the relation

$$\left(\sqrt{a_o'} - \sqrt{b_o'}\right)^2 \leq a_1'b_1, \tag{3-6}$$

to ensure the positive realness of the function.

The even part of $Z_1(s)$ is given by

$$\text{Ev } Z_1(s) = \frac{N_1(-s^2)}{D_1(-s^2)},$$

or

$$\text{Ev } Z_1(s) = \frac{K'\left[s^4 + s^2\left\{\frac{K(a_o + b_o - a_1b_1) - R_o(2b_o - b_1^2)}{K'}\right\} + \frac{b_o(a_oK - b_oR_o)}{K'}\right]}{s^4 + (2b_o - b_1^2)s^2 + b_o^2}. \tag{3-7}$$

From Figure 1, it may be seen that the even part numerator of $Z_1(s)$ can be written as

$$N(-s^2) = (s - s_1)(s + s_1)(s - \bar{s}_1)(s + \bar{s}_1),$$

or as

$$N(-s^2) = s^4 - 2(\sigma_1^2 - w_1^2)s^2 + (\sigma_1^2 - w_1^2)^2 + 4\sigma_1^2w_1^2, \tag{3-8}$$

when the substitution

$$s_1 = \sigma_1 + jw_1$$

is made. Comparison of the coefficients of s in Eq. (3-7) with the like coefficients of Eq. (3-8) yields the equations

$$-2(\sigma_1^2 - w_1^2)(K - R_o) = K(a_o + b_o - a_1 b_1) - R_o(2b_o - b_1^2), \qquad (3\text{-}9)$$

and

$$(\sigma_1^2 - w_1^2)^2(K - R_o) = b_o(a_o K - b_o R_o). \qquad (3\text{-}10)$$

The similtaneous solution of Eqs. (3-9) and (3-10) for σ_1 and w_1 leads to the following expressions,

$$\sigma_1 = \sqrt{\left[\frac{b_o(a_o K - b_o R_o)}{4(K - R_o)}\right]^{1/2} - \frac{K(a_o + b_o - a_1 b_1) - R_o(2b_o - b_1^2)}{4(K - R_o)}}, \qquad (3\text{-}11)$$

$$w_1 = \sqrt{\left[\frac{b_o(a_o K - b_o R_o)}{4(K - R_o)}\right]^{1/2} + \frac{K(a_o + b_o - a_1 b_1) - R(2b_o - b_1^2)}{4(K - R_o)}}. \qquad (3\text{-}12)$$

Equations (3-11) and (3-12) are the desired form for the real and imaginary parts, respectively, of the even part zero of $Z_1(s)$ located in the first quadrant of the complex plane.

Expressions for U_1 and V_1, the real and imaginary parts of $Z_1(s)$ evaluated at s_1, respectively, may be found by substituting $s_1 = \sigma_1 + jw_1$ into Eq. (3-1) and separating out the real and imaginary parts. The expressions obtained are

$$U_1 = \frac{K(\sigma_1^2 - w_1^2 + a_1\sigma_1 + a_o)(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_o) + Kw_1^2(2\sigma_1 + a_1)(2\sigma_1 + b_1)}{(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_o)^2 + w_1^2(2\sigma_1 + b_1)^2} - R_o, \qquad (3\text{-}13)$$

$$V_1 = \frac{w_1 K(2\sigma_1 + a_1)(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_o) - w_1 K(2\sigma_1 + b_1)(\sigma_1^2 - w_1^2 + a_1\sigma_1 + a_o)}{(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_o)^2 + w_1^2(2\sigma_1 + b_1)^2}. \qquad (3\text{-}14)$$

If the expressions contained in Eqs. (3-11) and (3-12) are substituted into Eqs. (3-13) and (3-14) for σ_1 and w_1, respectively, then the equations for U_1 and V_1 become explicit functions of the resistance R_o.

The relations for the parameters contained in the conditions (2-5) to (2-10) have now been determined. The next step is to find the resistance, R_o, which, when extracted from the given function, Z(s), will yield a positive real remainder function, $Z_1(s)$, that belong to the subclass of functions that satisfies one of the conditions given in Eqs. (2-5) and (2-10).

Case I: $\sigma_1 = w_1$.

When Eq. (3-11) is equated to Eq. (3-12) and the resulting eqation solved for R_o, the result obtained, in terms of the coefficients and the gain constant of Z(s), is

$$R_1 = \frac{K(a_o + b_o - a_1 b_1)}{(2b_o - b_1^2)}. \qquad (3\text{-}15)$$

The value of R_o that solves this particular condition is denoted as R_1.

Case II: $\sigma_1 U_1 + w_1 V_1 = 0, \quad V_1 < 0.$

Substitution of Eqs. (3-13) and (3-14) into Eq. (2-6) yields

$$\begin{aligned} &\sigma_1 K(\sigma_1^2 - w_1^2 + a_1\sigma_1 + a_o)(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_o) \\ &\qquad + \sigma_1^2 K w_1(2\sigma_1 + a_1)(2\sigma_1 + b_1) \\ &\quad - R_o\sigma_1(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_o)^2 - R_o\sigma_1 w_1^2(2\sigma_1 + b_1)^2 \\ &+ w_1^2 K(2\sigma_1 + a_1)(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_o) \end{aligned}$$

$$- w_1^2K(2\sigma_1 + b_1)(\sigma_1^2 - w_1^2 + a_1\sigma_1 + a_o) = 0. \qquad (3\text{-}16)$$

The substitution of Eqs. (3-11) and (3-12), for σ_1 and w_1, respectively, into Eq. (3-16) yields the result

$$R_2 = \frac{K(a_1b_1 - a_o + b_o)}{b_1^2}, \qquad (3\text{-}17)$$

when solved for the resistance R_o. The value of the resistance to be subtracted in order to satisfy the specified condition is designated as R_2.

The mathematical manipulations between Eqs. (3-16) and (3-17) are contained in Appendix B.

<u>Case III: $\sigma_1U_1 - w_1V_1 = 0, \quad V_1 > 0.$</u>

The expressions for U_1 and V_1 contained in Eqs. (3-13) and (3-14), respectively, may be substituted into Eq. (2-7) yielding

$$\begin{aligned}
&\sigma_1K(\sigma_1^2 - w_1^2 + a_1\sigma_1 + a_o)(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_o) \\
&+ \sigma_1w_1^2K(2\sigma_1 + a_1)(2\sigma_1 + b_1) \\
&- \sigma_1R_o(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_o)^2 - \sigma_1w_1^2R_o(2\sigma_1 + b_1)^2 \\
&- w_1^2K(2\sigma_1 + a_1)(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_o) \\
&+ w_1^2K(2\sigma_1 + b_1)(\sigma_1^2 - w_1^2 + a_1\sigma_1 + a_o) = 0. \qquad (3\text{-}18)
\end{aligned}$$

Equation (3-18) may be solved for R_o by substituting the appropriate relations for σ_1 and w_1. The resulting formula for the resistance is

$$R_3 = \frac{K(a_1b_1 + a_o - b_o)}{b_1^2}. \qquad (3\text{-}19)$$

Where R_3 is defined as that value of R_o which causes $Z_1(s)$ to be in the subclass of functions that satisfies the relation, $\sigma_1 U_1 - w_1 V_1 = 0$, at its even part zero.

The details of the derivation of Eq. (3-19) are contained in Appendix C.

Case IV: $V_1 = 0$.

From Eq. (3-14) it may be seen that the imaginary part of $Z_1(s_1)$ can be made equal to zero in two cases. These cases are,

$$1.\quad (2\sigma_1 + a_1)(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_o) - (2\sigma_1 + b_1)(\sigma_1^2 - w_1^2 + a_1\sigma_1 + a_o) = 0, \qquad (3\text{-}20)$$

and

$$2.\quad w_1 = 0. \qquad (3\text{-}21)$$

After substitution into Eq. (3-20) for σ_1 and w_1, it becomes obvious that there is no value of R_o which will satisfy this equation. Equation (3-20) may be satisfied, regardless of the value of R_o, only if the coefficients of $Z(s)$ obey the equality

$$a_o = b_o. \qquad (3\text{-}22)$$

This is demostrated in Appendix D. Consequently, a sufficient condition that the given function be of the subclass of functions which has V_1 set to zero is given by Eq. (3-22). In this cas, resistance need not be subtracted since the function automatically satisfies the condition V_1 equal to zero.

However, a biquadratic function with the coefficients

a_o and b_o equal may be easily realized by a four element network consisting of one inductor, one capacitor, and two resistors, regardless of the numerical value of the other coefficients, a_1 and b_1. Consequently, a simpler realization is availiable than that produced by the proposed technique.

The resistance required to meet the second case may be found by solving Eq. (3-22) for R_o. The expression that is obtained is identical to the expression for the minimum resistance of Z(s) except that the formula holds for an opposite set of inequalities. This appears in a discussion of the minimum resistance of a function and is given in Appendix E. The resistance required for condition (3-22) to hold is;

$$R_4 = \frac{K}{b_1(4b_o - b_1^2)}\left[b_1(a_o + b_o) + a_1(2b_o - b_1^2) + 2\sqrt{b_o(a_o - b_o)^2 + b_o(a_1 - b_1)(a_o b_1 - a_1 b_o)}\right], \tag{3-23}$$

or, if $b_1^2 - 4b_o$ equals zero, the solution for R_4 then reduces to the following,

$$R_4 = \frac{K}{b_1^2}\left[\frac{(a_o - b_o)^2 + a_1 b_1(a_1 b_1 - 2a_o - 2b_o)}{a_1 b_1 - 2(a_o - b_o)}\right]. \tag{3-24}$$

The formulas shown in Eqs. (3-23) and (3-24) are valid only if the following sets of conditions hold,

$$a_o > b_o, \tag{3-25}$$

and

$$a_1 b_1 \geq (a_o - b_o) + b_1^2, \tag{3-26}$$

or if,

$$a_o < b_o, \tag{3-27}$$

and

$$a_1 b_1 \geq (b_o - a_o) + b_1^2 a_o / b_o. \qquad (3\text{-}27)$$

However, it is shown in Appendix E that if the coefficients of the function satisfy either one of the above two sets of conditions, then a five element realization is possible (four if the equality is satisfied).

The proposed method will realize the two cases, in general, with six elements and seven elements, respectively. Consequently, the subclass of functions which satisfy the condition V_1 equal to zero may be realized by techniques which yield more economical realizations than the new method. Anattempt to obtain this subclass by this method would not be justified.

Case V: $\sigma_1 = 0$, $Z_1(0) = 4Z_1(\infty)$.

The condition that the real part σ_1 be zero is a requirement that $Z_1(s)$ be a minimum function. In terms of the coefficients of $Z_1(s)$, the requirement is

$$\left(\sqrt{a_o'} - \sqrt{b_o}\,\right)^2 = a_1' b_1. \qquad (3\text{-}28)$$

Also, in terms of the coefficients of $Z_1(s)$, the requirement that $Z_1(0)$ be equal to $4Z_1(\infty)$ is

$$a_o' = 4b_o. \qquad (3\text{-}29)$$

Substitution of Eq. (3-29) into Eq. (3-28) yields

$$a_o' = 4b_o = 4a_1' b_1. \qquad (3\text{-}30)$$

Equation (3-30) may be solved for R_o equal to R_5, and the result is