GENERALIZATIONS IN THE THEORY OF NILPOTENT GROUPS

BY

D.S.GRANT

A THESIS

PRESENTED TO THE

FACULTY OF GRADUATE STUDIES AND RESEARCH

OF THE

UNIVERSITY OF MANITOBA

IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

AUGUST 1966

TABLE OF CONTENTS

ABSTRACT

INTRODUCTION

In the second edition of his book[1], Zassenhaus showed some directions in which material in the text could be extended and generalized. This paper deals with the extensions and generalizations outlined for the theory of nilpotent groups.

For finite groups the notion of a central element is generalized to the notion of a weakly central element and is developed to the point where a new characterization of nilpotence is obtained. Also with respect to weakly central elements, a radical of a finite group, the nilradical, is isolated.

In the infinite case nilpotence and local nilpotence are developed. For an arbitrary infinite group a radical, the Hirsch radical, appears and is the counterpart to the nilradical for the finite case. The property of an element being periodic is used to show that for a locally nilpotent group, its torsion subgroup is, in fact, a radical.

Further, a locally nilpotent group which has the added property of being torsion-free is shown to have an equivalent property which, when extended slightly, is used to define the rank of a general group. It is shown that the rank so defined is a fairly independent number. Properties of periodic and torsion-free elements are developed when needed.

---

1. Zassenhaus, H. The Theory of Groups, Chelsea, 1958

Finally, formal sums over a ring are set up and some properties, such as local nilpotence, of various subsets of these sums are determined.

CHAPTER I

(BACKGROUND DEFINITIONS AND RESULTS)

In this chapter we enumerate the definitions, identities and results which are basic to the development of the material to follow. Of singular importance in the theory of nilpotent groups is the notion of a commutator.

If a and b are arbitrary elements of a group G, then the commutator, $[a,b]$, of a and b is defined by:

$$[a,b] = a^{-1}b^{-1}ab\,. \tag{1}$$

If, further, we define

$$a^b = b^{-1}ab \tag{2}$$

and

$$[[x_1,x_2,\ldots,x_{n-1}],x_n] = [x_1,x_2,\ldots,x_n] \tag{3}$$

then some immediate consequences of (1) are:

$$a[a,b] = a^b \tag{4}$$

$$[a,b]^{-1} = [b,a] \tag{5}$$

$$[a,bc] = [a,c][a,b][a,b,c] \tag{6}$$

$$= [a,c][a,b]^c \tag{7}$$

$$[ab,c] = [a,c][a,c,b][b,c] \tag{8}$$

$$= [a,c]^b[b,c]\,. \tag{9}$$

These commutator identities are easily verifiable by expanding the right hand sides.

If A and B are subsets of an arbitrary group G, then the subgroup generated by all commutators $[a,b]$ with a in A, b in B, is denoted by $[A,B]$. In symbols

$$[A,B] = \langle [a,b] : a \in A, \ b \in B \rangle. \tag{10}$$

By virtue of property (5) we have

$$[A,B] = [B,A]. \tag{11}$$

If H and K are two normal subgroups of the group G, and H is a subgroup of K, then K/H is said to be a central factor of G if and only if K/H is a subgroup of the centre of G/H i.e., $K/H \leq Z(G/H)$. Now $K/H \leq Z(G/H)$ if and only if $[g,k] \in H$ for any $g \in G$ and $k \in K$, consequently a characterization of the central factor is that K/H is a central factor of G if and only if $[K,G] \leq H$.

A group G is said to be nilpotent if and only if it has a finite series of normal subgroups with each factor a central factor. That is, G has a finite series

$$G = G_0 \geq G_1 \geq \ \ldots \ \geq G_s = E \tag{12}$$

where $G_i \trianglelefteq G$ for all i and $G_i/G_{i+1} \leq Z(G/G_{i+1})$ for $i = 0,\ldots,s-1$. The series (12) is called a central series of G and its length is defined to be s. If $s_1, s_2, \ldots$ are the lengths of various central series of G, then the class of G is defined to be $\min\{s_1, s_2, \ldots\}$.

The upper (or ascending) central series of a group G,

$$E = Z^0(G) \leq Z^1(G) \leq \ \ldots \tag{13}$$

is defined by:

$$Z^0(G) = E$$

$$Z^{i+1}(G) = Z(G/Z^i).$$

If (12) is any central series of the group G, then by induction on i it can be shown[1] that

---

1. Scott, W.R. <u>Group Theory</u>, Prentice-Hall, 1964

$$G_{s-i} \leqslant Z^i(G) \tag{14}$$

and the following characterization of nilpotence is a direct consequence of (14).

Theorem 1 A group G is nilpotent of class c if and only if G has an upper central series

$$E = Z^0(G) \leqslant Z^1(G) \leqslant \dots \leqslant Z^c(G) = G$$

of length c.

The lower (or descending) central series of the group G,

$$G = Z_1(G) \geqslant Z_2(G) \geqslant \dots \tag{15}$$

is defined recursively by:

$$Z_1(G) = G$$
$$Z_{i+1}(G) = [Z_i(G), G] .$$

Again[1] by induction on i, if (12) is any central series for the group G we have

$$Z_{i+1}(G) \leqslant G_i \qquad i = 0,1,\dots,s \tag{16}$$

and, as a consequence of (14) and (16), the following.

Theorem 2 The group G is nilpotent of class c if and only if G has a lower central series

$$G = Z_1(G) \geqslant Z_2(G) \geqslant \dots \geqslant Z_{c+1}(G) = E$$

of length c.

The property of being nilpotent is an hereditary one. That is subgroups and factor groups of a nilpotent group are themselves nilpotent. The product of two normal nilpotent subgroups of a group is also nilpotent. In order to prove this we need:

---

1. Scott, W.R. Group Theory, Prentice-Hall, 1964

<u>Lemma</u> 3 If A, B and K are normal subgroups of the group G, then

$$[KA,B] = [K,B][A,B]$$

and

$$[A,KB] = [A,B][K,A] .$$

Proof: Since $[A,B]$ is a normal subgroup of G we have $[K,B][A,B]$ is a subgroup.

Now $[KA,B] = \langle [ka,b] : k \in K, a \in A, b \in B \rangle$

and $[ka,b] = [k,b]^a [a,b]$ by (9).

Also $[k,b]^a \in [K,B]$ since $[K,B] \trianglelefteq G$, and $[a,b] \in [A,B]$

therefore $[ka,b] \in [K,B][A,B]$

that is $[KA,B] \leq [K,B][A,B]$.

Conversely, the generators $[ka,b]$ of $[K,B]$ are contained in $[KA,B]$ since $[k,b] = [ka,b]$ with $a = 1$. Therefore $[K,B] \leq [KA,B]$.

Similarly $[A,B] \leq [KA,B]$ and so $[K,B][A,B] \leq [KA,B]$ due to the normality of $[A,B]$ in $[KA,B]$. Therefore $[KA,B] = [K,B][A,B]$.

A similar argument yields

$$[A,KB] = [A,B][K,A] .$$

<u>Theorem</u> 4 If A and B are any two normal subgroups of the group G, then

$$Z_i(AB) \leq Z_i(B) \{Z_1(A) \cap Z_{i-1}(B)\}\{Z_2(A) \cap Z_{i-2}(B)\} \ldots \{Z_{i-1}(A) \cap Z_1(B)\} Z_i(A) .$$

Proof: Define the weight of the commutator (3) to be n. Then by repeated application of lemma 3, the commutator of weight i in AB is equal to a product of commutators of weight i in F_j where F_j is an A or a B.

That is

$$[AB,AB,\ldots,AB] = \prod [F_1,F_2,\ldots,F_i] \qquad F_j = A \text{ or } B \quad j = 1,\ldots,i.$$

Consider one factor $[F_1,F_2,\ldots,F_i]$ in the product.

Case 1: $F_j = A$ or $F_j = B$ for all j

then $[A,A,\ldots,A] = Z_i(A)$

or $[B,B,\ldots,B] = Z_i(B)$.

Case 2: There are j A's where $0 < j < i$.

Then there are $i-j$ B's, and since $[A,B] \leq A \cap B$ for $j = 1$

we have $[F_1,\ldots,F_i] \leq A = Z_1(A)$. Assume true for j,

that is $[F_1,\ldots,F_i] \leq Z_j(A)$.

For $j+1$ reduce $[F_1,\ldots,F_i]$ to $[Z_j(A),A,B,\ldots]$ to get

$[F_1,F_2,\ldots,F_i] \leq Z_{j+1}(A)$. An identical argument shows that it holds for B also and so the conclusion of the theorem is true.

Corollary: If A and B are normal subgroups of the group G and both are nilpotent, then AB is nilpotent.

Proof: If A and B have class a and b respectively, then for $i \geqslant a+b$ all the terms on the right hand side of the expansion for $Z_i(AB)$ in theorem 4 are the identity. Therefore AB is nilpotent by theorem 2.

CHAPTER II

(A CHARACTERIZATION OF NILPOTENCE)

An element of the group G is said to be a central element if it lies in the centre, $Z(G)$, of G, that is, if it commutes with all the elements of G. In other words x is a central element of G if and only if $[a,x] = 1$ for all a contained in G. It is this notion which is now generalized and used to give another characterization of nilpotence.

We begin with

<u>Definition</u> 1
$$[x,y;1] = [x,y]$$
$$[x,y;n] = [[x,y;n-1],y].$$

<u>Definition</u> 2 The element g in the group G is called weakly central if for every element x in G there exists an integer n such that $[x,g;n] = 1$.

An immediate consequence of this definition is the following, which gives a relationship between nilpotence and weak centrality.

<u>Theorem</u> 1 If the element g belongs to a nilpotent normal subgroup N of the group G, then g is weakly central.

Proof: Let N be nilpotent of class r-1 and let the lower central series of N be

$$N = Z_1(N) > Z_2(N) > \dots > Z_r(N) = E.$$

Let x be contained in G and g in N, then since N is normal

$$(g^{-1})^x \in N^x = N.$$

Therefore since N is a group $(g^{-1})^x g \in N$,

i.e., $[x,g;1] \in N = Z_1(N)$.

Now assume $[x,g;i] \in Z_i(N)$

then $[x,g;i+1] = [[x,g;i],g] \in [Z_i(N),N] = Z_{i+1}(N)$.

Therefore $[x,g;r] = 1$ since $Z_r(N) = E$.

This theorem, together with the next three lemmas, will allow us to characterize a finite nilpotent group as one, all of whose elements are weakly central.

Lemma 2 If in a finite group G the intersection of the p-Sylow subgroup S with any different p-Sylow subgroup is the identity E, then for $s \in S$, $s \neq 1$ and $x \in G$ we have $[x,s] \in N_G(S)$ if and only if $x \in N_G(S)$. $N_G(S)$ is the normalizer of S in G.

Proof: Clearly for $x \in N_G(S)$ $[x,s] \in N_G(S)$ for any $s \in S$.

Suppose for $s \in S$, $x \in G$ that $[x,s] \in N_G(S)$

then $$S^{x^{-1}s^{-1}xs} = S$$

hence $$S^{x^{-1}s^{-1}x} = S$$

and so $$x^{-1}s^{-1}x \in S^x \cap N_G(S).$$

Since $N_G(S)$ contains no elements of order p except those in S, it follows that $x^{-1}s^{-1}x \in S$. Therefore $S = S^x$.

Lemma 3 If for a prime p the elements of p-power order of a group G are weakly central, then the intersection of two different p-Sylow subgroups is the identity E.

Proof:

By induction on the order of G, let G be a minimal counterexample. Let D be the maximal intersection of its p-Sylow subgroups. If $N_G(D)$ is a proper subgroup of G then the lemma is true for this group. That is, for the p-Sylow subgroups Q_1 and Q_2 of $N_G(D)$ we have $Q_1 \cap Q_2 = E$. But $Q_1 \cap Q_2 = D$ and $D \neq E$, so we have a contradiction.

Therefore D is a normal subgroup of G.

Now consider G/D. The lemma is true for this group. Let Q be a p-Sylow subgroup of G/D, then for a p-Sylow subgroup P of G/D, $P \neq Q$, we have $Q \cap P = E$. All the elements of Q are weakly central, that is $[x,q;n] = 1$ for all $x \in G/D$, $q \in Q$, n finite. Therefore

$[x,q;n] = [[x,q;n-1],q] \in N_{G/D}(Q)$.

By lemma 2, since $q \in Q$, $[x,q;n-1] \in N_{G/D}(Q)$.

By induction $x \in N_{G/D}(Q)$.

Therefore Q is normal in G/D. But $Q = S/D$ where S is a p-Sylow subgroup of G. Consequently S is normal in G.

Contradiction! Therefore $D = E$.

<u>Lemma</u> 4 If the elements of p-power order of a group G are weakly central, then they form a normal subgroup.

Proof:

Let S be a p-Sylow subgroup of the group G, then by lemma 3 for a different p-Sylow subgroup P we have $S \cap P = E$. Therefore for $x \in G$ and $s \in S$, $[x,s] \in N_G(S)$ implies $x \in N_G(S)$ by lemma 2. The elements of S being of p-power order are weakly central, therefore $[x,s;n] = 1$ for finite n and all x in G.

Hence $[[x,s;n-1],s] \in N_G(S)$

and so $[x,s;n-1] \in N_G(S)$.

Repeating the process yields $x \in N_G(S)$ for all $x \in G$,

i.e., $G = N_G(S)$.

We are now in a position to prove the main result of the chapter.

<u>Theorem</u> 5 A finite group G is nilpotent if and only if every element of G is weakly central.

Proof:

If G is nilpotent, then by theorem 1 since G is a normal nilpotent subgroup all elements are weakly central.

Now assume all the elements of G are weakly central. Then by lemma 4 all p-Sylow subgroups, for all primes p, are normal in G. Hence G is nilpotent.

CHAPTER III

(THE NILRADICAL)

A radical, $Rad_P(G)$, of a group G may be defined in general to be a unique maximal normal P-subgroup of G where P is some group property, and $Rad_P(G/Rad_P(G))$ is the identity E. Thus in a nilpotent group G, for a prime p, a p-Sylow subgroup is a radical of G with respect to being a p-group. In short a radical contains all elements which have a certain property.

Now we define a set called the nilradical and determine that it is a radical.

<u>Definition</u> 1 Let $W = \{x \in G: \text{ x is weakly central}\}$ and define $R(G) = \{g \in G: gx \text{ and } g^{-1}x \in W \text{ for all } x \in W\}$. The set R(G) is called the nilradical of G.

<u>Theorem</u> 1 The nilradical R(G) of the group G is a characteristic subgroup of G.

Proof:

Since $1x$ and $1^{-1}x$ are contained in W for all x in W, we have that 1 is contained in R(G) and hence that R(G) is not empty.

If $g \in R(G)$ then $gx = (g^{-1})^{-1}x \in W$ for all $x \in W$ and $g^{-1}x \in W$ for all $x \in W$, therefore $g^{-1} \in R(G)$.

If $g,h \in R(G)$ then $g^{-1}x$ and $hx \in W$ for all $x \in W$. Consequently $h^{-1}(g^{-1}x) = (gh)^{-1}x$ and $g(hx) = (gh)x$ are contained in W for all $x \in W$. Therefore $gh \in R(G)$ and we have that R(G) is a subgroup of G.

To show that R(G) is characteristic, let a be an automorphism of G, $x \in W$, then

$$[g,x]^a = [g^a,x^a] \quad \text{for } g \in G.$$

By induction $[g,x;n]^a = [g^a,x^a;n]$ for all n.

Let $x \in W$ then for any $g \in G$ we have that $g = h^a$ for some $h \in G$. Also $[h,x;n] = 1$ for some n. Therefore $[h,x;n]^a = 1^a = 1$, hence $[h^a,x^a;n] = [g,x^a;n] = 1$ and $x^a \in W$. Thus $W^a \subseteq W$. Since automorphisms are invertible $W \subseteq W^a$ and so $W = W^a$.

If $g \in R(G)$ then $gx \in W$ for all $x \in W$ and

$$(gx)^a = g^a x^a \in W.$$

As x varies over all the elements of W so does x^a.

Therefore $g^a \in R(G)$ and hence $R(G)$ is characteristic in G.

Lemma 2 If N is a normal subgroup of the group G, then for $i = 0,1,2,\ldots$ given $a \in N$, $x \in G$, $n_i \in Z_i(N)$ there exists $n_{i+1} \in Z_{i+1}(N)$ such that

$$[n_i, ax;k] = [n_i,x;k]\, n_{i+1} \quad \text{for } k = 1,2,\ldots$$

where we let $n_0 \in G$.

Proof:

Since N is normal in G we have $[n_0,a]^x \in N$, and since $Z_{i+1}(N)$ is characteristic we have $[n_i,a]^x \in Z_{i+1}(N)$ for $i \geq 1$.

For any i induct on k. For k = 1

$$\begin{aligned} [n_i, ax;k] &= [n_i, ax] \\ &= [n_i,x]\,[n_i,a]^x \\ &= [n_i,x]\, n_{i+1} \quad \text{since } [n_i,a]^x \in Z_{i+1}(N). \end{aligned}$$

Now assume true for k, i.e.,

$$[n_i,ax;k] = [n_i,x;k]\, n_{i+1} \quad \text{for some } n_{i+1} \in Z_{i+1}(N)$$

then

$$\begin{aligned} [n_i,ax;k+1] &= [[n_i,ax;k],ax] \\ &= [[n_i,x;k]\, n'_{i+1}, ax] \quad \text{where } n'_{i+1} \in Z_{i+1}(N) \end{aligned}$$

$$= [s_k n'_{i+1}, ax] \qquad \text{where } s_k = [n_i, x; k]$$
$$= [s_k, ax][s_k, ax, n'_{i+1}][n'_{i+1}, ax]$$
$$= [s_k, x][s_k, a][s_k, a, x][s_k, ax, n'_{i+1}][n'_{i+1}, ax]$$
$$= [n_i, x; k+1]\, n_{i+1}$$

where $n_{i+1} = [s_k, a][s_k, a, x][s_k, ax, n'_{i+1}][n'_{i+1}, ax] \in Z_{i+1}(N)$.

<u>Theorem</u> 3 If N is a normal subgroup of the group G and N is nilpotent, then N is contained in the nilradical R(G).

Proof:

Suppose the class of N is r. Now for any $x \in W$ we have that for any $g \in G$ there exists a finite k such that

$$[g, x; k] = 1.$$

Then for an arbitrary $a \in N$ we have, by repeated application of lemma 2,

$$\begin{aligned}
[g, ax; (r+1)k] &= [[g, ax; k], ax; rk] \\
&= [[g, x; k]n_1, ax; rk] \qquad \text{where } n_1 \in Z_1(N) \\
&= [n_1, ax; rk] \\
&= [[n_1, ax; k], ax; (r-1)k] \\
&= [[n_1, x; k]n_2, ax; (r-1)k] \qquad n_2 \in Z_2(N) \\
&\ \ \vdots \\
&= n_{r+1} = 1 \qquad \text{where } n_{r+1} \in Z_{r+1}(N).
\end{aligned}$$

Equivalently if a is any element of N then $ax \in W$ for all $x \in W$. By definition the elements of N are contained in R(G) and so N is a subgroup of the nilradical.

<u>Theorem</u> 4 For a finite group G the nilradical R(G) is the maximal normal nilpotent subgroup of G.

Proof:

By theorem 1 we have R(G) normal in G. Every element of R(G) is weakly central and therefore $R(G)$ is nilpotent by theorem 5 of chapter II. Lemma 3 above states that all normal nilpotent subgroups of G are contained in $R(G)$. Consequently $R(G)$ is the maximal normal nilpotent subgroup of G.

To say that the subgroup H of the group G is subnormal in G means that there is a finite chain of subgroups

$$H = H_1 \leq H_2 \leq \dots \leq H_n = G$$

beginning with H and ending with G such that each is normal in the succeeding one. We now identify the nilradical further.

<u>Theorem</u> 5 The nilradical $R(G)$ of a finite group G consists of all the elements that generate a subnormal subgroup of the full group.

Proof:

Let N be a subnormal subgroup of G generated by an element of G. Then we have

$$N = N_r \lhd N_{r-1} \lhd \dots \lhd N_1 \lhd N_0 = G$$

Since $R(N_{r-i})$ is a characteristic subgroup of N_{r-i} and N_{r-i} is normal in $N_{r-(i+1)}$ we have that $R(N_{r-i}) \leq R(N_{r-(i+1)})$ for $i = 0,1,\dots,r-1$. Now N_r is cyclic, hence nilpotent and thus $N_r \leq R(N_{r-1})$. Consequently $N_r \leq R(N_0) = R(G)$.

Now let N be generated by an element from $R(G)$. Since $R(G)$ is nilpotent and finite, the chain

$$N \lhd N_{R(G)}(N) \lhd N_{R(G)}(N_{R(G)}(N)) \lhd \dots$$

must be finite and terminate in $R(G)$. But $R(G)$ is normal in G, hence N is subnormal in G.

CHAPTER IV

(PROPERTIES OF INFINITE GROUPS)

So far we have considered essentially only finite groups. Less restrictive conditions than finiteness are now adopted. In particular the maximal condition for subgroups and the condition of finite generation will replace finiteness.

The lower central series for a general nilpotent group is investigated further and used to show the connection between the above conditions and nilpotence.

The following lemma is needed in the sequal.

Lemma 1 If N is a normal subgroup of the group G and A and B are two subgroups of G satisfying

$$A \cap N = B \cap N, \; AN = BN, \; A \leq B$$

then $A = B$.

Proof:

If $x \in B$, then $x \in BN$. Now $BN = AN$ therefore $x = an$ for some $a \in A$ and $n \in N$. Hence $n = a^{-1}x \in B$ since $A \leq B$. Consequently $n \in B \cap N = A \cap N$, that is $n \in A$.

Therefore $x = an \in A$ and so $B \leq A$.

Definition 1 The group G is said to satisfy the maximal condition if in every increasing sequence of subgroups $A_1 < A_2 < \dots$ there exists an index after which all members of the sequence are equal.

An alternative to this definition and one which is sometimes easier to apply is that a group satisfies the maximal condition if and only if in every set of subgroups of the group there is a subgroup contained in no other subgroup of the set.

In order to illustrate this, suppose the maximal condition holds for the group G. Let $\{A_1,\ldots,A_n,\ldots\}$ be a set of subgroups of G. If A_{i_1} say is contained in another, A_{i_2}, then we have $A_{i_1} < A_{i_2}$. If A_{i_2} is contained in A_{i_3}, then we have $A_{i_1} < A_{i_2} < A_{i_3}$. Continuing in this way we get an increasing sequence of subgroups. The maximal condition states that the sequence terminates and so there exists a subgroup which contains no other.

Now suppose we have a chain of subgroups $A_1 < A_2 < \ldots$. Let $S = \{A_i\}$ be the collection of all subgroups of the chain. Now there exists one, say A_n, which is contained in no other, that is $A_n = A_{n+i}$ for $i = 1,2,\ldots$, and so the maximal condition holds.

<u>Theorem</u> 2 A group G satisfies the maximal condition if and only if every subgroup of G is finitely generated.

Proof:

Let H be a subgroup of G and suppose H is not finitely generated. Choose h_1 in H and form $\langle h_1\rangle$. Choose h_2 in H, $h_2 \neq h_1$, and form $\langle h_1,h_2\rangle$ to get $\langle h_1\rangle \leq \langle h_1,h_2\rangle$. Continue in this way to get the infinite series $\langle h_1\rangle \leq \langle h_1,h_2\rangle \leq \langle h_1,h_2,h_3\rangle \leq \ldots$ which contradicts the maximal condition for G. Therefore H is finitely generated.

Suppose that every subgroup is finitely generated. Let $H_1 \leq H_2 \leq \ldots$ be an ascending chain of subgroups in G. Let $H = \{b : b \in H_i \text{ for some } i\}$. Then H is a subgroup of G for if $b \in H_i$ and $c \in H_j$, then $bc \in H_k$ where $k = \max\{i,j\}$ hence the product is closed. Now H is finitely generated by hypothesis so let $H = \langle h_1,h_2,\ldots,h_n\rangle$. Let H_{j_k} be the first H_i containing h_k for $k = 1,2,\ldots,n$. If $m = \max\{j_1,\ldots,j_n\}$ then $H = H_m = H_i$ for $i \geq m$.

<u>Theorem</u> 3 The maximal condition is an hereditary one. That is if the group G satisfies the maximal condition then subgroups and factor groups also satisfy it.

Proof:

The relation "is a subgroup of" is transitive, therefore any set of subgroups of a subgroup H of G, as a set of subgroups of G, will contain a maximal element in G and hence in H. Consequently H satisfies the maximal condition.

If K is a subgroup of the factor group G/N then it will be the image of a unique subgroup H, containing N, under the natural homomorphism. Since H is finitely generated, K will be also. Therefore all subgroups of G/N are finitely generated. By theorem 2 we have G/N satisfying the maximal condition.

<u>Theorem</u> 4 If the normal subgroup N of the group G, and the factor group G/N satisfy the maximal condition, then G satisfies the maximal condition.

Proof:

Let S be a set of subgroups of G and consider the set of subgroups $\{K \cap N : K \in S\}$ of N. Since N satisfies the maximal condition this set has a maximal element A. Now consider the set of subgroups $\{KN/N : K \in S, K \cap N = A\}$ of G/N. We know G/N satisfies the maximal condition so this set has a maximal element B. That is, there exists a $K \in S$ such that $K \cap N = A$ and $KN/N = B$. This K is maximal in S, for if not then there exists a $K' \in S$ such that $K < K'$, $K' \cap N = A$, and $K'N/N = B$. By lemma 1 we have $K = K'$ which is a contradiction. Therefore K is maximal in S and hence G satisfies the maximal condition.

<u>Theorem</u> 5 Every finite group and every cyclic group satisfies the maximal condition.

Proof:

For a finite group G there are a finite number of subgroups. In any set S of subgroups of G let $K \in S$ have maximal order. Since the order of K divides the order of G, K must be maximal.

For cyclic groups we have that every subgroup of a cyclic group is cyclic, therefore finitely generated. Consequently it satisfies the maximal condition by theorem 2.

<u>Theorem</u> 6 If, for a group G, a normal chain

$$G = N_0 \rhd N_1 \rhd \quad \ldots \quad \rhd N_s = E$$

exists such that N_i/N_{i+1} is either finite or infinitely cyclic, then G satisfies the maximal condition.

Proof:

Clearly N_s satisfies the maximal condition. Assume N_{i+1} satisfies the maximal condition. Now N_i/N_{i+1} being finite or infinitely cyclic implies N_i/N_{i+1} satisfies the maximal condition by the previous theorem. Therefore N_i satisfies the maximal condition by theorem 4. By induction G does also.

<u>Theorem</u> 7 If the group G is finitely generated and abelian, then it satisfies the maximal condition.

Proof:

Let $G = \langle a_1, a_2, \ldots, a_n \rangle$ and define

$N_i = \langle a_1, a_2, \ldots, a_i \rangle$ for $i = 1, 2, \ldots, n$. Then we have

$$G = N_n \rhd N_{n-1} \rhd \quad \ldots \quad \rhd N_1 \rhd \quad E$$

where the N_i are normal since G is abelian. Also N_i/N_{i-1} is isomorphic to $\langle a_i \rangle$, that is the factors are infinite cyclic. By theorem 6 we have that G satisfies the maximal condition.

Theorem 8 If the group G is generated by the set K, then $Z_i(G)$ is generated by all commutations $[k_1 \ldots k_i]$ of weight i, where the k_j run independently over K, and $Z_{i+1}(G)$; i.e.,

$$Z_i(G) = \langle [k_1,\ldots k_i] , Z_{i+1}(G) : k_j \in K \rangle.$$

Proof:
For $i = 1$ we have $G = Z_1(G) = \langle k_1, Z_2(G) \rangle$ and we have a basis for induction. Assume $Z_i(G) = \langle [k_1 \ldots k_i] , Z_{i+1}(G) \rangle$.
By definition $Z_{i+1}(G) = [Z_i(G), G] = \langle [g_i, b] : g_i \in Z_i(G), b \in G \rangle$.
By hypothesis $g_i = \prod (\prod [k_1, \ldots, k_i]^{\pm 1}) g_{i+1}$ where $g_{i+1} \in Z_{i+1}(G)$.
Let $a = \prod [k_1 \ldots k_i]^{\pm 1}$ then a typical generator of $Z_{i+1}(G)$ is of the form

$$[ag_{i+1}, b] = [a,b][a,b,g_{i+1}][g_{i+1},b]$$
$$= [a,b] \quad \text{modulo } Z_{i+2}(G).$$

Now $$1 = [aa^{-1},b] = [a,b][a,b,a^{-1}][a^{-1},b]$$
$$= [a,b][a^{-1},b] \mod Z_{i+2}(G),$$

that is, $$[a,b]^{-1} = [a^{-1},b] \mod Z_{i+2}(G).$$
Similarly $$[a,b]^{-1} = [a,b^{-1}] \mod Z_{i+2}(G)$$
and $$[ab,c] = [a,c][b,c] \mod Z_{i+2}(G)$$
$$[a,bc] = [a,b][a,c] \mod Z_{i+2}(G).$$

Now induct on the sum of the length of a and b.
If both a and b are of length 1 then $a = [k_1 \ldots k_i]^{\pm 1}$ and $b = k_{i+1}^{\pm 1}$; hence $[a,b] = [[k_1 \ldots k_i], k_{i+1}]^{\pm 1} \mod Z_{i+2}(G)$ and the theorem is true. Assume true for length a + length b $< m$.
For length a + length b = m we have $a = a_1 a_2$ and $b = b_1 b_2$ where

$$\text{length } a_1 + \text{length } b_1 < m$$
$$\text{length } a_1 + \text{length } b_2 < m$$
$$\text{length } a_2 + \text{length } b_1 < m$$
$$\text{length } a_2 + \text{length } b_2 < m$$

and $[a,b] = [a_1 a_2, b_1 b_2]$

$$= [a_1,b_2][a_2,b_2][a_1,b_1][a_2,b_1] \mod Z_{i+2}(G).$$

Corollary: If $G = \langle K\rangle$ is nilpotent then $Z_i(G) = \langle [g_1 \ldots g_i] : g\text{'s} \in G\rangle$.

Proof:

Since G is nilpotent, $Z_{c+1}(G) = E$ for some c and so $Z_c(G) = \langle [k_1 \ldots k_c] : k\text{'s} \in K\rangle$. That is, for i = 1, $Z_{c-i+1}(G) = \langle [g_1 \ldots g_{c-i}]\rangle$. Assume true for $< i$, then

$$Z_{c-i+1} = \langle [k_1, \ldots k_{c-i}], Z_{c-i+2}\rangle$$
$$= \langle [k_1 \ldots k_{c-i}], [m_1, \ldots m_{c-i}, m_{c-i+1}]\rangle$$

but $[m_1 \ldots m_{c-i+1}] = [[m_1 m_2], \ldots m_{c-i+1}]$, i.e. commutator of length c-i.

<u>Theorem</u> 9 If G is finitely generated and nilpotent, then all factors of the lower central series are finitely generated.

Proof:

By the previous result, if $G = \langle K\rangle$ where K is finite, then $Z_i(G)/Z_{i+1}(G) = \langle [k_1, k_2, \ldots, k_i] : k\text{'s vary over } K\rangle$. Clearly there are a finite number of generators for $Z_i(G)/Z_{i+1}(G)$.

<u>Theorem</u> 10 If G is a finitely generated nilpotent group, then G satisfies the maximal condition.

Proof:

Let G be nilpotent of class c+1. Now Z_{c+1} is normal in Z_c and satisfies the maximal condition. Also Z_c/Z_{c+1} is finitely generated and abelian, and hence by theorem 4 satisfies the maximal condition.

Assume Z_{c-i+1} satisfies the maximal condition. By theorem 9 we have Z_{c-i}/Z_{c-i+1} is finitely generated and since it is contained in the centre of G/Z_{c-i+1} it is abelian, and hence by theorem 7 satisfies the maximal condition. Therefore Z_{c-i} satisfies the maximal condition by theorem 4. By induction, G satisfies the maximal condition.

CHAPTER V

(LOCAL NILPOTENCE AND THE HIRSCH RADICAL)

In this chapter the condition of local nilpotence is considered and several properties illustrated. The counterpart to the nilradical is the Hirsch radical, developed here, and is shown to be the nilradical when the group is finite.

To begin with we consider a more general concept.

Definition 1 Let P be a group property. A group G is said to be a local P group, or locally P, if every finitely generated subgroup of G has the property P.

Theorem 1 Every subgroup H of a local P group G is a local P group.

Proof:

The finitely generated subgroups of H, as subgroups of G, have the property P since G is locally P. Hence H is a local P group.

Theorem 2 Every local P-subgroup of G can be embedded in a maximal local P-subgroup of G.

Proof:

Let $G_a \leq G_b \leq \ldots$ be a series of local P-subgroups and $\{G_a\}_{a \in L}$ the set of these. Let $H = \bigcup_{a \in L} G_a$. Let $g_1, g_2, \ldots, g_r \in H$, then there exists an $a \in L$ such that $g_1, g_2, \ldots, g_r \in G_a$ and $\langle g_1, \ldots, g_r \rangle \leq G_a$. Therefore $\langle g_1, \ldots, g_r \rangle$ has property P since G_a is a local P-group. Consequently H is a local P-subgroup and a bound for the chain. By Zorn's lemma a maximal element exists.

When the group property P is nilpotence, we have then that subgroups of locally nilpotent groups are locally nilpotent, and that a locally nilpotent subgroup can be embedded in a maximal one. To continue the investigation of locally nilpotent groups we need the two lemmas:

Lemma 3 If K and L are non-empty subsets of the group G, $U = \langle [K,L], L \rangle$, V the smallest normal subgroup of U containing $[K,L]$, W the smallest normal subgroup of $\langle K,L \rangle$ containing $[K,L]$, then

$$\langle V,K \rangle \trianglelefteq \langle V,K,L \rangle,$$
$$V \leq W \leq \langle V,K \rangle,$$
$$\langle W,K \rangle \trianglelefteq \langle K,L \rangle,$$
$$\langle W,L \rangle \trianglelefteq \langle K,L \rangle.$$

Proof:

A typical generator of $\langle V,K \rangle$ is vk where $v \in V$, $k \in K$. V and K are both contained in $\langle V,K \rangle$ and hence are both contained in the normalizer of $\langle V,K \rangle$. Also, if $t \in L$, then

$$(vk)^t = v^t k^t$$
$$= v^t k [k,t] \in \langle V,K \rangle.$$

Since the conjugate of a generator is in $\langle V,K \rangle$ the conjugate of a product is in $\langle V,K \rangle$ also. Clearly $(v^{-1})^t$ and $(k^{-1})^t$ are in $\langle V,K \rangle$, therefore L is in the normalizer of $\langle V,K \rangle$.

Hence $\langle V,K \rangle \trianglelefteq \langle V,K,L \rangle$.

Now $\langle V,K,L \rangle = \langle K,L \rangle$ and $[K,L] \leq \langle V,K \rangle \trianglelefteq \langle K,L \rangle$ so by the minimality of W, $W \leq \langle V,K \rangle$.

Also $W \cap U \trianglelefteq U$ and so $V \leq W$, by the minimality of V.

Consider a typical generator wk of $\langle W,K \rangle$ where $w \in W$, $k \in K$. For $t \in L$ we have

$$(wk)^t = w^t k^t$$
$$= w^t k [k,t] \in \langle W,K \rangle.$$

Again, the conjugate of a generator of $\langle W,K\rangle$ being in $\langle W,K\rangle$ implies that the conjugate of a product of generators of $\langle W,K\rangle$ is in $\langle W,K\rangle$ and hence that L is contained in the normalizer of $\langle W,K\rangle$. Consequently $\langle W,K\rangle \trianglelefteq \langle K,L\rangle$. By symmetry $\langle W,L\rangle \trianglelefteq \langle K,L\rangle$.

<u>Lemma</u> 4 If K and L are finite subsets respectively of the locally nilpotent subgroups A and B of the group G, then U, V, W, $\langle W,K\rangle$, $\langle W,L\rangle$, $\langle K,L\rangle$ are finitely generated and nilpotent, where U, V, W are as in lemma 3.

Proof:

Clearly $U = \langle [K,L],L\rangle$ is finitely generated. Since A and B are normal in G, $[A,B] \leq A \cap B$ and hence $U \leq B$. Therefore U is nilpotent.

By theorem 10 of chapter IV, U satisfies the maximal condition. Therefore V, being a subgroup of U, is finitely generated and nilpotent.

Now $(A \cap B) \cap U = A \cap U$ and $A \cap U \trianglelefteq U$. Since V is the smallest normal subgroup of U containing $[K,L]$ we have $V \leq A \cap U$, i.e., $V \leq A$. Therefore $\langle V,K\rangle \leq A$ and is finitely generated. This implies $\langle V,K\rangle$ is nilpotent and so W is nilpotent and finitely generated.

Since W and K are finitely generated $\langle W,K\rangle$ is finitely generated, and $\langle W,K\rangle \leq A$ hence nilpotent.

Similarly $B \cap \langle K,L\rangle \trianglelefteq \langle K,L\rangle$, but W is the smallest subgroup of $\langle K,L\rangle$ containing $[K,L]$. Therefore $W \leq B$ and so $\langle W,L\rangle$ being finitely generated is nilpotent.

Since $\langle W,K\rangle$ and $\langle W,L\rangle$ are nilpotent and normal in $\langle K,L\rangle$ we have by the corollary to theorem 4 of chapter I that

$\langle W,K\rangle\langle W,L\rangle$ is nilpotent. That is

$$\langle K,L\rangle = \langle W,K\rangle\langle W,L\rangle$$

is nilpotent.

We are now in a position to prove:

Theorem 5 Any two locally nilpotent normal subgroups of a group generate a locally nilpotent subgroup.

Proof:

Let A and B be normal locally nilpotent subgroups of the group G. Let $\{a_1b_1,\ldots,a_nb_n : a_i \in A,\ b_i \in B\}$ be any finite set in AB. Let $K = \{a_1,\ldots,a_n\}$, $L = \{b_1,\ldots,b_n\}$. By lemma 4 $\langle K,L\rangle$ is nilpotent. $\langle a_1b_1, a_2b_2,\ldots,a_nb_n\rangle \leq \langle K,L\rangle$ and is therefore nilpotent. Consequently AB is locally nilpotent.

If a group G possesses a unique maximal normal locally nilpotent subgroup, this subgroup is called the Hirsch radical of the group and is denoted by H(G). That the Hirsch radical exists is brought out in:

Theorem 6 For every group G, the Hirsch radical H(G) exists.

Proof:

Theorem 2 guarantees the existence of a maximal normal locally nilpotent subgroup. If A and B are both maximal normal locally nilpotent subgroups, then by theorem 5 we know AB is also. Consequently a maximal normal locally nilpotent subgroup is unique. This is the Hirsch radical.

It was mentioned in the introduction that there was a connection between the Hirsch radical and the nilradical. We conclude our study of the Hirsch radical by considering this relationship.

Theorem 7 The Hirsch radical H(G) of the group G is contained in the nilradical R(G) of G.

Proof:

An element of $H(G)$ generates a nilpotent subgroup. This subgroup is subnormal in G. Therefore it is contained in $R(G)$ by theorem 5 of chapter III. Hence

$$H(G) \leq R(G).$$

Theorem 8 If the group G satisfies the maximal condition for subgroups, then the Hirsch radical is its maximal normal nilpotent subgroup.

Proof:

If G satisfies the maximal condition for subgroups, then every subgroup is finitely generated by theorem 2 of chapter IV. In this case locally nilpotent is merely nilpotent. By theorem 6 we have $H(G)$ is the maximal normal nilpotent subgroup.

We conclude the chapter with the generalization to locally nilpotent groups two properties which hold for nilpotent groups.

Theorem 9 The normalizer $N(M)$ of a maximal locally nilpotent subgroup M of a group G is its own normalizer.

Proof:

Let $g \in N(N(M))$. Then M and M^g are normal locally nilpotent subgroups of $N(M)$. By theorem 5 we have M and M^g generate a locally nilpotent subgroup. Since M is maximal $M = M^g$ and so $N(M) = N(N(M))$.

Theorem 10 A group G in which every proper subgroup is properly contained in its own normalizer is locally nilpotent.

Proof:

Let M be a maximal locally nilpotent subgroup of G. Since $M < N(M)$ implies $G = N(M)$ by theorem 9 we have that $M \triangleleft G$. That is, every maximal locally nilpotent subgroup of G is normal in G. By theorem 5 they generate a locally nilpotent group, i.e., $G = H(G)$, and so G is locally nilpotent.

CHAPTER VI

(LOCALLY NILPOTENT GROUPS CONT'D)

In this chapter we continue the discussion of locally nilpotent and finitely generated nilpotent groups. A second radical, the torsion subgroup, is determined for the former and a necessary and sufficient condition for the torsion free property is determined for the latter. This condition is generalized to define the rank of a group which in turn is characterized. The chapter is concluded with a property of locally nilpotent groups.

To say that a group is periodic means that each of its elements has finite order. The group is torsion-free if no element other than the identity has finite order. We begin with some elementary properties of these notions.

<u>Theorem</u> 1 Every subgroup and every factor group of a periodic group G is periodic. Further, if N and G/N are periodic, then G is periodic.

Proof:

If H is a subgroup of the periodic group G, then the elements of H, as elements of G, have finite order. Therefore H is periodic.

For factor groups consider the natural mapping of G onto G/N. The pre-images of the elements of G/N have finite order, and since products are preserved the elements of G/N have finite order. Therefore G/N is periodic.

To prove the last statement let $f: G \longrightarrow F/N$ be the natural mapping. For every $g \in G$ there is an image $g^f \in G/N$. If $(g^f)^n = 1$, then $(g^n)^f = 1$, that is $g^n \in N$. But N is periodic so there is an r such that $(g^n)^r = g^{nr} = 1$ and nr is finite. Consequently G is periodic.

Theorem 2 If N is a normal subgroup of the group G, and H a subgroup of G, and N and H both periodic, then HN is a periodic subgroup of G.

Proof:

Clearly HN is a subgroup of G, and N is normal in HN, and HN/N is isomorphic to $H/H \cap N$. Now $H/H \cap N$ is periodic by theorem 1 and hence HN/N is periodic. Again by theorem 1 we have HN periodic.

Theorem 3 In every group G there is precisely one maximal normal periodic subgroup T(G) called the torsion subgroup.

Proof:

If T_1 and T_2 are normal periodic subgroups of G, then by theorem 1 we have T_1T_2 a normal periodic subgroup of G. Therefore there exists at most one maximal normal periodic subgroup of G. Since the identity subgroup E is a normal periodic subgroup, there exists at least one.

Theorem 4 If the normal subgroup N of the group G is periodic, then $T(G/N) = T(G)/N$.

Proof:

The subgroup N being normal and periodic implies that N is contained in T(G) since T(G) is maximal. Under the natural mapping $f: G \to G/N$ we have that

$$f: T(G) \to T(G)N/N = T(G)/N.$$

By theorem 1 we have T(G)/N contained in T(G/N).

For containment the other way suppose there exists a $g \in G$ such that $g \notin T(G)$ and $g^f \in T(G/N)$. Then $(g^f)^n = 1$ for a finite n. Therefore $(g^n)^f = 1$ and so g^n is contained in N. But N is periodic, therefore $(g^n)^r = g^{nr} = 1$ for a finite r. Hence $g \in T(G)$ which is a contradiction. Consequently

$$T(G)/N = T(G/N).$$

We now turn our attention to torsion free groups. The existence of such groups is brought out in the following theorem.

Theorem 5 Every free group F is torsion free.

Proof:

Let F be free on $X = \{x_1, x_2, \ldots\}$ and $w \in F$. Then $w = a_1 a_2 \ldots a_t$ where each $a_i = x_j$ or x_j^{-1}, and w is reduced as written.

Induct on the length, $|w|$, of w.

If $|w| = 1$, or zero for the identity, the theorem is true.

Assume true for words of length $< t$.

For $|w| = t$, if $a_1 \neq a_t^{-1}$ then no reduction occurs in multiplying and so the theorem is true.

If $a_1 = a_t^{-1}$, consider w^{a_1}. Now $w^{a_1} = a_1^{-1} w a_1$ which has length t-2, and so the theorem is true for w^{a_1}. Since an element and its conjugate have the same order, the proof is complete.

Theorem 6 Every subgroup H of a torsion free group G is torsion free.

Proof:

If H is a subgroup of G, and G is torsion free, then the elements of H as elements of G have infinite order. Therefore H is torsion free.

Lemma 7 If N is a normal subgroup of the group G, and G/N is torsion free, then T(G) is a subgroup of N.

Proof:

Consider the natural mapping $f: G \to G/N$. For any $g \in G$ the order of g is greater that or equal to the order of its image. Since G/N is torsion free all elements of finite order are contained in the kernel N. Therefore $T(G) \leq N$.

<u>Theorem</u> 8 If N and G/N are torsion free, then G is torsion free.

Proof:

Now G/N being torsion free implies $T(G)$ is a subgroup of N by the previous lemma. But N being torsion free implies $T(G) = E$, and so G itself is torsion free.

We now turn our attention to locally nilpotent groups and determine what role the periodic part plays.

<u>Lemma</u> 9 If G is locally nilpotent, H and K periodic subgroups of G, then $\langle H,K \rangle$ is periodic.

Proof:

Let g be an arbitrary element of $\langle H,K \rangle$. Then g is a product of elements, say $h_1,\ldots,h_n,k_1,\ldots,k_m$ where the $h_i \in H$, $k_i \in K$.

Let $L = \langle h_1,\ldots,h_n,k_1,\ldots,k_m \rangle$ then L is nilpotent since L is a finitely generated subgroup of G. Now all elements of finite order in a nilpotent group form a normal subgroup[1], say $L' \trianglelefteq L$. But h_i and k_j are contained in L' for all i,j and so $L' = L$. That is, all elements of L are periodic. Since g is contained in L we have that $\langle H,K \rangle$ is periodic.

<u>Lemma</u> 10 If the group G is locally nilpotent, H a subgroup of G and H periodic, then H is contained in $T(G)$.

Proof:

Let $\{H_a\}_{a \in L}$ be a chain of periodic subgroups of G, and let $\bigcup_{a \in L} H_a = K$. For every $x \in K$ there is an $a \in L$ such that $x \in H_a$ and thus x is periodic. Thus K is an upper bound for the chain. By Zorn's lemma there is a maximal periodic subgroup M. Now

---

1. Scott, W.R. <u>Group Theory</u>, Prentice-Hall, 1964, Page 143.

G is locally nilpotent and so by the previous lemma M is unique. Also M is normal in G and hence $M = T(G)$.

<u>Theorem</u> 11 If G is locally nilpotent then $G/T(G)$ is torsion free.

Proof:

Let $aT \in G/T(G)$ and let aT have period n. Let $H \leqslant G$ be the preimage of $K = \langle aT \rangle \leqslant G/T(G)$ with respect to the canonical mapping. Then $T(G)$ is a subgroup of H if $a \notin T(G)$. But for any $h \in H$, if $h \longmapsto k$, then $h^m \longmapsto k^m = 1$ for some m. That is $h^m \in T(G)$. Hence $(h^m)^r = h^{mr} = 1$ for some finite r. Therefore H is periodic and so H is a subgroup of $T(G)$. Contradiction! Consequently G/T is torsion free.

Symbolically this theorem states that if the group G is locally nilpotent then $T(G/T(G)) = E$, that is $T(G)$ is another radical for locally nilpotent groups.

<u>Lemma</u> 12 If $g \in G$ and $h \in Z^{i+1}$ then

$$[g,h]^n = [g,h^n] \bmod Z^{i-1}.$$

Proof:

For any $g \in G$ and any $h \in Z^{i+1}$, we have $[g,h] \in Z^i$.

Now $[g,h^1] = [g,h]^1 \bmod Z^{i-1}$ so assume $[g,h^{n-1}] = [g,h]^{n-1} \bmod Z^{i-1}$.

Then

$$\begin{aligned}[g,h^n] &= [g,h^{n-1}h] \\ &= [g,h][g,h,h^{n-1}][g,h^{n-1}] \\ &= [g,h][g,h^{n-1}] \bmod Z^{i-1} \\ &= [g,h]^n \bmod Z^{i-1}.\end{aligned}$$

<u>Theorem</u> 13 If the centre $Z^1 = Z(G)$ of the group G is torsion free, then all the members of the upper central series of G are torsion free.

Proof:
Since $Z^1 = Z^1/Z^0$ we have that Z^1/Z^0 is torsion free. Assume that Z^i/Z^{i-1} is torsion free. If Z^{i-1}/Z^i is not torsion free then there exists a $y \in Z^{i+1}$, with $y \notin Z^i$, such that $y^n \in Z^i$ for some n. Therefore

$$\begin{aligned}[g,y]^n &= [g,y^n] \\ &= 1 \mod Z^{i-1}.\end{aligned}$$

But Z^i/Z^{i-1} is torsion free and hence $[g,y]^n = 1 \mod Z^{i-1}$ implies $[g,y] = 1$, i.e., $yZ^{i-1} \in Z(G/Z^{i-1}) = Z^i/Z^{i-1}$. That is $y \in Z^i$ which is a contradiction. Therefore Z^{i+1}/Z^i is torsion free. By induction all factors of the upper central series are torsion free. Again by induction assume Z^i is torsion free. Then Z^i torsion free and Z^{i+1}/Z^i torsion free implies Z^{i+1} is torsion free by theorem 8. Therefore all members of the upper central series are torsion free.

<u>Theorem</u> 14 A finitely generated abelian group is torsion free if and only if it is a free abelian group.

Proof:
Let A be a finitely generated, torsion-free, abelian group. Then, for some n, A is the direct product of the n infinite cyclic groups $C_i = \langle a_i \rangle$ $i = 1,2,\ldots,n$. Therefore A is free abelian.

If A is free abelian, then A is torsion free by lemma 1.

<u>Theorem</u> 15 A finitely generated nilpotent group G is torsion free if and only if there is a finite chain of normal subgroups

$$G = N_0 \geqslant N_1 \geqslant \cdots \geqslant N_r = E$$

such that the factor groups N_i/N_{i+1} are infinite cyclic for $i = 0,1\ldots,r-1$.

Proof:

Assume that G has a finite chain of normal subgroups

$$G = N_0 \geqslant N_1 \geqslant \dots \geqslant N_r = E$$

with each factor infinite cyclic.

Now $N_{r-1} = N_{r-1}/N_r$ is infinite cyclic and hence torsion free.

Assume that N_i is torsion free.

N_{i-1}/N_i being infinite cyclic is torsion free. Therefore N_{i-1} is torsion free by theorem 8. By induction G is torsion free.

Let G be torsion free and let

$$G = Z^c > Z^{c-1} > \dots > Z^0 = E$$

be the upper central series for G. Since G is finitely generated, all the elements of this central series are finitely generated, and all the factors of the series are finitely generated by theorem 10 of chapter IV. Also, G torsion free implies Z^i is torsion free for all i.

Consider Z^1. It is abelian, torsion free, and finitely generated.

Let $Z^1 = \langle a_1, \dots, a_n \rangle$

then $Z^1 \triangleright \langle a_1, \dots, a_{n-1} \rangle \triangleright \langle a_1, \dots, a_{n-2} \rangle \triangleright \dots \triangleright \langle a_1 \rangle \triangleright E$

is a normal series for Z^1 and the factors are infinite cyclic.

Now Z^i is a subgroup of G and since G is torsion free, Z^i is torsion free by theorem 6. But $Z^i = Z(G/Z^{i-1})$ and so $Z(G/Z^{i-1})$ is torsion free. Hence by theorem 13 we have G/Z^{i-1} torsion free. Therefore Z^i/Z^{i-1}, being a subgroup of G/Z^{i-1}, is torsion free. Also Z^i/Z^{i-1} is finitely generated and abelian, and so by theorem 15 we have the normal series

$$Z^i/Z^{i-1} > N_1/Z^{i-1} > \dots > N_r/Z^{i-1} > Z^{i-1}/Z^{i-1} = E$$

with infinite cyclic factors. For an arbitrary subgroup H of G which satisfies the conditions $Z^{i-1} \leqslant H$ and $H/Z^{i-1} \leqslant Z^i/Z^{i-1}$, we have that

H is normal in G. Hence we can refine the central series of G to a normal series with infinite cyclic factors.

We depart from nilpotent groups as such to look at a concept which is derived from the normal chain of theorem 15, that is:

Definition 1 A group G is called a group of finite rank if there is a finite normal chain

$$G = N_0 \geqslant N_1 \geqslant \dots \geqslant N_s = E$$

for which each factor group N_i/N_{i+1} is either periodic or infinite cyclic.

Theorem 16 The number $r = r(G)$ of infinite cyclic factor groups N_i/N_{i+1} is independent of the choice of the normal chain by which it is determined.

Proof:

The Schreier theorem states that two normal chains can be refined so that the lengths are the same and factors are isomorphic in some order.

Suppose the pair $A \geqslant B$ occurred in a chain, and this was refined to $A \geqslant H \geqslant B$.

Case 1: A/B is periodic.

Then H/B is a subgroup of A/B and so H/B is periodic.

Also A/H is isomorphic to $(A/B)/(H/B)$ and so A/H is periodic.

Consequently any refinement between two elements of a chain whose factor is periodic does not produce an infinite cyclic factor.

Case 2: A/B is infinite cyclic.

Then H/B, being a subgroup of A/B is infinite cyclic. Also $(A/B)/(H/B)$ is isomorphic to A/H, therefore A/H is finite.

Hence A/H is periodic. Consequently any refinement between two elements of a chain whose factor is infinite cyclic neither adds infinite cyclic factors nor deletes the existing one. Therefore r is invariant.

Theorem 17 Every subgroup and every factor group of a group G of rank r is of rank at most r.

Proof:
Let G have rank r and let H be a subgroup of G.

Let
$$G = G_0 \geqslant G_1 \geqslant \dots \geqslant G_s = E$$
be a normal series for G. Then
$$H = H_0 \geqslant H_1 \geqslant \dots \geqslant H_s = E$$
is a normal series for H, where
$$H_i = H \cap G_i .$$
We also have that
$$H_i/H_{i+1} = H \cap G_i/H \cap G_{i+1} \cong (H \cap G_i)G_{i+1}/G_{i+1} \leqslant G_i/G_{i+1}$$
and so G_i/G_{i+1} infinite cyclic or periodic implies H_i/H_{i+1} is infinite cyclic or periodic. Therefore the rank of H is at most equal to the rank of G.

For factor groups let
$$G = G_0 \geqslant G_1 \geqslant \dots \geqslant G_s = E$$
be a normal series for G. Then if N is a normal subgroup of G there is an i such that N is contained in G_i but is not contained in G_{i+1}. Then
$$G/N = G_0/N \geqslant G_1/N \geqslant \dots \geqslant G_i/N \geqslant N/N = E$$
is a normal series for G/N.

Since $(G_j/N)/(G_{j+1}/N)$ is isomorphic to G_j/G_{j+1} we have that the rank of G/N is at most equal to the rank of G.

Theorem 18 If N is a normal subgroup of the group G, the rank of N is r_1, the rank of G/N is r_2, then the rank of G is r_1+r_2.

Proof:

Let

$$G/N = G_0/N \geqslant G_1/N \geqslant \dots \geqslant G_s/N = E$$

and
$$N = G_s \geqslant N_1 \geqslant \dots \geqslant N_t = E$$

be appropriate chains for G/N and N respectively. Then

$$G = G_0 \geqslant G_1 \geqslant \dots \geqslant G_s = N \geqslant N_1 \geqslant \dots \geqslant N_t = E$$

is a normal chain for G which contains precisely r_1+r_2 infinite cyclic factors. Therefore the rank of G is r_1+r_2.

Theorem 19 If the group G has finite rank r(G), and if T(G) is the torsion subgroup of G, then

$$r(G) = r(G/T(G)).$$

Proof:

By the previous theorem we have that

$$r(G) = r(T(G)) + r(G/T(G))$$

But $r(T(G)) = 0$ since T(G) is periodic. Hence the result.

Definition 2 The elements $a_1, a_2, \dots, a_n$ of a group G are called independent if a_i, for $i = 1,2,\dots,n$, is of infinite order modulo the smallest normal subgroup U_{i-1} of $\langle a_1, \dots, a_i \rangle$ containing the elements $a_1, a_2, \dots, a_{i-1}$.

Theorem 20 A group G has rank n if and only if it has n independent elements.

Proof:

Induct on the rank of G. If $r(G) = 1$ then the normal chain

$$G = G_0 \geqslant G_1 \geqslant \dots \geqslant G_s = E$$

has precisely one infinite cyclic factor, say G_i/G_{i+1}.

That is, for $a \in G_i$ and $a \notin G_{i+1}$ we have $\langle aG_{i+1} \rangle$ is infinite cyclic. So a is an element of infinite order, thus a is an independent element.

If there was another, say b, let U be the appropriate normal subgroup of $\langle a,b \rangle$ containing a such that b is of infinite order modulo U.

Now $r(\langle a \rangle) = 1$ implies $r(U) \geqslant 1$ by theorem 17, and

$r(\langle a,b \rangle /U) \geqslant 1$ since $r(\langle bU \rangle) = 1$ and $\langle bU \rangle \leq \langle a,b \rangle /U$.

Therefore
$$\begin{aligned} r(\langle a,b \rangle) &= r(\langle a,b \rangle /U) + r(U) \\ &\geqslant 1 + 1 = 2 \end{aligned}$$

Hence $r(G) \geqslant 2$ which is a contradiction.

Assume true for $r(G) = n-1$.

If G has rank n, then the normal chain

$$G = G_0 \geqslant G_1 \geqslant \dots \geqslant G_s = E$$

has n infinite cyclic factors. Let G_i/G_{i+1} be the first such. Then for an $a_n \in G_i$ and $a_n \notin G_{i+1}$ we have $\langle a_n G_{i+1} \rangle$ is infinite cyclic.

Now $r(G_i/G_{i+1}) = 1$ and $r(G_i) = n$ implies $r(G_{i+1}) = n-1$.

Let $a_1, a_2, \dots, a_{n-1}$ be the n-1 independent elements of G_{i+1} guaranteed by hypothesis and consider $\langle a_1, a_2, \dots, a_n \rangle$.

Let $U = G_{i+1} \cap \langle a_1, \dots, a_n \rangle$.

Then $a_1, a_2, \dots, a_{n-1}$ are contained in U and a_n has infinite order modulo U. Also, if A is any normal subgroup of $\langle a_1, \dots, a_n \rangle$ containing $a_1, \dots, a_{n-1}$ then $A \cap U$ is normal, contains $a_1, \dots, a_{n-1}$, and a_n still has infinite order modulo $A \cap U$. Consequently a_n has infinite order modulo the smallest normal subgroup of $\langle a_1, \dots, a_n \rangle$ containing $a_1, \dots, a_{n-1}$. Therefore G has n independent elements. By the same argument as before, more than n would imply that G had rank greater than n and so G has precisely n independent elements.

For the converse of the theorem induct on the number of independent elements. If there is one independent element, say a, then $\langle a \rangle$ is infinite cyclic and $r(\langle a \rangle) = 1$. Since $\langle a \rangle \leqslant G$, $r(G) \geqslant 1$. Now $r(G)$ can not be greater than 1 for this would imply that G has more than 1 independent element. Consequently $r(G) = 1$. Assume true for n-1 independent elements. Suppose G has the n independent elements $\{a_1, \ldots, a_n\}$. Let U be the smallest normal subgroup of $\langle a_1, \ldots, a_n \rangle$ containing $a_1, \ldots, a_{n-1}$, then $r(U) = n-1$ by hypothesis, and $r(\langle a_1, \ldots, a_n \rangle / U) \geqslant 1$ since it has an infinite cyclic subgroup $\langle a_n U \rangle$. Hence

$$\begin{aligned} r(\langle a_1, \ldots, a_n \rangle) &= r(\langle a_1, \ldots, a_n \rangle / U) + r(U) \\ &\geqslant n \end{aligned}$$

Therefore $r(G) \geqslant n$. But $r(G) > n$ implies there are more than n independent elements for G. Consequently $r(G) = n$.

With this theorem on the equivalence of the rank of a group and the number of independent elements, we leave rank and return to nilpotence. We conclude the chapter with one more property of locally nilpotent groups. To arrive at the intended result we need the three lemmas:

<u>Lemma</u> 21 If A and B are subgroups of the group G, then $[A,B]$ is a normal subgroup of $\langle A,B \rangle$.

Proof:

For $a \in A$, $b,c \in B$ we have

$$[a,bc] = [a,c][a,b]^c$$

and so $[a,b]^c$ is contained in $[A,B]$.

Therefore B is contained in the normalizer of $[A,B]$ in $\langle A,B\rangle$.

Similarly, for $a,c \in A$ and $b \in B$

$$[ac,b] = [a,b]^c [c,b]$$

so that $[a,b]^c = [ac,b][c,b]^{-1}$ is contained in $[A,B]$

i.e., $A \leq N_{\langle A,B\rangle}([A,B])$.

Therefore $[A,B] \trianglelefteq \langle A,B\rangle$.

Lemma 22 If B is a subgroup of $[A,B]$ and $\langle A,B\rangle$ is nilpotent, then B = E.

Proof:

We have

$$B \leq [A,B] \leq [\langle A,B\rangle, \langle A,B\rangle] = Z_2(\langle A,B\rangle)$$

so assume that $B \leq Z_i(\langle A,B\rangle)$. Then

$$Z_{i+1}(\langle A,B\rangle) = [\langle A,B\rangle, Z_i(\langle A,B\rangle)] \geq [A,B] \geq B$$

that is B is contained in $Z_i(\langle A,B\rangle)$ for all i. But $\langle A,B\rangle$ is nilpotent, hence for some finite r

$$B \leq Z_r(\langle A,B\rangle) = E.$$

Lemma 23 If B is finitely generated, $B \neq E$ and $B \leq [A,B]$, then $\langle A,B\rangle$ is not locally nilpotent.

Proof:

Assume $\langle A,B\rangle$ is locally nilpotent and let $B = \langle g_1, g_2 \ldots, g_n\rangle$. Then since $B \leq [A,B]$ we have that each g_i for $i = 1,\ldots,n$ is a product of commutations of the form $[a,b]$ with $a \in A$, $b \in B$. Let S be generated by those $a \in A$ which appear in at least one of these n products, then S is finitely generated, $B \leq [S,B] \leq \langle S,B\rangle$ and $\langle S,B\rangle$ is nilpotent since it is finitely generated.

Therefore $B = E$ which is a contradiction. Hence $\langle A,B \rangle$ is not locally nilpotent.

Theorem 24 If N is a minimal normal subgroup of the locally nilpotent group G, then $N \leq Z(G)$.

Proof:

Assume $N \not\leq Z(G)$, then $N \cap Z(G) = E$ since N is minimal. Let $n \in N$, $n \neq 1$, and let $g \in G$ be such that $[n,g] \neq 1$. Let $B = \langle [n,g] \rangle$. Now $[B,G] \trianglelefteq \langle B,G \rangle = G$ and $[B,G] \leq N$. Therefore $[B,G] = N$ due to the minimality of N. Hence $1 \neq B \leq [B,G] \leq \langle B,G \rangle = G$ implies that G is not locally nilpotent. Contradiction! Therefore $B = E$ and so $N \leq Z(G)$.

CHAPTER VII

(FORMAL MATRICES)

In this chapter sets of formal sums, or matrices, are constructed. S-rings are defined since for the most part these sets turn out to be rings, and are considered as such. We look at some subrings and ideals, and finally one particular set which forms a group. This group is locally nilpotent, and conditions for it to be nilpotent are examined.

We begin with the definition of an S-ring.

Definition 1 Let S be a ring. A ring R is called an S-ring if for any $m \in S$, $a \in R$, the products ma and am are defined and contained in R, and the operation satisfies the following:

i) $m(a + b) = ma + mb$ $\qquad (a + b)m = am + bm$

ii) $(m + n)a = ma + na$ $\qquad a(m + n) = am + an$

iii) $(mn)a = m(na)$ $\qquad a(mn) = (am)n$

iv) $m(ab) = (ma)b$, $(ab)m = a(bm)$, $(am)b = a(mb)$

v) $(ma)n = m(an)$

for all $m,n \in S$, and all $a,b \in R$.

Definition 2 For any set L, and any ring S, the ring $M(S,L)$ of row finite matrices over L with coefficients in S is defined as the set of all formal sums $(m_{ik}) = \sum\sum m_{ik}e_{ik}$, where for every ordered pair (i,k) of elements from L there is assigned a matrix unit e_{ik}, and where the $m_{ik} \in S$ are such that for any fixed row index i all but a finite number of the $m_{ik} = 0$. Operations are defined by:

i) $\sum\sum m_{ik}e_{ik} + \sum\sum n_{ik}e_{ik} = \sum\sum(m_{ik} + n_{ik})e_{ik}$

ii) $\sum\sum m_{ik}e_{ik} \cdot \sum\sum n_{ik}e_{ik} = \sum\sum(\sum_j m_{ij}n_{jk})e_{ik}$

iii) $m\sum\sum m_{ik}e_{ik} = \sum\sum mm_{ik}e_{ik}$

iv) $\sum\sum m_{ik}e_{ik} \cdot m = \sum\sum m_{ik}me_{ik}$

Theorem 1 $M = M(S,L)$ is an S-ring. If $1 \in S$ then $e = \sum\sum d_{ik}e_{ik}$ (where $d_{ik} = 0$ if $i \neq k$, $d_{ik} = 1$ if $i = k$) is the unit for M, and the mapping $f:S \longrightarrow M$ defined by $f(m) = me$ is an S-isomorphism of S into M.

Proof:

By (iii) and (iv) of definition 2 if $a \in M$, then ma and $am \in M$ for $m \in S$. If $a,b \in M$, $m \in S$, then

$$\begin{aligned} m(a + b) &= m(\sum\sum m_{ik}e_{ik} + \sum\sum n_{ik}e_{ik}) \\ &= m. \sum\sum(m_{ik} + n_{ik})e_{ik} \\ &= \sum\sum m(m_{ik} + n_{ik})e_{ik} \\ &= \sum\sum(mm_{ik} + mn_{ik})e_{ik} \\ &= \sum\sum mm_{ik}e_{ik} + \sum\sum mn_{ik}e_{ik} \\ &= ma + mb. \end{aligned}$$

Similarly for $(a+b)m = am + bm$.

Also
$$\begin{aligned} (m + n)a &= (m + n)\sum\sum m_{ik}e_{ik} \\ &= \sum\sum(m + n)m_{ik}e_{ik} \\ &= \sum\sum(mm_{ik} + nm_{ik})e_{ik} \\ &= ma + na. \end{aligned}$$

Similarly for $a(m + n) = am + an$.

We also have $(mn)a = m(na)$ and $a(mn) = (am)n$.

Now for $a,b \in M$, $m \in S$

$$\begin{aligned} m(ab) &= m(\sum\sum m_{ik}e_{ik} \cdot \sum\sum n_{ik}e_{ik}) \\ &= m \sum\sum (\sum_j m_{ij}n_{jk})e_{ik} \\ &= \sum\sum (m \sum_j m_{ij}n_{jk})e_{ik} \\ &= \sum\sum (\sum_j mm_{ij}n_{jk})e_{ik} \\ &= \sum\sum mm_{ik}e_{ik} \cdot \sum\sum n_{ik}e_{ik} \\ &= (ma)b. \end{aligned}$$

Similarly for $(ab)m = a(bm)$, $(am)b = a(mb)$, and $(ma)n = m(an)$.

Thus M is an S-ring.

Let $1 \in S$ and $e = \sum\sum d_{ik}e_{ik}$. For any $a \in M$ we have

$$\begin{aligned} ea &= \sum\sum d_{ik}e_{ik} \cdot \sum\sum m_{ik}e_{ik} \\ &= \sum\sum (\sum_j d_{ij}m_{jk})e_{ik} \\ &= \sum\sum m_{ik}e_{ik} \\ &= a. \end{aligned}$$

Similarly $ae = a$ and so e is the unit for M.

The mapping $f(m) = me$ is an S-isomorphism between S and M considered as modules since if $f(m) = me$, and $f(n) = ne$, then

$f(m + n) = (m + n)e = me + ne = f(m) + f(n)$.

Also $f(m) = 0$ if and only if $me = 0$, if and only if $\sum\sum md_{ik}e_{ik} = 0$, if and only if $m = 0$. That is, the kernel of f is $\{0\}$.

Finally $f(mn) = (mn)e = m(ne) = mf(n)$ for every $m,n \in S$.

Let R be a subset of L, and $M(S,R,L) = \{(m_{ik}) \in M(S,L) : m_{ik} = 0$ when i or $k \notin R\}$. We then have:

<u>Theorem</u> 2 $M(S,R,L)$ is an S-subring of $M(S,L)$, and $M(S,R,L)$ is S-isomorphic to $M(S,R)$.

Proof:
By theorem 1 $M(S,R)$ is an S-ring. By definition 2, if $a = \sum\sum m_{ik}e_{ik}$ is contained in $M(S,R)$ then $a \in M(S,R,L)$ since $\sum\sum m_{ik}e_{ik} \in M(S,R,L)$ when i,k are both contained in R. So we can define the mapping $f:M(S,R) \longrightarrow M(S,R,L)$ by $f(a) = a$. Then $f(a + b) = a + b = f(a) + f(b)$ and f is a homomorphism. The kernel of f is the zero element since $f(a) = 0$ implies that $a = 0$. Also $f(ma) = ma = mf(a)$ and consequently f is an S-isomorphism. Therefore $M(S,R,L)$ is an S-subring of $M(S,L)$, and $M(S,R,L)$ is S-isomorphic to $M(S,R)$.

Initially L was considered as an arbitrary set. We consider L now as a partially ordered set under the relation $\leqslant$. Define the set $T(S,L)$ of all triangular matrices by $T(S,L) = \{(m_{ik}): m_{ik} = 0 \text{ if } i \nleqslant k\}$. We then show:

<u>Theorem</u> 3 $T(S,L)$ is an S-subring of $M(S,L)$.

Proof:
We have $(m_{ik}) + (n_{ik}) = (m_{ik} + n_{ik}) = (p_{ik})$ and $p_{ik} = 0$ if $i \leqslant k$ since if $i \leqslant k$, then $m_{ik} = 0 = n_{ik}$. Consequently addition is closed.

Also $(m_{ik})(n_{ik}) = (p_{ik})$ where $p_{ik} = \sum\limits_j m_{ik}n_{ik}$,

and $p_{ik} = 0$ if $i \nleqslant k$ since

$$m_{ij}n_{jk} = \begin{cases} 0, \text{ for } i < j \text{ implies } n_{jk} = 0 \\ 0, \text{ for } i > j \text{ implies } m_{ij} = 0 \\ 0, \text{ for } i = j \text{ implies } n_{jk} = 0 \end{cases}$$

Therefore multiplication is closed. The other properties readily follow so $T(S,L)$ is an S-subring of $M(S,L)$.

<u>Theorem</u> 4 The set $N(S,L) \subseteq T(S,L)$ of matrices with all but a finite number of columns vanishing, and with the property that $m_{ik} \neq 0$ implies $i < k$ is a left ideal of $T(S,L)$. For every $a \in N(S,L)$ there exists an integer n such that $a^n = 0$.

Proof:

Let $(m_{ik}) \in T(S,L)$ and $(a_{ik}) \in N(S,L)$, then

$$(m_{ik})(a_{ik}) = \sum\sum(\sum_j m_{ij}a_{jk})e_{ik}.$$

If $\sum_j m_{ij}a_{jk} \neq 0$ then there is a j such that $m_{ij}a_{jk} \neq 0$.

Therefore $j < k$ and $i \leq j$, that is $i < k$. Hence $ma \in N(S,L)$ for $a \in N(S,L)$, $m \in T(S,L)$.

Let $a = (a_{ik}) \in N(S,L)$, then $a^2 = \sum\sum(\sum_j a_{ij}a_{jk})e_{ik}$ and the only terms which survive are those for which $a_{ik} \neq 0$ originally, and where $i < j < k$. Now

$$a^3 = \sum\sum(\sum_r a_{ir}\sum_j a_{rj}a_{jk})e_{ik}$$

$$= \sum\sum(\sum_r\sum_j a_{ir}a_{rj}a_{jk})e_{ik}$$

and the terms which survive have $i < r < j < k$. If there are n columns (the number of columns is finite) then at the end of the n-1[st] step, i.e., at a^n, at least one of the strict inequalities will be violated. That is $a^n = 0$.

<u>Theorem</u> 5 The set $G = \{e + a: a \in N(S,L)\}$ is a group.

Proof:

We have $(e + a)(e + b) = e + a + b + ab = e + c$ where $c \in N(S,L)$, hence multiplication is closed. Also, $e + 0$ is the unit since

$(e + 0)(e + a) = e + 0 + a + 0a = e + a$. Multiplication is associative and the inverse for $e + a$ is $e - a + a^2 - a^3 + \ldots + (-1)^{n-1}a^{n-1}$.

<u>Theorem</u> 6 The group $G = e + N(S,L)$ is locally nilpotent.

Proof:

Let $H = \langle e + a_1, e + a_2, \ldots, e + a_n \rangle \leq G$.

Since there are a finite number of generators of H, that is, a finite number of $a_i \in N(S,L)$, there will be a finite number of finite chains with elements in L.

Let $E < H_1 < H_2 < \ldots < H$ be defined by $H_j = \{e + a$: the indices appearing in a are from the last $j + 1$ elements in each chain$\}$. Now each H_j is normal in H, and H_{j+1}/H_j is abelian, or is contained in $Z(H/H_j)$. Therefore we have a finite central series for H.

<u>Theorem</u> 7 The group $G = e + N(S,L)$ is not nilpotent if there are arbitrary long chains in the partially ordered set L. However, if the length of properly increasing chains is bounded, then G is nilpotent.

Proof:

If there are arbitrary long chains in L, then by the previous theorem a central series of arbitrary length may be constructed, hence G is not nilpotent.

For the second part of the theorem, we have that the number of row indices in G is finite, and to each row index there is a chain. Since each chain is bounded this guarantees finiteness. By the previous theorem G is nilpotent.

<u>Theorem</u> 8 Let P be a partially ordered subset of the partially ordered set L. If any element of L is properly contained in an element of S and if all elements of L not in S are equivalent, then the subgroup $H = e + N(S,P) = e + N(S,L) \cap M(S,P,L)$ of the group $G = e + N(S,L)$ is its own normalizer in G.

Proof:

Let $a \in S$ and $a \in L$. Then every element of G which is not in H is of the form $e + g$ where $g = (m_{ik})$ has an $m_{ak} \neq 0$ for some $k \in S$. Let $e + h \in H$, then $(e + h)(e + g) = e + h + g + hg \notin H$ since the m_{ak} is retained in the product. Also $(e + g)(e + h) \notin H$ for the same reason.

Let n be such that $g^n = 0$. Then $e - g + \ldots (-1)^n g^{n-1}$ is the inverse of $e + g$ and m_{ak} appears in the inverse. Therefore $(e + g)^{-1}(e + h)(e + g) \notin H$ for any $e + g \notin H$. Hence H is its own normalizer.

BIBLIOGRAPHY

1. Hall, M. The Theory of Groups, Macmillan, 1964.
2. Hall, P. Nilpotent Groups, Printed Notes, Canadian Mathematical Congress, 1957.
3. Kurosh, A.G. The Theory of Groups, Vol. I & II, Chelsea, 1960.
4. Scott, W.R. Group Theory, Prentice-Hall, 1964.
5. Zassenhaus, H. The Theory of Groups, Chelsea, 1958.