CATEGORICAL CHARACTERIZATIONS
OF CERTAIN
ALGEBRAIC AND TOPOLOGICAL CONSTRUCTIONS

BY

JOHN EDWARD KLASSEN

A THESIS

PRESENTED TO THE

FACULTY OF GRADUATE STUDIES AND RESEARCH

OF THE

UNIVERSITY OF MANITOBA

IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

AUGUST 1966

THE WRITER WISHES TO EXPRESS SINCERE APPRECIATION TO DR. K. W. ARMSTRONG OF THE DEPARTMENT OF MATHEMATICS FOR HIS INTEREST AND GUIDANCE.

TABLE OF CONTENTS

INTRODUCTION

The concepts of direct products, direct sums, free products, direct limits, and inverse limits, occur frequently in algebra and topology, and in order to clarify the interrelationship of these concepts in various algebraic and topological systems we define them in a categorical manner and attempt to derive as many properties as possible that are independent of the category, and for those concepts that cannot be proven counter-examples are sought. Special acknowledgements are due to S. MacLane and W. Burgess whose papers were used as source material.

CHAPTER II

BASIC DEFINITIONS OF CATEGORY THEORY

Definition 1. A category C is a class of objects A, B, - - - together with a family of disjoint sets $\mathrm{Hom}_C(A,B)$ one for each ordered pair (A,B) of objects. Write $f\colon A \to B$ for $f \in \mathrm{Hom}_C(A,B)$ and call f a morphism of C with domain A and codomain B. Assume a rule which assigns to each pair of morphisms $f\colon A \to B$, $g\colon B \to C$ a unique morphism $gof\colon A \to C$ called the product morphism with gof defined only if the codomain of f is the domain of g. In addition we have two axioms:

A.1 If $f\colon A \to B$, $g\colon B \to C$ and $h\colon C \to D$, then $h(gf) = (hg)f$.

A.2 To each object B there exists a morphism $1_B\colon B \to B$ such that $1_B f = f$ and $g 1_B = g$ for $f\colon A \to B$, $g\colon B \to C$.

Remarks. The objects of C will be denoted by $\mathrm{Ob}(C)$ and the family of disjoint sets by $\mathrm{Hom}(C)$.

A category may be completely described by its morphisms, ignoring the objects.

Let C be a class of "morphisms," f, g, h, in which a composite gof is sometimes defined, call a morphism u an identity of C if $uof = f$ whenever uof is defined and $gou = g$ whenever gou is defined.

The axioms are:

B.1 The product $h(gf)$ is defined iff the product $(gh)f$ is defined. When either is defined they are equal. This triple product will be written hgf.

B.2. The triple product hgf is defined whenever both products hg and gf are defined.

B.3. For each morphism f of C there exist identities u and u' such that $u'f$ and fu are defined.

Remarks. The two definitions are equivalent. However in the actual constructions of various categories we usually use the first definition while in the examination of abstract categories the second one is often used.

Definition 2. In a category C a morphism $e : A \longrightarrow B$ is said to be invertible (or an equivalence) if there is a morphism $e' : B \longrightarrow A$ with $e'e = 1_A$ and $ee' = 1_B$. If e' exists it is unique and is written $e' = e^{-1}$.

Definition 3. Two objects of a category C are said to be equivalent if there exists an invertible $e: A \longrightarrow B$.

Definition 4. A morphism $k: A \longrightarrow B$ is said to be monic in C if it is left cancellable; i.e., if $k\varphi = k\eta$ implies $\varphi = \eta$.

Definition 5. Dually a morphism $k: A \longrightarrow B$ is said to be epic in C if it is right cancellable; i.e., if $\varphi k = \eta k$ implies $\varphi = \eta$.

Definition 6. An object T is said to be terminal in C if to each object A there is exactly one morphism $h: A \longrightarrow T$.

Definition 7. Dually, an object I in C is said to be initial if to each object A there is exactly one morphism $K : I \longrightarrow A$.

Remarks. If T is terminal then the only morphism taking $T \longrightarrow T$ is the identity morphism. Furthermore, any two terminal objects in C are equivalent, since we then have a unique $\alpha : T \to T'$ (T, T' both terminal) and a unique $\beta : T' \to T$. Then $\beta\alpha : T \to T$ and $\beta\alpha$ must necessarily be the identity morphism on T. Similarly $\alpha\beta$ is then the identity morphism on T' and hence α and β are invertible with $\alpha = \beta^{-1}$, $\beta = \alpha^{-1}$.

Dually if I is an initial object of C then the identity morphism is the only morphism taking $I \to I$. Analogously two initial objects of C are equivalent.

Examples of Categories

1. The category Ens of sets has as objects all sets T, S, - - and as morphisms all functions from S to T. In this category the monics are the injections and the epics are the surjections. This assertions will be proved at a later stage.

2. The category Gr of groups has as objects all groups and morphisms all group homomorphisms. The monic morphisms are the monomorphisms and the epics are the epimorphisms.

3. The category Ab of all abelian groups has as objects all abelian groups and morphisms all abelian group homomorphisms. Again the monics are monomorphisms and the epics are group epimorphisms.

4. The category Mon of all monoids has as objects all monoids A, B - - and as morphisms all monoid homomorphisms. Again monics are monomorphisms and epics are epimorphisms.

5. The category Ens_* of pointed sets. By a pointed set we mean a non-empty set P with a selected element x called the "base point" of P. A morphism $f: P \to Q$ of pointed sets is a function on the set P to

the set Q which carries base points to base points.

6. The category Top_* denotes the category of pointed topological spaces. The objects are topological spaces with a designated base point x and the morphisms are continuous maps $f\colon X \longrightarrow Y$ which send the base point of X into the base point of Y.

8. Let P be any partially ordered set. Let the elements of P be the objects of the category P^* with the set $Hom_p(a,b)$ either empty or consisting of exactly one element (when $a \leq b$).

<u>Definition 8</u>. For any category C, the category Morph (C) has as objects the morphisms $f\colon A \to B$ of C and as morphisms $m\colon f \longrightarrow f'$ the pairs $m = (a,b)$ of morphisms $a\colon A \longrightarrow A'$ and $b\colon B \longrightarrow B'$ of C such that the square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"] & B' \end{tikzcd}$$

commutes.

The composition of two morphisms is formed by pasting the first square on top of the second and erasing the junction.

<u>Definition 9</u>. A functor is a map of categories. More explicitly a covariant functor $F\colon C' \longrightarrow C$ consists of an object function F and a mapping function also written F. The object function assigns to each object B of C' an object $F(B)$ of C; the mapping function assigns to each morphism $f\colon B \longrightarrow B'$ of (C') a morphism $F(f)\colon F(B) \longrightarrow F(B')$ of (C) in such a way that $F(1_b) = 1_{F(b)}$ and $F(gf) = (F(g))(F(f))$ whenever gf is defined.

<u>Definition 10</u>. Each category C determines an opposite category C^{op}.

The objects of C^{op} are the same objects as C while the morphisms $f^*: B \to A$ of C^{op} are in one-one correspondence with morphisms $f: A \to B$ of C. $f^* g^* = (gf)^*$ is defined in C^{op} when gf is defined in C. Note that the co-domain of f^* is the domain of f and f^* is monic iff f is epic.

Definition 11. A covariant functor $G: B^{op} \to C$ is called a contravariant functor on the category B to C. We have $G(1_B) = 1_{G(B)}$ $G(gf) = (G(f))\ (G(g))$.

Definition 13. A subcategory D of C consists of a subclass of $Ob(c)$ and a subclass of $Hom(C)$ denoted by $Ob(D)$ and $Hom(D)$ respectively, such that

1) if $A \in Ob(D)$, then $1_A \in Hom(D)$.
2) if $\alpha, \beta \in Hom(D)$ and $\alpha\beta$ is defined in C, then $\alpha\beta \in Hom(D)$.
3) if $\alpha \in Hom(D)$ and $\alpha: A \to B$, then $A, B \in Ob(D)$.

Definition 14. The subcategory D of C is said to be full if $Hom_C(A,B) = Hom_D(A,B)$ for any $A, B \in Ob(D)$.

Example. Abelian groups and abelian group homomorphisms form a full subcategory of the category of groups and homomorphisms.

Lemma 1. Let $C = \langle Ob, M \rangle$ be a category where Ob = objects and M = morphisms. Let $P \subseteq Ob$ and let N be the set of all morphisms of objects in P when these are considered as being in C. Then $\langle P, N \rangle$ is a full subcategory of C.

Proof: Follows trivially from the definitions.

CHAPTER III

MONO AND EPI IN CERTAIN CATEGORIES

We now wish to investigate the equivalence of mono and one-one and also the equivalence of epic and onto. That is, we wish to establish in which categories these are equivalent concepts.

Definition 1. A concrete category is one whose objects are sets and whose maps are a subclass of the class of set functions.

Theorem 1. In every concrete category one to one implies onto.

Proof: Let $f: A \to B$ be one to one and $g,h: C \to A$ be such that $fog = foh$. If $g(x) \neq h(x)$ for some $x \in C$ then $(fog)(x) = f(g(x)) \neq f(h(x)) = (foh)(x)$ by definition of one-one. Q.E.D.

Theorem 2. In the category of sets mono implies one-one.

Proof: We prove the contra-positive. Let g be a map which is not one-one. Let $g: S_1 \to S_2$ where S_1 and S_2 are sets, and $g(s_1) = g(s_2)$ but $s_1 \neq s_2$. Now let $h, k: T \to S_1$ be two maps which map T onto s_1 and s_2 respectively. Then $goh = gok$ but $h \neq k$. Q.E.D.

Theorem 3. In the category of pointed sets mono implies one-one.

Proof: Again we prove the contra-positive. Let g be a map which is not one-one and let $g: (S_1, x_1) \to (S_2, x_2)$ be such that $g(x_1) = x_2$ and $g(s_1) = g(s_2)$ with $s_1 \neq s_2$. Let $h,k: (T,t) \to (S_1, x_1)$ such that $h(t) = k(t) = x_1$ and $h(T-t) = s_1$ and $k(T-t) = s_2$. Then $goh = gok$ but $h \neq k$. Q.E.D.

Theorem 4. In the category of topological spaces mono implies one-one.

Proof: Let S_1 and S_2 be two topological spaces and assume $f: S_1 \to S_2$

is a continuous map such that $f(s) = f(s')$ and $s \neq s'$. Let T be any topological space from the category and let $g,h : T \to S_1$ be the constant maps onto s and s' respectively. Then g,h are certainly continuous and $f \circ g = f \circ h$ but $g \neq h$. Q.E.D.

<u>Definition 2</u>. Let S be an arbitrarily given set. By a free object on the set S in a particular algebraic system we mean an object F in that system together with a morphism $f: S \to F$ such that for every morphism $g: S \to X$ from the set S into another object X of the system there exists a unique morphism $h: F \to X$ satisfying the commutativity relation $h \circ f = g$ for the following triangle:

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "g"'] & & F \arrow[dl, "h"] \\ & X & \end{tikzcd}$$

(See [6] pages 30)

<u>Definition 3</u>. An algebraic system is called equationally defined if its structure is defined by a family of identities or equations. (or operations).

Consider a particular equationally defined class of algebraic systems called σ-systems which are defined by the operations f_1, f_2 - - (not necessarily countable) such that f_β is an $n(\beta)$ - ary operation, $n(\beta)$ finite, and some set of identities involving these operations.

Given a set of symbols $\{\alpha_1, \alpha_2 \cdots\}$ a free σ -system may be constructed with these symbols as generators. The process is to consider all expressions built from the generators and the operations and then make only the identifications required by the given set of identities. The resulting set of equivalence classes can then be considered as a σ -system.

Theorem 5. In the category of all σ -systems with σ -homomorphisms as maps, mono is equivalent to one-one.

Proof: Let A,B be σ -systems and f: $A \to B$ a σ -homomorphism. Then if $f(a) = f(a')$ and $a \neq a'$ we can proceed as follows: Denote by F the free σ-system on one generator x. Define $g,h : F \to A$ by $g(x) = a$, $h(x) = a'$. Since g and h are defined on the generator of F they can be extended uniquely to σ -homomorphisms. Then $f \circ g = f \circ h$ but $g \neq h$. Q.E.D.

Some examples of equationally defined systems are groups, rings, R-modules and monoids.

Theorem 6. In a concrete category a map is epi if it is onto.

Proof: Let f: $A \to B$ be onto. Let $g,h : B \to C$ be such that $g \circ f = h \circ f$. Now suppose there exists $b \in B$ such that $g(b) \neq h(b)$. Then since f is onto take a pre-image of b. Let it be a. Then $gf(a) = g(f(a)) \neq h(f(a)) = h\,f(a)$. Contradiction! Therefore onto implies epi.

Theorem 7. In the category of sets epi is equivalent to onto.

Proof: Let $g: S_1 \to S_2$ be not onto. Let $h: S_2 \to S_3$ such that $h(s) = s_3$ for all $s \in S_2$,
$k: S_2 \to S_3$ such that $k(s) = s_3{}'$ for all $s \notin g(S_1)$ and $k(s) = s_3$ for all $s \in g(S_1)$ with $s_3{}' \neq s_3$.
Then $h \circ g = k \circ g$ but $g \neq k$. Q.E.D.

Theorem 8. In the category of pointed sets epi means onto.

Proof: Let $g:(S_1,x_1) \longrightarrow (S_2,x_2)$ be not onto.

Let $h:(S_2,x_2) \longrightarrow (S_3,x_3)$ be such that $h(s) = s_3$ for all $s \in (S_2 - x_2)$. $h(x_2) = x_3$.

Let $k:(S_2,x_2) \longrightarrow (S_3,x_3)$ such that $k(s) = s_3$ for all $s \in g(S_1) - x_2$ and $k(s) = s_3{}'$ for all $s \in S_2 - g(S_1) - x_2$ and $k(x_2) = x_3$.

Then $hog = kog$ but $h \neq k$. Q.E.D.

Theorem 9. In the category of all topological spaces epi implies onto.

Proof: If $f: S \longrightarrow U$ is not onto. Let $U' = \{a,b\}$ with the trivial topology. Let $g,h: U \longrightarrow U'$ be given by $g(t) = a$ for all $t \in U$ and $h(t) = a$ if $t \in f(S)$ and $h(t) = b$ if $t \notin f(S)$. Then $gof = hof$ but $h \neq k$. Q.E.D.

Theorem 10. In the category of all groups epi implies onto.

Proof: Assume $f: H \longrightarrow G$ (H and G groups) is not onto and form the free product of G with itself with the subgroup $f(H)$ amalgamated

$K = G *_{f(H)} G$

Define $g,h: G \longrightarrow K$ by g mapping G onto the first copy of G in K and h mapping G onto the second copy of G in K. By construction of the amalgamated product these maps coincide on $f(H)$. Then $gof = hof$ but $g \neq h$. Q.E.D.

Remarks. The question of mono and epi has not been completely answered. There are apparently some categories where the answers are not known, (e.g., epi implying onto in the category of rings.)

CHAPTER IV

PRODUCTS AND CO-PRODUCTS IN CATEGORIES

We now wish to formalize the concepts of direct products, cartesian products, sum of spaces, direct sum, and free products. Along with various constructions we shall examine also certain characterizations of monic and epic morphisms.

Definition 1. Let C be any category. Define the category Π C as having objects $A = \langle I, \{ A_i \mid i \in I \} \rangle$ where I is any indexing set and $\{A_i\}_{i \in I}$ is a collection of objects from C indexed by I, and morphisms $\varphi = \langle \varphi', \{ \varphi_i \mid i \in I \} \rangle : A \to B$ where φ' is a set map taking $I \to J$ (both indexing sets) and a collection of morphisms $\{\varphi_i\}_{i \in I}$ in C where $\varphi_i : A_i \to B_{\varphi'(i)}$. Define a composition of morphisms as $\varphi_1 \circ \varphi_2 = \langle \varphi_1' \circ \varphi_2', \{ \varphi_{1\varphi_2'(i)} \circ \varphi_{2i} \mid i \in I \} \rangle$

Definition 2. Now define the sub-category C^I of Π C where C^I has as objects all those objects of Π C which have the same indexing set I, and as morphisms all admissible morphisms between such objects in Π C having the identity map on I. Then C^I is a subcategory of Π C by lemma 1 of chapter II.

Definition 3. Then define the sub-category C' of C^I which has as objects all those objects of C^I where the collection $\{A_i\}_{i \in I}$ is such that each A_i is equal to a fixed object A of C, and as morphisms all admissible morphisms between such objects in C^I. Then C' is a full sub-category of C^I by lemma 1 of Chapter II.

Remarks. We can consider C as being imbedded in C', for let F be the covariant functor taking $C \rightarrow C'$ where F is defined by

$F(A) \rightarrow \langle I, \{A_i \mid i \in I\}\rangle$ where $A_i = A$ for all $i \in I$. If $A, B \in C$, and $\varphi_{AB}: A \rightarrow B$,

$F(\varphi_{AB}) \rightarrow \langle \varphi', \{\varphi_i \mid i \in I\}\rangle$ where φ' is the identity map on I and $\varphi_i = \varphi_j = \varphi$ for all i and j.

We will often denote objects in categories C^1 and C^I by $\{A_i\}$ $i \in I$ when it is understood that we are using a fixed indexing set I and only the identity maps on I.

Definition 4. Let D^* denote the sub-category of Morph (πC) which has as objects those objects of Morph (πC) which when considered as morphisms in πC have as their domain an object in C' and as their co-domain an object in C^I, and morphisms the ordered pair of morphisms in C' and C^I respectively which make the following diagram commute.

$$\begin{tikzcd}
\langle I, \{A_i \mid i \in I\}\rangle \arrow[r, "\varphi"] \arrow[d, "\eta"'] & \langle I, \{B_i \mid i \in I\}\rangle \arrow[d, "\gamma"] \\
\langle I, \{C_i \mid i \in I\}\rangle \arrow[r, "\psi"] & \langle I, \{D_i \mid i \in I\}\rangle
\end{tikzcd}$$

This clearly forms a sub-category of Morph (πC).

Definition 5. Define the sub-category D_A of D^* to consist of those objects which as morphisms in πC have a fixed element A of C^I as their co-domain, and as morphisms those pairs $\langle \varphi, Id\rangle \in D^*$ where Id is the identity map on A and $\varphi = \langle I, \{\varphi_i \mid i \in I\}\rangle$ is such that $\varphi_i = \varphi^*$ (fixed) for all $i \in I$.

Definition 6. Dually define the category G^* to be the sub-category of Morph (πC) which has objects those morphisms in πC having domain in C^I and co-domain in C'. Define the sub-category G_A^* of G^* analogously to D_A^* which now has objects those morphisms in πC having domain fixed in C^I and co-domain in C'. The morphisms of these categories are defined similarly to those above.

Definition 7. Define a categorical product to be a terminal object in D_A^*.

Definition 8. Define a categorical co-product to be an initial object in G_A^*.

Theorem 1. Any two co-products are equivalent.

Proof: Consider two co-products $\alpha = \langle \alpha', \{\alpha_i \mid i \in I\} \rangle$ and $\beta = \langle \beta', \{\beta_i \mid i \in I\} \rangle$

$$\begin{array}{ccc}
\langle I, \{A_i \mid i \in I\} \rangle & \longrightarrow & \langle I, \{P_i \mid i \in I\} \rangle = P \\
Id \downarrow & \beta = \langle \beta', \{\beta_i \mid i \in I\} \rangle & \downarrow \varphi \\
\langle I, \{A_i \mid i \in I\} \rangle & \longrightarrow & \langle I, \{M_i \mid i \in I\} \rangle = M \\
Id \downarrow & \alpha = \langle \alpha', \{\alpha_i \mid i \in I\} \rangle & \downarrow \eta \\
\langle I, \{A_i \mid i \in I\} \rangle & \longrightarrow & \langle I, \{P_i \mid i \in I\} \rangle = P
\end{array}$$

where $P_i = P$ for all $i \in I$ and Id denotes the identity map. Then both φ and η are unique by the definition of co-product. Then since the identity morphism on $\langle I, \{P_i \mid i \in I\} \rangle$ is certainly an admissible morphism in the large square. Therefore $\eta\varphi = 1_P$. Similarly $\varphi\eta = 1_M$. Therefore $\eta = \varphi^{-1}$ and α and β are equivalent. Q.E.D.

Theorem 2. Any two products are equivalent.

Proof: Dual argument to proof of theorem 1.

Lemma 1. If α, β are morphisms and $\alpha\beta$ is defined and monic, then β is monic.

Proof: Consider $\beta \circ \eta = \beta \circ \varphi$. We wish to show that this implies $\eta = \varphi$.

If $\beta\eta = \beta\varphi$ then $\alpha(\beta\eta) = \alpha(\beta\varphi)$;
i.e., $(\alpha\beta)\eta = (\alpha\beta)\varphi$. But by assumption $\alpha\beta$ was monic. Hence $\eta = \varphi$. Q.E.D.

Lemma 2. If α, β are morphisms and $\alpha\beta$ is defined and epic then α is epic.

Proof: Consider $\eta\alpha = \varphi\alpha$. We wish to show that this implies $\eta = \varphi$. If $\eta\alpha = \varphi\alpha$ then $(\eta\alpha)\beta = (\varphi\alpha)\beta$ since $\alpha\beta$ is defined. Therefore $\eta(\alpha\beta) = \varphi(\alpha\beta)$ but $\alpha\beta$ is epic by assumption. Therefore $\eta = \varphi$. Q.E.D.

Theorem 3. Let $\varphi : \langle I, \{A_i \mid i \in I\}\rangle \to \langle I, \{B_i \mid i \in I\}\rangle$ be initial in G_A^* where $\varphi = \langle \varphi', \{\varphi_i \mid i \in I\}\rangle$. Then each φ_i is monic in C if $\mathrm{Hom}_C(A,B)$ is not empty for all pairs (A,B).

Proof: Consider

$$\begin{array}{ccc}
\langle I, \{A_i \mid i \in I\}\rangle & \xrightarrow{\varphi} & \langle I, \{B_i \mid i \in I\}\rangle \\
{\scriptstyle I\alpha}\downarrow & & \downarrow{\scriptstyle \eta} \\
\langle I, \{A_i \mid i \in I\}\rangle & \xrightarrow{\psi} & \langle I, \{C_i \mid i \in I\}\rangle
\end{array}$$

where $C_i = A_j$ for all $i \in I$ and some fixed $j \in I$, and $\psi = \langle \psi', \{\psi_i \mid i \in I\}\rangle$ is a morphism such that $\psi_j : A_j \to C_j$ is the identity morphism on A_j and φ_i is any other admissible morphism for $i \neq j$. Then there exists a unique morphism $\gamma = \langle Id, \eta\rangle : \varphi \to \psi$ such that the above diagram commutes. Then we must have $\eta\varphi = \psi$ which implies that $\eta_j \cdot \varphi_j = \psi_j$. But $\psi_j = 1_{A_j}$ and since ψ_j is certainly monic then by lemma 1 we have φ_j monic. Q.E.D.

Remarks. In the categories of sets, pointed sets, and topological spaces the constant maps guarantee that $\mathrm{Hom}_C(A,B) \neq \phi$. In algebraic systems the 0 homomorphisms guarantee this.

Theorem 4. If each φ_i is monic in C and φ' is monic in the category of sets then φ is monic in $\overline{\Pi}$ C.

Proof: Let $\varphi : \langle I, \{ C_i \mid i \in I \} \rangle \to \langle J, \{ D_j \mid j \in J \} \rangle$

where $\varphi = \langle \varphi', \{ \varphi_i : C_i \to D_{\varphi'(i)} \mid i \in I \} \rangle$

We wish to show that $\varphi \circ \eta = \varphi \circ \gamma$ implies that $\eta = \gamma$ where

$\eta = \langle \eta', \{ \eta_k \mid k \in K \} \rangle$

$\gamma = \langle \gamma', \{ \gamma_k \mid k \in K \} \rangle$

Now since $\varphi \circ \eta = \varphi \circ \gamma$ by assumption then $\varphi' \circ \eta' = \varphi' \circ \gamma'$

but φ' is monic by assumption. Hence $\eta' = \gamma'$

Also $\varphi_{\eta'(k)} \eta_k = \varphi_{\gamma'(k)} \gamma_k$

but $\eta'(k) = \gamma'(k)$ Therefore $\varphi_{\eta'(k)} \eta_k = \varphi_{\eta'(k)} \gamma_k$

and since $\varphi_{\eta'(k)}$ was assumed to be monic, then $\eta_k = \gamma_k$ for all $k \in K$.

Therefore $\eta = \gamma$ i.e., φ is monic.

Theorem 5. If φ is monic in $\overline{\Pi}$ C, then each φ_i is monic in C.

Proof: Suppose that

$$\eta'' : A_i \to B_j$$
$$\gamma'' : A_i \to B_j$$

such that $\varphi_j \circ \eta'' = \varphi_j \circ \gamma''$

Now consider A_i as being in $\overline{\Pi}$ C indexed by an indexing set of cardinality one. That is, we extend η'' to a morphism in $\overline{\Pi}$ C by letting $\eta = \langle \eta', \eta'' \rangle$ where $\eta' : I \to J$ and card I = 1 and $\eta'' : A_1 = A_i \to B_j$. Similarly extend γ'' to a morphism in $\overline{\Pi}$ C. We then have:

$$\begin{array}{ccccc} I & \xrightarrow{\eta'} & J & \xrightarrow{\varphi'} & K \\ i & \longrightarrow & \eta'(i) = \gamma'(i) = j & \longrightarrow & \varphi'(\eta'(i)) = k \end{array}$$

Then certainly $\varphi\eta = \varphi\gamma$. But φ is monic by assumption which implies that $\eta = \gamma$ which in turn implies that $\eta'' = \gamma''$, i.e., φ_j is monic. A similar argument works for any φ_i. Q.E.D.

Remarks. φ monic in πC does not imply that φ' is monic in the category of sets. For consider:

η' : $I \longrightarrow J$

γ' : $I \longrightarrow J$ where J is of cardinality 2, and

φ' : $J \longrightarrow K$ where card K = 1.

Let the objects of C being indexed be sets with the following structure:

$$I \longrightarrow J \longrightarrow K$$

$$\begin{tikzcd} & A_1 \arrow[r, "\text{INJECTION}"] & A_1 \cup A_2 \\ \{B_i\}\ i \in I \arrow[ur] \arrow[dr] & & \\ & A_2 \arrow[uur, "\text{INJECTION}"'] & \end{tikzcd}$$

(DISJOINT UNION)

Now suppose that $\varphi\eta = \varphi\gamma$. Then since $\varphi_2(A_2) \cap \varphi_1(A_1) = \phi$ each $\eta_i = \gamma_i$. Hence φ is monic but certainly φ' is not monic since in the category of sets monic is equivalent to one-one and $\varphi'(1) = \varphi'(2)$.

Theorem 6. If φ' is epic in the category of sets and each φ_i is epic in C then φ is epic in πC.

Proof: Suppose $\eta\varphi = \gamma\varphi$. We wish to show that then necessarily $\eta = \gamma$. Now $\eta' \cdot \varphi' = \gamma' \cdot \varphi'$ and by assumption φ' is epic. Therefore $\eta' = \gamma'$.

Also $\eta_{\varphi'(i)}\varphi_i = \gamma_{\varphi'(i)}\varphi_i$. But each φ_i is assumed to be epic. Therefore $\eta_{\varphi'(i)} = \gamma_{\varphi'(i)}$ for all i. Hence $\eta = \gamma$ and φ is epic.

Q.E.D.

<u>Theorem 7</u>. If ω is epic in Π C then each ω_i is epic in C.

<u>Proof</u>: Suppose that $\eta'' \circ \omega_j = \gamma'' \circ \omega_j$ in C where $\omega' : I \to J$.

Then extend η'' and γ'' to morphisms in Π C as follows.

$\eta' : J \to J$ is the identity map on J and each

$\eta_{\omega'(i)} : B_{\omega'(i)} \to B_{\omega'(i)}$ is the identity map on each

$B_{\omega'(i)}$ except η'' which takes $B_{\omega'(j)} \to C_{\omega'(j)}$. Extend γ''

to Π C similarly. Then clearly $\eta\,\omega = \gamma \circ \omega$ and therefore

$\eta = \gamma$ since by assumption ω is epic in Π C. Hence

$\eta'' = \gamma''$, i.e., ω_j is epic. Q.E.D.

We now give actual constructions of products and co-products in certain categories.

(a) Partially ordered sets.

Consider the category D_A^* where $\{A_i\}_{i \in I}$ is a collection of objects from the category P^* defined in Chapter II. Then consider a terminal object in this category.

$$\langle I, \{B_i \mid i \in I\} \rangle \xrightarrow{\pi = \langle \pi', \{\pi_i\} \rangle} \langle I, \{A_i \mid i \in I\} \rangle$$

Let $\omega = \langle \omega', \{\omega_i\} \rangle$ be any other object in this category. Then by definition there exists a unique morphism $\eta : \omega \to \pi$ such that the following diagram is commutative.

$$\begin{tikzcd}
\langle I, \{B_i \mid i \in I\} \rangle \arrow[r, "\pi"] & \langle I, \{A_i \mid i \in I\} \rangle \\
\langle I, \{C_i \mid i \in I\} \rangle \arrow[u, "\gamma"] \arrow[r, "\omega"] & \langle I, \{A_i \mid i \in I\} \rangle \arrow[u, "Id"']
\end{tikzcd}$$

Now since we have a map $\pi_i : B_i \to A_i$ for all $i \in I$ where B_i is fixed, then B_i must be such that $B_i \leq A_i$ for all $i \in I$. Similarly $C_i \leq A_i$ for all $i \in I$. But since $\gamma : C_i \to B_i$ then $C_i \leq B_i$. Hence B_i is the greatest lower bound for the collection $\{A_i\}$. Dually the co-product is the unique map into the least upper bound of the collection $\{A_i\}$.

(b) Category of sets.

Let $\langle I, \{S_i \mid i \in I\}\rangle$ be an object in C^I. Let $T = \dot{\bigcup}_{i \in I} S_i$ and let $f = \langle Id, \{f_i \mid i \in I\}\rangle$ be such that each f_i is the injection map of S_i into T.

Theorem 8. $f : \langle I, \{S_i \mid i \in I\}\rangle \longrightarrow \langle I, \{T_i \mid i \in I\}\rangle$ where $T_i = T$ for all $i \in I$ is the co-product in this category.

Proof: Let $\alpha : \langle I, \{S_i \mid i \in I\}\rangle \rightarrow \langle I, \{M_i \mid i \in I\}\rangle$ be any other object.

Then consider:

$$\begin{tikzcd}
\{T\} \arrow[d, "\varphi"'] & & \{S_i\}_{i \in I} \arrow[ll, "f = \{f_i\}"'] \arrow[d, "Id"] \\
\{M\} & & \{S_i\}_{i \in I} \arrow[ll, "\alpha = \{\alpha_i\}"']
\end{tikzcd}$$

and define $\varphi : T \longrightarrow M$ by

$\varphi(t) = \alpha_i(s_i)$ where s_i is the unique element of S_i such that $f_i(s_i) = t$.

Because of the properties of T and the injection maps $\{f_i\}$ we know that for each $t \in T$ there exists a unique i and s_i such that $f_i(s_i) = t$. Then φ is clearly an admissible morphism and also $\varphi f_i = \alpha_i$ for all $i \in I$.

Also, because of the uniqueness of the pre-image of any $t \in T$ we clearly have φ being unique. Q.E.D.

Now let $\{A_i\}_{i \in I}$ be an object in C^I and let $P = \prod A_i$ be the cartesian product of the A_i. Let $p_i : P \longrightarrow A_i$ be the projection maps.

<u>Theorem 9</u>. $p : \langle I, \{A_i \mid i \in I\}\rangle \rightarrow \langle I, \{P_i \mid i \in I\}\rangle$ where $p = \langle I\alpha, \{\rho_i\}\rangle$ and $P_i = P$ for all $i \in I$ is the product.

<u>Proof</u>: Consider

$$\begin{tikzcd} P \arrow[r, "p = \{\rho_i\}"] & \{A_i\} \\ S \arrow[u, "\varphi"] \arrow[r, "\alpha = \{\alpha_i\}"] & \{A_i\} \arrow[u, "I\alpha"'] \end{tikzcd}$$

where $\alpha : \langle I, \{S_i \mid i \in I\}\rangle \rightarrow \langle I, \{A_i\} \ i \in I\}\rangle$ is any other object.

Define φ as $\varphi\ (s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s), \text{-- --} \alpha_i(s) \text{ -- --})$

Then clearly φ is an admissible morphism and further $\rho_i \varphi = \alpha_i$ for all $i \in I$. Also, φ is unique for suppose there exists a $\eta : B \rightarrow P$ satisfying the conditions. Then $\rho_i \eta = f_i$ for all $i \in I$.

Therefore $\eta\ (s) = (\alpha_1(s), \alpha_2(s), \text{-- --}\ \alpha_i(s), \text{-- --})$

for all $s \in S$. i.e., $\varphi = \eta$. Q.E.D.

(c) Category of pointed sets.

Let $\{(S_i, x_i)\}_{i \in I}$ be an object in C^I, where each (S_i, x_i) is a pointed set. Let $T = \dot{\bigcup\limits_{i \in I}} (S_i - x_i) \cup (x_0)$ and define the injection maps $f_i : S_i \rightarrow T$ by

$$f_i\ (s_i) = s_i \qquad s_i \neq x_i$$

$$f_i\ (x_i) = x_0$$

<u>Theorem 10</u>. $f : \langle I, \{S_i, x_i \mid i \in I\}\rangle \rightarrow \langle I, \{T_i \mid i \in I\}\rangle$ where $f = \langle Id, \{f_i\}\rangle$ and $T_i = T$ for all $i \in I$ is the co-product in this category.

<u>Proof</u>: Let (U, u) be any other pointed set and consider

$$\begin{array}{ccc} (S_i, x_i)\ i \in I & \xrightarrow{f = \{f_i\}} & (T, x_0) \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle \varphi} \\ (S_i, x_i)\ i \in I & \xrightarrow{\alpha = \{\alpha_i\}} & (U, \mu) \end{array}$$

where $\alpha = \langle Id, \{\alpha_i\} \rangle$ is any other object in the category. Again, because of the properties of T and the f_i's we have for every $t \in T$, $t \neq x_0$ a unique i and s_i such that $f_i(s_i) = t$.

Then define φ by

$$\varphi(t) = \alpha_i(s_i) \text{ for } t \neq x_0$$

$$\varphi(x_0) = \mu$$

φ is clearly an admissible morphism and further $\varphi \cdot f_i = \alpha_i$ for all $i \in I$.

Also, φ is unique because of the uniqueness of a pre-image for every $t \in T$, $t \neq x_0$; i.e., if there exists another map η such that $\eta \circ f_i = \alpha_i$ then $\eta(t) = \alpha_i(s_i) = \varphi(t)$ for all $t \neq x_0$ and $\eta(x_0) = \mu = \varphi(x_0)$. Therefore φ is certainly unique. Q.E.D.

Let (S_1, x_1) $i \in I$ be an object in C^I where each (S_i, x_i) is a pointed set. Let P^* be the cartesian product of the collection and let p_i be the projection map of P^* on to (S_i, x_i).

<u>Theorem 11.</u> $p : \langle I, \{P_i^* \mid i \in I\} \rangle \rightarrow \langle I, \{(S_i\ x_i) \mid i \in I\} \rangle$ where $p = \langle Id, \{p_i\} \rangle$ and $p_i^* = P^*$ for all $i \in I$ is the product in this category.

<u>Proof</u>: Let $f : \langle I, \{(T, t_0) \mid i \in I\} \rangle \longrightarrow \langle I, \{(S_i\ x_i) \mid i \in I\} \rangle$ be another object in D_A^* where $f = \langle Id, \{(f_i)\} \rangle$ such that $f_i : (T, t_0) \longrightarrow (S_i, x_i)$ and $f_i(t_0) = x_i$.

$$\begin{array}{ccc} P^* & \xrightarrow{\rho = \{\rho_i\}} & \{(S_i, x_i)\}\ i \in I \\ \uparrow \varphi & & \uparrow Id \\ (T, t_0) & \xrightarrow{f = \{f_i\}} & \{(S_i, x_i)\}\ i \in I \end{array}$$

where φ is defined by

$$\varphi(t) = (f_1(t), f_2(t), \ -- \quad f_i(t), \ -- \ )$$

Then certainly φ is an admissible morphism and also $p_i \varphi = f_i$.

Now φ is unique for suppose there exists a η satisfying the conditions that η is an admissible morphism and $p_i\eta = f_i$ for all $i \in I$.

Then $p_i\eta(t) = f_i(t)$ for all $i \in I$.

i.e., $\eta(t) = (f_1(t), f_2(t), \ -- \ f_i(t), \ -- \ )$

i.e., $\eta(t) = \varphi(t)$ for all $t \in T$.

Therefore φ is unique. Q.E.D.

(d) Category of Toplogical Spaces.

Let $\{T_i\}_{i \in I}$ be an object in C^I where each T_i is a topological space.

Let $U = \dot{\cup}\, T_i$ where each T_i is considered as a set and then define the finest topology on U which makes each injection map $\rho_i : T_i \to U$ continuous.

Theorem 12. $\rho : \langle I, \{T_i \mid i \in I\}\rangle \to \langle I, \{U_i\}\ i \in I\}\rangle$ where $U_i = U$ for all $i \in I$ and $p = \langle Id, \{p_i\}\rangle$ is the co-product in this category.

Proof: Let $f: \langle I, \{T_i\}\ i \in I\}\rangle \to \langle I, \{S_i \mid i \in I\}\rangle$ where $f = \langle Id, \{f_i\}\rangle$ be any other object in $\mathcal{G}_A^*$. Then consider

$$\begin{tikzcd}
\{T_i\}_{i \in I} \arrow[r, "P = \{p_i\}"] \arrow[d, "I\alpha"'] & U \arrow[d, "\varphi"] \\
\{T_i\}_{i \in I} \arrow[r, "f = \{f_i\}"] & S
\end{tikzcd}$$

where φ is defined as

$$\varphi\ (u) = f_i(t_i)$$

where t_i is the unique pre-image of u. Because of the properties of U there exists a unique i and t_i for each $u \in U$, such that $P_i(t_i) = u$. Then clearly $\varphi P_i = f_i$ and since each f_i is continuous by assumption, then φ is continuous (see N. Bourbaki, Topologie Generale, Chapters I & II, page 31 proposition 6). Now φ is unique, for suppose there exists another admissible morphism η exhibiting these properties. Then $\eta p_i = f_i$ for all $i \in I$. Then because of the uniqueness of the pre-image of any $u \in U$, η must be defined exactly as φ. Therefore φ is unique. Q.E.D.

Now let $\langle I, \{T_i \mid i \in I\}\rangle$ be an object in C^I where each T_i is a topological space. Let $T^* = \prod_{i \in I} T_i$ be the cartesian product of the spaces with the product topology defined on T^*. Then each of the projection maps $p_i\colon T^* \to T_i$ is continuous.

<u>Theorem 13</u>. $p\colon \langle I, \{T_i^* \mid i \in I\}\rangle \to \langle I, \{T_i \mid i \in I\}\rangle$ where $p = \langle I\alpha, \{p_i\}\rangle$ and each $T_i^* = T^*$ is the product.

<u>Proof</u>: Let $f\colon \langle I, \{S_i \mid i \in I\}\rangle \to \langle I, \{T_i \mid i \in I\}\rangle$ where $S_i = S$ for all i is a topological space, be another object of D_A^*.

Then consider:

$$\begin{array}{ccc} T^* & \xrightarrow{P = \{p_i\}} & \{T_i\} \quad i \in I \\ \varphi \uparrow & & \uparrow Id \\ S & \xrightarrow{f = \{f_i\}} & \{T_i\} \quad i \in I \end{array}$$

where φ is defined as

$$\varphi(s) = (f_1(s), f_2(s), \text{- -} f_i(s), \text{- -})$$

Then clearly $p_i \varphi = f_i$ for all $i \in I$ and since each p_i and f_i are continuous, then φ is continuous (see N. Bourbaki, Topologie Generale, Chapters I & II, pages 28-29, proposition 4). Then φ is also unique, for suppose there exists an admissible morphism η satisfying the conditions that

1) η is continuous

2) $p_i \eta = f_i$ for all $i \in I$

Then $p_i \eta(x) = f_i(x)$ which implies that

$$\eta(x) = (f_1(x), f_2(x), \text{- -} f_i(x) \text{- -}) = \varphi(x)$$

for all $x \in S$. Therefore φ is unique. Q.E.D.

(e) Category of groups.

Let $\langle I, \{ G_i \mid i \in I\}\rangle$ be an object in C^I where each G_i is a group. Let FP be the free product of the G_i's. Let p_i be the injection group, homomorphism taking $G_i \longrightarrow FP$ defined by $p_i(a_i) =$ word of length one with that element as the unique entry and $p_i(1_{G_i}) =$ null word.

<u>Theorem 14.</u> $p : \langle I, \{G_i \mid i \in I\}\rangle \longrightarrow \langle I, \{(FP)_i \mid i \in I\}\rangle$ where $p = \langle Id, \{p_i\}\rangle$ and $(FP)_i = FP$ for all $i \in I$ is a co-product.

<u>Proof</u>: Let $f: \langle I, \{G_i \mid i \in I\}\rangle \longrightarrow \langle I, \{H_i \mid i \in I\rangle$ where $f = \langle Id, \{f_i\}\rangle$ and $H_i = H$ (a fixed group) for all $i \in I$ be another object in $G_A{}^*$. Then consider:

$$\begin{tikzcd} \{A_i\}\ i \in I \arrow[r, "p = \{p_i\}"] \arrow[d, "1_d"'] & FP \arrow[d, "\varphi"] \\ \{A_i\}\ i \in I \arrow[r, "f = \{f_i\}"] & H \end{tikzcd}$$

where φ is defined by

$$\varphi\ (x_1 x_2 - - x_j - - x_n) = f_1(x)\ f_2(x)\ f_3(x)\ - - f_n(x)$$

Then φ is a group homomorphism since

$$\begin{aligned} \varphi\ (\text{null word}) &= f_1\ (1_{A_1})\ f_2\ (1_{A_2})\ - - f_n\ (1_{A_n}) \\ &= 1_B \bullet 1_B \circ 1_B\ - - \circ 1_B \\ &= 1_B \end{aligned}$$

$$\begin{aligned} \varphi \{(x_1 - - x_n)(x_1'\ x_2'\ - - x_m')\} &= \varphi (x_1\ - - x_n x_1'\ - - x_m') \\ &= f_1(x_1)\ f_2(x_2)\ - - f_n(x_n)\ f_1(x_1')\ - - f_m(x_m') \\ &= (f_1(x_1)\ - - f_n(x_n))\ (f_1(x_1')\ - - f_m(x_m')) \\ &= \varphi\ (x_1\ - - x_n)\ \varphi\ (x_1'\ - - x_m') \end{aligned}$$

Further, φ is unique for if η is another group homomorphism satisfying $\eta\, p_i = f_i$ for all $i \in I$, then $\eta\ (x_i) = f_i(x_i)$ and hence

$$\begin{aligned} \eta\ (x_1\ - - x_n) &= \eta(x_1)\ - - \eta\ (x_n) \\ &= f_1(x_1)\ - - f_n(x_n) \\ &= \varphi\ (x_1\ - - x_n) \end{aligned}$$

i.e., φ is unique. Q.E.D.

Now let $\langle I, \{G_i \mid i \in I\}\rangle$ be an object in C^I where each G_i is a group. Let $G^* = \prod G_i$ be the direct product of the G_i. Define p_i to be the projection homomorphism taking $G^* \longrightarrow G_i$.

Theorem 15. $p: \langle I, \{G_i^* \mid i \in I\}\rangle \to \langle I, \{G_i \mid i \in I\}\rangle$ where $p = \langle 1_d, \{p_i\}\rangle$ and each $G_i^* = G^*$, is the product.

Proof: Let $g: \langle I, \{H_i \mid i \in I\}\rangle \to \langle I, \{G_i \mid i \in I\}\rangle$ where $g = \langle 1_d, \{g_i\}\rangle$ and $H_i = H$ (fixed group) for all $i \in I$, be another object of D_A^*. Then consider:

$$\begin{tikzcd} G^* \arrow[r, "P = \{p_i\}"] & \{G_i\}_{i \in I} \\ H \arrow[u, "\varphi"] \arrow[r, "g = \{g_i\}"] & \{G_i\}_{i \in I} \arrow[u, "I_d"'] \end{tikzcd}$$

where φ is defined by

$$\varphi(h) = (g_1(h), g_2(h), -- g_i(h), --)$$

Then $\varphi(1_H) - (1_{G_1}, 1_{G_2}, -- 1_{G_i}, --) = 1_{G^*}$

and $\varphi(hk) = (g_1(hk), g_2(hk), -- g_i(hk), --)$

$$= (g_1(h)\ g_1(k), g_2(h)\ g_2(k), --, g_i(h)\ g_i(k), --)$$

$$= (g_1(h), g_2(h), -- g_i(h), --)\ (g_1(k), g_2(k), --, g_i(k) --)$$

$$= \varphi(h)\ \varphi(k)$$

Therefore φ is certainly a group homomorphism. Also clearly

$p_i \varphi = g_i$ for all $i \in I$.

Now φ is unique, for suppose there exists another group homomorphism $\eta : H \longrightarrow G$ such that $p_i \eta = g_i$ for all $i \in I$.

Then $p_i \eta(h) = g_i(h)$ which implies that

$$\eta(h) = (g_1(h), g_2(h), --, g_i(h), --)$$
$$= \varphi(h)$$

Therefore $\eta = \varphi$, i.e., φ is unique. Q.E.D.

(f) Category of Abelian groups.

Let $\{A_i\}_{i \in I}$ be an object in C^I where each A_i is an abelian group. Let $S = \coprod A_i$ be the direct sum of the A_i's. Let λ_j be the injective homomorphism taking $A_j \longrightarrow \coprod A_i$ defined by

$\lambda_j(a_j) = (o, o --, a_j, o, \ldots)$.

<u>Theorem 16.</u> $\lambda : \langle I, \{A_i \mid i \in I\}\rangle \to \langle I, \{S_i \mid i \in I\}\rangle$ where $\lambda = \langle Id, \{\lambda_i\}\rangle$ and $S_i = S$ for all $i \in I$, is the co-product.

<u>Proof:</u> Let $g: \langle I, \{A_i \mid i \in I\}\rangle \to \langle I, \{H_i \mid i \in I\}\rangle$ where $g = \langle Id, \{g_i\}\rangle$ g_i an abelian group homomorphism for all i, $H_i = H$ (fixed abelian group) for all i be another object in G_A^*.

Then consider:

$$\begin{tikzcd} \{A_i\}_{i \in I} \arrow[r, "\lambda = \{\lambda_i\}"] \arrow[d, "Id"'] & S \arrow[d, "\varphi"] \\ \{A_i\}_{i \in I} \arrow[r, "g = \{g_i\}"] & H \end{tikzcd}$$

where φ is defined as

$$\varphi\,(a_1, a_2, - - a_i, o, o --) = g_1(a_1) + g_2(a_2) - - + g_i(a_i)$$

Then φ is an abelian group homomorphism since $\varphi\,(o, o, - - o) = 0$ and

$$\varphi\,\{(a_1, a_2 - - a_i, o\text{-}o\text{-}) + (a_1^*, a_2^*, - - a_j^*, o, o --)\}$$

$$= \varphi(a_1+a_1^*, a_2+a_2^*, - - a_i+a_i^*, a_{i+1}^*, - - a_j^*, o, o - -)$$

$$= g, (a_1+a_1^*) + - - + g_i(a_i+a_i^*) + g_{i+1}(a_{i+1}^*) + - - + g_j(a_j^*)$$

$$= g_1(a_1) + g_1(a_1^*) + - - + g_i(a_i) + g_i(a_i^*) + - - + g_j(a_j^*)$$

$$= g_1(a_1) + g_2(a_2) + - - + g_i(a_i) + g_1(a_1^*) + - - + g_j(a_j^*)$$

$$= \varphi(a_1, a_2, - - a_i, o, o -) + \varphi(a_1^*, a_2^*, - - a_j^*, o, o, --)$$

Further φ is unique for suppose there exists an abelian group homomorphism satisfying the condition that $\eta\lambda_i = g_i$ for all $i \in I$.

Then,

$$\eta\,(a_1, - - a_i, o - -) = \eta(a_1, o, o -) + \eta\,(o, a_2, o, o - -)$$
$$+ - - + \eta\,(o, o, - - o, a_i, o, - -)$$

$$= \eta\lambda_1(a_1) + \eta\lambda_2(a_2) - - + \eta\lambda_i(a_i)$$

$$= g_1(a_1) + g_2(a_2) + - \quad g_i(a_i)$$

$$= \varphi(a_1, - - a_i, o -)$$

Therefore φ is unique. Q.E.D.

Now let $< I, \{ A_i \mid i \in I \}>$ be an object in C^I where each A_i is an abelian group. Let $P = \prod A_i$ be the direct product of the A_i's. Then let $p_i : P \to A_i$ be the projection homomorphisms of P onto A_i.

Theorem 17. $p: < I, \{ P_i \mid i \in I \}> \to < I, \{ A_i \mid i \in I \}>$ where $p = < I_d, \{ p_i \} >$ and $P_i = P$ for all $i \in I$ is the product.

Proof: Similar proof as to that for groups.

Remarks. The construction of the co-product for groups involved the free product while that for abelian groups involved the direct sum. This is because the free product of abelian groups is itself no longer an abelian group.

(g) Category of monoids.

The constrution of the product and co-product are identical with that for groups.

We now wish to examine further epic and monic maps for products and co-products.

Theorem 18. Consider two co-products, one in the category G_A^* and the other in G_B^*, and consider a morphism θ between these two objects where $\theta = <\{ \theta_i \}, \varphi >$ and each θ_i is epic. Then φ is epic.

Proof: Diagramatically we have:

$$\begin{tikzcd}
A_1 \arrow[d, "\theta_1"'] \arrow[rrrr, bend left, "\rho_1"] & A_2 \arrow[d, "\theta_2"'] & A_3 \arrow[d] & & A^* \arrow[d, "\varphi"] \\
B_1 & B_2 & B_3 \arrow[rr, bend right, "\rho_3'"'] & & B^* \arrow[d, "\eta, \gamma"] \\
& & & & T
\end{tikzcd}$$

where T is any other object in C and where $\eta\varphi = \gamma\varphi$. We wish to show that $\eta = \gamma$. Note that the square above must be commutative in the sense that $\varphi\rho_i = \rho_i'\theta_i$ because each object is a co-product. Now define the map from A_i into T to be $\eta\omega\rho_i = \gamma\varphi\rho_i$ since $\eta\omega = \gamma\omega$. Then because of the commutativity defined above we have $\eta\rho_i'\theta_i = \eta\varphi\rho_i = \gamma\omega\rho_i = \gamma\rho_i'\theta_i$. Now by assumption each θ_i is epic. Therefore $\eta\rho_i' = \gamma\rho_i'$.

Hence we have a unique set of maps from the collection $\{B_i\}_{i \in I}$ into P for η and γ. Since $\rho' = \{\rho_i'\}$ is initial, then we must have a unique map $B^* \to T$ for the collection of maps $\rho' = \{\rho_i'\}$. Therefore $\eta = \gamma$. i.e. φ is epic. Q.E.D.

Dually for two products one in category D_A^* and one in D_B^* and a morphism θ between these two objects where $\theta = \langle\{\theta_i\}, \omega\rangle$ and each θ_i is monic, then φ is monic.

CHAPTER V

DIRECT LIMITS IN CATEGORIES

We now want to examine the concepts of direct limit categorically. Again it will be seen that the concept of a direct limit can be categorically characterized. Certain questions regarding monic and epic morphisms will also be answered.

Definition 1. A directed set M is a set having a relation $\leq$ defined on it, such that

1) if $\alpha \in M$ then $\alpha \leq \alpha$

2) if $\alpha \leq \beta$, $\beta \leq \gamma$ then $\alpha \leq \gamma$

3) if $\alpha, \beta \in M$ then there exists a $\gamma \in M$ such that $\alpha \leq \gamma$, $\beta \leq \gamma$.

Definition 2. A map $\phi : M \longrightarrow M'$ where M and M' are directed sets is said to be order preserving, if whenever $\alpha \leq \beta$ in M then $\phi(\alpha) \leq \phi(\beta)$ in M'.

Definition 3. Let πC be the category with objects $\langle M, \{A_\alpha \mid \alpha \in M\}\rangle$ where M is a directed set and A_α is an object in C for all $\alpha \in M$ with the added condition that if $\alpha \leq \beta$ in M then there is a unique morphism π_α^β in C such that $\pi_\alpha^\beta : A_\alpha \longrightarrow A_\beta$ and morphisms $\varphi = \langle \varphi', \{\varphi_\alpha \mid \alpha \in M\}\rangle$ where φ' is an order preserving map between directed sets M and M' and $\varphi_\alpha : A_\alpha \longrightarrow B_{\varphi'(\alpha)}$ such that if in M then the following diagram commutes:

$$\begin{tikzcd}
A_\alpha \arrow[r, "\pi_\alpha^\beta"] \arrow[d, "\varphi_\alpha"'] & A_\beta \arrow[d, "\varphi_\beta"] \\
B_{\varphi'(\alpha)} \arrow[r, "\pi_{\varphi'(\alpha)}^{\varphi'(\beta)}"] & B_{\varphi'(\beta)}
\end{tikzcd}$$

Remarks. The collection of maps $\{\pi_\alpha^\beta\}$ agree in the sense that if $\alpha \leq \beta \leq \gamma$ in M then $\pi_\beta^\gamma \pi_\alpha^\beta = \pi_\alpha^\gamma$

Define a composition of maps as

$$\varphi_1 \circ \varphi_2 = < \varphi_1' \circ \varphi_2', \{ \varphi_{1\varphi_2'(\alpha)} \circ \varphi_{2\alpha} \mid \alpha \in m,\ \varphi'(\alpha) \in m' \} >$$

Then define the category Morph ($\underset{\to}{\pi}$C) analogously to Morph (πC).

Theorem 1. If φ' is monic in the category of sets and each φ_α is monic in C then φ is monic in $\underset{\to}{\pi}$ C.

Proof: Let $\varphi : < M, \{ C_\alpha \mid \alpha \in M > \to < M' \{ D_\beta \mid \beta \in M' \} >$

We wish to show that $\varphi \circ \eta = \varphi \circ \gamma$ implies that $\eta = \gamma$.

Since $\varphi \circ \eta = \varphi \circ \gamma$ then $\varphi' \eta' = \varphi' \gamma'$ which implies that $\eta' = \gamma'$ since φ' was assumed to be monic.

Also $\varphi_{\eta'(\alpha)} \eta_\alpha = \varphi_{\gamma'(\alpha)} \gamma_\alpha$ but $\eta'(\alpha) = \gamma'(\alpha)$. Therefore $\varphi_{\eta'(\alpha)} \eta_\alpha = \varphi_{\eta'(\alpha)} \gamma_\alpha$ but φ_α was assumed to be monic for each $\alpha \in m$

Hence $\eta_\alpha = \gamma_\alpha$ for all $\alpha \in m''$. Therefore since $\eta' = \gamma'$ and $\eta_\alpha = \gamma_\alpha$ we have $\eta = \gamma$. Therefore φ is monic. Q.E.D.

Theorem 2. If φ is monic in $\underset{\to}{\pi}$ C then each φ_α is monic in C.

Proof: Suppose $\varphi_\alpha \circ \eta'' = \varphi_\alpha \circ \gamma''$. We wish to show that $\eta'' = \gamma''$.

Let $\eta'' : A_\beta \to B_\alpha$

$\gamma'' : A_\beta \to B_\alpha$

We extend η'' and γ'' to $\underset{\to}{\pi}$ C. We do this as follows. Suppose $\varphi' : M \to M'$. Then let η' and γ' both be the identity maps on M, and let each η_α be the identity map on B_α for $\eta_\alpha \neq \eta''$. Similarly define each γ_α. Then $\varphi \circ \eta = \varphi \circ \gamma$ which implies that $\eta = \gamma$ which in turn implies that $\eta_\alpha = \gamma_\alpha$ for all $\alpha \in$ M. Therefore φ_α is monic. This construction works for all φ_α. Q.E.D.

A dual type of argument shows that φ is epic in $\underset{\to}{\pi}$ C if φ' and each φ_α are monic, and if φ is epic in $\underrightarrow{\pi}\, C$ then each φ_α is monic in C.

We now construct a new subcategory of $\underrightarrow{\pi}\, C$.

<u>Definition 4</u>. Let C^M be the subcategory of $\underrightarrow{\pi}\, C$ consisting of those objects of $\underrightarrow{\pi}\, C$ indexed by the same directed set M, and all morphisms in between such objects which consist partly of the identity maps on the directed set M.

Let F be the covariant functor where

$$F : C \longrightarrow C^M \quad \text{defined by}$$

$$F(A) \to \langle M, \{A_\alpha \mid \alpha \in M\}\rangle \qquad \text{where } A_\alpha = A \text{ for all } \alpha \in M$$

$$F(\varphi_{AB}) \to \psi : \langle M, \{A_\alpha \mid \alpha \in M\}\rangle \to \langle M, \{B_\alpha \mid \alpha \in M\}\rangle$$

Note that F is a one-one functor in the sense that there is a one-one correspondence between objects and a one-one correspondence between morphisms.

<u>Definition 5</u>. Define the image of C under the functor F to be C^*. Then C^* is a subcategory of C^M .

<u>Remarks</u>. Note that for any object in C^* each π_α^β is the identity morphism on A_α .

<u>Definition 6</u>. Let D^* be the subcategory of Morph ($\underrightarrow{\pi}\, C$) which has as objects all those objects of Morph ($\underrightarrow{\pi}\, C$) which when considered as morphisms in $\underrightarrow{\pi}\, C$ have domain in C^M and codomain in C^*, and as morphisms all admissible morphisms between such objects in Morph ($\underrightarrow{\pi}\, C$).

<u>Definition 7</u>. Define the subcategory $D_A{}^*$ of D^* to consist of those objects of D^* which as morphisms in $\underrightarrow{\pi}\, C$ have as their domain a fixed object A of C^M and as morphisms those pairs $\langle \mathrm{Id}, \varphi\rangle \in D^*$ where Id is

the identity morphism on A and where $\varphi = \langle I, \{\varphi_\alpha \mid \alpha \in M\}\rangle$ is such that $\varphi_\alpha = \varphi^*$ (fixed) for all $\alpha \in M$.

Definition 8. Define the direct limit written $\varinjlim$ to be an object $\langle M, \{B_\alpha \mid \alpha \in M\}\rangle$ in C^* which makes the morphism $\varphi : \langle M, \{A_\alpha \mid \alpha \in M\}\rangle \to \langle M, \{B_\alpha \mid \alpha \in M\}\rangle$ in D_A^* initial

Theorem 3. Any two direct limits are equivalent.

Proof: Consider two initial objects of D_A^* which contain these direct limits.

$$\begin{array}{ccc}
\langle M, \{A_\alpha \mid \alpha \in M\}\rangle & \longrightarrow & \langle M, \{B_\alpha \mid \alpha \in M\}\rangle \\
\downarrow \scriptstyle{Id} & & \downarrow \scriptstyle{\varphi} \\
\langle M, \{A_\alpha \mid \alpha \in M\}\rangle & \longrightarrow & \langle M, \{C_\alpha \mid \alpha \in M\}\rangle \\
\downarrow \scriptstyle{Id} & & \downarrow \scriptstyle{\eta} \\
\langle M, \{A_\alpha \mid \alpha \in M\}\rangle & \longrightarrow & \langle M, \{B_\alpha \mid \alpha \in M\}\rangle
\end{array}$$

where $\langle M, \{B_\alpha \mid \alpha \in M\}\rangle$ and $\langle M, \{C_\alpha \mid \alpha \in M\}\rangle$ are both direct limits, say $\varinjlim 1$ and $\varinjlim 2$. Since both objects are initial then there exists unique morphisms φ and η which make the diagrams commutative. Hence in the large square $\eta \circ \varphi$ is the unique morphism making this commute. But certainly the identity morphism is an admissible morphism. Therefore $\eta \circ \varphi$ = identity morphism on $\varinjlim 1$. Similarly $\varphi \circ \eta$ = identity morphism on $\varinjlim 2$. Therefore $\eta = \varphi^{-1}$ and $\varinjlim 1$ and $\varinjlim 2$ are equivalent. Q.E.D.

We now construct the direct limit in some familiar categories.

(a) Category of Sets.

Let $\{S_\alpha\}_{\alpha \in M}$ be a collection of sets indexed by a directed set M and let π_α^β be the map from $S_\alpha \longrightarrow S_\beta$ if $\alpha \leq \beta$ in M. Let

$T = \dot{\cup}\, S_\alpha$ and define an equivalence relation on T as follows.

$g_\alpha \sim g_\beta$ iff there exists a $\gamma \in \mathcal{M}$ such that

$\alpha \leq \gamma$, $\beta \leq \gamma$ and $\pi_\alpha^\gamma(g_\alpha) = \pi_\beta^\gamma(g_\beta) = g_\gamma$

This is certainly an equivalence relation; i.e.

1) $g_\alpha \sim g_\beta$ implies $g_\beta \sim g_\alpha$

2) $g_\alpha \sim g_\alpha$

3) $g_\alpha \sim g_\beta$, $g_\beta \sim g_\gamma \Rightarrow g_\alpha \sim g_\gamma$

Now consider T modulo this equivalence relation $(\equiv)$. Denote this by $T/\equiv$.

Then define $\pi_\alpha : S_\alpha \to T/\equiv$ by $\pi_\alpha : g_\alpha \to [g_\alpha]$ where $[g_\alpha]$ is the equivalence class of g_α. π_α is certainly a set map. Two maps π_α and π_β agree in the sense that if $\pi_\alpha^\beta : S_\alpha \to S_\beta$ then

$$\begin{tikzcd} S_\alpha \arrow[r, "\pi_\alpha^\beta"] \arrow[dr, "\pi_\alpha"'] & S_\beta \arrow[d, "\pi_\beta"] \\ & T/\equiv \end{tikzcd}$$

commutes

Let $T^* = T/\equiv$.

<u>Theorem 4.</u> $\langle M, \{T_\alpha \mid \alpha \in M\}\rangle$ where $T_\alpha = T^*$ for all $\alpha \in M$ is the direct limit.

<u>Proof</u>: We wish to show that

$$\pi : \langle M, \{S_\alpha \mid \alpha \in M\}\rangle \to \langle M, \{T_\alpha^* \mid \alpha \in M\}\rangle$$

where $\pi = \langle Id, \{\pi_\alpha\}\rangle$ is initial in D_A^*.

Consider

1)
$$\begin{tikzcd} \langle M, \{S_\alpha \mid \alpha \in M\}\rangle \arrow[r, "\pi = \langle Id, \{\pi_\alpha\}\rangle"] \arrow[d, "Id"'] & \langle M, \{T_\alpha^* \mid \alpha \in M\}\rangle \arrow[d] \\ \langle M, \{S_\alpha \mid \alpha \in M\}\rangle \arrow[r, "f = \langle Id, \{f_\alpha\}\rangle"] & \langle M, \{U_\alpha \mid \alpha \in M\}\rangle \end{tikzcd}$$

where $\cup_\alpha = \cup$ (fixed) for all $\alpha \in M$, and f is any other object in D_A^*.

Then each f_α is such that it must agree with the π_α^β, i.e.

2)
$$\begin{tikzcd} S_\alpha \arrow[r, "\pi_\alpha^\beta"] \arrow[dr, "f_\alpha"'] & S_\beta \arrow[d, "f_\beta"] \\ & U \end{tikzcd}$$
commutes.

Then define as follows:

$$\varphi[g_\alpha] = f_\alpha(g_\alpha)$$

Because of the commutativity in (2) φ is a well defined map, and also

$$\varphi\, \pi_\alpha = f_\alpha \quad \text{for all } \alpha \in M.$$

Now φ is unique, for suppose that there exists a morphism η satisfying the condition that $\eta\, \pi_\alpha = f_\alpha$ for all $\alpha \in M$.

Then $\eta\, \pi_\alpha(g_\alpha) = f_\alpha(g_\alpha)$

i.e., $\eta[g_\alpha] = f_\alpha(g_\alpha) = \varphi[g_\alpha]$ for every equivalence class $[g_\alpha]$.

Therefore $\varphi = \eta$. Q.E.D.

(b) Category of Pointed Sets.

Let $(S_\alpha, x_\alpha)_{\alpha \in M}$ be a collection of pointed sets with mappings $\pi_\alpha^\beta : S_\alpha \rightarrow S_\beta$, such that $\pi_\alpha^\beta(x_\alpha) = x_\beta$ if $\alpha \leq \beta$ in M. Again take the disjoint union and form equivalence classes as before. One equivalence class will contain all the elements x_α. Then reduce by this equivalence relation and set

$$(T, t_0) = \dot{\cup}\, (S_\alpha, x_\alpha)/\equiv$$

where t_0 corresponds to $[x_\alpha]$.

Then define $\pi_\alpha : (S_\alpha, x_\alpha) \longrightarrow (T, t_0)$ by

$\pi_\alpha(g_\alpha) = [g_\alpha]$. Certainly π_α is a well defined map.

Further if $\pi_\alpha^\beta : (S_\alpha, x_\alpha) \longrightarrow (S_\beta, x_\beta)$ then π_α and

π_β agree in the sense that the diagram

$$\begin{tikzcd} (S_\alpha, x_\alpha) \arrow[r, "\pi_\alpha^\beta"] \arrow[dr, "\pi_\alpha"'] & (S_\beta, x_\beta) \arrow[d, "\pi_\beta"] \\ & (T, t_0) \end{tikzcd}$$

commutes.

Theorem 5. $\langle M, \{(T, t_0)_\alpha \mid \alpha \in M\}\rangle$ where $(T, t_0)_\alpha = (T, t_0)$ for every $\alpha \in M$, is the direct limit.

Proof: Let $f = \langle Id, \{f_\alpha\}\rangle : \langle M, \{(S_\alpha, x_\alpha) \mid \alpha \in M\}\rangle$

$\longrightarrow \langle M, \{(P, p_0)_\alpha \mid \alpha \in M\}\rangle$

where $(P, p_0)_\alpha = (P, p_0)$ (a fixed directed set) for every $\alpha \in M$, be any other object in D_A^*.

Then the diagram

$$\begin{tikzcd} (S_\alpha, x_\alpha) \arrow[r, "\pi_\alpha^\beta"] \arrow[dr, "f_\alpha"'] & (S_\beta, x_\beta) \arrow[d, "f_\beta"] \\ & (P, p_0) \end{tikzcd}$$

necessarily commutes.

Now consider

$$\begin{tikzcd} \langle M, \{(S_\alpha, x_\alpha) \mid \alpha \in M\}\rangle \arrow[r, "\pi = \{\pi_\alpha\}"] \arrow[d, "Id"'] & \langle M, \{(T, t_0)_\alpha \mid \alpha \in M\}\rangle \arrow[d, "\varphi"] \\ \langle M, \{(S_\alpha, x_\alpha) \mid \alpha \in M\}\rangle \arrow[r, "f = \{f_\alpha\}"] & \langle M, \{(P, p_0)_\alpha \mid \alpha \in M\}\rangle \end{tikzcd}$$

where φ is defined as

$$\varphi[g_\alpha] = f_\alpha(g_\alpha), \quad \varphi(t_0) = p_0$$

Then the proof follows through as for sets. Q.E.D.

(c) Category of Topological Spaces.

Let $\{X_\alpha\}_{\alpha \in M}$ be a collection of topological spaces with continuous maps $\pi_\alpha^\beta : X_\alpha \to X_\beta$ if $\alpha \leq \beta$ in M, such $\pi_\beta^\gamma \pi_\alpha^\beta = \pi_\alpha^\gamma$ if $\alpha \leq \beta \leq \gamma$ in M.

Now let $\dot{\cup} X_\alpha$ be the disjoint union of the sets X_α and define an equivalence relation as in sets. Let $T = \dot{\cup} X_\alpha / \equiv$. Define π_α as before. Now define the finest topology on T which makes each of the π_α continuous where π_α and π_β agree in the previously described sense. Now if P is any other topological space then if $f_\alpha : S_\alpha \to P$ for all $\alpha \in M$, then the diagram

$$\begin{tikzcd} S \arrow[r, "\pi_\alpha^\beta"] \arrow[dr, "f_\alpha"'] & S \arrow[d, "f_\beta"] \\ & P \end{tikzcd}$$

commutes if $\alpha \leq \beta$ in M.

<u>Theorem 6</u>. $\langle M, \{T_\alpha \mid \alpha \in M\}\rangle$ where $T_\alpha = T$ for all $\alpha \in M$ is the direct limit.

<u>Proof</u>: Let $f = \langle I_d, \{f_\alpha\}\rangle : \langle M, \{X_\alpha \mid \alpha \in M\}\rangle \to \langle M, \{P_\alpha \mid \alpha \in M\}\rangle$ where $P_\alpha = P$ for each $\alpha \in M$ be any other object in D_A^*.

Consider

$$\begin{tikzcd} \langle M, \{X_\alpha \mid \alpha \in M\}\rangle \arrow[r, "\pi = \{\pi_\alpha\}"] \arrow[d, "I_d"'] & \langle M, \{T_\alpha \mid \alpha \in M\}\rangle \arrow[d, "\varphi"] \\ \langle M, \{X_\alpha \mid \alpha \in M\}\rangle \arrow[r, "f = \{f_\alpha\}"] & \langle M, \{P_\alpha \mid \alpha \in M\}\rangle \end{tikzcd}$$

where φ is defined as follows:

$$\varphi\{g_\alpha\} = f_\alpha(g_\alpha)$$

Then clearly $\varphi\,\pi_\alpha = f_\alpha$ for all $\alpha \in M$.

Hence φ is continuous (see reference to N. Bourbaki on Page 22.)

Now by an argument analogous to that for sets we see that φ is unique. Q.E.D.

(d) Category of Groups.

Let $\{G_\alpha\}_{\alpha \in M}$ be a collection of groups and let π_α^β be a homomorphism from $G_\alpha \to G_\beta$ provided $\alpha \leq \beta$ in M, such that $\pi_\beta^\gamma \pi_\alpha^\beta = \pi_\alpha^\gamma$ if $\alpha \leq \beta \leq \gamma$ in M.

We define an equivalence relation as follows:

We say $g_\alpha \in G_\alpha$ is equivalent to $g_\beta \in G_\beta$ if there exists a $\gamma \geq \alpha, \beta$ such that $\pi_\alpha^\gamma(g_\alpha) = \pi_\beta^\gamma(g_\beta) = g_\gamma$. This relation divides the elements $g_\alpha \in G_\alpha$ for all α into disjoint equivalence classes. The product of two equivalence classes is found by taking the equivalence class of the product of two representatives of these classes in the same group. This is always possible because of the directedness of M. Then let G^∞ be this set of equivalence classes. These classes clearly form a group. We define the injection homomorphism π_α by $\pi_\alpha(g_\alpha) = [g_\alpha]$. Certainly π_α is a homomorphism and further π_α and π_β agree in the sense previously described.

Now let H be any other group and let $\{g_\alpha\}: G_\alpha \to H$ be a collection of homomorphisms such that

$$\begin{tikzcd} G_\alpha \arrow[r, "\pi_\alpha^\beta"] \arrow[dr, "g_\alpha"'] & G_\beta \arrow[d, "g_\beta"] \\ & H \end{tikzcd}$$

commutes if $\alpha \leq \beta$ in M.

<u>Theorem 7.</u> $< M, \{G_\alpha^\infty \mid \alpha \in M\}$ where $G_\alpha^\infty = G^\infty$ for all $\alpha \in M$ is a direct limit.

<u>Proof:</u> Let $g = < id, \{g_\alpha\} > : < M, \{G_\alpha \mid \alpha \in M\} > \to < M, \{H_\alpha \mid \alpha \in M\} >$ where $H_\alpha = H$ for all $\alpha \in M$ be any other object in D_A^*.

Consider:

$$\begin{array}{ccc} < M, \{G_\alpha \mid \alpha \in M\} > & \xrightarrow{\pi = \{\pi_\alpha\}} & < M, \{G_\alpha^\infty \mid \alpha \in M\} > \\ \Big\downarrow id & & \Big\downarrow \varphi \\ < M, \{G_\alpha \mid \alpha \in M\} > & \xrightarrow{f = \{f_\alpha\}} & < M, \{H_\alpha \mid \alpha \in M\} > \end{array}$$

where φ is defined as follows:

$$\varphi : G^\infty \to H \text{ such that } \varphi[g_\alpha] = f_\alpha(g_\alpha)$$

Then certainly φ is a group homomorphism for

$$\varphi[1_\alpha] = f_\alpha(1_\alpha) = 1_H$$

$$\begin{aligned} \varphi[g_\alpha g_\beta] &= \varphi[g_{\gamma_1} \cdot g_{\gamma_2}] \quad \text{where } \gamma_1, \gamma_2 \in G \\ &= f_\gamma(g_{\gamma_1} \cdot g_{\gamma_2}) = f_\gamma(g_{\gamma_1}) f_\gamma(g_{\gamma_2}) \\ &= \varphi[g_{\gamma_1}] \varphi[g_{\gamma_2}] \\ &= \varphi[g_\alpha] \varphi[g_\beta] \end{aligned}$$

and $\varphi \pi_\alpha = f_\alpha$ for $\alpha \in M$.

Again φ is unique by arguments similar to those for sets.

(e) <u>Category of Monoids.</u>

Construction and proof of theorem follows exactly as in the case for groups.

(f) <u>Category of Abelian Groups.</u>

Let $\{A_\alpha\}_{\alpha \in M}$ be a collection of abelian groups indexed by M and let π_α^β be the admissible homomorphism from $A_\alpha \to A_\beta$ if $\alpha \leq \beta$ in M. Now define an equivalence relation on the elements of G_α for all $\alpha \in M$

as was done in the case for groups. The sum of two equivalence classes is defined to be $[g_\alpha] + [g_\beta] = [g_{\gamma_1} + g_{\gamma_2}]$ where g_{γ_1} and g_{γ_2} are representatives of $[g_\alpha]$ and $[g_\beta]$ in the same abelian group A_γ. This is possible because of the directedness of M. Then clearly the set of equivalence classes form an abelian group. Denote the set of equivalence classes by A^∞. Define the injection maps $\pi_\alpha : A_\alpha \to A^\infty$ by $\pi_\alpha(a_\alpha) = [a_\alpha]$. Clearly this is a homomorphism for $\pi_\alpha(1_\alpha) = [1_\alpha]$ and

$$\pi_\alpha(g_{\alpha_1} + g_{\alpha_2}) = [g_{\alpha_1} + g_{\alpha_2}] = [g_{\alpha_1}] + [g_{\alpha_2}]$$
$$= \pi_\alpha(g_{\alpha_1}) + \pi_\alpha(g_{\alpha_2})$$

Further π_α and π_β agree in the sense that if $\pi_\alpha^\beta : A_\alpha \to A_\beta$ for $\alpha \leq \beta$ in M, then the diagram

$$\begin{tikzcd}
A_\alpha \arrow[r, "\pi_\alpha^\beta"] \arrow[dr, "\pi_\alpha"'] & A_\beta \arrow[d, "\pi_\beta"] \\
 & A^\infty
\end{tikzcd}$$

commutes.

<u>Theorem 8</u>. $\langle M, \{A^\infty_\alpha \mid \alpha \in M\}\rangle$ where $A^\infty_\alpha = A^\infty$ for all $\alpha \in M$ is a direct limit.

<u>Proof</u>: Let $f = \langle Id, \{f_\alpha\}\rangle : \langle M, \{A_\alpha \mid \alpha \in M\}\rangle \to \langle M, \{G_\alpha \mid \alpha \in M\}\rangle$ where $G_\alpha = G$ (fixed abelian group) for all $\alpha \in M$ be any other object in D_A^*.

Then consider:

$$\begin{tikzcd}
\langle M, \{A_\alpha \mid \alpha \in M\}\rangle \arrow[r, "\pi = \{\pi_\alpha\}"] \arrow[d, "Id"'] & \langle M, \{A^\infty_\alpha \mid \alpha \in M\}\rangle \arrow[d, "\varphi"] \\
\langle M, \{A_\alpha \mid \alpha \in M\}\rangle \arrow[r, "f = \{f_\alpha\}"] & \langle M, \{G_\alpha \mid \alpha \in M\}\rangle
\end{tikzcd}$$

Note that again each f_α is an abelian group homomorphism such that $\alpha \leq \beta$ in M then

$$\begin{tikzcd} A_\alpha \arrow[r, "\pi_\alpha^\beta"] \arrow[dr, "f_\alpha"'] & A_\beta \arrow[d, "f_\beta"] \\ & G \end{tikzcd}$$

commutes.

Define $\varphi : A^\infty \to G$ by

$\varphi[a_\alpha] = f_\alpha(a_\alpha)$. Under this definition φ is certainly a homomorphism.

$$\varphi[1_\alpha] = f_\alpha(1_\alpha) = 1_G$$

$$\begin{aligned} \varphi[g_\alpha + g_\beta] &= \varphi([g_\alpha] + [g_\beta]) = \varphi([g_{\gamma_1}] + [g_{\gamma_2}]) \\ &= f_\gamma((g)_{\gamma_1} + (g)_{\gamma_2}) = f_\gamma(g)_{\gamma_1} + f_\gamma(g)_{\gamma_2} \\ &= \varphi[g_\alpha] + \varphi[g_\beta] \end{aligned}$$

Further $\varphi\pi_\alpha = f_\alpha$ for all $\alpha \in$ M from the manner in which φ was defined.

Again, φ is unique by arguments similar to those for sets. Q.E.D.

We shall now examine further the question of monic and epic maps in $\varinjlim$.

<u>Theorem 9</u>. Consider:

$$\begin{tikzcd} X_\alpha \arrow[r, "\pi_\alpha^\beta"'] \arrow[rrrr, bend left, "\pi_\alpha"] & X_\beta \arrow[r, "\pi_\beta^\gamma"'] & X_\gamma \arrow[r] & \cdots \arrow[r] & \varinjlim \end{tikzcd}$$

If each π_α^β is epic then each π_α is epic.

<u>Proof</u>: Suppose $\eta \circ \pi_\alpha = \varphi \circ \pi_\alpha$. We wish to show that then $\eta = \varphi$.

Consider any other object S from the same category and let $\eta, \varphi : \varinjlim \to S$ be admissible morphisms. Now the π_α's must agree in the sense that $\pi_\alpha = \pi_\beta \pi_\alpha^\beta$. Then since each π_α^β is epic by assumption we have

$$\eta \, \pi_\beta \, \pi_\alpha^\beta = \varphi \, \pi_\beta \, \pi_\alpha^\beta \quad \text{implies} \quad \eta \, \pi_\beta = \varphi \, \pi_\beta$$

Now define morphisms $f_\alpha : X_\alpha \longrightarrow S$ by $f_\alpha = \eta \, \pi_\alpha = \varphi \, \pi_\alpha$ for all α contained in the chain. Then since $\pi = \{\pi_\alpha\}$ is initial in D_A^* then there exists a unique map $\varphi : \underrightarrow{\lim} \longrightarrow S$ such that the diagram

$$\begin{tikzcd}
S_\alpha \arrow[r] \arrow[rrrr, bend left, "\pi_\alpha"] \arrow[drrrr, "f_\alpha"'] & S_\beta \arrow[r] \arrow[drrr, "f_\beta"] & S_\gamma \arrow[r] & \cdots \arrow[r] & \underrightarrow{\lim} \arrow[d, "\varphi"] \\
 & & & & S
\end{tikzcd}$$

commutes.

Therefore $\varphi = \eta$, i.e., π_α is epic.

<u>Theorem 10</u>. In those categories in which the particular construction given previously is admissible and in which mono is equivalent to one-one then each π_α^β monic implies that each π_α is monic.

<u>Proof</u>: Consider

$$\begin{tikzcd}
S_\alpha \arrow[r, "\pi_\alpha^\beta"'] \arrow[rrrr, bend left, "\pi_\alpha"] & S_\beta \arrow[r, "\pi_\beta^\gamma"'] & S_\gamma \arrow[r] & \cdots \arrow[r] & \underrightarrow{\lim}
\end{tikzcd}$$

Suppose that s_1 and s_2 are two elements of S_α such that

$\pi_\alpha(s_1) = \pi_\alpha(s_2)$. Then there exists a $\gamma \in M$ such that $\gamma \geq \alpha$ and $\pi_\alpha^\gamma(s_1) = \pi_\alpha^\gamma(s_2)$ but since each π_α^γ is one-one by assumption then $s_1 = s_2$; i.e., π_α is one-one. Q.E.D.

<u>Remarks</u>. If the hypothesis of the previous theorem is weakened by requiring that each morphism π_α^β is monic only and/or that the particular construction of the $\underrightarrow{\lim}$ is not admissible then it seems to be an open question as to whether each π_α is monic.

<u>Theorem 11.</u> Suppose that $\Theta = \langle \{\theta_\alpha\}, \varphi \rangle$ is an admissable morphism in Morph $(\underrightarrow{\Pi}\ \mathbb{C})$ such that the domain and co-domain are both initial objects for subcategories D_S^* and D_K^* of Morph $(\underrightarrow{\Pi}\ \mathbb{C})$. Then if each Θ_α is epic then $\varphi = \varinjlim (\Theta_\alpha)$ is epic, where $\varphi = \varinjlim (\Theta_\alpha)$ is defined to be the unique map from $\varinjlim (S_\alpha) \longrightarrow \varinjlim ((K_\alpha))$ such that

$$\pi'_\alpha \Theta_\alpha = \varphi\, \pi_\alpha \quad .$$

<u>Proof:</u> Consider

$$\begin{tikzcd}
S_\alpha \arrow[r] \arrow[d, "\Theta_\alpha"'] & S_\beta \arrow[r] \arrow[d, "\Theta_\beta"'] \arrow[rr, bend left, "\pi_\beta"] & \cdots \arrow[r] & \varinjlim (S_\alpha) \arrow[d, "\varphi = \varinjlim (\Theta_\alpha)"] \\
K_\alpha \arrow[r] & K_\beta \arrow[r] \arrow[rr, bend left, "\pi'_\beta"] & \cdots \arrow[r] & \varinjlim (K_\alpha) \arrow[d, "\eta,\gamma"] \\
 & & & T
\end{tikzcd}$$

where each Θ_α is epic.

Let T be any other object in $\mathbb{C}$ such that η and γ are two admissable morphisms from $\varinjlim (K_\alpha)$ into T such that $\eta\,\pi_\alpha = \gamma\,\pi_\alpha$. Note that the above diagram must be commutative. Now define the map from S_β into T to be $\eta\varphi\pi_\beta = \gamma\varphi\pi_\beta$ since $\eta\varphi = \gamma\varphi$ by the characterization of $\varinjlim (S_\alpha)$. Because of the commutativity of the above diagram we have $\eta\,\pi'_\beta \Theta_\beta = \eta\varphi\pi_\beta = \gamma\varphi\pi_\beta = \gamma\pi'_\beta\Theta_\beta$. Since Θ_β is epic by assumption we have $\eta\pi'_\beta = \gamma\pi'_\beta$ for all $\beta \in M$. Hence we have a unique set of morphisms from $\{K_\alpha\}$ into T. Then because $\Pi' = \{\pi'_\alpha\}$ is an initial object in D_K^* we must have $\gamma = \eta$, i.e., π_α is epic for all $\alpha \in M$. Q.E.D.

<u>Remarks.</u> Whether φ is monic if each Θ_α is monic seems to be an open question, a lthough dually from examples which will be referred to in the next cha pter it seems safe to assume that this is not the case.

CHAPTER VI

INVERSE LIMITS IN CATEGORIES

The concept of an inverse limit written $\varprojlim$ is well known in various topological and algebraic structures. We shall now proceed to give a categorical definition to $\varprojlim$ and give the construction in various categories. Again certain questions regarding monic and epic maps will be answered.

Definition 1. We now construct a new category $\underleftarrow{\Pi}\, C$ from C which has objects $< M', \{ A_\alpha \mid \alpha \in M' \} >$ where M' is a directed set and each A_α is an object from C with the added condition that if $\alpha \leq \beta$ in M' then there exists a unique morphism $\pi_\alpha^\beta : A_\beta \rightarrow A_\alpha$ such that if $\alpha \leq \beta \leq \gamma$ in M' then $\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma$, and having morphisms $\varphi = < \varphi', \{ \omega_\alpha \mid \alpha \in m \} >$ where $\varphi' : M' \rightarrow M''$ (both directed sets) is an order preserving map and each $\omega_\alpha : A_\alpha \rightarrow B_{\varphi'(\alpha)}$ such that if $\alpha \leq \beta$ in M' then the diagram

$$\begin{tikzcd} A_\beta \arrow[r, "\pi_\alpha^\beta"] \arrow[d, "\omega_\alpha"'] & A_\alpha \arrow[d, "\omega_\beta"] \\ B_{\varphi'(\beta)} \arrow[r, "\pi_{\varphi'(\alpha)}^{\varphi'(\beta)}"] & B_{\varphi'(\alpha)} \end{tikzcd}$$

commutes.

Define a composition of maps

$$\omega_1 \circ \omega_2 = < \omega_1' \circ \omega_2', \{ \omega_{1,\omega_2'(\alpha)} \, \omega_{2,\alpha} \mid \alpha \in m', \omega_2'(\alpha) \in m'' \} >$$

Then form the category morph $(\underleftarrow{\Pi}\, C)$.

Definition 2. Define $C^{M'}$ to be the subcategory of $\underleftarrow{\Pi}\, C$ which has as objects all those objects of $\underleftarrow{\Pi}\, C$ which are indexed by a fixed directed indexing set M' and as morphisms all admissible morphisms between such objects in

C which consist partly of the identity map on M'.

Definition 3. Define C^o to be the subcategory of $C^{M'}$ which has as objects all those objects of $C^{M'}$ $\langle M', \{A_\alpha \mid \alpha \in M\} \rangle$ for which $A_\alpha = A_\beta$ for all $\alpha, \beta \in M'$ and all admissible morphisms between such objects in $C^{M'}$, such that if $\varphi = \langle Id, \{\varphi_\alpha\} \rangle$ has $\varphi_\alpha = \varphi_\beta$ for all $\alpha, \beta \in M'$

Let F be the one-one covariant functor where

$$F : C \longrightarrow C^o \text{ defined by}$$

$$F(B) \longrightarrow \langle M', \{B_\alpha \mid \alpha \in M'\} \rangle$$

$$F(\varphi_{AB}) \longrightarrow \eta = \langle Id, \langle \varphi_\alpha \mid \alpha \in M' \rangle \rangle \text{ where}$$

$$\varphi_\alpha : A \longrightarrow B \text{ for all } \alpha \in M'.$$

Then we can consider C as being naturally imbedded in $\underleftarrow{\pi}\, C$.

Definition 4. Let K be the subcategory of Morph ($\underleftarrow{\pi}\, C$) which has as objects those objects of Morph ($\underleftarrow{\pi}\, C$) which when considered as morphisms in $\underleftarrow{\pi}\, C$ have their domain in C^o and co-domain in $C^{M'}$.

Definition 5. Define the subcategory K_A of K to consist of those objects of K which as morphisms in $\underleftarrow{\pi}\, C$ have as their co-domain a fixed object A of $C^{M'}$ and morphisms those pairs $\langle \varphi, Id \rangle$ where Id = identity on A and $\varphi = \langle I, \{\varphi_\alpha \mid \alpha \in M'\} \rangle$ is such that $\varphi_\alpha = \varphi^*$ (fixed) for all $\alpha \in M'$.

Definition 6. Define the inverse limit written $\underleftarrow{\lim}$ to be an object $\langle M', \{A_\alpha \mid \alpha \in M'\} \rangle$ of C^o which makes the object $\pi = \langle Id, \{\pi_\alpha\} \rangle : \langle M', \{A_\alpha \mid \alpha \in M'\} \rangle \rightarrow \langle M', \{B_\alpha \mid \alpha \in M'\} \rangle$ of K_A terminal.

Theorem 1. Any two $\underleftarrow{\lim}$ are equivalent.

Proof: Dual argument for theorem 6, chapter 5.

We now give the construction of the $\varprojlim$ in some familiar categories.

(a) Category of Sets.

Let $\{X_\alpha\}_{\alpha \in M'}$ be a collection of sets indexed by the directed set M' with unique morphisms $\pi_\alpha^\beta : X_\beta \longrightarrow X_\alpha$ if $\alpha \leq \beta$ in M' and $\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma$ if $\alpha \leq \beta \leq \gamma$ in M'.

Let X^∞ be the subset of the direct product πX_α consisting of those functions $x = \{x_\alpha\}$ such that for each relation $\alpha \leq \beta$ in M' we have $\pi_\alpha^\beta (x_\beta) = x_\alpha$. The projection maps π_β are defined by $\pi_\beta : X^\infty \longrightarrow X_\beta$ such that $\pi_\beta(x) = x_\beta$. Then each π_β is certainly an admissible morphism.

<u>Theorem 2</u>. $\langle M', \{X_\alpha^\infty \mid \alpha \in M'\}\rangle$ where each $X_\alpha^\infty = X^\infty$ is an inverse limit.

<u>Proof</u>: We wish to show that

$$\pi = \langle Id, \{\pi_\alpha\}\rangle : \langle M', \{X_\alpha^\infty \mid \alpha \in M'\}\rangle \longrightarrow \langle M', \{X_\alpha \mid \alpha \in M'\}\rangle$$

is an initial object of the category K_X.

Let $f = \langle Id, \{f_\alpha\}\rangle : \langle M', \{T_\alpha \mid \alpha \in M'\}\rangle \to \langle M', \{X_\alpha \mid \alpha \in M'\}\rangle$ be any other object in the category K_X where each f_α is such that

1)
$$\begin{tikzcd} X_\alpha & & X_\beta \arrow[ll, "\pi_\alpha^\beta"'] \\ T \arrow[u, "f_\alpha"] \arrow[urr, "f_\beta"'] & & \end{tikzcd}$$

commutes for each $\alpha \leq \beta$ in M'.

Then consider:

$$\begin{tikzcd} \langle M', \{X_\alpha^\infty \mid \alpha \in M'\}\rangle \arrow[r, "\pi = \{\pi_\alpha\}"] & \langle M', \{X_\alpha \mid \alpha \in M'\}\rangle \\ \langle M', \{T_\alpha \mid \alpha \in M'\}\rangle \arrow[u, "\varphi"] \arrow[r] & \langle M', \{X_\alpha^\infty \mid \alpha \in M'\}\rangle \arrow[u, "Id"'] \end{tikzcd}$$

and define $\varphi : \langle M', \{T_\alpha \mid \alpha \in M'\}\rangle \to \langle M', \{X_\alpha^\infty \mid \alpha \in M'\rangle$ by

$\varphi : T \to X^{\infty}$ such that

$\varphi(t) = x = \{x_\alpha\}$ where $x_\alpha = f_\alpha(t)$

Then φ is certainly an admissible morphism since $\varphi(t)$ is a unique element of X^{∞} because of the commutativity of 1). Also $\pi_\alpha \varphi = f_\alpha$ for all $\alpha \in M'$. Clearly φ is unique for suppose there exists a morphism η satisfying the condition that $\pi_\alpha \eta = f_\alpha$ for all $\alpha \in M'$, then $\eta(t) = x = \{x_\alpha\}$ such that $x_\alpha = f_\alpha(t)$, i.e., $\eta(t) = \{f_\alpha(t)\} = \varphi(t)$. Q.E.D.

(b) Category of Pointed Sets.

Construction and proof of theorem similar to those for the category of sets.

(c) Category of Topological Spaces.

Let $\{T_\alpha\}_{\alpha \in M'}$ be a collection of topological spaces indexed by the directed set M' with morphisms consisting of continuous maps $\pi_\alpha^\beta : T_\beta \to T_\alpha$ if $\alpha \leq \beta$ in M' and such that $\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma$.

Then define T^{∞} as for sets. T^{∞} is a subspace of the direct product having the topology induced on it as a subspace of the direct product. Then each projection map $\pi_\alpha : T^{\infty} \to X_\alpha$ defined as in sets is continuous.

<u>Theorem 3</u>. $\langle M', \{T_\alpha^{\infty} \mid \alpha \in M'\}\rangle$ where $T_\alpha^{\infty} = T^{\infty}$ for all $\alpha \in M'$ is a $\underleftarrow{\lim}$.

<u>Proof</u>: Let $f = \langle Id, \{f_\alpha\}\rangle : \langle M', \{S_\alpha \mid \alpha \in M\}\rangle \to \langle M', \{T_\alpha \mid \alpha \in M'\}\rangle$ where $S_\alpha = S$ (fixed topological space) for all $\alpha \in M'$, and the collection $\{f_\alpha\}$ is such that

$$\begin{tikzcd} T_\alpha & & T_\beta \arrow[ll, "\pi_\alpha^\beta"'] \\ & S \arrow[ul, "f_\alpha"] \arrow[ur, "f_\beta"'] & \end{tikzcd}$$

commutes if $\alpha \leq \beta$ in M'.

Consider:

$$\begin{tikzcd} \langle M', \{T_\alpha^\infty : \alpha \in M'\}\rangle \arrow[r, "\pi = \{\pi_\alpha\}"] & \langle M', \{T_\alpha \mid \alpha \in M'\}\rangle \\ \langle M', \{S_\alpha \mid \alpha \in M'\}\rangle \arrow[u, "\varphi"] \arrow[r, "f = \{f_\alpha\}"] & \langle M', \{T_\alpha \mid \alpha \in M'\}\rangle \arrow[u, "Id"'] \end{tikzcd}$$

where φ is defined as

$\varphi: S \longrightarrow T^\infty$ by $\varphi(s) = x = \{x_\alpha\}$ such that

$\pi_\alpha(x) = f_\alpha(s)$,

i.e., $\varphi(s) = x = \{f_\alpha(s)\}$

Then $\pi_\alpha \varphi = f_\alpha$ for all $\alpha \in M'$ and since each f_α is continuous by assumption then φ is continuous (see reference to N. Bourbaki, page 23).

Again φ is unique by arguments similar to the one for sets. Q.E.D.

(d) Category of Groups.

Let $\{G_\alpha\}_{\alpha \in M'}$ be a collection of groups indexed by a directed set M' such that π_α^β is the unique group homomorphism from $G_\beta \longrightarrow G_\alpha$ if $\alpha \leq \beta$ in M' and $\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma$. Define G^∞ to be a subset of the direct product consisting of those functions $x = \{x_\alpha\}$ such that for each relation $\alpha \leq \beta$ in M $\pi_\alpha^\beta(x_\beta) = x_\alpha$. G^∞ is then a group. Define the projection homomorphisms

$\pi_\beta : G^\infty \longrightarrow G_\beta$ by $\pi_\beta(x) = x_\beta$.

<u>Theorem 4</u>. $\langle M', \{G_\alpha^\infty \mid \alpha \in M'\}\rangle$ where $G_\alpha^\infty = G^\infty$ for all $\alpha \in M'$ is

an inverse limit.

<u>Proof</u>: Let $f = \langle I\alpha, \{f_\alpha\} \rangle : \langle M', \{H_\alpha | \alpha \in M'\} \rangle \to \langle M', \{G_\alpha | \alpha \in M'\} \rangle$ where $H_\alpha = H$ (fixed group) for all $\alpha \in M'$ be any other object in K_G where again $\{f_\alpha\}$ is such that the diagram

1)
$$\begin{tikzcd} G_\alpha & & G_\beta \arrow[ll, "\pi_\alpha^\beta"'] \\ & H \arrow[ul, "f_\alpha"] \arrow[ur, "f_\beta"'] & \end{tikzcd}$$
commutes.

Consider:

$$\begin{tikzcd} \langle M', \{G^\infty_\alpha | \alpha \in M'\} \rangle \arrow[r] & \langle M', \{G_\alpha | \alpha \in M'\} \rangle \\ \langle M', \{H_\alpha | \alpha \in M'\} \rangle \arrow[u, "\varphi"] \arrow[r] & \langle M', \{G_\alpha | \alpha \in M'\} \rangle \arrow[u, "I\alpha"'] \end{tikzcd}$$

where φ is defined as $\varphi : H \to G^\infty$ by

$$\varphi(h) = x = \{f_\alpha(h)\}$$

Then φ is certainly a group homomorphism since

$$\varphi\{f_\alpha(h)\} \in G^\infty \text{ by (1) and}$$

$$\varphi(1_H) = 1_{G^\infty}$$

$$\begin{aligned} \varphi(g_1 g_2) &= \{f_\alpha(g_1 g_2)\} = \{f_\alpha(g_1)\, f_\alpha(g_2)\} \\ &= \{f_\alpha(g_1)\}\{f_\alpha(g_2)\} \\ &= \varphi(g_1)\, \varphi(g_2) \end{aligned}$$

Further, $\pi_\alpha \varphi = f_\alpha$ for all $\alpha \in M'$.

Now φ is unique, for suppose there exists a homomorphism η satisfying the condition that $\pi_\alpha \eta = f_\alpha$ for all $\alpha \in M'$. Then $\eta(h) = x = \{f_\alpha(h)\}$ because of the way π_α was defined. But $\{f_\alpha(h)\} = \varphi(h)$, i.e., $\varphi(h) = \eta(h)$ true for all

$h \in H$.

Therefore $\eta = \varphi$, i.e., φ is unique. Q.E.D.

The construction of the $\varprojlim$ and proof of the theorems for monoids and abelian groups are similar.

We now wish to answer a few questions regarding epic and monic maps in the inverse limit.

Theorem 5. Suppose that $\theta = \langle \{\theta_\alpha\}, \varphi \rangle$ is an admissible morphism in Morph ($\varprojlim \Pi C$) such that the domain and co-domain are both initial objects for subcategories K_A and K_B. Then if each θ_α is monic then $\varphi = \varprojlim(\theta_\alpha)$ is monic, **where $\varprojlim(\theta_\alpha)$ is dual to $\varinjlim(\theta_\alpha)$.**

Proof: Diagrammatically we have

$$\begin{tikzcd}
\varprojlim(A_\alpha) \arrow[rr, bend left, "\pi_\alpha"] \arrow[d, "\varphi = \varprojlim(\theta_\alpha)"'] & & A_\alpha \arrow[d, "\theta_\alpha"] & A_\beta \arrow[l] \arrow[d, "\theta_\beta"] & A_\gamma \arrow[l] & \arrow[l] \\
\varprojlim(B_\alpha) \arrow[rr, bend left, "\pi'_\alpha"] & & B_\alpha & B_\beta \arrow[l] & B_\gamma \arrow[l] & \arrow[l]
\end{tikzcd}$$

Then the proof is a dual argument to the proof of theorem 11, chapter V.

Theorem 6. Consider $\begin{tikzcd} \varprojlim(S_\alpha) \arrow[r, bend left, "\pi_\alpha"] & S_\alpha & S_\beta \arrow[l, "\pi_\alpha^\beta"'] & \arrow[l] \end{tikzcd}$

Then if each π_α^β is monic then each π_α is monic.

Proof: Dual argument to theorem 9, chapter V.

Remarks. In theorem 5 each θ_α being epic does not imply that $\varphi = \varprojlim(\theta_\alpha)$ is epic. We refer the reader to N. Bourbaki, Theorie Des Ensembles, Livre 1, Chapitre III, pp. 37, exercise 32, for the outline of a counter example. Further, each π_α^β being epic in theorem 6 does not imply that each π_α is epic. The reader is again referred to the same exercise.

It seems safe to assume that dually each θ_α monic in theorem 11, chapter V does not imply that $\varphi = \varinjlim(\theta_\alpha)$ is monic.

BIBLIOGRAPHY

1 Bourbaki, N. Topologie Generale. Livre III, Chapitre I & II. Herman, Paris.

__________. Algebre. Livre II, Chapitre II. Herman, Paris.

__________. Theorie Des Ensembles. Livre I, Chapitre III. Herman, Paris.

2 Burgess, W. "The Meaning of Mono and Epi in Some Familiar Categories," Canadian Math. Bulletin, Vol. 8, No. 6, 1965.

3 Cohn, P.M. Universal Algebra. New York, 1965.

4 Eilenberg, S., and N. Steenrod. Foundations of Algebraic Topology. Princeton, 1952.

5 Freyd, P. Abelian Categories. New York, 1964.

6 Hu, S.T. Elements of Modern Algebra. San Francisco, 1965.

7 Kurosh, A.G. The Theory of Groups (two volumes). New York, 1955.

8. Lang, S. Algebra. New York, 1965.

9. MacLane, S. Homology. New York, 1963.

__________. "Categorical Algebra," Bulletin of the American Math. Society. Vol. II, No. 1, January 1965.

Remarks. We can consider C as being imbedded in C', for let F be the covariant functor taking $C \longrightarrow C'$ where F is defined by

$F(A)$ $\langle I, \{A_i \mid i \in I\}\rangle$ where $A_i = A$ for all $i \in I$. If $A, B \in C$, and $\varphi_{AB}: A \to B$,

$F(\varphi_{AB}) \rightarrow \langle \varphi', \{\varphi_i \mid i \in I\}\rangle$ where φ' is the identity map on I and $\varphi_i = \varphi_j = \varphi$ for all i and j.

We will often denote objects in categories C^i and C^I by $\{A_i\}\ i \in I$ when it is understood that we are using a fixed indexing set I and only the identity maps on I.

Definition 4. Let D^* denote the sub-category of Morph (π C) which has as objects those objects of Morph (π C) which when considered as morphisms in π C have as their domain an object in C' and as their co-domain an object in C^I, and morphisms the ordered pair of morphisms in C' and C^I respectively which make the following diagram commute.

$$\begin{tikzcd}
\langle I, \{A_i \mid i \in I\}\rangle \arrow[r, "\varphi"] \arrow[d, "\eta"'] & \langle I, \{B_i \mid i \in I\}\rangle \arrow[d, "\gamma"] \\
\langle I, \{C_i \mid i \in I\}\rangle \arrow[r, "\psi"] & \langle I, \{D_i \mid i \in I\}\rangle
\end{tikzcd}$$

This clearly forms a sub-category of Morph (π C).

Definition 5. Define the sub-category D_A of D^* to consist of those objects which as morphisms in π C have a fixed element A of C^I as their co-domain, and as morphisms those pairs $\langle \varphi, Id\rangle \in D^*$ where Id is the identity map on A and $\varphi = \langle I, \{\varphi_i \mid i \in I\}\rangle$ is such that $\varphi_i = \varphi^*$ (fixed) for all $i \in I$.

Definition 6. Dually define the category G^* to be the sub-category of Morph (π C) which has objects those morphisms in π C having domain in C^I and co-domain in C'. Define the sub-category G_A^* of G^* analogously