CENTRAL COLLINEATIONS OF FINITE PROJECTIVE PLANES

by

RUSSELL GRANT WOODS

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ABSTRACT

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It is well-known that the structure of the finite projective plane is determined to a great extent by the structure of the collineation group of the plane. In this thesis certain assumptions are made concerning the nature of the action of the collineation group considered as a permutation group on the points and lines of the plane. Assumptions are also made concerning the number and nature of the central collineations that occur in the collineation group, and the way in which these assumptions determine the structure of the plane is investigated. The approach used is that employed in recent papers of Piper and Wagner. In order to carry out this investigation, a development of the elementary theory of the finite projective plane and of aspects of the theory of permutation groups is also given.

The writer wishes to express his thanks to Dr. W. J. Jonsson of the Department of Mathematics for his advice and helpfulness, to Mrs. Elaine Halliday for her patience during the typing of this thesis, and to the late Tom Lehrer, one of whose songs provided the basic idea of the thesis.

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INTRODUCTION

The purpose of this thesis is to develop several recent results concerning the structure of the projective plane. It is shown that the structure of the plane can be deduced to a great extent from a knowledge of the structure of the collineation group of the plane, and more particularly from a knowledge of the properties of the central collineations of the plane.

The thesis is divided into five chapters. In the first chapter a number of elementary properties of central collineations are obtained. In the second chapter the method of co-ordinatizing the projective plane by means of a ternary ring is developed, and the important theorem that a projective plane is alternative if and only if its ternary ring is an alternative field is proved. The third chapter is devoted to a study of finite alternative fields, and it is shown that all finite alternative fields are commutative fields. This, coupled with the result of Chapter 2, yields the important result that all finite alternative planes are Desarguesian.

In the fourth chapter a number of combinatorial theorems, many relying heavily on the theory of permutation groups, are proved. In addition, a purely grouptheoretic result (theorem 4.1) is obtained. The results of Chapter 4 are used repeatedly in Chapter 5.

Chapter 5 is essentially a synthesis of the results of several recent papers of Wagner and Piper (10), (11), and (13). Certain conditions imposed on the collineation group of the projective plane are shown to be sufficient to ensure that the plane is Desarguesian; conditions under which the plane is a translation plane, or the dual of a translation plane, are also found. Thus the structure of the projective plane is shown to be determined to a great extent by the properties of the collineation group of the plane considered as a permutation group on the points and lines of the plane.

FOREWORD

In the body of this thesis it is assumed that the reader is familiar with the terminology and notation of projective geometry, and with the basic properties of the projective plane. The purpose of this foreword is to summarize these basic properties and to define notation not defined elsewhere. The results quoted below can be found in Pickert (9) and in Hall (5).

A projective plane π is a triple $(\mathcal{P}, \mathcal{L}, \varepsilon)$ consisting of a set $\mathcal{P}$ whose elements are called points, a collection $\mathcal{L}$ of distinguished subsets of $\mathcal{P}$, and the set-theoretic membership relation ε relating elements of $\mathcal{P}$ and elements of $\mathcal{L}$. The elements of $\mathcal{L}$ are called lines. If ℓ is a line and P is a point, then "$P \varepsilon \ell$" is defined to mean that P is a member of the distinguished subset ℓ of $\mathcal{P}$. Geometrical language is used throughout; hence "P is on ℓ", "P belongs to ℓ", "ℓ is a line through P", "ℓ contains P", "P is incident with ℓ", and "P is a point of ℓ", are all phrases meaning "$P \varepsilon \ell$". If $P_1, \cdots, P_n$ are all on the same line ℓ, then the points $P_1, \cdots, P_n$ are said to be collinear, and this is symbolized by writing $\equiv P_1, \cdots, P_n$. Similarly, if lines $\ell_1, \cdots, \ell_n$

all pass through the same point P, then the lines $\ell_1, \ldots, \ell_n$ are said to be concurrent. On occasion the symbol ε will be used in its more general sense of denoting set membership. The use of the symbol will always be clear from the context.

A projective plane π obeys the following axioms of incidence :

(1) If $P_1 \varepsilon \mathcal{P}$, $P_2 \varepsilon \mathcal{P}$, $P_1 \neq P_2$, then there exists exactly one line $\ell \varepsilon \mathcal{L}$ such that $P_1 \varepsilon \ell$, $P_2 \varepsilon \ell$.

(2) If $\ell_1 \varepsilon \mathcal{L}$, $\ell_2 \varepsilon \mathcal{L}$, $\ell_1 \neq \ell_2$, then there exists exactly one $P \varepsilon \mathcal{P}$ such that $P \varepsilon \ell_1$, $P \varepsilon \ell_2$.

(3) There exist four distinct points of $\mathcal{P}$, no three of which are collinear.

It immediately follows that there exist four distinct lines of $\mathcal{L}$, no three concurrent. Since a knowledge of two distinct points P_1 and P_2 on a line uniquely determines the line, we shall often denote by P_1P_2 the (unique) line containing both P_1 and P_2. Similarly, if ℓ_1 and ℓ_2 are distinct lines, $\ell_1 \cap \ell_2$ will denote the unique point incident with each.

Suppose that the number of points of a projective plane π is finite (such a plane is called a finite projective plane). Then the following statements are shown to be equivalent:

(1) One line contains exactly (n+1) points.

(2) One point is on exactly (n+1) lines.

(3) Every line contains exactly (n+1) points.

(4) Every point is on exactly (n+1) lines.

(5) There are exactly (n^2+n+1) points in $\mathcal{P}$.

(6) There are exactly (n^2+n+1) lines in $\mathcal{L}$.

These equivalences will be used repeatedly. The <u>order</u> of a finite projective plane π will be said to be n if some line of π contains exactly (n+1) points.

Although the lines of π were defined to be distinguished subsets of the points of π, it is evident from the axioms of incidence that an equivalent characterization of the plane can be obtained by considering the lines to be the primitive elements and defining the points of π to be distinguished subsets of the lines of π; thus a point could be considered to be the set of all lines passing through it. Consequently if the triple $\pi = (\mathcal{P}, \mathcal{L}, \varepsilon)$ is a projective plane, the triple $\pi^* = (\mathcal{L}, \mathcal{P}, \varepsilon^*)$ is also a projective plane where the binary relation ε is defined by

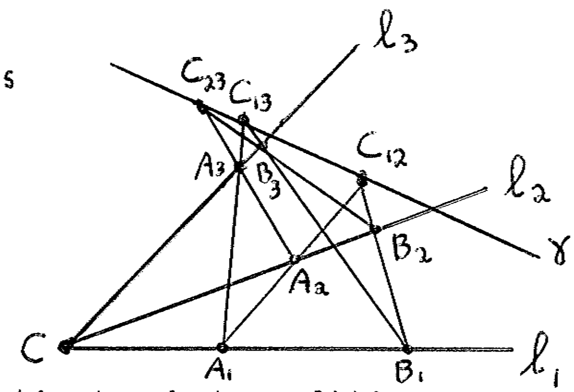
$$\ell \, \varepsilon^* P \Longleftrightarrow P \varepsilon \ell \quad \text{for all } P \varepsilon \mathcal{P},\ \ell \varepsilon \mathcal{L}.$$

That π^* is indeed a projective plane can be verified by noting that π^* satisfies the axioms of incidence for a projective plane. π^* is called the projective plane dual to π. It is evident that $(\pi^*)^* = \pi$. Thus every statement about a projective plane π can be "dualized" to a statement about π^* by interchanging the roles of points and lines and replacing ε by ε^*. It follows that if an assertion A is true of a projective plane π, the "dual" assertion A^* will be true of π^*. More generally, if all projective planes satisfying hypotheses H have property K, then all projective planes satisfying the dual hypotheses H^* will have the dual property K^*. This "principle of duality" will be used repeatedly throughout the thesis.

If π is a projective plane and ℓ is a line of π, then by the affine plane π_ℓ we shall mean the projective plane π with the line ℓ and the points thereof deleted. The line ℓ will be called the "line at infinity". Points not on ℓ and lines distinct from ℓ will be called affine points and lines. The concept of the affine plane will be used chiefly to facilitate notation and to aid in the co-ordinatization of the projective plane (see chapter II).

Let C be a point of π and γ a line of π. Let ℓ_1, ℓ_2, ℓ_3 be three arbitrary distinct lines through C (and $\neq \gamma$) and let A_i and B_i be two distinct points of $\ell_i - \{C\}$ $(i = 1,2,3)$. If, for all such A_i, B_i, and ℓ_i, $(A_1A_3 \cap B_1B_3)\varepsilon\gamma$ and $(A_1A_2 \cap B_1B_2)\varepsilon\gamma$ together imply that $(A_2A_3 \cap B_2B_3)\varepsilon\gamma$, then we shall say that Desargues'

(C,γ) theorem holds. If Desargues (C,γ) theorem holds for all points C and lines γ of π, then π will be said to be Desarguesian. The fundamental problem of this thesis will be to investigate what conditions determine the number of point-line pairs (C,γ) for which Desargues′ (C,γ) theorem holds in a given projective plane.



The theory of groups, and in particular the theory of permutation groups, is used extensively throughout the thesis. A self-contained development of the theory of permutation groups appears in chapter IV, and several abstract group theoretical results are proved there as well. However, it is assumed that the reader is familiar with elementary abstract group theory, and with the standard notation employed in that subject. The results used can be found, for instance, in Hall (5).

Lemmas and theorems are numbered independently. Thus for example there is both a lemma 4.4 and a theorem 4.4, and these are distinct.

CHAPTER I

ELEMENTARY PROPERTIES OF COLLINEATIONS

In this chapter several elementary lemmas and theorems about collineations will be proved. Continual reference to these will be made throughout the rest of the paper.

Lemma 1.1 (i) The product of two collineations is a collineation.

(ii) The inverse of a collineation is a collineation.

Proof: (i) Let π be a projective plane with a point set $\mathcal{P}$ and a line set $\mathcal{L}$. Let σ and τ be two collineations of π.

Define a mapping $\sigma\tau$ as follows:

$$P \varepsilon \mathcal{P} \Longrightarrow P^{(\sigma\tau)} = (P^{\sigma})^{\tau}$$
$$\ell \varepsilon \mathcal{L} \Longrightarrow \ell^{(\sigma\tau)} = (\ell^{\sigma})^{\tau}.$$

As $\mathcal{P} \xrightarrow{\sigma} \mathcal{P}$, $\mathcal{L} \xrightarrow{\sigma} \mathcal{L}$ are one-to-one onto mappings and as τ is similarly one-to-one onto, $\sigma\tau$ is a one-to-one onto mapping of $\mathcal{P} \longrightarrow \mathcal{P}$ and $\mathcal{L} \longrightarrow \mathcal{L}$.

To demonstrate that the mapping $\sigma\tau$ preserves incidence, suppose that for $P \varepsilon \mathcal{P}$ and $\ell \varepsilon \mathcal{L}$, $P \varepsilon \ell$. Then $P^{\sigma} \varepsilon \ell^{\sigma}$ (as σ is a collineation), and similarly $(P^{\sigma})^{\tau} \varepsilon (\ell^{\sigma})^{\tau}$ (as τ is a collineation). By definition of $\sigma\tau$, this implies that $P^{(\sigma\tau)} \varepsilon \ell^{(\sigma\tau)}$. Hence

$$P \varepsilon \ell \Longrightarrow P^{(\sigma\tau)} \varepsilon \ell^{(\sigma\tau)};$$

thus $\sigma\tau$ preserves incidence and by definition is a collineation.

(ii) Let σ be a collineation of π projective plane π.

Define a mapping $\mathcal{P} \xrightarrow{\sigma^{-1}} \mathcal{P}$ and $\mathcal{L} \xrightarrow{\sigma^{-1}} \mathcal{L}$ by

$$P^{\sigma^{-1}} = Q \iff Q^{\sigma} = P \quad (P, Q \varepsilon \mathcal{P})$$

$$\ell^{\sigma^{-1}} = m \iff m^{\sigma} = \ell \quad (\ell, m \varepsilon \mathcal{L}).$$

Then σ^{-1} is a one-to-one onto mapping, since σ is. In addition, σ^{-1} preserves incidence; for suppose that it did not. Then there exist $P \varepsilon \mathcal{P}$ and $\ell \varepsilon \mathcal{L}$ such that

$$P \varepsilon \ell \text{ but } P^{\sigma^{-1}} \not\varepsilon \ell^{\sigma^{-1}} .$$

But as σ is a collineation, it preserves non-incidence; hence

$$(P^{\sigma^{-1}})^{\sigma} \not\varepsilon (\ell^{\sigma^{-1}})^{\sigma} ;$$

$$P \not\varepsilon \ell \quad \text{(from the definition of } \sigma^{-1}).$$

This contradicts the assumption that $P \varepsilon \ell$, and thus σ^{-1} is an incidence-preserving mapping and hence a collineation. It is the inverse of σ since by definition of σ^{-1}, the mappings $\sigma\sigma^{-1}$ and $\sigma^{-1}\sigma$ fix π elementwise.

<u>Corollary</u>: The set of all collineations of a projective plane π forms a group.

<u>Proof</u>: This follows from the theorem and from the associativity of mappings.

<u>Definition</u>: The trivial collineation (also called the identity collineation) is the collineation that fixes every point and line of the plane.

<u>Lemma 1.2</u> Let π be a projective plane and σ a non-trivial collineation of π. Let there exist a line $\ell \epsilon \pi$ such that σ fixes every point on ℓ. Then there exists a point $A \epsilon \pi$ such that σ fixes every line through A.

<u>Proof</u>: Pick an arbitrary point $P \epsilon \pi$ such that $P \not\epsilon \ell$, and consider the point P^σ. Then $P^\sigma \not\epsilon \ell$; for otherwise $(P^\sigma)^\sigma = P^\sigma$, and application of σ^{-1} gives $P^\sigma = P$, which implies that $P \epsilon \ell$, contrary to hypothesis. There are now two cases:

(i) $P^\sigma \neq P$. Then $PP^\sigma \cap \ell$ is a well-defined point which we will denote as Q.

Now $\equiv P, P^\sigma, Q$;

thus $(PQ)^\sigma = P^\sigma Q^\sigma = P^\sigma Q$ (as $Q \epsilon \ell$)

$= PQ$

and thus the line PQ is fixed by σ.

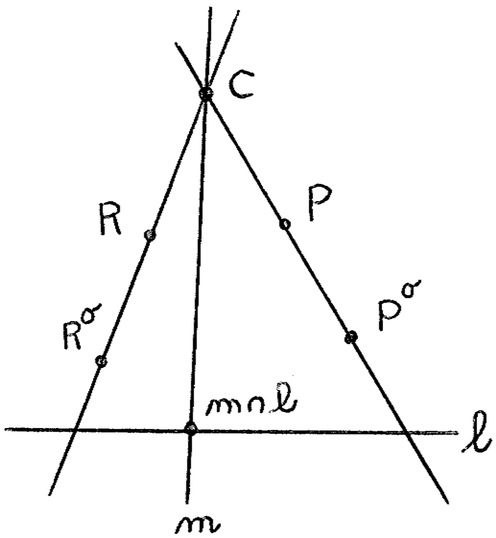
Pick an arbitrary point R, $R \not\epsilon \ell$, $R \not\epsilon PQ$, and consider the line RR^σ (assuming that $R \neq R^\sigma$). It too is fixed by σ, by the above argument; thus the point

$RR^\sigma \cap PP^\sigma$ is also fixed by σ.

Let $RR^\sigma \cap PP^\sigma = C$.

(ii) $P^\sigma = P$ (or $R^\sigma = R$). In this case substitute P (or R) in place of C in the following argument.

Again there are two cases; either $C \notin \ell$ or $C \in \ell$.



Case (i)

(i) Suppose $C \notin \ell$. Let m be an arbitrary line through C. Then $m \cap \ell \neq C$, and we have

$$m = (m \cap \ell)C.$$

Thus

$$m^\sigma = [(m \cap \ell)C]^\sigma = (m \cap \ell)^\sigma C^\sigma = (m \cap \ell)C = m.$$

Thus all lines through C are fixed by σ, and C is the desired point A.

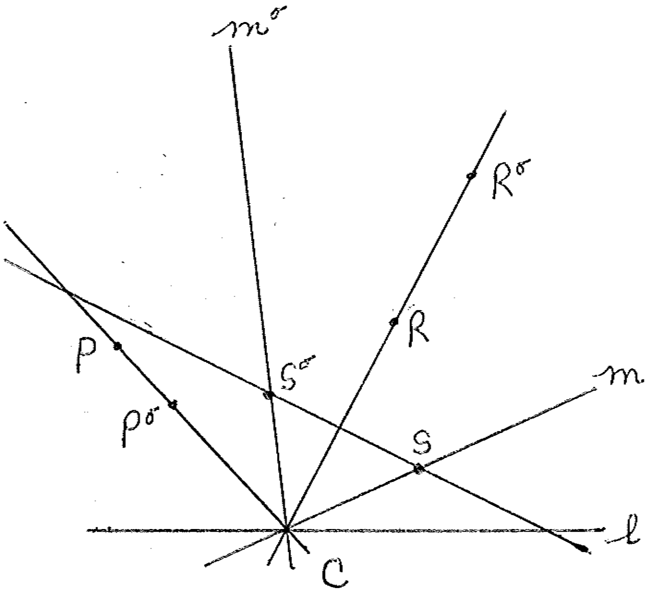
(ii) Suppose $C \in \ell$. Let m be an arbitrary line through C and let S be a point on m ($S \neq C$). Then $S^\sigma \in m^\sigma$, and by the argument used above, SS^σ is fixed by σ. Hence if $S^\sigma \in m$, we have

$$m = CS$$

$$m^\sigma = C^\sigma S^\sigma = CS^\sigma = m \qquad (\text{as } \ell^\sigma = \ell, \text{ and if } m \neq \ell,\ S \notin \ell \text{ implies } S^\sigma \notin \ell)$$

and m is fixed by σ. If $S^\sigma \notin m$, then $SS^\sigma \neq m$ and $SS^\sigma \notin m^\sigma$. Thus

$$SS^\sigma \cap PP^\sigma \neq SS^\sigma \cap RR^\sigma \quad (\text{see diagram})$$



and so $SS^\sigma \cap PP^\sigma$ and $SS^\sigma \cap RR^\sigma$ are distinct points $\notin \ell$ and fixed by σ. By the argument of case (i), it follows that all lines through each point are fixed, and thus σ fixes all points and lines of π. This contradicts the assumption that σ is non-trivial; hence $S^\sigma \notin m$ is impossible and all lines through C are fixed.

The dual of this theorem is also true:

<u>Corollary</u>: If σ is a collineation of π and P a point of π such that σ fixes all lines through P, then there exists a line ℓ of π such that σ fixes all points on ℓ.

<u>Definition</u>: A collineation σ that fixes all points on the line ℓ and all lines through the point C is called a (C,ℓ)-collineation, or a central collineation. ℓ is called the axis of the collineation, and C is called its centre.

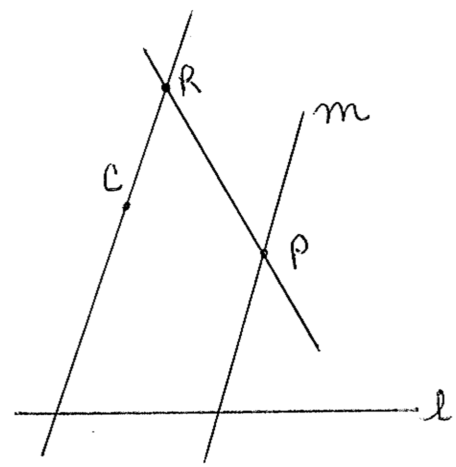
If $C \in \ell$, then σ is called a (C,ℓ)-elation; if $C \notin \ell$, then σ is called a (C,ℓ)-homology.

<u>Lemma 1.3</u> A (C,ℓ)-collineation σ that fixes a point P, $P \neq C$ and $P \notin \ell$, is the identity collineation.

<u>Proof</u>: Let m be an arbitrary line through P. Then m is of the form PQ, where $Q = m \cap \ell$ and hence $Q \neq P$. Thus

$$m^\sigma = (PQ)^\sigma = P^\sigma Q^\sigma = PQ$$

since P and Q are both fixed points of σ. The two distinct points P and C then have the property that a line through either of them is fixed. Let R be a point not on CP. Then $R = PR \cap CR$ and $R^\sigma = (PR \cap CR)^\sigma = PR \cap CR = R$. Thus all points of the plane not on CP are fixed by σ. A similar argument in which P is replaced by $\bar{P}$, where $\bar{P} \notin CP$, $\bar{P} \in \ell$, shows that all points on CP are fixed by σ. Hence σ fixes all points of the plane, and as it preserves incidence, it fixes all lines of the plane.



Hence $$\sigma = 1.$$

By the principle of duality, we have the

<u>Corollary</u> A (C, ℓ)-collineation σ that fixes a line m, $m \neq \ell$ and $C \notin m$, is the identity collineation.

<u>Lemma 1.4</u> A (C, ℓ)-collineation σ is determined by the image under σ of a point P (or, by the principle of duality, a line m) if $P \neq C$ and $P \notin \ell$ (dually, $m \neq \ell$ and $C \notin m$).

<u>Proof</u>: Let σ_1 and σ_2 be two (C, ℓ)-collineations. By lemma 1, $\sigma_1 \sigma_2^{-1}$ is also a (C, ℓ)-collineation.

Let P be a point such that $P^{\sigma_1} = P^{\sigma_2}$ ($P \neq C$, $P \notin \ell$).
Then

$$P^{\sigma_1 \sigma_2^{-1}} = P,$$

and by the previous lemma, $\sigma_1 \sigma_2^{-1} = 1$.
As inverses are unique, this means that $\sigma_1 = \sigma_2$, which proves the lemma.

<u>Lemma 1.5</u> Let σ_1 be a (C_1, ℓ)-elation and σ_2 a (C_2, ℓ)-elation. Then either one of the following occurs:

(i) $C_1 = C_2$ and $\sigma_1 \sigma_2$ is a (C_1, ℓ)-elation.

(ii) $C_1 \neq C_2$ and $\sigma_1 \sigma_2$ is a (C_3, ℓ)-elation for some point C_3 with $C_1 \neq C_3 \neq C_2$.

<u>Proof</u>: (i) If P is a point and m a line such that $P \varepsilon \ell$ and $C \varepsilon m$, then

$$P^{\sigma_1} = P = P^{\sigma_2}, \quad m^{\sigma_1} = m = m^{\sigma_2} ;$$

hence $P^{\sigma_1 \sigma_2} = P$ and $m^{\sigma_1 \sigma_2} = m$. Thus $\sigma_1 \sigma_2$ is a (C_1, ℓ)-elation.

(ii) As both σ_1 and σ_2 fix each point on ℓ, $\sigma_1 \sigma_2$ does. Hence by lemma 2, there exists a point C_3 such that $\sigma_1 \sigma_2$ fixes all lines through C_3. In order to prove that $C_3 \varepsilon \ell$, it suffices to show that $\sigma_1 \sigma_2$ fixes no point of $\pi - \{\ell\}$. If $P \notin \ell$ and $P^{\sigma_1 \sigma_2} = P$, then $P^{\sigma_1} = P^{\sigma_2^{-1}}$. But σ_2^{-1} is evidently a (C_2, ℓ)-collineation, so $P^{\sigma_1} \varepsilon C_1 P$ and $P^{\sigma_1} \varepsilon C_2 P$. Thus as $\not\equiv C_1, C_2, P$, it follows that $P^{\sigma_1} = P^{\sigma_2^{-1}} = P$; thus both σ_1 and σ_2 are trivial (i.e. are the identity collineation) by lemma 1.3, contrary to hypothesis. Hence $\sigma_1 \sigma_2$ fixes only points on , and is thus an elation.

If $C_3 = C_1$, then $\sigma_2 = \sigma_1^{-1}(\sigma_1\sigma_2)$ is a (C_1, ℓ)-elation (by case (i)), contradicting the hypothesis that $C_1 \neq C_2$. Thus $C_1 \neq C_3 \neq C_2$.

Corollary I: The set of all (C_1, ℓ)-elations forms a group, provided that the identity collineation is counted as a (C, ℓ)-elation for all point-line pairs (C, ℓ).

Corollary II: The set of all elations with a given axis ℓ forms a group, provided that the identity collineation is considered to be such an elation.

Definition: A projective plane π is said to be (C, ℓ)-transitive if, for arbitrary points $P, Q \notin \ell$ such that $\equiv P, Q, C$, and $P \neq C \neq Q$ there is a (C, ℓ)-collineation σ such that $P^\sigma = Q$.

Lemma 1.6 Let π be a projective plane that is (C, γ)-transitive, and let σ be a (C, γ)-collineation and φ an arbitrary collineation. Then $\varphi^{-1}\sigma\varphi$ is a $(C^\varphi, \gamma^\varphi)$-collineation and π is $(C^\varphi, \gamma^\varphi)$-transitive.

Proof: Let $P\varepsilon\gamma^\varphi$; then $P^{\varphi^{-1}}\varepsilon\gamma$, and $P^{\varphi^{-1}\sigma}\varepsilon\gamma^\sigma = \gamma$ (as σ is a (C,γ)-collineation). Hence $P^{\varphi^{-1}\sigma\varphi}\varepsilon\gamma^\varphi$.
But as $P^{\varphi^{-1}}\varepsilon\gamma$, $P^{\varphi^{-1}\sigma} = P^{\varphi^{-1}}$ as σ fixes points on γ.
Hence $P^{\varphi^{-1}\sigma\varphi} = P^{\varphi^{-1}\varphi} = P$, i.e. $\varphi^{-1}\sigma\varphi$ fixes points on γ^φ.
The dual argument gives that $\varphi^{-1}\sigma\varphi$ fixes all lines through C^φ, and hence $\varphi^{-1}\sigma\varphi$ is a $(C^\varphi,\gamma^\varphi)$-collineation.

Pick distinct points A and B in the plane, arbitrary except that $\equiv A,B,C^{\varphi}$, $A \neq C^{\varphi}$, $B \neq C^{\varphi}$, and $A,B \notin \;^{\varphi}$. As φ^{-1} is a collineation, this means that $\equiv A^{\varphi^{-1}}, B^{\varphi^{-1}}, C$. As the plane is (C, γ)-transitive, there exists a (C, γ)-collineation σ such that $(A^{\varphi^{-1}})^{\sigma} = B^{\varphi^{-1}} \therefore A^{\varphi^{-1}\sigma\varphi} = B^{\varphi^{-1}\varphi} = B$. But $\varphi^{-1}\sigma\varphi$ is a $(C^{\varphi}, \gamma^{\varphi})$-collineation; hence, as A and B were arbitrary, the plane is $(C^{\varphi}, \gamma^{\varphi})$-transitive.

__Definition__: (a) A projective plane π is said to be a translation plane with respect to the line ℓ if π is (C, ℓ)-transitive for all points C on ℓ.

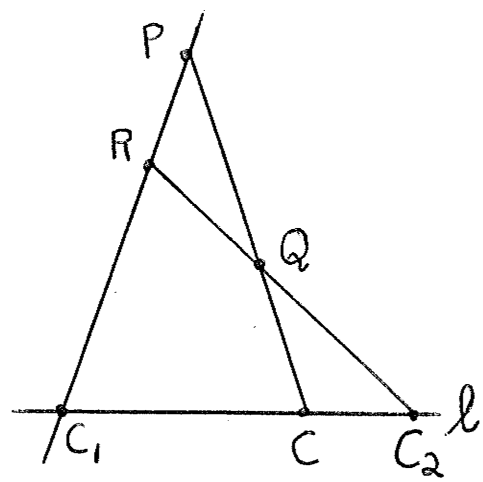
(b) The projective plane π is said to be the dual of a translation plane with respect to the point P if π is (P, ℓ)-transitive for all lines through P.

(c) If π is a translation plane with respect to a line ℓ, then the group of all elations with axis ℓ is called the translation group of π.

__Lemma 1.7__ Let π be a projective plane containing a line ℓ and distinct points C_1 and C_2 on ℓ. If π is (C_1, ℓ)- and (C_2, ℓ)-transitive, then it is a translation plane with respect to ℓ.

__Proof__: Let C be an arbitrary point on ℓ, $C_1 \neq C \neq C_2$. Let P and Q be arbitrary points of π such that $\equiv P,Q,C$ and $P,Q \notin \ell$. As $C_1 \neq C_2$, the point $C_1P \cap C_2Q$ is well-defined; denote it by R.

As π is (C_1, ℓ)-transitive and as $\equiv P,R,C_1$, there exists a (C_1, ℓ)-elation σ_1 such that $P^{\sigma_1} = R$. Similarly there exists a (C_2, ℓ)-elation σ_2 such that $R^{\sigma_2} = Q$. Hence $P^{\sigma_1\sigma_2} = Q$. But by lemma 1.5, $\sigma_1\sigma_2$ is a (C, ℓ)-elation, and evidently it maps P onto R. Hence as C was arbitrary in π, and as P and Q were arbitrary points satisfying $\equiv P,Q,C$, it follows that π is (C, ℓ)-transitive for all $C \varepsilon \ell$. Hence π is a translation plane with respect to ℓ.



Lemma 1.8 Let π be a translation plane with respect to the line ℓ. If α is a collineation then π is a translation plane with respect to ℓ^{α}.

Proof: Let C be an arbitrary point of ℓ^{α}. Then there exists a point $\bar{C} \varepsilon \ell$ such that $\bar{C}^{\alpha} = C$. As π is a translation plane with respect to ℓ, π is $(\bar{C}, \ell)$-transitive. Then by lemma 1.6, π is $(\bar{C}^{\alpha}, \ell^{\alpha})$-transitive, i.e. (C, ℓ^{α})-transitive. As C was arbitrary on ℓ^{α}, π is a translation plane with respect to ℓ^{α}.

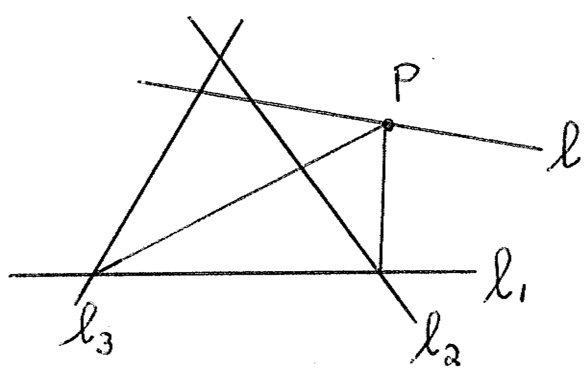
Corollary: If π is a translation plane with respect to two lines ℓ_1 and ℓ_2 intersecting at a point P, then it is a translation plane with respect to every line of the plane that passes through P.

__Proof:__ Let m be an arbitrary line through P ($\ell_1 \neq m$). As π is a translation plane with respect to ℓ_1, there is a (C, ℓ_1)-elation σ ($C \neq P$) such that $\ell_2^{\sigma} = m$. By lemma 1.8, since π is a translation plane with respect to ℓ_2, it is also a translation plane with respect to m. As m was arbitrary, π is a translation plane with respect to all lines through P.

__Definition:__ An alternative plane π is a projective plane that is a translation plane with respect to every line of the plane.

__Lemma 1.9__ If π is a translation plane with respect to three non-concurrent lines ℓ_1, ℓ_2, ℓ_3, then it is an alternative plane.

__Proof:__ Let ℓ be an arbitrary line of π. If ℓ passes through any of $\ell_1 \cap \ell_2$, $\ell_1 \cap \ell_3$, $\ell_2 \cap \ell_3$, then by the corollary of lemma 1.8, π is a translation plane with respect to ℓ . If ℓ passes through none of these, choose an arbitrary point $P \in \ell$, and without loss of generality, assume $P \notin \ell_1$.



Then as above, π is a translation plane with respect to the distinct lines $(\ell_1 \cap \ell_2)P$ and

$(\ell_1 \cap \ell_3)P$. Thus it is a translation plane with respect to all lines through P, and in particular ℓ. As ℓ was arbitrary, π must be an alternative plane.

<u>Theorem 1.1</u> Let π be a projective plane and let ℓ be a line of π. Let there exist non-trivial elations σ_1 and σ_2 with axis ℓ and with centres C_1 and C_2, $C_1 \neq C_2$. Then $G(\ell)$, the group of all elations with axis ℓ, is either infinite abelian or elementary abelian.

<u>Proof</u>: We first prove that $G(\ell)$ is abelian. By lemma 1.4, it suffices to show that for an arbitrary point $A \notin \ell$, $A^{\sigma_1\sigma_2} = A^{\sigma_2\sigma_1}$.

Since σ_2 fixes all lines through C_2,

$$(A^{\sigma_1}C_2)^{\sigma_2} = A^{\sigma_1}C_2$$

But $(A^{\sigma_1}C_2)^{\sigma_2} = A^{\sigma_1\sigma_2}C_2$

and so $\equiv A^{\sigma_1}, A^{\sigma_1\sigma_2}, C_2$.

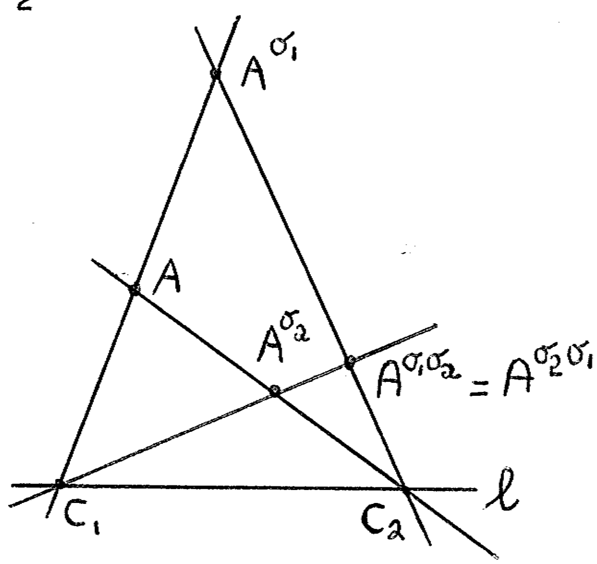
However, $\equiv A^{\sigma_1}, A, C_1$

and hence $\equiv A^{\sigma_1\sigma_2}, A^{\sigma_2}, C_1$.

Thus as $A^{\sigma_2}C_1 \neq A^{\sigma_1}C_2$ (as $C_1 \neq C_2$), it follows that

$$A^{\sigma_1\sigma_2} = C_1A^{\sigma_2} \cap C_2A^{\sigma_1}$$

Analogously $A^{\sigma_2\sigma_1} = C_2A^{\sigma_1} \cap C_1A^{\sigma_2}$
$= A^{\sigma_1\sigma_2}$.



Thus $\sigma_1\sigma_2 = \sigma_2\sigma_1$. Evidently by the same argument, any two elations with distinct centres and the same axis ℓ will commute. Further, two distinct elations of

$G(\mathcal{L})$ with the same centre will commute; for let σ_1 and σ_1^* be $(C_1, \mathcal{L})$-elations. Then by lemma 5, $\sigma_1^*\sigma_2$ and $\sigma_1\sigma_2$ have centres $\neq C_1, C_2$; thus by the above

$$\begin{aligned}\sigma_2(\sigma_1\sigma_1^*) &= (\sigma_1\sigma_1^*)\sigma_2 = \sigma_1(\sigma_1^*\sigma_2)\\ &= (\sigma_1^*\sigma_2)\sigma_1 = (\sigma_2\sigma_1^*)\sigma_1\\ &= \sigma_2(\sigma_1^*\sigma_1).\end{aligned}$$

Multiplying on the left by σ_2^{-1},

$$\sigma_1\sigma_1^* = \sigma_1^*\sigma_1.$$

Thus $G(\mathcal{L})$ is abelian as claimed.

If $G(\mathcal{L})$ has an element of finite order, then it has an element σ_1 with centre C_1 of prime order p. If σ_2 is an arbitrary non-trivial element of $G(\mathcal{L})$ with centre $C_2 \neq C_1$, then

$$\begin{aligned}(\sigma_1\sigma_2)^p &= \sigma_1^p\sigma_2^p \qquad \text{(as } G(\mathcal{L}) \text{ is abelian)}\\ &= \sigma_2^p.\end{aligned}$$

As any power of an elation has the same centre as the elation, $(\sigma_1\sigma_2)^p$ has centre C_2, as σ_2^p has. This contradicts lemma 1.5 unless $(\sigma_1\sigma_2)^p = 1$, i.e. unless

$$\sigma_2^p = 1.$$

Thus all elations of $G(\mathcal{L})$ with centre $\neq C_1$, are of order p. By extension of the above argument with σ_2 playing the role of σ_1, all elations with centre C_1 are of order p as well. Hence $G(\mathcal{L})$ is an elementary abelian group.

By the principle of duality we obtain the following

<u>Corollary</u>: Let π be a projective plane and P a point of π. Let there exist non-trivial elations σ_1 and σ_2 with centre P and axes ℓ_1 and ℓ_2, $\ell_1 \neq \ell_2$. Then G(P), the group of all elations with centre P, is either infinite abelian or elementary abelian.

<u>Definition</u>: An involution is a (C, ℓ)-collineation σ such that $\sigma \neq 1$ but $\sigma^2 = 1$. If σ is an elation (homology) of order 2, it is said to be an involutory elation (homology).

<u>Theorem 1.2</u> Let σ be an involution of a projective plane π of order n. Then if n is even, σ is an elation; if n is odd, σ is a homology.

<u>Proof</u>: Let σ have axis ℓ and centre C, and let m be any line $\neq \ell$ such that $C \varepsilon m$. Then σ interchanges points of $m - \{C \cup (m \cap \ell)\}$ in pairs; thus $m - \{C \cup (m \cap \ell)\}$, considered as a point set, has an even number of points. If σ is an elation, then $C = m \cap \ell$ and $m - \{C \cup (m \cap \ell)\}$ contains n points; hence n is even. If σ is a homology, then $C \neq m \cap \ell$ and $m - \{C \cup (m \cap \ell)\}$ contains n-1 points. Thus n-1 is even, i.e. n is odd.

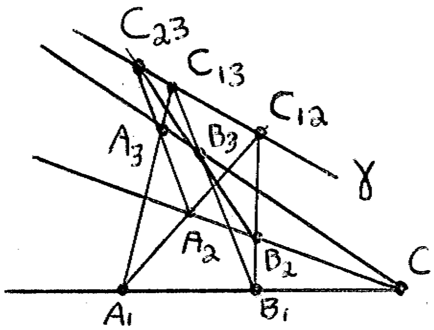
<u>Theorem 1.3</u> Desargues' (C, γ) theorem holds in a projective plane if and only if the plane is (C, γ)-transitive.

<u>Proof</u>: First suppose that for a particular line γ and point C the plane is (C, γ)-transitive. Let

C, A_1, A_2, A_3, B_1, B_2, B_3 be seven distinct points such that $\equiv C, A_1, B_1$, $\equiv C, A_2, B_2$, and $\equiv C, A_3, B_3$. Suppose that $C_{12} = A_1A_2 \cap B_1B_2 \varepsilon \gamma$ and that $C_{13} = A_1A_3 \cap B_1B_3 \varepsilon \gamma$. It must be shown that $C_{23} = A_2A_3 \cap B_2B_3$ also lies on γ .

From the above conditions it follows that γ does not pass through A_i or B_i $(i = 1,2,3)$, so as there exists (C,γ)-transitivity, there exists a (C,γ)-collineation σ such that $A_1{}^{\sigma} = B_1$.

Thus
$$A_2{}^{\sigma} = (C_{12}A_1 \cap CA_2)^{\sigma}$$
$$= C_{12}{}^{\sigma}A_1{}^{\sigma} \cap CA_2$$
$$= C_{12}B_1 \cap CA_2 = B_2$$



A similar argument shows that $A_3{}^{\sigma} = B_3$.

Thus

$$A_2A_3 \cap \gamma = (A_2A_3 \cap \gamma)^{\sigma} = A_2{}^{\sigma}A_3{}^{\sigma} \cap \gamma = B_2B_3 \cap \gamma .$$

Hence $A_2A_3 \cap B_2B_3 \varepsilon \gamma$, and Desargues' (C,γ) theorem holds.

Conversely, assume that Desargues' (C,γ) theorem holds for a particular point-line pair C and γ. Let A_1 and B_1 be any pair of distinct points $\neq C$ and not on γ such that $\equiv A_1, B_1, C$. Construct a mapping σ, defined on the affine plane (obtained by considering CA_1B_1 to be the line at infinity) as follows:

$$A_1{}^{\sigma} = B_1$$
$$C^{\sigma} = C$$