A DISCUSSION
OF THE FUNDAMENTAL THEOREM
OF GAME THEORY

BY

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ABSTRACT

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Many proofs of the fundamental theorem of game theory are known, with varying restrictions on the payoff function and strategy sets. In this work, a survey of various types of proof is presented so that a reader without background in game theory may discover the relation of the fundamental theorem to other mathematical and social disciplines.

The great advantage of symmetrization is demonstrated in proving the fundamental theorem. Algorithms are presented, both for solution on digital and analogue computer. The latter is shown to simplify drastically if symmetrization is employed.

Finally ensues a brief discussion of the practical adequacy of matrix games in solving real life situations.

THE AUTHOR WISHES TO EXPRESS SINCERE
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TABLE OF CONTENTS

CHAPTER		PAGE
	INTRODUCTION..............................	1
I	GENERAL THEORY OF GAMES..................	4
	1.1 INTRODUCTION......................	4
	1.2.1 The GENERAL N-PERSON GAME.........	4
	1.2.2.GAMES IN EXTENSIVE FORM...........	6
	1.2.3 GAMES IN NORMAL FORM...............	7
	1.2.4 SOLUTION OF A GAME................	8
	1.3.1 THE TWO-PERSON ZERO-SUM GAME......	9
	1.3.2 THE SYMMETRIC GAME................	10
	1.4 CONCLUSION........................	10
II	THE TWO-PERSON ZERO-SUM GAME.............	11
	2.1 DEFINITION AND TERMINOLOGY........	11
	2.2 DEFINITION........................	11
	2.3 THEOREM...........................	12
	2.4 THEOREM...........................	13
	2.5 DEFINITION........................	13
	2.6 DEFINITIONS AND THEOREM...........	14
	2.7 DEFINITION AND THEOREMS-MIN,MAX,&c	15
	2.8 THEOREM...........................	18
	2.9 DEFINITIONS-SYMMETRIZATION........	19
	2.10 THEOREM..........................	20
	2.11 CONCLUSION.......................	21
III	MATRIX GAMES............................	22
	3.1 DEFINITION-MATRIX GAME............	22
	3.2 DEFINITION-FUNDAMENTAL THEOREM....	22
	3.3 DEFINITION-SYMMETRIZATION.........	23
	3.4.1 THEOREM-SEPARATING HYPERPLANE.....	24
	3.4.2 LEMMA-SYMMETRIC GAMES.............	26
	3.4.3 THEOREM-SYMMETRIC GAMES...........	27
	3.4.4 TEMMA-SYMMETRIC GAMES.............	30
	3.5 FUNDAMENTAL THEOREM-ALGEBRAIC.....	30

CHAPTER		PAGE
III		
	3.6.1 THEOREM OF THE ALTERNATIVE........	30
	3.6.2 ANOTHER ALGEBRAIC PROOF...........	31
	3.7 GEOMETRIC CONSIDERATIONS..........	32
IV	ALGORITHMIC PROOFS OF MAIN THEOREM........	35
	4.1 DEFINITION-ALGORITHMIC PROOF......	35
	4.2.1 LINEAR PROGRAMMING................	35
	4.2.2 TYPES OF PROGRAM..................	36
	4.2.3 FUNDAMENTAL DUALITY THEOREM.......	37
	4.2.4 MATRIX GAME PROGRAM...............	40
	4.3. SIMPLEX PROOF....................	42
	4.4.1 LEMMA.............................	48
	4.4.2 ANALYTIC PROOF-SYMMETRIC GAME.....	49
	4.4.3 ANALYTIC PROOF-ARBITRARY MATRIX...	51
V	LESS RESTRICTED PROOFS....................	54
	5.1 THEOREM...........................	54
	5.2 DEFINITIONS-TOPOLOGY &c...........	57
	5.2.1 TRANSLATION OF KNESER............	58
	5.3 INDUCTION ON AN AFFINE FAMILY.....	62
	5.3.1 FUNDAMENTAL THEOREM...............	66
	5.4 THEOREM-PECK and DULMAGE..........	66
	5.5 TOPOLOGICAL PROOF.................	72
	5.6 CONCLUDING COMMENTS...............	74
VI	CLOSING COMMENTS..........................	75
	6.1 FUNCTION OF GAME THEORY...........	75
	6.2 CONJECTURE-MATRIX GAMES...........	75
	NOTATION..................................	77
	BIBLIOGRAPHY..............................	78

INTRODUCTION

The "game of skill" as opposed to the "game of chance", and the concept of a "strategy" in playing such a game, were described by E. Borel (5) as early as 1921. Although he later recognized the importance of the "symmetric game" (6) he was unable to prove the "fundamental theorem", conjecturing the impossibility of such theorem. Therefore, in spite of many published claims for his priority in the theory of games, he evidently did not establish its foundations. All the same, because his concept of a "mixed strategy" was very clear, he did point the way to applying parts of classical gambling theory to games of skill.

In 1928, J. von Neumann, a giant of modern mathematics, wrote his classic paper in which for the first time the "fundamental theorem" was proved under quite general assumptions. (A complete translation of this classic is found in Contributions to the theory of Games, Vol. 4, Ann. Math. Stud. 24, Princeton, 1950) At that time, von Neumann had not seen Borel's papers, so it may be rightly said that he laid the foundations of the modern theory of games.

For some years the new discipline did not attract a wide following. Von Neumann had written in German, and his style made difficult reading; he had not yet acquired his international reputation; his original proof was both topological and non-elementary; and World War II had not been fought. This last is not at all incongruous, since the vast logistic problems encountered during the War greatly stimulated the study of games and gamelike situations.

Twenty-five years later the same author, writing in English, joined forces with the renowned Economist, Oskar Morgenstern, to produce the magnum opus THEORY OF GAMES AND ECONOMIC BEHAVIOUR (39). Because both authors were so famous, the book was avidly studied, and a place for game theory in the modern world was assured.

Economist and Mathematician have been working busily ever since to flesh out the skeleton of the subject. In the process, game theory has been applied to all types of real life situation, often with inadequate justification, as ably discussed by Rapoport (43). Such misuse occurred because some individuals got the impression that the mathematical theory of games of conflict must necessarily apply to every type of conflict situation in real life. As will be seen later, the application of game theory is actually rather restricted by the requirement that rewards or payoffs be intercomparable. Because of this, some mathematicians view game theory as shallow. A perusal of some papers from the Bibliography (8,13-16,22 for example) should dispel this notion, since however pedestrian the game theory itself might be, it has led to interesting results in topology and the theory of convex cones.

Motivation for this thesis came during the author's exposure to a course in linear programming. Cogitation on the subject led to the feeling that in practical cases the finite theory will suffice, even if the strategy sets seem unbounded. Continuation of this trend of thought led quite naturally to a summary of available proofs of the fundamental theorem and comparison of their similarities and differences. As a result, the original impression was strengthened. Although many workers in game theory will not agree with it on grounds of lack of generality, the concept should nevertheless prove useful to workers in other fields, Economics for example, where game theory is seen rather as a tool to be used than as an end in itself.

CHAPTER I

GENERAL THEORY OF GAMES

1.1 INTRODUCTION General N-person games are qualitatively described. Zero-sum games are defined and the treatment of arbitrary games reduced to that of the zero-sum game. Extensive and intensive or normal forms of a game are defined and contrasted. The meaning of a solution is discussed. Finite and infinite two-person games are defined and justification given for concentrating on the zero-sum form of the two-person game. Symmetric games are defined and symmetrization discussed.

1.2.1 THE GENERAL N-PERSON GAME Many kinds of conflict situation found in real life may be reduced to the following mathematical abstraction: an integral number N of players compete against each other in an attempt to maximize utility . The utility might be money, commodities, power, pleasure &c. but in any case must be measurable in some objective way to permit intercomparison of utilities among the various players. If chance or nature has an effect on the outcome of the situation, it is counted as one of the players. Certain rules govern the outworking of the conflict situation. At each stage of the process there are certain courses of action available to each player. An exhaustive prescription of how the player shall act at each such stage, based on all that has gone before during the conflict, is a strategy. While it might difficult to predict it, there is no difficulty in extending this idea to a strategy for nature or chance. In essence, by conducting himself in a certain manner all through the conflict situation, a player chooses one strategy. Thus in all N strategies are chosen, one for each player, including nature. All these are effective in governing the outcome of the conflict situation. So the <u>payoff</u> to each player or his increase in utility as a result of the conflict is a function of N variables- the players' strategies. An essential feature of the above is that each player has only one decision to make during the conflict situation- a manner of play, or a policy to guide him. The details are fore-ordained by his choice of strategy.

Mathematical abstraction is straightforward. A <u>game</u> is a set of rules governing a conflict situation. A <u>play</u> is one particular outworking of the conflict in accordance with the rules. Each player's <u>move</u> is his occasion to choose a strategy for the game. Each player's <u>choice</u> is the strategy actually chosen during a play. N <u>payoff functions</u> of the N strategy choices exist, one for each player. After each play, the N function values corresponding to the choices measure the players' gains of utility as <u>payoffs</u>. If after each play the initial stock of utility remains within the circle of players, the game is <u>zero-sum</u>, that is, the algebraic sum of the payoffs is zero. If not, and an outside agency upsets the internal stock of utility, the game is non-zero sum.

Since this is very common in real life, it is useful to exhibit that the two forms of game are not essentially different. Into the N-person non-zero sum game is introduced another player, the dump, having one strategy and whose payoff is always exactly sufficient to make the game zero-sum. Since this converts the original N-person game to an N + 1 person zero-sum game, all games may be treated provided a valid general theory is available for the N-person zero-sum game.

1.2.2 GAMES IN EXTENSIVE FORM Following the detailed structure of the players' strategies through to the end of the game, considering the information available at each stage and the choices available there, yields a game in extensive form. Such games have not been very avidly pursued in the past, because the complexities mount up very rapidly, and render even a fairly straightforward game like checkers, inaccessible to the fastest computers. A player has either perfect information or incomplete information. Chess exemplifies the former, in which each player is completely informed of the other players' choices up until making his own at each stage of the play. Bridge exemplifies the latter, where not only is the player ignorant of his opponent's resources, but also of half his own. (Bridge is properly a two-person game, in which each player is bifurcated, each half holding half the resources)

It seems intuitively obvious that a "best" way of playing a game of perfect information exists, that is, there is a best possible strategy for each player. Von Neumann and Morgenstern (op cit) have shown that this is true in the two-person zero-sum game. Unfortunately, most real conflict situations are not of this type, and in general one works with probabilities of other players' choices rather than with the certain knowledge of them.

If a player knows in advance which strategies will be used by the others, he turns this knowledge to advantage in making his own choice. If the choices are not proclaimed in advance, but rather recorded secretly, there is the possibility of espionage. Therefore the only way a player can be sure of concealing his choice from the other players is to conceal it from himself! A way of doing this is to specify an a priori probability distribution over the available strategies and leave the actual choice in a given play to a suitably constructed mechanism. Such a probability distribution is denoted a <u>mixed strategy</u> for the player. Although this prevents certainty of winning a particular play, it presents the possibility that over a sufficiently large number of plays, one mixed strategy may give rise to the best obtainable average winning.

1.2.3 <u>GAMES IN NORMAL FORM</u> Von Neumann and Morgenstern (op cit) showed that the entire cumbersome structure of the game in extensive form could be reduced to the following intensive or normal form:

Each player chooses a mixed strategy after which he receives the payoff corresponding to the choices made by all the players.

The general N-person game is complicated because any number of players from two to N - 1 may band together, pool their resources, and play to maximize their common utility. At the first glance, this seems merely to decrease the number of players, but because many coalitions are possible in general , and are not unique, the problem is much more difficult. This fact of coalitions makes it difficult to give a precise meaning to solution of an N-person game.

1.2.4 SOLUTION OF A GAME There are two rather different interpretations of solution of an N-person zero-sum game. That of von Neumann and Morgenstern (op cit) is based on the idea of imputation. An imputation is a way of dividing up the spoils among the members of a coalition in such a way that each player receives as much or more in coalition as he would have individually. The solution of the game is then the set of imputations. Because this concept does not yield a unique answer, it has not been very satisfying. A more attractive definition is given by Nash J.(36,37). An equilibrium point is an N-tuple of mixed strategies, one for each player, such that no player can improve his payoff by modifying his mixed strategy independently of the others. There is an element of uniqueness that the previous definition lacked, because there is no incentive for players to modify their play if they are in possession of equilibrium point strategies. One further advantage is that this definition is valid without any change in the case of the two-person zero-sum game.

1.3.1 THE TWO-PERSON ZERO-SUM GAME Since this is merely the general game already discussed with N replaced by two, a great deal of 1.2 carries over directly. It is best to note only that coalitions cease to have meaning for this game, and that each player receives all that the other loses. A very concise definition of this game will be given in mathematical terms at the beginning of Chapter II. Because it has been exhaustively investigated and is a beautiful example of a mathematical theory, the remainder of the thesis will be devoted to the fundamental theorem of the two-person zero-sum game.

There is nothing to prevent considering games with infinite numbers of pure strategies for each of the two players. Indeed, one could conceive of such a situation in the general case. Such games are known as *infinite*, whereas if only a finite number of pure strategies is available to each player the game is *finite*. The *extended game* in which the mixed strategies enter is evidently always infinite, since probability distributions may vary in infinitely many ways.

1.3.2 <u>THE SYMMETRIC GAME</u> A two-person game is <u>symmetric</u> if its payoff functions are skew-symmetric in the strategies. An actual game is of this nature if each player must prepare himself in advance to play either side. In chess, the privilege of playing the white pieces is governed by the choice of white or black from two pawns concealed behind the back of one of the players in his two hands. Hence roughly half the time, a player has to be prepared to play white, and the remainder he must be ready to play black. There is thus a certain symmetry in the situation.

Sometimes there is no such symmetry inherent in a game. In this case it is possible to make use of one of two <u>symmetrizations</u>. These both have as object, to reduce the problem of playing a zero-sum two-person game to that of playing a <u>symmetric</u> game. (A symmetric game, is as a consequence of its definition, zero-sum) One of these due to von Neumann (8) formalizes the game situation just described. The other, due to D.Gale,H.W.Kuhn and A.W.Tucker (20) is more elegant in yielding smaller composite strategy sets, but is less appealing to the intuition. Additionally it does not seem to generalize to the infinite game, whereas the former experiences no difficulty in so doing.

1.4 <u>CONCLUSION</u> In view of the preceding discussion, the pages following will review selected proofs of the fundamental theorem, emphasizing its great mathematical interest.

CHAPTER II

THE TWO-PERSON ZERO-SUM GAME

2.1.1 *DEFINITION* A *two-person zero-sum game* $G=(X,Y;f)$, referred to hereafter simply as *game*, consists of a set X, a set Y, and a real-valued function f defined on the Cartesian Product $X \times Y$.

2.1.2 *TERMINOLOGY* An element x of X is a *pure strategy* for the *maximizing player* P. An element y of Y is a pure strategy for the *minimizing player* p. f is the *payoff function*. $f(x,y)$ is the *payoff* for a *play* for which P makes the *choice* x, and p makes the choice y, paid *by* p *to* P. Since P's *gain* is p's *loss*, the sum of the payoffs is zero. If v exists such that P is sure of winning at least v, while p is sure of not losing more than v, then v is the *value* of the game. Pure strategies $\underline{x}$ and $\underline{y}$ achieving this, are *optimal*. The triple $(\underline{x},\underline{y};v)$ is a *solution* of G.

Note that this game is *fair*, in the usual sense, if one fixed first move is appended to G, namely the payment of v by P to p.

2.2 *DEFINITION* If X and Y are arbitrary sets, and f a real-valued function defined on their Cartesian Product, and

$$f(x,\underline{y}) \leq f(\underline{x},y)$$

for all x in X and all y in Y, then f has a *saddle point* at $(\underline{x},\underline{y})$ and $w=f(\underline{x},\underline{y})$ is the corresponding *saddle value*.

Existence of a saddle point in the payoff function is intimately associated with the state of information of the game. Von Neumann and Morgenstern (op cit) show that every game with perfect infommation has a saddle point.

2.3 <u>THEOREM</u> A game has a solution if and only if its payoff function has a saddle point.

<u>PROOF</u> IF- $f(x,\underline{y}) \leq f(\underline{x},y)$ for all (x,y) in $X \times Y$,

then $f(\underline{x},\underline{y}) \leq f(\underline{x},y)$ and $f(x,\underline{y}) \leq f(\underline{x},\underline{y})$

so $f(x,\underline{y}) \leq f(\underline{x},\underline{y}) \leq f(\underline{x},y)$ But $w = f(\underline{x},\underline{y})$.

Therefore $f(x,\underline{y}) \leq w \leq f(\underline{x},y)$ or verbally:

no strategy of P played against p's <u>saddle strategy</u> can win P more than w, whereas no strategy of p played against P's saddle strategy can cost p less than w. Hence by 2.1.2 the game has solution $(\underline{x},\underline{y};w)$.

ONLY IF- $(\underline{x},\underline{y};w)$ solves the game, therefore P wins at least w by playing $\underline{x}$, and p loses no more than w by playing $\underline{y}$. But P's gain when p plays strategy y against $\underline{x}$ is $f(\underline{x},y)$ and p's loss when P plays strategy x against $\underline{y}$ is $f(x,\underline{y})$.

Therefore $f(x,\underline{y}) \leq f(\underline{x},y)$

because $f(x,\underline{y}) \leq w \leq f(\underline{x},y)$ so that by 2.2

f has a saddle point. Q.E.D.

It is very easy to construct a game with no solution in the above sense. Let $X=Y=(1,2)$, $f(1,1)=f(2,2)=0$ and $f(1,2)=f(2,1)=v\neq 0$. For definiteness suppose $0 < v$. Then if P plays first, p may hold his loss to zero, whereas if p plays first, P can gain exactly v. So no solution exists in the above sense. Meanwhile, the question presents itself, if a game has more than one solution, does it also have more than one value?

2.4 <u>THEOREM</u> The value of a game is unique.

<u>PROOF</u> Suppose two solutions of G are $(\underline{x},\underline{y};\underline{v})$ and $(x',y';v')$.
Thus by 2.3 $(\underline{x},\underline{y})$ and (x',y') are saddle points.
Thus by 2.2 $f(x,\underline{y}) \leq f(\underline{x},y)$ and $f(x,y') \leq f(x',y)$
for all x in X and y in Y.

In particular,
$$f(x',\underline{y}) \leq f(\underline{x},\underline{y}) = \underline{v}$$
$$f(\underline{x},\underline{y}) \leq f(\underline{x},y')$$
and
$$f(\underline{x},y') \leq f(x',y') = v'$$
$$f(x',y') \leq f(x',\underline{y})$$

Combining, $f(x',\underline{y}) \leq f(\underline{x},\underline{y}) = \underline{v} \leq f(\underline{x},y') \leq f(x',y') = v' \leq f(x',\underline{y})$.
But the extremes of this string of inequalities are identical so that the equality must be taken all the way through. In particular $\underline{v} = v'$ so the value of the game is unique. Q.E.D.

2.5 <u>DEFINITION</u> A <u>mixed strategy</u> x^* for P is a real-valued probability distribution defined over x in X such that
$$0 \leq x^*(x) \quad \text{and} \quad \sum_{x \text{ in } X} x^*(x) = 1.$$
A mixed strategy y^* for p is defined similarly, the sum being taken over y in Y.

The number $x^*(x)$ is the a priori probability that P will choose pure strategy x.

2.6.1 <u>DEFINITION</u> Related to $G=(X,Y;f)$ is the <u>extended two-person zero-sum game</u> $G^*=(X^*,Y^*;f^*)$ referred to hereafter simply as the <u>extended game</u>.

X^* and Y^* are the sets of all mixed strategies x^* and y^* respectively, and are called the <u>extended strategy sets</u> or the <u>mixed strategy sets</u>. Since the pure strategy x' is obtained by setting $x^*(x')= 0$ and $x^*(x)= 0$ for $x\neq x'$, all pure strategies are contained in the mixed strategy sets, and X^* and Y^* are commonly referred to simply as the <u>strategy sets</u>. The <u>extended payoff function</u> f^* often referred to simply as the <u>payoff function</u> is the <u>mathematical expectation</u> of P from p. That is, $f^*(x^*,y^*)$ represents the long term expected gain to P when he plays x^* and p plays y^*. Thus straightforwardly:

$$f^*(x^*,y^*) = \sum_{x \text{ in } X} \sum_{y \text{ in } Y} x^*(x)y^*(y)f(x,y)$$ and

$x^*(x)=0$ (resp. $y^*(y)=0$) for all but a finite number of $x\in X$ (resp. $y\in Y$).

2.6.2 <u>DEFINITION</u> G has a <u>solution in mixed strategies</u>, hereafter referred to simply as <u>solution</u>, if and only if G^* has a solution.

2.6.3 <u>THEOREM</u> $(\underline{x}^*,\underline{y}^*;v)$ is a solution of G if and only if $f^*(x,\underline{y}^*) \leq f^*(\underline{x}^*,y)$ for all pure strategies x and y.

<u>PROOF</u> ONLY IF- G has a solution, so G^* has a solution by 2.6.2 implying that G^* has a saddle point from 2.3

Hence, $f^*(x^*,\underline{y}^*) \leq f^*(\underline{x}^*,\underline{y}^*) \leq f^*(\underline{x}^*,y^*)$ by 2.2 for all x^* in X^* and y^* in Y^*. But as noted in 2.6.1 these include all the pure strategies, proving the theorem.

IF– $\quad f^*(x,\underline{y}^*) \leq f^*(\underline{x}^*,y) \qquad x$ in $X \qquad y$ in Y

Then $\quad x^*(x)f^*(x,\underline{y}^*) \leq f^*(\underline{x}^*,y)x^*(x) \qquad$ " $\qquad$ "

Sum over $x \quad f^*(x^*,\underline{y}^*) \leq f^*(\underline{x}^*,y) \qquad x^*$in $X^* \qquad$ "

since $x^*(x)$ sum to unity.

Thus $\quad f^*(x^*,\underline{y}^*)y^*(y) \leq y^*(y)f^*(\underline{x}^*,y) \qquad$ " $\qquad$ "

Sum over $y \quad f^*(x^*,\underline{y}^*) \leq f^*(\underline{x}^*,y^*) \qquad$ " $\qquad y^*$in Y^*

Which exhibits a saddle point for G^*, whence a solution for G^*, whence a solution in mixed strategies for G. Q.E.D.

Hereafter, as in the literature, the word game will usually be taken to mean G^* unless otherwise specified.

2.7.1 DEFINITION If for fixed x in X, $f(x,y)$ always has a greatest lower bound with respect to y in Y, and for fixed y in Y, $f(x,y)$ always has a least upper bound with respect to x in X, then:

$Bf(y) = \text{lub } f(x,y) = \sup f(x,y)$ re x in X

$bf(x) = \text{glb } f(x,y) = \inf f(x,y)$ re y in Y

$bBf = \inf Bf(y) = \inf \sup f$

$Bbf = \sup bf(x) = \sup \inf f$

It is possible that these limits will actually be attained by function values of f. In that case, sup becomes max, inf becomes min, sup inf becomes maxmin and inf sup becomes minmax as follows:

$Mf(y) = \max f(x,y)$ re x in X

$mf(x) = \min f(x,y)$ re y in Y

$mMf = \text{minmax } f$

$Mmf = \text{maxmin } f$

There is no difficulty remembering which set goes with which operation, since both max and sup are characteristic of the maximizing player and logically take place over X. Hence it is not necessary to specify the strategy set when using the preceding abbreviations.

2.7.2 THEOREM If Bbf and bBf exist, then $Bbf \leq bBf$.

PROOF $f(x,y) \leq Bf(y)$

So $bf(x) \leq Bf(y)$ showing that the set of $bf(x)$ is bounded above and that of $Bf(y)$ bounded below.

Therefore $Bbf \leq Bf(y)$

And $Bbf \leq bBf$ Q.E.D.

COROLLARY If Mmf and mMf exist, then $Mmf \leq mMf$.

2.7.3 THEOREM $Bbf = bBf$ if and only if there exists a point $(\underline{x},\underline{y})$ in X x Y such that for any positive e there exists a positive δ so that

$f(x,y') \leq f(x',y) + e$ whenever $/x'-\underline{x}/ \leq \delta$ and $/y'-y/ \leq \delta$

PROOF ONLY IF- Assuming $Bbf = bBf$, then by 2.7.1 and the definition of a limit there exists a point $(\underline{x},\underline{y})$ in X x Y for which the following inequalities hold:

$$bBf \leqq Bf(y') \leqq bBf + \tfrac{1}{2}e$$

$$Bbf - \tfrac{1}{2}e \leqq bf(x') \leqq Bbf$$

whenever $/x'-\underline{x}/ \leqq \delta$ and $/y'-\underline{y}/ \leqq \delta$

But from 2.7.2 for all x and y:

$$f(x,y') \leqq Bf(y') \leqq bBf + \tfrac{1}{2}e \text{ (from above)}$$

$$Bbf - \tfrac{1}{2}e \leqq bf(x') \leqq f(x',y) \text{ (from above)}$$

Combining these inequalities:

$$-\tfrac{1}{2}e+f(x,y') \leqq Bf(y')-\tfrac{1}{2}e \leqq bBf=Bbf \leqq bf(x')+\tfrac{1}{2}e \leqq f(x',y)+\tfrac{1}{2}e$$

Whence $f(x,y') \leqq f(x',y) + e$ with the stated restrictions on x',y' with respect to $\underline{x}$ and $\underline{y}$. So this point $(\underline{x},\underline{y})$ is the one whose existence is required by the theorem.

IF- Assuming the existence of the point as in the enunciation of the theorem, and applying 2.7.1 yields:

$$f(x,y') \leqq f(x',y) + e$$

$$Bf(y') \leqq bf(x') + e$$

$$bBf \leqq Bbf + e$$

But $Bbf \leqq bBf$ by 2.7.2

Therefore $Bbf \leqq bBf \leqq Bbf + e$

But since e is purely arbitrary, being only required to be positive, therefore the only way this inequality can be always valid is if $Bbf = bBf$ Q.E.D.

<u>COROLLARY</u> If mMf and Mmf both exist, mMf=Mmf if and only if f has a saddlepoint.

<u>PROOF</u> The only difference from the previous theorem is that $e = 0$. But by 2.2 the enunciation of the theorem is precisely the condition for a saddle point in f. Q.E.D.

Since this could be rephrased as minmax = maxmin if and only if the game has a saddlepoint, the origin of the term Minimax Theorem referring to the fundamental theorem of game theory should be clear.

2.8 THEOREM If $G = (X,Y;f)$ has the solution $(\underline{x},\underline{y};v)$ and w is an arbitrary constant, then $G' = (X,Y;f+w)$ has the solution $(\underline{x},\underline{y};v+w)$.

PROOF Let $f+w = f'$. Then $f'(x,y) = f(x,y) + w$.
By 2.3 and the hypothesis, f has a saddlepoint:

$$f(x,\underline{y}) \leq v \leq f(\underline{x},y)$$

So $$f(x,\underline{y}) + w \leq v + w \leq f(\underline{x},y) + w$$

Or $$f'(x,\underline{y}) \leq v+w \leq f'(\underline{x},y)$$ which tells us that G' has the same saddle point as G and hence by 2.2 and 2.3 has the solution $(\underline{x},\underline{y};\ v+w)$ Q.E.D.

The solvability and the optimal strategies of a game are not affected by the addition of an arbitrary constant to the payoff function.

COROLLARY The value of a game depends continuously on the payoff function.

PROOF Let G and G' have f and f' such that:

$$/\ f(x,y) - f'(x,y)\ / \leq e, \text{ a positive number.}$$

Then $$f(x,y) - e \leq f(x,y) \leq f(x,y) + e \quad \text{for all } x,y.$$

By 2.8 $$v - e \leq v' \leq v + e$$

Or $$/v-v'/ \leq e$$ Q.E.D.

A small change in payoff function generates a small change in the payoff.

2.9.1 <u>DEFINITION</u> A game $G = (X,Y;f)$ is <u>symmetric</u> if $X = Y$ and f is <u>skew-symmetric</u>. IE $f(a,b) = -f(b,a)$.

Both players have equal resources and the skew-symmetry of f shows that identical strategy choices will result in a payoff of zero. So if G has a solution, its value is zero.

2.9.2 <u>DEFINITION</u> $-G-$ is the <u>symmetrization</u> of G if $-G-$ is symmetric, and G has a solution if and only if $-G-$ has a solution.

See 1.3.2 . Here only one of the symmetrizations will be defined, consideration of the other waiting until matrix games have been introduced.

2.9.3 <u>DEFINITION</u> <u>Von Neumann's symmetrization</u> of a game $G = (X,Y;f)$ is $(X\text{x}Y, X\text{x}Y; -f-)$ where

$$-f-(z,z') = f(x,y') - f(x',y)$$

Both players have pure strategies $z = (x,y)$ in the same strategy set $X \text{ x } Y$. The definition of $-f-$ makes it automatically skew-symmetric. Mixed strategies z^* in $X^*\text{x}Y^*$ may be introduced without difficulty. Since each player must be prepared in advance to play either side, his mixed strategy consists of choosing probability distributions $x^*(x)$ and $y^*(y)$. Since these choices are independent, the probability of choosing a particular pair of probability distributions in X and Y is the product of the individual probabilities. That is, $z^* = x^*\,y^*$. It remains to be seen whether the other conditions of 2.5 are satisfied.

Since $x^*(x)$ and $y^*(y)$ are never negative, neither is their product, so $0 \leq z^*(z)$ for all z, which is the first condition.

$$\text{But} \quad \sum_z z^*(z) = \sum_y \left(y^*(y) \sum_x x^*(x) \right)$$

$$= \sum_y y^*(y)$$

$$= 1, \text{ since } x^* \text{ and } y^* \text{ satisfy 2.5}$$

Therefore z^* satisfies 2.5 and a bona fide mixed strategy has been obtained for the symmetrization.

2.10 <u>THEOREM</u> G has a solution if and only if its von Neumann symmetrization has a solution.

<u>PROOF</u> ONLY IF- Let $(\underline{x}^*, \underline{y}^*; v)$ solve G. Then by 2.6.3

$$f(x,\underline{y}^*) \leq f(\underline{x}^*,y) \text{ for all pure strategies } x,y.$$

Thus $\quad -f-(\underline{z}^*,z) = f(\underline{x}^*,y) - f(x,\underline{y}^*)$ is never negative for any pure strategy z of the symmetrization. Taking any mixed strategy z^* of the symmetrization:

$$-f-(\underline{z}^*,z) = \sum_z -f-(\underline{z}^*,z)z^*(z) \quad \text{and} \quad 0 \leq z^*(z)$$

Therefore $\quad 0 \leq -f-(\underline{z}^*,z^*)$ for all mixed strategies of the symmetrization. But -f- is skew-symmetric, therefore:

$$-f-(z'^*,\underline{z}^*) \leq 0 \leq -f-(\underline{z}^*,z'^*) \quad \text{for all } z'^* \text{ and } z^*.$$

Hence -f- has a saddlepoint at $(\underline{z}^*,\underline{z}^*)$; by 2.3 the symmetrization has the solution $(\underline{z}^*,\underline{z}^*;0)$.

IF- the symmetrization has the solution mentioned, then -f- has a saddle point by 2.3 and setting $z'^* = z^*$

yields $\quad 0 \leq -f-(\underline{z}^*, z^*) \quad$ for all z^*.

So $\quad 0 \leq f(\underline{x}^*, y^*) - f(x^*, \underline{y}^*)$ for all x^* and y^*.

Or $\quad f(x^*, \underline{y}^*) \leq f(\underline{x}^*, y^*) \quad$ " " " "

This is the condition for a saddlepoint in G^* by 2.2 and by 2.6.2 G has a solution in mixed strategies. Q.E.D.

2.11 CONCLUSION These have been results needed for the thesis, and readily derivable for general strategy sets. In the next chapter will be treated a narrower, but more tractible subject- matrix games.

CHAPTER III

MATRIX GAMES

3.1 DEFINITION A matrix game is a two-person zero-sum game with finite strategy sets.

The name arises because the payoff function can be set up in matrix form. Suppose X contains m elements and Y contains n elements. Within these sets, the pure strategies may be put into one-to-one correspondance with the first m and n positive integers, respectively, so that $f(x,y) = f(i,j) = a_{ij}$ for $i = 1,m$ and $j = 1,n$. $A = (a_{ij})$ is a matrix whose general element is the payoff when P plays pure strategy i against p's pure strategy j. But $u_i A v^j = a_i v^j = a_{ij}$, so that the mixed strategies now appear in the form of m-vectors x for P and n-vectors y for p, with the additional requirement that $xu = yv = 1$. If this is a valid interpretation, then the payoff when the mixed strategies x and y are played should be exactly the mathematical expectation value. Let $x = (x_i)$, $y = (y^j)$, $i=1,m; j=1,n;$ $xu = yv = 1$. Then $xAy = x_i \; a_{ij} \; y^j$ which by 2.6.1 is exactly the payoff in the extended game G*.

3.2 DEFINITION The fundamental theorem or minimax theorem of game theory may be stated in two equivalent ways:

Every matrix game has a solution in mixed strategies.

In every extended matrix game is found a saddle point.

The proofs of this theorem fall into two general categories: existence proofs, and algorithmic proofs. The former type only proves the theorem. In accomplishing this the latter furnished the optimal vectors, the value of the game or both. Each type of proof could be further subdivided as to method: topological, analytic &c. A representative selection of these proofs will be given and some of the similarities and differences between them will be noted as well as a brief indication of the sort of situation each proof best suits. Von Neumann's first two proofs made use of difficult topology, and many years elapsed before Ville's "elementary" proof appeared. A proof is elementary if it involves only operations in an ordered field requiring addition, subtraction, multiplication, division and decision as to whether a number exceeds, equals, or falls short of zero; decisions in a finite set of numbers exhibited one at a time.

3.3 DEFINITION Gale, Kuhn and Tucker's symmetrization (20) of the game with matrix A is the game whose matrix is

$$S = \begin{pmatrix} 0 & A & -1 \\ -A^t & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

where the 0 and 1 in each instance refer to submatrices containing all zeros and all ones sufficient to fill out S to $(m+n+1) \times (m+n+1)$.

When any confusion might arise, this symmetrization will have matrix S_2 to distinguish it from the previous one which shall have matrix S_1. Of course, this accords with the chronological order of their appearance, so no difficulty will be experienced in keeping them sorted out.

This symmetrization has a similar interpretation to the previous one as is seen by the manner in which A and its transpose enter the matrix. At first it seems that the marginal row and column of zeros and ones are superfluous, but they are found to be necessary in order that the vectors' components satisfying the symmetrization be forced to add up to unity to meet the mixed strategy condition.

3.4.1 THEOREM OF THE SEPARATING HYPERPLANE For a given $m \times n$ matrix A and a given n-vector b, exactly one of the following vector equations has a solution:

$$(1) \qquad xA = b \quad 0 \leqq x$$

$$(2) \qquad 0 \leqq Ay \quad yb < 0$$

This theorem is presented as Theorem 2.6 in D. Gale (18) and is very powerful for working with linear inequalities, but the proof in Gale's book is an uninspiring dull exercise in induction. There is another method available, first proposed about a decade ago by Motzkin, and lately revived by Hoffman and Mc Andrew (25). A modification of this method will be used here to establish the theorem.

PROOF Suppose that (1) and (2) are both true. Then:

$0 \leq xAy = by = yb \leq 0$ which is a contradiction.

Hence (1) and (2) are mutually exclusive possibilities.

Now suppose that (2) is false, which means essentially that $Ay \leq 0$ and $yb \leq 0$. If this is not so because of the sign of y, use its negative. Now consider the function

$f(y) = yb + \sum_i \exp(-a_i y)$ which can be visualized as a manifold in an n+1 space where the other n dimensions represent the components of y. Evidently f(y) is a continuous surface in such a space. It therefore attains its bounds over the set of y for which $yb \leq 0$. But the term yb is bounded above by zero, and the summation is bounded below by zero, since all its terms are exponentials. Furthermore, exponentials go to infinity much faster than quadratic terms of the type found in yb. There is therefore no question that the two components of f(y) might cancel each other off for all values of y. Hence f(y) is bounded below, hence has a greatest lower bound, and by the argument above assumes this as a minimum. Suppose a vector for which such minimum is attained is y_0. Let the first partial derivative with respect to the j'th component of y be denoted by a superscript ' and a subscript j. Then, $f'_j = by'_j + \sum_i -a\, y'_j \exp(-a_i y)$

and $y'_j = v^j_j$ so $f'_j = bv_j + \sum_i -a_i v^j \exp(-a_i y)$

$$f'_j = b^j - \sum_i a_{ij} \exp(-a_i y)$$

and the necessary condition for the existence of a minimum of $f(y)$ for the vector y_o is that all the first partials vanish there, yielding the n equations:

$$b^j = x_i \; a_{ij} \;, \qquad i=1,m$$ in which $\exp(-a_i y)$ has been replaced by x_i. But $x_p \; a_{pq} = (xA)^q$ from which

$$b^j = (xA)^j$$

or $$b = xA$$ which is the first half of (1) satisfied. But since all the components of x are in the form of real exponentials, it follows that no such component can be negative. Hence $0 \leqq x$ which is the other half of (1). This proves the theorem. Q.E.D.

3.4.2 <u>LEMMA</u> If A is skew-symmetric then the set of inequalities: $0 \leqq xA$, $0 \leqq x$, $xu=1$ always has a solution.

<u>PROOF</u> Let $b > 0$. Then x must be semipositive, since $x = 0$ implies $xA = 0$. Similarly, $y = 0$ implies $yb = 0$, so y may be taken semipositive. Both x and y may be considered normalized (strategies); if not, divide through by xu and yv respectively. Now the alternatives become:

$$0 \leq b=xA, \; 0 \leqq x, \; xu=1 \quad \text{or} \quad 0 \leqq Ay, \; yb= -1.$$

But since $0 \leq b$, no component of y can be positive, so reverse the second alternative by letting $z = -y$.

Then $0 \leq xA$, $0 \leqq x$, $xu = 1$ or $Az \leqq 0$, $zb = 1$, $0 \leqq z$

$$0 \leq xA, \; 0 \leqq x, \; xu = 1 \quad \text{or} \quad zA^t \leqq 0, \; 0 \leq z, \; zb=1$$ since at least one component of z is positive. Also the transpose of A is its negative because of the skew-symmetry. Hence:

$0 \leqq xA,\ 0 \leqq x,\ xu = 1$ or $-zA \leqq 0,\ 0 \leqq z,\ zb=1$

" " " or 0 "zA " "

Taking $b=v$ it is seen that both these alternatives are subsumed under: $0 \leqq xA$, x semipositive, $xu = 1$ Q.E.D.

3.4.3 THEOREM The S_2 symmetrization of a game has a solution if and only if the game has a solution.

PROOF By 2.8 the solvability of a game is not disturbed by adding a constant to the payoff function. Therefore a large enough constant is added to every element of the matrix to assure that there are no negative elements left, in fact, that the smallest element is now positive. Hence the value of the game is also positive.

If $(\underline{x}, \underline{y}\,; w)$ solves the original game, then $(\underline{x}, \underline{y})$ is a saddle point in the extended game. Therefore by 2.3

$a_i \underline{y} \leqq w \leqq a^j \underline{x}$ for $i=1,m; j=1,n$; $xu=yv=1$. But a_i and a^j are positive vectors, and the strategies semipositive, therefore the left and right hand sides of the above inequality are both positive. Therefore:

$$0 \leqq wu - A\underline{y}$$

$$0 \leqq A\underline{x} - wv \qquad \text{and} \quad \underline{x}u = \underline{y}v = 1$$

Consider the $m+n+1$ vector $\underline{z} = (\underline{x}, \underline{y}, w)$ and form both its products with S_2: $\underline{z}\, S_2 = S_2^t z = -S_2 \underline{z}$ and

$$\underline{z}\, S_2 = (\ -\underline{y}A^t + wu,\ \underline{x}A - wv,\ -\underline{x}u + \underline{y}v\ )$$

$$= (\ wu - A\underline{y},\ \underline{x}\, A - wv,\ 0)$$

and from the preceding set of inequalities, every component of this vector is either positive or zero. It is therefore semipositive and for any vector z which is semipositive the product $\underline{z}\, S_2\, z$ is not negative. Hence $z'\, S_2\, \underline{z}$ is not positive and the following inequality holds

$$z'\, S_2\, \underline{z} \leqq \underline{z}\, S_2\, z \quad \text{for all semipositive vectors}$$

and in particular for any mixed strategies of the symmetrization. But the sum of the components of $\underline{z}$ is $2 + w$. Hence $\underline{z}'' = \dfrac{\underline{z}}{2 + w}$ is an optimal strategy for the symmetrization and its value is zero.

* * *

Assume now that the symmetrization has the solution $(\underline{z}'', \underline{z}''\ ;\ 0)$. Because of the way S_2 is partitioned, it is natural to write $\underline{z}'' = (\underline{x}^*, \underline{y}^*, w)$ where the first is an m-vector, the second an n-vector, and the third a scalar. From 2.6.3 applied to symmetric games follows:

$$\underline{z}''\, S_2\, z \quad \text{not negative for any pure strategy } z.$$

Letting the z be equal to each k'th unit vector in turn yields the condition $0 \leqq \underline{z}''\, S_2$ and the condition that the components of $\underline{z}''$ add up to unity yields

$$\underline{x}^*u + \underline{y}^*v + w = 1.$$

Expanding the first of these yields

$$0 \leqq -A^t\underline{y}^* + wu$$

$$0 \leqq \underline{x}^*A - wv \qquad \text{two vector inequalities}$$

and $$0 \leqq -\underline{x}^*u + \underline{y}^*v \qquad \text{a scalar inequality}$$

and writing out components:

$$0 \leqq w - a_i\underline{y}^*$$

$$0 \leqq a^j\underline{x}^* - w \qquad \text{for } i=1,m;\, j=1,n$$

which combines into $a_i\underline{y}^* \leqq w \leqq a^j\underline{x}^*$ for all i and j. This is the condition for $\dfrac{\underline{x}^*}{\underline{x}^*u}$ and $\dfrac{\underline{y}^*}{\underline{y}^*v}$ to be optimal mixed strategies for the game with matrix A, provided that neither $\underline{x}^*u$ nor $\underline{y}^*v$ is equal to zero. It is at this juncture that the extra marginal row and column in the symmetrization matrix become indispensable. Note that $\underline{x}^*u \leqq \underline{y}^*v$ and $\underline{x}^*$ and $\underline{y}^*$ are non-negative since $\underline{z}''$ being a strategy was semipositive. If, then, $\underline{y}^*v = 0$, $\underline{x}^*u = 0$ also which is only possible if $\underline{x}^* = 0$. But since $0 \leqq \underline{x}^* A - wv$ and w is not negative since it was a component of $\underline{z}''$, therefore $w = 0$. But this implies that $\underline{x}^*u + \underline{y}^*v + w = 0$, which is a contradiction, since this sum is the sum of the components of a mixed strategy $\underline{z}''$ for the symmetrization and by definition is equal to one. Therefore $\underline{y}^*v \not\leqq 0$.

Now suppose that $w = 0$. But $-A\underline{y}^* + wu$ has no negative components, therefore $A\underline{y}^*$ has no positive component. Thus $xA\underline{y}^* \leqq 0$ for every strategy x of the game with matrix A. But this means that the value of that game is also negative or zero, when in fact the value was made positive by adding a sufficiently large positive constant to all terms of the matrix. Here, then, is another contradiction, and it is impossible that $w = 0$.

Combining the two vector inequalities from the bottom of page 28, yields $w\underline{y}^*v \leqq \underline{x}^* A\, \underline{y}^* \leqq w\underline{x}^*u$ and since w is positive, $\underline{y}^*v \leqq \underline{x}^*u$, and since it is already known that the inequality holds in the opposite sense, then must be true that $\underline{x}^*u = \underline{y}^*v = r$, say. Consider now $\underline{x}^*/r$ and $\underline{y}^*/r$ whose components sum to unity. The previous inequality becomes $a_i\underline{y}^*/r \leq w/r \leq a^j\underline{x}^*/r$ which is precisely the condition that the game with matrix A have the solution $(\underline{x}^*/r \,,\, \underline{y}^*/r \,;\, w/r)$ Q.E.D.

3.4.4 LEMMA Every symmetric matrix game has a solution.

PROOF From 3.4.2 applied to the skew-symmetric matrix S: $0 \leq x\,S$, $0 \leq x$, $xu = 1$ always has a solution. The second pair of conditions guarantees that this solution is a mixed strategy, and the first condition is that for x to be optimal for S. Q.E.D.

3.5 ALGEBRAIC PROOF OF FUNDAMENTAL THEOREM Every matrix game has a solution in mixed strategies.

PROOF From 2.10 or 3.4.3 every matrix game is equivalent to a symmetric matrix game. From 3.4.4 every such game has a solution. Q.E.D.

This is an uncommonly compact proof, but it should not be forgotten that several pages of straight algebra were involved in it. Proof without symmetrization is not so neat, as will be shown now by way of comparison.

3.6.1 THEOREM OF THE ALTERNATIVE FOR MATRICES Straight from 3.4.1 come the pairs of alternatives:

EITHER	OR
$xA = b \qquad 0 \leqq x$	$0 \leqq Ay \qquad yb < 0$
$(x,z)\begin{pmatrix}A\\I_n\end{pmatrix} = b \qquad 0 \leqq x \qquad 0 \leqq z$	$\begin{pmatrix}A\\I_n\end{pmatrix}y$ non-negative $\quad yb < 0$
$xA \leqq b \qquad 0 \leqq x$	(Ay,y) " " "
" "	$0 \leqq Ay \qquad 0 \leqq y \qquad yb < 0$
$x(A,-u) \leqq (0,-1) \qquad 0 \leqq x$	$0 \leqq (A,-u)\begin{pmatrix}y\\k\end{pmatrix} \qquad (0,-1)\begin{pmatrix}y\\k\end{pmatrix} < 0$
	$0 \leqq y \qquad 0 < k$
$xA \leqq 0 \leqq x$ and $-xu \leqq -1$	$Ay-ku$ non-negative, also y $-k < 0$
$xA \leqq 0 \qquad 0 < x$	$0 < Ay \qquad y$ non-negative
$Ay \leqq 0 \qquad 0 < y$	$0 < xA \qquad x$ " "
$(A,I_n)\begin{pmatrix}y\\z\end{pmatrix} \leqq 0 \qquad 0<y \qquad 0<z$	$0 < x(A,I_n) \qquad 0 \leqq x$
$Ay + z \leqq 0$ " "	$0 < (xA,x)$ "
$Ay \leqq 0 \qquad 0 < y$	$0 < xA \qquad 0 < x.$

Q.E.D.

3.6.2 <u>ANOTHER ALGEBRAIC PROOF</u> Every matrix game has a solution in mixed strategies.

<u>PROOF</u> Let the game matrix be A and consider the set of payoffs xAy for which $Bf(y) = \sup xAy$ and $bf(x) = \inf xAy$ whence $\inf Bf(y) = bBf$ and $\sup bf(x) = Bbf$. From 2.7.2 $Bbf \leqq bBf$.

Assume that $Bbf < bBf$ and form a new matrix

$$A' = A - \frac{(bBf + Bbf)}{2}\, I_{mn}$$

for which is true

(1) $Bbf' = Bbf - \frac{1}{2}(Bbf + bBf) = \frac{1}{2}(Bbf - bBf) < 0$

(2) $bBf' = bBf - \frac{1}{2}(Bbf + bBf) = \frac{1}{2}(bBf - Bbf) > 0$

by 2.8 . Applying 3.6.1 to the matrix A' suppose the first alternative to be true where the vector involved is a strategy: $A'y' \leqq 0$, $0 \leq y'$, $y'v = 1$. Then for any strategy x' the payoff $x'A'y' \leq 0$ because any strategy is semipositive. From this $Bf'(y') \leq 0$ and $bBf' \leq 0$ in contradiction to (2).

If on the other hand the second alternative is taken, then $0 \leq x'A'$, $0 \leq x'$, $x'u = 1$, so that x' is a strategy. Then for any strategy y' the payoff satisfies $0 \leq x'A'y'$, from which $bf'(x')$ and Bbf' are positive, contradicting (1). Since neither alternative holds, the assumption that $Bbf < bBf$ must be wrong, and the alternative of equality must be valid.

Finally, for a matrix game there exist mixed strategies for which the bounds are attained so that

$$Mmf = mMf \qquad \text{Q.E.D.}$$

3.7 <u>GEOMETRIC CONSIDERATIONS</u> The outline of a geometric proof, essentially Ville's, as presented in D. Gale (18) will now be presented.

<u>PROOF</u> Let $A = (a^j)$ and consider the set of columns of A as a finite set of n points in m-space. Let -A- be the smallest convex set containing all the a^j, $j=1,n$.

That is, $-A- = (x ; x = Ay , yv = 1)$. Let $z = w u$, the m-vector all of whose components are w. Define a new convex set $-A_z- = (x ; x = Ay - z , yv = 1)$ which comes from $-A-$ by subtracting z from every vector in it. But since z is equally inclined to the axes in the m-space, the formation of this new set is geometrically the same as sliding the entire set $-A-$ along the equally inclined line towards the negative orthant. Then for large enough w the entire set $-A-$ may be slid into the negative orthant as all the points in it are finitely far from the origin. Conversely, for a sufficiently negative value of w, the entire set will be in the positive orthant. Therefore at some intermediate value of $w = w_o$, $-A-$ will contain exactly one point contiguous to the negative orthant, namely the origin. In algebraic language, there exists y^* such that $A\, y^* - w_o\, u \leqq 0$. On multiplying through by x, any strategy for P, it is found that

$$xAy^* - w_o xu \leqq 0$$

$$x\, A\, y^* \leqq w_o\, x\, u \quad \text{for all strategies } x.$$

This is precisely the statement that the minimising player may restrict his loss not to exceed w_o, since for a strategy x, it is true that $xu = 1$.

Since the intersection of $-A_z-$ and the negative orthant contains only one point, there is a hyperplane separating them. A vector x perpendicular to the hyperplane, pointing away from the negative orthant, will form an obtuse or right angle with any vector from the negative orthant and an acute or right angle with any vector from the positive orthant. Hence x is semi-positive. Therefore $x^* = x/xu$ is a strategy for P. Since $-A_{\bar{z_0}}$ lies all on the same side of the separating hyperplane as does x^*, the inner product of the latter and any element of the former must be positive or zero.

Algebraically $0 \leqq x^*(Ay - z_0)$

$$0 \leqq x^*Ay - x^*z_0$$

$$w_0 x^*u \leqq x^*Ay$$

$$w_0 \leqq x^*Ay \quad \text{for } y \text{ any mixed strategy for p.}$$

Combining this result with the previous

$$xAy^* \leqq w_0 \leqq x^*Ay \quad \text{for all mixed strategies}$$

of the original game. By 2.3 therefore the game has the solution $(x^*, y^* ; w_0)$ Q.E.D.

There is a great deal of intuitive appeal to this proof, as it is so easily by diagrams and models in two and three dimensions. Probably it is the best of all the proofs to present in a first course in Game Theory for practical people.

The next chapter will deal with algorithmic proofs of the fundamental theorem by linear programming and differential equations.

CHAPTER IV

ALGORITHMIC PROOFS OF THE FUNDAMENTAL THEOREM

4.1 DEFINITION An algorithmic proof of the fundamental theorem is any proof which shows the truth of the theorem by actually producing the optimal strategies, the value, or both. For most practical work, these are much more useful than the existence proofs. In the final chapter will be shown that practical games do have solutions, regardless of the apparent complexity of the strategy sets. Therefore the interest of game theory to a practical man lies more particularly in actually determining strategies or value. These might be vital for use in a decision procedure for example.

4.2.1 LINEAR PROGRAMMING is a method of solving certain maximum and minimum problems which do not yield to differential calculus. The word "linear" refers to the constraint imposed on the variables involved by requiring that they satisfy certain linear equations or inequalities. To any such problem in which we are especially interested, denoted the primal problem, corresponds another related problem, denoted the dual problem. A necessary and sufficient condition for either of these problems to have a solution is that the other have a solution. Here, then, is a similarity to game theory, where if P has an optimal strategy, so does p and the game has a value.

4.2.2 <u>TYPES OF PROGRAM</u> <u>Standard maximum:</u>

Given: m x n matrix A, determine a non-negative x, to

maximize xc , constrained by $xA \leqq b$.

<u>Dual:</u> determine a non-negative y, to

minimize yb , constrained by $c \leqq Ay$.

<u>Canonical maximum:</u>

Given: m x n matrix A, determine a non-negative x, to

maximize xc , constrained by $xA = b$.

<u>Dual:</u> determine a non-negative y, to

minimize yb , with y unconstrained.

<u>General maximum:</u>

Given: m x n matrix A, two sets $M = (x ; x = 1, m)$ and $N = (x ; x = 1 , n)$ of which S and T are subsets respectively, determine x such that:

xc is maximum

$0 \leqq x_i$ for $i \in S$

$xa^j \leqq b_j$ for $j \in T$

$xa^j = b_j$ for $j \in N-T$

<u>Dual:</u> determine y such that:

yb is minimum

$0 \leqq y_j$ for $j \in T$

$c_i \leqq a_i y$ for $i \in S$

$c_i = a_i y$ for $i \in M-S$

A few minutes work leads to the conclusion that the first two types are special cases of the third. In fact, as shown in standard works on the subject, the three types are readily convertible one to another.

If x^* and y^* solve the primal and dual standard maximum problems, then: $cx^* \leq x^*Ay^* \leq y^*b$, $0 \leq x^*$; $0 \leq y^*$. If it turns out that $x^*c = y^*b$, then:

$xAy^* \leq by^* = x^*c \leq x^*Ay$ for all non-negative x and y. This looks suspiciously like a saddlepoint situation and bears further investigation. The following theorem will be proved only for the standard maximum problem, since the types of programs are interchangeable anyway and from the above remark, the standard problem is the one which bids fair to be useful in discussing game theory.

4.2.3 <u>FUNDAMENTAL DUALITY THEOREM</u> If the primal and dual standard maximum problems are both <u>feasible</u> (ie there are vectors satisfying the constraints), then optimal vectors exist for both programs, and the value of the primal maximum is the same as that of the dual minimum. If one of the programs is infeasible, then neither program has an optimal vector.

<u>PROOF</u> Suppose both programs feasible, that is:

$xA \leq b$, $0 \leq x$ and $c \leq Ay$, $0 \leq y$ for some vectors x and y. Then:

(1) $xc \leq xAy \leq yb$ for all feasible x and y.

Since this automatically puts bounds on xc and yb, the maximum and minimum problems are both solved if it ever happens that $yb - xc \leq 0$, in which case (1) tells us that , say, $x^*c = y^*b$. Let x and y now refer to any other two feasible vectors. The following inequality results: $xc \leqq xAy^* \leqq y^*b = x^*c \leqq x^*Ay \leqq yb$ which shows that x^*c is never smaller than any other xc and y^*b never larger than another yb. That is, both primal and dual have been solved. Therefore $x^*c = y^*b$ is the optimality criterion for feasible vectors.

The truth of the duality theorem therefore depends on whether the following have a solution:

$xA \leqq b \qquad c \leqq Ay \qquad xc = yb \qquad$ or

$\text{"} \quad \text{"} \qquad -Ay \leqq -c \qquad yb - xc \leqq 0 \qquad$ (from (1) above)

$\text{"} \quad \text{"} \qquad -yA^t \leqq -c \qquad \text{"} \; \text{"} \; \text{"}$

$x'A' \leqq b' \qquad 0 \leqq x'$

where $x' = (x,y)$
$$A' = \begin{pmatrix} A & 0_m & -c \\ 0_n & -A^t & b \end{pmatrix} \qquad b' = (b, -c, 0)$$

$x'A' + z = b' \qquad 0 \leqq x' \qquad 0 \leqq z$

$x'' A'' = b' \qquad 0 \leqq x''$

where $x'' = (x', z)$
$$A'' = \begin{pmatrix} A' \\ v_{m+n+1} \end{pmatrix}$$

If this last has no solution, then by alternative (2) of 3.4.1 the following have a solution:

$0 \leqq A''y'' \qquad y''b' \leqq 0 \qquad$ or in terms of A'

$0 \leqq (A'y'', y'') \qquad \text{"} \qquad$ which tells this

$0 \leqq A'y'' \qquad 0 \leqq y'' \qquad yb - zc \leqq 0 \qquad$ where $y'' = (y, z, w)$.

$0 \leqq Ay - cw \qquad 0 \leqq -A^t z + wb \qquad 0 \leqq y, z \qquad yb - zc \leqq 0 \leqq w$

Two possible cases now present themselves, either $w = 0$ or w is positive. If it is zero the following inequality is valid: $zc \leqq zAy \leqq 0 \leqq xAy \leqq yb \leqq zc$ for any feasible vectors x and y. The result follows from the enunciation of the primal and dual problems and the preceding consequence of the theorem of the separating hyperplane. Taking the extreme members of this inequality $zc < zc$, a contradiction. Hence $w \not< 0$.

If so, then $c \leqq A\,y/w$, $z/w\ A \leqq b$, $yb < zc$ since w is positive. From the first two of these follows that z/w and y/w are feasible vectors for the primal and dual problems (non-negative, since y and z both are) But from the top of the preceding page this implies that $zc/w < yb/w$ whence $zc < yb$. But this results in the inequality $yb < zc \leqq yb$, another contradiction.

Conjecturing that the standard maximum problem may have feasible vectors without having a solution leads to contradiction, so the opposite must be true- if the primal and dual programs have feasible vectors, then they have optimal vectors and a common <u>value</u>.

* * *

Now suppose that the primal maximum problem has no feasible vector. Then by the discussion on page 38, the following have a solution: $0 \leqq Az \quad 0 \leqq z \quad zb < 0$.
If the dual also has no feasible vector, the problem is trivial. Suppose then that $c \leqq Ay$, $0 \leqq y$ have a solution. Take some positive number w : $c \leqq A(wz + y)$ which states that $wz + y$ is a feasible vector of the dual for all positive w. But $(wz + y)b = w(zb) + yb < yb$ because $zb < 0$. This says that no minimum exists for the dual

program, since by taking w sufficiently large it is possible to make $(wz + y)b$ as negative as desired.

If the dual is infeasible, then exactly the same procedure establishes:

$$0 \leqq Ay - c \qquad \text{has no non-negative solution}$$

$$-Ay + c \leqq 0 \qquad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"}$$

$$y(-A^t) \leqq -c \qquad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"}$$

Therefore $(-A^t)z \geqq 0$, $(-c)z < 0$, $0 \leqq z$ have a solution.

$$z\,A \leqq 0 \qquad 0 < zc \qquad 0 \leqq z \quad \text{"} \quad \text{"}$$

Again if the primal is infeasible, the problem is trivial. Assume therefore $x\,A \leqq b$, $0 \leqq x$ for some x. Taking a positive w as before: $(x + wz)\,A \leqq b$ expressing the feasibility of $x + wz$ for the primal problem. But $xc < (x + wz)c$ for all positive w since $0 < zc$. There is hence no optimal vector for the primal problem, and the theorem is complete. Q.E.D.

It is very easy to pursue this and get another existence proof for a matrix game with positive payoff function. By 2.8 the result will be valid for arbitrary matrix games.

4.2.4 <u>MATRIX GAME PROGRAM</u> Given: A a matrix of positive entries. $0 < a_{ij}$ $i=1,m;\ j=1,n$. Determine x and y to maximize yv and minimize xu subject to the constraints

$$A\,y \leqq u \qquad \text{and} \qquad x\,A \geqq v.$$

Let $\bar{a}$ and $\underline{a}$ be the maximum and minimum matrix elements respectively. Then ua^j = sum of m matrix elements of which the smallest is $\underline{a}$. Also $a_i v$ = sum of n matrix elements of which the largest is $\bar{a}$. That is $m\underline{a} \leqq ua^j$, $a_i v \leqq n\bar{a}$. Whence $Av \leqq n\bar{a}u$ and $m\underline{a}v \leqq uA$ which state that

$v/n\bar{a}$ and $u/m\underline{a}$ are feasible vectors for the problem. Hence by 4.2.3 there exist optimal vectors x^* and y^* such that $y^*v = x^*Ay^* = x^*u = w$. Further these are semipositive, because if either were actually zero it would not satisfy the constraints. Therefore x^*/w and y^*/w are strategies. Inserting in the constraint inequalities:

$$vy/w \leqq x^*Ay/w \quad \text{and} \quad xAy^*/w \leqq xu/w$$

for all strategies x and y. But therefore $xu = yv = 1$. Hence $x\,A\,y' \leqq 1/w \leqq x'\,A\,y$ where $y' = y^*/w$, $x' = x^*/w$ for all strategies x and y. But this shows that the game has the solution $(x', y'; 1/w)$. Q.E.D.

This illustrates that if the standard maximum linear program may be solved systematically, the optimal vectors and value of a matrix game may be determined with equal facility. A method for doing this exists, is known as the Simplex Method, and since its invention in 1949 has become almost classic. Basically it is Gauss-Jordan elimination in matrices, but with a decision procedure incorporated which tells which substitution to make and at what stage to do so. The solution now presented makes use of nothing more sophisticated than the idea of inverting a matrix.

4.3 SIMPLEX PROOF From 2.7.1 and 2.7.2 applied to matrix games follows: $mf(x) \leq Mmf \leq mMf \leq Mf(y)$ so that the truth of the fundamental theorem hinges on finding x^* and y^* such that $mf(x^*) = Mf(y^*)$ since in that case $Mmf = mMf$. But $a_i y \leq Mf(y)$ $i=1,m$ and $mf(x) \leq xa^j$ $j=1,n$ by definition of the bounds. They are also recognized as the constraints of a standard maximum primal problem and its dual. The condition $mf(x^*) = Mf(y^*)$ imposes simultaneous maximization and minimization in these problems. For game purposes the solution of this program must yield strategies, the condition for which is eg, $0 \leq x$, $xu=1$. So the following seven conditions formulate a standard maximum problem and its dual, the solution to which will provide the optimal strategies and the value of the game with payoff matrix A:

(1) $a_i y \leq Mf(y^*)$ $\quad i=1,m$

(2) $-xa^j \leq -mf(x^*)$ $\quad j=1,n$

(3) $mf(x^*) = Mf(y^*)$

(4) $0 \leq x$

(5) $0 \leq y$

(6) $xu = 1$

(7) $yv = 1$

The theorem in this form merely states that the above seven conditions may always be satisfied.

PROOF The augmented matrix corresponding to A is

$$A' = \begin{pmatrix} 0 & v & O_m \\ -u & A & I_m \end{pmatrix}$$

It is convenient for the purpose of this proof to arrange the row vectors in this (m+1) x (m+n+1) such that $a_{i1} \leq a_{m1}$ in A for i=1,m. Having done this, it is possible to write $A' = (a_i')$ i=0,m or $A' = (a'^0, a'^j, u'^i)$, i=1,m;j=1,n. A set of m+1 columns of A' determines a square matrix B. B is a basis if:

(a) $a'^0 = b^0$

(b) B is non-singular; hence $BB^{-1}=B^{-1}B=I_{m+1}$

(c) $b_i^- \succ 0$, i=1,m.

Condition (c) provides a way of ordering the rows of B^{-1} by comparison with the null vector and is necessary at a later stage in order to proceed straight to a solution without getting caught in a circular argument. It will now be proved that a basis exists. Consider the matrix:

$$B_0=(a'^0,a'^1,u'^1\ldots u'^{m-1})= \begin{pmatrix} 0 & 1 & 0_{m-1} \\ -u^{m-1} & a^* & I_{m-1} \\ -1 & a_{m1} & 0_{m-1} \end{pmatrix}$$

where u^{m-1} is the (m-1)x 1 vector of ones, a^* a (m-1)x 1 vector, 0_{m-1} a 1 x (m-1) zero vector and I_{m-1} the (m-1) x (m-1) unity matrix. In the latter form, inversion is easily performed by inspection:

$$B_0^{-1} = \begin{pmatrix} a_{m1} & 0_{m-1} & -1 \\ 1 & 0_{m-1} & 0 \\ a_{m1}u^{m-1}-a^* & I_{m-1} & -u^{m-1} \end{pmatrix}$$

Checking this, it is seen that (a) is automatically satisfied by the definition of B_0. Further (b) is satisfied since the inverse of the B_0 has actually been found. $b_1^- \succ 0$ since its first term is one, and the others because $a_{i1} \leqq a_{m1}$ and the I_{m-1} in the last m-1 rows of B_o^-. Therefore (c) is valid and it has been proved that a basis exists.

Let $B = (a'^o, a''^p)$ $p=1,m$; where all the $a''^p e (a'^j, u'^i)$ $i=1,m; j=1,n$ be a basis, of which was shown that at least one exists. Its inverse $B^{-1}=(b_i^-) = (b^{-j})$ $i=0,m; j=0,m$ is $(m+1) \times (m+1)$. Set $b_o^-=(m',-x)$ and $b^{-o}=(M',y')$. Then under certain conditions on the matrix B, x and the vector $y=(y',0_{n-m})$ are optimal mixed strategies for the game.

Because $BB^{-1}= I_{m+1}$, $b_o^- a'^o=1=(m',-x)(0,-u)= xu$ satisfying (6). But B is a basis, so from (c) follows that b^{-o} has no negative component except maybe the first. So $0 \leqq y'$ whence $0 \leqq y$. But $Bb^{-o}=(1,0_m)$ and equating the first components yields $yv = 1$, satisfying (5) and (7). Equating the other components: $-M' + a''_{ip}y'_p = 0$, $p=1,m$. The latter sum is composed of two parts according as $p \in (i)$ or (j). Therefore $-M' + a_{ij}y'_j + a''_{ip}y'_p = 0$. But the a''_{ip} in the second sum are either 1 or 0, and y' non-negative, therefore the second sum is not negative. Also the first sum represents $(Ay)_i$ since the last n-m components of y are zero. Thus $(Ay)_i - M' \leqq 0$ for all i, satisfying (1). Also $M'=m'$ from their definitions, whence (3) is also satisfied. This is as far as can be got in solving the program without adding additional restrictions on B to elevate it to the status of <u>optimal basis</u>, a basis for which (2) should also be satisfied, and (4).

From $B^{-1}B = I_{m+1}$, $b_o^- a''^p = m' - xa^j$ if $p \in (j)$ and $b_o^- a''^p = 0 - xu_i$ for $a''^p \in (u'^i)$. Satisfying (2) requires $b_o^-\ a'^j \leqq 0$ for all a'^j entering the basis B as a''^p and (4) requires $b_o^-\ u'^i \leqq 0$ for all u'^i entering B as a''^p. Since it would be rare to find such a fortunate turn of events first try, the question presents itself: is there a way of,changing B one step at a time, until a basis is reached for which the foregoing are true? To answer this, note that in the expansion of all other terms of $B^{-1}B$ are found $b_i^-\ a''^p = 0$ for all $i \neq p$, in particular $b_o^- a''^p = 0$ for all p. There are exactly m such so among the m+n scalar products $b_o^-\ a'^j$ and $b_o^-\ u'^i$ remain at most n non-zero numbers. If none of these exceeds zero, the basis is optimal and the problem is solved.

Suppose then that B is not optimal. Then $0 \angle b_o^- a''^p$ for at least one p. Take that for which it is largest and introduce into the basis B. Compute $w = B^{-1} a''^q$ where q satisfies the previous condition. $0 \angle w_o$ on this account. If w_o is the only positive component of w then $w_i \leqq 0$ $i=1,m$. But $a''^q = B\ w = w_o a'^o + w_i\ a''^i, i=1,m$. So $a'^o = 1/w_o\ (a''^q + (-w_i)\ a''^i)$ $i=1,m$. But $0 \leqq -w_i$ and a'^o has been expressed as a positive linear combination of columns of A' all of which are headed with 0 or 1. This implies that 0 is a positive linear combination of zeros and ones since 0 is the first component of a'^o and is absurd. Therefore w has at least one other positive component, from which is obtained the replacement criterion.

Consider $\dfrac{b_r^-}{w_r} \prec \dfrac{b_i^-}{w_i}$ for $0 \angle w_i$ and $i \neq 0$ and replace b^r. Note that $0 \prec b_i^- = w_i/w_r \; b_r^-$. Denote the new matrix with b^r replaced by a''^q as C. For the rule given to be effective, C must be a basis and have one more than before of nonpositive scalar products $c_o^- \; a''^p$.

Note that $a'^o = c^o$ since it was excluded from the replacement criterion. Therefore (a) is satisfied. To prove (b), it suffices to exhibit an inverse for C. Consider D defined by:

$$d_i = (1-l_{ir})(b_i^- - w_i/w_r \; b_r^-) + (l_{ir}/w_r) \; b_r^-$$

where l_{ir} is a Kronecker delta. Noting that:

$w_i = b_i^- a''^q$ and $w_r = b_r^- \; a''^q$ this becomes:

$$d_i = (1-l_{ir}) \left(b_i^- - \frac{b_i^- a''^q}{b_r^- a''^q} \; b_r^-\right) + \frac{l_{ir}}{b_r^- \; a''^q} \; b_r^-$$

Since $b_i^- a'^o = l_{io}$ for all i because $BB^{-1} = I$ and $r \neq 0$, the first column of DC is that of I_{m+1} as should be. And $b_r^- \; a''^p = l_{rp} \; b_r^- \; a''^q$, since for all $a''^p \neq a''^q$ the scalar products are members of I whereas for the vector entering the basis the scalar product is the corresponding member of the vector w. Therefore:

$$d_i a''^p = (1-l_{ir})\left(b_i^- a''^p - \frac{b_i^- a''^q}{b_r^- a''^q} \; b_r^- a''^p\right) + \frac{l_{ir} \; b_r^- a''^p}{b_r^- \; a''^q}$$

$$= (1-l_{ir})\left(l_{ip} - \frac{w_i}{w_r} \; l_{rp} \; w_r\right) + \frac{l_{ir} \; l_{rp} \; w_r}{w_r}$$

When $i \neq r$ and $a''^p \neq a''^q$, then $d_i a''^p = 1_{ip}$ as is right for a unit matrix and in the row corresponding to the replaced column of B, $d_r a''^q = 1_{rr} 1_{rq} = 1_{rq}$ so the element on the diagonal is again one and all the others zeros. This shows that d_i are the rows of the inverse of C, or $D = C^{-1}$, so (b) is satisfied.

For all d_i, $i \neq 0$, it follows from the top of page 46 and the fact that B was a basis, that d_i is lexicographically positive, satisfying (c) so that C is indeed a new basis.

Further, since $d_o = b_o^- - w_o/w_r \; b_r^-$ and w_o and w_r both positive, and b_r^- lexicographically positive, therefore $d_o \prec b_o^-$ and for any vector removed from B:

$$d_o \; a''^k \angle \; b_o^- \; a''^k = 0$$ since it is expressly forbidden to remove the first column. This means that a vector removed from a basis can never again qualify to enter a basis under the replacement criterion, and it is impossible to have cycling of bases.

Now $d_o a''^q = b_o^- a''^q - w_o/w_r \;\; b_r^- a''^q$; $b_o^- a''^q = w_o$ and $b_r^- a''^q = w_r$ by the replacement criterion. Therefore $d_o a''^q = w_o - w_o/w_r \;\; w_r = 0$, so one more scalar product satisfies the optimality criterion, so that C is an improvement over B basewise. But there are precisely ${}^{m+n}C_m$ possible bases and with the worst possible luck, the process would terminate after that many steps. Hence an optimal basis always exists and may be found. Q.E.D.

The fundamental theorem is thus established, and the value of the game and the optimal strategies appear almost as fringe benefits. The proof is somewhat long but uses very little sophisticated knowledge and is very straightforward to program in a digital computer, since available matrix inversion programs may be pieced into a scheme incorporating the replacement criterion. In the next section is described an algorithm equally suited to analogue computation.

4.4.1 LEMMA For a non-negative vector x in n-space:

$$xx \leq (xv)^2 \leq nxx$$

PROOF By induction on n. For one-vectors the equality holds throughout, hence there is a starting-point. Assume true for n=k; for n=(k+1) the three expressions become: $xx + x_{k+1}^2$

$$(xv)^2 + 2x_{k+1}\, xv + x_{k+1}^2$$

$$(k+1)(xx + x_{k+1}^2)$$

Forming differences from left to right:

$$xx + x_{k+1}^2 - (xv)^2 - 2\, x_{k+1}xv - x_{k+1}^2 \leq 0 \quad \text{by}$$

the inductive assumption and the non-negative property of the (k+1) vector.

$(xv)^2 + 2\, x_{k+1}xv + x_{k+1}^2 - kxx - xx - kx_{k+1}^2 - x_{k+1}^2 =$

$(xv)^2 - kxx - (xx - 2\, x_{k+1}xv + kx_{k+1}^2) =$

$(\, (xv)^2 - kxx\,) - (\, x - x_{k+1}v)^2$ since vv = k by the definition of the unit k-vector. The first of these expressions is not positive by the inductive assumption so the whole difference is also not positive. This established the inequality for (k+1) vectors also and the lemma is proved. Q.E.D.

4.4.2 ANALYTIC PROOF; SYMMETRIC GAME with n x n matrix $S=-S^t$. Consider n-vectors y and $z = Sy$. Let x be the n-vector with components $x_j = \max(0, z_j)$. Let t be a variable continuous with respect to time.

Consider the system of n differential equations:

$$D_t y = x - (xv) y$$

with initial conditions $0 \leqq y_0$, $0 \leqq x_0$, $y_0 v = 1$, $0 \leqq x_0 v$. Having started as a strategy, will y remain a strategy? Suppose that for some value of t, $y_k=0$. Then at that time and for the same component $D_t y_k = x_k$ which is never negative. Therefore $0 \leqq y$ for all t.

If $yv = 1$ at any value of t, then

$$vD_t y = vx - (xv)(vy) = xv(1-vy) = 0 = D_t(vy)$$

so yv remains constantly unity for all t. Hence if y starts out as a strategy, it remains a strategy.

For a meaningful solution to be obtained, it is necessary to show that x and xv both approach zero with increasing t fast enough to assure that the rate of change of y becomes nil. If at any value of t, $x=0$ the process has stopped, so it suffices to consider $x \neq 0$.

Note that $D_t z = S\ D_t y = Sx - (xv)(Sy) = Sx - (xv)z$

$$D_t z_k = s_k D_t y = s_k x - (xv) s_k y = s_k(x - xv\ y)$$

and $$v(x - xv\ y) = vx - xv\ vy = xv(1 - vy) = 0$$

and since every component of v is one, this means that

$$x - xv\ y \text{ is a null vector.}$$

So suppose that $0 \angle z_k$. Then $x_k = 0$ and remains so because the negative component of z cannot change by the above. Thus $D_t x_j = D_t z_j$ for positive z_j.

$$D_t(x_j^2) = 2\ x_j\ D_t x_j = 2\ x_j\ D_t z_j$$ because

when $z_j \angle 0$, $x_j = 0$ and nothing is contributed. So then

$$D_t(x_j^2) = 2\ x_j (Sx)_j - 2\ xv\ x_j (Sy)_j$$

and summing over the index $j=1,n$

$$D_t(xx) = 2\ xSx - 2\ xv\ xSy$$

$$= -2\ xv\ xSy$$ because $xSx = 0$ by

the skew-symmetry of S. But $Sy = z$, and $xx = xz$ since every time z has a negative component, x has a zero to counteract it. Therefore:

$$D_t(xx) = -2\ xv\ xx$$ whence

$$D_t(\ln xx) = -2\ xv$$ to approximate

the solution of which recourse is had to 4.4.1 in the form:

$$(xx)^{\frac{1}{2}} \leqq xv \leqq (nxx)^{\frac{1}{2}}$$

$$+2\ (xx)^{\frac{1}{2}}(xx) \leqq 2xx\ xv \leqq 2\ xx\ n^{\frac{1}{2}}(xx)^{\frac{1}{2}}$$

Changing sign, reversing inequalities, and replacing xv by $-\frac{1}{2} D_t(\ln xx)$:

$$1 \leqq -\tfrac{1}{2}\ (xx)^{-3/2}\ D_t(xx) \leqq n^{\frac{1}{2}}$$

of which the solution starting from $t = 0$ is:

$$\frac{x_o x_o}{n(1+t(x_o x_o)^{\frac{1}{2}})^2} \leqq xx \leqq \frac{x_o x_o}{(1+t(x_o x_o)^{\frac{1}{2}})^2}$$

showing that xx indeed approaches zero with increasing t. Noting 4.4.1 again, it is seen that xv also goes to zero. But x has no negative components, so the only way this can happen is for x to approach zero. In the original system of differential equations, then , the rate of change of y goes to zero with increasing t.

Because $0 \leqq y_j \leqq 1$, a compact range, limit points y_j^* exist for increasing t, such that $x^* = 0$ according to the last inequality. This implies that $z^* = S\, y^* \leqq 0$ since $z \leqq x$ always. Furthermore, y has always remained a strategy, so that y^* is a strategy. Playing any other strategy y for P against y^* for p and vice versa:

$$ySy^* = Sy^*\, y = y^*S^t y = -y^*Sy \leqq 0$$
$$ySy^* \leqq 0 \leqq y^*Sy$$

which is the condition for S to have solution $(y^*, y^*; 0)$. Since $yv=1$ at all times, the actual number of independent differential equations is $n-1$, and the process can always be started by letting $y_o = v_1$. Q.E.D.

4.4.3 <u>ANALYTIC PROOF; ARBITRARY MATRIX GAME</u> Since by the above proof, symmetric games are solvable by algorithm, arbitrary games may be handled through their von Neumann symmetrization. Of course, the other one could be used too, but this one yields a system with fewer differential equations.

If G has the m x n matrix A, and S_1 the mn x mn matrix S, then a pure strategy for P is the choice of (q,r) where q is a maximizing strategy and r is a minimising strategy in G. A pure strategy for p is, similarly, (s,t). If P plays q against t he wins a_{qt} and r against s loses him a_{sr} so that $s_{ij} = a_{qt} - a_{sr}$. By lemma 3.4.2 therefore, $0 \leqq zS$, $0 \leqq z$, $zw = 1$ for some z, where w is the vector of ones and dimension mn. Replace $zS = - Sz$ in the first of these by the skew-symmetry of S.

Expanding, $\sum_{s,t} (a_{qt}z_{st} - a_{sr}z_{st}) \leqq 0$ for all q and r.

Therefore $$\sum_t a_{qt} \sum_s z_{st} \leqq \sum_s a_{sr} \sum_t z_{st}$$

of form $$\sum_t a_{qt}\, y_t \leqq \sum_s a_{sr}\, x_s$$

or $$Ay \leqq xA \qquad \text{where} \quad xu = yv = zw = 1.$$

And since $0 \leqq z$, it follows that $0 \leqq x$ and $0 \leqq y$ which qualifies x and y as optimal strategies for G.

Now a system of mn differential equations:

$$D_t z = X - (XV)\, z$$

where $U = S\,z$, $X = (\ \max(0, U_{ij})\ )$. But because $U = (a_i y - xa^j)$ from $Ay - xA = S\,z$, the treatment of 4.4.2 applies, with X , XX, XV depending only on the m+n indices of G instead of on the mn indices of S_1.

Reëxpressing these differential equations in component form and summing selectively:

$$\sum_i D_t z_{ij} = \sum_i X_{ij} - (XV) \sum_i z_{ij}$$

$$\sum_j D_t z_{ij} = \sum_j X_{ij} - (XV) \sum_j z_{ij}$$

$$D_t y_j = X^*_j - (XV) y_j$$

$$D_t x_i = X^*_i - (XV) x_i$$

a system of m+n differential equations, the simultaneous solution of which system furnishes x^* and y^* optimal vectors for G. Because $xu = yv = 1$ always, only m+n-2 of the differential equations are independent. Further, it is also possible to start the process by taking $x_o = u_1$ and $y_o = v_1$. Q.E.D.

This algorithm should be very easily set up on an analogue computer, and by comparison with the previous one, should be relatively insensitive to small errors in the equipment. See in this connection N. Mendelsohn (35) on ill-conditioning in matrices.

All the proofs to this point have relied rather heavily on restricting the strategy sets of the game under consideration. In the next chapter will be given some less restricted proofs of the fundamental theorem.

CHAPTER V

LESS RESTRICTED PROOFS

5.1 THEOREM $G = (X,Y;f)$ has a value if f is bounded and for $0 \leq \epsilon$, there exists a finite set $(x'_1, \ldots x'_m)$ in X such that for any (x,y) in $X \times Y$ the particular mixed strategy x' involving only the previously mentioned finite set satisfies: $f(x,y)-\epsilon \leq f(x',y)$.

PROOF The set of vectors $u_i = f(x'_i,y) = u_i(y)$ (y in Y) is in R_m and bounded since f is bounded. Therefore among these are finitely many $u(y'_j)$; $j=1,n$ such that for every y in Y : $|u(y) - u(y'_j)| \leq \epsilon$ for at least one value of j. $|u_i(y) - u(y'_j)| \leq \epsilon$ follows by taking the i'th component of the difference vector, whose length cannot exceed the length of the vector itself. This holds for each such length, ie for $i=1,m$ and a particular j. So $|f(x'_i,y) - f(x'_i,y'_j)| \leq \epsilon, i=1,m$.

The sets $X_o=(x'_i)$ and $Y_o=(y'_j)$, $i=1,m$; $j=1,n$, define a related matrix game G_o, which is known to have a solution by 3.5 . Let the optimal strategies be x'^*, y'^*, and x', y' any other mixed strategies in G_o. By 2.3 and

2.6.2	$f(x',y'^*) \leq w_o \leq f(x'^*,y')$	and by	
hypothesis	$f(x,y'_j) \leq f(x',y'_j) + \epsilon$	$j=1,n$	
Summing over j:	$f(x,y'^*) \leq f(x',y'^*) + \epsilon$	for all x in X.	
Or	$f(x,y'^*) \leq w_o + \epsilon$	for all x in X.	
IE	$bf(x) \leq w_o + \epsilon$	for all x in X.	

Hence $\quad Bbf \leqq w_0 + e.$

But $/ f(x_i', y) - f(x_i', y_j')/ \leqq e$, $i = 1,m$, all y in Y.

So $\quad f(x_i', y_j') \leqq f(x_i', y) + e \qquad$ " " " " " " " "

Forming mixed strategies over the x_i':

$f(x'^*, y_j') \leqq f(x'^*, y) + e \qquad$ for all y in Y.

But $\quad f(x', y'^*) \leqq w_0 \leqq f(x'^*, y') \qquad$ for all x', y'.

So $\quad f(x_i', y'^*) \leqq w_0 \leqq f(x'^*, y_j') \qquad$ in particular.

Hence $\quad w_0 \leqq f(x'^*, y) + e \qquad$ for all y in Y.

Or $\quad w_0 - e \leqq Bf(y) \qquad$ " " " " " " "

Hence $\quad w_0 - e \leqq bBf.$

IE $\quad -bBf \leqq e - w_0$ to which we add the inequality at the top of the page to get:

$Bbf - bBf \leqq 2e$ which is twice an arbitrarily small number, hence arbitrarily small. This can only be if $Bbf = bBf$. Therefore G has a value and there exist strategies approximating this value with arbitrary precision. Q.E.D.

5.1.1 <u>COROLLARY</u> A game with X finite has a value and P has a strategy attaining the value against all strategies of p.

<u>PROOF</u> 5.1 applies with $(x_i') = X$, $i=1,m$. Hence G has a value. Let $X = (x_i)$, $i=1,m$ and consider $G^* = (X^*, Y^*; f^*)$ where X^*, Y^* are the convex hulls of X, Y respectively, and f^* is f made <u>affine</u> in both variables. IE for $0 \leqq a \leqq 1$ and

$a + b = 1$, $f^*(az+bw,y) = a\, f(z,y) + b\, f(w,y)$

or $f^*(x,az+bw) = a\, f(x,z) + b\, f(x,w)$

Then $Bbf^* = w = \sup bf^*(x^*)$ where x^* in X^*.

In X^* exists a sequence of mixed strategies x^{*k} such that $\lim f^*(x^{*k},y) = w$ as k goes to ∞. Because no component of a mixed strategy can exceed one, these vectors all have their ends in the m-dimensional unit cube in the positive orthant with one vertex at the origin. Thus a convergent subsequence exists which approaches a vector x^{**}, $0 \leqq x^{**}$, $ux^{**}=1$. By the affine property of f^*, $\lim f^*(x^{*k},y) = f^*(x^{**},y)$ for all y. But by definition of a greatest lower bound;

$bf^*(x^*) \leqq f(x^{*k},y)$ for all y. Hence

$w \leqq \lim f(x^{**},y)$ which exhibits the existence of an optimal strategy for P. Q.E.D.

5.1.2 <u>COROLLARY</u> If f is continuous, and X,Y are bounded closed subsets of R_m, R_n respectively, G has a value.

<u>PROOF</u> Since f is a continuous function on a closed, bounded Cartesian Product, it is uniformly continuous. So a d exists corresponding to $0 \leqq e$ such that:

$|f(x,y) - f(x',y')| \leqq e$ provided that:

$|(x,y) - (x',y')| \leqq d$.

Take (x_i) in X, $i=1,m$ such that for every x in X there is at least one x_i for which $|x-x_i| \leqq d$. This is possible because X is bounded and contains the boundary.

Then $/ (x,y) - (x_i,y) / = / x - x_i / \angle d$ for all y, so that by a previous result:

$$/ f(x,y) - f(x_i,y) / \angle e$$

which assures that G has a value, by 5.1   Q.E.D.

5.2 DEFINITIONS A *topology* is a set of sets containing the intersection of any two of its members, and the union of sets in any subset. The union of all its members is the *space* of the topology. This space and the topology together constitute a *topological space*. A point is a *limit point* of a set of points X, if every open set containing it also contains a point of X distinct from it. The set X is *compact* if every infinite subset of X has at least one limit point in it. X is *separable* if it has a countable subset P such that every point of X is either a point of P or a limit point of P. *Borel's Theorem* states that a necessary and sufficient condition for a space X in which each point is a closed set to be compact is that every set of open sets covering X contains a finite subset also covering X. ** A real-valued function f(x) defined over members of X is *upper semi-continuous* at x_o e X if for any positive e there exists an open set U containing x_o such that $f(x) \angle f(x_o) + e$ for all x e U. *Affine* has been defined in 5.1.1 . If $F=(f_i)$ $i=1,m$ is a family of functions, then F*, the family of all possible affine combinations of the f_i, is the *convex family* generated by F. A function is said to be *concave* if

** This is certainly true of a bounded subset of an n-dimensional Euclidean space.

for $0 \leq k \leq 1$, $kf(a) + (1-k)f(b) \leq f(\, ka + (1-k)b\,)$ for all a and b in the domain of f. If the inequality points the other way, then f is convex. Note that when the equality holds, the function is affine. A real, linear space is a set of elements x,y,z ... not necessarily countable, for which $a(bx) = (ab)x$, $(a+b)x = ax + bx$, $a(x + y) = ax + ay$, and $1x = x$, where x and y are any elements of the space, and a, b are in the field of real numbers.

* * *

The following paper is an original, although short, contribution to the art which has been used by many subsequent investigators as a basis for new results. Because no translation has been available to date, this translation is included for the benefit of those who wish to acquire an acquaintance and have not much knowledge of Mathematical French:

5.2.1 TRANSLATION OF KNESER (30)

MATHEMATICAL ANALYSIS.- On a Fundamental Theorem of the Theory of Games. Note(Session of June 4,1952) by Mr. Hellmuth Kneser, presented by Mr. Joseph Pérès.

A generalisation of von Neumann's fundamental theorem in the theory of games(Math. Ann., 100, 1928, p.295-320; J. von Neumann and O. Morgenstern, THEORY OF GAMES AND ECONOMIC BEHAVIOUR, 1944)

<u>THEOREM</u> (N').- Let:

(1) K and L be two convex spaces (for example two convex regions of vector spaces) on the field of real numbers;

(2) f(x,y) be a function, linear in x and y, for $x \in K$ and $y \in L$.

(3) K be compact (by "compact" we do not understand that K be separable, but only that Borel's theorem be valid), for a topology in which each function f(x,y) for fixed $y \in L$ is upper semi-continuous.

Then one has :

$$\sup_{x \in K} \inf_{y \in L} f(x,y) = \inf_{y \in L} \max_{x \in K} f(x,y)$$

This is a quite broad generalization of von Neumann's fundamental theorem in the theory of games. Other generalizations have been given by Messrs. J. Ville (E. Borel et al. TRAITE DU CALCUL DES PROBABILITES ET DE SES APPLICATIONS, 2, 1938, No. 5), A. Wald (Ann. Math. 46, 1945, pp. 281-286), S. Karlin (Ann. Math. Stud. No. 24, 1950), and others.

The theorem (N') is demonstrated by means of three lemmas:

<u>LEMMA I</u>.- Let f and g be two linear functions, upper semicontinuous in the convex, compact space K, and let the $\min(f(x),g(x)) \leq 0$ for all $x \in K$. Then can be found $\rho \geq 0$ and $\sigma \geq 0$, such that $\rho + \sigma = 1$ and $\rho f(x) + \sigma g(x) \leq 0$ for all $x \in K$.

To prove it let $M(N)$ be the region of K in which $0 \leq f(x)$ ($0 \leq g(x)$). In $M(N)$ one has $g(x) < 0$ ($f(x) < 0$). The regions M and N are compact, without a point in common. If M or N is empty one takes $\rho = 1 - \sigma = 1$ or 0. Otherwise, setting:

$$\max_{x \in M} \frac{f(x)}{-g(x)} = \frac{f(p)}{-g(p)} = \alpha \geq 0$$

$$\max_{x \in N} \frac{g(x)}{-f(x)} = \frac{g(q)}{-f(q)} = \beta \geq 0$$

and calculating $g(x) < 0$ at the point where the segment pq meets the hyperplane $f(x) = 0$, one finds $\alpha\beta \leq 1$. Let $\gamma > \alpha$, $\delta > \beta$, $\gamma\delta = 1$; then one can take

$$\rho = \frac{1}{1+\gamma} = \frac{\delta}{\delta+1}, \quad \sigma = \frac{\gamma}{1+\gamma} = \frac{1}{\delta+1} .$$

<u>LEMMA</u> 2(n).- Let $f_1(x), \ldots, f_n(x)$ be linear and upper semicontinuous functions in the convex and compact space K, and let $\min f_k(x) < 0$, $k=1,n$; for all $x \in K$. Then can be found $\rho_k \geq 0$ ($k=1,n$), such that $\sum \rho_k = 1$ and $h(x) = \sum \rho_k f_k(x) < 0$ for all $x \in K$.

The lemma 2(1) is trivial. Suppose then that $1 \leq n$ and the lemma 2(n-1) holds. One can then apply 2(n-1) to the region $0 \leq f_n(x)$ of K (if it is not empty, in which case h would be taken $= f_n$), and to the functions $f_1, \ldots f_{n-1}$. Let $g = \sum_1^{n-1} \sigma_j f_j$ be the function obtained. Applying lemma I to K and the functions g and f_n, one finds $h = \rho g + \sigma f_n$, that is to say $\rho_k = \rho\sigma_k$, $k=1,n-1$;

and $\rho_n = \sigma$.

<u>LEMMA</u> 3.- Under the hypotheses of theorem (N') either:

(a) There is an $x \in K$ such that $0 \leq f(x,y)$ for all $y \in L$;

(b) There is a $y \in L$ such that $f(x,y) < 0$ for all $x \in K$.

Let us suppose (a) false; which means to say: to each $x \in K$ can be assigned a $y = p(x) \in L$ such that $f(x,p(x)) < 0$. Let Y_y be the set of $x \in K$ which give $f(x,y) < 0$; it is an open set in K. As $x \in Y_{p(x)}$, $K = \bigcup_{x \in K} Y_{p(x)}$; a finite number of points $y_k = p(x_k)$ are found such that $K = \bigcup_{k=1,n} Y_{y_k}$, that is to say for all $x \in K$ $\min_k f(x,y_k) < 0$. From the lemma 2 it follows that $f(x,y) < 0$ for all $x \in K$ if one takes $y = \sum \rho_k y_k$. Therefore (b) is valid if (a) isn't. Q.E.D.

Now, to demonstrate theorem (N') lemma 3 is applied to the function $f(x,y)-c$ which is itself upper semi-continuous in x and linear in x and y. If (a) holds there is an $x_o \in K$ such that the function $\phi(x) = \inf_{y \in L} f(x,y)$ satisfies $c \leq \phi(x_o)$; then

(a') $c \leq \sup_{x \in K} \inf_{y \in L} f(x,y) = \sup_{x \in K} \phi(x) = A$

If (b) holds there is a $y_o \in L$ such that the function $\psi(y) = \max f(x,y)$ satisfies $\psi(y_o) < c$; then

(b') $B = \inf \psi(y) = \inf \max f(x,y) < c$

Taking $c = B$ makes (b') impossible, so that (a') holds, that is to say $B \leqq A$. The proof of the inequality $A \leqq B$ being elementary, the theorem (N') is established.

Let us remark, finally, that sup may be replaced by max in the enunciation of the theorem.

Kneser's work seems to have inspired a number of authors (13,15,41 for example) and a couple of examples will now be given.

5.3 <u>INDUCTION ON AN AFFINE FAMILY</u> due to J.E.L. Peck(41).

<u>THEOREM</u> (N') - Let:

(i) K be a convex subset of a real, linear space;

(ii) $F = (f_i)$ be the family of functions $f_i(x)$ affine in x, for $x \in K$;

(iii) $F^* = (x:x=m_i f_i, i=1,n; f_i \in F, 0 \leqq m_i, mv=1)$ be the convex family generated by F.

Then
$$\sup_K \min_{F^*} f(x) = \min_{F^*} \sup_K f(x)$$

<u>PROOF</u> For $x \in K$ and $f \in F$, $f(x) \leqq \sup_K f(x)$

therefore in particular $f_j(x) \leqq$ " " $j=1,n$. But
$$f^*(x) = m_j f_j(x) \leqq m_j(\sup_K f(x))_j = mv \sup_K f(x) = \sup_K f(x)$$

So that $f^*(x) \leqq \sup_K f(x)$ for all $x \in K$.

In particular $\sup_K f^*(x) \leqq \sup_K f(x)$ " " "

But the right-hand member is actually attained, so by definition of f^*, $\sup_K f(x) \leqq \sup_K f^*(x)$ so that equality prevails and $\sup f(x) = \sup f^*(x)$. Returning to the previous inequality and making the above replacement

$$f^*(x) \leq \sup f^*(x)$$

$$\min f^*(x) \leq \min \sup f^*(x)$$

$$\sup_K \min_{F^*} f^*(x) \leq \min_{F^*} \sup_K f^*(x)$$

But $\min f^*(x)$ is got by taking $f^*(x) = 1 \times \min f_j(x)$.

Therefore $\min_{F^*} f^*(x) = \min_F f(x)$ for all $x \in K$.

Hence also their least upper bounds

$$\sup_K \min_F f(x) = \sup_K \min_{F^*} f^*(x)$$

Let this common value be $w \leq \min_{F^*} \sup_K f^*(x)$

The theorem is evidently meaningless if w is infinite.

Let $- \infty \leq w \leq \min_{F^*} \sup_K f^*(x)$

Equality will always hold if a finite w and $g^* \in F^*$ can be found such that $g^*(x) \leq w$ and $\sup_K g^*(x) \leq w$. These will be exhibited with the aid of two lemmas.

<u>LEMMA I</u>- For $F = (f_o, f_1)$, let $f^*_m = mf_1 + (1-m)f_o$. Hence $F^* = (f^*_m\colon 0 \leq m \leq 1)$. Let $M(x) = (m\colon f^*_m(x) \leq w, x \in K)$. There are three possibilities:

I. $w \leq f_o(x)$ and $w \leq f_1(x)$ for all $x \in K$. Then $f^*_m(x)$ exceeds $mw + (1-m)w = w$ for all m and all $x \in K$. Also $w \leq \min_{F^*} f^*(x)$ and $w \leq \sup_K \min_{F^*} f^*(x)$ which contradicts the definition of w. Thus this possibility is thrown out.

II. $f_o(x) \leq w$ and $f_1(x) \leq w$ for all $x \in K$. Then $f^*_m(x) \leq w$ for all m and the lemma is true trivially.

III. There is now left the really interesting case, where $w \leqq$ one of $f_i(x)$, $i=0,1$; and the other $f_i(x) \leqq w$ at each point $x \in K$. If in the latter inequality, for some x, equality prevails, then $M(x) = (0)$ or (1). If on the other hand, the equality is barred, then $M(x)$ consists of a continuum of values in $0 \leqq m \leqq 1$.

Cases II and III conform with the definition of w, so for every $x \in K$, $M(x)$ exists in $0 \leqq m \leqq 1$ and contains one endpoint (III) or both endpoints (II). If $\bigcap_K M(x) = \emptyset$, the lemma is not true, since there is no m for which $f^*_m(x) \leqq w$ for all $x \in K$. Therefore it is necessary to exhibit one point in the intersection. Only case III leads to non-trivial considerations. Let $x = a$, and $x = b$ where:

$0 \in M(a) \quad 0 \leqq m \leqq 1 \qquad\qquad 1 \in M(b) \quad 0 \leqq m \leqq 1$

Then:

$$
\begin{array}{llll}
(1) & f^*_0(a) = 0f_1(a) + (1-0)f_0(a) = f_0(a) \leqq w & & 0 \in M(a) \\
 & f^*_1(a) = 1f_1(a) + (1-1)f_0(a) = f_1(a) > w & & 1 \text{ not} \\
 & f^*_0(b) = 0f_1(b) + (1-0)f_0(b) = f_0(b) > w & & 0 \notin M(b) \\
 & f^*_1(b) = 1f_1(b) + (1-1)f_0(b) = f_1(b) \leqq w & & 1 \text{ is}
\end{array}
$$

A constant convex combination will now be found in $a \leqq m \leqq b$ and its constant value proved $\leqq w$.

Since $f^*_m(x) = mf_1(x) + (1-m)f_0(x)$ by definition, then in order for the convex combination to be constant:

$$(2) \quad f^*_m(a) = mf_1(a)+(1-m)f_0(a) = f^*_m(b) = mf_1(b)+(1-m)f_0(b)$$

It is easily verified from (1) that the solution m_0

of (2) satisfies $0 \leq m \leq 1$. To confirm $f^*_{m_o}$ as constant:

$$\begin{aligned} f^*_{m_o} &= f^*_{m_o}(a+k(b-a)) \quad \text{where } 0 \leq k(x) \leq 1 \\ &= f^*_{m_o}(a) + kf^*_{m_o}(b) - kf^*_{m_o}(a) \\ &= f^*_{m_o}(a) \qquad \text{from (2), proving} \end{aligned}$$

the function constant in $a \leq x \leq b$.

Now consider the point $c \in a \leq x \leq b$, such that $f_o(c) = f_1(c)$, whose existence is guaranteed by the continuity of the functions and (1). Now $f_{m_o}{}^*(c) = f_o(c) = f_1(c)$ because the former is constant in the interval. Also suppose that $w \leq f^*_{m_o}(c)=f_o(c)=f_1(c)$ which contradicts the hypothesis that at least one of the functions is $\leq w$ for all x. Hence $f^*_{m_o}(x) \leq w$, and since all convex combinations have the same value for $x = c$, w does not fall short of the least upper bound of such, establishing the lemma for two functions.

<u>LEMMA</u> 2(n) The lemma 2(2) just having been proved, assume the lemma 2(n-1). Let $F = (f_o, \ldots f_n)$ and $K_o=(x: f_o(x) > w)$ this w being appropriate to F. K_o is a convex subset of K since f_o is affine. If $K_o = \emptyset$ the lemma is trivially true. Assume therefore that it has at least one element. Let $F_o = (f_1, \ldots f_n)$. By 2(n-1) there exists p^* in F^*_o such that $p^*(x) \leq w$, $x \in K$. Also $\sup_{K_o} \min_{F_o} f(x) \leq w$ because f_o might conceivably be the largest function in F. But $\min(f_o(x), p^*(x)) \leq w$, $x \in K$ because in K_o, $p^* \leq w$ and out of K_o, $f \leq w$. Hence the lemma is established by applying lemma I to the family (f_o, p^*)

having two members. Let the resulting function be g^*.

<u>THEOREM</u>(N') The existence of $g^* \in F^*$ such that $g^*(x) \leq w$ for all $x \in K$ allows the conclusion that the theorem is valid.

Q.E.D.

5.3.1 <u>FUNDAMENTAL THEOREM</u> Let $G = (X, Y; f)$ be a game with Y finite. Let $G^* = (X^*, Y^*; f^*)$ be the extended game. Then X^* is convex by the definition of a mixed strategy, f^* is affine by definition of expectation value, so the family of functions $f_y(x) = f^*(x,y)$ satisfies the hypothesis of the preceding theorem, with $x \in X^*$ and $y \in Y$, the last being a finite set.

Hence $$\sup_{X^*} \min_{Y^*} f^*(x,y) = \min_{Y^*} \sup_{X^*} f^*(x,y)$$

so that G has a value and P has an optimal strategy by the method of 5.1.1

Q.E.D.

5.4 <u>THEOREM</u> due to J.E.L. Peck and A.L. Dulmage (42). Let (1) Y be a subset of a real, linear space; -Y- its linear extension; -X- a convex subset of a real , linear space;

(2) f be a real-valued function, concave in x and convex in y, for $x \in$ -X- and $y \in$ -Y- ;

(3) -X- be compact in a topology for which $f(x,y)$ is upper semicontinuous in x for every $y \in Y$;(no axiom of separation is assumed)

Then $$\inf_{Y} \sup_{X} f(x,y) \leq \sup_{X} \inf_{Y} f(x,y)$$

The theorem is established by means of three lemmas:

<u>LEMMA I</u> If f and g are real, concave functions, upper semi-continuous on the convex, compact set K and if $\min(f(x), g(x)) \leq 0$ for all $x \in K$; then for all $x \in K$ exist ρ and σ satisfying $0 \leq \rho, \sigma$ and $\rho + \sigma = 1$ such that $\rho f(x) + \sigma g(x) \leq 0$.

Let $M(N)$ be the subset(s) of K for which $0 \leq f(x)$ $(0 \leq g(x))$. M and N are compact and disjoint since the hypothesis concerning $\min(f,g)$ prevents both functions from being simultaneously positive. If$(M(N) = \emptyset$, take $\rho = 1$, or 0, a trivial situation. If $x \notin M \cup N$, take any σ in $0 \leq \sigma \leq 1$ and another trivial situation ensues since any convex combination of negative functions is also negative. Otherwise, set:

$$\max_{M} \frac{f(x)}{-g(x)} = \frac{f(p)}{-g(p)} = \alpha \geq 0,$$

$$\max_{N} \frac{g(x)}{-f(x)} = \frac{g(q)}{-f(q)} = \beta \geq 0.$$

Because f and g are upper semi-continuous over K, hence over M and N, $p \in M$ and $q \in N$ satisfying the above do exist.

Now $f(q) \leq 0$ and $0 \leq f(p)$ by definition of M. Hence $k\, f(p) + (1-k)\, f(q) = 0$, $0 \leq k$ for some k.

By definition of a concave function:

$$0 = kf(p) + (1-k)f(q) \leq f(\ kp + (1-k)q\ )$$

Therefore $kp + (1-k)q \in M$ and since M and N are disjoint $kp + (1-k)q \notin N$. From the definition of N

$$g(\ kp + (1-k)q\ ) < 0 \quad \text{and } g \text{ is concave, so:}$$

$$kg(p) + (1-k)g(q) \leq g(\ kp + (1-k)q\ ) < 0.$$

From the definitions of α, β, and k:

$$f(p) = -\alpha g(p)\ ,\quad g(q) = -\beta f(q),\quad f(q) = \frac{kf(p)}{k-1}$$

while from the preceeding inequality $kg(p) < (k-1)g(q)$.

Hence
$$\begin{aligned} kg(p) &< (k-1)g(q) \\ \text{" "} &< (1-k)\beta f(q) \\ \text{" "} &< \frac{(1-k)\beta kf(p)}{k-1} = \frac{(k-1)\beta\alpha\, kg(p)}{k-1} \end{aligned}$$

But $g(p) < 0$ by definition of N and $0 < k$, so $kg(p) < 0$.

Hence $\alpha\beta < 1$.

Let $\alpha < \gamma$, $\beta < \delta$, $\gamma\delta = 1$, and take:

$$\rho = \frac{1}{1+\gamma} = \frac{\delta}{\delta+1} \qquad \sigma = \frac{\gamma}{1+\gamma} = \frac{1}{\delta+1}$$

If $x \in M$, $x \notin N$, so $0 \leq f(x)$ and $g(x) < 0$ hence

$$\begin{aligned} \rho f(x) + \sigma g(x) &\leq \rho(-\alpha g(x)\ ) + \sigma g(x) \\ &= g(x)\ \frac{(\ -\alpha + \gamma\ )}{1+\gamma} \end{aligned}$$

The fraction has both parts positive and $g(x) < 0$, so the combination is negative, proving the lemma if $x \in M$.

If $x \in N$, $x \notin M$, so $0 \leq g(x)$ and $f(x) < 0$ hence

$$\begin{aligned} \rho f(x) + \sigma g(x) &\leq \rho f(x) + \sigma(\ -\beta f(x)) \\ &= f(x)\ \frac{(\ \delta - \beta\ )}{1+\delta} < 0 \end{aligned}$$

Therefore the lemma is valid for x in N. But since it has been shown true for x in M and is trivially true for x in neither, it is true for all x in K.

<u>LEMMA</u> 2(n)- Let f_i, $i=1,n$ be a set of real concave functions on the compact, convex set K, for which $\min f_i(x) < 0$ for all x in K. Then exist ρ_i, $i=1,n$ such that $0 \leq \rho_i$ $\sum \rho_i = 1$, $h(x) = \sum \rho_i f_i(x) < 0$ for all x in K.

Lemma 2(1) is trivial. Assume $1 < n$ and lemma 2(n-1). If $(x: f_n(x) < 0) \neq \emptyset$, take $h = f_n$. If $(x: 0 \leq f_n(x)) \neq \emptyset$, apply 2(n-1) since this is a subset of K and hence compact and convex. Let $g = \sum \sigma_i f_i$ result, concave and upper semi-continuous, since all its components are. Also $g(x) < 0$ everywhere so the conditions of 2(2) are satisfied by g and f_n. Hence by Lemma I, $h = \rho g + \sigma f_n < 0$ and the result is obtained with $\rho_i = \rho \sigma_i$, $i=1,n-1$; $\rho_n = \sigma$

<u>LEMMA</u> 3- Under the hypotheses of theorem 5.4 either:

(a) there is $x \in \mathcal{X}$ such that $0 \leq f(x,y)$ for $y \in Y$

(b) there is $y \in Y$ such that $f(x,y) < 0$ for $x \in \mathcal{X}$

Suppose (a) false. Let $y = p(x) \in Y$; $f(x,p(x)) < 0$. Let $Y_y = (x: x \in \mathcal{X}, f(x,y) < 0)$, an open set in $\mathcal{X}$ since $f(x,y)$ is upper semi-continuous in x by hypothesis. But every $x \in Y_{p(x)}$, so $\mathcal{X}$ is covered by $\bigcup_{x \in \mathcal{X}} Y_{p(x)}$ and since $\mathcal{X}$ is compact, a finite number of $y_i = p(x_i)$ exist such that $\mathcal{X} = \bigcup_{k=1,n} Y_{y_k}$. In each Y_{y_k} it is true that $f(x,y_k) \leq 0$ for x in -X- so lemma 2(n) yields a negative mean $h(x) = \sum_1^n \rho_k f(x,y_k)$. Consider the function value of y: $y = \sum_1^n \rho_k y_k$ and because f(x,y) is convex in y:

$$f(x,y) \leq \sum_1^n \rho_k f(x,y_k) \leq 0 \quad \text{because } 0 \leq \rho_k.$$

This proves (b) valid if (a) is not. Q.E.D.

$f(x,y) - c$ has all the properties of f(x,y) so lemma 3 may be applied to it. If (a) is true, then for $x_o \in$ -X-, $\phi(x) = \inf f(x,y)$ satisfies $c \leq \phi(x_o)$ so:

$$\text{(a')} \qquad c \leq \sup_{-X-} \inf_{Y} f(x,y) = \sup_{-X-} \phi(x) = A$$

If (b) is true, then for $y_o \in$ -Y-, $\psi(y) = \max_{-X-} f(x,y)$ satisfies $\psi(y_o) \leq c$. Max is used rather than sup, because f(x,y) is upper semi-continuous in x and attains its upper bound. Then:

$$\text{(b')} \qquad B = \inf_{-Y-} \psi(y) = \inf_{-Y-} \max_{-X-} f(x,y) \leq c$$

Taking $c = B$ makes (b') impossible so (a') holds, whence $B \leq A$ proving the theorem.

5.4.1 <u>COROLLARY</u> 2.7.2 states $A \leq B$ always. Therefore

$$\sup_{-X-} \inf_{-Y-} f(x,y) = \inf_{-Y-} \sup_{-X-} f(x,y)$$

with Y = -Y-, proving the fundamental theorem of game theory with -X- and -Y- mixed strategy spaces.

5.4.2 <u>COROLLARY</u> If f(x,y) is linear in y for $y \in$ -Y-, then $$f(x, \sum k_i y_i) = \sum k_i f(x,y_i) \quad \text{so that}$$

$$\begin{aligned}\inf_{-Y-} f(x,y) &= \inf_{-Y-} \sum k_i f(x,y_i) \\ &= \sum k_i \inf_{Y} f(x,y_i) \\ &= \inf_{Y} f(x,y_i) \sum k_i \\ &= \inf_{Y} f(x,y)\end{aligned}$$

so that once again

$$\sup_{-X-} \inf_{-Y-} f(x,y) = \inf_{-Y-} \sup_{-X-} f(x,y)$$

5.4.3 <u>COROLLARY</u> If $f(x,y)$ defined on $-X- \times Y$ is extended to $-X- \times -Y-$ by linearity in y, and for all y in $-Y-$ exist $0 \leq \phi(y)$ and unrestricted $\psi(y)$ such that $-X-$ is compact in a topology in which $\phi(y)f(x,y) + \psi(y)$ is concave and upper semi-continuous in x, then:

$$\sup_{-X-} \inf_{-Y-} f(x,y) = \inf_{-Y-} \sup_{-X-} f(x,y)$$

Let $g(x,k) = \phi(k)f(x,k) + \psi(k)$. $-X-$ is a convex subset of a real, linear space and hence contains non-trivially an infinite number of elements. Hence by W. Sierpinski (47) every infinite subset of $-X-$ has a non-empty derived set in the topology for which $-X-$ is compact (Page 34), that is the one in which $g(x,k)$ is upper semi-continuous in x. Defining such a subset by restricting $g(x,k)$ generates a subset identical to one generated by restricting $f(x,k)$. Hence every infinite subset of $-X-$ has a non-empty derived set in the topology for which $f(x,k)$ is upper semi-continuous in x.

But $k \in Y$, hence 5.4 applies. The extension of f by linearity satisfies the requirements of 5.4.2 thus establishing this corollary. In fact, if both inf and sup of f(x,y) over -X- are finite, using notation from 2.7.1:

$$bg(y) = \emptyset(y)bf(y) + \not\mu(y)$$

$$Bg(y) = \emptyset(y)Bf(y) + \not\mu(y)$$

Since g was not specified exactly, set $Bg(y)=1$, $bg(y)=0$ for every y in Y. Then

$$\not\mu(y) = -bf(y) \quad (Bf(y) - bf(y))^{-1}$$

$$\emptyset(y) = (Bf(y) - bf(y))^{-1}$$

and the game (-X-,Y;f) possesses a value if and only if (-X-,Y;g) has one. This last might be termed the associated normalized game. Q.E.D.

H. Nikaido (40) and Wald (50,51) are two authors whose work is generalized by the above theorem and corollaries. See the original paper for details. Ky Fan(15) has developed a number of minimax theorems not involving linear spaces using this same general method.

5.5 <u>TOPOLOGICAL PROOF</u> due to J. Nash(37).

<u>KAKUTANI'S FIXED POINT THEOREM</u> Since the proof of this is far outside the purvue of this thesis, the result only will be given:

If $x \longrightarrow \emptyset(x)$ is an upper semi-continuous point-to-set mapping of an r-dimensional closed simplex S into K(S) then there exists an $x_o \in S$ such that $x_o \in \emptyset(x_o)$.

<u>NASH'S THEOREM</u> A game with n players, each with finite pure strategy set, and definite payoffs to each player for each n-tuple of pure strategies. Mixed strategies are discrete probability distributions and the payoffs expectation values.

An n-tuple <u>counters</u> another if the strategy for each player in the <u>countering n-tuple</u> yields the highest expectation value possible against the other players' strategies in the <u>countered n-tuple</u>. If an n-tuple counters itself, it is <u>self-countering</u>, and is in some degree optimal because no one player alone has anything to gain in varying his strategy singly. There is hence a certain stability associated with this situation and another term for such an n-tuple is <u>equilibrium point</u>.

The mapping defined by each n-tuple going into the set of all its countering n-tuples is a one-to-many or point-to-set mapping of the product space into itself. Each set of countering points is convex by definition of a mixed strategy and the payoffs are continuous functions on the product space, which causes the graph of the mapping to be closed. For if (z_j) and (u_j) are sequences with limits z^* and u^* in the product space, and for every k, u_k counters z_k, then u^* counters z^*. So the mapping precisely conforms to the Kakutani Theorem. By this theorem some n-tuple has a point in its image identical to itself, hence has itself as a countering strategy, hence is an equilibrium point. Q.E.D.

Since the definition of an equilibrium point when n=2 is the same as a pair of optimum strategies, this theorem is also a very brief, very elegant proof of the fundamental theorem.

In general, this theorem guarantees at least one equilibrium point. There could be several, underlying the fact that only for the two-person zero-sum game is the concept of value defined. This property of the general game raises the exciting possibility of competition-preserving mutually beneficial coalitions of all the players working in concert.

The same author has written a paper proving this theorem directly by the Brouwer Fixed Point Theorem.

5.6 <u>CONCLUDING COMMENTS</u> It is seen in this chapter that the fundamental theorem may be proved in many direct and indirect fashions. Some of these proofs are quite surprising at first glance. Common to all has been the fact that a more comprehensive theorem was derived first and the fundamental theorem of game theory out of it. This seems to be one of the good things about the theory of games to date- it stimulates fundamental work in many related disciplines.

CHAPTER VI

CLOSING COMMENTS

6.1 FUNCTION OF GAME THEORY To date game theory has not succeeded in revising the foundations of economic theory or in giving the solution to any really worthwhile problem from the real world. Even parlour board games are beyond its present scope. This has caused some to wonder what the ultimate function of game theory may be.

In itself it is a beautiful example of a mathematical model. But there is another aspect-the way in which it brings together various disciplines of mathematics in solving seemingly unrelated problems. This has been noted all through the thesis. Combinatorics, analysis, topology all are brought to bear on the problem. Perhaps, then, in the future game theory will bring about a sort of super-unification of mathematics somewhat as Quantum Mechanics unified the theory of infinite matrices with that of integral equations because of the different attacks by Heisenberg and Schrödinger.

6.2 CONJECTURE the theory of matrix games will always be adequate for the computation of optimal strategies and value of games arising out of real-life situations.

This follows from a consideration of 5.1 . Any strategy set for a game arising out of the real world must be recorded before anything can be done with it.

Even the sum total storage capacity of all the world's computers is finite for this purpose. A firm, faced with the necessity of playing a game with infinite strategy sets would try to select a finite number of them in such a way as not to miss out some which could have a drastic effect on the outcome of the game. In doing this, of course, the game loses its general character and the methods applicable to matrix games will work very well.

This note is not meant to disparage the investigation of more general situations. Some of the happy results of such investigation have already been described. But it should be emphasized that these excursions deep into technicalities belong to the province of the theoretical mathematician. The practical man calculating optimal strategies should go ahead and use the theory of the matrix game, confident that an astute selection of strategies will prevent him from going far astray.

NOTATION

The notation used is principally that of D.Gale (18) with changes where necessary to avoid confusion. Its chief feature is that column and row vectors are seldom explicitly distinguished, the dimension serving to prevent ambiguity.

u and v , as vectors, have ones for all components.

u_i " v_j, " " , have one in the i'th (j'th) position and zeros elsewhere

$/x/$ is the absolute value of the scalar x

$/x/$ is the length of the vector x

$A = (a_{ij}) = (a_i) = (a^j)$ is the matrix with rows a_i, columns a^j, and element a_{ij} at the intersection of such.

Occasionally, tensor summation is used, e.g. $xA = x_i a_i$, $i=1,m$; where x has components x_i.

$0 \leqq x$, or x non-negative means $0 \leqq x_i$, all i.

$0 \leq x$, or x semi-positive means $0 \leqq x$, $x \neq 0$.

$0 < x$, or x positive means $0 < x_i$, all i.

xc is the inner product of vectors x and c.

A^t is the transpose of the matrix A.

$A^{-1} = (a_i^-) = (a^{-j})$ is its inverse.

$0 \prec x$ means the first non-zero component of x is positive.

D_t is the operation of differentiation with respect to the variable t.

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