ELEMENTARY PARTICLES AND THEIR INTERACTIONS

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ABSTRACT

This thesis does not present results which were previously unknown in physics, but attempts to present a thorough introduction to a field which previously had been difficult of access to beginning graduate students because detailed explanations of basic methods had been lacking. The subject matter covered includes a general discussion of the Lagrangian formalism for physical fields, with detailed examples of several fields illustrating specific properties. Thus the complex field introduces charge and the vector field introduces spin. The method of second quantization, which forms the basis of the quantum mechanical discussion of interactions for elementary particles, is, it is hoped, formulated in such a way that the method is clear and that the self-consistency of the technique, with postulates explicitly stated, is evident. The relationship between invariance properties of a system and conservation laws is worked out, in detail for energy, momentum, charge, and angular momentum, with a discussion of spin, both classical and quantum mechanical. Two second quantization procedures are discussed corresponding to Bose-Einstein and to Fermi-Dirac types of particles, along with an introductory discussion of Dirac theory. Finally, the application of perturbation theory to the second quantized fields is discussed, with brief illustrative examples, as a method of obtaining results which can be subjected to experimental test.

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INTRODUCTION

The main object of this thesis is pedagogical. It was felt that the subject of the interactions of elementary particles had not appeared in a form suitable for study by students with the equivalent of an Honours degree in Physics, and yet that the subject did not have inherent difficulties sufficient to justify its neglect at this level.

The interaction of the electromagnetic field with elementary particles has received much attention in the modern literature of physics, and has had fairly adequate treatment. Because of this, and even more because of the limitations of time available in preparing an M. Sc. thesis, the subject of the electromagnetic field in quantum phenomena has been omitted. Instead, the emphasis has been placed upon discussion of the various types of fields which have been postulated as accounting for nuclear forces. ~~meson field~~. These fields may be generally categorized as Bose-Einstein fields, due to the nature of the statistics obeyed by their quanta. Of equal importance, both to the subject and to the thesis, is a discussion of the Fermi-Dirac field, whose quanta are the commonest elementary particles: electrons, protons and neutrons.

The necessity of quantizing the radiation field in considering sub-microscopic phenomena is one of the oldest facts of quantum physics, dating back to Planck's quantum hypothesis. Since phenomena are observed in which particles of one type are transformed into particles of another type, perhaps with different masses as in β-decay for example, it is desirable to have a theory in which the creation and annihilation arise naturally, and the development of such a theory is not too difficult, for the quantization of the radiation field in which photons are produced or absorbed, is available as a guide. The quantum mechanical equations are taken as the analogue of the classical electromagnetic field equations. The resultant quantization, which is discussed in detail in what follows, is called second quantization.

It should be emphasized that the second quantized field theory of elementary particles, with the concept of particle creations and annihilations is significant mainly in considering systems of interacting particles. The theory also shows certain qualitative properties of systems of particles, particularly with respect to conservation laws. The theory in the form presented is not original. It is a form possessing much beauty in its logical consistency and in its clear relation to classical field mechanics. Unfortunately, calculations based on the theory and utilizing the perturbation theory of quantum mechanics, generally give only qualitative agreement with experiment, and the theory relies upon experimental checks at perhaps more points than is desirable. In particular, the reader may feel, after reading the section on interactions (Chapter V), that an experimental determination of which form to choose for the interaction for every type of interaction, requires less of the theory than might be hoped for.

Even the theory of the interaction of radiation with matter, on which a tremendous quantity of experimental evidence is available, has not been satisfactorily developed. The techniques are not available for arriving at experimentally verifiable results even for many relatively simple cases. Such difficulties are even more pronounced in dealing with other types of interactions for which relatively little data are available. There is nevertheless a twofold value in presenting this quantitatively unsatisfactory theory. Firstly, the theory is qualitatively largely valid, and almost certainly has features which will be of lasting importance. Secondly, the theory presented forms the prerequisite knowledge for understanding present-day attempts to arrive at a quantitatively valid form.

Time has not permitted as detailed a discussion of interactions as would ideally be desired. More examples of common interactions could well have been discussed, and some more general questions could also have received more detailed treatment. It was felt to be of prime importance, however, to set down the underlying principles of second quantization in sufficient detail and clarity. It is hoped that the subject of interactions has been treated well enough that the reader can progress to the study of standard textbooks and current papers, confident in a familiarity with the principles involved.

The thesis assumes familiarity with classical mechanics at a level comparable to that occurring in Goldstein's "Classical Mechanics", and with a first course in non-relativistic quantum mechanics.

The following books are particularly pertinent references for this subject and are referred to in the text by author's surname only.

Schiff, L. I: "Quantum Mechanics", (McGraw-Hill, 1949), for general quantum theory of particles.

Schweber, S. S., Bethe, H. A., and de Hoffmann, F: "Mesons and Fields", Vol. I: Fields, (Row, Peterson, 1955). for an outline of the subject and for a helpful discussion of Dirac theory (relativistic electron theory).

Wentzel, G: "Quantum Theory of Fields", (Interscience, New York, 1949). This is a standard book on the subject, but is too difficult for a student to whom the ideas are new. Some of the central ideas of the theory are not sufficiently emphasized, and some of the arguments, while correct, do not proceed so as to have intuitive appeal: rather than indicating the method of arriving at a result, the result is merely set down and shown to be correct.

Goldstein, H: "Classical Mechanics" (Addison-Wesley, 1951).
An advanced text designed to prepare the student for modern theoretical physics. Contains also an adequate introduction to special relativity.

In the text, the Einstein summation convention has been used consistently, repeated lower case Greek subscripts indicating summation over indices (1,2,3,4) unless otherwise indicated, and lower case Roman subscripts indicating summation over indices (1,2,3). The symbol ∂_μ has generally been employed to indicate $\frac{\partial}{\partial x_\mu}$. The superscript star $*$ indicates Hermitian adjoint, which for a number or a function is the complex conjugate quantity.

CHAPTER I

THE SCALAR KLEIN-GORDON FIELD

Lagrangian Formalism for Fields

The method of quantum field theory to be discussed proceeds in essentially two stages. In the first step the equations of motion of the field are found using the Hamiltonian method for mechanics as applied to fields. In the second step the field is quantized by first finding a canonical set of variables describing the field and then representing these variables by non-commuting Hermitian operators. The result is that the quantized field has mathematical properties which allow it to be described as a system of particles.

In mechanics, Hamilton's principle is written

(1.1) $$\delta \int \mathcal{L}\, dt = 0.$$

where $\mathcal{L}$ is the Lagrangian for the mechanical system and δ is an operator effecting a variation of the generalized coordinates which is arbitrary except at the limits of integration, over which it is zero. The necessary and sufficient condition for equation (1.1) to be satisfied is a set of differential equations called Lagrange's equations. For a field the Lagrangian $\mathcal{L}$ is defined by

$$\mathcal{L} = \int_V L\, dV.$$

where L is the Lagrangian density, and V is the volume of space under consideration containing the physical field. L is a function of one or more quantities ϕ called the field functions, and of their time and space derivatives. Since ϕ depends on the time and space coordinates, so does L, whereas the total Lagrangian $\mathcal{L}$ for the field depends only on the values of ϕ and its derivatives on the surface bounding V, that is on the limits of the integral $\int_V LdV$. For simplicity L is assumed to depend only on the first partial derivatives of the ϕ's with respect to position and time. Then Hamilton's principle for a field is:

$$(1.2) \qquad \delta\int \mathcal{L}\, dt = \delta\int L\, d_4x = 0$$

where $d_4x = dx_1dx_2dx_3dx_4$ is the four dimensional volume element, x_4 being (ict). The variation δ of the integral means infinitesimal changes of the ϕ's and their derivatives anywhere in V or in time except at the limits, and Hamilton's principle then requires L to be such that the integral has an extremum value. From the form (1.2) it is evident that if L is invariant under four-space rotations (proper Lorentz transformations), then since d_4x is likewise invariant, the field equations resulting from Hamilton's principle will be Lorentz-invariant.

The result of Hamilton's principle for fields is the well-known Euler-Lagrange equations, which arise in the following way. The variation δ behaves essentially like a differential operator, but care must be taken in deciding the functional dependence of the quantities involved. If L is a function of the ϕ's and the $\partial_\nu\phi$'s, but not explicitly a function of the

x_ν's, as will be the case here, then:

$$\delta \int L\, d_4x = \int \delta L\, d_4x = \int \left\{ \frac{\partial L}{\partial \phi} \delta\phi + \frac{\partial L}{\partial(\partial_\nu \phi)} \delta(\partial_\nu \phi) \right\} d_4x$$

$$= \int \left\{ \frac{\partial L}{\partial \phi} \delta\phi + \partial_\nu \left[\frac{\partial L}{\partial(\partial_\nu \phi)} \cdot \delta\phi \right] - \partial_\nu \left[\frac{\partial L}{\partial(\partial_\nu \phi)} \right] \delta\phi \right\} d_4x.$$

(1.3) $$= \int \left\{ \frac{\partial L}{\partial \phi} - \partial_\nu \left[\frac{\partial L}{\partial(\partial_\nu \phi)} \right] \right\} \delta\phi\, d_4x + \int_S \left[\frac{\partial L}{\partial(\partial_\nu \phi)} \delta\phi \right]_n d_4S$$

where the following relationships have been used:

$$\delta(\partial_\nu \phi) = \partial_\nu(\delta\phi).$$

and $$\frac{\partial L}{\partial(\partial_\nu \phi)} \partial_\nu(\delta\phi) = \partial_\nu \left[\frac{\partial L}{\partial(\partial_\nu \phi)} \delta\phi \right] - \partial_\nu \left[\frac{\partial L}{\partial(\partial_\nu \phi)} \right] \delta\phi.$$

and a four-dimensional Gauss' theorem:

$$\int \partial_\nu \left[\frac{\partial L}{\partial(\partial_\nu \phi)} \delta\phi \right] d_4x = \int_S \left[\frac{\partial L}{\partial(\partial_\nu \phi)} \delta\phi \right]_n d_4S,$$

the subscript n indicating the component of the four-vector normal to the four-surface S bounding V. Since $\delta\phi$ is zero over the entire surface, this integral is zero. Since $\delta\phi$ is arbitrary except on S, $\delta\int L d_4x = 0$ from equation (1.3) has as a sufficient condition:

(1.4) $$\frac{\partial L}{\partial \phi} = \partial_\nu \left[\frac{\partial L}{\partial(\partial_\nu \phi)} \right].$$

which is the Euler-Lagrange equation. If L depends on several field functions, the generalization of the above derivation, which is straightforward, results in one such equation for each field function ϕ.

In deciding on the form of the Lagrangian density for a field the main criteria are Lorentz invariance, as mentioned, and simplicity. For simplicity, derivatives of order greater than one are generally not used in the Lagrangian density. If the field equations are known, as, for example, Maxwell's equations for the electromagnetic field, L is so formed that Hamilton's principle leads to the correct equations. If the field equations are unknown, the predictions arising from the use of various simple forms of L are compared with experimental results, and on this basis a selection of the best form of L may be possible.

As an example, the field obeying the Klein-Gordon equation will be considered. The transition from the classical equation $p^2/2m = E$ for a free particle, where $\underline{p}$ is momentum, m mass and E energy, to the quantum mechanical equation describing the motion of an elementary particle:

$$\frac{\hbar}{2m}\nabla^2\psi = -i\hbar\frac{\partial\psi}{\partial t}.$$

by replacing $\underline{p}$ by $-i\hbar\nabla$ and E by $i\hbar\frac{\partial}{\partial t}$, and interpreting $p^2/2m = E$ as an operator equation, should be familiar to the reader. The corresponding relativistic equation $E^2 = p^2c^2 + m^2c^4$ leads, by the same substitutions for $\underline{p}$ and E and interpretation as operators, to the Klein-Gordon equation

$$-\hbar^2\ddot{\phi} = -\hbar^2c^2\nabla^2\phi + m^2c^4\phi.$$

which in relativistic notation is:

$$\partial_\nu \partial_\nu \phi = \left(\frac{m^2c^2}{\hbar^2}\right) \phi. \tag{1.5}$$

In this equation the operand ϕ will be thought of as describing a field which after quantization will be found to have properties equivalent to those of a system of particles. Such operands, having different covariance properties[1], or obeying different field equations, will be found to describe particles having different properties. A scalar field function ϕ obeying

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[1] The covariance property of an expression denotes its behaviour under rotation of four-space axes (proper Lorentz transformation), and under reflection of three-space axes. Common types of covariants are:

<u>Scalar</u>: unchanged under rotations or reflections.

<u>Pseudoscalar</u>: changes algebraic sign under reflection of three-space axes. Otherwise it behaves as a scalar.

<u>Vector</u>: If a vector before and after Lorentz transformation is denoted by $Y \equiv \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$ and $Y' \equiv \begin{pmatrix} y'_1 \\ y'_2 \\ y'_3 \\ y'_4 \end{pmatrix}$ respectively, then $Y' = AY$ (where A is the four-by-four matrix of the Lorentz transformation) denotes the behaviour of a vector under such transformation.

<u>Pseudovector</u>: The three-space components of a pseudovector do not change under reflection of three-space axes. (Those of a vector change sign). The pseudovector behaves like a vector

the Klein-Gordon equation will first be considered. A suitable Lagrangian density is

$$(1.6) \qquad L = g_s\left(\partial_\nu \phi . \partial_\nu \phi + \frac{m^2c^2}{\hbar^2} \phi^2 \right)$$

where g_s is a constant to be assigned later, for then:

$$\frac{\partial L}{\partial \phi} = 2g_s \frac{m^2c^2}{\hbar^2} \phi \qquad \text{and} \qquad \partial_\nu \left[\frac{\partial L}{\partial(\partial_\nu \phi)} \right] = 2g_s \partial_\nu \partial_\nu \phi .$$

so that the Euler-Lagrange equation is the same as the Klein-Gordon equation.

The detailed behaviour of the field is studied on the basis of densities of observable quantities such as energy. In the mechanics of a system of particles, if the Lagrangian $\mathcal{L}$ is

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under four-space rotations.

<u>Tensor</u>: Let $a_{\mu\nu}$ represent the elements of a Lorentz transformation. Then a tensor of r th rank $T_{\alpha\beta\ldots\gamma}$ where γ is the r th index, is defined as transforming under Lorentz transformation by the rule:

$$T'_{\alpha\beta\ldots\gamma} = a_{\alpha\mu} a_{\beta\nu} \cdots a_{\gamma\rho} T_{\mu\nu\ldots\rho}$$

Another covariant, the spinor, will be discussed in connection with the Dirac field.

a function of generalized coordinates q_α having conjugate generalized momenta $p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}$, the Hamiltonian $\mathcal{H}$, equal to the energy of the system if $\frac{\partial \mathcal{L}}{\partial t} = 0$, is defined by

$$\mathcal{H} = p_\alpha \dot{q}_\alpha - \mathcal{L}.$$

Analogously for a field theory, in which the Lagrangian density L plays the central rôle, the values of the ϕ's at every point in the field are taken as generalized coordinates, and the conjugate momenta are defined by

$$\pi = \frac{\partial L}{\partial \dot{\phi}}$$

at each point. Then the energy density of a field at a point is defined by $H = \pi \dot{\phi} - L$, and the total energy of the field by $\mathcal{H} = \int_V H dV$. If the total energy $\mathcal{H}$ is conserved the energy density H must satisfy a continuity equation[2].

(1.7) $$\frac{\partial H}{\partial t} + \nabla . \underset{\sim}{S} = 0$$

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2 If a quantity $\mathcal{H}$ in a volume V, having density H, is conserved, then $\frac{d\mathcal{H}}{dt} = \int \frac{\partial H}{\partial t} dV$ is the rate of change of $\mathcal{H}$ in V. This must equal the inward flux of $\mathcal{H}$ through the surface σ of V. If $\underset{\sim}{S}$ is the flux density of $\mathcal{H}$ then

$$\frac{d\mathcal{H}}{dt} = -\int_\sigma \underset{\sim}{S} . d\underset{\sim}{\sigma} = \int_V (- \nabla . \underset{\sim}{S}) \, dV = \int_V \frac{\partial H}{\partial t} dV.$$

where $\underline{S}$ is the energy flux density. An attempt is therefore made to set $\frac{\partial H}{\partial t}$ in the form of the divergence of a vector. Now:

$$H = \pi\dot{\phi} - L = \frac{\partial L}{\partial \dot{\phi}} . \dot{\phi} - L .$$

Therefore:

(1.8) $$\frac{\partial H}{\partial t} = \dot{\phi}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\dot{\phi}}\right) + \frac{\partial L}{\partial \dot{\phi}} . \ddot{\phi} - \frac{\partial L}{\partial t}$$

It should be recalled that L is taken as explicitly dependent only on ϕ and $\partial_\nu\phi$ and H on π and ϕ . If L and H depended explicitly on the x_ν's as well, the symbols $\frac{\partial L}{\partial x_\nu}$ and $\frac{\partial H}{\partial x_\nu}$ would be ambiguous. A distinction would have to be made between the dependence of L and H on the x_ν's through ϕ , and the explicit dependence. Here:

(1.9) $$\frac{\partial L}{\partial t} = \frac{\partial L}{\partial(\partial_j\phi)} . \frac{\partial}{\partial t}(\partial_j\phi) + \frac{\partial L}{\partial\dot{\phi}} . \ddot{\phi} + \frac{\partial L}{\partial\phi} . \dot{\phi} .$$

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where $d\underline{\sigma}$ is a vector normal to σ in the outward direction. For arbitrary volume V the last equation above implies:

$$\frac{\partial H}{\partial t} + \nabla . \underline{S} = 0 .$$

Such an equation is called a continuity equation.

The terms in $\ddot{\phi}\frac{\partial L}{\partial \dot{\phi}}$ cancel, and equations (1.8) and (1.9) combine to form:

$$\frac{\partial H}{\partial t} = \dot{\phi}\left\{\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi}\right\} - \frac{\partial L}{\partial(\partial_j \phi)} \cdot \partial_j(\dot{\phi}).$$

Substituting for $\left\{\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi}\right\}$ from the Euler-Lagrange equation (1.4), written in three-dimensional notation,

$$\frac{\partial L}{\partial \phi} = \partial_j\left[\frac{\partial L}{\partial(\partial_j \phi)}\right] + \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\phi}}\right).$$

$\frac{\partial H}{\partial t}$. can be written:

$$(1.10) \qquad \frac{\partial H}{\partial t} = -\dot{\phi}\,\partial_j\left[\frac{\partial L}{\partial(\partial_j \phi)}\right] - \partial_j(\dot{\phi})\left[\frac{\partial L}{\partial(\partial_j \phi)}\right] = -\partial_j\left[\dot{\phi}\,\frac{\partial L}{\partial(\partial_j \phi)}\right].$$

One concludes that

$$S_j = \dot{\phi}\,\frac{\partial L}{\partial(\partial_j \phi)}.$$

in the continuity equation (1.7) for energy.

Continuity equations can often be found from the invariance properties of L. Lorentz invariance has already been mentioned. An especially simple form of Lorentz transformation will be considered, namely translation of four-space axes. Suppose:

$$x'_\sigma = x_\sigma + \varepsilon_\sigma.$$

for one value of σ, where ε_σ is an infinitesimal constant. Invariance of L means that at any given point in the field the value of L will not be changed by referring to the new coordinates x'_σ. This requirement is written

$$\delta L = 0 .$$

For such a transformation the reader can verify that the change produced in ϕ and $\partial_\nu \phi$, where ϕ is a scalar or a vector component, is zero. If L depends explicitly on ϕ, $\partial_\nu \phi$, and on the x_ν's, then:

$$\delta L = \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial(\partial_\nu \phi)} . \delta(\partial_\nu \phi) + \frac{\partial L}{\partial(x_\nu)} . \delta x_\nu = 0 .$$

where $\frac{\partial L}{\partial(x_\nu)}$ indicates the partial derivative taken with ϕ and $\partial_\nu \phi$ held constant. The above equation is trivial, since $\delta \phi = \delta(\partial_\nu \phi) = 0$, and x_ν, being a vector component, is also unchanged at the field point by translation of axes. In order to obtain a non-trivial result, let us consider the transformation in two stages. In the first stage the point of observation is shifted relative to the field, parallel to the x axis: a new point of the field is considered. The variation of L in this process is

$$\delta_1 L = \left[L(x_\sigma + \varepsilon_\sigma) - L(x_\sigma) \right] = \varepsilon_\sigma \frac{\partial L}{\partial x_\sigma} \qquad \text{(no summation over } \sigma \text{)}.$$

To complete the translation the physical field is now shifted until the previous point of observation coincides with the new point. The variation of L in this process is the reverse of $\delta_1 L$, for the value of L at the point of observation reverts to the value of L at the initial point. The net variation of L is zero, as it must be, with the field translated in the x_σ direction a distance ε_σ relative to the original frame of reference. The second variation is

$$\delta_2 L = [L(x_\sigma) - L(x_\sigma + \varepsilon_\sigma)] = [L(x'_\sigma - \varepsilon_\sigma) - L(x'_\sigma)] = -\varepsilon_\sigma \frac{\partial L}{\partial x'_\sigma} .$$

The result $\delta L = \delta_1 L + \delta_2 L = 0$ again seems trivial, since

$\frac{\partial L}{\partial x'_\sigma} = \frac{\partial L}{\partial x_\sigma}$. What we are seeking is to set δL in the form of a four-divergence, which being zero will denote a continuity equation: a conservation law. Now

$$\frac{\partial L}{\partial x_\sigma} = \partial_\nu (L \delta_{\nu\sigma}) = \frac{\partial L}{\partial \phi} . \partial_\sigma \phi + \frac{\partial L}{\partial(\partial_\nu \phi)} . \partial_\sigma (\partial_\nu \phi) + \frac{\partial L}{\partial (x_\sigma)} .$$

which from the Euler-Lagrange equation (1.4) is:

$$\partial_\nu (L \delta_{\nu\sigma}) = \partial_\nu \left[\partial_\sigma \phi \cdot \frac{\partial L}{\partial(\partial_\nu \phi)} \right] + \frac{\partial L}{\partial(x_\sigma)} .$$

Substituting into $\delta L = 0$ for one of the $\frac{\partial L}{\partial x_\sigma}$ terms:

$$\varepsilon_\sigma \left\{ \partial_\nu (L \delta_{\nu\sigma}) - \partial_\nu \left[\partial_\sigma \phi \cdot \frac{\partial L}{\partial(\partial_\nu \phi)} \right] - \frac{\partial L}{\partial(x_\sigma)} \right\} = 0 . \quad \text{(no summation over } \sigma \text{)}.$$

Invariance of L under translation of axes therefore implies nothing more than could have been determined from evaluating

$(\partial_\sigma L)$. If $\frac{\partial L}{\partial(x_\nu)}$ is zero, the invariance implies a continuity equation:

(1.11) $$\partial_\nu T_{\sigma\nu} = 0$$

where $T_{\sigma\nu} = \left\{ \partial_\sigma \phi . \frac{\partial L}{\partial(\partial_\nu \phi)} - L\,\delta_{\sigma\nu} \right\}$. It should be emphasized that the conservation laws are always present in the theory, but they may not always be so obvious as in this case where they follow simply from differentiating the Lagrangian density. Consideration of the invariance properties of L then allow a systematic determination of continuity equations. Consideration of angular momentum in a later chapter will illustrate this point.

The conserved quantities corresponding to the continuity equation (1.11) will be shown to be energy and momentum. Invariance under translation in time implies conservation of energy, for with $\sigma = 4$ equation (1.11) is simply the energy continuity equation (1.10). Energy density H, defined as $\left(\dot{\phi} \frac{\partial L}{\partial \dot{\phi}} - L \right)$ is given by T_{44}. Then since energy and (-ic) times momentum form a relativistic four-vector [3], momentum density $\underline{G}$ must be given by

$$G_j = -\frac{1}{ic} T_{j4} = -\partial_j \phi . \frac{\partial L}{\partial \dot{\phi}} .$$

[3] C. F. Goldstein, p. 203.

Momentum conservation, following from equation (1.11) with σ = 1, 2, or 3, is a consequence of invariance under translation of spatial axes. $T_{\mu\nu}$ is called the canonical energy-momentum tensor, and obeys a continuity law provided $\frac{\partial L}{\partial(x_\nu)} = 0$. The latter condition is applied whenever possible in forming L so that energy and momentum conservation may hold for the field.

Field Quantization and the Particle Properties of a Field.

The second stage in the development of the quantum field theory has now been reached. The generalized coordinates consisting of the values of ϕ at every point are not convenient, for they form a non-denumerable set. One method of obtaining a denumerable set is to expand ϕ in a Fourier series throughout the volume V. For simplicity V will be taken to be a cube of side length ℓ. For a field extending through the whole of space ℓ approaches infinite length. Let

$$\phi = N \sum_{\underline{k}} (q_{ke} \cos \underline{k}.\underline{r} + q_{k} \sin \underline{k}.\underline{r}).$$

where N is a constant, $\underline{r}$ is the position vector relative to axes with origin at one corner of the cube V, and $\underline{k}$ is a wave number or propagation vector with j th component equal to $\left(\frac{2\pi n_j}{\ell}\right)$, where each of the n_j's ranges over all positive integers and zero: in this way the trigonometric functions form a complete orthogonal set, for:

$$\int_V \sin \underline{k}.\underline{r} \sin \underline{k}'.\underline{r}\, dV = \int_V \cos \underline{k}.\underline{r} \cos \underline{k}'.\underline{r}\, dV = \frac{1}{2} \ell^3 \delta_{\underline{k}\underline{k}'}$$

$$\text{and} \quad \int_V \sin \underline{k}.\underline{r} \cos \underline{k}'.\underline{r}\, dV = 0.$$

The time dependence of ϕ occurs in the q_{kr} $(r = e,o)$ expansion coefficients. These coefficients form a suitable set of generalized coordinates because to specify them at any time is to specify the field ϕ completely at that time. The corresponding generalized momenta p_{kr} are defined by

$$p_{kr} = \frac{\partial \mathcal{L}}{\partial \dot{q}_{kr}}.$$

For the scalar field obeying the Klein-Gordon equation:

$$\mathcal{L} = \int_V L\, dV = \int_V g_5 \left(\partial_\nu \phi . \partial_\nu \phi + \frac{m^2c^2}{\hbar^2}\phi^2\right) dV = \int_V g_5 \left[(\nabla\phi)^2 - \frac{1}{c^2}\dot{\phi}^2 + \frac{m^2c^2}{\hbar^2}\phi^2\right] dV.$$

Substituting the Fourier expansion for ϕ:

$$\int_V (\nabla\phi)^2 dV = N^2 \int_V \sum_{\underline{k},\underline{k}'} (-\underline{k}\, q_{ke} \sin \underline{k}.\underline{r} + \underline{k}\, q_{ko} \cos \underline{k}.\underline{r})(-\underline{k}'\, q_{k'e} \sin \underline{k}'.\underline{r} + \ldots)\, dV$$

$$= \frac{N^2 \ell^3}{2} \sum_{\substack{k \\ r=e,o}} k^2 q_{kr}^2 .$$

having used the orthogonality relations. Similarly:

$$-\frac{1}{c^2}\int_V \dot{\phi}^2 dV = -\frac{N^2}{c^2}\int_V \sum_{\underline{k},\underline{k}'} (\dot{q}_{ke} \cos \underline{k}.\underline{r} + \dot{q}_{ko} \sin \underline{k}.\underline{r})(\dot{q}_{k'e} \cos \underline{k}'.\underline{r} + \ldots)\, dV$$

$$= -\frac{N^2 \ell^3}{2c^2} \sum_{\substack{k \\ r=e,o}} \dot{q}_{kr}^2 .$$

$$\text{and} \quad \frac{m^2c^2}{\hbar^2}\int_V \phi^2 dV = \left(\frac{m^2c^2N^2\ell^3}{2\hbar^2}\right) \sum_{\substack{k \\ r=e,o}} (q_{kr}^2).$$

Collecting these results, $\mathcal{L}$ is evaluated as:

$$\mathcal{L} = \left(-\frac{N^2 \ell^3 g_s}{2c^2}\right) \sum_{\substack{k \\ r=e,o}} (\dot{q}_{kr}^2 - \omega_k^2 q_{kr}^2)$$

where $c^2 \left(k^2 + \frac{m^2c^2}{h^2}\right)$ has been written (ω_k^2). For this field the generalized momenta are:

$$p_{kr} = \frac{\partial \mathcal{L}}{\partial \dot{q}_{kr}} = \left(-\frac{2N^2\ell^3 g_s}{2c^2}\right) \dot{q}_{kr} .$$

The choice of $N = \sqrt{2}\,\ell^{-\frac{3}{2}}$ normalizes the expansion functions ($N \cos \underline{k}\cdot\underline{r}$) and ($N \sin \underline{k}\cdot\underline{r}$) in V, and the choice of $g_s = -\frac{1}{2}c^2$ is convenient in that with this choice p_{kr} is simply $\dot{q}_{kr}$. Then $\mathcal{L}$ has the form:

$$\mathcal{L} = \frac{1}{2} \sum_{k,r} (p_{kr}^2 - \omega_k^2 q_{kr}^2)$$

so that $\mathcal{H}$ for the field is:

$$\mathcal{H} = \sum_{k,r} p_{kr}\dot{q}_{kr} - \mathcal{L} = \sum_{k,r} (p_{kr}^2 - \tfrac{1}{2} p_{kr}^2 + \tfrac{1}{2}\omega_{kr}^2 q_{kr}^2)$$

$$= \frac{1}{2}\sum_{k,r} (p_{kr}^2 + \omega_k^2 q_{kr}^2).$$

The reader should verify that this result is the same as obtained from:

$$\mathcal{H} = \int_V H\, dV = \int_V T_{44}\, dV = \int_V \left(\dot{\phi}\frac{\partial L}{\partial \dot{\phi}} - L\right) dV$$

by substitution of the Fourier series for ϕ into T_{44} and integration. It is also left to the reader to verify that the field equation, $\nabla^2\phi - \frac{1}{c^2}\ddot{\phi} = \frac{m^2c^2}{\hbar^2}\phi$, under the substitution of the Fourier series and using the orthogonality conditions, yields the same field equations for the q's as do Hamilton's equations: $\dot{p}_{kr} = -\frac{\partial \mathcal{H}}{\partial q_{kr}}$, and $\dot{q}_{kr} = \frac{\partial \mathcal{H}}{\partial p_{kr}}$.

The form of $\mathcal{H}$ is that of a sum over k of harmonic oscillators of angular frequency ω_k and mass unity.

The quantization procedure, as stated previously is now to consider the p's and q's as Hermitian operators instead of as functions. The operators are assumed to have the commutation relations:

$$(q_{kr}p_{k'r'} - p_{k'r'}q_{kr}) \equiv [q_{kr}, p_{k'r'}] = i\hbar\,\delta_{kk'}\,\delta_{rr'}.$$

all other quantities commuting. This is called second quantization. The first quantization consists of assuming momentum and position operators for a particle, having the same commutation rule with positional coordinates x, y, and z represented by multiplicative operators and momentum by $(-i\hbar\nabla)$. We have seen that in this way the equation $E^2 = m^2c^4 + p^2c^2$ becomes the Klein-Gordon equation. The properties of $\mathcal{H}$ will first be discussed in the light of second quantization. To this end it will be instructive to discuss the properties of the quantum mechanical Hamiltonian of a simple harmonic oscillator.

The operator $\mathcal{H} = \frac{1}{2}(p^2 + \omega^2 q^2)$, where p and q are Hermitian operators such that $(qp-pq) \equiv [q, p] = i\hbar$, suggests the factorization:

$$(p - i\omega q)(p + i\omega q) = p^2 + \omega^2 q^2 + i\omega[p, q].$$

(1.12) or $$AA^* = 2\mathcal{H} + \hbar\omega.$$

~~having written~~ where $A = (p - i\omega q)$ and $A^* = (p + i\omega q)$.

(1.13) Similarly: $$A^*A = 2\mathcal{H} - \hbar\omega.$$

From these two equations:

$$\mathcal{H} = \frac{1}{4}(A^*A + AA^*).$$

and $$(AA^* - A^*A) = [A, A^*] = 2\hbar\omega.$$

Also, multiplying the whole of equation (1.12) on the right by A and of (1.13) on the left by A and subtracting:

$$0 = 2(\mathcal{H}A - A\mathcal{H}) + 2\hbar\omega A.$$

which can be rewritten:

$$\mathcal{H}A = A\mathcal{H} - \hbar\omega A.$$

Similarly operating on the left with A^* on (1.12) and on the right on (1.13),

$$\mathcal{H}A^* = A^*\mathcal{H} + \hbar\omega A^*.$$

Suppose now that $\mathcal{H}$ operating on a wave function ψ_E has an eigenvalue E. Then:

$$\mathcal{H}\psi_E = \frac{1}{4}(A^*A + AA^*)\psi_E = E\psi_E.$$

Now consider the operator $\mathcal{H}A$ operating on the same function:

$$\mathcal{H}A\psi_E = A\mathcal{H}\psi_E - \hbar\omega A\psi_E = AE\psi_E - \hbar\omega A\psi_E = (E-\hbar\omega)A\psi_E.$$

and similarly;

$$\mathcal{H}A^*\psi_E = A^*\mathcal{H}\psi_E + \hbar\omega A^*\psi_E = A^*E\psi_E + \hbar\omega A^*\psi_E = (E+\hbar\omega)A^*\psi_E.$$

These results indicate that if E is an eigenvalue of $\mathcal{H}$ then so is $(E\pm\hbar\omega)$, and that A and A^* are ladder operators for obtaining the corresponding eigenfunctions $(A^*\psi_E)$ and $(A\psi_E)$. $\mathcal{H}$ therefore has eigenvalues differing by units of $\hbar\omega$. Now the energy of a quantum mechanical oscillator must be positive[4]. Then there must exist some eigenfunction of $\mathcal{H}$ such that A operating in it gives zero, for otherwise the energy eigenvalues would not be bounded below: for any eigenvalue E' there would be another, $(E'-\hbar\omega)$. Let ψ_M be the eigenfunction belonging to the lowest eigenvalue E_M. Then

$$A\psi_M = 0,$$

and

$$\mathcal{H}\psi_M = E_M\psi_M = \frac{1}{4}(A^*A + AA^*)\psi_M = \frac{1}{4}A^*(A\psi_M) + \frac{1}{4}AA^*\psi_M$$
$$= \frac{1}{4}A^*(0) + \frac{1}{4}(A^*A + 2\hbar\omega)\psi_M = 0 + \frac{1}{2}\hbar\omega\,\psi_M.$$

---

4 $\mathcal{H} = p^2 + \omega^2 q^2$ where p and q are Hermitian operators. p and q therefore have real eigenvalues and so p^2 and q^2 have positive or zero eigenvalues. Since p and q are non-commuting operators, they cannot both give zero operating on a given wave function. $\mathcal{H}$ can therefore not have an eigenvalue zero.

Therefore $E_M = \frac{1}{2}\hbar\omega$, and the energy eigenvalues are $(N + \frac{1}{2})\hbar\omega$ where N is any positive integer or zero.

Returning to the scalar field the Hamiltonian operator

$$\mathcal{H} = \frac{1}{2}\sum_{k,r}(p_{kr}^2 + \omega_k^2 q_{kr}^2)$$

must have, eigenvalues:

$$E = \sum_{k,r}(N_{k,r} + \tfrac{1}{2})\hbar\omega_k.$$

where for each k and r, N_{kr} may have one of the values 0,1,2,3,...
The energy E may be written:

$$E = \sum_{k,r}(N_{k,r}\hbar\omega_k) + \frac{1}{2}\sum_{k,r}\hbar\omega_k.$$

The second sum is infinite in magnitude, for k has infinitely many values. This difficulty can be eliminated by proper choice of the order of terms in $\mathcal{H}$ before second quantization. Had $\mathcal{H}$ been written in the form:

(1.14) $$\mathcal{H} = \frac{1}{2}\sum_{k,r}(p_{k,r}^2 + \omega_k^2 q_{k,r}^2) = \frac{1}{2}\sum_{k,r}(p_{kr} + i\omega_k q_{kr})(p_{kr} - i\omega_k q_{kr}).$$

and then the p's and q's interpreted as non-commuting operators, the Hamiltonian operator would have had the form:

$$\mathcal{H} = \frac{1}{2}(A^*A) = \frac{1}{2}(p^2 + \omega^2 q^2) + \frac{1}{2}i\omega[q,p]$$

$$= \frac{1}{2}(p^2 + \omega^2 q^2) - \frac{\hbar\omega}{2}$$

Now $\frac{1}{2}(p^2 + \omega^2 q^2)$ still has eigenvalues $(N + \frac{1}{2})\hbar\omega$ so that $\mathcal{H}$ will have eigenvalues:

$$E = \left(N + \frac{1}{2}\right)\hbar\omega - \frac{\hbar\omega}{2} = N\hbar\omega.$$

Two forms which are equivalent when p and q commute may not be equivalent when p and q commute do not commute. The choice of the form of $\mathcal{H}$ prior to quantization is then made in such a way that the result is meaningful. If the form (1.14) is used the energy of the field is a sum of terms of the form $(N_{k\nu}\hbar\omega_k)$, where $\omega_k = \sqrt{\frac{m^2c^4}{\hbar^2} + k^2c^2}$. According to the de Broglie postulate, there is associated with a particle of energy E a wave of angular frequency ω defined by $\omega = E/\hbar$. If then we interpret the energy $(N_{k\nu}\hbar\omega_k)$ as representing $N_{k\nu}$ particles of energy $\hbar\omega_k$, the field energy is a sum over k of such terms.

Instead of a Fourier expansion in trigonometric functions, complex exponentials could as well have been used. This approach has certain advantages which will be demonstrated. Let ϕ be written:

$$\phi = \frac{1}{\ell^{3/2}} \sum_k \left(q_k e^{i\underline{k}.\underline{r}} + q_k^* e^{-i\underline{k}.\underline{r}} \right).$$

where ϕ is the real scalar field function, and where now $\underline{k} = \frac{2\pi}{\ell}(n_x, n_y, n_z)$ is such that n_x, n_y, and n_z take on all negative, as well as positive, values, and zero. The q's are different from those used previously, and are in general complex time dependent quantities. The orthonormality condition now is:

$$\frac{1}{\ell^3}\int_V e^{i\underline{k}.\underline{r}} e^{i\underline{k}'.\underline{r}} \, dV = \delta_{\underline{k},-\underline{k}'}.$$

The field energy, as calculated from $H = T_{44} = \left(\dot{\phi}\frac{\partial L}{\partial\dot{\phi}} - L\right)$ where

$L = -\frac{c^2}{2}\left[(\nabla\phi)^2 - \frac{1}{c^2}\dot{\phi}^2 + \frac{m^2c^2}{\hbar^2}\phi^2\right]$, is:

$$\mathcal{H} = \int_V H\,dV = \frac{1}{2}\int_V \left[\dot{\phi}^2 + c^2(\nabla\phi)^2 + \frac{m^2c^4}{\hbar^2}\right] dV.$$

Substituting the exponential expansion of ϕ:

$$\mathcal{H} = \frac{1}{2\ell^3}\int_V dV\left\{\frac{m^2c^4}{\hbar^2}\sum_{k,k'}(q_k e^{ik\cdot r} + q_k^* e^{-ik\cdot r})(q_{k'}e^{ik'\cdot r} + q_{k'}^* e^{-ik'\cdot r})\right.$$
$$+ c^2\sum_{k,k'}(i\underline{k}q_k e^{ik\cdot r} - i\underline{k}q_k^* e^{-ik\cdot r})(i\underline{k}'q_{k'}e^{ik'\cdot r} - i\underline{k}'q_{k'}^* e^{-ik'\cdot r})$$
$$+ \sum_{k,k'}(\dot{q}_k e^{ik\cdot r} + \dot{q}_k^* e^{-ik\cdot r})(\dot{q}_{k'}e^{ik'\cdot r} + \dot{q}_{k'}^* e^{-ik'\cdot r}).$$

Making use of the orthogonality condition, and recalling that in the sum, for every term in $\underline{k}$ there is now a term in $-\underline{k}$:

$$\mathcal{H} = \frac{1}{2}\sum_k\left(\frac{m^2c^4}{\hbar^2}\right)(q_k q_{-k} + q_k^* q_{-k}^* + 2q_k^* q_k)$$
$$+ c^2k^2(q_k q_{-k} + q_k^* q_{-k}^* + 2q_k^* q_k)$$
$$+ (\dot{q}_k\dot{q}_{-k} + \dot{q}_k^*\dot{q}_{-k}^* + 2\dot{q}_k^*\dot{q}_k).$$

$$(1.15)\quad = \frac{1}{2}\sum_k(\dot{q}_k\dot{q}_{-k} + \dot{q}_k^*\dot{q}_{-k}^* + 2\dot{q}_k^*\dot{q}_k) + \omega_k^2(q_k q_{-k} + q_k^* q_{-k}^* + 2q_k^* q_k).$$

The q_k's and q_k^*'s are not suitable as generalized coordinates since they are not independent. A transformation is sought from the q_k's and q_k^*'s to P_k's and Q_k's such that the latter are canonical variables. In attempting to find such a transformation, an examination of the field equation might be helpful. On substitution of the complex exponential series for ϕ the field equation:

$$\nabla^2 \phi - \frac{1}{c^2}\ddot{\phi} = \frac{m^2c^2}{\hbar^2}\phi$$

becomes:

$$\sum_{k'}\left(-k'^2 q_{k'} e^{i\underline{k}'.\underline{r}} - k'^2 q_{k'}^{*} e^{-i\underline{k}'.\underline{r}} - \frac{1}{c^2}\ddot{q}_{k'} e^{i\underline{k}'.\underline{r}} - \frac{1}{c^2}\ddot{q}_{k'}^{*} e^{-i\underline{k}'.\underline{r}}\right)$$

$$= \frac{m^2c^2}{\hbar^2}\sum_{k'}\left(q_{k'} e^{i\underline{k}'.\underline{r}} + q_{k'}^{*} e^{-i\underline{k}'.\underline{r}}\right).$$

Multiplying both sides of this equation by $e^{-i\underline{k}.\underline{r}}$ and integrating over V, using the orthogonality relation, the equation reduces to:

$$(\ddot{q}_k + \omega_k^2 q_k) = -(\ddot{q}_{-k}^{*} + \omega_k^2 q_{-k}^{*}).$$

Similarly multiplication by $e^{i\underline{k}.\underline{r}}$ and integration gives:

$$(\ddot{q}_k^{*} + \omega_k^2 q_k^{*}) = -(\ddot{q}_{-k} + \omega_k^2 q_{-k}^{*}).$$

These equations do not determine uniquely the time dependence of the q's, there being essentially two equations for the four quantities q_k, q_k^{*}, q_{-k}, q_{-k}^{*}. It is consistent to take the time dependence of q_k to be $e^{-i\omega_k t}$, for if this is done a given term $q_k e^{+i\underline{k}.\underline{r}}$ in the expansion of ϕ represents a single plane wave. If q_k were taken to have for its time dependence a linear combination of $e^{-i\omega_k t}$ and $e^{i\omega_k t}$, a term $q_k e^{i\underline{k}.\underline{r}}$ would be a superposition of two waves propogating in opposite directions. If

$$\dot{q}_k = -i\omega_k q_k.$$

and

$$\dot{q}_k^{*} = +i\omega_k q_k^{*}.$$

Then $\dot{q}_k \dot{q}_{-k} = -\omega_k^2 q_k q_{-k}$ and $\dot{q}_k^* \dot{q}_{-k}^* = -\omega_k^2 q_k^* q_{-k}^*$, and $\dot{q}_k^* \dot{q}_k = \omega_k^2 q_k^* q_k$. Making use of these relationships the Hamiltonian (1.15) becomes:

$$\mathcal{H} = \sum_k 2\omega_k^2 q_k^* q_k .$$

This is reminiscent of the form (1.14) obtained using the trigonometric Fourier series for ϕ from which

$$\mathcal{H} = \frac{1}{2} \sum_{k,r} (p_{kr}^2 + \omega_k^2 q_{kr}^2) = \frac{1}{2} \sum_{k,r} (p_{kr} + i\omega_k q_{kr})(p_{kr} - i\omega_k q_{kr}).$$

$$= \frac{1}{2} \sum_{k,r} A_{kr}^* A_{kr} .$$

where after quantization $(A_{kr}^* \cdot A_{kr})$ has eigenvalues $(2N_{kr} \hbar\omega_k)$ with N_{kr} a positive integer or zero. A reasonable attempt to transform to canonical variables is to set:

$$\sqrt{2}\,\omega_k q_k = \frac{1}{\sqrt{2}} A_k = \frac{1}{\sqrt{2}} (P_k - i\omega_k Q_k). \tag{1.16}$$

Then $\mathcal{H}$ becomes:

$$\mathcal{H} = \sum_k 2\omega_k^2 q_k^* q_k = \frac{1}{2} \sum_k (P_k^2 + \omega_k^2 Q_k^2). \tag{1.17}$$

That the P_k's and Q_k's satisfy the Hamiltonian equations $\dot{P}_k = -\frac{\partial \mathcal{H}}{\partial Q_k}$, and $\dot{Q}_k = \frac{\partial \mathcal{H}}{\partial P_k}$ and are therefore canonical variables follows from $\dot{q}_k = -i\omega_k q_k$, for from equation (1.16) and its complex conjugate:

$$P_k = \omega_k (q_k + q_k^*).$$

$$Q_k = i (q_k - q_k^*).$$

so that: $\dot{P}_k = -i\omega_k^2(q_k - q_k^*) = -\omega_k^2 Q_k$

and: $\dot{Q}_k = \omega_k(q_k + q_k^*) = P_k$.

in agreement with $-\frac{\partial \mathcal{H}}{\partial Q_k} = -\omega_k^2 Q_k$ and $\frac{\partial \mathcal{H}}{\partial P_k} = P_k$, from equation (1.17). Therefore, with a transformation such as equation (1.16), the quantization procedure may be carried out as before.

However, a more elegant statement of the quantization procedure now appears possible. From considering the coefficients in a trigonometric expansion of ϕ as generalized coordinates the eigenvalues of $\mathcal{H}$ for the quantized field are known to be $(\sum_k \hbar\omega_k N_k)$ that is, the eigenvalues of an operator $(\frac{1}{2}\sum_k A_k^* A_k)$, where from equations (1.12) and (1.13):

$$[A_k, A_{k'}^*] = 2\hbar\omega_k \delta_{kk'}.$$

In view of the interpretation of quantized field energy as equivalent to that of a sum of particles having energies $(h\omega_k)$, it is convenient to define a number of particles operator having eigenvalues N_k. Such an operator is $(a_k^* a_k) = \left(\frac{A_k^* A_k}{2\hbar\omega_k}\right)$, where a_k is $\left(\frac{A_k}{\sqrt{2\hbar\omega_k}}\right)$. The Hamiltonian then has the form $\mathcal{H} = \sum_k (a_k^* a_k)\hbar\omega_k$, where $[a_k, a_{k'}^*] = \delta_{kk'}$. Instead of transforming to canonical variables when using the exponential series for ϕ, and then quantizing, and introducing the operators a_k and a_k^*, the Hamiltonian (1.16) can be written directly in the form:

$$\mathcal{H} = \sum_k 2\omega_k^2 q_k^* q_k = \sum_k \hbar\omega_k a_k^* a_k.$$

by the transformation

(1.18) $$q_k = \sqrt{\frac{\hbar}{2\omega_k}} \cdot a_k .$$

Quantization then consists of considering the a_k's and a_k^*'s as operators obeying the commutation rule $[a_k, a_{k'}^*] = \delta_{kk'}$. The energy eigenvalues are then $\left(\sum_k N_k \hbar \omega_k\right)$.

The operators a_k and a_k^* are called annihilation and creation operators respectively, for the following reason. Let ψ_N be such that:

$$a^* a \, \psi_N = N \psi_N .$$

Then by an argument similar to that used in showing the ladder properties of the A's following equation (1.13)

$$a^* a (a \psi_N) = (N-1) \psi_N .$$

This is the mathematical statement that the operator a operating on an eigenfunction of the number of particles operator, reduces the eigenvalue by one in forming a new eigenfunction. Since the eigenvalue N represents the number of particles, a reduces the number of particles by one. Similarly a^* increases N by one.

It is of interest to see whether the definition of field momentum

$$\mathcal{P}_j = \int_V G_j \, dV = \int_V \left(-\partial_j \phi \cdot \frac{\partial L}{\partial \dot{\phi}}\right) dV .$$

agrees with the interpretation that the field energy is equivalent to that of a sum over k of N_k particles, each of energy $\hbar\omega_k$.

For the real scalar field:

$$\nabla\phi \frac{\partial L}{\partial \dot{\phi}} = g_s\left(-\frac{1}{c^2}\right) 2\dot{\phi}\,\nabla\phi = \dot{\phi}\,\nabla\phi .$$

Substitution of the complex exponential series for ϕ leads to:

$$\underline{g} = -\int_V (\nabla\phi \cdot \dot{\phi})\, dV$$

$$= -\frac{1}{c^2}\int \sum_{k,k'} \left(i\underline{k}q_k e^{i\underline{k}.\underline{r}} - i\underline{k}q_k^* e^{-i\underline{k}.\underline{r}}\right)\left(\dot{q}_{k'} e^{i\underline{k}'.\underline{r}} + \dot{q}_{k'}^* e^{-i\underline{k}'.\underline{r}}\right) dV.$$

$$= i \sum \underline{k}\left(q_k\dot{q}_{-k} + q_k\dot{q}_k^* - q_k^*\dot{q}_k - q_k^*\dot{q}_{-k}^*\right).$$

Since $\dot{q}_k$ is taken to be $-i\omega_k q_k$ this reduces to:

$$\underline{g} = -\sum_k \underline{k}\left(\omega_k q_k q_{-k} + \omega_k q_k^* q_{-k}^* - 2\omega_k q_k^* q_k\right).$$

The terms $\omega_k(\underline{k}q_k q_{-k} + \underline{k}q_k^* q_{-k}^*)$, for $\underline{k} = +\underline{k}'$ and $\underline{k} = -\underline{k}'$ cancel in summation over k. The field momentum is then:

$$\underline{g} = 2\sum_k \underline{k}\,\omega_k q_k^* q_k .$$

Substituting from equation (1.18) for q_k^* and q_k:

$$\underline{g} = \sum_k (2\omega_k \underline{k}) \frac{\hbar}{2\omega_k} \cdot a_k^* a_k .$$

$$\underline{g} = \sum_k \hbar\underline{k}\, a_k^* a_k .$$

After quantization, $(a_k^* a_k)$ is an operator having eigenvalues N_k which are positive integers or zero. This supports the interpretation of the quantized field as equivalent to an integral number of quanta, for it says that the field momentum is that of a sum over k of N_k particles having momentum $(+\hbar\underline{k})$. The energy $\hbar\omega_k = \hbar c\sqrt{k^2 + \frac{m^2c^2}{\hbar^2}}$ agrees with the relativistic expression for the energy of a particle having momentum $\underline{p} = (+\hbar\underline{k})$ and rest mass m:

$$\hbar c\sqrt{k^2 + \frac{m^2c^2}{\hbar^2}} = \sqrt{(\hbar\underline{k})^2c^2 + m^2c^4} = \sqrt{p^2c^2 + m^2c^4}$$

Fields obeying the Klein-Gordon equation are called meson fields, and the quanta are called mesons. The theory of the pseudoscalar meson field, whose properties differ from those of the scalar field just discussed only when interactions with other fields are considered, is found to give fair qualitative agreement with π-meson experiments. No quantitatively satisfactory theory for mesons has yet been developed.

CHAPTER II

CHARGED MESONS: THE COMPLEX FIELD.

If a type of particle has two charge states, the field theory should show this property. The scalar field just discussed describes a single type of particle. If some of the particles are charged positively and some negatively a field function having two independent parts is needed: a complex ϕ has such a property. Instead of the real and imaginary parts of ϕ, the functions ϕ and ϕ^* will be used to describe the charged field. If both ϕ and ϕ^* are required to satisfy the Klein-Gordon equation then it will be possible to describe the two types of quanta which appear on second quantization as oppositely charged mesons.

The Lagrangian density

$$L = -c^2\left(\partial_\nu \phi \, \partial_\nu \phi^* + \frac{m^2c^2}{\hbar^2} \phi \phi^* \right).$$

results, as the reader should verify, in the Euler-Lagrange equations for ϕ and ϕ^* both being Klein-Gordon equations. Following the same argument as for the real scalar field, the Hamiltonian density for this field is:

$$H = T_{44} = \left(c^2 \nabla \phi^* . \nabla \phi + \frac{m^2c^4}{\hbar^2} \phi^* \phi + \dot{\phi}^* \dot{\phi} \right).$$

Expanding ϕ and ϕ^* in complex Fourier series:

$$\phi = l^{-3/2} \sum_k \left(q_k e^{i\underline{k}.\underline{r}} + b_k e^{-i\underline{k}.\underline{r}} \right).$$

$$\phi^* = l^{-3/2} \sum_k \left(q_k^* e^{-i\underline{k}.\underline{r}} + b_k^* e^{i\underline{k}.\underline{r}} \right).$$

where q_k and b_k are independent complex functions of time, the Hamiltonian is:

$$\mathcal{H} = \int_V H\,dV = \sum_k \Big\{ \omega_k^2 (q_k q_k^* + q_k b_{-k}^* + b_k q_{-k}^* + b_k b_k^*) + (\dot{q}_k \dot{q}_k^* + \dot{q}_k \dot{b}_{-k}^* + \dot{b}_k \dot{q}_{-k}^* + \dot{b}_k \dot{b}_k^*) \Big\}.$$

For the Klein-Gordon equation for ϕ, substitution of the Fourier expansion and separation of orthogonal terms gives:

$$(\ddot{q}_k + \ddot{b}_{-k}) = -\omega_k^2 (q_k + b_{-k}).$$

and similarly for ϕ^*:

$$(\ddot{q}_k^* + \ddot{b}_{-k}^*) = -\omega_k^2 (q_k^* + b_{-k}^*).$$

Then it is consistent to take:

(2.1)
$$\dot{q}_k = -i\omega_k q_k \, ; \qquad \dot{b}_k = +i\omega_k b_k .$$
$$\dot{q}_k^* = +i\omega_k \dot{q}_k^* \, ; \qquad \dot{b}_k^* = -i\omega_k b_k^* .$$

It then follows that:

$$\dot{q}_k \dot{q}_k^* = \omega_k^2 \dot{q}_k^* q_k \, ; \qquad \dot{q}_k \dot{b}_{-k}^* = -\omega_k^2 q_k b_{-k}^* .$$
$$\dot{b}_k \dot{q}_{-k}^* = -\omega_k^2 q_{-k}^* b_k \, ; \qquad \dot{b}_k \dot{b}_k^* = \omega_k^2 b_k^* b_k .$$

Using these relations the field energy becomes:

$$\mathcal{H} = 2 \sum_k \omega_k^2 (q_k^* q_k + b_k^* b_k).$$

If $a_k^{(+)}$; $a_k^{(+)*}$; $a_k^{(-)}$; $a_k^{(-)*}$ are defined by:

(2.2) $$a_k^{(+)} = \sqrt{\frac{2\omega_k}{\hbar}} \cdot q_k \; ; \qquad a_k^{(-)} = \sqrt{\frac{2\omega_k}{\hbar}} \cdot b_k .$$

then $\mathcal{H}$ takes the form $\mathcal{H} = \sum_{\underline{k}} \hbar\omega_k \left(a_k^{(+)*} a_k^{(+)} + a_k^{(-)*} a_k^{(-)} \right)$,

and the quantization procedure is to consider the $a_k^{(+)}$'s and $a_k^{(-)}$'s as operators obeying the commutation rule:

$$\left[a_k^{(r)} , a_{k'}^{(r')*} \right] = \delta_{kk'} \cdot \delta_{r,r'} .$$

$\mathcal{H}$ as an operator then has eigenvalues:

$$\sum_k \hbar\omega_k \left[N_k^{(+)} + N_k^{(-)} \right].$$

The field momentum is calculated in a similar way.

$$\underline{\mathcal{G}} = \int_V \underline{G}\, dV = \int_V \left(- \nabla\phi \frac{\partial L}{\partial \dot{\phi}} - \nabla\phi^* \frac{\partial L}{\partial \dot{\phi}^*} \right) dV$$

$$= \int_V - \left[(\nabla\phi)\, \dot{\phi}^* + (\nabla\phi^*)\, \dot{\phi} \right] dV$$

$$= -\frac{1}{\ell^3} \int_V \sum_{k,k'} \Big\{ \left(i\underline{k}\, q_k e^{i\underline{k}.\underline{r}} - i\underline{k}\, b_k e^{-i\underline{k}.\underline{r}} \right) \left(\dot{q}_{k'}^* e^{-i\underline{k}'.\underline{r}} + \dot{b}_{k'}^* e^{i\underline{k}'.\underline{r}} \right)$$

$$+ \left(-i\underline{k}\, q_k^* e^{-i\underline{k}.\underline{r}} + i\underline{k}\, b_k^* e^{i\underline{k}.\underline{r}} \right) \left(\dot{q}_{k'} e^{i\underline{k}'.\underline{r}} + \dot{b}_{k'} e^{-i\underline{k}'.\underline{r}} \right) \Big\} dV$$

$$= -\sum_k \Big\{ \left(i\underline{k}\, q_k \dot{q}_k^* + i\underline{k}\, q_k \dot{b}_{-k}^* - i\underline{k}\, b_k \dot{q}_{-k}^* - i\underline{k}\, b_k \dot{b}_k^* \right)$$

$$+ \left(-i\underline{k}\, q_k^* \dot{q}_k - i\underline{k}\, q_k^* \dot{b}_{-k} + i\underline{k}\, b_k^* \dot{q}_{-k} + i\underline{k}\, b_k^* \dot{b}_k \right) \Big\} .$$

Making use of the relations (2.1) for $\dot{q}_k$, $\dot{b}_k$:

$$\underline{\mathcal{G}} = \sum_{\underline{k}} 2\omega_k \underline{k} \Big\{ q_k^* q_k + b_k^* b_k + q_k b_{-k}^* + q_{-k} b_k^*$$

$$+ b_k q_{-k}^* + b_{-k} q_k^* \Big\} .$$

Each of the last two bracketed sets cancel in the summation over $\underline{k}$, in the sense that when $\underline{k} = \underline{k}'$:

$$\omega_k \underline{k}(q_k b_{-k}^* + q_{-k} b_k^*) = \omega_{k'} \underline{k}'(q_{k'} b_{-k'}^* + q_{-k'} b_{k'}^*)$$

and when $\underline{k} = -k'$ it is:

$$\omega_{k'}(-\underline{k}')(q_{-k'} b_{k'}^* + q_{k'} b_{-k'}^*) = -\omega_{k'} \underline{k}'(q_{k'} b_{-k'}^* + q_{-k'} b_{k'}^*).$$

and similarly for the other term. Finally:

$$\underline{\mathcal{G}} = 2 \sum_k \omega_k (q_k^* q_k + b_k^* b_k).$$

Substituting for q_k and b_k from equations (2.2):

$$\underline{\mathcal{G}} = \sum_k \hbar \underline{k} (a_k^{(+)*} a_k^{(+)} + a_k^{(-)*} a_k^{(-)}).$$

which has eigenvalues $\sum_k \hbar \underline{k} (N_k^{(+)} + N_k^{(-)})$.

From these calculations it is evident that for the complex scalar field there are two independent sets of occupation numbers available for every momentum state ($\hbar \underline{k}$).

The conservation of charge for closed physical systems is well-known experimentally, and for a field is expressed by a continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \underline{J} = 0$$

where ρ is charge density and $\underline{J}$ is current density. If interaction of the field in question with the electromagnetic field is not to be used in determining the forms of ρ and $\underline{J}$, the relationship between conservation laws and invariance properties may be helpful, as previously was the case for energy-momentum conservation. In Schrodinger quantum mechanics, the content of the theory is embodied in expressions of the form $\psi^*(\text{Operator})\psi$, and such expressions are unchanged by a transformation $\psi' = \psi . e^{i\alpha}$, where α is a real constant: the wave function is said to be arbitrary up to a constant factor of modulus unity. Since the Klein-Gordon equation is a relativistic extension of the Schrodinger equation, it is reasonable to postulate that the content of the field theory will be invariant under such transformations. This postulate will be formulated as follows: For a transformation

$$e^{i\alpha} = 1 + \frac{i}{1!}\alpha + \frac{i^2}{2!}\alpha^2 + \ldots$$

let α be an infinitesimal. Any transformation $e^{i\beta}$ can be considered as a succession of infinitesimal transformations. Then:

$$\phi' = \phi e^{i\alpha} \approx \phi(1 + i\alpha).$$

This transformation changes the form of L. Let

$$\begin{aligned} &L(\phi', \partial_\nu \phi', \phi^{*\prime}, \partial_\nu \phi^{*\prime}) \\ &= L(\phi, \partial_\nu \phi, \phi^*, \partial_\nu \phi^*) + \delta L(\phi, \partial_\nu \phi, \phi^*, \partial_\nu \phi^*). \end{aligned}$$

Since δL is the change in L resulting from the transformation, the postulate is that $\delta L = 0$, for in the field theory L embodies the physical content, and so should have such invariance. If δL can be written as a four divergence $\partial_\nu J_\nu$, then $\delta L = 0$ will constitute a continuity equation $\partial_\nu J_\nu = 0$, which might apply to charge conservation. The four vector J_ν would be identified with ($\underline{J}$, i cρ). Evaluating the variation of L:

$$\delta L = \frac{\partial L}{\partial \phi}\,\delta\phi + \frac{\partial L}{\partial \phi^*}\,\delta\phi^* + \frac{\partial L}{\partial(\partial_\nu \phi)}\,\delta(\partial_\nu\phi) + \frac{\partial L}{\partial(\partial_\nu \phi^*)}\cdot\delta(\partial_\nu\phi^*).$$

Now $\delta\phi = (\phi' - \phi) = (1 + i\alpha)\phi - \phi = i\alpha\phi,$

and similarly since $(\phi')^* = (1 - i\alpha)\phi^*$, it follows that

$\delta\phi^* = -i\alpha\phi^*$; also $\delta(\partial_\nu\phi) = \partial_\nu(\delta\phi) = i\alpha\,\partial_\nu\phi.$

and $\delta(\partial_\nu\phi^*) = -i\alpha\,\partial_\nu\phi^*$. Then using these relationships

$$\delta L = i\alpha\left\{\frac{\partial L}{\partial\phi}\phi - \frac{\partial L}{\partial\phi^*}\phi^* + \frac{\partial L}{\partial(\partial_\nu\phi)}\cdot\partial_\nu\phi - \frac{\partial L}{\partial(\partial_\nu\phi^*)}\cdot\partial_\nu\phi^*\right\}$$

Substituting for $\frac{\partial L}{\partial\phi}$ and $\frac{\partial L}{\partial\phi^*}$ from the Euler-Lagrange equations:

$$\delta L = i\alpha\left\{\phi\,\partial_\nu\left[\frac{\partial L}{\partial(\partial_\nu\phi)}\right] - \phi^*\,\partial_\nu\left[\frac{\partial L}{\partial(\partial_\nu\phi^*)}\right] + \frac{\partial L}{\partial(\partial_\nu\phi)}\,\partial_\nu\phi - \frac{\partial L}{\partial(\partial_\nu\phi^*)}\cdot\partial_\nu\phi^*\right\}$$

$$= \partial_\nu\left\{i\alpha\left[\phi\frac{\partial L}{\partial(\partial_\nu\phi)} - \phi^*\frac{\partial L}{\partial(\partial_\nu\phi^*)}\right]\right\}$$

Having found δL equal to a four divergence, $J_\nu \equiv$ ($\underline{J}$, icρ) is taken to be proportional to

$$\left\{\phi\frac{\partial L}{\partial(\partial_\nu\phi)} - \phi^*\frac{\partial L}{\partial(\partial_\nu\phi^*)}\right\}.$$

That J_ν actually represents the four-current density can only be seen by considering the system of the electromagnetic field and the Klein-Gordon field in interaction.[5] The field equations for

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[5] See for example Schweber, et. al., p. 118.

such a system then include the Maxwell equations, with an expression of the form just given for J_ν occurring in the position of the four-current density in Maxwell's equations. We conclude therefore that the charge density ρ is given by:

$$\rho = \frac{J_4}{ic} = i\varepsilon\left\{ \phi \frac{\partial L}{\partial \dot{\phi}} - \phi^* \frac{\partial L}{\partial \dot{\phi}^*}\right\}.$$

where ε is some real constant.

The total charge for the field can now be calculated using the complex Fourier expansions for ϕ and ϕ^*.

$$Q = \int_V \rho\, dV = i\varepsilon \int_V (\phi \dot{\phi}^* - \phi^* \dot{\phi})\, dV$$

$$= \frac{i\varepsilon}{\ell^3}\int_V \sum_{\underset{\sim}{k},\underset{\sim}{k}'} \Big\{ (q_k e^{i\underset{\sim}{k}.\underset{\sim}{r}} + b_k e^{-i\underset{\sim}{k}.\underset{\sim}{r}})(\dot{q}_{k'}^* e^{-i\underset{\sim}{k}'.\underset{\sim}{r}} + \dot{b}_{k'}^* e^{i\underset{\sim}{k}'.\underset{\sim}{r}})$$

$$- (q_k^* e^{-i\underset{\sim}{k}.\underset{\sim}{r}} + b_k^* e^{i\underset{\sim}{k}.\underset{\sim}{r}})(\dot{q}_{k'} e^{i\underset{\sim}{k}'.\underset{\sim}{r}} + \dot{b}_{k'} e^{-i\underset{\sim}{k}'.\underset{\sim}{r}})\Big\}\, dV$$

$$= i\varepsilon \sum_{\underset{\sim}{k}} \Big\{ (q_k \dot{q}_k^* + q_k \dot{b}_{-k}^* + b_k \dot{q}_{-k}^* + b_k \dot{b}_k^*)$$

$$- (q_k^* \dot{q}_k + q_k^* \dot{b}_{-k} + b_k^* \dot{q}_{-k} + b_k^* \dot{b}_k)\Big\}$$

which reduces, using equations (2.1), to:

$$Q = -2\varepsilon \sum_{\underset{\sim}{k}} \omega_k \{ q_k^* q_k - b_k^* b_k \} + \varepsilon \sum_{\underset{\sim}{k}} \omega_k (q_k b_{-k}^* - b_k^* q_{-k})$$

$$+ \varepsilon \sum_{\underset{\sim}{k}} \omega_k (q_k^* b_{-k} - b_k q_{-k}^*).$$

On summation over k each of the last two sums is zero, and Q is:

$$Q = -2\varepsilon \sum_{\underset{\sim}{k}} \omega_k (q_k^* q_k - b_k^* b_k).$$

Transforming to $a_k^{(+)}$ and $a_k^{(-)}$ according to equations (2.2):

$$Q = (-\hbar\varepsilon) \sum_{\underline{k}} \left[a_k^{(+)*} a_k^{(+)} - a_k^{(-)*} a_k^{(-)} \right]$$

If the particles described by the field are to have charges $\pm$e where e is a positive number representing the charge on a proton, and if $\left(a_k^{(+)*} a_k^{(+)}\right)$ is to have eigenvalues which are the occupation numbers for positively charged particles in the momentum state ($+\hbar\underline{k}$) then the number ε must be:

$$\varepsilon = \left(-\frac{e}{\hbar}\right).$$

This amounts to an experimental determination of the constant ε for the total charge of the field quanta is the difference of two charges, $\left[(-\varepsilon\hbar)\sum_{k} N_k^{(+)}\right]$, and $\left[(-\varepsilon\hbar)\sum_{k} N_k^{(-)}\right]$, and experimentally for a system of oppositely charged particles the charge is e times the difference between the total number of positively charged particles and that of negatively charged particles. It is therefore possible to identify one set of occupation numbers for the complex field with positively charged quanta and the other with negatively charged quanta.

CHAPTER III

THE VECTOR MESON FIELD AND THE CONCEPT OF SPIN.

The Vector Field.

Fields described by real and complex scalar field functions have been discussed as an illustration of the appearance in field theory of quanta having zero, positive, or negative charge. No particle has yet been found to correspond to the theoretical quanta of the vector meson field. Nevertheless, it is fruitful to investigate this field in detail, for it is related to the electromagnetic field whose properties are of such great importance. The vector meson theory, having quanta of non-zero rest mass, allows a simpler discussion of field angular momentum, or of the angular momenta of associated quanta, than does the electromagnetic field theory.

The vector meson field is described by four quantities, ϕ_1, ϕ_2, ϕ_3, and ϕ_4, which form a four-vector. Uncharged quanta will be assumed for simplicity, so that ϕ_j ($j = 1,2,3$) will be real, and ϕ_4 will be pure imaginary, as for the four-vector of space-time position x_1, x_2 and x_3 have been taken as real, and x_4 as ict. Each component of the field will satisfy the Klein-Gordon equation:

$$\partial_\nu \partial_\nu \phi_\lambda - \frac{m^2c^2}{\hbar^2} \phi_\lambda = 0 \quad ; \qquad \lambda = 1,2,3,4.$$

If we anticipate that the field quanta will have mass m, as has occurred for the scalar field, we see that a field having quanta with rest mass zero would obey the equations:

$$\partial_\nu \partial_\nu \phi_\lambda = 0 ; \quad \lambda = 1, 2, 3, 4.$$

Then ϕ_λ could be identified with the four vector potential $A_\lambda \equiv (\underline{A}, i\phi)$ of the electromagnetic field, for these are precisely the wave equations:

$$\nabla^2 \phi - \frac{1}{c^2} \ddot{\phi} = 0.$$
$$\nabla^2 \underline{A} - \frac{1}{c^2} \ddot{\underline{A}} = 0.$$

which follow from Maxwell's equations in the absence of charges or currents. For the vector meson field, the Lagrangian density

$$L = g_v[(\partial_\nu \phi_\lambda - \partial_\lambda \phi_\nu)(\partial_\nu \phi_\lambda - \partial_\lambda \phi_\nu) + \frac{2m^2c^2}{\hbar^2} \phi_\nu \phi_\nu].$$

leads to the Klein-Gordon equation, for:

$$\frac{\partial L}{\partial \phi_\nu} = 4g_v \frac{m^2 c^2}{\hbar^2} \phi_\nu. \qquad \text{and:}$$

$$\partial_{\nu'} \left[\frac{\partial L}{\partial(\partial_{\nu'} \phi_{\lambda'})} \right] = 2g_v \, \partial_{\nu'} \left\{ (\partial_\nu \phi_\lambda - \partial_\lambda \phi_\nu)(\delta_{\nu\nu'} \delta_{\lambda\lambda'} - \delta_{\nu'\lambda} \delta_{\lambda'\nu}) \right\}$$

$$(3.1) \qquad = 4g_v \, \partial_{\nu'} (\partial_{\nu'} \phi_{\lambda'} - \partial_{\lambda'} \phi_{\nu'})$$

$$(3.2) \qquad = 4g_v \, \partial_\nu \partial_\nu \phi_{\lambda'} - 4g_v \, \partial_{\lambda'} (\partial_\nu \phi_\nu).$$

This does not appear to give the Klein-Gordon equation unless $\partial_{\lambda'} (\partial_\nu \phi_\nu) = 0$. The equation $\partial_\nu \phi_\nu = 0$ is the same as the

Lorentz condition:

$$\nabla . \underline{A} + \frac{1}{c} \dot{\phi} = \partial_\nu \phi_\nu = 0.$$

in electromagnetic theory. For the vector field this condition is a consequence of the Euler-Lagrange equations, for if the latter are written, from equation (3.1) as:

$$\frac{m^2c^2}{\hbar^2} \phi_\lambda = \partial_\nu (\partial_\nu \phi_\lambda - \partial_\lambda \phi_\nu),$$

then taking the four divergence:

$$\frac{m^2c^2}{\hbar^2} \partial_\lambda \phi_\lambda = \partial_\lambda \partial_\nu \partial_\nu \phi_\lambda - \partial_\nu \partial_\lambda \partial_\lambda \phi_\nu .$$

The right hand side of this equation is zero since λ and ν, being dummy indices indicating the same summation, can have their rôles interchanged. It follows that if m is not zero, then $\partial_\nu \phi_\nu = 0$. The Euler-Lagrange equations are then, from equation (3.2), Klein-Gordon equations.

The field energy for the vector field is calculated in the usual way. Let:

$$\underline{\phi} = \frac{1}{\ell^{3/2}} \sum_{\underline{k}} (\underline{q}_{\underline{k}} e^{i\underline{k}.\underline{r}} + \underline{q}_{\underline{k}}^* e^{-i\underline{k}.\underline{r}})$$

and

$$\phi_4 = \frac{i}{\ell^{3/2}} \sum_{\underline{k}} (b_k e^{i\underline{k}.\underline{r}} + b_k^* e^{-i\underline{k}.\underline{r}}).$$

In three-dimensional vector notation:

$$L = 2g_v \left[(\nabla \times \underline{\phi})^2 + \frac{m^2c^2}{\hbar^2} \phi^2 - \frac{1}{c^2}(ic\nabla\phi_4 - \dot{\underline{\phi}})^2 + \frac{m^2c^2}{\hbar^2} \phi_4^2 \right]$$

Then $\dfrac{\partial L}{\partial \dot{\phi}_4} = 0$ and $\dfrac{\partial L}{\partial \dot{\phi}_j} = \dfrac{4 g_v}{c^2}(ic\,\partial_j \phi_4 - \dot{\phi}_j)$.

and $$\mathcal{H} = \left(\dot{\phi}_\nu \frac{\partial L}{\partial \dot{\phi}_\nu} - L \right)$$

$$= \frac{2 g_v}{c^2}\left\{ c^2(\nabla\phi_4)^2 + \dot{\underline{\phi}}^2 + c^2(\nabla \times \underline{\phi})^2 + \frac{m^2c^4}{\hbar^2}(\underline{\phi}^2 + \phi_4^2)\right\}.$$

In evaluating $\mathcal{H}$ the term $\int_V (\nabla \times \underline{\phi})^2\, dV$ merits special attention. Using the relation:

$$\nabla \times \underline{q}_k e^{i\underline{k}.\underline{r}} = -\underline{q}_k \times \nabla e^{i\underline{k}.\underline{r}} = i\underline{k} \times \underline{q}_k e^{i\underline{k}.\underline{r}}.$$

$$(\nabla \times \underline{\phi})^2 = i^2 \sum_{\underline{k},\underline{k}'} (\underline{k}\times\underline{q}_k e^{i\underline{k}.\underline{r}} - \underline{k}\times\underline{q}_k^{*} e^{-i\underline{k}.\underline{r}}).$$
$$.(\underline{k}'\times\underline{q}_{k'} e^{i\underline{k}'.\underline{r}} - \underline{k}'\times\underline{q}_{k'}^{*} e^{-i\underline{k}'.\underline{r}}).$$

If $\underline{q}_k$ is referred to axes which are right handed and orthogonal and whose third direction coincides with the direction of $\underline{k}$, then $(\underline{k} \times \underline{q}_k)$ has no third component. Dot products of the form $(\underline{k} \times \underline{q}_k).(\underline{k}' \times \underline{q}_{k'})\, e^{i(\underline{k}+\underline{k}').\underline{r}}$ on integration involve the dot product of two vectors in directions transverse to $\underline{k}$, and are therefore expressible in terms of transverse components: that is as a sum over two components only.

$$\int_V (\nabla\times\underline{\phi})^2 dV = i^2 \sum_{\substack{k \\ r=1,2}} k^2(-q_{kr}q_{-kr} - q_{kr}q_{kr}^{*} - q_{kr}^{*}q_{kr} - q_{kr}^{*}q_{-kr}^{*})$$

Evaluation of all the other terms in $\mathcal{H}$ is straightforward, and gives:

$$(3.3)\quad \mathcal{H} = -\frac{2q_v}{c^2}\sum_{\underline{k}}\Big\{ k^2c^2(-b_k b_{-k} - b_k b_k^* - b_k^* b_k - b_k^* b_{-k}^*)$$

$$+(\dot{\underline{q}}_k\cdot\dot{\underline{q}}_{-k} + \dot{\underline{q}}_k\cdot\dot{\underline{q}}_k^* + \dot{\underline{q}}_k^*\cdot\dot{\underline{q}}_k + \dot{\underline{q}}_k^*\cdot\dot{\underline{q}}_{-k}^*)$$

$$+\sum_{r=1,2} k^2c^2(q_{kr}\,q_{-kr} + q_{kr}\,q_{kr}^* + q_{kr}^*\,q_{kr} + q_{kr}^*\,q_{-kr}^*)$$

$$+\frac{m^2c^4}{\hbar^2}(\underline{q}_k\cdot\underline{q}_{-k} + \underline{q}_k^*\cdot\underline{q}_k + \underline{q}_k\cdot\underline{q}_k^* + \underline{q}_k^*\cdot\underline{q}_{-k}^*)$$

$$-\frac{m^2c^4}{\hbar^2}(b_k b_{-k} + b_k^* b_k + b_k b_k^* + b_k^* b_{-k}^*)\Big\}$$

Since $\underline{\phi}$ and ϕ_4 satisfy the Klein-Gordon equation it can be shown as for the scalar field that the equations:

$$(3.4)\quad \begin{aligned} \dot{\underline{q}}_k &= -i\omega_k \underline{q}_k\,; & \dot{b}_k &= -i\omega_k b_k. \\ \dot{\underline{q}}_k^* &= i\omega_k \underline{q}_k^*\,; & \dot{b}_k^* &= i\omega_k b_k^*. \end{aligned}$$

are consistent. The further condition $\partial_\nu \phi_\nu = 0$ following from the field equations allows elimination of one of the field functions. Substitution of the Fourier series for ϕ_ν into:

$$\partial_\nu \phi_\nu = \nabla\cdot\underline{\phi} + \frac{1}{ic}\dot{\phi}_4 = 0.$$

and separation of orthogonal terms gives:

$$(3.5)\quad i\underline{k}.(\underline{q}_k + \underline{q}_{-k}^*) + \frac{1}{c}(\dot{b}_k + \dot{b}_{-k}^*) = 0.$$

If $\underline{q}_k$ are vectors relative to axes whose third direction coincides with $\underline{k}$ then $\underline{k}.\underline{q}_k = kq_{k3}$. Equation (3.5) may be written consistently as: $\dot{b}_k = -ick\, q_{k3}$.

and $\dot{b}_k^* = +ick\, q_{k3}^*$.

which from equations (3.4) give:

(3.6) $$b_k = \frac{ck}{\omega_k} q_{k3} \quad ; \qquad b_k^* = \frac{ck}{\omega_k} q_{k3}^*.$$

Eliminating b_k and b_k^* from the Hamiltonian (3.3) and using equations (3.4):

$$\mathcal{H} = -\frac{2g_v}{c^2} \sum_{k} \Big\{ -c^2k^2\left(2q_{k3}^* q_{k3} - q_{k3}q_{-k3} - q_{k3}^* q_{-k3}^*\right)$$

$$+\left(-\omega_k^2 + \frac{m^2c^4}{\hbar^2}\right)\left(\mathbf{q}_k \cdot \mathbf{q}_{-k} + \mathbf{q}_k^* \cdot \mathbf{q}_{-k}^*\right)$$

$$+\left(\omega_k^2 + \frac{m^2c^4}{\hbar^2}\right)\left(2\,\mathbf{q}_k^* \cdot \mathbf{q}_k\right)$$

$$+ c^2k^2 \sum_{r=1,2} \left(2q_{kr}^* q_{kr} + q_{kr}q_{-kr} + q_{kr}^* q_{-kr}^*\right) \Big\}$$

$$= -\frac{2g_v}{c^2} \sum_{\substack{k \\ r=1,2}} 4\omega_k^2 q_{kr}^* q_{kr} + \left(\omega_k^2 + \frac{m^2c^4}{\hbar^2} - k^2c^2\right) 2q_{k3}^* q_{k3}.$$

Finally $\mathcal{H}$ has the form:

$$\mathcal{H} = 4 \sum_{\substack{k \\ r=1,2}} \left(\omega_k^2 q_{kr}^* q_{kr} + \frac{m^2c^4}{\hbar^2} q_{k3}^* q_{k3}\right).$$

where g_v has been set equal to $(-\frac{1}{2}c^2)$. If a_k's are defined by:

$$a_{kr} = 2\sqrt{\frac{\omega_k}{\hbar}} \cdot q_{kr} \; ; \; r = 1,2 \; ; \quad a_{k3} = 2\frac{mc^2}{\hbar} \cdot \frac{1}{\sqrt{\hbar\omega_k}} \cdot q_{k3}.$$

$$[a_{ks}, a_{k's'}^*] = \delta_{kk'}\delta_{ss'}.$$

Then $\mathcal{H} = \sum_{\substack{k \\ r=1,2,3}} \hbar \omega_k a_{kr}^* a_{kr}$ has eigenvalues $\sum_k \hbar \omega_k \left(\sum_{r=1}^{3} N_{kr} \right)$.

Similarly the calculation of field momentum gives:

$$\mathcal{P} = \sum_{\underline{k}} \hbar \underline{k} \left(\sum_{r=1}^{3} (a_{kr}^* a_{kr}) \right).$$

The vector meson field is therefore seen to be described by three independent sets of occupation numbers.

Field Angular Momentum.

Energy and momentum conservation have been shown to follow from invariance of a field under time and space translations. As in the classical mechanics of particles[6], angular momentum conservation for fields should follow from invariance under rotations of axes, that is, from Lorentz invariance. It will be shown that a conservation law does follow from Lorentz invariance and that the conserved quantity in the case of a scalar field agrees with the conventional definition of angular momentum $\underline{\mathcal{M}}$.

$$\mathcal{M}_{ij} = \int_V M_{ij}\, dV = \int_V (x_i G_j - x_j G_i)\, dV.$$

where i and j are two of 1, 2, 3 in cyclic permutations. That $\underline{\mathcal{M}}$ as defined through its density by $(\underline{r} \times \underline{G})$ is not conserved for the vector field will be shown, but should not be too surprising when one realizes that under rotations a scalar ϕ is unchanged,

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[6] See Goldstein, p.p. 258-263

whereas a vector ϕ_J is changed, so that the scalar field angular momentum is likely an especially simple case of a more general angular momentum quantity. This general angular momentum will be shown to consist of the orbital angular momentum $\underline{\mathcal{M}}$ plus a further term $\underline{\mathcal{S}}$. $\underline{\mathcal{S}}$ is called the spin, or intrinsic angular momentum, for it will be shown that in the quantized theory a particle having no linear momentum, and hence no orbital angular momentum still has a spin angular momentum.

Before considering the consequences of Lorentz invariance, we shall investigate the condition for orbital angular momentum conservation.

$$M_{ij} = (x_i G_j - x_j G_i) = -\frac{1}{ic}(x_i T_{j4} - x_j T_{i4}).$$

If this definition is generalized to

$$M_{\alpha\beta\gamma} = -\frac{1}{ic}(x_\alpha T_{\beta\gamma} - x_\beta T_{\alpha\gamma}). \tag{3.7}$$

then orbital angular momentum conservation requires

$$\partial_\gamma M_{\alpha\beta\gamma} = 0.$$

Now:

$$\partial_\gamma M_{\alpha\beta\gamma} = -\frac{1}{ic}(x_\alpha \partial_\gamma T_{\beta\gamma} - x_\beta \partial_\gamma T_{\alpha\gamma} + T_{\beta\gamma}\delta_{\gamma\alpha} - T_{\alpha\gamma}\delta_{\gamma\beta}).$$

and since $\partial_\gamma T_{\nu\gamma} = 0$ for energy-momentum conservation:

$$\partial_\gamma M_{\alpha\beta\gamma} = \frac{1}{ic}(T_{\alpha\beta} - T_{\beta\alpha}). \tag{3.8}$$

If the energy-momentum tensor is symmetric, orbital angular momentum will be conserved, otherwise not. The reader should verify that for the scalar field $T_{\beta\alpha} = T_{\alpha\beta}$, but that for the vector field $T_{\alpha\beta}$ is not symmetric, and so $\underline{\mathcal{M}}$ is not a constant of the motion. For the vector field $(T_{\beta\alpha} - T_{\alpha\beta})$ may be set equal to a four-divergence, $\partial_\gamma(\phi_\beta \partial_\alpha \phi_\gamma - \phi_\alpha \partial_\beta \phi_\gamma)$, as the reader should verify. The quantity $(\phi_\beta \partial_\alpha \phi_\gamma - \phi_\alpha \partial_\beta \phi_\gamma)$ may be used as a density $\underline{S}'$ and integrated over V using the exponential Fourier series to give a quantity $\underline{\mathcal{S}}'$. On quantization, however, the components of $\underline{\mathcal{S}}'$ do not have the quantum mechanical angular momentum commutation rule $[\mathcal{S}_i', \mathcal{S}_j'] = i\hbar \mathcal{S}_k'$; (i, j, k) = (1,23) or cyclic permutations. $\underline{\mathcal{S}}'$ is therefore not spin.

The general expression for angular momentum density will now be deduced. Since L is assumed invariant under Lorentz transformation, the variation of L under such transformation will be calculated[7], excluding space-time translations which have been discussed in considering energy-momentum conservation. Equating this variation to zero will then lead to a continuity equation for angular momentum.

If an infinitesimal Lorentz transformation is described by:

$$x_\mu' = a_{\mu\nu} x_\nu .$$

[7] See Appendix by Jauch in Wentzel, p. 218.

then for the vector field:

$$\phi'_\mu = a_{\mu\nu}\,\phi_\mu.$$

where the $a_{\mu\nu}$ differ infinitesimally from unity for $\mu = \nu$, and infinitesimally from zero for $\mu \neq \nu$. The variation may be considered as a sum of two variations, the first of which produces a translation of the field, and the second a rotation about the point of observation. We have seen in considering the energy-momentum conservation, that the variation due to translation is zero. In requiring δL to be zero, it is then sufficient to consider the variation produced by rotation at the point of observation as zero. In this case for the four-vectors x_μ and ϕ_μ, the variation is:

$$\delta x_\mu = x'_\mu - x_\mu = (a_{\mu\nu} - \delta_{\mu\nu})\,x_\nu = w_{\mu\nu}\,x_\nu$$

and similarly $\quad \delta\phi_\mu = w_{\mu\nu}\,\phi_\nu.$

where $\delta_{\mu\nu}$ is the Kronecker delta and $(a_{\mu\nu} - \delta_{\mu\nu})$ has been written $w_{\mu\nu}$. The variation of L can now be evaluated at the fixed point under rotation:

$$(3.9)\quad \delta L = \frac{\partial L}{\partial \phi_\mu}.\delta\phi_\mu + \frac{\partial L}{\partial(\partial_\sigma \phi_\mu)}\,\delta(\partial_\sigma\phi_\mu).$$

assuming L does not depend explicitly on the x_ν's.

$$(3.10)\quad \text{Now: } \delta(\partial_\sigma\phi_\mu) = \partial_\sigma(\delta\phi_\mu) + (\delta\partial_\sigma)\,\phi_\mu.$$

$$(3.11)\quad \text{And: } (\delta\partial_\sigma) = \partial'_\sigma - \partial_\sigma = \frac{\partial}{\partial x'_\sigma} - \frac{\partial}{\partial x_\sigma} = \frac{\partial}{\partial x_\rho}.\frac{\partial x_\rho}{\partial x'_\sigma} - \frac{\partial}{\partial x_\sigma} = \frac{\partial}{\partial x_\rho}\left(\frac{\partial x_\rho}{\partial x'_\sigma} - \delta_{\rho\sigma}\right).$$

For the Lorentz transformation $x'_\mu = a_{\mu\nu} x_\nu$ the inverse transformation is $x_\nu = a_{\mu\nu} x'_\mu$, so that

$$\frac{\partial x_\rho}{\partial x'_\sigma} = a_{\sigma\rho}. \tag{3.12}$$

Combining equations (3.9), (3.10), (3.11), and (3.12), and using the fact that $\delta\phi_\mu = w_{\mu\nu}\phi_\nu$:

$$\delta L = \frac{\partial L}{\partial \phi_\mu}(w_{\mu\rho}\phi_\rho) + \frac{\partial L}{\partial(\partial_\sigma \phi_\mu)}\left[w_{\mu\rho}\partial_\sigma \phi_\rho + w_{\sigma\rho}\partial_\rho \phi_\mu\right].$$

Substituting from the Euler-Lagrange equations for $\frac{\partial L}{\partial \phi_\mu}$:

$$\delta L = w_{\mu\rho}\partial_\sigma\left[\frac{\partial L}{\partial(\partial_\sigma \phi_\mu)}\phi_\rho\right] + w_{\sigma\rho}(\partial_\rho \phi_\mu)\frac{\partial L}{\partial(\partial_\sigma \phi_\mu)}$$

In the last term, interchanging the rôles of σ and μ,

$$\delta L = w_{\mu\rho}\left\{\partial_\sigma\left[\frac{\partial L}{\partial(\partial_\sigma \phi_\mu)}\phi_\rho\right] + (\partial_\rho \phi_\sigma)\frac{\partial L}{\partial(\partial_\mu \phi_\sigma)}\right\} = 0. \tag{3.13}$$

This is not a continuity equation such as was obtained from δL under the transformation $\psi' = e^{i\alpha}\psi$ in discussing charge. It is however a condition on L which leads to a continuity equation. First $w_{\mu\rho}$ must be eliminated. For the Lorentz transformation, the orthogonality condition:

$$a_{\mu\nu}a_{\lambda\nu} = \delta_{\mu\lambda}.$$

in terms of the $w_{\mu\nu}$'s may be written:

$$(w_{\mu\nu} + \delta_{\mu\nu})(w_{\lambda\nu} + \delta_{\lambda\nu}) = \delta_{\mu\lambda}.$$

Since the $w_{\mu\nu}$'s are infinitesimals, $(w_{\mu\nu}w_{\lambda\nu})$ is an infinitesimal of higher order than the other terms and may be dropped.

Then:

$$w_{\mu\lambda} = -w_{\lambda\mu}.$$

The $w_{\mu\rho}$ are arbitrary except for their property of being infinitesimals, and this anti-symmetry property. Suppose $w_{mn} = -w_{nm} = \varepsilon$, and $w_{\mu\nu} = 0$ if (μ,ν) is not (m,n). Then in summation

$$w_{\mu\rho}\Gamma_{\mu\rho} = \varepsilon\left(\Gamma_{mn} - \Gamma_{nm}\right) = 0$$

Since ε is not zero, $\Gamma_{mn} = \Gamma_{nm}$ for every element of $\Gamma_{\mu\nu}$: the latter is a symmetrical tensor. Applying this reasoning to equation (3.13) which has the form: $w_{\mu\rho}\Gamma_{\mu\rho} = 0$, writing $\Gamma_{\mu\rho} = \Gamma_{\rho\mu}$:

$$\partial_\sigma\left[\frac{\partial L}{\partial(\partial_\sigma\phi_\mu)}\phi_\rho\right] + (\partial_\rho\phi_\sigma)\frac{\partial L}{\partial(\partial_\mu\phi_\sigma)} = \partial_\sigma\left[\frac{\partial L}{\partial(\partial_\sigma\phi_\rho)}\phi_\mu\right] + (\partial_\mu\phi_\sigma)\frac{\partial L}{\partial(\partial_\rho\phi_\sigma)}$$

which is:

$$\partial_\sigma\left\{\frac{\partial L}{\partial(\partial_\sigma\phi_\mu)}\phi_\rho - \frac{\partial L}{\partial(\partial_\sigma\phi_\rho)}\phi_\mu\right\} = \left\{(\partial_\mu\phi_\sigma)\frac{\partial L}{\partial(\partial_\rho\phi_\sigma)} - (\partial_\rho\phi_\sigma)\frac{\partial L}{\partial(\partial_\mu\phi_\sigma)}\right\} \tag{3.14}$$

From the definition $T_{\mu\rho} = \left\{(\partial_\mu\phi_\sigma)\frac{\partial L}{\partial(\partial_\rho\phi_\sigma)} - L\,\delta_{\mu\rho}\right\}$, the right hand side of equation (3.14) is $(T_{\mu\rho} - T_{\rho\mu})$ which in turn is equal to $(ic\,\partial_\gamma M_{\mu\rho\gamma})$ from equation (3.8). Using the definition (3.7) for $M_{\mu\rho\gamma}$, equation (3.14) becomes:

$$\partial_\sigma\left\{\frac{\partial L}{\partial(\partial_\sigma\phi_\mu)}.\phi_\rho - \frac{\partial L}{\partial(\partial_\sigma\phi_\rho)}\phi_\mu\right\} = -\frac{\partial_\sigma}{ic}\left\{ic\,x_\mu T_{\rho\sigma} - ic\,x_\rho T_{\mu\sigma}\right\}$$

which has the form:

$$\partial_\sigma\left\{M_{\mu\rho\sigma} + S_{\mu\rho\sigma}\right\} = 0.$$

where $S_{\mu\rho\sigma}$ is called the spin tensor, and is defined by:

$$S_{\mu\rho\sigma} = \frac{1}{ic}\left\{ \phi_\mu \frac{\partial L}{\partial(\partial_\sigma \phi_\rho)} - \phi_\rho \frac{\partial L}{\partial(\partial_\sigma \phi_\mu)} \right\}$$

The total angular momentum, consisting of orbital plus spin angular momentum, is therefore conserved for the vector field.

In order to investigate the spin of the vector meson the field spin must be calculated and the theory quantized. For the field:

(3.15) $\underline{\mathcal{S}} = \int_V \underline{S}\, dV$ where $S_{ij4} = \left\{ \phi_i \frac{\partial L}{\partial(\dot{\phi}_j)} - \phi_j \frac{\partial L}{\partial(\dot{\phi}_i)} \right\}$.

Using the vector field Lagrangian density:

$$S_{ij4} = \phi_j [2(ic\,\partial_i \phi_4 - \dot{\phi}_i)] - \phi_i [2(ic\,\partial_j \phi_4 - \dot{\phi}_j)]$$

Then: $\underline{S} = 2\,\underline{\phi} \times (\underline{\dot{\phi}} - ic\nabla \phi_4)$.

and substituting for $\underline{\phi}$ and ϕ_4:

$$\underline{\mathcal{S}} = \frac{2}{\ell^{3/2}} \int_V dV \sum_{k,k'} (\underline{q}_k e^{i\underline{k}.\underline{r}} + \underline{q}_k^* e^{-i\underline{k}.\underline{r}}) \times$$

$$\times (\underline{\dot{q}}_{k'} e^{i\underline{k}'.\underline{r}} + \underline{\dot{q}}_{k'}^* e^{-i\underline{k}'.\underline{r}} + ic\underline{k}' b_{k'} e^{i\underline{k}'.\underline{r}} - ic\underline{k}' b_{k'}^* e^{-i\underline{k}'.\underline{r}})$$

$$= 2\sum_k \{ -ic(\underline{q}_k \times \underline{k}) b_{-k} - ic(\underline{q}_k \times \underline{k}) b_k^* + (\underline{q}_k \times \underline{\dot{q}}_{-k}) + (\underline{q}_k \times \underline{\dot{q}}_k^*)$$

$$+ ic(\underline{q}_k^* \times \underline{k}) b_k + ic(\underline{q}_k^* \times \underline{k}) b_{-k}^* + (\underline{q}_k^* \times \underline{\dot{q}}_k) + (\underline{q}_k^* \times \underline{\dot{q}}_{-k}^*) \}.$$

Substituting from equations (3.4) and (3.6) for $\dot{q}_k$ and for b_k:

(3.15) $$\underline{\mathcal{S}} = -2\sum_k \frac{ic^2 k}{\omega_k} \left[(q_{-k3} - q_{k3}^*)(\underline{q}_k \times \underline{k}) + (q_{k3} - q_{-k3}^*)(\underline{q}_k^* \times \underline{k}) \right]$$

$$-2\sum_k i\omega_k \left[(\underline{q}_k \times \underline{q}_{-k}) - (\underline{q}_k^* \times \underline{q}_{-k}^*) + 2(\underline{q}_k^* \times \underline{q}_k) \right]$$

The terms in $(q_{\underline{k}} \times q_{\underline{-k}})$ and $(q^*_{\underline{-k}} \times q^*_{\underline{-k}})$ cancel separately on summation over k. The above form for $\underline{\mathcal{S}}$ is not such that the eigenvalues after quantization can be deduced in the way in which energy and momentum eigenvalues have been found. The reason is that the complex Fourier expansion, which is really an expansion in free-particle momentum eigenfunctions, separates the quantities $\mathcal{H}$, $\underline{\mathcal{P}}$ and $\underline{\mathcal{S}}$ into the contributions from given momentum eigenstates, whereas the angular momentum eigenstates do not correspond to given values of linear momentum. In fact, $\underline{\mathcal{S}}$ is not even a constant of the motion, so that in the quantized theory $\underline{\mathcal{S}}$ will not commute with $\mathcal{H}$ and it will therefore not be possible to find a representation in which $\mathcal{H}$ and $\underline{\mathcal{S}}$ have simultaneous eigenvalues. In contrast, in discussing $\mathcal{H}$ and $\underline{\mathcal{P}}$ we have tacitly assumed a representation in which $\mathcal{H}$ and $\underline{\mathcal{P}}$ have simultaneous eigenfunctions. In spite of the fact that $\underline{\mathcal{M}}$ and $\underline{\mathcal{S}}$ are not convenient forms in the q's and b's, if one quantizes by transforming to the a's and a^*'s, the commutators for the components of $\underline{\mathcal{M}}$, calculated using the commutation rule for the a's and a^*'s, will be found to be angular momentum relations:

$$[\mathcal{M}_{ij}, \mathcal{M}_{jk}] = i\hbar \mathcal{M}_{ki}$$

where (i, j, k) is a cyclic permutation of (1,2,3). The same will be found for spin, as will be proven in a special case. The proofs of these statements in general are straightforward, but are too tedious to warrant the space here.

There is, however, a conservation law which can only be proven conveniently from the form (3.15) for $\underline{\mathcal{S}}$. A particle having momentum ($\hbar\underline{k}$) has no orbital angular momentum in the direction $\underline{k}$. If

$$(\underline{\mathcal{M}} + \underline{\mathcal{S}}) = \sum_{\underline{k}} (\underline{\mathcal{M}}_k + \underline{\mathcal{S}}_k).$$

and if $\underline{\mathcal{M}}_k$ and $\underline{\mathcal{S}}_k$ are referred to axes whose three-direction is in the direction of $\underline{k}$, then $\mathcal{M}_{k3} = 0$. Suppose that only particles of type k are considered. Since angular momentum including spin is conserved, each component, and in particular the three-component $(\mathcal{M}_{k3} + \mathcal{S}_{k3})$, must be conserved separately. Since $\mathcal{M}_{k3} = 0$ it follows that $\mathcal{S}_{k3}$, the longitudinal component of spin, is a constant of the motion. This result can be verified quantum mechanically by evaluating the commutator of the Hamiltonian with the k and (-k) terms of $\mathcal{S}_3$ in equation (3.15) after quantization. The latter expression is:

$$(\underline{\mathcal{S}}_k + \underline{\mathcal{S}}_{-k})_3 = -4i\,\omega_k \left[(\underline{q}_k^* \times \underline{q}_k) + (\underline{q}_{-k}^* \times \underline{q}_{-k}) \right]_3 .$$

since the terms having the cross product with $\underline{k}$ have no three-component. Quantizing as for the Hamiltonian, with

$$a_{kr} = 2\sqrt{\frac{\omega_k}{\hbar}} \cdot q_{kr} \ ; \ r = 1,2$$

$$(\underline{\mathcal{S}}_k + \underline{\mathcal{S}}_{-k})_3 = i\hbar \sum_{\underline{k}' = \pm \underline{k}} \left\{ a_{k'1}^* a_{k'2} - a_{k'2}^* a_{k'1} \right\}.$$

In evaluating the commutator, we need consider only the terms of $\mathcal{H}$ in k and (-k), corresponding to the transverse field components, for:

$$[\, a_{kr} \,, a_{k's}^*] = \delta_{kk'}\, \delta_{rs} .$$

These terms are:

$$(\mathcal{H}_k + \mathcal{H}_{-k})_{12} = \sum_{\substack{k'=k,-k \\ r=1,2}} (a_{k'r}^* a_{k'r}).$$

The commutator is then:

$$[(\mathcal{S}_k + \mathcal{S}_{-k})_3, (\mathcal{H}_k + \mathcal{H}_{-k})_{12}]$$

$$= i\hbar \sum_{k'=\pm k} [(a_{k'1}^* a_{k'2} - a_{k'2}^* a_{k'1}), (a_{k'1}^* a_{k'1} + a_{k'2}^* a_{k'2})]$$

$$= i\hbar \sum_{k'} \{(a_1^* a_2 a_1^* a_1 - a_1^* a_1 a_1^* a_2) + (a_1^* a_2 a_2^* a_2 - a_2^* a_2 a_1^* a_2)$$

$$- (a_2^* a_1 a_1^* a_1 - a_1^* a_1 a_2^* a_1) - (a_2^* a_1 a_2^* a_2 - a_2^* a_2 a_2^* a_1)\}$$

$$= i\hbar \sum_{k'} \{[a_1^* a_1^* a_1 - a_1^*(a_1^* a_1 + 1)] a_2 + [a_2 a_2^* a_2 - (a_2 a_2^* - 1) a_2] a_1^*$$

$$- [a_1 a_1^* a_1 - (a_1 a_1^* - 1) a_1] a_2^*] - [a_2^* a_2^* a_2 - a_2^*(a_2^* a_2 + 1)] a_1^*\}$$

$$= i\hbar \sum_{k'} \{(-a_1^*) a_2 + (+a_2) a_1^* - (a_1) a_2^* - (-a_2^*) a_1\} = 0.$$

The quantum mechanical conservation law for the longitudinal component of spin has therefore been proven using only the commutation rules for the operators a_{kr} and a_{kr}^*.

In order to find the intrinsic angular momentum of a vector meson, we consider a quantized field for which $N_k = \delta_{k0}$. This means that there is in the field region only one particle, and this particle has zero linear momentum. The spin, which comprises the total angular momentum of a particle having no linear momentum, is:

$$\underline{s}_0 = -4i\omega_0(\underline{q}_0^* \times \underline{q}_0).$$

from equation (3.15) with $\underline{k}$ equal to zero. As in setting up $\mathcal{H}$ for the field, the quantization procedure follows elimination of the $\dot{q}$'s, and ordering products with q^*'s preceding q's. The substitution

$$a_{0r} = 2\sqrt{\frac{\omega_0}{\hbar}} \cdot q_{0r}\,;\; r = 1,2\,;\; a_{03} = \frac{2mc^2}{\hbar\sqrt{\hbar\omega_0}} \cdot q_{03}.$$

where $\omega_0 = c^2\sqrt{0 + \frac{m^2c^2}{\hbar^2}} = \frac{mc^2}{\hbar}$.

and where a_{0r} (r = 1,2,3) are operators such that

$$[a_{0r}, a_{0r'}^*] = \delta_{rr'}.$$

then gives the quantum mechanical spin operator:

$$\underline{s}_0 = -4i\,\frac{mc^2}{\hbar}\Big[\frac{\hbar^2}{4mc^2}(a_{02}^* a_{03} - a_{03}^* a_{02})\,\underline{\varepsilon}_1$$

$$+\frac{\hbar^2}{4mc^2}(a_{03}^* a_{01} - a_{01}^* a_{03})\,\underline{\varepsilon}_2 + \frac{\hbar^2}{4mc^2}(a_{01}^* a_{02} - a_{02}^* a_{01})\,\underline{\varepsilon}_3\Big].$$

First the formal angular momentum property of $\underline{s}_0$ will be proven. For example:

$$[s_{01}, s_{02}] = -i\hbar\big[(a_{02}^* a_{03} - a_{03}^* a_{02}),(a_{03}^* a_{01} - a_{01}^* a_{03})\big]$$

$$= -i\hbar\big[(a_{02}^* a_{03} a_{03}^* a_{01} - a_{03}^* a_{01} a_{02}^* a_{03}) - (a_{02}^* a_{03} a_{01}^* a_{03} - a_{01}^* a_{02} a_{02}^* a_{03})$$

$$-(a_{03}^* a_{02} a_{03}^* a_{01} - a_{03}^* a_{01} a_{03}^* a_{02}) + (a_{03}^* a_{02} a_{01}^* a_{03} - a_{01}^* a_{03} a_{03}^* a_{02})\big]$$

$$= -i\hbar\left\{a_{01}a_{02}^{*}\left[a_{03}a_{03}^{*} - (a_{03}a_{03}^{*} - 1)\right] + a_{02}a_{01}^{*}\left[a_{03}^{*}a_{03} - (a_{03}^{*}a_{03} + 1)\right]\right\}$$

$$= -i\hbar\left\{-a_{01}^{*}a_{02} + a_{02}^{*}a_{01}\right\}$$

$$= i\hbar\, s_{03}.$$

A similar result holds true for the other two components:

$$[s_{0i}, s_{0j}] = i\hbar\, s_{0m}.$$

where (i,j,m) is a cyclic permutation of $(1,2,3)$. From the properties of angular momentum operators $|\underline{s}_0|^2$ has eigenvalues of the form $s(s+1)\hbar^2$. The spin of the particle described by the corresponding eigenfunction is defined to be s. Now:

$$|\underline{s}_0|^2 = -\hbar^2 \sum_{i,j=1}^{3} (a_{0i}^{*}a_{0j} - a_{0j}^{*}a_{0i})^2.$$

with summation such that $i \neq j$, and i and j are always in cyclic order $(1,2,3)$. The fact that there is assumed to be one particle only in the state with $k = 0$ may be written:

$$\sum_{r=1}^{3} N_{0r} = 1.$$

that is, two of the N_{0r} are zero, and one of them is unity. Then:

$$|\underline{s}_0|^2 = -\hbar^2 \sum_{i,j} (a_{0i}^{*}a_{0j}\, a_{0i}^{*}a_{0j} + a_{0j}^{*}a_{0i}\, a_{0j}^{*}a_{0i}$$

$$- a_{0i}^{*}a_{0j}\, a_{0j}^{*}a_{0i} - a_{0j}^{*}a_{0i}\, a_{0i}^{*}a_{0j})$$

$$= -\hbar^2 \sum_{i,j} \left[(a_{oi}^* a_{oi}^*)(a_{oj} a_{oj}) + (a_{oj}^* a_{oj}^*)(a_{oi} a_{oi}) \right. \tag{3.16}$$

$$\left. - (a_{oi}^* a_{oi})(a_{oj} a_{oj}^*) - (a_{oj}^* a_{oj})(a_{oi} a_{oi}^*) \right]. \tag{3.17}$$

The terms in line (3.16) operating on a system for which $N_{oj} = 1$, and $N_{oi} = 0$ are identically zero, for the state function operated on by a_{oj} becomes that for no particles, and operation again with a_{oj} gives identically zero. Operation with a_{oi} once gives identically zero. This is a necessary condition that the energy eigenvalues be bounded below. The terms in line (3.17) have eigenvalues:

$$|\underline{s}_o|^2 = \sum_{i,j} \hbar^2 \left[N_{oi}(N_{oj}+1) + N_{oj}(N_{oi}+1) \right]$$

since $(a_{oi} a_{oi}^*) = (a_{oi}^* a_{oi} + 1)$ and $a_{oi}^* a_{oi}$ has eigenvalues N_{oi}. Since not both of N_{oi} and N_{oj} are different from zero:

$$|\underline{s}_o|^2 = \hbar^2 \sum_{i,j} (N_{oi} + N_{oj}) = \hbar^2 (1+1) = 2\hbar^2.$$

Equating this eigenvalue to $s(s+1)\hbar^2$, the spin must be $s = 1$. Therefore the vector meson could be observed to have any ~~o~~ one of the spin values 0, ± 1 in a given direction. It has now been shown that the quanta of a field described by three independent vector components possess three possible intrinsic angular momentum states, observable in the non-relativistic limit.

It should not be thought, however, that the number of field components determine the spin of the quanta. For the complex scalar field we have seen that the two field functions are related to the charges of the quanta. In the next section a field described by four independent complex functions will be shown to have two possible spin values.

Before leaving the subject of angular momentum it should be proved that the quanta of the scalar field have zero spin. If ϕ and ϕ^* are scalars, then for an infinitesimal Lorentz transformation, $\delta\phi$ and $\delta\phi^*$ are zero at a fixed point, so that referring back to equation (3.9):

$$\delta L = \frac{\partial L}{\partial(\partial_\sigma \phi)} (\delta \partial_\sigma) \phi + \frac{\partial L}{\partial(\partial_\sigma \phi^*)} (\delta \partial_\sigma) \phi^* .$$

It follows by the same argument as for the vector field that:

$$w_{\sigma\rho} \left\{ \frac{\partial L}{\partial(\partial_\sigma \phi)} . \partial_\rho \phi + \frac{\partial L}{\partial(\partial_\sigma \phi^*)} . \partial_\rho \phi^* \right\} = 0.$$

and from the antisymmetry of $w_{\sigma\rho}$:

$$\left\{ \frac{\partial L}{\partial(\partial_\sigma \phi)} . \partial_\rho \phi + \frac{\partial L}{\partial(\partial_\sigma \phi^*)} \partial_\rho \phi^* \right\} = \left\{ \frac{\partial L}{\partial(\partial_\rho \phi)} \partial_\sigma \phi + \frac{\partial L}{\partial(\partial_\rho \phi^*)} \partial_\sigma \phi^* \right\}$$

If ϕ is real the statement is trivial and gives no information. If ϕ is complex the statement is equivalent to saying that $T_{\mu\nu}$ is symmetric, and therefore that orbital angular momentum is conserved, since

$$\partial_\gamma M_{\alpha\beta\gamma} = \frac{1}{ic} (T_{\alpha\beta} - T_{\beta\alpha}).$$

The condition $\delta L = 0$ under infinitesimal Lorentz transformation therefore shows that the conserved angular momentum is the same as the orbital angular momentum: there is no spin.

CHAPTER IV.

THE DIRAC PARTICLES: THE SPINOR FIELD

The most common elementary particles, electrons, protons, and neutrons, are described by the Dirac theory. The development of the Dirac equation is indicated in the following argument. For a causal quantum theory obeying a superposition principle, the differential equation describing the system should be linear and should involve only first time derivatives. The Schrodinger equation has these properties. However, for a covariant, relativistic theory, time and space coordinates should enter on an equal basis: first derivatives only with respect to spatial coordinates should occur. If the quantum mechanical momentum operator is represented by $(-i\hbar\nabla)$, and the energy operator by $(i\hbar\frac{\partial}{\partial t})$, the Hamiltonian operator replacing the classical expression

$$H = (p^2c^2 + m^2c^4)^{1/2}$$

will be of the form:

$$H = c\,\underline{\alpha}\cdot\underline{p} + \beta mc^2 .$$

where clearly $\underline{\alpha}$ and β cannot be ordinary numbers. Assuming $\underline{\alpha}$ and $\underline{\beta}$ commute with ordinary numbers and with coordinates and momenta, their algebra can be deduced from considering:

$$H^2 = (c^2p^2 + m^2c^4) = (c\,\underline{\alpha}\cdot\underline{p} + \beta mc^2)^2$$

or:

$$m^2c^4 + c^2p_1^2 + c^2p_2^2 + c^2p_3^2 = c^2\alpha_j^2p_j^2 + c^2(\alpha_1p_1\alpha_2p_2 + \alpha_2p_2\alpha_1p_1)$$

$$+ c^2(\alpha_1p_1\alpha_3p_3 + \alpha_3p_3\alpha_1p_1) + c^2(\alpha_2p_2\alpha_3p_3 + \alpha_3p_3\alpha_2p_2).$$

$$+ mc^3(\underline{\alpha}\cdot\underline{p}\beta + \beta\underline{\alpha}\cdot\underline{p}) + \beta^2m^2c^4.$$

With $\underline{p}$ chosen to be zero, we see that:

$$\beta^2 = 1.$$

In the remaining expression, the choice of any two of (p_1, p_2, p_3) equal to zero shows that:

$$\alpha_j^2 = 1.$$

$$(\alpha_j\beta + \beta\alpha_j) = 0.$$

satisfies the equation resulting. With these results, the choice of any one of (p_1, p_2, p_3) zero gives the result:

$$(\alpha_i\alpha_j + \alpha_j\alpha_i) = 0.$$

Since α_j^2, β are both one, as operators α_j and β have eigenvalues $\pm$ 1, and since they have only these real eigenvalues, they are Hermitian. Furthermore the α_j and β anticommute in pairs. The anticommutator is written:

$$(\alpha_j\beta + \beta\alpha_j) = [\alpha_j, \beta]_+.$$

Returning to the Hamiltonian operator involving $\underline{\alpha}$ and β, the

Dirac equation for free particles is:

(4.1) $$H\psi = (c\underline{\alpha}.\underline{p} + mc^2\beta)\psi = i\hbar\frac{\partial\psi}{\partial t}.$$

or setting $p = i\hbar\nabla$ and multiplying on the left with β:

(4.2) $$-i\hbar c\beta\underline{\alpha}.\nabla + \frac{\hbar c}{ic}\beta\frac{\partial\psi}{\partial t} + \beta^2 mc^2\psi = 0.$$

A convenient notation is to set $-i\beta\alpha_j = \gamma_j$ and $\beta = \gamma_4$. Then (4.2) becomes:

(4.3) $$\gamma_\rho\partial_\rho\psi + \frac{mc}{\hbar}\psi = 0.$$

The operators $\underline{\alpha}$ and β may be represented by four by four matrices:

(4.4) $$\underline{\alpha} = \begin{pmatrix} 0 & \underline{\sigma}' \\ \underline{\sigma}' & 0 \end{pmatrix} ; \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

where $\underline{\sigma}'$ are the Pauli spin matrices:

(4.5) $$\sigma_1' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma_2' = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \sigma_3' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and I and O are two by two identity and zero matrices respectively. The notation indicates that α_j has σ_j' in the appropriate position. The reader should verify that the matrices so defined for $\underline{\alpha}$ and β have the required anticommutation relations and that $\alpha_j^2 = \beta^2 = 1$. If this matrix representation is used for $\underline{\alpha}$ and β, the state functions or operands must be four row matrices: they are taken to be column matrices.

Invariance of equation (4.3) under Lorentz transformation[8]

(4.6)
$$x_\rho' = a_{\rho\nu} x_\nu .$$
$$\partial_\rho' = a_{\rho\nu} \partial_\nu .$$

may be obtained by requiring

(4.7)
$$\gamma_\mu \partial_\mu' \psi' + \frac{mc}{\hbar} \beta^2 \psi' = 0.$$

where ψ' is defined by some transformation

(4.8)
$$\psi' = S\psi,$$

where S is assumed to have an inverse. S is called a spinor transformation, and quantities which transform accoring to (4.8) are called spinors. The Dirac equation could alternatively be made invariant by requiring the γ_μ's to transform as vector components. ψ would then be a column matrix of scalar quantities, but the form of the equation would change under Lorentz transformation in the sense that the γ-matrices would have a different form in each different Lorentz frame.

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[8] Only Lorentz transformations not involving reflection of time axes will be considered. For discussion of time reflections in Dirac theory, see R. H. Good, Jr: "Properties of the Dirac Matrices", Rev. Mod. Phys., 27, No. 2, 187 - 211, April, 1955.

The spinor transformation rule is generally assumed, and will be followed here. The properties of S will now be discussed.[9] Substituting equations (4.6) and (4.8) into (4.7):

$$\gamma_\mu a_{\mu\nu} \partial_\nu (S\psi) + \frac{mc}{\hbar}(S\psi) = 0.$$

Multiplication on the left by S^{-1} gives:

$$(a_{\mu\nu} S^{-1} \gamma_\mu S)\, \partial_\nu \psi + \frac{mc}{\hbar} \psi = 0.$$

Comparing with equation (4.3) it follows that:

$$(a_{\mu\nu} S^{-1} \gamma_\mu S) = \gamma_\nu,$$

or, multiplying on the left by $a_{\lambda\nu}$ and summing, from the orthogonality relation $a_{\lambda\mu} a_{\nu\mu} = \delta_{\lambda\nu}$:

(4.9) $$(S^{-1} \gamma_\lambda S) = a_{\lambda\nu} \gamma_\nu.$$

This matrix equation determines the spinor transformation S, and relates it to the Lorentz transformation. Since the operators $\underline{\alpha}$ and β are Hermitian, the Hermitian adjoint of the Dirac equation (4.3) should be valid. The Hermitian adjoint of a matrix B, which is the complex conjugate of its transpose, will be written B^*. Then from (4.3)

(4.10) $$(\partial_\mu \psi)^* \gamma_\mu^* + \frac{mc}{\hbar} \psi^* = 0.$$

since for matrices B and C:

$$(BC)^* = C^* B^*.$$

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9 See, for example, Good, loc. cit.

Now $\gamma_j = -i\beta\alpha_j$ and $\gamma_4 = \beta$ so that:

$$\gamma_j^* = +i\alpha_j^*\beta^* = i\alpha_j\beta = -i\beta\alpha_j \quad \text{and} \quad \gamma_4^* = \beta.$$

It follows that the γ's as defined are Hermitian. Also:

$$(\partial_4\psi)^* = \frac{1}{(-i)c}\frac{\partial\psi^*}{\partial t} = -\partial_4\psi^*.$$

Making use of these relations, equation (4.10) becomes:

$$+\partial_j\psi^*(\beta\cdot\beta)(i\alpha_j\beta) - \partial_4\psi^*\gamma_4 + \frac{mc}{\hbar}\psi^* = 0.$$

where $\beta^2 = 1$ has been inserted in the first term. Multiplying this last equation on the right by β and defining

$$\psi^\dagger = \psi^*\beta.$$

we have for the Hermitian adjoint Dirac equation:

(4.11) $$\partial_\mu\psi^\dagger\gamma_\mu - \frac{mc}{\hbar}\psi^\dagger = 0.$$

The requirement of invariance of the form of the γ_μ's under Lorentz transformation, applied to this equation, gives:

(4.12) $$\partial_\mu'(\psi^\dagger)'\gamma_\mu - \frac{mc}{\hbar}(\psi^\dagger)' = 0.$$

From this we can, by comparison with (4.11), obtain a form for $(\psi^\dagger)'$, which compared with

$$(\psi^\dagger)' = (\psi^*)'\beta = (\psi')^*\beta = (S\psi)^*\beta = \psi^*S^*\beta.$$

should give a condition on S^*. Now $\partial_\mu' = a_{\mu\nu}\partial_\nu$, Now equation (4.12) has the form:

$$\partial_\nu (\psi^\dagger)' a_{\mu\nu} \gamma_\mu - \frac{mc}{\hbar} (\psi^\dagger)' = 0.$$

and we have:

$$S^{-1} \gamma_\mu S = a_{\mu\lambda} \gamma_\lambda .$$

To obtain $a_{\mu\nu} \gamma_\mu$, multiply the latter equation on the left by S, on the right by S^{-1} and by $a_{\mu\nu}$ and sum over μ. Then:

$$a_{\mu\nu} \gamma_\mu = a_{\mu\nu} a_{\mu\lambda} S \gamma_\lambda S^{-1}.$$

From the orthogonality of the Lorentz transformation

$$a_{\mu\nu} a_{\mu\lambda} = \delta_{\nu\lambda}$$

we have:

$$a_{\mu\nu} \gamma_\mu = S \gamma_\nu S^{-1}.$$

Substituting this into (4.12) then gives:

$$\partial_\nu (\psi^\dagger)' (S \gamma_\nu S^{-1}) - \frac{mc}{\hbar} (\psi^\dagger)' = 0,$$

and multiplication on the right by S leads to the conclusion that, for the equation to be the same as (4.11), the condition is:

$$(\psi^{\dagger}) = (\psi^{\dagger})' S$$

or
$$\psi^{\dagger} S^{-1} = \psi^{*}\beta\, S^{-1} = (\psi^{\dagger})'.$$

From the spinor transformation $\psi' = S\psi$ we had the result:

$$(\psi^{\dagger})' = \psi^{*} S^{*}\beta .$$

It follows that S must have the property:

(4.13) $$S^{*}\beta = \beta S^{-1}.$$

This property will be of use in a later discussion.

For the field theory of Dirac spinors, the Lagrangian density

(4.14) $$L = -\frac{\hbar c}{2}\psi^{\dagger}\left(\gamma_{\mu}\partial_{\mu} + \frac{mc}{\hbar}\right)\psi - \frac{\hbar c}{2}\left[-(\partial_{\mu}\psi^{\dagger})\gamma_{\mu} + \frac{mc}{\hbar}\psi^{\dagger}\right]\psi$$

leads by Hamilton's principle to the field equations (4.3) and (4.11). In this the spinors ψ and $\psi^{\dagger}$ will be taken as the field functions. That in the matrix representation each consists of four functions, and that one should expect not two but eight sets of occupation numbers, will be shown to be correct at a later stage. Viewing the Dirac equation in its general form, with no particular representation in mind, only leads at this stage to field functions ψ and $\psi^{\dagger}$. From the choice (4.14) for L, the Euler-Lagrange equations are

$$\frac{\partial L}{\partial \psi} = -mc^{2}\psi^{\dagger} + \frac{\hbar c}{2}(\partial_{\mu}\psi^{\dagger})\gamma_{\mu}.$$

$$\partial_{\nu}\frac{\partial L}{\partial(\partial_{\nu}\psi)} = -\frac{\hbar c}{2}\,.\,\partial_{\nu}\psi^{\dagger}\gamma_{\mu}.$$

whence: $\partial_\mu \psi^\dagger \gamma_\mu - \frac{mc}{\hbar} \psi^\dagger = 0$

in agreement with (4.11). Similarly equation (4.3) follows from

$$\frac{\partial L}{\partial \psi^\dagger} = \partial_\nu \frac{\partial L}{\partial(\partial_\nu \psi^\dagger)}.$$

It should be noted that L is defined in equation (4.14) is zero as a consequence of the field equations. This illustrates the importance of L as a form rather than as a number.

The argument advanced in Chapter I to show that invariance of L under translations of four-space axes implies energy and momentum continuity equations is not dependent on the nature of the field functions ψ and $\psi^\dagger$, as reëxamination of the argument will convince the reader. One must, in evaluating δL, write $\left(\frac{\partial L}{\partial x_\nu} - \frac{\partial L}{\partial x_\nu}\right)$ in such a way that equating the result to zero is not a trivial form. The tensor

$$T_{\mu\nu} = \partial_\mu \psi^\dagger \frac{\partial L}{\partial(\partial_\nu \psi^\dagger)} + \frac{\partial L}{\partial(\partial_\nu \psi)} . \partial_\mu \psi - L\,\delta_{\mu\nu}.$$

therefore obeys a continuity equation for the Dirac field. The energy-momentum tensor for the Dirac field, using the field equation (4.3) and (4.11) is:

$$T_{\mu\nu} = \frac{\hbar c}{2}\left[(\partial_\mu \psi^\dagger)\gamma_\nu \psi - \psi^\dagger \gamma_\nu \partial_\mu \psi\right]. \tag{4.15}$$

The field Hamiltonian is then:

$$\mathcal{H} = \int_V H\,dV = \int_V T_{44}\,dV$$

$$= \frac{\hbar c}{2}\int_V \left[(\partial_4 \psi^\dagger)\gamma_4 \psi - \psi^\dagger \gamma_4 \partial_4 \psi \right] dV$$

Making use of the field equations in the form:

$$\gamma_4 \partial_4 \psi = -\frac{mc}{\hbar}\psi - \gamma_j \partial_j \psi$$

and
$$\partial_4 \psi^\dagger \gamma_4 = \frac{mc}{\hbar}\psi^\dagger - \partial_j \psi^\dagger \gamma_j ,$$

and writing $\psi^\dagger \gamma_4 = \psi^* \beta^2 = \psi^*$, $\mathcal{H}$ becomes:

$$\mathcal{H} = \frac{\hbar c}{2}\int_V dV \left[\left(\frac{mc}{\hbar}\psi^\dagger - \partial_j \psi^\dagger \gamma_j\right)\psi - \psi^\dagger\left(-\frac{mc}{\hbar} - \gamma_j \partial_j\right)\psi \right]$$

$$= \frac{\hbar c}{2}\int_V dV \left[2\frac{mc}{\hbar}\psi^\dagger\psi - \partial_j(\psi^\dagger \gamma_j \psi) + 2\psi^\dagger \gamma_j \partial_j \psi \right].$$

Transforming

$$\int_V \partial_j (\psi^\dagger \gamma_j \psi)\, dV.$$

to a surface integral, if ψ is required to be periodic over the bounding surface of the region V, or to vanish on the surface, the integral is zero. Using the definition of the γ's, $\mathcal{H}$ then reduces to:

(4.16) $$\mathcal{H} = \int_V \psi^* (-i\hbar c\, \underline{\alpha}\cdot\nabla + mc^2\beta)\psi\, dV$$

This is the expression in the first quantized theory for the expected average in V of the operator $H = + c\underline{\alpha}\cdot\underline{p} + \beta mc^2$ for the state ψ of the system. Similarly the field momentum is given by:

$$\mathcal{P}_j = \int_V G_j \, dV = \int_V -\frac{1}{ic} T_{j4} \, dV$$

$$= \frac{i\hbar}{2} \int_V [(\partial_j \psi^\dagger)\gamma_4 \psi - \psi^\dagger \gamma_4 \partial_j \psi] \, dV$$

$$= \frac{i\hbar}{2} \int_V [\partial_j(\psi^\dagger \gamma_4 \psi) - 2\psi^\dagger \gamma_4 \partial_j \psi] dV.$$

which from the periodicity of ψ, using Gauss' theorem again on the first term, reduces to:

$$(4.17) \qquad \mathcal{P}_j = \int_V \psi^*(-i\hbar\partial_j)\psi \, dV.$$

The expression for the spin of the Dirac field will now be deduced. It is clear from equation (4.15) defining $T_{\mu\nu}$ that the energy-momentum tensor is not symmetric, and it follows as shown in Chapter III that the orbital angular momentum $\underline{\mathcal{M}} = \int_V (\underline{r}\times\underline{G}).\, dV$ is not a conserved quantity (see equation 3.8). The argument deducing angular momentum conservation including spin from Lorentz invariance was carried out only for scalar and vector fields. For the spinor field under an infinitesimal Lorentz transformation,

$$\delta x_\mu = w_{\mu\nu} x_\nu.$$

as in Chapter III. The spinor transformation is:

(4.18) $$\delta\psi = \psi' - \psi = S\psi^{\dagger} - \psi = (S - I)\,\psi = \delta S\psi.$$

where δS is an infinitesimal transformation equal to the difference of S from the identity transformation I. Similarly:

$$\delta\psi^{\dagger} = \psi'^{\dagger} - \psi^{\dagger} = \psi^{\dagger} S^{-1} - \psi^{\dagger} = \psi^{\dagger}(S^{-1} - I) = \psi^{\dagger}(\delta S^{-1}).$$

If $S = (I + \delta S)$, then to first order in infinitesimal quantities, $S^{-1} = (I - \delta S)$ satisfied the requirement

$$S S^{-1} = I.$$

the terms linear in δS cancelling. It follows that

$$\delta S^{-1} = (S^{-1} - I) = -\delta S.$$

and therefore that

(4.19) $$\delta\psi^{\dagger} = -\psi^{\dagger}\delta S.$$

Then evaluating the variation of L at a fixed point:

$$\delta L = \frac{\partial L}{\partial\psi}\delta\psi + \delta\psi^{\dagger}\frac{\partial L}{\partial\psi^{\dagger}} + \frac{\partial L}{\partial(\partial_\sigma\psi)}\,\delta(\partial_\sigma\psi) + \delta(\partial_\sigma\psi^{\dagger})\frac{\partial L}{\partial(\partial_\sigma\psi^{\dagger})}.$$

Using (4.18), (4.19), and (3.11):

$$\delta L = \frac{\partial L}{\partial\psi}\cdot\delta S\,\psi - \psi^{\dagger}\delta S\cdot\frac{\partial L}{\partial\psi^{\dagger}} + \frac{\partial L}{\partial(\partial_\sigma\psi)}\,\partial_\sigma(\delta S\,\psi) - \partial_\sigma(\psi^{\dagger}\delta S)\frac{\partial L}{\partial(\partial_\sigma\psi^{\dagger})}$$

$$+\, w_{\sigma\rho}\left[\frac{\partial L}{\partial(\partial_\sigma\psi)}\cdot\partial_\rho\psi - (\partial_\rho\psi^{\dagger})\frac{\partial L}{\partial(\partial_\sigma\psi^{\dagger})}\right].$$

which from the Euler-Lagrange equations and the definition of $T_{\mu\nu}$ is:

(4.20) $$\delta L = \partial_\sigma \left\{ \frac{\partial L}{\partial(\partial_\sigma \psi)} \delta S \psi - \psi^\dagger \delta S \frac{\partial L}{\partial(\partial_\sigma \psi^\dagger)} \right\} + w_{\sigma\rho} \left(T_{\rho\sigma} + L\, \delta_{\rho\sigma} \right).$$

As in the discussion of angular momentum for the vector field, we wish to express δS as a linear combination of the $w_{\sigma\rho}$'s, for δS is to be a first order infinitesimal.

$$\delta S = w_{\sigma\rho} \Lambda_{\sigma\rho}.$$

Now $\Lambda_{\sigma\rho}$ must be a four-by-four matrix if we think of S as such, and may be taken antisymmetrical in the indices σ and ρ, since $w_{\sigma\rho}$ is antisymmetrical and interchange of dummy indices should not change δS. The trial form

(4.21) $$\delta S = A\, \omega_{\sigma\rho} (\gamma_\sigma \gamma_\rho - \gamma_\rho \gamma_\sigma)$$

is therefore reasonable, where A is a constant to be determined. The condition set on S by a Lorentz transformation $a_{\mu\nu}$ is

$$S^{-1} \gamma_\mu S = a_{\mu\nu} \gamma_\nu .$$

from equation (4.9). Substituting $S^{-1} = (I - \delta S)$ and $S = (I + \delta S)$, and $a_{\mu\nu} = (\delta_{\mu\nu} + w_{\mu\nu})$, and considering only first order infinitesimals:

$$(I - \delta S)\, \gamma_\mu (I + \delta S) = (w_{\mu\nu} + \delta_{\mu\nu})\, \gamma_\nu$$

(4.22) or: $$\gamma_\mu \delta S - \delta S \gamma_\mu = w_{\mu\nu} \gamma_\nu .$$

Substituting the form (4.21) for δS and using the relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu},$$

which follow from the properties of $\underline{\alpha}$ and β, the condition (4.22) on δS becomes:

$$A w_{\sigma\rho} \{ \gamma_\mu (\gamma_\sigma \gamma_\rho - \gamma_\rho \gamma_\sigma) - (\gamma_\sigma \gamma_\rho - \gamma_\rho \gamma_\sigma) \gamma_\mu \} = w_{\mu\nu} \gamma_\nu$$

$$= A w_{\sigma\rho} \{ (-\gamma_\sigma \gamma_\mu + 2\delta_{\sigma\mu}) \gamma_\rho - (-\gamma_\rho \gamma_\mu + 2\delta_{\rho\mu}) \gamma_\sigma - \gamma_\sigma (-\gamma_\mu \gamma_\rho + 2\delta_{\mu\rho}) + \gamma_\rho (-\gamma_\mu \gamma_\sigma + 2\delta_{\mu\sigma}) \}$$

$$= A \{ 2 w_{\mu\rho} \gamma_\rho - 2 w_{\sigma\mu} \gamma_\sigma - 2 w_{\sigma\mu} \gamma_\sigma + 2 w_{\mu\rho} \gamma_\rho \}$$

$$= 8A\, w_{\mu\nu} \gamma_\nu .$$

where the property $w_{\mu\nu} = -w_{\nu\mu}$ has been used, and all summed indices have been represented by ν. It follows that (8A) must equal one. Then δS in (4.21) may be written:

$$\delta S = \frac{1}{8} w_{\sigma\rho} (\gamma_\sigma \gamma_\rho - \gamma_\rho \gamma_\sigma)$$

Schweber shows[10] that this form for δS is uniquely determined by equation (4.22) subject to the normalization (determinant of δS) = 1.

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[10] Schweber, et. al: p. 26.

Returning to the calculation of δL, equation (4.20) now becomes:

$$\delta L = \frac{w_{\sigma\rho}}{8}\left\{ \partial_\nu\left[\frac{\partial L}{\partial(\partial_\nu\psi)}.(\gamma_\sigma\gamma_\rho - \gamma_\rho\gamma_\sigma)\psi - \psi^\dagger(\gamma_\sigma\gamma_\rho - \gamma_\rho\gamma_\sigma)\frac{\partial L}{\partial(\partial_\nu\psi^\dagger)}\right] + T_{\rho\sigma} + L\,\delta_{\rho\sigma}\right\}$$

and from the antisymmetry of $w_{\sigma\rho}$, if $\delta L = 0$, the tensor in $\{\ \}$ brackets must be symmetric. That is:

$$\partial_\nu\left\{ \frac{\partial L}{\partial(\partial_\nu\psi)}\cdot\frac{1}{4}(\gamma_\sigma\gamma_\rho - \gamma_\rho\gamma_\sigma)\psi - \psi^\dagger.\frac{1}{4}(\gamma_\sigma\gamma_\rho - \gamma_\rho\gamma_\sigma)\frac{\partial L}{\partial(\partial_\nu\psi^\dagger)}\right\} = (T_{\sigma\rho} - T_{\rho\sigma}).$$

which has the form:

$$\partial_\nu(S_{\rho\sigma\nu} + M_{\rho\sigma\nu}) = 0$$

where

$$S_{\rho\sigma\nu} = \frac{1}{4ic}\left\{ \psi^\dagger(\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho)\frac{\partial L}{\partial(\partial_\nu\psi^\dagger)} - \frac{\partial L}{\partial(\partial_\nu\psi)}(\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho)\psi\right\}.$$

since $(T_{\sigma\rho} - T_{\rho\sigma}) = -ic\,\partial_\gamma(M_{\rho\sigma\gamma})$.

The spin density S_{ij4} is therefore:

$$S_{ij} = \frac{\hbar c}{4ic}\left\{ \psi^\dagger(\gamma_i\gamma_j - \gamma_j\gamma_i)\cdot\frac{1}{2}\gamma_4\psi + \frac{1}{2}\psi^\dagger\gamma_4(\gamma_i\gamma_j - \gamma_j\gamma_i)\psi\right\}.$$

and since γ_4 anticommutes with γ_j:

$$S_{ij} = \frac{\hbar}{4i}\left\{ \psi^*(\gamma_i\gamma_j - \gamma_j\gamma_i)\psi\right\}.$$

Using the definition $\gamma_j = -i\beta\alpha_j$ and the anticommutation rules between the α_j and β in pairs, S_{ij} can be written:

$$\underline{S} = \frac{\hbar}{4i}\left\{\psi^*(\underline{\alpha}\times\underline{\alpha})\psi\right\}$$

where $(\underline{\alpha}\times\underline{\alpha})$ is the ordinary vector cross product. Using the matrix representation (4.4) and (4.5) for $\underline{\alpha}$, and defining

$\underline{\sigma} = \begin{pmatrix} \underline{\sigma}' & 0 \\ 0 & \underline{\sigma}' \end{pmatrix}$, then $(\underline{\alpha}\times\underline{\alpha}) = (\underline{\sigma}\times\underline{\sigma})$, and since $(\underline{\sigma}'\times\underline{\sigma}') = 2i\underline{\sigma}'$, it follows that:

$$\underline{S} = \frac{\hbar}{2}\left(\psi^*\underline{\sigma}\psi\right).$$

The spin of the field is then:

$$\underline{\mathcal{S}} = \int_V \underline{S}\,dV = \frac{\hbar}{2}\int_V\left(\psi^*\underline{\sigma}\psi\right)dV. \tag{4.23}$$

The matrix $\left(\frac{\hbar}{2}\underline{\sigma}\right)$ is therefore the spin operator in the first quantized theory.

As a first step toward second quantization, ψ and ψ^t must be expanded in Fourier series. Now ψ is to be a solution of the Dirac equation: that is, an energy eigenfunction. Its Fourier series must then be an expansion in a complete set of free particle energy eigenfunctions. For this purpose a non-degenerate set of eigenfunctions will be found by considering the eigenfunctions of operators commuting with the Hamiltonian. If the γ's are written as four-by-four matrices and ψ as a four component spinor, the Dirac equation

$$\gamma_\mu\partial_\nu\psi + \frac{mc}{\hbar}\psi = 0$$

and its Hermitian adjoint are each really four equations. Now suppose that ψ_k is a momentum eigenfunction. Then:

$$-i\hbar\nabla\psi_k = (\hbar \underline{k})\psi_k.$$

and ψ_k must have the form:

$$\psi_k = u_k e^{i\underline{k}.\underline{r}}$$

where u_k is a spinor not depending on position. If ψ_k is to be simultaneously an energy eigenfunction, from equations (4.2) and the momentum eigenfunction property:

$$-i\hbar c\,\underline{\alpha}.\nabla\psi_k + \beta mc^2\psi_k - E_k\psi_k = 0.$$

assuming $i\hbar\dfrac{\partial\psi_k}{\partial t} = E_k\psi_k$. This reduces to:

$$(\hbar c\,\underline{\alpha}.\underline{k} + \beta mc^2 - E_k)u_k = 0.$$

If, using the matrices (4.4) for $\underline{\alpha}$ and β, one writes down the four equations for the four components of u_k, then the condition that a non-trivial solution exists is:[11]

$$(E_k^2 - \hbar^2c^2k^2 - m^2c^4)^2 = 0.$$

This is an equation of fourth degree in E_k, having two double roots corresponding to:

$$E_k = \pm(m^2c^4 + \hbar^2c^2k^2)^{1/2}.$$

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[11] c.f. Schiff, p. 315.

It is to be expected that there are four linearly independent solutions u_k, and this is found to be true. To remove the twofold degeneracy of the energy eigenvalues which still remains after considering simultaneous momentum eigenfunctions, a third operator is sought which commutes with p and H and for which an eigenfunction of p and H will have two different eigenvalues. Commutation with H implies that the operator is a constant of the motion. Two ~~such~~ constants we know to be the angular momentum including spin, and the longitudinal component of spin. Neamtan[12] has shown that the latter operator, $(\underline{\sigma}.\underline{p})$, is convenient to resolve the degeneracy. From the discussion of spin in Chapter III we know that $(\underline{\sigma}.\underline{p})$ and H commute, and that $(\underline{\sigma}.\underline{p})$ commutes with p follows from:

$$(-i\hbar\nabla)\,\underline{\sigma}.(-i\hbar\nabla) = \underline{\sigma}.(-i\hbar\nabla)(-i\hbar\nabla).$$

For given eigenvalues E_k of H and $(\hbar\underline{k})$ of $\underline{p}$, the eigenvalues of $(\underline{\sigma}.\underline{p})$ are found from writing in matrix form the eigenvalue equation

$$\underline{\sigma}.(-i\hbar\nabla)\,\psi_k = (\hbar\sigma)\,\psi_k.$$

where $(\hbar\sigma)$ is the eigenvalue to be determined. Since $(-i\hbar\nabla)\psi_k = (\hbar\underline{k})\,\psi_k = (\hbar\underline{k})\,u_k\,e^{i\underline{k}.\underline{r}}$, the eigenvalue equation

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12 S. M. Neamtan: Am. J. Phys., 20, 450 (1952).

is:

$$\underline{\sigma}\cdot(-i\hbar\nabla)\psi_k = \hbar\cdot\begin{pmatrix} k_3 & k_1 - ik_2 & 0 & 0 \\ k_1 + ik_2 & -k_3 & 0 & 0 \\ 0 & 0 & k_3 & k_1 - ik_2 \\ 0 & 0 & k_1 + ik_2 & -k_3 \end{pmatrix}\begin{pmatrix} u_k^{(1)} \\ u_k^{(2)} \\ u_k^{(3)} \\ u_k^{(4)} \end{pmatrix} = (\hbar\sigma)\begin{pmatrix} u_k^{(1)} \\ u_k^{(2)} \\ u_k^{(3)} \\ u_k^{(4)} \end{pmatrix}$$

The determinant of the coefficients of the four equations resulting is:

$$[-k_3^2 + \sigma^2 - k_1^2 - k_2^2]^2$$

which must be zero for a non-trivial solution. Then:

$$\sigma^2 = k_1^2 + k_2^2 + k_3^2 = k^2$$

or:

$$(\hbar\sigma) = \pm\hbar k.$$

It follows that the four linearly independent wave functions ψ_k satisfying the Dirac equation and having a given momentum eigenvalue ($\hbar\underline{k}$) may be separated into two pairs, each pair corresponding to a given sign of

$$E_k = \pm(m^2c^4 + \hbar^2c^2k^2)^{1/2} = \pm\varepsilon_k.$$

and that a given pair may be distinguished as corresponding to particles having spin parallel or antiparallel to the momentum ($\hbar\underline{k}$). The orthogonality of eigenfunctions of ($\underline{\sigma}\cdot\underline{p}$) belonging to different eigenvalues follows from the fact that $(\underline{\sigma}.\underline{p})$ is Hermitian. That $\underline{\sigma}$ is Hermitian follows from the fact that each component has real eigenvalues, since $\sigma_j^2 = 1$. Let u_{ks} and u_{kr} correspond to eigenfunctions of $(\underline{\sigma}.\underline{p})$ belonging to different eigenvalues $(\hbar\, \sigma_{r,s})$. From the Hermiticity of $(\underline{\sigma}.\underline{p})$:

$$\underline{\sigma}.\underline{k}\; u_{ks} = \sigma_s u_{ks}. \tag{4.24}$$

$$u_{kr}^* \;\underline{\sigma}.\underline{k} = \sigma_r u_{kr}^*. \tag{4.25}$$

assuming u_{ks} and u_{kr} correspond to eigenfunctions of $\underline{p}$ with eigenvalue $(\hbar k)$. Multiplying (4.24) on the left by u_{kr}^* and (4.25) on the right by u_{ks} and subtracting gives the result

$$(\sigma_s - \sigma_r)\, u_{kr}^*\, u_{ks} = 0.$$

Now if k is not zero, $\sigma_{s,r}$ are not zero, and if $r \neq s$, then $(\sigma_s - \sigma_r)$ is not zero, so that:

$$u_{kr}^*\, u_{ks} = 0.$$

Similarly eigenfunctions of H belonging to different eigenvalues are orthogonal, so that if the spinors are normalized:

$$u_{kr}^*\, u_{ks} = \delta_{rs}.$$

where r and s can have four values each.

For the Fourier expansion of ψ we now write

$$\psi = \frac{1}{\ell^{3/2}} \sum_{\substack{k \\ s=1,2}} \left(q_{ks+}\, u_{ks+}\, e^{i\underline{k}.\underline{r}} + q_{ks-}\, u_{-ks-}\, e^{-i\underline{k}.\underline{r}} \right) \tag{4.26}$$

where $u_{ks+}e^{i\underline{k}.\underline{r}}$ is a simultaneous eigenfunction of H, $\underline{p}$, and $(\underline{\sigma}.\underline{p})$ belonging to eigenvalues $E_k = +\mathcal{E}_k$, $\underline{p} = (\hbar\underline{k})$, and the summation over s covering both eigenvalues of $(\underline{\sigma}.\underline{p})$. Similarly $u_{-ks-}e^{+i\underline{k}.\underline{r}}$ is an eigenfunction of H with $E_k = -\mathcal{E}_k$. The sum over $\underline{k}$ is the same as in earlier chapters where we consider a region cubical in shape with sides of length ℓ. Similarly

$$(4.27) \qquad \psi^{\dagger} = \frac{1}{\ell^{3/2}} \sum_{\underline{k}} \left(q_{ks+}^{*} \, u_{ks+}^{\dagger} \, e^{-i\underline{k}.\underline{r}} + q_{ks-}^{*} \, u_{-ks-}^{\dagger} \, e^{+i\underline{k}.\underline{r}} \right)$$

It is clear that $u_{ks+} \sim e^{-i\varepsilon t/\hbar}$ and $u_{-ks-} \sim e^{+i\varepsilon t/\hbar}$ since they satisfy $i\hbar\dfrac{\partial u_{\pm}}{\partial t} = \pm\varepsilon u_{\pm}$, and that the expansions (4.26) and (4.27) are plane-wave expansions forming a complete set. Using these expansions the values of $\mathcal{H}$, $\underline{\mathcal{G}}$, $\underline{\mathcal{S}}$, and $|\underline{\mathcal{S}}_0|^2$ will be worked out and the second quantization effected. For the Hamiltonian:

$$\mathcal{H} = \int_V \psi^{*}(-i\hbar c\underline{\alpha}.\nabla + mc^2\beta)\psi \, dV.$$

and substituting the Fourier series for ψ and ψ^{*} and using the fact that

$$(-i\hbar c\underline{\alpha}.\nabla + mc^2\beta)\, u_{\pm} = \pm\varepsilon u_{\pm},$$

it follows that:

$$\mathcal{H} = \frac{1}{\ell^3}\int_V \sum_{k,k'} \sum_{s,s'} \left[q_{ks+}^{*} \, u_{ks+}^{*} \, e^{-i\underline{k}.\underline{r}} + q_{ks-}^{*} \, u_{-ks-}^{*} \, e^{+i\underline{k}.\underline{r}} \right] \times$$

$$\times \left[q_{k's'+} \, \varepsilon_{k'} u_{k's'+} \, e^{i\underline{k}'.\underline{r}} + q_{k's'-} \, u_{-k's'-}(-\varepsilon_{k'}) \, e^{-i\underline{k}'.\underline{r}} \right] dV.$$

Using the relationships:

$$u_{ks}^{*} u_{ks'} = \delta_{ss'}.$$

and $$u_{k+}^{*} u_{k+} = 1 \quad ; \quad u_{k+}^{*} u_{k-} = 0.$$

and $$\frac{1}{\ell^3}\int_V e^{i(\underline{k}-\underline{k}').\underline{r}}\, dV = \delta_{\underline{k},\underline{k}'}.$$

$\mathcal{H}$ reduces to:

$$\mathcal{H} = \sum_{\underline{k},s} \varepsilon_k \left(q_{ks+}^{*} q_{ks+} - q_{ks-}^{*} q_{ks-} \right). \tag{4.28}$$

As in the meson field theory, we here wish to quantize by considering the q's and q^*'s as **non-commuting** operators. We know, however, that the relations

$$[\, a_{kst} , a_{k's't'}^{*} \,] = \delta_{kk'}\, \delta_{ss'}\, \delta_{tt'}. \tag{4.29}$$

where $t = \pm$, lead to number of particles operators $N_{kst} = (a_{kst}^{*} a_{kst})$ having as eigenvalues all positive integers and zero. It is known experimentally, however, that two electrons having the same energy, momentum, and spin cannot exist in a closed system: this is the Pauli exclusion principle, and applies to all particles obeying the Dirac equation. The principle states, in other words, that the operator N_{kst} can only have eigenvalues zero or one. This may be expressed by the equation

$$N_{kst}(N_{kst} - 1) = a_{kst}^{*} a_{kst} (a_{kst}^{*} a_{kst} - 1) = 0. \tag{4.30}$$

If the a's and a^*'s are to have the interpretation of creation and annihilation operators, they must have the properties:

(4.31) $$a_{kst}\, a_{kst} = 0\ ; \qquad a_{kst}^*\, a_{kst}^* = 0.$$

for a_k operating twice on a state function for which N_k is one or zero must give identically zero, and a_k^* must likewise, for otherwise N_k could have a value of at least two. Operators a_k and a_k^* obeying (4.29) will satisfy (4.30) of course, but will not in general satisfy (4.31). However, operator rules:

(4.32) $$(a_{kst}\, a_{k's't'}^* + a_{k's't'}^*\, a_{kst}) = [a_{kst}, a_{k's't'}^*]_+ = \delta_{kk'}\, \delta_{ss'}\, \delta_{tt'}.$$

$$[a_{kst}, a_{k's't'}]_+ = [a_{k'st}^*, a_{k's't'}^*]_+ = 0.$$

allow all three consequences (4.30) and (4.31), of the Pauli principle, to be satisfied. Accordingly, the quantization procedure for the Dirac field involves replacing the expansion coefficients q_{kst} by operators a_{kst} which satisfy the anticommutation rules (4.32). The reader should verify that if Ψ_k is an eigenfunction of $\mathcal{H}$ such that

$$\mathcal{H}\,\Psi_k = \varepsilon_k\, a_k^*\, a_k\, \Psi_k = N_k\, \varepsilon_k\, \Psi_k.$$

where N_k is zero or one, then from the rules (4.32) it follows that $(a_k^*\, \Psi_k)$ is an eigenfunction of $\mathcal{H}$ belonging to an eigenvalue ε_k, and that a_k^* is an operator which either increases by one or leaves unchanged the number of particles N_k depending on whether N_k is zero or one respectively. Similarly a_k reduces by one or leave unchanged the number of particles in the state $\underline{k}$.

After quantization the Hamiltonian (4.28) has the form:

$$\mathcal{H} = \sum_{k,s} \varepsilon_k (a_{ks+}^* a_{ks+} - a_{ks-}^* a_{ks-})$$

with eigenvalues:

$$\mathcal{H} = \sum_{k,s} \varepsilon_k (N_{ks+} - N_{ks-})$$

This form for $\mathcal{H}$ allows a negative energy for the system of free particles described by the second quantized field. Such a situation is not in agreement with the conventional notion that free particles can have only positive energy $\varepsilon_k = +(m^2c^4 + \hbar^2c^2k^2)^{1/2}$. Indeed, what we have taken to be the Hamiltonian operator in the first quantized theory, $(-i\hbar c\underline{\alpha}\cdot\nabla + mc^2\beta)$, allows negative energy eigenvalues also. Now the field Hamiltonian operator can be rewritten, using the anticommutation rules, to be:

$$\mathcal{H} = \sum_{k,s} \varepsilon_k (a_{ks+}^* a_{ks+} + a_{ks-} a_{ks-}^* - 2).$$

where the 2 results from summation over s. Now since the a's obey anticommutation rules, it is possible to write

$$a_{ks-} = b_{ks-}^* \; ; \qquad a_{ks-}^* = b_{ks-} ,$$

and still have $(b_{ks-}^{*}\ b_{ks-})$ a suitable number of particles operator. With this transformation $\mathcal{H}$ is the sum of two terms, the first of which can only be positive or zero:

$$\sum_{k,s} \varepsilon_k (a_{ks+}^{*} a_{ks+} + b_{ks-}^{*} b_{ks-}) \tag{4.33}$$

and the second of which is negatively infinite. The latter term is dropped in the theory as being a fixed quantity which is unobservable. The expression (4.33) is taken to be the Hamiltonian operator.

A picturesque terminology has been developed, called the Dirac hole theory, to describe the second quantized Dirac field. Considering the form,

$$\mathcal{H} = \sum_{k,s} \varepsilon_k (a_{ks+}^{*} a_{ks+} - a_{ks-}^{*} a_{ks-})$$

one says that the field energy is equivalent to that of one set of particles, described by the a_{ks+}'s, of positive energy, plus that of another set having negative energy and described by the a_{ks-}'s. The form

$$\mathcal{H} = \sum_{k,s} \varepsilon_k (a_{ks+}^{*} a_{ks+} + b_{ks-}^{*} b_{ks-} - 2)$$

then indicates that the vacuum state energy is negatively inifinite, corresponding to all the negative energy states being filled, since $a_{ks-}^{*} a_{ks-} = (1 - b_{ks-}^{*}\ b_{ks-})$ which is one if $(b_{ks-}^{*}\ b_{ks-}) = 0$. It is differences from this vacuum state which are observable. Thus the annihilation of a negative energy particle, indicated by a_{ks}, corresponds to the creation of a positive energy particle of the type described by the b_{ks-}'s, since $a_{ks-} = b^*_{ks-}$. The annihilated negative energy particle leaves a hole in the normally (vacuum state) filled set of negative energy states. It is this hole, this absence of a negative energy particle which is observable as a positive energy particle. Writing $\mathcal{H}$ in the form (4.33) eliminates the need for the "hole" artifice and allows one to talk of two types of particles, each having positive energy in the free state. In what follows, the additional terms introduced in going from $(-a^*_{ks-}\, a_{ks-})$ to $(+\, b^*_{ks-}\, b_{ks-})$ will always be dropped.

The reader should verify that the field momentum

$$\underline{\mathcal{G}} = \int_V \psi^*(-i\hbar\nabla)\psi\, dV,$$

on substitution of the expansions (4.26) and (4.27) for ψ and ψ^*, using the fact that

$$-i\hbar\nabla\left(u_{ks}\, e^{i\underline{k}\cdot\underline{r}}\right) = \hbar\underline{k}\, u_{ks}\, e^{i\underline{k}\cdot\underline{r}}.$$

reduces to:

$$\text{(4.34a)} \qquad \underline{\mathcal{G}} = \sum_{ks}\left(q^*_{ks+}\, q_{ks+} - q^*_{ks-}\, q_{ks-}\right)\hbar\underline{k}.$$

Quantizing by replacing q_{ks+} by a_{ks+} and q_{ks-} by b^*_{ks-}, and requiring the anticommutation rules (4.32) for the operators a

and b, then leads to:

$$(4.34) \qquad \underline{\mathcal{G}} = \sum_{k,s} \hbar \underline{k} \left(a^*_{ks+} a_{ks+} + b^*_{ks-} b_{ks-} \right).$$

where a term $(-2\sum_k \hbar \underline{k})$ has been dropped, as in renormalizing $\mathcal{H}$ previously. The field momentum then has eigenvalues

$$\underline{\mathcal{G}} = \sum_{k,s} \hbar \underline{k} \left[N^{(-)}_{ks} + N^{(+)}_{ks} \right]$$

where $(a^*_{ks+} a_{ks+})$ has eigenvalues a $N^{(-)}_{ks}$ and $(b^*_{ks-} b_{ks-})$ has eigenvalues $N^{(+)}_{ks}$. The choice of signs for superscripts will be clear later. ~~The field momentum may be written:~~

$$\cancel{\underline{\mathcal{G}} = \sum_{k,s} \hbar \underline{k} \left[N^{(+)}_{ks} + N^{(-)}_{-ks} \right]}.$$

The field momentum is contributed by ~~that is,~~ a sum over $\underline{k}$ and s of particles of two types, each having momentum $(\hbar\underline{k})$. It is interesting to notice that the absence of a negative energy particle of momentum $(\hbar\underline{k})$, corresponding to a "hole" with momentum $(\hbar\underline{k})$, corresponds to a positive energy particle having momentum $(-\hbar\underline{k})$, for if

$$a^*_{ks-} a_{ks-} = 0.$$

then $(b^*_{ks-} b_{ks-}) = 1$, and from the form of (4.34), the term in (KS-) contributes momentum $(-\hbar\underline{k})$. In the hole theory the sum $(2\sum_k \hbar\underline{k})$ is not dropped, but it cancels for the summation is over pairs of opposite terms.

If one refers back to Chapter II, to the discussion of charge conservation following from invariance under transformations of the type:

$$\psi' = \psi e^{i\alpha}.$$

it will be seen that the argument is not dependent on the covariance property of the field function. We can therefore write for the charge of the Dirac field:

$$Q = -\frac{ie}{\hbar}\int_V \left(\frac{\partial L}{\partial \dot{\psi}}\psi - \psi^{\dagger}\frac{\partial L}{\partial \dot{\psi}^{\dagger}} \right) dV$$

which from the Lagrangian density (4.14) for the Dirac field is:

$$Q = -e\int_V \left[-\tfrac{1}{2}\psi^{\dagger}\beta\psi - \psi^{\dagger}(\tfrac{1}{2}\beta\psi) \right] dV$$

$$= e\int_V (\psi^{*}\psi)\, dV.$$

Inserting the Fourier expansions for ψ and ψ^{*}:

$$Q = e\sum_{ks}(q^{*}_{ks+}q_{ks+} + q^{*}_{ks-}q_{ks-}).$$

Quantization then gives:

(4.35) $$Q = e\sum_{ks}(a^{*}_{ks+}a_{ks+} - b^{*}_{ks-}b_{ks-}).$$

In Chapter II the choice of e as the charge on a proton rather than that on an electron was arbitrary: it simply decided which of two independent sets of occupation numbers was to describe positively charged particles. For this discussion we let e be the charge on the electron, since it has been customary to think of the positive energy particles of the hole theory as being electrons. It is clear from (4.35) that the two sets of occupation numbers defined by a's and by b's respectively, describe particles having opposite charge. The positively charged particles are positrons. In the hole theory, the absence of a negative energy electron corresponds to the presence of a positively charged particle. In (4.35) a term

$$e \sum_{ks} (n_{ks})$$

where n_{ks} is one for all k and s, has been discarded: this negatively infinite charge represents the charge of the vacuum state in the hole theory, in which all negative energy states are filled with electrons. The extension of this theory to cover other charged particles, such as protons, obeying Fermi-Dirac statistics is obvious. The importance of the negative energy particles, or antiparticles, in the case of uncharged Fermions such as neutrons and neutrinos, is less clear.

As in the vector meson theory, the spin of the Dirac field does not have a useful form when a momentum eigenfunction expansion is used. The spin is calculated to be:

$$\underline{S} = \frac{\hbar}{2} \int_V (\psi^* \underline{\sigma} \psi)\, dV$$

$$= \frac{\hbar}{2} \sum_{k,s} \Big\{ q_{ks+}^* q_{ks+} (u_{ks+}^* \underline{\sigma}\, u_{ks+}) + q_{ks+}^* q_{ks-} (u_{ks+}^* \underline{\sigma}\, u_{-ks-})$$

$$+ q_{ks-}^* q_{ks-} (u_{ks-}^* \underline{\sigma}\, u_{-ks-}) + q_{ks-}^* q_{ks+} (u_{-ks-}^* \underline{\sigma}\, u_{ks+}) \Big\}$$

which on quantization has the form:

$$\underline{\mathcal{S}} = \frac{\hbar}{2}\sum_{ks}\Big\{a^*_{ks+}a_{ks+}(u^*_{ks+}\,\underline{\sigma}\,u_{ks+}) + a^*_{ks+}b^*_{ks-}(u^*_{ks+}\underline{\sigma}\,\underline{u}_{ks-})$$

$$+ b_{ks-}a_{ks+}(\underline{u}^*_{ks-}\,\underline{\sigma}\,u_{ks+}) - b^*_{ks-}b_{ks-}(\underline{u}^*_{ks-}\,\underline{\sigma}\,\underline{u}_{ks-})\Big\}.$$

To determine the spin of the electron, we consider:

$$\sum_s N^{(-)}_{os} = 1;\quad N^{(+)}_{os} = 0;\quad N^{(\pm)}_{ks} = 0 \text{ for } k \neq 0.$$

that is, we consider only one electron, and that one at rest.

Making use of the orthogonality relations

$$u^*_{kst}\,u_{k's't'} = \delta_{ss'}\,\delta_{kk'}\,\delta_{tt'}.$$

and the fact that $(b^*_{ks-}\ b_{ks-}) = N^{(+)}_{ks} = 0$ for this discussion

$|\underline{\mathcal{S}}|^2$ reduces to:

$$|\underline{\mathcal{S}}_o|^2 = \frac{\hbar^2}{4}\sum_{k,s}\Big\{(a^*_{os+}a_{os+})^2(u^*_{os+}\,\underline{\sigma}\,u_{os+})\cdot(u^*_{os+}\,\underline{\sigma}\,u_{os+})$$

$$+ a^*_{os+}a_{os+}a^*_{os+}b^*_{os-}(u^*_{os+}\,\underline{\sigma}\,u_{os+})\cdot(u^*_{os+}\,\underline{\sigma}\,u_{os-})$$

$$+ b_{os-}a_{os+}a^*_{os+}a_{os+}(u^*_{os-}\,\underline{\sigma}\,u_{os+})\cdot(u^*_{os+}\,\underline{\sigma}\,u_{os+})$$

$$+ b_{ks-}a_{ks+}a^*_{ks+}b^*_{ks-}(\underline{u}^*_{ks-}\,\underline{\sigma}\,u_{ks+})\cdot(u^*_{ks+}\,\underline{\sigma}\,\underline{u}_{ks-})\Big\}.$$

Using the relations: $u^*_{ost}\,u_{ost} = u_{ost}\,u^*_{ost} = 1$, and since $\sigma_j^2 = 1$, $(\underline{\sigma}\cdot\underline{\sigma}) = 3$, the first term in $|\underline{\mathcal{S}}_o|^2$ reduces to

$$\frac{\hbar^2}{4}\sum_s N^{(+)}_{os}\cdot 3. \tag{4.36}$$

and the second and third terms are zero, for

$$(u_{os+}^{*}\,\underline{\sigma}\,u_{os+})\cdot(u_{os+}^{*}\,\underline{\sigma}\,u_{os-}) = (u_{os+}^{*}\,\underline{\sigma}\cdot\underline{\sigma}\,u_{os-})$$
$$= 3\,(u_{os+}^{*}\,u_{os-}) = 3\,(0).$$

The last term is:

$$\frac{\hbar^2}{4}\sum_{k,s} 3(b_{ks-}\,b_{ks-}^{*})(a_{ks+}\,a_{ks+}^{*}).$$

and from the anticommutation rule would seem to be infinite, equal to:

$$\frac{\hbar^2}{4}\sum_{k,s} 3(n_{ks} - b_{ks-}^{*}\,b_{ks-})(n_{ks} - a_{ks+}^{*}\,a_{ks+})$$

(4.37)

$$= \frac{\hbar}{4}\sum_{ks} 3(n_{ks} - 0)(n_{ks} - 0).$$

where $n_{ks} = 1$ for all k and s. This infinite term should be dropped, however, for if one works out $|\underline{s}|^2$ before quantization, then the quantization procedure involves replacing $(a_{ks-}^{*}\,a_{ks-}\ \ )$ by $(-b_{ks-}^{*}\,b_{ks-})$ rather than by $(1-b_{ks-}^{*}\,b_{ks-})$. In this case the first factor in (4.37) is zero for each term of the sum. From (4.36) and the requirement of only one electron present, $\sum_{s} N_{os}^{(-)} = 1$, it follows that the value of $|\underline{s}_o|^2$ is:

$$|\underline{s}_o|^2 = \hbar^2\,s(s+1) = \frac{3}{4}\,\hbar^2.$$

where s, the spin of the electron, must then be 1/2. The same argument, leading to the same result, is valid for the antiparticle.

CHAPTER V

INTERACTION OF ELEMENTARY PARTICLES

The final test of a physical theory is the calculation of some result which can be obtained experimentally. In processes involving elementary particles, the average lifetime of a state of the system, or the calculation of a cross-section, are the most common sort of quantitative check of a theory. For the calculation of these quantities, a technique called perturbation theory, has been developed. For a quantum mechanical system involving the interaction of various particles or fields, it is assumed that the Hamiltonian operator can be written in the form:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}'.$$

where $\mathcal{H}_0$ is the Hamiltonian operator of the system neglecting any interaction between the various elements of the system. The notation

$$\langle f | \mathcal{H}' | i \rangle$$

will be used to indicate the matrix element of the operator between an initial state i and a final state f. The result of perturbation theory is to show[13] that the transition probability per unit time from the initial state i to the final state f of the system is given to first order by:

$$w(E) = \frac{2\pi}{\hbar} \rho_F(E) \left| \langle f | \mathcal{H}' | i \rangle \right|^2.$$

[13] See, for example, Schiff, pp. 189-196.

where $\rho_F(E)$ is the density of final states in the vicinity of the energy E_0 corresponding to energy conservation for the system, and where $\langle f|\mathcal{H}'|i\rangle$ is evaluated for the final state corresponding to energy conservation for the process. Implicit in this result is the assumption that $\mathcal{H}'$ and $\rho_F(E)$ vary negligibly over the time period required for the perturbation.

A cross section for a process is defined as the number of particles passing to the final state per unit time, per unit incident flux. Cross sections are observed in experiments in which a beam of particles enters an interaction, and in which the resultant particles are observed. Now a transition probability per unit time may be used to represent the density of particles appearing in a final state per unit time, divided by the density of incident particles. Now the density of incident particles is the incident flux divided by the incident velocity. The number of particles appearing in the final states is the density of particles times the volume of the system. From these considerations it follows that the cross-section σ' for a process is related to the transition probability per unit time by:

$$\sigma' = \frac{V}{v} \, w(E)$$

where $V = \ell^3$ is the volume of the quantized system, and v is the velocity of incident particles.

The average lifetime of a quantum mechanical state is calculated on the assumption that the number of transitions away from the state in a given time (the decay rate) is proportional to the number of particles in that state. Quantitatively:

$$\frac{dN}{dt} = -\lambda N.$$

where λ is a positive constant called the decay constant, and N is the number of particles in the given state at time t. Then to first order approximation,

$$N = N_o e^{-\lambda t} = N_o(1-\lambda t)$$

having taken the first two terms of the Taylor expansion for $e^{-\lambda t}$ and having let N_o be the number of particles in the initial state at time t = 0. λ then represents the transition probability per unit time since λt gives the fractional decrease in N_o during time t. Now the average lifetime defined by:

$$\tau_{av}\int_{N_o}^{0} dN = \int_{N_o}^{0} t\,dN.$$

is given by:

$$\tau_{av} = -\frac{1}{N_o}\int_{t=0}^{\infty} t N_o(-\lambda)\,e^{-\lambda t}dt = \frac{1}{\lambda}.$$

The average lifetime is therefore the reciprocal of the transition probability per unit time. The half-life of a state defined by:

$$\frac{N_o}{2} = N_o e^{-\lambda(\tau_{1/2})}.$$

is then given by:

$$\tau_{1/2} = \frac{\log_e 2}{\lambda} = \frac{0.693}{\lambda}$$

The half-life as defined is the time required for the number of particles in a state to diminish by half.

Decay of the π - Meson.

The second quantized field theory can be applied to interactions, and used to calculate transition probabilities per unit time by writing the Lagrangian density in the form:

$$L = L_0 + L'.$$

where L_0 is the Lagrangian density of the fields without interaction, and L' accounts for the interactions between the fields. In this section the decay of the π- meson will be used as an example of the development of the theory. The decay may be written:

$$\pi^+ = \mu^+ + \nu.$$

where μ^+ represents a positively charged μ- meson, and ν a neutrino. For this process, there are three fields involved, so that L_0 has the form:

$$L_0 = L_\pi + L_\mu + L_\nu.$$

where from Chapter I,

$$(5.1) \qquad L_\pi = -c^2\left(\partial_\rho \phi^* \partial_\rho \phi + \frac{m_\pi c}{\hbar} \phi^* \phi\right).$$

assuming that the π- meson is best described by a complex pseudoscalar field function ϕ, obeying the Klein-Gordon equation. It is known experimentally that the μ- meson and neutrino are

Dirac particles. Then:

$$(5.2)\quad L_{\mu} = -\frac{\hbar c}{2}\left\{ \psi^{\dagger}\left(\gamma_{\rho}\partial_{\rho} + \frac{mc}{\hbar}\right)\psi + \left(-\partial_{\rho}\psi^{\dagger}\gamma_{\rho} + \frac{mc}{\hbar}\psi^{\dagger}\right)\psi \right\}.$$

and, assuming that the rest mass of the neutrino is zero:

$$(5.3)\quad L_{\nu} = -\frac{\hbar c}{2}\left\{ \chi^{\dagger}(\gamma_{\rho}\partial_{\rho}\chi) - (\partial_{\rho}\chi^{\dagger})\gamma_{\rho}\chi \right\}$$

We have written m_{π} and m for the rest masses of π- and μ- mesons respectively, and ϕ, ψ, and χ to represent the field functions of π- meson, μ- meson, and neutrino respectively.

Since each of L_{μ}, L_{π}, and L_{ν} is Lorentz invariant, L' must be chosen to be Lorentz invariant also, if L for the whole system is to be so. This, and simplicity of form, are the two main criteria in setting up the interaction term of the Lagrangian density, and a comparison of calculated values for half-lives, for example, with experimental data, helps decide whether a given form is acceptable or not. For simplicity L' will be set up containing each field function only linearly, and not having derivatives of the field functions. The form,

$$L' = \phi\,\psi^{\dagger}\chi$$

is not Lorentz invariant, however, since ϕ is a pseudoscalar and $(\psi^{\dagger}\chi)$ is a scalar.

For the sake of the general problem of fields in interaction it is important to known how to construct the various covariant forms using field functions of different types. Lorentz invariants can then be formed by combining pairs of terms having the same covariance property: the product of two scalars or of two pseudoscalars, the inner product of two vectors, or of two pseudovectors, or the contracted product ($T_{\rho\sigma} T'_{\rho\sigma}$) of two tensors, form scalar quantities. The following table lists some of the simplest covariant forms which can be set up for different types of field functions. The ϕ's are all assumed to satisfy the Klein-Gordon equation.

	Field	Scalar	Pseudoscalar	Vector	Pseudovector
Covariant Form	Scalar	ϕ	$\phi^*\phi$	$\phi_\rho \phi_\rho$	$\phi_\rho \phi_\rho$
	Pseudoscalar	—	ϕ	—	—
	Vector	$\partial_\rho \phi$	—	ϕ_ρ	—
	Pseudovector	—	$\partial_\rho \phi$	—	ϕ_ρ
	Second Rank Tensor	$\partial_\rho \partial_\sigma \phi$	—	$\partial_\rho \phi_\sigma$	—

It should be noted that several of the terms in the above table are non-linear in the field functions. Non-linear theories are notoriously difficult in calculating such results as cross-sections. Several other terms in the table involve derivatives of the field functions: interaction Lagrangian density terms involving such covariants are said to have derivative coupling. Derivative coupling is not commonly used. For the vector and pseudovector field, one cannot set up a useful scalar or pseudoscalar from $\partial_\rho \phi_\rho$, for this quantity vanishes for the vector meson field (see Chapter III).

Covariants constructed from spinors are also needed, and the simplest forms, not involving derivatives of the field functions will now be listed.

Scalar: In Chapter IV it was shown that a spinor which transforms under Lorentz transformation according to the relationship

$$\psi' = S\psi$$

obeys also the transformation:

$$(\psi^\dagger)' = \psi^\dagger S^{-1}.$$

It follows that ($\psi^\dagger\psi$) is a scalar, for:

$$(\psi^\dagger\psi)' = (\psi^\dagger S^{-1})(S\psi) = \psi^\dagger\psi.$$

This is the property required of a scalar.

Vector. Since the spinor transformation is related to the Lorentz transformation by:

$$S^{-1}\gamma_\mu S = a_{\mu\nu}\gamma_\nu,$$

the quantity ($\psi^\dagger\gamma_\mu\psi$) transforms as a vector:

$$(\psi^\dagger\gamma_\mu\psi)' = (\psi^\dagger S^{-1})\gamma_\mu(S\psi) = a_{\mu\nu}(\psi^\dagger\gamma_\nu\psi).$$

Tensor of Second Rank:

$$(\psi^\dagger\gamma_\mu\gamma_\nu\psi)' = (\psi^\dagger S^{-1})\gamma_\mu(SS^{-1})\gamma_\nu(S\psi)$$

$$= a_{\mu\sigma}a_{\nu\rho}(\psi^\dagger\gamma_\sigma\gamma_\rho\psi).$$

where $(S\ S^{-1}) = 1$ has been inserted.

Pseudoscalar: It will be shown that $(\psi^\dagger i\gamma_5\psi)$, where $\gamma_5 = (\gamma_1\gamma_2\gamma_3\gamma_4)$

transforms as a pseudoscalar.

$$(\psi^\dagger i\gamma_5\psi)' = i(\psi^\dagger S^{-1})\gamma_5(S\psi) = i\psi^\dagger S^{-1}\gamma_1\gamma_2\gamma_3\gamma_4 S\psi$$

Inserting $(S\,S^{-1})$ between each pair of γ's and using the property $S^{-1}\gamma_\mu S = a_{\mu\nu}\gamma_\nu$ this reduces to:

$$(5.4) \qquad a_{1\lambda}a_{2\mu}a_{3\nu}a_{4\sigma}\,\psi^\dagger i\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\sigma\psi.$$

which can be written:

$$(5.5) \qquad \frac{1}{4!}\,\varepsilon_{\lambda'\mu'\nu'\sigma'}\,a_{\lambda'\lambda}a_{\mu'\mu}a_{\nu'\nu}a_{\sigma'\sigma}\,\psi^\dagger(i\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\sigma)\psi.$$

where $\varepsilon_{\lambda'\mu'\nu'\sigma'}$ is a symbol which is zero if $(\lambda'\mu'\nu'\sigma')$ is not the set (1,2,3,4), and is plus or minus one as $(\lambda'\mu'\nu'\sigma')$ is an even or an odd permutation respectively of the order (1,2,3,4). There are no terms in the sum (5.4) having two subscripts of the γ's the same, for suppose $\lambda=\mu$. There are four such terms, with λ = 1,2,3, and 4 respectively. Now $\gamma_\lambda^2 = 1$. Then the contribution to the sum (5.4) from the terms with $\lambda=\mu$ is:

$$a_{1\lambda}a_{2\lambda}a_{3\nu}a_{4\sigma}\cdot\gamma_\nu\gamma_\sigma.$$

which is zero from the orthogonality property for the Lorentz transformation, that $a_{\mu\lambda}a_{\mu'\lambda} = \delta_{\mu\mu'}$. All terms of (5.5) for which $(\lambda'\mu'\nu'\sigma')$ are even permutations of (1,2,3,4) can be put in the order (1,2,3,4) since an even number of permutations of the indices (λ,μ,ν,σ) changes nothing due to the anticommutation of the γ's. All sets of $(\lambda'\mu'\nu'\sigma')$ which are odd permutations of (1,2,34) can likewise be set in the order (1,2,3,4) introducing an odd power of (-1) due to anticommutation, which is cancelled by the value (-1) of $\varepsilon_{\lambda'\mu'\nu'\sigma'}$. The factor $(\frac{1}{4!})$ accounts for the fact that in the sum over $(\lambda'\mu'\nu'\sigma')$ there are (4!) non-zero terms. Now $\gamma_\lambda \gamma_\mu \gamma_\nu \gamma_\sigma$ with all of $(\lambda\mu\nu\sigma)$ different can be written:

$$\varepsilon_{\lambda\mu\nu\sigma}\, \gamma_1 \gamma_2 \gamma_3 \gamma_4 = \varepsilon_{\lambda\mu\nu\sigma}\, \gamma_5 .$$

because of the anticommutation of the γ's. It follows that (5.4) can be written:

$$a_{1\lambda} a_{2\mu} a_{3\nu} a_{4\sigma} \gamma_\lambda \gamma_\mu \gamma_\nu \gamma_\sigma = \frac{1}{4!} \varepsilon_{\lambda\mu\nu\sigma} \varepsilon_{\lambda'\mu'\nu'\sigma'} a_{\lambda\lambda'} a_{\mu\mu'} a_{\nu\nu'} a_{\sigma\sigma'} \gamma_5 .$$

$$= (\det a)\, \gamma_5 .$$

where (deta) is the determinant of the Lorentz transformation[14]. For four-space rotations, (deta) is +1, and for three-space reflections $a_{ij} = -\delta_{ij}$, $a_{\mu 4} = \delta_{\mu 4}$, so that (deta) $= (-1)^3 = -1$. It follows that

$$(\psi^\dagger i \gamma_5 \psi)' = (\det a)(\psi^\dagger i \gamma_5 \psi).$$

and that $(i\psi^\dagger \gamma_5 \psi)$ is a pseudoscalar.

Pseudovector

$$(\psi^\dagger i \gamma_5 \gamma_\rho \psi)' = i(\psi^\dagger S^{-1}) \gamma_5 (S S^{-1}) \gamma_\rho (S\psi)$$

$$= i\psi^\dagger (S^{-1}\gamma_5 S)(S^{-1}\gamma_\rho S)\psi = (\det a)\, a_{\rho\sigma} (\psi^\dagger i \gamma_5 \gamma_\sigma \psi).$$

---

[14] That $\frac{1}{4!} \varepsilon_{\lambda\mu\nu\sigma} \varepsilon_{\lambda'\mu'\nu'\sigma'} a_{\lambda\lambda'} a_{\mu\mu'} a_{\nu\nu'} a_{\sigma\sigma'} = \varepsilon_{\lambda'\mu'\nu'\sigma'} a_{1\lambda'} a_{2\mu'} a_{3\nu'} a_{4\sigma'}$ is equal to (deta) is proven in most texts on tensor analysis. See for example: Lass: "Vector and Tensor Analysis", p.263, (McGraw-Hill, New York, 1950).

which proves the pseudovector character of ($\psi^{\dagger} i \gamma_5 \gamma_\rho \psi$).
The following table summarizes the Dirac covariants.

Covariant	Spinor function.
Scalar	$\psi^{\dagger}\psi$
Pseudoscalar	$\psi^{\dagger} i \gamma_5 \psi$
Vector	$\psi^{\dagger} \gamma_\rho \psi$
Pseudovector	$\psi^{\dagger} i \gamma_5 \gamma_\rho \psi$
Second Rank Tensor	$\psi^{\dagger} \gamma_\rho \gamma_\sigma \psi$.

We are now equipped to set up the interaction term for the π - meson decay problem. Since the π - meson field function is taken to be a pseudoscalar, a Lorentz invariant quantity involving the μ - meson spinor ψ and the neutrino spinor χ , is:

$$(5.6) \qquad L' = g\, \phi(\psi^{\dagger} i \gamma_5 \chi).$$

where g is a constant to be experimentally determined, and is called the coupling constant for the three fields here considered in interaction. The most general interaction term using only the three field functions, each linearly, is:

$$(5.7) \quad L' = g\left\{ \phi\, \psi^{\dagger} i \gamma_5 \chi - \chi^{\dagger} i \gamma_5 \psi \phi^{*} + \phi \chi^{\dagger} i \gamma_5 \psi - \psi^{\dagger} i \gamma_5 \chi \phi^{*} \right\}$$

which is Hermitian, for a purpose which will appear later. To proceed to second quantization we shall have to evaluate $\int_V T_{44} dV$ with appropriate Fourier series inserted for the field functions, and using the latter form of L' this is a formidable task. Much of the work involved can be eliminated by considering the specific process of decay of one positive π- meson at rest into one positive μ- meson and one neutrino. The Fourier series for the field functions are:

$$\phi = \frac{1}{\ell^{3/2}} \sum_{k} q_k(\pi)\, e^{i\underline{k}.\underline{r}} + b_k(\pi)\, e^{-i\underline{k}.\underline{r}}.$$

$$\psi = \frac{1}{\ell^{3/2}} \sum_{k,s} q_{ks+}(\mu)\, u_{ks+}(\mu) e^{i\underline{k}.\underline{r}} + q_{ks-}(\mu)\, \underline{u}_{ks-}(\mu)\, e^{-i\underline{k}.\underline{r}}.$$

$$\chi = \frac{1}{\ell^{3/2}} \sum_{ks} q_{ks+}(\nu)\, u_{ks+}(\nu) e^{i\underline{k}.\underline{r}} + q_{ks+}(\nu)\, \underline{u}_{-ks-}(\nu)\, e^{-i\underline{k}.\underline{r}}.$$

and their Hermitian conjugates. In the second quantized theory the following interpretation is given to the operators which replace the q-coefficients,

	π^+	π^-	μ^+	μ^-	ν	$\bar{\nu}$.
creation	$q_k^*(\pi)$	$b_k^*(\pi)$	$q_{ks+}^*(\mu)$	$q_{ks-}(\mu)$	$q_{ks+}^*(\nu)$	$q_{ks-}(\nu)$
annihilation	$q_k(\pi)$	$b_k(\pi)$	$q_{ks+}(\mu)$	$q_{ks-}^*(\mu)$	$q_{ks+}(\nu)$	$q_{ks-}^*(\nu)$.

where $\bar{\nu}$ represents the antineutrino. The application of these operators to an initial state in which there is one π^+- meson only, gives zero except for the creation operators and for the annihilation operator corresponding to $q_o(\pi)$. Now we are ultimately interested in the matrix element of $\mathcal{H}'$ between and initial state Ψ_i and a final state Ψ_f, where Ψ_f describes the presence of one μ^+- meson and one neutrino. The matrix element may be written symbolically in the form of an inner product

$$(\Psi_f^*, \mathcal{H}'\Psi_i).$$

Recalling that the relationship between an operator O and its Hermitian adjoint O^* is expressed by:

$$(\Psi_f^*, O\,\Psi_i) = ((O^*\Psi_f)^*, \Psi_i)$$

it follows that all of the creation operators following from the above table, will cause the term of $\mathcal{H}'$ in which they occur to have zero matrix element, for our choice of final state, except for $q^*_{ks+}(\mu)$ and one of $q^*_{ks+}(\nu)$ and $q_{ks-}(\nu)$. For example:

$$(\Psi_f^*, a_k^{(+)*}(\pi)\,\Psi_i) = ((a_k^{(+)}(\pi)\,\Psi_f)^*, \Psi_i)$$

is zero since Ψ_f contains no π^+- mesons, so that the π^+-annihilation operator applied to Ψ_f gives identically zero. Similarly:

$$(\Psi_f^*, a_{ks}^{(+)*}(\mu)\,\Psi_i) = ((a_{ks}^{(+)}(\mu)\,\Psi_f)^*, \Psi_i)$$

and since Ψ_f is not a state devoid of μ^+- mesons, $a_{ks}^{(+)}(\mu)$ operating on it is not necessarily zero. Since we must have the terms in $q^*_{ks+}(\mu)$, we must have $\psi^\dagger$ in L', and therefore χ rather than $\chi^\dagger$, in order to form a covariant. This specifies that it is $q_{ks-}(\nu)$ rather that $q^*_{ks+}(\nu)$ which must appear, and corresponds to a process in which an antineutrino is produced. In our theory, there is no physical distinction between the neutrino and the antineutrino. In calculating $\mathcal{H}'$, we may, for this problem, use the following partial Fourier expansions:

$$\phi = \frac{1}{\ell^{3/2}}\, q_0(\pi).$$

(5.8)
$$\psi^\dagger = \frac{1}{\ell^{3/2}} \sum_{ks} q^*_{ks+}(\mu)\, u^\dagger_{ks+}(\mu)\, e^{-i\underline{k}.\underline{r}}$$

$$\chi = \frac{1}{\ell^{3/2}} \sum_{ks} q_{ks-}(\nu)\, u_{-ks-}(\nu)\, e^{i\underline{k}.\underline{r}}.$$

Before calculating $\mathcal{H}'$, it will be interesting to find the field equations for fields in interaction. For this purpose the general form (5.7) for L' will be used. If Φ_α represents the set of field functions describing the three fields here in interaction:

$$\Phi_\alpha \sim (\phi, \phi^*, \psi, \psi^\dagger, \chi, \chi^\dagger).$$

the Euler-Lagrange equations are

$$\frac{\partial L}{\partial \Phi_\alpha} = \partial_\nu \frac{\partial L}{\partial(\partial_\nu \Phi_\alpha)}.$$

Then from the forms (5.1), (5.2), and (5.3) contributing to L_0, and from L', for which $\dfrac{\partial L'}{\partial(\partial_\nu \Phi_\alpha)}$ is zero for all Φ_α, L' not depending on derivatives of the field functions, the following field equations result:

$$\left(\partial_\rho \partial_\rho - \frac{m_\pi c}{\hbar}\right) \phi^* = -\frac{g}{c^2}\left(\psi^\dagger i\gamma_5 \chi + \chi^\dagger i\gamma_5 \psi\right)$$

$$\left(\partial_\rho \partial_\rho - \frac{m_\pi c}{\hbar}\right) \phi = +\frac{g}{c^2}\left(\psi^\dagger i\gamma_5 \chi + \chi^T i\gamma_5 \psi\right).$$

$$\left(\gamma_\rho \partial_\rho - \frac{mc}{\hbar}\right) \psi^\dagger = \frac{g}{\hbar c}\left(\phi^* \chi^\dagger i\gamma_5 - \phi \chi^T i\gamma_5\right).$$

$$\left(\gamma_\rho \partial_\rho + \frac{mc}{\hbar}\right) \psi = \frac{g}{\hbar c}\left(\phi - \phi^*\right) i\gamma_5 \chi.$$

$$\gamma_\rho \partial_\rho \chi^\dagger = -\frac{g}{\hbar c} (\phi - \phi^*) \psi^\dagger i \gamma_5 .$$

$$\gamma_\rho \partial_\rho \chi = \frac{g}{\hbar c} (\phi - \phi^*) i \gamma_5 \psi$$

By analogy with electromagnetic theory, in which the field equations $\partial_\rho \partial_\rho A_\sigma = 0$ in the absence of sources of four-current, and are $\partial_\rho \partial_\rho A_\sigma = J_\sigma$ in the presence of sources, the non-zero terms on the right hand side of the above field equations are called source terms. The source terms are a consequence of the interaction of the fields.

It should be pointed out that the conservation laws for energy, momentum, and charge, as derived in the earlier chapters, hold true for fields in interaction, for the derivations were not dependent upon the form of the Lagrangian density except in requiring $\frac{\partial L}{\partial (x_\rho)} = 0$. The reader should verify that the conservation of angular momentum including spin for interacting spinor and tensor fields (where tensor fields include the scalar, vector, pseudoscalar, pseudovector, and any rank of covariant tensor fields), follows directly, with the definition of spin simply a sum of those for Lagrangian densities depending on spinor and on tensor field functions respectively. The fact that the conservation laws and the corresponding definitions of conserved quantities in terms of a Lagrangian density, hold for such a variety of systems, indicates the value of the Lagrangian formalism. If a Lagrangian density for a field system can be found, it is a routine matter to deduce the form of the constants of the field motion.

As a first step to calculating the average lifetime for π - mesons decaying to μ - mesons and neutrinos, we must obtain the second quantized operator $\mathcal{H}'$. Having written:

$$L = L_0 + L',$$

the energy momentum tensor has a form:

$$T_{\rho\sigma} = \sum_{\alpha}\left(\partial_\rho \Phi_\alpha \frac{\partial L_0}{\partial(\partial_\sigma \Phi_\alpha)} - L_0 \delta_{\rho\sigma}\right) + \left(\partial_\rho \Phi_\alpha \frac{\partial L'}{\partial(\partial_\sigma \Phi_\alpha)} - L'\delta_{\rho\sigma}\right)$$

$$= T^{(0)}_{\rho\sigma} + T'_{\rho\sigma}.$$

Now $\dfrac{\partial L'}{\partial(\partial_\sigma \Phi_\alpha)} = 0$ as noted before, so that $T'_{\rho\sigma} = -L'\delta_{\rho\sigma}$. and therefore $H' = T'_{44} = -L'$. Using the shortened form (5.6) for L', and the partial Fourier series (5.8) for the field functions, $\mathcal{H}'$ is:

$$\mathcal{H}' = -\frac{g}{\ell^{9/2}} \int_V dV \sum_{\underline{k}_\mu, \underline{k}'_\nu, s, s'} q_0(\pi) q^*_{ks+}(\mu) q_{k's'-}(\nu)\, u^\dagger_{ks+}(\mu)\, i\gamma_5\, u_{-k's'-}(\nu)\, e^{i(-\underline{k}'_\nu - \underline{k}_\mu)\cdot \underline{r}}$$

Now: $$\int_V e^{i(-\underline{k}'_\nu - \underline{k}_\mu)\cdot\underline{r}}\, dV = \ell^3\, \delta_{\underline{k}'_\nu, \underline{k}_\mu}.$$

That is, only terms of the sum contribute to $\mathcal{H}'$ which conserve momentum for the process, for in the second quantized theory $(\hbar \underline{k}_\mu)$ and $(\hbar \underline{k}_\nu)$ correspond to the momenta of μ - meson and neutrino respectively. Then:

$$\mathcal{H}' = \frac{-g}{\ell^{3/2}} \sum_{\underline{k},s,s'} q_o(\pi)\, q^*_{ks+}(\mu)\, q_{-ks'-}(\nu) [u^\dagger_{ks+}(\mu)\, i\gamma_5\, u_{+ks'-}(\nu)]$$

Quantization consists of the replacements:

$$q_o(\pi) \rightarrow \sqrt{\frac{\hbar}{2\omega_o}} \cdot a_o^{(+)}(\pi) = \frac{\hbar}{c(2m_\pi)^{1/2}} \cdot a_o^{(+)}(\pi).$$

$$q^*_{ks+}(\mu) \rightarrow a^{(+)*}_{ks}(\mu).$$

$$q_{-ks'-}(\nu) \longrightarrow a^*_{-ks'}(\bar{\nu}).$$

Finally $\mathcal{H}'$ has the operator form:

$$(5.9)\quad \mathcal{H}' = \frac{-\hbar g\, a_o^{(+)}(\pi)}{c(2m_\pi)^{1/2}\ell^{3/2}} \sum_{k,s,s'} [u^\dagger_{ks+}(\mu)\, i\gamma_5\, u_{+ks'-}(\nu)]\, a^{(+)*}_{ks}(\mu)\, a^*_{-ks'}(\bar{\nu}).$$

The summation over $\underline{k}$ and s accounts for all possible directions for the decay products as well as spins. The transi tion probability per unit time defined by

$$\frac{2\pi}{\hbar} \rho_F \left| \langle f | \mathcal{H}' | i \rangle \right|^2.$$

will not depend on the momentum direction or spins of the decay products. In evaluating

$$|\langle f | \mathcal{H}' | i \rangle|^2 = (\Psi_f^*, \mathcal{H}' \Psi_i)(\Psi_i^*, \mathcal{H}'^* \Psi_f).$$

if Ψ_i is normalized, $\Psi_i \Psi_i^* = 1$, and the double sum resulting from $(\mathcal{H}', \mathcal{H}'^*)$ reduces, due to orthogonality of spinors for a given type of particle corresponding to different spins and different momenta, to a single sum over $\underline{k}$ and s. In this way the reader should verify that the transition probability can be calculated for transition to given momentum and spin states, and then a summation performed. At this stage it is evident that making L′ Hermitian in (5.7) renders the operator $\mathcal{H}'$ likewise Hermitian.

The sum (5.9) is over all magnitudes as well as directions of $\underline{k}$, but the transition probability per unit time involves evaluating $\langle f|\mathcal{H}'|i\rangle$ for the momentum which corresponds to energy conservation for the process. For this, $\underline{k}$ must satisfy the equation:

$$(5.10) \qquad m_\pi c^2 = [m^2c^4 + (\hbar \underline{k})^2 c^2]^{1/2} + |(-\hbar\underline{k})|c .$$

which is a condition only on $|\underline{k}|$: the terms in the summation over direction of $\underline{k}$ still remain. Solving equation (5.10) for k we find:

$$(5.11) \qquad k = \frac{(m_\pi^2 - m^2)c}{2\hbar m_\pi}.$$

There remains the calculation of the density of final states in the vicinity of final state energy equal to $(m_\pi c^2)$, that is, equal to the initial energy. The final state energy

$$E = E_k(\mu) + E_k(\nu) = (m^2c^4 + \hbar^2k^2c^2)^{1/2} + \hbar ck$$

where k has the above value, is a function of $|\underline{k}|$ only. The density of final states times dE is the number of final states embraced by an energy range (E, E + dE). dE is related to dk by:

(5.12) $$dE = \hbar c \left(\frac{\hbar c k}{E_k(\mu)} + 1 \right) dk.$$

Since the final states are limited to a fixed magnitude of $\underline{k}$ it is convenient to calculate the number of final states in the momentum range (k, k + dk). If the latter number is dN, then:

(5.13). $$dN = \rho_F(E)\, dE.$$

Now $\underline{k}$ is defined by

$$\underline{k} \equiv \frac{2\pi}{l} (n_{k1}, n_{k2}, n_{k3}).$$

where n_{kj} are integers or zero. If instead of coordinate space we consider a $\underline{k}$-space, the distance between values of k_j or between cubical lattice points, is $\frac{2\pi}{l}$. If the volume V is large the number of lattice points, or states, is approximately equal to the number of lattice cubes. In this case the volume per state is $\left(\frac{2\pi}{l}\right)^3$. For a fixed value of $\underline{k}$ one needs consider only a spherical shell in k-space, bounded by the surfaces of radii k and (k + dk). The volume of the shell is $(4\pi k^2 dk)$ and the number of final states contained is

$$dN = \frac{4\pi k^2 dk}{\left(\frac{2\pi}{l}\right)^3} = \frac{l^3 k^2 dk}{2\pi^2}.$$

It follows from equations (5.12) and (5.13) that:

$$\rho_F(E) = \frac{\ell^3}{2\pi^2} \cdot \frac{k^2}{\hbar c} \cdot \frac{E_k(\mu)}{m_\pi c^2}$$

Since at the appropriate value of k, $E_k(\mu) + \hbar ck = m_\pi c^2$. Substitution of the value of (5.11) for k, and simplification leads to

$$w(k) = \frac{2\pi}{\hbar} \rho_F(E) \left| \langle f | \mathcal{H}' | i \rangle \right|^2$$

$$= \frac{2\pi}{\hbar} \cdot \frac{\ell^3 k^2 E_k(\mu)}{2\pi^2 \hbar c^3 m_\pi} \cdot \frac{g^2}{\ell^3} \cdot \frac{\hbar^2}{2 m_\pi c^2} \left| \langle f | \cdots | i \rangle \right|^2.$$

$$= \frac{g^2 (m_\pi^2 - m^2)^2 (m_\pi^2 + m^2)}{8\pi \hbar^2 c\, m_\pi^5} \left| \langle f | \sum_{\substack{k \\ s,s'}} \left[u^\dagger_{ks+}(\mu)\, i\gamma_5\, u_{+ks'-}(\nu) \right] a_0^{(+)}(\pi)\, a_{ks}^{(+)*}(\mu)\, a^*_{-ks'}(\bar{\nu}) | i \rangle \right|^2$$

where summation over $\underline{k}$ is over direction only, with magnitude fixed by equation (5.11). The value of $w(k)$ can be used to calculate the half-life of the π -meson for decay from rest to a neutrino and a μ^+- meson. (See Appendix, page 117.)

Neutron-Proton Scattering: Virtual States.

The problem of scattering of charged particles by other charged particles, due to the electromagnetic interaction between them, was one of the earliest of quantum mechanical problems. As the study of nucleons progressed, it became evident that another type of force was present between some elementary particles, such as the force which produced the scattering of neutrons by protons. Experiments showed that this force was of short range relative to the Coulomb-type force, and the development of a theory by Yukawa, based on a force of the type $\frac{e^{-\mu r}}{r}$, where μ is a positive constant and r the distance from the force centre, led to the concept of a field whose quanta had non-zero rest mass, as contrasted with the electromagnetic field, whose quanta are massless. The quanta of the nuclear field were called mesons, and nuclear interactions were thought of as involving an interchange of quanta, as electromagnetic interactions in the quantized theory are described by interchanges of photons. In recent years the π - meson has been associated with nuclear forces, and consequently a second quantized field theory of nucleon-nucleon scattering must include, in the interaction term, the field functions for the nucleons and the π - meson field function.

If ϕ, ψ, and χ represent the field functions for the π - meson, the neutron, and the proton respectively, then as in the discussion of π - meson decay, since nucleons are Dirac particles, we choose an interaction term of the form:

$$L' = g\left\{ \phi^* \psi^\dagger i\gamma_5 \chi + \phi^* \chi^\dagger i\gamma_5 \psi - \phi \psi^\dagger i\gamma_5\chi - \phi\chi^\dagger i\gamma_5\psi \right\} \tag{5.14}$$

Since we are not considering derivative coupling, $T'_{\mu\nu}$ reduces to $(-L'\delta_{\mu\nu})$, so that the energy of the interaction is:

$$\mathcal{H}' = \int_V (-L')\, dV.$$

It is easily verified that the above form for L' leads to zero-valued matrix elements for neutron-proton scattering in first order. For such a process the initial and final states are identically zero when operated upon by annihilation operators for π- mesons, since they both correspond to occupation numbers $N_k(\pi) = 0$ for all $\underline{k}$. The terms in ϕ, in the second quantized theory, give $a_k^{(+)}$ and $a_k^{(-)}$, both of which operating on the initial state function are zero.

$$(\Psi_f^*, a_k^{(\pm)} \Psi_i) = 0.$$

Similarly the terms in ϕ^* provide $a_k^{(\pm)*}$, and

$$(\Psi_f^*, a_k^{(\pm)*}\Psi_i) = ((a_k^{(\pm)} \Psi_f)^*, \Psi_i) = 0.$$

Since first order transitions cannot occur, second order perturbation theory must be applied.[15] The transition probability per unit time is then given by

$$w(E) = \frac{2\pi}{\hbar} \rho_F(E) \left| \sum_n \frac{\langle f|\mathcal{H}'|n\rangle\langle n|\mathcal{H}'|i\rangle}{(E_0 - E_n)} \right|^2$$

where n numbers intermediate states, for which energy is not conserved with the initial state. The intermediate states are not necessarily devoid of π - mesons, and therefore matrix elements of the type $\langle n|\mathcal{H}'|i\rangle$ and $\langle f|\mathcal{H}'|n\rangle$ are not necessarily zero. In order that the qualitative features of the theory may be most clearly brought out we shall consider processes into

[15] Schiff: pp. 195-6.

which enter only positive π- mesons, neutrons, and protons. We shall therefore use the partial Fourier expansions:

$$\phi = \frac{1}{\ell^{3/2}} \sum_k q_k(\pi) e^{i\underline{k}.\underline{r}}.$$

$$\psi = \frac{1}{\ell^{3/2}} \sum_{k,s} q_{ks+}(N)\, u_{ks+}(N)\, e^{i\underline{k}.\underline{r}}$$

$$\chi = \frac{1}{\ell^{3/2}} \sum_{k,s} q_{ks+}(P)\, u_{ks+}(P)\, e^{i\underline{k}.\underline{r}}$$

and their Hermitian conjugates. We will also postulate charge conservation between initial and intermediate states. This postulate is necessary because the interaction Lagrangian density is not invariant under infinitesimal gauge transformations of the type

$$\Phi_\alpha' = \Phi_\alpha e^{i\beta} \approx \Phi_\alpha(1+i\beta)$$

where Φ_α is the set (ϕ, ψ, χ), and β is an infinitesimal real number, for under such transformation L' has terms

$$\phi^*(1-i\beta)\psi^\dagger(1-i\beta)\, i\gamma_5 \chi(1+i\beta)$$

$$= \phi^*\psi^\dagger i\gamma_5\chi(1-i\beta)(1+\beta^2)$$

$$\approx \phi^*\psi^\dagger i\gamma_5\chi(1-i\beta).$$

having dropped terms in powers of β beyond the first. The variation of L' is then of the form $(i\beta L')$, which cannot be incorporated in a continuity equation. An elegant formalism involving the concept of isotopic spin space has been developed, by Kemmer, called the symmetric meson theory,[16] in which it is possible to set up gauge invariant interaction terms. The postulate of charge conservation at every stage of the interaction process, while less elegant, serves the same purpose as gauge invariance. With this requirement and using the above partial Fourier series, only the terms

$$(5.15) \qquad g\left(\phi^{*}\psi^{\dagger} i\gamma_5 \chi - \phi\, \chi^{\dagger} i\gamma_5 \psi\right)$$

of the Lagrangian density (5.14) need be considered. For an initial state describing a proton at rest and a neutron with momentum $\underline{k}_0$, on second quantization the first term of (5.15) contributes terms having non-zero matrix elements, of the form:

$$\frac{g}{\ell^{3/2}}\,\frac{\hbar}{c(2m_\pi)^{1/2}}\; a^{*}_{-k}(\pi)\, a^{*}_{ks}(N)\, a_{os'}(P)\left[u^{\dagger}_{ks+}(N)\, i\gamma_5\, u_{os+}(P)\right].$$

The matrix element of such an operator indicates the process:

$$(5.16) \qquad P(o) \longrightarrow N(\underline{k}) + \pi^{+}(-\underline{k}).$$

where the bracketed quantities are momenta. Momentum is conserved in the process because of

$$\mathcal{H}' = \int_V (-L')\,dV.$$

Only terms of the Fourier expansion which correspond to momentum conservation have a non-zero integral over V. That energy is

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[16] For reference and discussion, see Wentzel, p. 63.

not conserved in the above "virtual" process can be seen from the energy balance equation:

$$m_P c^2 = (m_N^2 c^4 + \hbar^2 c^2 k^2)^{1/2} + (m_\pi^2 c^4 + \hbar^2 c^2 k^2)^{1/2}$$

which cannot be satisfied by any value of k, even k = 0, since:

$$m_P = 1836.6\ m_e;\ m_N = 1839.1\ m_e;\ m_\pi = 273.3\ m_e$$

where m_e is the mass of the electron. Similarly the second term of (5.15) having non-zero matrix elements only between intermediate states n and the final state, contributes through terms of the form

$$\frac{g}{\ell^{3/2}} \frac{\hbar}{c(2m_\pi)^{1/2}}\, a_{-\underline{k}}(\pi)\, a_{k_0 s''}(N)\, a^*_{\underline{k}_0-\underline{k},s'''}(P) \left[u^\dagger_{\underline{k}_0-\underline{k},s'''}(P)\, i\gamma_5\, u_{k_0,s''}(N) \right]$$

corresponding to virtual processes of the type:

$$(5.17) \qquad \pi^+(-\underline{k}) + N(\underline{k}_0) \rightarrow P(\underline{k}_0 - \underline{k}).$$

In the calculation of transition probability per unit time, the product $\langle f|\mathcal{H}'|n\rangle\langle n|\mathcal{H}'|i\rangle$ then describes a process which is the combination of (5.16) and (5.17), namely:

$$P(o) + N(\underline{k}_0) \rightarrow N(\underline{k}_0) + N(\underline{k}) + \pi^+(-\underline{k}) \rightarrow N(\underline{k}) + P(\underline{k}_0 - \underline{k}).$$

The scattering may be picturesquely described by considering the proton at rest, under the influence of the incoming neutron, to emit a π^+-meson and become a neutron. The incident neutron then absorbs the π^+-meson, becoming a proton. The scattering involves the exchange of a virtual meson.

The matrix elements used in calculating $w(E)$ must be such that energy is conserved between initial and final states. This serves to determine both the magnitude and direction of (-k), the intermediate meson momentum , for $\underline{k}$ must satisfy the energy conservation equation:

$$m_p c^2 + [m_N^2 c^4 + \hbar^2 c^2 k_0^2]^{1/2} = [m_N^2 c^4 + \hbar^2 c^2 k^2]^{1/2} + [m_p^2 c^4 + \hbar^2 c^2 (\underline{k}_0 - \underline{k})^2]^{1/2}$$

For initial and final states which are specified completely as to spins and momenta of initial and final particles, the summation over n, the intermediate states is just a summation over intermediate spins.

β - Decay.

The π - meson decay is an example of two-particle decay. An example of three-particle decay will now be discussed, namely the decay of a neutron into an electron, a proton, and a neutrino. The process is known to occur spontaneously for free neutrons and for neutrons in certain nuclear configurations. For the process

$$N \rightarrow P + e^- + \bar{\nu}.$$

an interaction between neutron, proton, electron, and neutrino fields is required. Let ψ , χ , ξ , η represent the corresponding field functions: all are Dirac type particles. The interaction in which we are interested may be represented by a form:

$$L' = g(\chi^{\dagger} \mathcal{O} \psi)(\xi^{\dagger} \mathcal{O} \eta) \tag{5.18}$$

where $\mathcal{O}$ represents some combination of Dirac matrices such that $(\chi^{\dagger} \mathcal{O} \psi)$ is a covariant, and the product of the two factors in L' is such as to produce a Lorentz invariant quantity. Other terms similar to (5.18), describing the interaction of the four fields in question are also possible, but if the initial state is to have only one neutron and the final state one proton, one electron, and one neutrino, the other terms, such as $(\psi^{\dagger} \mathcal{O} \chi)(\xi^{\dagger} \mathcal{O} \eta)$, in the second quantized form will have zero matrix elements between such states. That the term (5.18) is suitable to describe the process of β-decay follows when one notes that η contains, in second quantized form, both the annihilation operators for neutrinos, and the creation operators for antineutrinos. As far as our theory is concerned, there is no distinction between neutrinos and antineutrinos, both being uncharged.

The calculation of decay rates using different operators $\mathcal{O}$ in L' in general leads to different results. Experimental data on nuclear β-decay as now interpreted indicates that a linear combination of scalar, tensor, and pseudoscalar couplings, and no others, is needed for the theory to agree with observations[17]. The relative

[17] C.f. Konopinski and Langer: Ann. Rev. Nuc. Sci., 2, 261-304, (1953).

strengths of each of the components (scalar, tensor, and pseudoscalar) have not yet been determined, because it is difficult to relate the simple theory as presented here to the problem of a complex nuclear structure undergoing β-decay[18]. For example, the electromagnetic interaction between β-particle (electron) and the positively charged nucleus in general is not negligable, though it will not be discussed here. Two of the most commonly used interaction forms are the Fermi type, where $\mathcal{O}$ is either 1 or γ_μ (scalar or vector interaction), and the Gamow-Teller type, in which $\mathcal{O}$ is $(i\gamma_5\gamma_\mu)$ or $(\gamma_\mu\gamma_\nu)$, (pseudo-vector or tensor interaction). Without specifying the operator $\mathcal{O}$, we can write down the form of the operator $\mathcal{H}'$ in the second quantized theory. For this, only the following partial Fourier series need to be considered:

$$\chi^\dagger = \ell^{-3/2} \sum_{k,s} u^\dagger_{ks+}(P)\, a^*_{ks}(P)\, e^{-i\underline{k}\cdot\underline{r}}$$

$$\psi = \ell^{-3/2} \sum_{k,s} u_{os+}(N)\, a_{os}(N).$$

$$\xi^\dagger = \ell^{-3/2} \sum_{k,s} u^\dagger_{ks+}(e^-)\, a^*_{ks}(e^-)\, e^{-i\underline{k}\cdot\underline{r}}$$

$$\eta = \ell^{-3/2} \sum_{ks} u_{-ks-}(\nu)\, a^*_{ks}(\bar{\nu})\, e^{-i\underline{k}\cdot\underline{r}}$$

Substitution into

$$\mathcal{H}' = \int_V \left[-q(\chi^\dagger \theta \psi)(\xi^\dagger \theta \eta)\right] dV$$

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18 For a lucid though brief discussion of the problems involved see Fermi: "Elementary Particles", p. 39, (Yale, 1951). This book is particularly valuable for the discussion of principles of elementary particle theory, although it assumes some knowledge of the details of the subject. The reader of this thesis would profit from the further discussion of interactions in Fermi's book.

results in terms of the form:

$$a^*_{k_1 s_1}(P)\, a_{0 s_2}(N)\, a^*_{k_3 s_3}(e^-)\, a^*_{k_4 s_4} \left[u^\dagger_{k_1 s_1 +}\, \theta\, u_{0 s_2 +}\right]\left[u^\dagger_{k_3 s_3 +}\, \theta\, u_{k_4 s_4 -}\right]$$

where $(\underline{k}_1 + \underline{k}_3 + \underline{k}_4) = 0.$

To calculate the transition probability per unit time for decay of a neutron at rest one must evaluate $\frac{2\pi}{\hbar} \rho_F |\langle f|\mathcal{H}'|i\rangle|^2$ where the initial state is that of a neutron at rest with specified spin, and the final state is with an electron, a proton, and an antineutrino, each having specified spins and momenta, such that energy is conserved with the initial state. Averaging over the initial states, which in this case involves summation over spin states and division by two since there are two equally probable such states, and summation over all final states, then gives the transition probability per unit time, which is the reciprocal of the average lifetime.

CONCLUSION.

It should not be thought that this thesis claims to present a balanced discussion of its title subject. Great emphasis has been given to the introductory concepts of field theory and of second quantization, while the subject of interactions has been only sketchily covered. Had time permitted, a detailed discussion of the interactions of fields and the effects of quantization in the interactions would have been desirable. Such a treatment would have required more space than the first four chapters of the thesis now fill. However, this lack is not viewed as an important shortcoming: the literature lacks clear discussion of the introductory concepts covered in the first four chapters, without which a detailed study of interactions is impossible. It is hoped that this thesis provides such background as to enable the reader to study in detail interactions of the sort outlined in Chapter V.

A more serious lack is felt to be the absence of the concepts of isotopic spin and of isotopic spin space, for these ideas are also difficult to understand from the general literature of physics, but are assuming a role of basic importance in the nascent theories of the "new" elementary particles: hyperons and heavy mesons.[19] The

[19] For phenomenological discussion see Bethe et al: "Mesons and Fields", Vol. 2. The most thorough and successful theoretical attempts to date are due to D'Espagnat and Prentki: Nuclear Physics, 1, 33, (1956), and Phys. Rev. (to be published). The subject of the new particles is growing at a fast rate, and only perusal of the most recent issues of the journals of physics can assure one of up-to-date information.

omission of isotopic spin, as with other major omissions from this thesis, is due to the practical limitations of time available for writing the thesis.

It should be reiterated that the theory as here presented is not considered to be of general validity for elementary particles. The subject of elementary particles and their interactions, including electromagnetic interactions, is really in its early stages, and the theory given here forms a basis for many of the theoretical attempts now in progress. In some places the theory works fairly well, in many others it clearly does not apply: it may at best be a first approximation to a full theory of the subject. This thesis, then, has been written as a contribution to the pedagogical needs of the immediate future, without in any way purporting to present a finalized theory.

APPENDIX.

The calculation of the transition probability per unit time for π-meson decay will be completed here. The expression given on page 105 is:

$$(A.1)\quad w(k) = \frac{g^2(m_\pi^2 - m^2)^2(m_\pi^2 + m^2)}{8\pi\hbar^2 c\, m_\pi^5}\left|\sum_{\substack{k\\ ss'}} \langle f|a_0^{(+)}(\pi)\,a_{ks}^{(+)*}(\mu)\,a^*_{-ks'_-}(\bar\nu)|i\rangle \left[u_{ks}^{(+)\dagger}(\mu)\,i\gamma_5\, u_{+ks'_-}(\nu)\right]\right|^2$$

where summation over k is over direction only, the magnitude of k being fixed according to equation (5.11). For the moment, if we assume that the final state function specifies the momentum direction and the spins s and s', then only one term of the sum has non-zero matrix element. Then w(k) has the form:

$$w(\underline{k},s,s') = \text{const.}\left|\left[u_{ks}^{(+)\dagger}(\mu)\,i\gamma_5\, u_{+ks'_-}(\nu)\right]\langle f|a_0^{(+)}(\pi)\,a_{ks}^{(+)*}(\mu)\,a^*_{-ks'_-}(\bar\nu)|i\rangle\right|^2.$$

The matrix element contributes a factor of modulus unity, as can be seen from the following considerations. We will assume that initial and final state functions are normalized. Then in inner product notation: $(\Psi_i^*, \Psi_i) = 1$. Also: $a_{ks}^{(+)*}(\mu)\,\Psi_i = c_\mu \Psi_f$, where c_μ is a constant to be determined, for the creation operator $a_{ks}^{(+)*}(\mu)$ operating on the initial state function gives a constant times the final state function, as the latter describes a state having one more μ-meson of type $ks^{(+)}$ than does Ψ_i. We also require that: $(\Psi_f^*, \Psi_f) = 1$, for normalization. It follows that:

$$([c_\mu \Psi_f]^*, c_\mu \Psi_f) = |c_\mu|^2 = ([a_{ks}^{(+)*}(\mu)\,\Psi_i]^*, a_{ks}^{(+)*}(\mu)\,\Psi_i)$$

$$= (\Psi_i^*, a_{ks}^{(+)}(\mu)\, a_{ks}^{(+)*}(\mu)\,\Psi_i).$$

From the anticommutation of the a's and from the fact that the number-of-μ-mesons operator $[a_{ks}^{(+)*}(\mu)\, a_{ks}^{(+)}(\mu)]$ gives zero operating on Ψ_i, the last equation reduces to:

$$|c_\mu|^2 = (\Psi_i^*, [1-0]\,\Psi_i) = (\Psi_i^*, \Psi) = 1.$$

Similarly the antineutrino creation operator introduces, in the matrix element, a constant factor of modulus unity. The reader should show that the same is true for the π-meson annihilation operator, using the <u>commutation</u> of the $a_0^{(+)}(\pi)$ and $a_0^{(+)*}(\pi)$ and thefact that the number-of-π-mesons operator on Ψ_i gives unity. w(k) then reduces to:

$$w(\underline{k}, s, s') = \text{const} \times 1 \times |\, u_{ks}^{(+)\dagger}(\mu)\, i\gamma_5\, u_{+ks'_-}(\nu)|^2$$

(A.2) $$= \text{const}\,[u_{+ks'_-}^{*}(\nu)(-i\gamma_5)\gamma_4\, u_{ks}^{(+)*}(\mu)][u_{ks}^{(+)\dagger}(\mu)\, i\gamma_5\, u_{+ks'_-}(\nu)].$$

since $u^\dagger = u^*\beta = u^*\gamma_4$, and since γ_5 is Hermitian.

Since we are not interested in the spins of the resultant particles (to observe these would vastly complicate an experiment), we must add up the contributions to w(k) from all possible spin states. The method of performing such summations is adequately discribed in Schweber, pps. 49-53, and will be applied in the following without discussion. From equation (A.2) we are interested in:

(A.3) $$\sum_{ss'} w(\underline{k}, s, s') = \text{const} \sum_{ss'} [u_{+ks'_-}^{*}(\nu)(-i\gamma_5)\gamma_4\, u_{ks}^{(+)}(\mu)][u_{ks}^{(+)\dagger}(\mu)\, i\gamma_5\, u_{+ks'_-}(\nu)].$$

The summation is extended over all four spin states of (μ), (two positive energy, and two negative energy), by introducing the projection operator:

$$\frac{\epsilon_\mu + (\hbar c\, \underline{\alpha}\cdot\underline{k} + mc^2\beta)}{2\epsilon_\mu} = \left(\frac{\epsilon_\mu + H_\mu}{2\epsilon_\mu}\right).$$

where:

(A.4) $$\epsilon_\mu = +(m^2c^4 + \hbar^2c^2k^2)^{1/2},$$

k being fixed by equation (5.11). This projection operator is suitable because the non-spatially-dependent spinors $u^{(+)}_{ks}(\mu)$ satisfy the equation:

$$[c\underline{\alpha}.(\hbar\underline{k}) + mc^2\beta]\, u^{(+)}_{ks}(\mu) = \epsilon_\mu\, u^{(+)}_{ks}(\mu).$$

The spin summations (A.3) then reduce to:

$$\sum_{ss'} w(\underline{k},s,s') = \sum_{s'} \text{const.}\left[u^{*}_{+ks'-}(\nu)(-i\gamma_5)\gamma_4\left(\frac{\epsilon_\mu + H_\mu}{2\epsilon_\mu}\right)\gamma_4 i\gamma_5\, u_{+ks'-}(\nu)\right].$$

Introducing the further projection operator

$$\frac{\epsilon_\nu - [\underline{\alpha}.(+\hbar c\underline{k})]}{2\epsilon_\nu} = \left(\frac{\epsilon_\nu - H_\nu}{2\epsilon_\nu}\right)$$

where

(A.5) $$\epsilon_\nu = (\hbar c k),$$

the spin summations reduce to the problem of evaluating the trace (sum of diagonal elements) of the matrix:

$$(-i\gamma_5)\gamma_4\left(\frac{\epsilon_\mu + H_\mu}{2\epsilon_\mu}\right)\gamma_4 i\gamma_5\left(\frac{\epsilon_\nu - H_\nu}{2\epsilon_\nu}\right).$$

That is, we must evaluate:

(A.6) $$\sum_{ss'} W(\underline{k},s,s') = \frac{\text{const}}{4\epsilon_\mu\epsilon_\nu}\, Tr\left\{\gamma_5\gamma_4(\epsilon_\mu + \hbar c\underline{k}.\underline{\alpha} + mc^2\beta)\gamma_4\gamma_5(\epsilon_\nu - \hbar c\,\underline{k}.\underline{\alpha})\right\}$$

Using the anticommutation in pairs of the γ's, including γ_5 and the definitions $\gamma_j = -i\beta\alpha_j$ $(j = 1,2,3)$; $\gamma_4 = \beta$, (A.6) becomes

(A.7) $$\sum_{ss'} w(\underline{k},s,s') = \frac{\text{const}}{4\epsilon_\mu\epsilon_\nu}\, Tr\left\{\gamma_5(\epsilon_\mu\gamma_4 + i\hbar c\gamma_j k_j + mc^2)(-\gamma_5)(\epsilon_\nu\gamma_4 - i\hbar c\gamma_j k_j)\right\}$$

$$= \frac{\text{const.}}{4\epsilon_\mu\epsilon_\nu}\, Tr\left\{(\epsilon_\mu\gamma_4 + i\hbar c\gamma_j k_j - mc^2)(\epsilon_\nu\gamma_4 - i\hbar c\,\gamma_j k_j)\right\}$$

since $\gamma_5^2 = I$. Using the matrix representation of α, β given in Chapter IV, it is easily demonstrated that

$$Tr(\gamma_4 \gamma_j) = Tr(\gamma_j \gamma_4) = 0;$$

$$Tr(\gamma_j) = Tr(\gamma_4) = 0,$$

and since $\gamma_\rho^2 = I$ $(\rho = 1,2,3,4,)$; $Tr(\gamma_\rho^{\,2}) = 4$, where I is the 4X4 identity matrix. (A.7) then becomes:

$$\text{(A.8)} \quad \sum_{ss'} w(\underline{k}, s, s') = \frac{\text{const.}}{4\epsilon_\mu \epsilon_\nu} Tr\left\{ \epsilon_\mu \epsilon_\nu \gamma_4^2 - \hbar^2 c^2 (\gamma_j k_j)^2 \right\}.$$

It is a straightforward problem to show that:

$$(\gamma_j k_j)^2 = (-i\beta \underline{\alpha}.\underline{k})(-i\beta \underline{\alpha}.\underline{k}) = (\underline{\alpha}.\underline{k})(\underline{\alpha}.\underline{k}) = k^2 I.$$

Finally, then, the spin summations (A.3) give, using (A.4), (A.5), and (A.8):

$$\text{(A.9)} \quad \sum_{ss'} w(\underline{k}, s, s') = \left(\frac{\epsilon_\mu \epsilon_\nu + \hbar^2 c^2 k^2}{\epsilon_\mu \epsilon_\nu} \right) = \left(\frac{\epsilon_\mu + \epsilon_\nu}{\epsilon_\mu} \right).$$

If we similarly are not to specify the momentum direction, we must sum over these. In the limit as the region under consideration goes to infinity, the summation over directions becomes an integration over solid angle. Since w(k) is independent of the direction of $\underline{k}$, this summation introduces a factor of 4π. Finally then, from (A.1) and (A.8):

$$\text{(A.10)} \quad w(k) = 4\pi . \frac{g^2 (m_\pi^2 - m^2)^2 (m_\pi^2 + m^2)}{8\hbar^2 c\, m_\pi^3} . \left(\frac{\epsilon_\mu + \epsilon_\nu}{\epsilon_\mu} \right),$$

which using (A.4), (A.5), and (5.11) reduces to:

$$w(k) = \frac{\pi g^2 (m_\pi^2 - m^2)^2}{\hbar^2 c\, m_\pi^3}.$$

It is desirable to discuss the dimensionality of the above expression. For H' we had:

$$H' = g\ \phi\ \psi^{\dagger} i \gamma_5 \chi.$$

which must have dimensions (energy)X(length)$^{-3}$. If ψ, ϕ, and χ have dimensions (length)$^{-3/2}$, then g must have dimensions (energy)X(length)$^{+3/2}$.

Then:

$$\frac{g^2 (m_\pi^2 - m^2)^2}{\hbar^2 c\, m_\pi^3}$$

has dimensions: $\left[\frac{(\text{energy})^2 X (\text{length})^3 X (\text{energy})}{(\text{energy})^2 X (\text{time})^2 X (\text{length})^3 X (\text{time})^{-3}}\right]$ = (energy)X(time).

where mass is taken as (energy)X(length)$^{-2}$X(time)^2, and ($\hbar$) $\sim$ (energy)X(time).

A factor $\left[1\ (\text{erg})^{-1} X (\text{sec.})^{-2}\right]$ in w(k) comes from $a_0^{(+)}(\pi$ in $|\langle f| \cdots |i\rangle|^2$.

If ϕ is of dimensions (length)$^{-3/2}$ then the Fourier coefficient $q_o(\pi)$ is non-dimensional. The q's and $a(\pi)$'s are related (see page 102) by:

$$q_o(\pi) \rightarrow \frac{\hbar}{c(2m)^{1/2}} \cdot a_o^{(+)}(\pi).$$

$a_o^{(+)}(\pi)$ therefore is an operator having dimensions (energy)$^{1/2}$X(energy)$^{-1}$X X(time)$^{-1}$ = (energy)$^{-1/2}$X(time)$^{-1}$, which in evaluating $|\langle f| \cdots |i\rangle|^2$ introduces (erg)$^{-1}$X(sec.)$^{-2}$ in the c.g.s. system of units. w(k) is therefore given only numerically by (A.10), but has dimensions (time)$^{-1}$, as a transition probability per unit time must have.