THE COLLINEATION GROUP

OF A

VEBLEN-WEDDERBURN NON-DESARGUESIAN PROJECTIVE GEOMETRY

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INTRODUCTION

In the latter part of the last century it was discovered that Desargues's theorem could not be proved in the plane from the axioms of incidence. Subsequently several non-Desarguesian plane projective geometries have been constructed.

It is the purpose of this report to investigate one such geometry from the point of view of collineation theory. The report is in four parts. Part I deals with the more important properties which distinguish Desarguesian from non-Desarguesian geometries. Part II deals with the special geometry discovered by Veblen and Wedderburn in [1]. Part III is concerned with the finding of the collineation group of this geometry and with some geometrical consequences. These geometrical consequences are special cases of some unpublished theorems of N. Mendelsohn. Part IV discusses the collineation group from the point of view of its abstract properties.

PART I.

DISTINCTION BETWEEN DESARGUESIAN AND NON-DESARGUESIAN GEOMETRY

A projective plane geometry is a mathematical system consisting of elements called points and lines respectively. These elements which are finite or infinite in number are connected by a relationship called "on" subject to the following conditions.

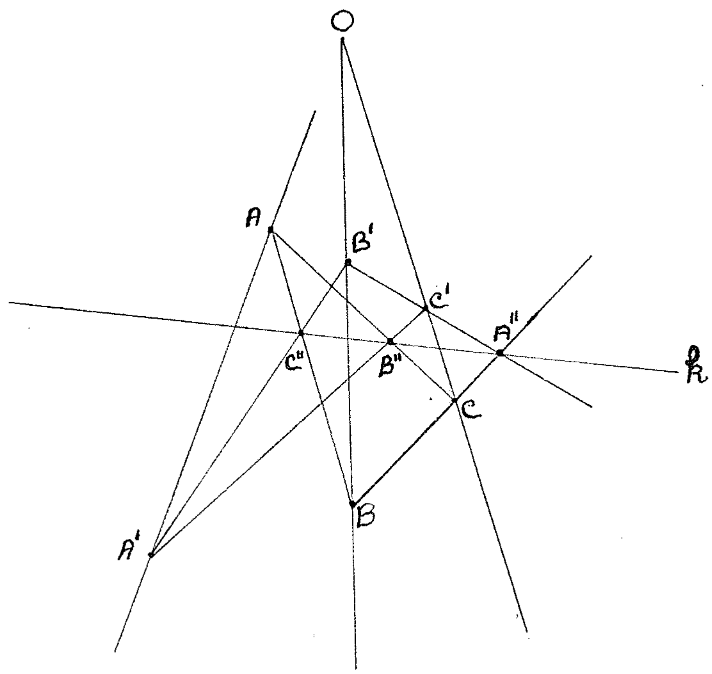
1. There is one and only one line on any two distinct points.
2. There is one and only one point which is on both of two distinct lines.
3. There are four points no three of which are on one line.

In projective geometry it is customary to denote lines by small letters of the alphabet and the points by capital letters of the alphabet. In cases where points P,Q,R are all on the same line "m" the expression 'P,Q,R are collinear' may be used. Similarly, if lines p,q,r are all on the same point P the expression 'p,q,r are concurrent' may be used. Furthermore the statement 'the line p passes through the point P' may be taken as a paraphrase of 'the point P is on the line p'.

Two point figures A,B,C,... and A',B',C',... are said to be centrally perspective from a point O (called the centre of perspectivity) if the lines AA',BB',CC',... are all on the point O. Two line figures a,b,c,... and a',b',c',... are said to be axially perspective on the line p (called the axis of perspectivity) if the points aa',bb',cc',... are all on p.

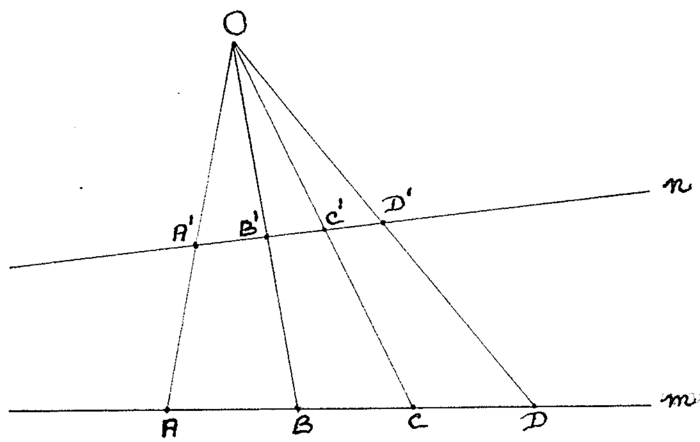
One of the most important theorems in projective geometry is that of Desargues which states that if two triangles are centrally perspective then they are axially perspective. At this point it is very important to note that for a projective geometry of two dimensions, Desargues's theorem is not an immediate consequence of the conditions of the first paragraph. In fact, there exist systems of points and lines for which all these conditions hold but for which Desargues's theorem is not generally true. Such a geometrical system is called non-Desarguesian. There are many examples of non-Desarguesian geometry, one of which, the special Veblen-Wedderburn system, will be described in part II of this paper.

It is a known fact that Desargues's theorem in the plane is equivalent to its converse. In other words, if Desargues's theorem is true for all centrally perspective pairs of triangles, its converse is true for all axially perspective pairs, and if Desargues's theorem fails for one pair of centrally perspective triangles, its converse fails for at least one pair of axially perspective triangles. A proof that Desargues's theorem implies its converse will now be given.



Consider the triangles ABC and A'B'C' which are axially perspective on the line k. i.e. AB meets A'B' at C", BC meets B'C' at A", CA meets C'A' at B", and A",B",C" are on k. Let the line CC' and the line BB' intersect in O. Apply Desargues's theorem to the triangles BB'C" and CC'B". The line BC" meets the line CB" in the point A. The line B'C" meets the line C'B" in the point A'. The line BB' meets the line CC' in the point O. Hence, since Desargues's theorem holds for the triangles BB'C" and CC'B", the points A,A', and O are collinear. Therefore the triangles ABC and A'B'C' are centrally perspective.

Now, suppose that in the plane there exists a set of points A,B,C,... on a line m, a set of points A',B',C',... on a line n distinct from m, and a point O not on m or n. If the lines AA',BB',CC',DD',... are all on the point O, the



relationship between the two figures is said to be a central perspectivity. This relationship can be expressed symbolically as follows:

$$m(A,B,C,\ldots) \overset{O}{\underset{\wedge}{-}} n(A',B',C',\ldots).$$

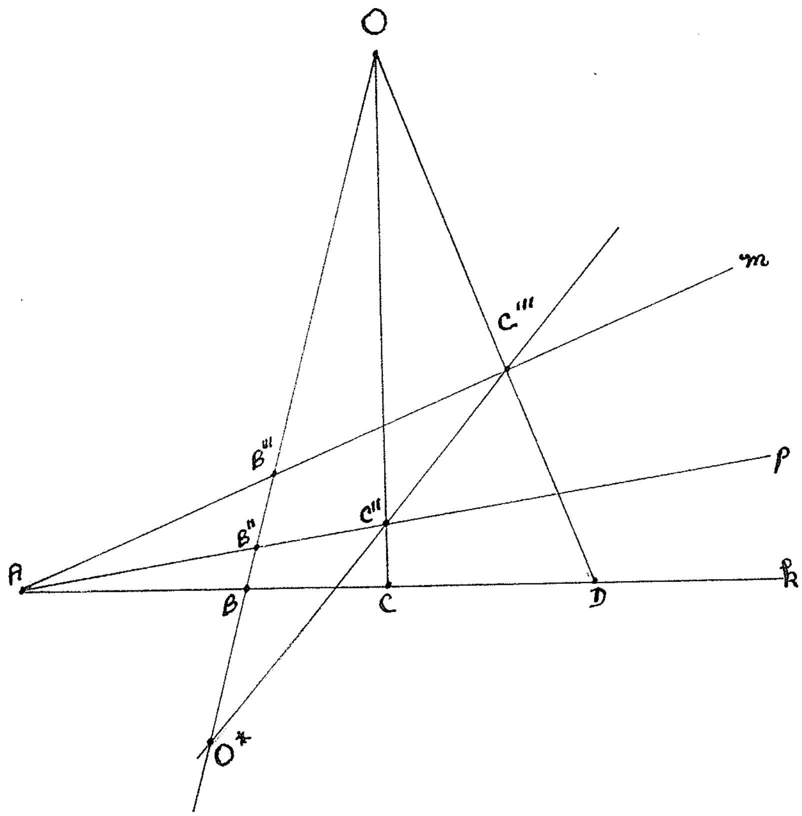
The points of a line m are said to be projectively related to those of a line n if the points of n are obtained from those of m as a result of a finite sequence of perspectivities. In other words

$$m(A,B,C,\ldots) \overset{O_1}{\underset{\wedge}{-}} m_1(A_1,B_1,C_1,\ldots) \ldots\ldots\ldots\ldots \overset{O'}{\wedge} n(A',B',C',\ldots).$$

In this case we will say

$$m(A,B,C,\ldots) \sim n(A',B',C',\ldots) .$$

Given any three points A,B,C on a line and any other three points A',B',C' on the same line there always exists a projective transformation (in this case a sequence consisting of two central perspectivities) which sends A into A', B into B', and C into C'. Further, if A,B,C,D are any four points on a line k, there exists a projective transformation which sends A into A, B into B, and C into D. For this latter case the following proof is given.



Let p,m be two arbitrary lines concurrent with k at the point A. Let O be any other point not on any of the lines m,p,k. Let OB and OC meet p at B'' and C'' respectively. Let OD meet m at C''' and OB meet m at B'''.

Let C''',C'' meet OB at O^*. Then

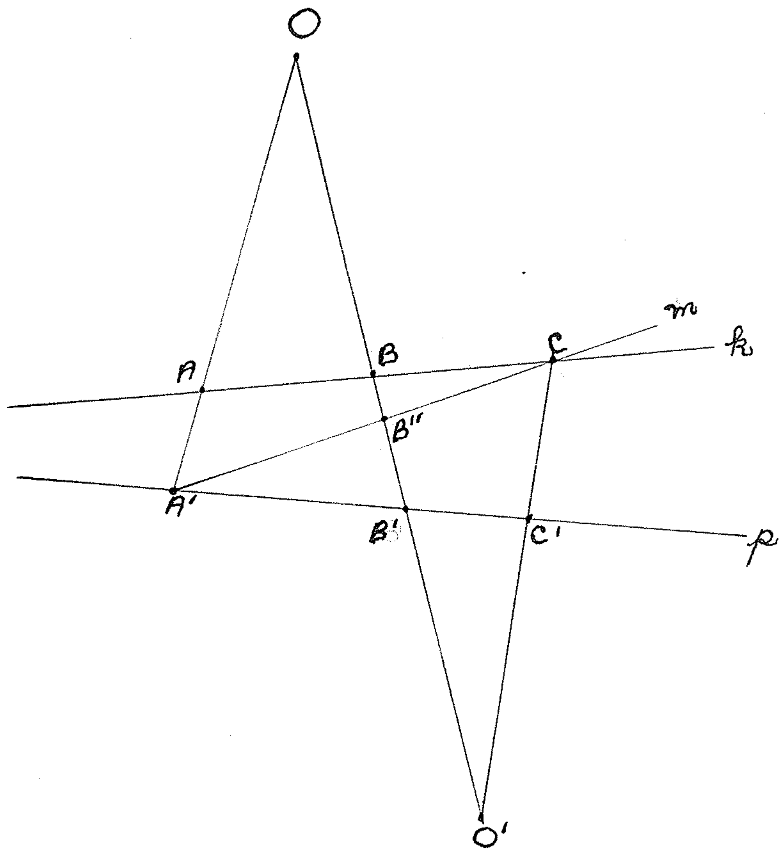
$$k(A,B,C) \overset{O}{\barwedge} p(A,B'',C'') \overset{O^*}{\barwedge} m(A,B''',C''') \overset{O}{\barwedge} k(A,B,D) .$$

Hence

$$k(A,B,C) \sim k(A,B,D) .$$

i.e. There exists a projective transformation which sends A into A, B into B, and C into D. By iteration of transformations of this type, any three distinct points on a line may be mapped into any other set of three points in the same line.

In the case of two distinct lines, a simpler proof can be given of the fact that any three points of the first line can be projectively related to any three points of the second in the following way.



Let A,B,C and A',B',C' be two sets of points on the lines k and p respectively. Let AA' and BB' meet at the point O, and let BB' and CC' meet in the point O'. Let m be the line A'C and let BB' meet m in B".

Then

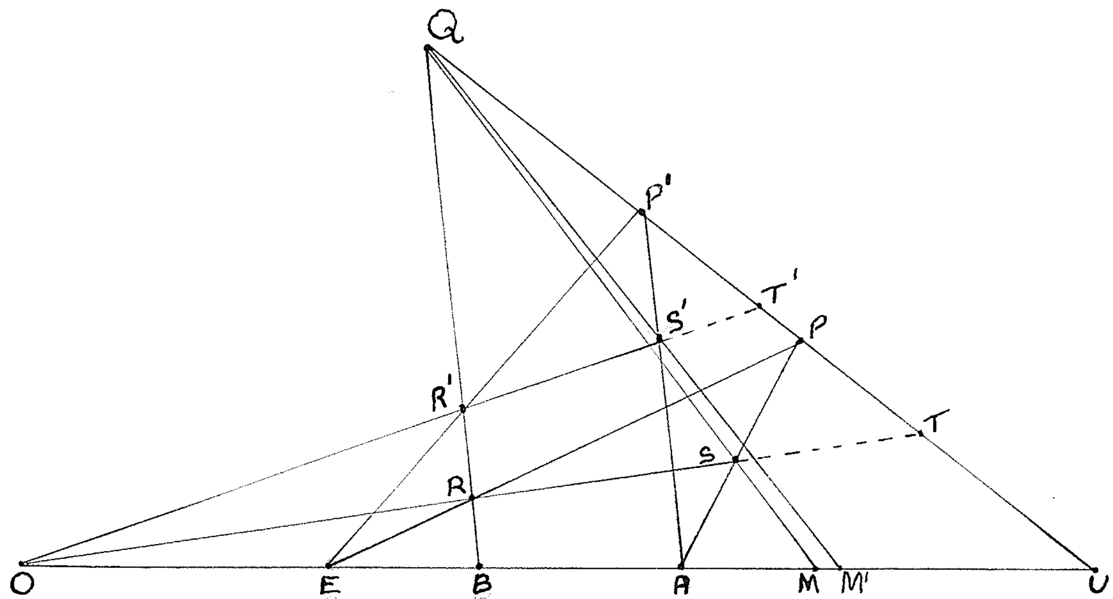
$$k(A,B,C) \overset{O}{\underset{\wedge}{-}} m(A',B'',C) \overset{O'}{\underset{\wedge}{-}} p(A',B',C') \quad .$$

Hence

$$k(A,B,C) \sim p(A',B',C') \quad .$$

i.e. There exists a projective transformation between the lines k and p which sends the points A into A', B into B', and C into C'.

It is a known property of plane projective geometries for which Desargues's and Pappus's[1] theorems are true that a projective transformation in a line reduces to the identity if three points are fixed. It will now be shown that if Desargues's theorem (and hence its converse) fails that there exists a line and four points on it, together with a projective transformation, which keeps three of these points fixed but which does not keep the fourth fixed.



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1. Pappus's theorem states:

If A,B,C are any three distinct points of a line m, and A',B',C' any three distinct points of another line m', the three points of intersection of pairs of lines AB'and A'B, BC' and B'C, CA' and C'A are collinear.

For let PRS and P'R'S' be two triangles for which the converse of Desargues's theorem fails. Let the line PS and the line P'S' meet at A; the line PR and the line P'R' at E; the line RS and the line R'S' at O. The points O,E,A are collinear. Let the line RR' and the line PP' meet at Q. Since the converse of Desargues's theorem fails, Q,S, and S' are not collinear. Let QS meet OE at M and QS' meet OE at M' where M is not equal to M'. Let R'S',RS and OE meet PP' at T',T and U respectively. Then

$$OMAU \overset{S}{\underset{\wedge}{\rightarrow}} TQPU \overset{R}{\underset{\wedge}{\rightarrow}} OBEU \overset{R'}{\underset{\wedge}{\rightarrow}} T'QP'U \overset{S'}{\underset{\wedge}{\rightarrow}} OM'AU \quad .$$

Hence

$$OMAU \sim OM'AU$$

Therefore there exists a projective transformation which sends O into O, A into A, U into U, but which sends M into M'.

A collineation of a plane is any one to one correspondence between the points of a plane which has the following properties.

1. The transformation has an inverse.
2. The images of collinear points are collinear points.
3. Any three collinear points are the images of three points which are collinear.

For geometries in which Desargues's theorem is true the transformations $\rho\, x_i' = \sum a_{ij}\, x_j$ for which the determinant

(a_{ij}) is not equal to zero are all collineations called projective collineations. In general, if the co-ordinate field has an automorphism φ the transformation $\rho x'_i = \sum a_{ij} \varphi(x_j)$ is also a collineation. For geometries in which Desargues's theorem fails, it is not always advantageous to distinguish between projective and non-projective collineations.

In non-Desarguesian geometries, it is useful to distinguish various special cases of Desargues's theorem. In the case where the centre of perspectivity is on the axis of perspectivity the theorem is usually referred to as the little Desargues's theorem. Other cases are of interest but these are not needed in the subsequent development.

PART II.

THE SPECIAL VEBLEN-WEDDERBURN NON-DESARGUESIAN GEOMETRY

This special geometry is defined in terms of a system of co-ordinates. These co-ordinates form a "near-field" which is a system of elements satisfying all the axioms of a field except for the commutative law of multiplication and the distributive law $(b+c)a = ba + ca$. A special near-field can be constructed as follows. Each element is an expression of the form $a+bj$ where a and b are elements of the Galois field of order three, and where $j^2 = 2$. If $b = 0$, then for all a, $a+bj$ is called a scalar element. If $b \neq 0$, the element $a+bj$ will be referred to as a non scalar element. The right distributive law will be replaced by the law $(a+bj)j = -j(a+bj)$ provided a and b are both distinct from zero. Any scalar element commutes with any other element under multiplication and hence also is distributive on the right. In this particular near-field, although multiplication is not commutative, it can be verified that either $\alpha\beta = \beta\alpha$ or $\alpha\beta = -\beta\alpha$. In what follows, the development will be restricted to this special near-field since in this case, computation is relatively simple because of the laws which hold. The multiplication table for the elements of the near-field (distinct from the elements zero and one) appears below.

	2	j	2j	1 + j	1 + 2j	2 + j	2 + 2j
2	1	2j	j	2 + 2j	2 + j	1 + 2j	1 + j
j	2j	2	1	2 + j	1 + j	2 + 2j	1 + 2j
2j	j	1	2	1 + 2j	2 + 2j	1 + j	2 + j
1 + j	2 + 2j	1 + 2j	2 + j	2	2j	j	1
1+2j	2 + j	2 + 2j	1 + j	j	2	1	2j
2 + j	1 + 2j	1 + j	2 + 2j	2j	1	2	j
2+2j	1 + j	2 + j	1 + 2j	1	j	2j	2

A point of this geometry is defined as one of the following systems of three co-ordinates:

(α) (a,b,1)

(β) (a,1,0)

(γ) (1,0,0)

where a and b are arbitrary elements of the near-field.

A line is defined as all points which satisfy an equation of one of the following forms:

(1) $x + ya + zb = 0$

(2) $y + zc = 0$

(3) $z = 0$

Since the left distributive law holds, a general point (x,y,z) can be represented by (kx,ky,kz) where $k \neq 0$. In other words, if a set of numbers (x,y,z) satisfies the equation of a line, the set (kx,ky,kz) will satisfy the same equation of a line. It is however, important to note that multiplication on the right of the co-ordinates of a point by a non-zero constant k is, in general, not valid. That this leads to a result which is not valid is shown by the following example. Consider the point with co-ordinates (1 + j, j, 1). Multiplying this set of co-ordinates on the right by 2 + j, we get the co-ordinates (j, 2 + 2j, 2 + j) and multiplying this last set on the left by 1 + 2j, we get (2 + 2j, 2j, 1) which is not the same as the first set of co-ordinates (1 + j, j, 1).

In order to show that the points and lines of this Veblen-Wedderburn geometry form a projective plane geometry, it is necessary to prove that

1. Any two distinct lines have one and only one point in common.
2. Any two distinct points have one and only one line in common.
3. There are four points A, B, C, D no three of which are on a line.

Proof of 1.

In order to show that any two distinct lines have one and only one point in common it is necessary to consider several possibilities. Hence the proof will be divided up into the following five cases.

a) Two distinct lines of type (1).

b) A line of type (1) and a line of type (2).

c) A line of type (1) and a line of type (3).

d) Two distinct lines of type (2).

e) A line of type (2) and a line of type (3).

a) Two distinct lines of type (1).

$$x + ya_1 + zb_1 = 0$$

$$x + ya_2 + zb_2 = 0$$

Upon eliminating x, the equations reduce to the form

$$y(a_1 - a_2) + z(b_1 - b_2) = 0$$

If $b_1 = b_2 = b$, then $a_1 \neq a_2$, and hence $y = 0$. Taking $z = 1$, $x = 2b$. Therefore the point common to these lines has co-ordinates of type (α).

If $a_1 = a_2 = a$, then $b_1 \neq b_2$, and hence $z = 0$. Therefore taking $y = 1$, $x = 2a$. The point common to these lines has co-ordinates of type (β).

If $a_1 \neq a_2$ and $b_1 \neq b_2$, take $z = 1$. Then $y = -(b_1 - b_2)(a_1 - a_2)^{-1}$ and $x = \{-(b_1 - b_2)(a_1 - a_2)^{-1}\}(a_1 + a_2) + (b_1 + b_2)$. Therefore the point common to these lines has co-ordinates of type (α).

b) A line of type (1) and a line of type (2).

$$x + ya_1 + zb_1 = 0$$

$$y + zb_2 = 0 \qquad \text{where } z \neq 0.$$

Take $z=1$, then $y=-b_2$, and hence $x=b_2a_1-b_1$. Therefore the point common to these two equations is of type (α).

c) A line of type (1) and a line of type (3).

$$x+ya_1+zb_1=0$$
$$z=0$$

Since $z=0$, take $y=1$, and hence $x=2a_1$. The common point then has co-ordinates of type (β).

d) Two lines of type (2).

$$y+zc_1=0$$
$$y+zc_2=0 \qquad \text{where } c_1\neq c_2.$$

Solving for z, the equation becomes

$$z(c_1-c_2)=0 \qquad \text{i.e.} \quad z=0.$$

Hence $y=0$ and $x=1$. Therefore the point common to these lines has co-ordinates of type (γ).

e) A line of type (2) and a line of type (3).

$$y+zc=0$$
$$z=0$$

The only solution is (1,0,0). Therefore the point common to these lines has co-ordinates of type (γ).

Proof of 2.

To show that any two distinct points have one and only one line in common, it is necessary to divide the proof into the following five cases.

a) Two points of type (α).

b) One point of type (α) and one point of type (β).

c) One point of type (α) and one point of type (γ).

d) Two points of type (β).

e) One point of type (β) and one point of type (γ).

a) Two points of type (α).

$$(a_1, b_1, 1)$$

$$(a_2, b_2, 1)$$

Consider the equation $x + y\alpha + z\beta = 0$. If this line is to be on $(a_1, b_1, 1)$ and $(a_2, b_2, 1)$ then

$$\left.\begin{aligned} a_1 + b_1\alpha + \beta &= 0 \\ \text{and} \quad a_2 + b_2\alpha + \beta &= 0 \end{aligned}\right\} \qquad \text{(i)}$$

Upon eliminating β the equations reduce to

$$b_1\alpha - b_2\alpha = a_2 - a_1 . \qquad \text{(ii)}$$

Since the right distributive law is not valid the left side of (ii) can not be further simplified. To establish whether there exists an α for which (ii) is valid, it is necessary to prove the following lemma.

<u>Lemma</u>: If $a \neq b$ then the equation $ax-bx=c$, where a,b,c are elements of the Veblen-Wedderburn near-field, has exactly one solution.

Proof-

Consider the nine elements $x_1, x_2, x_3, \ldots, x_9$ where these x_i are the elements of the Veblen-Wedderburn near-field F.

Form $ax_1-bx_1, ax_2-bx_2, \ldots, ax_9-bx_9$ which are now shown to be distinct.

For, suppose they are not distinct. Then for at least one pair of elements x_i and x_j with $i \neq j$

$$ax_i-bx_i = ax_j-bx_j .$$

i.e. $$a(x_i-x_j) = b(x_i-x_j).$$

Since $x_i \neq x_j$, then $a=b$ which contradicts the hypothesis that $a \neq b$. Hence the quantities ax_i-bx_i, where $i = 1,2,\ldots,9$ are all distinct.

Since there are nine elements of the form ax-bx in F, all the latter elements appear amongst those of the form ax-bx. Hence, there will be some x, say $x = \bar{x}$, such that $a\bar{x}-b\bar{x} = c$. This completes the proof of the lemma.

Hence in equation (ii) of section a) there exists an α say $\alpha = k$ such that $b_1k-b_2k = a_2-a_1$.

Solving for β in (i),

$$\beta = (a_1 + a_2) + (b_1 k + b_2 k).$$

Therefore the line on these points has the equation

$$x + yk + z\left\{(a_1 + a_2) + (b_1 k + b_2 k)\right\} = 0$$

which is an equation of type (1).

b) One point of type (α) and one point of type (β).

$$(a_1, b, 1)$$

$$(a_2, 1, 0)$$

Consider the equation $x + y\alpha + z\beta = 0$. If this line is to be on the above points then

$$a_1 + b\alpha + \beta = 0 \qquad \text{(i)}$$

and $$a_2 + \alpha = 0 \; . \qquad \text{(ii)}$$

Hence $\alpha = 2a_2$ and $\beta = 2a_1 + ba_2$.

The line on these points has the equation

$$x + y(2a_2) + z(2a_1 + ba_2) = 0 \; .$$

i.e. An equation of type (1).

c) One point of type (α) and one point of type (γ).

$$(a, b, 1)$$

$$(1, 0, 0)$$

It is immediately verified that these points lie on the line $y + z(2b) = 0$. Therefore the line on these points has the equation of type (2).

d) Two points of type (β).

$(a_1,1,0)$

$(a_2,1,0)$

It is immediately verified that these points lie on the line $z = 0$. Therefore the line on these points has an equation of type (3).

e) One point of type (β) and one point of type (γ).

$(a,1,0)$

$(1,0,0)$

It is immediately verified that these points lie on the line $z = 0$. Therefore the line on these points has an equation of type (3).

Proof of 3.

The four points (1,0,0),(0,1,0),(0,0,1), and (1,1,1) satisfy the required condition.

This completes the proof that the points and lines of the Veblen-Wedderburn geometry form a projective plane geometry.