BOUNDARY CONDITIONS IN THE MECHANICS OF RELATIVISTIC WAVE FIELDS

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A thesis submitted to the Faculty of Graduate Studies and Research of the University of Manitoba in partial fulfillment of the requirements for the degree of Master of Science.

University of Manitoba

1952

Acknowledgments

It is a pleasure to thank Professor S. M. Neamtan for suggesting the problems which led to this investigation and for his constant interest and aid in the developement of the investigation into its present form. His supervision is greatly appreciated. The writer also wishes to thank Professor H. R. Coish for suggesting the definitions of the generalized field quantities which are discussed.

Grateful acknowledgment is also due to the National Research Council of Canada for the award of a Bursary.

CONTENTS

SUMMARY

In the usual discussion of relativistic wave fields with variational principles, only the field equations are obtained from the variational principle. It is shown, in this investigation, that the form of the boundary conditions as well may be obtained from the action-principle for these fields. The boundary conditions which are allowed depend on the form of the Lagrangian density which is used in the action-principle. The scalar field, the vector field and the Dirac field are discussed. The usual Lagrangian densities, for these fields, do not contain second derivatives. By generalizing the form of the Lagrangian density, for the scalar and vector fields, to contain second derivatives, it is shown that more boundary conditions are allowed than in the usual formulation. In particular, the use of the generalized Lagrangian density allows the possibility of linear homogeneous boundary conditions. This generalization also modifies the definition of the conjugate momentum fields, the stress-energy tensor and the charge-current vector. Thus the conjugate momentum fields for the vector field are defined so that the time components of these fields do not vanish identically. A symmetry in formulation between the scalar and the vector fields is obtained. The classical (unquantized) formulation for the scalar and the vector fields is completed with the modified field quantities.

INTRODUCTION

1.1 Field Theories and Variational Methods

The motions of wave fields are usually described by variational methods. These are analytical methods in which the field is represented by a function - called the Lagrange function - which is employed in a variational principle. If the Lagrange function and the principle are properly chosen they may describe the motion of the field completely.

In relativistic field theories we discuss physical systems which consist of field functions defined in given regions of space-time. The definitions of these field functions are stated in the form of partial differential equations which must be satisfied by the field functions at every point of their space-time regions. The solutions of these differential equations are restricted by conditions existing at the space-time boundaries of the fields. Since, in describing the motion of the system, we are usually interested in the progression, through time, of the physical system, it is customary to specify that the space-time region for the field be open with regard to the time co-ordinate. Then the field is given meaning in the following way. We seek those solutions of the differential equation of the field which can be separated into a product of two terms, one of which is time-dependent only and the other space-dependent only. Then we require that the space-dependent part of these solutions satisfy, for all values of time, certain conditions which we specify at the boundary. If the field region is finite the solutions of the field equation form a denumerably infinite set, the so-called normal modes of the system. The equations of motion for the field are the set of equations of motion for these normal modes. If a variational principle is to describe a relativistic wave field we ask that it give meaning to this field, that is, we ask that one principle be a unifying basis from which the entire denumerably infinite set of equations of motion for the normal modes follows directly.[1] This is equivalent to saying that the variational principle be required to yield the differential equation of the field <u>and</u> the conditions which are to hold at the space boundary of the field.

In the discussion of field theories by variational principles the boundary conditions are not usually recognized as a consequence of the principle. The variational method usually involves a fundamental quantity called 'action' and the principle associated with the method is then the statement that this action is stationary. The discussion above has shown that if this action-principle is to describe any wave-field completely we will require that it yield the partial differential equations of the field and also the space boundary conditions which may accompany these field equations. The manner in which the action-principle does this is readily seen if we examine the variational methods applied to a simple wave field, the real scalar field.

We assume that we have a field function ψ defined in a region τ, in space-time. τ is to be a cylinder which is open in the direction of the time-co-ordinate, x_4 (= ict). The intersection of any x_4 = constant plane and τ forms a volume, v, in the three dimensional sub-space defined by the spatial co-ordinates, x_1, x_2, x_3. We require that this volume

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[1] Variational principles are discussed very generally in " The Variational Principles of Mechanics" by Cornelius Lanczos (University of Toronto Press).

be fixed for all values of the time-co-ordinate x_4. Let the two-dimensional surface, in (x_1, x_2, x_3), which encloses V, be S. S is the space boundary of the field and is fixed for all values of time.

We represent the field by a Lagrange function, $\mathcal{L}$, which we call the Lagrangian density. $\mathcal{L}$ is a function of ψ and its space-derivatives and time-derivatives. It will be shown, in a later section, that relativistic considerations restrict the form of the dependence of $\mathcal{L}$, on ψ, to the following:

$$\mathcal{L} = \mathcal{L}(\psi, \partial_\alpha \psi, \partial_\alpha \partial_\alpha \psi) \qquad \alpha = 1,2,3,4$$

where $\partial_\alpha \equiv \frac{\partial}{\partial x_\alpha}$, x_1, x_2, x_3 are Cartesian space-co-ordinates

and $\partial_\alpha \partial_\alpha \psi \equiv \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_3^2} + \frac{\partial^2 \psi}{\partial x_4^2}$

The notation, above, for the co-ordinates and for derivatives with respect to co-ordinates will be used throughout this investigation. The dummy index convention for the summation over repeated indices, as in $\partial_\alpha \partial_\alpha \psi$ above, will always be used unless otherwise stated. $\partial_\alpha \partial_\alpha \psi$ is frequently written $\Box \psi$ and called the Dalembertian of ψ.

In order to discuss the motion of the field with the aid of a variational principle, we define the action, I, for the field, in the following way:

$$I \equiv \int_{x_4^0}^{x_4^1} \int_V \mathcal{L} \, d\tau \qquad (1)$$

where x_4^0 and x_4^1 are two values of x_4

and $d\tau$ is the four-dimensional element of volume.

The action principle now states that, for arbitrary x_4^0 and x_4^1, the action, as defined in the above manner, is stationary in the following sense. Assume a variation, $\delta\psi$, in the functional dependence of ψ on its variables. Let $\delta\psi$ be completely arbitrary except at x_4^0 and x_4^1 where it vanishes, and on S where it is consistent with the boundary conditions for the field. Not only the variations $\delta\psi$ but also the variations $\delta\partial_\alpha\psi$ vanish at x_4^0 and x_4^1. However, only the vanishing of $\delta\psi$ and $\delta\partial_4\psi$, at x_4^0 and x_4^1, will be required in the variational procedure. If δI is the variation in the action, I, corresponding to the variation $\delta\psi$ in ψ, then the action principle may be stated, explicitly: for $\delta\psi$ which are variations about the correct functional dependence of ψ on its variables the corresponding δI vanishes. That is,

$$0 = \delta I = \delta \int_{x_4^0}^{x_4^1}\int_V \mathcal{L}\, d\tau = \int_{x_4^0}^{x_4^1}\int_V \delta\mathcal{L}\, d\tau \qquad (2)$$

If the variation, $\delta\psi$, is small we may expand $\delta\mathcal{L}$ in a Taylor's series, disregarding higher powers in $\delta\psi$ than the first.

Then $$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)}\partial_\alpha\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\gamma\partial_\gamma\psi)}\partial_\alpha\partial_\alpha\delta\psi$$

If this is substituted into the action principle the terms involving $\partial_\alpha\partial_\alpha(\delta\psi)$ and $\partial_\alpha(\delta\psi)$ may be integrated by parts to obtain:

$$0 = \delta I = \int_{x_4^0}^{x_4^1}\int_S \left\{\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\gamma\partial_\gamma\psi)}\partial_\alpha\delta\psi - \partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\gamma\partial_\gamma\psi)}\delta\psi\right\} n_\alpha\, dS\, dx_4$$

$$+ \int_{x_4^0}^{x_4^1}\int_V \left\{\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)} + \partial_\alpha\partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\gamma\partial_\gamma\psi)}\right\}\delta\psi\, d\tau$$

where n_α is the unit-normal to S.

Since $\delta\psi$ is completely arbitrary for the whole region V - S for all values of X_4 between x_4^0 and x_4^1, we may write:

$$\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)} + \partial_\alpha\partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\gamma\partial_\gamma\psi)} = 0$$

which is the Euler-Lagrange equation for the field.