LATER POINTS ON THE BASES OF THE CUBIC AND QUARTIC

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GENERAL INTRODUCTION

Sec. 1. The thesis considers some points in the derivation of bases for the cubic $x^3+3Hx+G = 0$ and for the quartic $x^4+Px^3+Qx^2+R = 0$. The method is an adaptation in particular, of certain general theorems due to N. R. Wilson. These theorems are in preparation for publication, and attack the general problem of deriving bases for algebraic number fields from an opposite viewpoint to the one adopted by the same author in his paper on "Integers and Bases of a Number Field."* There the process began with the elements of lowest degree, and methods were developed permitting gradual progress to the elements of higher degree. Practical application became more complicated the higher the degree of the element. The later theorems referred to, establish methods for obtaining the highest degree elements first and working back to the elements of lower degree. Where unpublished theorems are used they are quoted. The treatment falls naturally into two parts:-

PART 1. The Basis for the Cubic $x^3+3Hx+G = 0$; some points in its derivation.

PART 2. The Basis for the Quartic $x^4+Px^3+Qx^2+R=0$; some points in its derivation.

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Sec. 2

The form of equation used for the cubic was the usual $x^3+3Hx+G = 0$, the second degree term being eliminated. It is a convenient form in dealing with symmetric functions of the roots. It seemed desirable, however, because of the ideas underlying the later theorems, to manifest certain relations between f(x) and its derivative; hence the form $x^4+Px^3+Qx^2+R = 0$ was used, that transformation being assumed, which eliminates the term of the first degree. It is possible by a linear transformation, to eliminate all factors p such that $p^2/3H$ and p^3/G, from the coefficients 3H and G of $x^3+3Hx+G = 0$; similarly factors p such that p/P, p^2/Q, and p^4/R, from the coefficients P, Q, R, of $x^4+Px^3+Qx^2+R = 0$. When this has been done, the equation is said to be normal. This normalizing is assumed as done for the equations we deal with.

Sec. 3. Notation:-

1. The denominator of element of degree i-1 is represented throughout by P_i.
2. The letter "p" denotes any prime.
3. We read p/Q as "p divides Q" and $p\not/Q$ as "p does not divide Q".
4. The symbol $\left[\frac{m}{2}\right]$ means "the greatest integer in $\frac{m}{2}$ ".
5. We represent the discriminant by Δ.
6. Any other notation is used with an obvious meaning in each case, or references made to it when it is introduced.

PART 1.

The basis for the cubic $x^3+3Hx+G = 0$; some points in its derivation.

THE CUBIC

The equation is $f(x) = x^3+3Hx+G$

The elements are of the form 1, $\frac{x+a}{p^v}$ and $\frac{x^2+bx+c}{p^u}$

The following general relations hold between the denominators and symmetric functions of the roots. These are based on a theorem from a paper to be published shortly by N. R. Wilson. The theorem is: "The symmetric sum of the products of the squared differences of the roots of $f(x) = 0$, i at a time is divisible by $P_1^2P_2^2.....P_i^2$."

For the cubic $P_2^2/\Sigma(x_1-x_2)^2$ where the latter equals $-18H$; $P_2^2P_3^2/\Delta$ the discriminant where $\Delta = 3^3(4H^3-G^2) = 3^3\Delta'$

If $(\frac{x+a}{p^v})$ is an integer, then $(\frac{x+a}{p^u})^2$ is also an integer, $\therefore$ we must have $(\frac{x+a}{p^v})^2 \equiv \frac{K(x^2+bx+c)}{p^u}$ + integers. The coefficients of x must be equal, i.e. $p^u=Kp^{2v}$; hence $2v \leqslant u$. Also since $p^{2v}/18H$, $2v \leqslant \nu_1$, where ν_1 is the highest power of p in 18H. Finally since $P_2^2P_3^2/\Delta$, $2u+2v \leqslant \nu_2$ where ν_2 is the highest power of p in Δ.

We gather these results

(1)..(a) $2v \leqslant u$ (b) $2v \leqslant \nu_1$ (c) $2u+2v \leqslant \nu_2$

For $H \neq 0$, the G. C. D. of $f(x)$ and $f'(x)$ is $2Hx+G$. Repeated division of $f(x)$ by $2Hx+G$ gives $Hx-G$ as the other factor. We write $2Hx+G = q$ and $Hx-G = a$.

(2)..Then $2a \equiv 2Hx-2G = q-3G$

(3)..Further $8H^3f(x) \equiv 2aq^2+3\Delta' q-G\Delta' \equiv q^3-3Gq^2+3\Delta' q-G\Delta'$

Let α and β be numbers, $\alpha = \frac{2aq}{p^u}$ and $\beta = \frac{aq}{p^u}$.

Then except for p = 2, the conditions that α and β be integers are equivalent. For, since p^u and 2 are prime, we may find k and l such that $2k+p^u l = 1$

$$\therefore \quad = \frac{aq}{p^u}(2k+p^u l) = k\alpha + \text{a simple integer.}$$

From (3) we have $4a^2q^2+6\Delta'aq-2G\Delta'a = 0$

(5)... $$\text{i.e. } \alpha^2 + \frac{3\Delta'\alpha}{p^u} - \frac{2G\Delta'a}{p^{2u}} = 0$$

or $2a^2q^2+3\Delta'aq-G\Delta'a = 0$, giving

(6)... $$\beta^2 + \frac{3\Delta'\beta}{2p^u} - \frac{G\Delta'a}{2p^{2u}} = 0$$

From the paper quoted above, we have also that f(x) contains a squared factor, mod. p^w which we call q: q is also a simple factor mod. p^{2w}. For the cubic, q must be linear; so that $f(x) \equiv 0$ mod. p^{2w}, and $f'(x) \equiv 0$ mod. p^w, will have a common root. The rise p^w referred to here, is the last rise. Let this factor be $x-n_1$. Then the theorem, referred to the cubic case, reads:-

(7)... $n_1^3+3Hn_1+G \equiv 0$ mod. p^{2w}, $3(n_1^2+H) \equiv 0$ mod. p^w.

In particular, it is further shown that where a rise occurs at the second element, i.e. $\frac{x-n_1}{p^v}$, $x-n_1$ is a factor of f(x) mod. p^{3v}, of f'(x) mod. p^{2v} and of f"(x) mod. p^v: that we have $f(x) \equiv 0$ mod. p^{3v}, $f'(x) \equiv o$ mod. p^{2v}, and $\frac{1}{2}f'(x) \equiv 0$ mod. p^v, with a common solution. Referring to the case in hand, this reads:-

(8)... $n_1^3+3Hn_1+G \equiv 0$ mod. p^{3v}, $3(n_1^2+H) \equiv 0$ mod. p^{2v}, $3n_1 \equiv 0$ mod. p^v

(9)...The latter has the solution $n_1 \equiv 1$ when $p = 3$ and $v = 1$; $H \equiv -1$ mod. 3, $G \pm (3H+1) \equiv 0$ mod. 27; and requires that $3^6/\Delta$.

(10)...The solution $n_1 \equiv 0$ requires p^2/H and p^3/G if $p \neq 3$; that $3/H$ and $3^3/G$ if $p = 3$, the elements then being 1, $\frac{x}{3}$, $\frac{x^2}{3^2}$, unless $3^3/H$ and $3^6/G$.

(11)...If $n_1 \equiv 0$ is a solution for the conditions under (7), then p/H and p^2/G (unless the case comes under the heading(10.)).

For the discussion, we will use the following classification. We will let p^i and p^k denote the highest powers of p in H and G respectively. For $H \neq 0$, we will classify:

Case 1. $3i = 2k$ for $p \neq 2$; $3i-2 = 2k$ for $p = 2$

Case 2. $3i < 2k$ for $p \neq 2$; $3i-2 < 2k$ for $p = 2$

Case 3 $3i > 2k$ for $p \neq 2$; $3i-2 > 2k$ for $p = 2$

The transformation $x = p^k x'$ will leave these relations unaltered.

(A) The Cubic: $p \neq 2$ and $p \neq 3$

We will deal first with the cases where $p \neq 2$ and $p \neq 3$. We normalize so that i and k have their least values except where $3i>2k$ and $k \equiv 2$ mod. 3 when we arrange things so that $k = i+2$.

Case 1. (i.e. $3i = 2k$) $i = k = 0$ (since the equation is normal) and $p^{\frac{1}{2}}/\Delta'$. Hence by (5), $\alpha = \frac{2aq}{p^u}$ is an integer for $u = \left[\frac{\frac{1}{2}}{2}\right]$; and also $\beta = \frac{aq}{p^u}$.

The element $\frac{aq}{p^u}$ is equivalent to $\frac{(2Hx+G)(Hx-G)}{p^{u+2\frac{1}{2}}}$ where $u = \left[\frac{\frac{1}{2}}{2}\right]$

Case 2. (i.e. $3i<2k$), if $i = 0$, $k>0$ and $\therefore$ p^2/Δ only if $p = 3$. If $i = 1$, $k \geq 2$ $\therefore$ by (11) elements are 1, x, $\frac{x^2}{p}$, Here $\frac{x^2}{p}$ is equivalent to $\frac{(2Hx+G)(Hx-G)}{p^{u+2\frac{1}{2}}}$ where $u = \left[\frac{\frac{1}{2}}{2}\right]$

Case 3. (i.e. $3i>2k$), if $k \equiv 0$ or $k \equiv 1$ mod. 3, the normalized form will have $k = 0$ or $k = 1$, and there are no elements in p. If $k = 3m+2$, since $k = i+2$, we have $i = 3m = \frac{1}{2}$. Writing $x = p^m x'$, we obtain $x^3+3p^m hx'+p^2 g = 0$ where h and g are prime to p; basis 1, x', $\frac{x^2}{p}$ by (11), whence the original basis is 1, $\frac{x}{p^m}$, $\frac{x^2}{p^{2m+1}}$. The first is equivalent to $\frac{(2Hx+G)}{p^{4m}} = \frac{(2Hx+G)}{p^{\frac{1}{2}+v}}$ say. Since $\frac{1}{2} = 6m+4$, the latter is equivalent to $\frac{(2Hx+G)(Hx-G)}{p^{2\frac{1}{2}+u}}$ where $u = \left[\frac{\frac{1}{2}}{3}\right]$ and $v = \left[\frac{u}{2}\right]$.

(B) The Cubic: $p = 2$

If $p = 2$, we normalize the equation as before. In Case 1. (i.e. $3i+2 = 2k$) we have as the only possibility $i = 0$, $k = 1$, and $\nu = 1$.

Equation (6) i.e. $\beta^2 + \frac{3\Delta'\beta}{2p^u} - \frac{G\Delta'a}{2p^{2u}} = 0$ is satisfied for $\beta = \frac{aq}{2^m}$ where $m = \left[\frac{\nu}{2}\right]$ but not for $m = \left[\frac{\nu+2}{2}\right]$. (The extra 2 in the denominator of the second term is absorbed by the q factor of β and in the denominator of the third term by G). Because of the factor 2 in $2Hx+G = q$, β is equivalent to an element of the form $\frac{x^2+bx+c}{2^{m-1}}$, one under the maximum degree.

If there is an element of maximum degree in 2, let it be $\frac{a'q'}{2^m} = \frac{(x+d)(x+e)}{2^m}$. Let $q_1 = 2H(x+e)$ and $a_1 = H(x+d)$. The element is $\frac{2H^2(x+d)(x+e)}{2^{m+1}}$, i.e. $\frac{a_1q_1}{2^{m+1}}$. Now if this is an element of the basis, then, since $\frac{aq}{2^m}$ is an integer, we must have $\frac{aq}{2^m} = k\left(\frac{a_1q_1}{2^{m+1}}\right) +$ integers. The leading coefficients here must be equal, i.e. $\frac{2H^2}{2^m} = k\left(\frac{2H^2}{2^{m+1}}\right)$. Obviously $k = 2$; and multiplying by 2^m we get $aq = a_1q_1 + 2^m$(integers), or

$$aq \equiv a_1q_1 \text{ mod. } 2^m \quad \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

Further, the repeated factor q' is such that $q'/f(x)$ mod. 2^{2m} and $(q')^2/f(x)$ mod. 2^m by our general theorem, i.e.

$f(x) = a'(q')^2 + 2^m$(integers).

and $8H^3f(x) = 8H^3a'(q')^2 + 2^{m+3}$(integers).

Now $a_1 = Ha'$ and $q_1 = 2Hq'$ $\therefore$ $8H^3f(x) = 2a_1q_1^2 + 2^{m+3}$(integers)

i.e. from (3) $2aq^2 + 3\Delta'q - G\Delta' = 2a_1q_1^2 + 2^{m+3}$(integers).

From this, when $2m \geq m+3$, i.e. $m \geq 3$, we get $2aq^2 \equiv 2a_1q_1^2$ mod. 2^{m+3}.

or $aq^2 \equiv a_,q_,^2 \bmod. 2^{m+2}$ .....(b)

(When m = 1 or 2 we will have special cases). We have, from (b) and (a)

$$aq^2 \equiv a_,q_,^2 \bmod. 2^{m+2}$$

$$aq \equiv a_,q_, \bmod. 2^m$$

$aq^2 = q(aq) = a(a_,q_, - 2^m(\text{integer}_1)) = q_,(a_,q_,) + 2^{m+2}(\text{integer}_2)$

$\therefore a_,q_,(q-q_,) = -q(2^m)(\text{integer}_1) + 2^{m+2}(\text{integer}_2)$.

Since q and $q_,$ each contain one 2, we may write

$(q-q_,) = 2^m(\text{integer}_3)$, for 2^m is a factor of the right after division by 2, and $\therefore$ must be of the left and so of $(q-q_,)$, since the only 2 in $a_,q_,$ has been removed.

$$q \equiv q_, \bmod. 2^m \text{.....(c)}$$

Again, $q \not\equiv q \bmod. 2^{m+1}$, since in $\frac{a_,q_,}{2^{m+1}}$ we could then write q for $q_,$.

We may $\therefore$ take $q_, = q - 2^m$ since multiples of 2^{m+1} can be discarded from $q_,$.

We already have $aq^2 \equiv a_,q_,^2 \bmod. 2^{m+2} \equiv a_,(q-2^m)^2 \bmod. 2^{m+2}$

$\equiv a_,q^2 - 2^{m+1}a_,q + 2^{2m}a_, \bmod. 2^{m+2}$, $\therefore$

$q^2(a-a_,) \equiv -2^{m+1}a_,q + 2^{2m}a_, \bmod. 2^{m+2}$. Since q contains 2 to the first power, we may divide through by 2^2, getting

$$a - a_, = 2^m(\text{integers}) \quad \text{i.e.}$$

$$a \equiv a_, \bmod. 2^m$$

Now q' is a factor of $f(x) \bmod 2^{2m}$, and

i.e. $q_,$ is a factor of $2f(x) \bmod. 2^{2m+1}$

$\therefore$ $q_,$ is a factor of $8H^3f(x) \bmod. 2^{2m+3}$ i.e. $q-2^m$ is such a factor. Therefore $q=2^m$ is a solution of $q^3 - 3Gq^2 + 3\Delta' q - G\Delta' \equiv 0 \bmod. 2^{2m+3}$. Also $q'/f'(x) \bmod. 2^m$, i.e. $q_,/2f'(x) \bmod. 2^{m+1}$ i.e. $q - 2^m/4H^3f'(x) \bmod. 2^{m+2}$.

But $8H^3f'(x) = 3Q^2\cdot 2-6Gq\cdot 2+3\Delta'\cdot 2$

$\therefore$ $q = 2^m$ is a solution of $3q^2-6Gq+3\Delta' \equiv 0 \bmod. 2^{m+2}$.

We will let $\Delta' = n\cdot 2^{2m}$ (where n is odd). The foregoing conditions become on substituting,

$2^{3m}-3G2^{2m}+3(n\cdot 2^{2m})2^m-G(n\cdot 2^{2m}) \equiv 0 \bmod. 2^{2m+3}$, and

$3\cdot 2^{2m}-6G2^m-3(n\cdot 2^{2m}) \equiv 0 \bmod. 2^{m+2}$. These are equivalent to

$2^m-3G-3n\cdot 2^m-Gn \equiv 0 \bmod. 8$, and $3\cdot 2^m-6G+3n\cdot 2^m \equiv 0 \bmod. 4$.

The latter, by itself, would require only that $m \geqslant 1$, since 2/G and the other terms are $3\cdot 2^m(n+1)$ where n is odd, i.e. $3\cdot 2^{m+1}$(some rat. int.).

For $m \geqslant 3$, the former requires $-3G-Gn \equiv 0 \bmod. 8$, or $-G(n+3) \equiv 0 \bmod. 8$, i.e. $n+3 \equiv 0 \bmod. 4$ (since $k = 1$) and $n \equiv 1 \bmod. 4$.

For $m = 2$ we would still get the relations $2^m-3G+3n\cdot 2^m-Gn \equiv 0 \bmod. 8$, and $3\cdot 2^m-6G+3n\cdot 2^m \equiv 0 \bmod. 4$. Let $G = 2(2t+1)$ since $k = 1$. We have, from the first,

$2^2-3\cdot 2(2t+1)+3\cdot n\cdot 2^2-2(2t+1)n \equiv 0 \bmod. 8$, i.e.

$2-3(2t+1)+6n-n(2t+1) \equiv 0 \bmod. 4$, i.e. $-1-6t+6n-2nt-n \equiv 0 \bmod. 4$, i.e. $-1+5n-2t(n+3) \equiv 0 \bmod. 4$.

Since n is odd, $(n+3)$ is even $\therefore$ $-1+5n \equiv 0 \bmod. 4$, $\therefore$ $n \equiv 1 \bmod. 4$

$\therefore$ for $m \geqslant 2$ (i.e. $\nu_2 \geqslant 4$) we must have Δ' of the form $2^{2m}(4r+1)$ or Δ of the form $2^{2m}\cdot 27(4r+1)$, i.e. $2^{2m}(4s-1)$, for an element to have the maximum degree in 2. In such a case, the element is of the form $\frac{a'q'}{2^m}$ or $\frac{a_1q'}{2^m}$ (since $a_1 = Ha'$ and $H \not\equiv 0 \bmod. 2$) or $\frac{aq'}{2^m}$ (since $a \equiv a_1 \bmod. 2^m$), or $\frac{aq_1}{2^{m+1}}$ (since $q_1 = 2H(q')$), or $\frac{a(q-2^m)}{2^{m+1}}$ where $m = \left[\frac{\nu_2}{2}\right]$. If Δ is not of this form, there is no

element of highest degree in 2; and the element is $\frac{aq}{2^m}$ (of degree one less than the highest).

Numerical Example:

$x^3+9x+26 = 0$; $\Delta = 2^4 \cdot 3^3 \cdot 7^2$; $a \equiv Hx-G = 3x-26$, and $q \equiv 2Hx+G = 6x+26$. Our element is $\frac{a(q-2^m)}{2^{m+1}} = \frac{(3x-26)(6x+22)}{2^3}$ which is equivalent to $\frac{(x-2)(x+1)}{4}$.

For $m = 1$, i.e. $\nu_2 = 2$ or 3, the solution of (7) becomes: $n_1^3-3Hn_1-G \equiv 0 \bmod. 4$, and $3(n_1^2+H) \equiv 0 \bmod. 2$ since $p = 2$ and $w = 1$. For $n_1^2+H \equiv 0 \bmod. 2$, $n_1 \equiv 0$ is not a solution, since $H \not\equiv 0 \bmod. 2$. The solution $n_1 \equiv \pm 1$ is the only possible one, giving $H \equiv -1 \bmod. 2$ (or $H \equiv 1 \bmod. 2$). $n_1^3-3Hn_1-G \equiv 0 \bmod. 4$ gives $1+3H+4t+2 \equiv 0 \bmod. 4$ since G is even; i.e. $3+3H \equiv 0 \bmod. 4$, and $H \equiv -1 \bmod. 4$. In this event $\Delta \equiv 27(4H^3+G^2)$ becomes $27(4(4r-1)^3+(4t+2)^2)$, which is divisible by 2^4. Hence this case cannot arise.

In Case 2. (i.e. $3i+2<2k$), for a normal equation we have the possibilities $i = 0$, $k>1$; and $i = 1$, $k>2$. When $i = 0$ and $k>1$, $\Delta \equiv 27(4H^3+G^2) \equiv 0 \bmod. 4$, but $\not\equiv 0 \bmod. 8$

PART 2.

The basis for the Quartic $x^4+Px^3+Qx^2+R = 0$; some points in its derivation.

The form of the quartic used was $x^4+Px^3+Qx^2+R=0$

The elements are of the form $1, \frac{x+b}{P_2}, \frac{x^2+cx+d}{P_3}, \frac{x^3+ex^2+fx+g}{P_4}$

Certain relations were found, which exist between the denominators of these elements and symmetric functions of the roots. The theorem underlying these is from a paper in process of preparation by N.R.Wilson. The statement of the theorem is

"The symmetric sum of the products of the squared differences of the roots of $f(x)=0$, i at a time, is divisible by $P_1^2P_2^2P_3^2P_4^2.....P_i^2$".

We use this theorem to establish the following:-

(a) $P_2^2/\Sigma(x_1-x_2)^2$. We represent $\Sigma(x_1-x_2)^2$ by Δ_2

For the form of $f(x)$ used, it is found that $\Delta_2 = 3P^2-8Q$

In general, the method was simply to set up functions of P, Q, R, of appropriate weight, the coefficients of the possible terms entering then being found. In these functions P will have weight 1, Q weight 2, R weight 4.

The weight of Δ_2 is 2 $\therefore$ the function is of the form aP^2+bQ

The equation having roots ±1 ±1 is $x^4-2x^2+1=0$. For these roots the function here is 16; for this equation $P=0$, $Q=-2$, $R=1$. Hence $b=-8$.

Again, for roots $-1,-1,0,0$, the $f(x)$ is $x^4-2x^3+x^2$.

The function is found as 4; also $P=2$, $Q=1$, $R=0$.

Hence $4=4a-8$ and $a=3$, giving the function as $3P^2-8Q$.

(b.) $P_2^2P_3^2/\Sigma(x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^2$. We represent $(x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^2$ by Δ_3 and find $\Delta_3=2P^2Q^2-8Q^3+32QR-12P^2R$. The weight of Δ_3 is 6 $\therefore \Delta_3 = aP^2Q^2 + bQ^3 + cQR + dP^2R + eP^4Q$.

The following numerical work gives the coefficients:

Roots	Equation	P Q R	Function
$\pm1\pm1$	$x^4-2x^2+1 = 0$	0,-2, 1,	0
	$\therefore$ 4b + c = 0.....(1)		
$\pm1\pm2$	$x^4-5x^2+4 = 0$	0,-5, 4,	360
	$\therefore$ -125b - 20c = 360.....(2)		
$-1,-1, \frac{1+i}{2}, \frac{1-i}{2}$	$x^4+x^3-\frac{1}{2}x^2+\frac{1}{2}=0$	$1,-\frac{1}{2}, \frac{1}{2}$,	$-\frac{1}{2}(25)$
And mult. the resulting equation by 8 we get 2a-b-2c+4d-4e=-400..(3)			
$-1\pm i\sqrt{3}, -1\pm\frac{\sqrt{5}}{2}$,	$x^4+2x^3+x^2-1=0$	2, 1,-1,	16
	$\therefore$ 4a+b-c-4d+16e = 16.....(4)		
-1, -1, 0, 0	$x^4+2x^3+x^2=0$	2, 1, 0,	0
	$\therefore$ 4a+b+16e = 0.....(5)		

(1) and (2) give b =-8 and c = 32

(3) and (4) simplify to a+2d-2e =-22 and a-d-4e = 14 which give a-2e = 2 on eliminating d.

(5) becomes a-4e = 2, hence e=0, a=2, d=-12, giving the function as $2P^2Q^2-8Q^3+32QR-12P^2R$.

(c.) $P_2^2P_3^2P_4^2/\Sigma(x_1-x_2)^2(x_1-x_3)^2(x_1-x_4)^2(x_2-x_3)^2(x_2-x_4)^2(x_3-x_4)^2$. We represent this by Δ and find $\Delta=R(16Q^4-128Q^2R+256R^2+144QP^2R-4Q^3P^2-27P^4R$.

Let the roots of $x^4+Px^3+Qx^2+R = 0$.....(1).

be $x_1\ x_2\ x_3\ x_4$

Then the roots of $Rx^4+Qx^2+Px+1 = 0$, are $\frac{1}{x_1}\ \frac{1}{x_2}\ \frac{1}{x_3}\ \frac{1}{x_4}$

The discriminant of the quartic being of weight 12, the form it takes for $x^4 + \frac{Q}{R}x^2 + \frac{P}{R}x + \frac{1}{R} = 0$.....(2).

is $a(\frac{Q}{R})^4(\frac{1}{R})+b(\frac{1}{R})^3+c(\frac{Q}{R})^2(\frac{1}{R})^2+d(\frac{Q}{R})^3(\frac{P}{R})^2+e(\frac{P}{R})^4+f(\frac{Q}{R})(\frac{P}{R})^2(\frac{1}{R})+g(\frac{Q}{R})^6$

But also this discriminant $= (\frac{1}{x_1}-\frac{1}{x_2})^2$ etc. $= (\frac{x_1-x_2}{x_1x_2})^2$ etc. $= \frac{\Delta}{(x_1x_2x_3x_4)^6}$

$= \frac{\Delta}{R^6}$ where Δ is the discriminant of $x^4-Px^3-Qx^2-R = 0$

$\therefore \Delta = R^6$ (discriminant of (2).) $= aQ^4R-bR^3-cQ^2R^2-dP^2Q^3R-eP^4R^2-fP^2QR^2-g$

But obviously if R = 0 in (1.) we have 2 roots equal to 0

$\therefore$ R must be a factor of the discriminant, $\therefore$ g = 0

and the discriminant of (1.) is $aQ^4R-bR^3-cQ^2R^2-dP^2Q^3R-eP^4R^2-fP^2QR^2$

or $R(aQ^4-bR^2-cQ^2R-dP^2Q^3-eP^4R-fP^2QR)$

Roots	Equation	P Q R	Disc.
$\pm1,\pm1$,	$x^4-2x^2+1 = 0$	0,-2, 1,	0

$\therefore$ $16a+b+4c = 0$.....(1)

$\pm1,\pm i$,	$x^4-1 = 0$	0, 0,-1,	-256

$\therefore$ $b = 256$.....(2)

$\pm1,\pm2$,	$x^4-5x^2+4 = 0$	0,-5, 4,	5184

$\therefore$ $625a+16b+100c = 1296$.....(3)

$\frac{-1\pm i\sqrt{3}}{2}$, $1\pm i$,	$x^4-x^3+x^2+2 = 0$	-1, 1, 2,	2028

$\therefore$ $a+4b+2c+d+2e+2f = 1014$.....(4)

$-2,-2$, $1\pm i$,	$x^4+2x^3-2x^2+8 = 0$	2,-2, 8,	0

$\therefore$ $a+4b+2c-2d+8e-4f = 0$.....(5)

$\frac{-1\pm i\sqrt{3}}{2}$, $\frac{-1\pm\sqrt{5}}{2}$,	$x^4+2x^3+x^2-1 = 0$	2, 1,-1,	-240

$\therefore$ $a+b-c+4d-16e-4f = 240$.....(6)

Using (2.), (1.) and (3.) become $16a-4c = -256$ and $25a-4c = -112$.

Hence $a = 16$, $b = 256$, and $c = -128$.

(4) then becomes d+2e+2f = 230, and (5) becomes d-4e+2f = 392, from which e = -27.

(4) is now d+2f = 284, and (6) becomes d-f = -148. Hence d=-4 and f = 144, and the discriminant is

$$R(16Q^4+256R^2-128Q^2R-4P^2Q^3-27P^4R+144P^2QR)$$

A generalized form of the preceding theorem is

"The symmetric sum of the squared differences of the roots of f(x) = 0, i at a time and j at a time, is divisible by $(P_1^2P_2^2\ldots P_i^2)(P_1^2P_2^2\ldots P_j^2)$."

So generalized, the theorem may be used to obtain other relations analogous to the three preceding. These however, quickly become complicated, and only the two following were found.

(a). $P_2^4/\Sigma(x_1-x_2)^2(x_3-x_4)^2$. We represent $\Sigma(x_1-x_2)^2(x_3-x_4)^2$ by Δ_2' and find $\Delta_2' = 2Q^2-24R$. The weight=4; the form is $aP^4-bQ^2-cR-dP^2Q$

Roots	Equation	P Q R	Function
$\pm1,\pm i$	$x^4-1 = 0$	0, 0,-1,	-24
	$\therefore -c = -24$ ..... (1)		
$\pm1,\pm2$	$x^4-5x^2+4 = 0$	0,-5, 4,	146
	$\therefore 25b+4c = 146$ ..... (2)		
$\frac{-1\pm i\sqrt{3}}{2}, \frac{-1\pm\sqrt{5}}{2}$	$x^4+2x^3+x^2-1 = 0$	2, 1,-1,	-22
	$\therefore 16a+b-c+4d = -22$ ..... (3)		
$2, 2, 1, -\frac{1}{2}$	$x^4-\frac{9x^3}{2}+\frac{11x^2}{2}-2 = 0$	$-\frac{9}{2},\frac{11}{2},-2,$	$\frac{25}{2}$
	$\therefore (\frac{81}{16})^2a+\frac{121}{4}b-2c+\frac{891}{8}d = \frac{25}{2}$ ..... (4)		

From (1) c = 24; from (2) b = 2. Then (3) becomes 4a+d = 0 and (4) becomes 81a+22d = 0. Hence a = 0 and b = 0, and the function is $2Q^2+24R$. (Checked for $x^4-\frac{5}{3}x^3+\frac{25}{36}x^2-\frac{1}{36} = 0$, having roots $1,\frac{1}{2},\frac{1}{3},-\frac{1}{6}$.)

(b) $P_2^4/\Sigma(x_1-x_2)^4$. We represent $\Sigma(x_1-x_2)^4$ by Δ_2'' and find $\Delta_2'' = 3P^4+20Q^2-16R-16P^2Q$.

The expression, being of weight 4, will have the form $aP^4+bQ^2+cR+dP^2Q$.

Roots	Equation	P Q R	Function
$\pm1,\pm i$	$x^4-1 = 0$	0, 0,-1,	16
	$\therefore -c = 16$		.....(1)
$\pm1,\pm2$	$x^4-5x^2+4 = 0$	0,-5, 4,	436
	$\therefore 25b+4c = 436$		.....(2)
$\frac{-1\pm i\sqrt{3}}{2}, \frac{-1\pm\sqrt{5}}{2},$	$x^4+2x^3+x^2-1 = 0$	2, 1,-1,	20
	$\therefore 16a+b-c+4d = 20$		.....(3)
$2, 2, 1, -\frac{1}{2}$	$x^4-\frac{9}{2}x^3+\frac{11}{2}x^2-2 = 0$	$-\frac{9}{2},\frac{11}{2},-2,$	$\frac{1363}{16}$
	$\therefore \frac{(81)^2}{16}a+\frac{121}{4}b-2c+\frac{891}{8}d = \frac{1363}{16}$		.....(4)

From (1) $c = -16$, and hence (2) gives $b = 20$. Equation (3) then becomes $4a+d = -4$, and (4) becomes $81a+22d = -109$.

From the latter two $a = 3$, $d = -16$; hence the function is $3P^4+20Q^2-16R-16P^2Q$.

(Checked with equation $x^4-\frac{5}{3}x^3+\frac{25}{36}x^2-\frac{1}{36} = 0$, having roots $1,\frac{1}{2},\frac{1}{3},-\frac{1}{6}$).

These forms are gathered on the following page for reference.

GENERAL RELATIONS

These relations exist between the denominators of the elements of the basis for the field defined by $x^4+Px^3+Qx^2+R=0$ and symmetric functions of the roots.

P_2^2/Δ_2 where $\Delta_2 = 3P^2-8Q$

$P_2^2P_3^2/\Delta_3$ where $\Delta_3 = 2P^2Q^2-8Q^3+32QR-12P^2R$

$P_2^2P_3^2P_4^2/\Delta$ where $\Delta = R(16Q^4-128Q^2R+256R^2+144QP^2R-4Q^3P^2-27P^4R)$

P_2^4/Δ_2' where $\Delta_2' = 2Q^2+24R$

P_2^4/Δ_2'' where $\Delta_2'' = 3P^4+20Q^2-16R-16P^2Q$

The first element in all bases of the type we are considering, is 1. We suggest first the general question concerning the conditions which would govern a rise at the second element.

Let us suppose then, an integer of the form $\frac{x-a}{p^w}$. This must satisfy an irreducible equation of the type $y^4+by^3+cy^2+dy+e=0$, where b,c,d,e, are integers. If y were defined by an equation of lower degree, it would follow quickly that x could be so defined; this would contradict $x^4+Px^3+Qx^2+R=0$ as being irreducible.

Substituting for y in terms of x, and simplifying, we get a function of x of degree 4 equated to zero. This must equal f(x) identically, i. e.

$$f(x) \equiv x^4+Px^3+Qx^2+R \equiv (x-a)^4+p^wb(x-a)^3+p^{2w}c(x-a)^2+p^{3w}d(x-a)+p^{4w}e$$

Expanding f(x) by Taylor's Theorem, we have

$$f(x) = f(a+\overline{x-a}) = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{\lfloor 2}f''(a)+\frac{(x-a)^3}{\lfloor 3}f'''(a) \text{ etc.}$$

Equating coefficients, it follows that $\frac{f'''(a)}{\lfloor 3} = p^wb$, $\frac{f''(a)}{\lfloor 2} = p^{2w}c$, $f'(a) = p^{3w}d$, and $f(a) = p^{4w}e$. Hence $\frac{f''(a)}{\lfloor 3} \equiv 0 \bmod p^w$, $\frac{f''(a)}{\lfloor 2} \equiv 0 \bmod p^{2w}$, $f'(a) \equiv 0 \bmod p^{3w}$, $f(a) \equiv 0 \bmod p^{4w}$.

For the particular case we are considering, and taking the minimum w = 1, we must have at least

$$a^4+Pa^3+Qa^2+R \equiv 0 \bmod p^4 \quad \ldots\ldots(1)$$

$$4a^3+3Pa^2+2Qa \equiv 0 \bmod p^3 \quad \ldots\ldots(2)$$

$$6a^2+3Pa+Q \equiv 0 \bmod p^2 \quad \ldots\ldots(3)$$

$$4a+P \equiv 0 \bmod P \quad \ldots\ldots(4)$$

We subdivide:- 1. $p \neq 2$; 2. $p = 2$.

Case 1. $p \neq 2$

(i) $a \equiv 0$ mod. p. From (4) we get quickly that p/P; hence from (3) that p^2/Q; hence from (1) that p^4/R. These together contradict the normality of the field equation, ∴ this case cannot arise.

(ii) $a \not\equiv 0$ mod. p. We may divide (2) by "a" without disturbing the congruence since $a \not\equiv 0$ mod. p. This gives $4a^2+3Pa+2Q \equiv 0$ mod. p^3.....(5)
Multiplying (5) by 3 and (3) by 2, and subtracting, gives $3Pa+4Q \equiv 0$ mod. p^2 at least. Multiplying (4) by "a", and subtracting from (5) gives $2Pa+2Q \equiv 0$ mod. p at least. Combining these, we have $aP \equiv 0$ mod. p; whence p/a or p/P. If p/a we have a contradiction to our case; if p/P then (4) requires p/a which is the same contradiction.

Case 2. $p = 2$

(i) $a \equiv 0$ mod. 2. From (4) 2/P; hence from (3) $2^2/Q$; hence from (1), $2^4/R$. This case cannot arise since the field equation is normal.

(ii) $a \not\equiv 0$ mod. 2. i.e. the integer is $\frac{x\pm1}{2}$.
For a = 1 and p=2 exactly (i.e. no higher power of 2) the conditions become

$$1+P+Q+R \equiv 0 \text{ mod. } 2^4 \text{.....(1)}$$

$$4+3P+2Q \equiv 0 \text{ mod. } 2^3 \text{.....(2)}$$

$$6+3P+Q \equiv 0 \text{ mod. } 2^2 \text{.....(3)}$$

$$4+P \equiv 0 \text{ mod. } 2 \text{.....(4)}$$

From (4) $2/P$; hence by (3) $2/Q$; hence by (2) $2^2/P$.
Going back to (3), knowing $2^2/P$, it follows $2^2 \not| Q$ since $2^2 \not| 6$ $\therefore$ Q is an odd multiple of 2. Let $Q = 4n+2$.
It follows from (2) that $4+3P+8n+4 \equiv 0 \bmod. 2^3$, i.e, $2^3/P$.
The necessary and sufficient conditions under this heading are $P \equiv 0 \bmod. 8$, $Q \equiv 2 \bmod. 4$, $1+P+Q+R \equiv 0 \bmod. 16$.

We append an example. Let $P = 8$, $Q = 2$, $R = 21$.
Then (1) becomes 32, (2) becomes 32, (3) becomes 32, and (4) becomes 12. Our equation is

$$(\frac{x-1}{2})^4 + \frac{12}{2}(\frac{x-1}{2})^3 + \frac{32}{2^2}(\frac{x-1}{2})^2 + \frac{32}{2^3}(\frac{x-1}{2}) + \frac{32}{2^4} = 0$$

using $b = \frac{f''(a)}{\lfloor 3 \; p^w}$ etc. from the general relations of this type established. $(\frac{x-1}{2})$ is obviously an integer here since all the coefficients are integers. The equation simplifies to $x^4+8x^3+2x^2+21 = 0$.

(Note:- It still remains a question whether this field equation in which $(\frac{x-1}{2})$ is an integer, is irreducible. Possible roots are $\pm 3, \pm 7$, which are easily disproved. It was the need for an irreducible equation which prompted R=21, instead of the more obvious 5. Taking $R = 5$ gives rise to a reducible equation).

The conditions for $\frac{x-1}{2}$ being an integer are found to be the same as for $\frac{x-1}{2}$. Q is again an odd multiple of 2, 8/P, and $1+P+Q+R \equiv 0 \bmod. 16$. This last is equivalent to $1+P+Q+R-2P \equiv 0 \bmod. 16$, and since $16/2P$ we get $1+P+Q+R \equiv 0 \bmod. 16$ as before.

We have shown the possibility of integers of the form $\frac{x\pm1}{2}$. The question of a higher initial rise in 2 remains. Let the rise be 2^u

(i). $a\equiv 0 \bmod 2$ contradicts normality as before.

(ii). $a\not\equiv 0 \bmod 2$. The general conditions to be satisfied are:

$$a^4+Pa^3+Qa^2+R \equiv 0 \bmod 2^{4u} \quad \ldots\ldots\ldots(1)$$

$$4a^3+3Pa^2+2Qa \equiv 0 \bmod 2^{3u} \quad \ldots\ldots\ldots(2)$$

$$6a^2+3Pa+Q \equiv 0 \bmod 2^{2u} \quad \ldots\ldots\ldots(3)$$

$$4a+P \equiv 0 \bmod 2^{u} \quad \ldots\ldots\ldots(4)$$

From (4) $2^2/P$; hence from (3) $2/Q$ but $2^2 \not/ Q$ i.e. Q is of form $4n+2$. Substituting this in (2) we get

$$4a^3+3Pa^2+(8n+4)a \equiv 0 \bmod 2^{3u}$$

i.e. $4a(a^2+(2n+1))+3Pa^2 \equiv 0 \bmod 2^{3u}$

But "a" being odd, a^2+2n+1 is even, and $\therefore$ $8/P$.

Further, reasoning as before, we get $Pa\equiv 0 \bmod 2^u$, and "a" being odd, $2^u/P$. But the condition (4), i.e. $4a+P \equiv 0 \bmod 2^u$ will hold for no higher value of u than $u = 2$. For since $8/P$, it would necessarily follow that $8/4a$ for $u \geqslant 3$ which contradicts "a" being odd. Hence we have shown the only possible rise at the second element occurs when $p = 2$, and the maximum rise is $u = 2$. The integer is then of the form $\frac{x\pm1}{4}$ or $\frac{x\pm3}{4}$ and the necessary and sufficient conditions are (1), (2), (3), (4), above when $u = 2$.

The following is an example. Let the integer be $(\frac{x-1}{4})$.

Since 8/P, we take P = 8. Then (4) becomes 12. From (3) we must have 30 - Q ≡ 0 mod. 16, and from (2) 28-2Q ≡ 0 mod. 64, i.e. Q ≡ 2 mod. 16, and Q ≡-14 mod. 32. Taking Q = -14 satisfies both. From (1) R ≡ 5 mod. 256. We take R = 517. The field equation is now $(\frac{x-1}{4})^4 + \frac{12}{4}(\frac{x-1}{4})^3 + \frac{16}{4^2}(\frac{x-1}{4})^2 + \frac{512}{4^4} = 0$, in which $(\frac{x-1}{4})$ is obviously an integer. The equation simplifies to $x^4+8x^3-14x^2+517 = 0$. Possible roots are ±11, ±47, which are readily disproved ∴ the equation is irreducible. $\Delta_2' = 4^4(50)$ and hence P_2^4/Δ_2' as required. The terms $20Q^2-16R$ in Δ_2'' equal $4^4(17)$; the other terms contain 4^4 at sight; ∴ P_2^4/Δ_2'' also as required.

We do not carry this case any further, but go on to outline the other cases. The case p = 2 appears as a special case throughout, and would have to be considered separately. All others have rises not sooner than at the third element. In the general theorems cited, it is established that f(x) contains a repeated factor mod. p^w, which is also a simple factor mod. p^{2w}, where p^w is the last rise. In the quartic this repeated factor may be quadratic or linear. Hence we make this natural division:-

(A) Repeated factor quadratic.

(B) Repeated factor linear.

(A.) REPEATED FACTOR QUADRATIC

Under this heading, $f(x) = x^4+Px^3+Qx^2+R$, and $f'(x) = 4x^3+3Px^2+2Qx$, have a common quadratic factor.

$$4x^3+3Px^2+2Qx \;\big|\; 4x^4+4Px^3+4Qx^2+4R \;\big|\; x+P$$

$$4x^4+3Px^3+2Qx^2$$

$$Px^3+2Qx^2+4R$$

$$4Px^3+8Qx^2+16R$$

$$4Px^3+3P^2x^2+2PQx$$

$$x^2(8Q-3P^2)-2PQx+16R$$

The common quadratic must be $x^2(8Q-3P^2)-2PQx+16R$.

$f'(x) = x(4x^2+3Px+2Q)$ $\therefore$ the repeated factor must be of the form (1) $x(x+K)_{mod\,p}$ or must be (2) $4a^2+3Px+2Q$.

Case 1. We are considering a factor of the form $x(x+K)$, which divides $f(x)$ mod. p^{2u} and $f(x)$ mod. p^u, $p \neq 2$. Since $x(x+K)/f(x)$ mod. p^{2u}, $x/f(x)$ mod. p^{2u}, and $\therefore$ $R \equiv 0$ mod. p^{2u}. $(x(x+K))^2 = x^4+2Kx^3+K^2x^2$. Comparing this with $f(x)$, it follows that $P \equiv 2K$ mod. p^u, and that $Q \equiv K^2$ mod. p^u. From these we get $P^2 \equiv 4Q$ mod. p^u.

Now $P_2^2P_3^2/\Delta_3 = 2P^2Q^2-8Q^3+32QR-12P^2R$, and since $R \equiv 0$ mod. p^{2u} and $P_2 = 1$, we must have $P_3^2/2Q^2(P^2-4Q)$. We separate (i) p/P, (ii) $p\not/P$

(i) p/P. Since $P^2 \equiv 4Q$ mod. p^u, $p\neq 2$, p^2/Q, and $\therefore$ u cannot be ≥ 2. For then p/P, p^2/Q, and p^4/R, contradicting normality. The maximum u is $\therefore$ $u = 1$.

Because p/P and $2K \equiv P$ mod. p, then $K \equiv 0$ mod. p, and the element is $\frac{x^2}{p}$.

We test this by substituting from $y = \frac{x^2}{p}$ in f(x), getting $p^4y^4+(2Q-P^2)p^3y^3+(2R+Q^2)p^2y^2+2QRpy+R^2 = 0$. Since p^4 divides all coefficients, y answers the conditions for an integer.

The question of rise in going to the last element will be discussed later.

(ii) $p\not/P$; and p^{2u}/R.

Since $P^2 \equiv 4Q$ mod. p^u and $p\not/P$, then $p\not/Q$, and $\therefore$ $P_3^2/2Q^2(P^2-4Q)$ becomes p^{2u}/P^2-4Q. An examination of Δ gives p^{4u}/Δ if p^{2u}/P^2-4Q, hence this condition is satisfied. Using $P^2 \equiv 4Q$ mod. p^{2u} we may write

$$4f(x) \equiv 4x^4+4Px^3+P^2x^2+4R \equiv 0 \text{ mod. } p^{2u}$$

i.e. $\frac{(2x^2+Px)^2+4R}{p^{2u}}$ = a simple integer

i.e. $(\frac{2x^2+Px}{p^u})^2 + \frac{4R}{p^{2u}}$ = a simple integer.

Therefore $y = \frac{x(2x+P)}{p^u}$ is an integer, since $\frac{4R}{p^{2u}}$ is a rational integer.

The necessary and sufficient conditions are $p\not/P$, $p\not/Q$, p^{2u}/R, $P^2 \equiv 4Q$ mod. p^{2u}.

Numerical examples:-

Case 1. (i). Integer is $\frac{x^2}{p}$; p/P, p/Q, p^2/R. $y = \frac{x^2}{7}$ is an integer in the field defined by $x^4+7x^3+7x^2+49 = 0$, the equation in y being $y^4-5y^3+3y^2+2y+1 = 0$. It is easily shown $y = \frac{x^2}{7^2}$ is not an integer in this field.

Case 1. (ii). The integer is $\frac{x(2x+P)}{p^u}$; $p \nmid P$, $p \nmid Q$, p^{2u}/R, $P^2 \equiv 4Q \bmod. p^{2u}$. Take $u = 1$, $p = 5$, $R = 25$, $P = 1$, $Q = -6$. The field equation is $x^4+x^3-6x^2+25 = 0$, and $y = \frac{x(2x+1)}{5}$. $4f(x) = 4x^4+4x^3+x^2-25x^2+100 = 0$, $\therefore \frac{(2x^2+x)^2}{25} - \frac{25x^2-100}{25} = 0$ i.e. $y^2-x^2+4 = 0$, the necessary equation for y.

Case 2. When x is not a factor of the repeated factor q.

Then $q=4x^2+3Px+2Q$, and $q^2 \equiv 16x^4+24Px^3+(9P^2+16Q)x^2+12PQx+4Q^2$ and since $q^2/f(x) \bmod. p^u$, we must have: $12PQ \equiv 0 \bmod. p^u$; $4Q^2 \equiv 16R \bmod. p^u$ or $Q^2 \equiv 4R \bmod. p^u$ since we are excluding $p = 2$; $9P^2+16Q \equiv 16Q \bmod. p^u$ or $9P^2 \equiv 0 \bmod. p^u$; $24P \equiv 16P$ or $8P \equiv 0$, or $P \equiv 0 \bmod. p^u$.

$P_2^2P_3^2/2P^2Q^2-8Q^3+32QR+12P^2R = \Delta_3$ i.e. $p^{2u}/8Q(4R-Q^2)$.

We separate (i) p/Q, (ii) $p \nmid Q$

(i) p/Q. The highest power of p in Q is the first. For $Q^2 \equiv 4R \bmod. p^u$, and if p^2/Q then p^4/R which combined with $P \equiv 0 \bmod. p^u$, contradicts the normality of f(x).

$4x^2+3Px+2Q \quad 4x^4+4Px^3+4Qx^2+4R \quad x^2+Px+(8Q-3P^2)$

$4x^4+3Px^3+2Qx^2$

$Px^3+2Qx^2 \quad +4R$

$4Px^3+8Qx^2 \quad +16R$

$4Px^3+3P^2x^2+2PQx$

$x^2(8Q-3P^2)-2PQx+16R$

$4x^2(8Q+3P^2)-8PQx+64R$

$4x^2(8Q+3P^2)+3P(8Q-3P^2)x+2Q(8Q-3P^2)$

$(9P^3-32PQ)x+64R-16Q^2+6P^2Q$

This remainder must be $\equiv 0$ mod. p^{2u}. The coefficient of x yields $PQ \equiv 0$ mod. p^{2u}. The absolute term can be written $16(4R-Q^2)-6P^2Q$, which again gives $4R \equiv Q^2$ mod. p^{2u}. We wish to establish $\frac{q}{p^u}$ as an integer. We have $\frac{q^2}{p^{2u}} - \frac{16f(x)}{p^{2u}}$

$= \frac{8Px^3+qP^2x^2+12PQx+4(Q^2-4R)}{p^{2u}} \equiv \frac{8Px^3}{p^{2u}}$ - simple integers

$= \frac{2Px}{p^u}(\frac{q}{p^u}) - \frac{6P^2x^2+4PQ}{p^{2u}}$ + simple integers $= \frac{2Px}{p^u}(\frac{q}{p^u})$ + simple int.

i.e. $(\frac{q}{p^u})^2 - \frac{2Px}{p^u}(\frac{q}{p^u}) - 0 =$ simple integers $\therefore \frac{q}{p^u}$ is an integer since it satisfies the requisite type of equation.

The necessary and sufficient conditions are that p/Q but no higher power of p divides Q; p^2/R; and u is the greatest such that p^{2u}/PQ and p^{2u}/Q^2-4R if $u \geq 2$

(ii) If $p \nmid Q$ then $p \nmid R$ and $\frac{4x^2+3Px+2Q}{p^u}$ is an integer where u is the greatest such that p^{2u}/P and p^{2u}/Q^2-4R.

Numerical Examples:-

Case 2. (i). p/P, p/Q but no higher power of p than the first, p^2/R, p^{2u}/PQ, p^{2u}/Q^2-4R. We take $p = 7$, $u = 2$, $Q = 14$. if $u \geq 2$

Then $P = 7^3$ and $R = 49$. The irreducible field equation is $x^4+7^3x^3+14x^2+49 = 0$. Then $\frac{4x^2+3Px+2Q}{p^u}$ becomes $\frac{4x^2+3\cdot7^3+2\cdot14}{7^2}$. The centre term may be dropped. Multiplying by 37, subtracting $\frac{147x^2}{49}$ which is an integer, and reducing 1036 by 49, we get $y = \frac{x^2+7}{7^2}$. Substituting $x^2 = 49y-7$ in $f(x)$ we get $y^2+7xy-x = 0$. Hence y is an integer.