DECOMPOSITION METHODS FOR FINITE QUEUE NETWORKS WITH A NON-RENEWAL ARRIVAL PROCESS IN DISCRETE TIME

By

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Abstract

The purpose of this thesis is to develop a decomposition method for obtaining the queue length distributions of open, tandem and split queue networks with Markovian arrival processes, and finite intermediate queues. Equivalent geometric systems are also studied to determine if maintaining the relationship between the decomposed queues improves the results over existing methods.

This thesis contains an introduction, conclusion and three main sections: a literature review; a section outlining the exact and decomposition procedures for the tandem networks; and a section outlining the exact and decomposition procedures for the split networks. Neuts' [46] Matrix Geometric Method is adopted to provide exact results which are used to validate the approximate results.

It can be concluded that for tandem and split systems with Markovian arrival processes the decomposition method developed in this thesis is superior to existing methods which fail to represent the dependence between the isolated queues. The opposite is true for both configurations of the geometric systems. That is, existing methods which do not maintain the dependence in their decomposition approach produce equal or superior results. Therefore, it can be concluded that utilizing the approximation method which captures the relationship between the queues is not worth the extra effort for geometric systems.

Acknowledgment

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Sincerely,

Michelle L. Schamber

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1. INTRODUCTION

1.1 Background

People most often associate queues with line-ups in service facilities. This is only one of many examples where queues occur in every day life. In general, a queue occurs anytime a person or object waits for a server or resource. The majority of time spent in a queue is wasted time, therefore, there is a constant push to reduce queue times and increase service output. This is the drive behind research in queuing theory.

When several queues and resources must be visited in a row, this may be termed as a network of queues. An assembly line with multiple stations in series is an example of a queue network. When customers leave the network after service is completed, this is referred to as an open system. Closed systems occur when the customers circulate around and around and no new customers arrive. This type of system is often found in computer networks. This thesis is only concerned with open systems.

In reality the amount of space between queues is limited or finite. If the space becomes full no new customers may enter the waiting area. If, at the completion of service, there is no space in the next buffer the customer must wait with the server until there is enough room. The resource cannot serve a new customer until the previous one leaves, that is the resource is idle. This situation is referred to as blocking which can lead to a large reduction in productivity, and may cause customer dissatisfaction.

There has been a great deal of research done in the area of solving queue networks, however, the majority of work applies to systems with infinite buffers. That is, blocking will never occur. When blocking is considered most techniques are very restricted. That is, networks which they apply to are limited in size and configuration. In addition the majority are restricted to renewal, independent, arrival and service processes. This limits the application of the solution techniques. Therefore, there still remains the need to solve systems which better represent reality.

1.2 Objective

The purpose of this thesis is to develop an approximate method for obtaining the queue length distributions of open networks with a non-renewal arrival process, and finite intermediate buffers. The approximation method developed for these systems uses a decomposition approach. The networks explored include two stage tandem and split systems.

Examples of systems with finite intermediate buffers can be found in manufacturing and telecommunications. The blocking mechanisms which apply to these two industries are different, and solution methods usually apply to only one type. Manufacturing type blocking occurs at the completion of service when a customer finds the next buffer is full. The customer stays in the blocked queue, and the server remains idle until there is room in the downstream buffer. For example, in an assembly line when a part is finished being processed at the station it will proceed to the down stream buffer, if this buffer is full the part remains at the machine until space becomes available. Communication blocking occurs when, prior to service starting, there is insufficient room in the down stream buffer. The server will not start service until there is adequate space in the next buffer. In telecommunications there must be sufficient bandwidth available before data can be transmitted. The method developed in this thesis assumes manufacturing blocking, however it has been shown by Onvural and Perros [47] that the two blocking types are equivalent for systems with only two stations.

Most of the research to date has involved systems with renewal arrival and service processes. There are a vast number of papers which derive exact and/or approximate solution methods for various network configurations. Most approximation methods decompose the system into a set of single node queues and each are solved independently. The arrival process of a sub-level queue is dependent on the departure process of the previous queue, however in the majority of methods there is no relationship between the approximated arrival process and the departure process of the previous queue. The decomposition method developed in this thesis utilizes a Markovian arrival process (MAP) to account for the dependence between isolated queues. The MAP is a non-renewal arrival process which defines correlated arrivals. Details about the MAP can be found in [44] and [46].

To determine the validity of the approximate results, they are compared with those derived from an exact solution method which is also developed in this thesis.

1.3 Scope

This thesis contains an introduction, conclusion, and three main sections including: the literature review; a section exploring the tandem configuration; and a third one studying the split configuration. The literature review outlines a number of the solution methods derived over the past forty years. Sections 3 and 4 contain sub-sections which look at the networks with the non-renewal arrival process and the geometric version independently. The geometric systems are presented as special cases for the purpose of analyzing the effect of utilizing correlated arrivals in the decomposition.

2. LITERATURE REVIEW

2.1 Introduction

Over the past forty years there has been an extensive amount of research done in the area of open, finite queuing systems. Finite systems may be defined as those that contain one or more queues with finite buffers, and an infinite system is one in which all of the buffers are unrestricted in size. Research on finite systems has been fueled by the fact that most real life systems contain limited size buffers between 'work stations' and blocking may occur. Applying infinite case procedures does not always provide reasonable approximations for systems with blocking.

A great deal of exploration has been done in the area of finite tandem queues with two stations. These systems can often be solved exactly. For larger tandem systems the state space is often too large and they must be solved using approximation methods. Approximation techniques are also often required in the solution of split and merge configurations, as well as arbitrary configurations.

This literature review will outline some of the exact and approximate methods used to solve open, finite queuing networks of various topographies. A brief examination of closed queuing networks is also provided. When relevant, the papers referred to in this review utilize manufacturing blocking unless otherwise stated.

2.2 Open, Finite Queuing Networks

The literature for this group of networks is presented in three sub-sections: (1) tandem; (2) split and merge; and (3) arbitrary configurations.

2.2.1 Tandem Configurations

A tandem configuration is defined as a series of queues where the output of one becomes the input of the next downstream queue. A customer passes through each and every station in sequence. The first exploration into the effect of blocking on the maximum utilization of a tandem network was undertaken by Hunt [28]. He studied a two node system with exponential service times and Poisson arrivals. The first queue was defined as having an infinite buffer and several cases were looked at for the intermediate buffer sizes: (a) infinite; (b) zero; and (c) finite. Hunt also looked at the situation of unpaced production where the line moves at each station simultaneously. In addition to the utilization, he also found the steady state distribution of the number in the system by solving the balance equations.

A similar system with deterministic service times was studied by Avi-Itzhak and Yadin [4], who found the marginal steady state queue length distributions. Their method also required that the arrival times follow a Poisson process, and that the second station have no proceeding buffer. Avi-Itzhak [5] later went on to find the steady state waiting time distribution of a larger multi-server line. He also relaxed the condition of a Poisson arrival process in [4], and allowed for finite intermediate buffers. Both methods use the moment generating functions of the queue times and the number of customers in their procedure.

Hillier and Boling [27] introduced a method which produces exact results for larger finite systems with exponential or Erlang service times. To simplify the procedure, the first queue is assumed to be saturated, or always busy. They were able to find results for the steady state mean output rate and the mean number of customers in the system excluding queue 1. Neuts [45] diverged further from [28] and found the equilibrium conditions of a two station system with general service times at the first queue, and exponential at the second. Neuts' system also requires that external arrivals follow a Poisson process. The method uses an embedded semi-Markov process approach.

Konheim and Reiser [36] studied a finite system like Hunt [28] with communication blocking and feedback. They used a product form solution to find the steady state probabilities. Foster and Perros [18] found bounds for the mean blocking time at the first queue for a two or three node exponential system using a decomposition approach. The capacity of a tandem system with no intermediate buffers, Poisson arrivals, and dependent exponential service times was found by Pinedo and Wolff [49].

Langarias and Conolly [40] presented a method for solving the waiting time of an two node exponential multi-server system. They later expanded their work [41] for a three node single-server system with no intermediate buffers. Their methods are difficult to use and are limited to the first few moments.

Exact parameter values for two and three node systems with unreliable machines were found by Gershwin and Schick [20]. They defined a system with constant and equal service times at each machine, and geometric failure and repair times. In addition, the server at the first queue is defined to be always busy. Their method could be applied to larger systems, but the dimension of the Markov chain limits the applicability.

The final exact method for tandem queues presented in this review is by Buzacott and Kostelski [12]. They compared the effort of deriving the steady state probabilities using a modification of the Matrix Geometric method (MGM) by Neuts [46], and the recursive approach by Herzog, Woo and Chandy [26]. The network studied is a two node system with Coxian-2 service and inter-arrival times. Unlike the others presented so far, their first queue is defined as finite and the arrival process is shut off when the buffer becomes full so that no customers are lost. They found that the techniques required comparable effort and produced similar results, but the recursive technique is much faster.

The majority of approximation methods for tandem systems can be grouped into two categories: those that decompose the system into single nodes; and those that group them into 'paired' sub-lines. In a single node approach each queue has a revised arrival and service process and is solved in isolation. In the 'paired' approach the line is analyzed sequentially as two to three node sub-lines with revised arrival and service processes. The results from each sub-line are used in the solution of another pair. A number of papers which use single node approximations will be presented first.

Single Node Decomposition

Altiok [1], Hillier and Boling [27], and Perros and Altiok [48] each developed approximation techniques to solve exponential K-node finite tandem queuing systems. Each define the revised arrival and service processes differently. Altiok [1] defines each individual queue as a $M/C_2/1/(N+1)$, where as Perros and Altiok [48] define them as $M/PH_{K-i+1}/1/(N+1)$, where $i = 1, 2, \ldots, K$. The phase distribution accounts for blocking by all downstream nodes, and not just the next one as in [1]. Hillier and Boling [27] have the simplest definition of an individual node as a $M/M/1/(N+1)$ queue. Springer [51] sped up the method in [27], by defining a different set of equations to solve the parameters values. In each of the above cases the first queue is infinite.

Altiok [3] also found the marginal steady state probabilities of a similar system as in [48], but with phase service times as the original service process. The revised service process is also phase and each queue is solved using the MGM. In addition to a finite first queue he also looked at the cases of a finite queue 1, and one that is saturated. Jun and Perros

[33] extended the method in [3] for a general system by redefining the decomposed system as a set of $C_2/C_2/1/N$ queues.

The single node decomposition approach has also been used on finite systems with a variety of traits not included in those listed above. Konheim and Reiser [37] found the equilibrium parameter values of an exponential system with feedback and communication blocking. Kelly [34] found the throughput for a system with general service times, a saturated queue 1, and communication blocking. Caseau and Pujolle [14] studied the maximum throughput of an exponential system with dependent service times and intermediate arrivals to any queue.

Additional approximation techniques for systems with equal service times at each station, for a single part, have been developed by Avi-Itzhak and Halfin [6], Ziedins [58], and Avi-Itzhak [7]. [6] and [7] both define saturated systems. Recently Chao [15] studied a deterministic priority system with no intermediate buffer.

'Paired' Decomposition

Unlike the single node approach which can be extended for other configurations, the 'paired' approaches presented in this review can only be used on tandem systems. A number of approximation techniques which use this approach are present below.

Gershwin [21] extended his earlier work with Schick [20] for a system with more than three nodes. The line is analyzed as pairs of machines with revised service and arrival processes. The buffer size for each pair is the same as in the original case. Each two node sub-line is solved exactly using the method in [20]. Choong and Gershwin [16] extended this work further to include random processing times, as well as unequal service times. The methodology is similar to [21], but the results do not always converge. Dallery, David and Xie [17] simplified [21] by replacing the set of equations used to solve the parameter values with an equivalent one. This resulted in a faster algorithm which converged for every case tried. A further improvement was made when Gershwin [22] applied the modification from [17] to the method in [16]. This resulted in an even faster and simpler algorithm.

Brandwajn and Jow [10] studied an exponential system with dependent service times. They also looked at the case when the first queue is always busy. Each sub-line is defined as a two node M/M/1/N system, and analysis begins at the first pair and proceeds forward. When the end of the line is reached the parameters from the last pair are used to revise the processes of the first pair. The steady state probability for each pair is found, and the algorithm stops when the convergence criteria is met. Brandwajn and Sahai [11] later improved this method by utilizing a back-and-forth sweep. The algorithm now proceeds down the line and then back up again. This speeds up the technique but has little effect on the results.

The paired decomposition approach was applied by Gun and Makowski [25] to solve a tandem networks with phase service and inter-arrival times, and communication blocking. In addition, the machines are unreliable and feedback may occur. The paper presents an iterative procedure for finding the steady state queue lengths. The algorithm ends when the convergence criteria is met.

Yannopoulos and Alfa [56] applied a three node decomposition method to an exponential tandem network to find the steady state probabilities of the number in each queue and the joint probability distribution of the number in each triplet. This method produces better results than in [10], but with increased computational effort.

2.2.2 Split and Merge Configurations

A split configuration is defined as one in which a source queue feeds two or more second stage queues. The merge topology is the opposite, that is several queues feed one queue. A split queue becomes blocked when the destination queue's buffer is full. Not all buffers must be full for this to occur. In a merge queue all first stage queues can become blocked when the second stage queue is full. Unlike the tandem configuration there has been relatively little research done in this area. An approximation method for an exponential three node tandem, split, or merge system with communication blocking was developed by Boxma and Konheim [9]. Their technique produces the results of the marginal probability distributions, and is an extension of Jackson's [31] work. Jackson

[31] used the 2-phase distribution to represent a two stage exponential service system. He used each of the phases to represent service in each of the stations, however this defined a system in which service could only occur at one station at a time.

Altiok and Perros [2] went on to study larger, greater than three nodes, exponential split and merge systems. They replace the actual service time with an effective one, and decompose the system into individual nodes. The individual second stage queues of either system are represented as $M/M/1/(N_j + 1)$ queues, where j = the number of second stage queues and j = 1 for merge configurations. The first stage queues are transformed into a $M/PH_{Ki}/1$ queues, where i = number of first stage queues and i = 1 for the split case. An exponential merge configuration was also studied by Lee and Pollock [42]. Like in [2] they decomposed the network into individual queues, however they analyzed each as either an M/M/1/N queue or as an M/G/1/N queue. By declaring different definitions of the states they were able to produce better results than Altiok and Perros [2].

Kerbache and Smith [35] devised the Generalized Expansion Method (GEM) for tandem, split and merge queue networks. This procedure requires that all processes be renewal and the first two moments be known. Based on the same principles used by Kuehn [38] and Labetoulle and Pujolle [39] for arbitrary configurations, this technique places an artificial node between two adjacent nodes. The artificial node services blocked customers for the remainder of the service time at the next queue. There are three stages to the GEM: (i) network reconfiguration; (ii) parameter estimation; and (iii) feedback elimination at the artificial node. [39] provides a procedure for the estimation of the parameters of the individual GI/G/1/N queues.

Yannopoulos and Alfa [55] developed a quick and easy method for analyzing tandem, split and merge configurations with general arrival and service processes. The method decomposes the system and analyses each queue in isolation. Once decomposed the methods by Yao and Buzacott [57] and Gelenbe [19] are used to solve each node. This technique is much simpler than GEM which requires the solution of a system of non-linear equations.

2.2.3 Arbitrary Configurations

An arbitrary network is composed of a combination of the other three configurations to form one larger network. Arrivals and departures can occur at several different nodes. Jackson [29] first looked at the arbitrary configuration as a method of representing a machine shop with multi-machine stations. He derived the product form solution, where the joint steady state probability distribution is found as the product of the marginals, to produce exact results for the infinite exponential case. He later expanded this work [30] to include the case of dependent arrival and service processes.

A product form solution is limited to smaller networks with service distributions which have a rational Laplace transform, therefore approximation methods are needed for general systems. Kuehn [38] derived an approximation technique to find the mean waiting time at each infinite queue for a network with general service and inter-arrival processes. In addition his model allows for feedback at the completion of service. Whitt [53] later developed a software package, the Queuing Network Analyzer (QNA), to solve a multi-server version of the network. In the procedure each node is analyzed as a GI/G/m queue.

The need for finite system solutions led to the development of an approximation technique by Takahashi, Miyahara, and Hasegawa [52]. Their method finds the blocking probabilities and output of a finite exponential system. They decompose the network into individual M/M/1 queues and solve each individually. The parameters are found by solving a set of simultaneous non-linear equations. Jun and Perros [32] also studied the exponential system, but in their method each individual queue is analyzed as an M/PH/1 queue. An approximation method for the case with cyclic queues, or feedback, was derived by Lee and Pollock [43].

The Isolation Method by Labetoulle and Pujolle [39] applies to general finite networks where the inter-arrival and service processes are renewal. The Isolation Method, like many other methods, decomposes the system into individual queues with revised arrival and service processes. The parameter values of the individual GI/G/1/N queues are solved for iteratively.

In the Isolation Method the first queue is infinite, and no customers are lost to the system. This is not always a valid assumption. Shi [50] looked at the throughput and the mean waiting time at each queue in a general system with customer loss. His method is an extension of the QNA procedure derived in [53].

2.3 Closed Queuing Networks

A closed network is one in which there are no external arrivals or departures, therefore the number in the system remains fixed. This section briefly outlines some of the earlier papers which discuss closed queuing networks. The product form solution method [30] for open networks also applies to closed, exponential, arbitrary systems with infinite buffers. Gordon and Newell [23] first applied this procedure, however the closed case requires the calculation of a normalizing constant. Their procedure for calculating this value was inefficient and later improved upon with an algorithm developed by Buzen [13].

Baskett, Chandy, Muntz, and Palacios [8] extended [23] to include a system with multiple customer classes. Each of the customer classes are assumed to have the same priority level. They found that the product form solution still holds when they used service time probability distributions which had rational Laplace transforms. In addition they explored the effect of various service disciplines. All of the papers discussed so far have assumed FIFO service. The results in [8] illustrate that the method also holds for FIFO, LCFS, process sharing, and no queuing service disciplines.

Gordon and Newell [24] were the first to look at closed finite systems, and limited their study to tandem configurations. [24] assumed exponential service times, and equated the closed system to an open one with a random number of customers. However, they did not look at the steady state conditions and the usefulness of the paper is limited.

A great deal more research has been done in the area of closed queuing networks in recent years. This extends beyond the scope of the thesis and the work is not presented here.

3. TANDEM CONFIGURATIONS

3.1 Introduction

The first network configuration studied in this thesis is the tandem configuration. Queues in tandem are aligned in series and customers enter the buffer at the first station and proceed sequentially though all of the queues. A customer may only exit at the last station. The tandem networks presented in Section 3 utilize the following assumptions:

- There are only two queues in series
- Each queue has only one server
- The service times in the queues are independent
- Queue 1 has an infinite buffer
- Queue 2 has a finite buffer of size K2
- Service follows a first-in, first-out (FIFO) rule

Figure 3.1.1 provides a simple illustration of the tandem network discussed in this section.

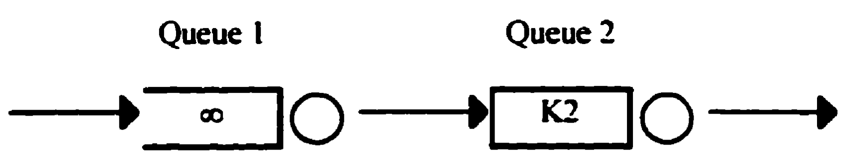

Figure 3.1.1: Tandem Network, Two Queues in Series

As presented in Section 2, all of the literature examined assume that the external arrival processes into their networks are renewal. This section develops an approximate solution method for a tandem network with related external arrivals. In addition to assuming a non-renewal arrival process, the decomposition method presented utilizes dependent arrivals for the isolated second stage queue. Results found using the approximate method are compared with those found using an exact solution approach. The exact method for the tandem system is also derived in this section.

This section begins with a presentation of an exact solution method for the tandem network with a non-renewal input process, and continues with the description of the decomposition method. A comparison between the two is then presented, as well as a comparison with the method derived by Yannopoulos and Alfa [55]. This section then continues on with a modification of the methods for a network with geometric arrival and service processes, as a special case of the non-renewal.

3.2 Networks With a Markovian Arrival Process

3.2.1 Exact Results

Introduction

The first tandem network to be examined is one with an external MAP and phase service distributions for each of the two queues. The first queue is defined as having an infinite buffer, and the second has a finite buffer. A FIFO service discipline governs both queues.

The arrival process into queue 2 is determined by the departure process of queue 1. The following parameters are used to define this system:

- **D0** = No departure matrix of MAP
- **D1** = Departure matrix of MAP
- $(\beta_1, \mathbf{S}_1)$ = phase service distribution of queue 1
- $(\beta_2, \mathbf{S}_2)$ = phase service distribution of queue 2
- K2 = buffer size of queue 2
- $(j = 0)$ = state when there are K2 + 1 customers in queue 2 but there is no blocking
- $(j = 1)$ = state when there are K2 + 1 customers in queue 2 and a customer leaving queue 1 is blocked from entering queue 2

Transition Matrix

The transition matrix which defines this system has two levels, one for each queue. The states of the primary level L_1 represent the number of customers in queue 1, for any number in queue 2. The states of the sub-level L_2 depict the number in the second queue. The size of the state spaces depends on the level. Queue 1 has an infinite state space and queue 2 is finite.

The following is the tri-diagonal transition matrix which defines this system:

P =

$$\begin{array}{cc} L_1 & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \cdot \end{array} & \left[\begin{array}{cccccc} \mathbf{B0} & \mathbf{B1} & & & & \\ \mathbf{C0} & \mathbf{A1} & \mathbf{A0} & & & \\ & \mathbf{A2} & \mathbf{A1} & \mathbf{A0} & & \\ & & \mathbf{A2} & \mathbf{A1} & \mathbf{A0} & \\ & & \cdot & \cdot & \cdot & \end{array} \right] \end{array}$$

Each of the sub-level matrices within **P** are defined in Appendix A.1

Solution Method

The infinite nature of **P** requires more complicated techniques, than their finite counterparts, to solve for the steady state queue length distributions. The Matrix Geometric Method (MGM) developed by Neuts [46] provides a relatively simple recursive method for finding the steady state vector $\bar{X}$ for a infinite transition matrix such as **P**. Once $\bar{X}$ is known the value of μ_q, the mean number in the system, can be calculated. The MGM results in exact solution values which can be used in comparison with the results of an approximated solution.

In order for steady state to be achieved, and the steady state parameters found, the system of interest must be stable. A system will be stable when the following is true:

$$\pi_A \beta > 1$$

where, for this system

$$\beta = 0\ \mathbf{A0}\ e + 1\ \mathbf{A1}\ e + 2\ \mathbf{A2}\ e$$
$$\pi_A\ e = 1$$
$$\pi_A\ \mathbf{A} = \pi_A$$
$$\mathbf{A} = \mathbf{A2} + \mathbf{A1} + \mathbf{A0}$$
$$\mathbf{A}\ e = e$$
$$e = \text{a column of 1's}$$

therefore

$$\pi_A\ \mathbf{A1}\ e + \pi_A\ 2\ \mathbf{A2}\ e > 1$$
$$\pi_A\ (\mathbf{A1} + \mathbf{A2})\ e + \pi_A\ \mathbf{A2}\ e > 1$$
$$\pi_A\ e - \pi_A\ \mathbf{A0}\ e + \pi_A\ \mathbf{A2}\ e > 1$$
$$\pi_A\ \mathbf{A2}\ e > \pi_A\ \mathbf{A0}\ e$$
$$\pi_A\ (\mathbf{A2} - \mathbf{A0})\ e > 0$$

A stability factor Q, where $Q = \pi_A\,(\mathbf{A2} - \mathbf{A0})\,e$, is defined so that Q is positive for a stable system. Although a calculation is required to check that the statement is true, it is important to remember that the external arrival rate, λ into any stable system must be less than the minimum service rate. For the tandem network with the non-renewal arrival process discussed in this section, λ must be less than the minimum service rate μ_1 or μ_2 where:

$$\lambda = \pi_D\,\mathbf{D1}\,e,\ \pi_D \text{ is the steady state vector of } \mathbf{D} = \mathbf{D0} + \mathbf{D1}$$
$$\mu_i = [\,\beta_i\,(\mathbf{I} - \mathbf{S}_i)^{-1}\,e\,]^{-1},\ i = 1 \text{ or } 2$$

To determine the mean number in both of the queues, μ_q, the marginal queue length distributions and the mean number in each of queues 1 and 2 , μ_{q1} and μ_{q2} respectively, must be calculated. In this scenario the number in the queue includes the customer in service, if any. The derivation of the formula for the mean number in queue 1 is as follows:

$$\mu_{q1} = \Sigma_i\,\Sigma_m\,i\,X_{i,m}$$
$$\mu_{q1} = \Sigma_i\,i\,\overline{X}_i\,e$$

given

$$\overline{X}_i = (X_{i,0}, X_{i,1}, X_{i,2}, \ldots, X_{i,K2}, X_{i,K2+1\,(j=0)}, X_{i,K2+1\,(j=1)})$$
$X_{i,m}$ = the probability of i customers in queue 1, and m customers in queue 2
$$\Sigma_m\,X_{i,m} = \overline{X}_i\,e$$

then

$$\mu_{q1} = \Sigma_i\,i\,(\overline{X}_1\,\mathbf{R}^{i-1})\,e$$
$$\mu_{q1} = \overline{X}_1\,\Sigma_i\,i\,\mathbf{R}^{i-1}\,e$$
$$\mu_{q1} = \overline{X}_1\,(\mathbf{I} - \mathbf{R})^{-2}\,e$$

where

$\mathbf{I}$ = identity matrix
$\mathbf{R}$ = rate matrix derived within the MGM solution procedure. Details about the derivation of $\mathbf{R}$ may be found in [46]

The following is the derivation for the mean number in queue 2:

$$\mu_{q2} = \sum_i \sum_m m\, X_{i,m}$$
$$\mu_{q2} = \sum_i [0\, X_{i,0} + 1\, X_{i,1} + 2\, X_{i,2} + \ldots + K2\, X_{i,K2} + (K2+1)\, X_{i,(K2+1)\,(j=0)} + (K2+1)\, X_{i,(K2+1)\,(j=1)}]$$
$$\mu_{q2} = \sum_i \overline{X}_i \mathbf{V}\, \mathbf{e}$$

where

$$V = \begin{bmatrix} 0 & & & & & & \\ & 1 & & & & & \\ & & 2 & & & & \\ & & & \ddots & & & \\ & & & & K2 & & \\ & & & & & K2+1 & \\ & & & & & & K2+1 \end{bmatrix}$$

then

$$\mu_{q2} = \overline{X}_0 \mathbf{V}\, \mathbf{e} + \sum_i (\overline{X}_1 \mathbf{R}^{i-1})\, \mathbf{V}\, \mathbf{e}$$
$$\mu_{q2} = \overline{X}_0 \mathbf{V}\, \mathbf{e} + \overline{X}_1 \sum_i \mathbf{R}^{i-1}\, \mathbf{V}\, \mathbf{e}$$
$$\mu_{q2} = \overline{X}_0 \mathbf{V}\, \mathbf{e} + \overline{X}_1 (\mathbf{I} - \mathbf{R})^{-1}\, \mathbf{V}\, \mathbf{e}$$

The mean number in the system is found as the sum of the number in each queue and may be summarized as follows:

$$\mu_q = \mu_{q1} + \mu_{q2}$$
$$\mu_q = \overline{X}_1 (\mathbf{I} - \mathbf{R})^{-2}\, \mathbf{e} + \overline{X}_0 \mathbf{V}\, \mathbf{e} + \overline{X}_1 (\mathbf{I} - \mathbf{R})^{-1}\, \mathbf{V}\, \mathbf{e}$$
$$\mu_q = \overline{X}_0 \mathbf{V}\, \mathbf{e} + \overline{X}_1 (\mathbf{I} - \mathbf{R})^{-1} [(\mathbf{I} - \mathbf{R})^{2} + \mathbf{V}]\, \mathbf{e}$$

3.2.2 Approximation Method

Introduction

The decomposition of a network of queues into isolated individual queues simplifies the method of calculating system parameters. A system of two queues in sequence, with a finite intermediate buffer, can easily be decomposed into two uncoupled queues: i) an isolated infinite queue; and ii) an isolated finite queue. However, the arrival process into the isolated queue 2 is not the same as the external arrival process into the network. The arrival process into the second queue is related to the departure rate of customers from the first queue. To maintain the relationship between the two queues a new MAP distribution, MAP*, with $\mathbf{D0}^*$ and $\mathbf{D1}^*$ based on the transition matrix of the network is used to approximate the arrival process into the second queue. The isolated second queue may now be described as a MAP/PH/1/K queue. The service distribution of the second queue is the same as in the combined system.

The arrival process into the first queue remains the same as the input process into the network, however the service distribution is different. The service rate must be modified to account for customers being blocked from entering queue 2. The new service rate depends on the probability of queue 2 being full, K2 in the buffer plus one in service. In other words, the service rate of queue 1 is related to the rate of service completions in queue 2. The infinite queue may now be defined as a MAP/PH/1/∞ queue.

Solution of Isolated Queue 2: MAP/PH/1/K

As mentioned on the previous page the service rate of the isolated queue 1 depends on queue 2. Departures from queue 1 may be blocked, with a blocking probability BP, from entering queue 2 if the buffer is full. Therefore, in order to solve for the first queue's system parameters the blocking probability of queue 2 must be found first.

To maintain the relationship between the isolated queues the arrival process into queue 2 is estimated with a MAP. The MAP captures the dependency between the departure rate of the first queue and the arrival rate into the second when the transition matrix $\mathbf{P}$ is divided into departure and no departure matrices. The new process for the isolated second stage queue is represented as MAP* with departure and no departure matrices $\mathbf{D0}^*$ and $\mathbf{D1}^*$. Due to the infinite nature of $\mathbf{P}$, $\mathbf{D0}^*$ and $\mathbf{D1}^*$ are also infinite. The first step of the approximation development was to truncate $\mathbf{D0}^*$ and $\mathbf{D1}^*$. This was done in order to simplify the procedure. If n is the number of blocks in $\mathbf{D0}^*$ and $\mathbf{D1}^*$, the approximated matrices are provided below:

$\mathbf{D0}^* =$

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \cdot \\ n \end{array} & \begin{bmatrix} \mathbf{B0} & \mathbf{B1} & & & \\ & \mathbf{A1}'' & \mathbf{A0} & & \\ & & & \cdot & \\ & & & & \mathbf{A1}'' + \mathbf{A0} \end{bmatrix} \end{array}$$

$\mathbf{D1}^{*} =$

$$
\begin{array}{c} \\ 1 \\ 2 \\ \ddots \\ n \end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & \cdots & n\text{-}1 & n \end{array} \\
\left[\begin{array}{ccccc}
 & & & & \\
\mathbf{C0} & \mathbf{A1'} & & & \\
 & & \ddots & & \\
 & & & \mathbf{A2} & \mathbf{A1'}
\end{array}\right]
\end{array}
$$

The matrices without a superscript are the same as those within the transition matrix **P**. Each of the new matrices within $\mathbf{D0}^{*}$ and $\mathbf{D1}^{*}$ are presented in Appendix A.1.2.

After running several numerical trials it was found that for every set of inputs there is a maximum desired number of blocks, n_{max}, in $\mathbf{D0}^{*}$ and $\mathbf{D1}^{*}$. That is, there is a point after which an increase in n has no effect on the steady state vector and the mean number in the queue. It was also observed that as the traffic intensity of queue 2, $\rho_2 = \lambda_2/\mu_2$, increases the value of n_{max} increases.

Once the approximated arrival process into queue 2 has been calculated the queue is analyzed as a MAP/PH/1/K queue. The transition matrix for our system is as follows:

$\mathbf{P}_{q2} =$

$$
\begin{array}{c|ccccccc}
 & 0 & 1 & 2 & 3 & K2-1 & K2 & K2+1 \\
\hline
0 & D0^{*} & D1^{*}\otimes\beta_2 & & & & & \\
1 & D0^{*}\otimes S^0_2 & D0^{*}\otimes S_2 + D1^{*}\otimes S^0_2\beta_2 & D1^{*}\otimes S_2 & & & & \\
2 & & D0^{*}\otimes S^0_2\beta_2 & D0^{*}\otimes S_2 + D1^{*}\otimes S^0_2\beta_2 & D1^{*}\otimes S_2 & & & \\
\cdot & & & & & & & \\
K2 & & & & & D0^{*}\otimes S^0_2\beta_2 & D0^{*}\otimes S_2 + D1^{*}\otimes S^0_2\beta_2 & D1^{*}\otimes S_2 \\
K2+1 & & & & & & D0^{*}\otimes S^0_2\beta_2 & D0^{*}\otimes S_2 + D1^{*}
\end{array}
$$

To calculate the mean queue length for this queue the steady state vector of the number in the system must be found. This may be done using several iterative methods for finite queues, including the Power method and Gauss-Seidel procedure. The formula for the mean queue length is then:

$$\mu_{q2} = \sum_{m=0}^{K2+1} m X_m^{(2)}$$

where

$X_m^{(2)}$ = the steady state probability of m customers in queue 2

Solution Method of Isolated Queue 1: MAP/PH/1/∞

As mentioned in the introduction the isolated queue 1 may be defined as a MAP/PH/1/∞ queue. The arrival rate into this queue is the same as the external arrival rate into the network, however the service rate is less than or equal to the original service rate. As the blocking probability increases there is a greater chance that a customer will be delayed in queue 1, decreasing the number of customers which can be served over time. In other words, the service rate decreases. If no blocking occurs the service rate remains the same. The new service distribution PH^* can be expressed as (β_1^*, S_1^*), where:

$$\beta_1^* = \begin{bmatrix} \beta_1 & 0 \end{bmatrix}$$
$$S_1^* = \begin{bmatrix} S_1 & S_1^0 \beta_2^* BP \\ 0 & S_2 \end{bmatrix}$$

and

- BP is the steady state probability of queue 2 being full

$$-\qquad \beta_2^{*} = \beta_2^{*}\,(\mathbf{S}_2 + S^0_2\,\beta_2)$$

Once the new service distribution is known the isolated queue is analyzed as a standard MAP/PH/1/∞ queue. The infinite transition matrix for queue 1 is as follows:

$\mathbf{P}_{q1} =$

$$\begin{array}{c} \\ 0 \\ 1 \\ \\ 2 \\ \\ 3 \\ \\ \ddots \end{array}
\begin{array}{c}
\begin{array}{cccccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad & \cdots \end{array} \\
\left[\begin{array}{cccccc}
D0 & D1\otimes\beta_1^{*} & & & & \\
D0\otimes S^0_1{}^{*} & D0\otimes S_1^{*} + & D1\otimes S_1^{*} & & & \\
 & D1\otimes S^0_1{}^{*}\beta_1^{*} & & & & \\
 & D0\otimes S^0_1{}^{*}\beta_1^{*} & D0\otimes S_1^{*} + & D1\otimes S_1^{*} & & \\
 & & D1\otimes S^0_1{}^{*}\beta_1^{*} & & & \\
 & & D0\otimes S^0_1{}^{*}\beta_1^{*} & D0\otimes S_1^{*} + & D1\otimes S_1^{*} & \\
 & & & D1\otimes S^0_1{}^{*}\beta_1^{*} & & \\
 & & & \ddots & \ddots & \ddots
\end{array}\right]
\end{array}$$

The infinite nature of the queue makes it more difficult to find the steady state vector than for queue 2. The MGM method may be employed to over come this difficulty. Once the steady state vector is found, the mean queue length can be calculated as follows:

$$\mu_{q1} = \textstyle\sum_i i\, X_i^{(1)}$$
$$\mu_{q1} = \textstyle\sum_i i\,(X_1^{(1)}\, \mathbf{R}^{i-1})$$
$$\mu_{q1} = X_1^{(1)} \textstyle\sum_i i\, \mathbf{R}^{i-1}$$
$$\mu_{q1} = X_1^{(1)}\, (\mathbf{I} - \mathbf{R})^{-2}$$

where

$X_i^{(1)}$ = the steady state probability of i customers in queue 1

When the probability of blocking, BP, increases the service rate decreases and the number waiting for service in queue 1 increases. When BP = 0, $(\beta_1^{*}, \mathbf{S}_1^{*}) = (\beta_1, \mathbf{S}_1)$ and the traffic intensity of queue 1, $\rho_1 = \lambda/\mu_1^{*}$, is a minimum.

3.2.3 Comparison of Combined and Isolated Results

Comparison of Methods Presented in This Section

Numerical trials were performed on the isolated and combined queues, and the results were compared. Each trial used the same inputs: **D0** and **D1**; (β_1, S_1); (β_2, S_2); and K2, and generated the same set of outputs. The values of the system parameters listed below were examined for each trial.

- Q = stability factor
- (β_1^*, S_1^*), modified service of queue 1
- λ_2 = arrival rate into queue 2
- ρ_1 = traffic intensity of queue 1
- ρ_2 = traffic intensity of queue 2
- μ_q = mean number in system
- μ_{q1} = mean number in queue 1
- μ_{q2} = mean number in queue 2
- % error of $\mu_q = | \mu_q(\text{combo}) - \mu_q(\text{iso}) | / \mu_q(\text{combo})$
- % error of $\mu_{q1} = | \mu_{q1}(\text{combo}) - \mu_{q1}(\text{iso}) | / \mu_{q1}(\text{combo})$
- % error of $\mu_{q2} = | \mu_{q2}(\text{combo}) - \mu_{q2}(\text{iso}) | / \mu_{q2}(\text{combo})$

Where

- mean number in each queue includes the customer in service
- (combo) represents combined system solution
- (iso) represents the isolated queue results

The results of the trials performed are presented in Appendix A.1.3 at the end of this paper.

Observations Related to Systems With Blocking

After examining the trial results it was determined that there exists a buffer size $K2_{max}$ so that the blocking probability BP is equal to zero, and the steady state vector remains the same for increases in the buffer size. It was also observed that as the service rate in queue 2 increased, and ρ_2 decreased, the value of $K2_{max}$ decreased. As the output from queue 2 increases the mean number in the queue 2 buffer decreases, therefore the maximum buffer requirements diminish. The same results occur as the departure rate from queue 1 decreases.

As BP decreases the mean number in queue 1 decreases as expected, however the mean number in queue 2 may increase if the stability of the second queue was low prior to decreasing the blocking probability. With low stability the mean queue size is nearly equal to K2 + 1. If the buffer size is increased the buffer will still remain nearly full, but it may not cause blocking. Therefore maximum number in the queue increases, and this increases the average.

It is important to note that the above observations are not unique to tandem systems with MAP arrivals. Reference to these observations will be made for the other systems studied later in this thesis.

Observations Related to the Application of the Decomposition Procedure

It would appear that the decomposition method will produce values for the mean number in the system within 30% of the actual results when the following is true:

1. The value of ρ_2 is less than 0.5, for any values of ρ_1
2. The number of blocks in $\mathbf{D0}^*$ and $\mathbf{D1}^*$ equal 2

There appears to be no direct relationship between the value of the percentage error and the values of the traffic intensities, and exceptions to the rule may occur. That is, there may be cases when traffic intensities outside the boundaries will produce good results. Some additional observations made about the results are as follows:

1. If $\rho_1 > \rho_2$, the approximate results are still within 30% if ρ_2 is less than two-thirds.
2. As ρ_2 decreases the total queue, and queue 1 percentage errors decrease. The opposite is true when K2 decreases.

Comparison With Yannopoulos & Alfa's Method

Although it would appear that the approximation method presented provides comparable findings for the marginal queue length distributions and the mean number in the system, it is necessary to determine if the results are better than those already found in literature. A comparison between the results provided in this thesis and those generated using the method presented by Yannopoulos and Alfa [55] is performed to provide some insights into the significance of the results.

The method in [55] was developed for networks with general arrival and service distributions for which the first two moments are known[1]. The paper provides two variations for tandem, split, and merge configurations. Yannopoulos & Alfa present results based on the single node approximation techniques by Yao & Buzacott [57] and Gelenbe [19]. Both methods were implemented and compared with the actual and approximated results provided in Appendix A.1.3. The following tables present the findings:

Table 3.2.1: Results of Mean Queue Length for Increasing ρ_1, $\rho_2 = 0.4$

ρ_1	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
0.2222	0.7619	0.7314	4.00%	0.3500	54.06%	0.3288	56.84%
0.2500	0.8000	0.7476	6.55%	0.3890	51.38%	0.3649	54.39%
0.2857	0.8534	0.7759	9.08%	0.4428	48.11%	0.4151	51.36%
0.3333	0.9336	0.8270	11.42%	0.5223	44.06%	0.4899	47.53%
0.4000	1.0675	0.9268	13.18%	0.6522	38.90%	0.6132	42.56%
0.6667	2.1490	1.9061	11.30%	1.5364	28.51%	1.4748	31.37%
0.8000	4.3798	3.8778	11.46%	3.8944	11.08%	3.8738	11.55%

Table 3.2.2: Results of Mean Queue Length for Increasing ρ_2, $\rho_1 = 0.4$

ρ_2	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
0.3333	0.9336	0.8403	9.99%	0.5226	44.02%	0.4900	47.51%
0.4000	1.0675	0.9268	13.18%	0.6522	38.90%	0.6132	42.56%
0.5000	1.3366	1.0832	18.96%	0.8732	34.67%	0.8362	37.44%
0.6667	2.1491	1.4276	33.57%	1.2486	41.90%	1.2425	42.19%
0.8000	4.3791	1.7672	59.64%	1.8390	58.01%	1.8588	57.55%

[1] The second moments of the MAP and phase distributions were calculated using the formulas presented in [44] and [46] respectively.

Table 3.2.3: Results of Mean Queue Length for Increasing ρ_2, $\rho_1 = 0.6667$

ρ_2	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
0.1000	2.0056	1.9623	2.16%	1.8885	5.84%	1.8700	6.76%
0.4000	2.1490	1.9061	11.30%	1.5364	28.51%	1.4748	31.37%
0.5000	2.4619	2.0245	17.77%	1.8626	24.34%	1.8046	26.70%
0.6667	3.5479	2.3226	34.54%	2.3011	35.14%	2.2986	35.21%
0.8000	3.8904	2.3933	38.48%	1.6185	58.40%	1.5849	59.26%

Table 3.2.4: Results of Mean Queue Length for Increasing ρ_2, $\rho_1 = 0.8$

ρ_2	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
0.4000	4.3798	3.8778	11.46%	3.8944	11.08%	3.8738	11.55%
0.6667	3.8904	3.1417	19.24%	1.9866	48.94%	1.8927	51.35%
0.8000	6.6366	3.7403	43.64%	2.5026	62.29%	2.4883	62.51%

To summarize, the decomposition method produced superior results in all cases tested. This was true for values which were both comparable and not comparable with the actual results. Therefore, it may be concluded that techniques which maintain the relationship between the isolated queues produce better results than those that do not, when external arrivals follow a non-renewal process.

3.3 Networks With a Geometric Arrival Process

3.3.1 Exact Results

Introduction

A special case of a network with a MAP arrival process and phase service in each of the queues is a geometric system. This is a simpler network to solve, however it has limited applications. In this section the network studied, as in the previous case, has an infinite input buffer and service follows a FIFO service discipline. However, it is now defined as having geometric arrival and service processes. The parameters used to define this system are as follows:

- p = probability of an arrival to queue 1
- $q = 1 - p$ = probability of no arrival to queue 1
- β_i = probability of a service completion in queue i, i = 1 or 2
- $\alpha_i = 1 - \beta_i$ = probability of no service completion in queue i, i = 1 or 2
- K2 = buffer size of queue 2
- $(j = 0)$ = state when there are K2 + 1 customers in queue 2 but there is no blocking
- $(j = 1)$ = state when there are K2 + 1 customers in queue 2 and a customer leaving queue 1 is blocked from entering queue 2

Transition Matrix

The discrete time transition matrix **P** which defines this system has the same tri-diagonal format as the one which characterizes the non-renewal system. However, the boundary conditions are reduced and the entries in each of the sub-matrices contain different values. The new sub-matrices within **P** are defined in Appendix A.2. The following is the tri-diagonal transition matrix which defines the geometric system:

$\mathbf{P} =$

$$\begin{array}{cc}
L_I & \begin{matrix} 0 & 1 & 2 & 3 & 4 & \cdots \end{matrix} \\
\begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ \ddots \end{matrix} & \begin{bmatrix} \mathbf{B0} & \mathbf{B1} & & & & \\ \mathbf{A2} & \mathbf{A1} & \mathbf{A0} & & & \\ & \mathbf{A2} & \mathbf{A1} & \mathbf{A0} & & \\ & & \mathbf{A2} & \mathbf{A1} & \mathbf{A0} & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}
\end{array}$$

Solution Method

Due to the simpler nature of this system, in addition to the number in the system it is also possible to derive the waiting time distribution. To calculate the distribution it is assumed that our customer of interest arrives when the system is at steady state. The arriving customer is only concerned about customers that are already in the system, as new arrivals will not increase their waiting time. An absorbing Markov chain defines a system with this condition.

The probability that the waiting time in the system is less than or equal to t, is the first component of the steady state vector $\bar{Y}$ of the absorbing system. This component is the value of the probability that all other customers have departed the queues. Since there are no arrivals after our customer, the point when the second queue empties is the point which our customer begins service in that queue. The steady state vector of the original system provides the values for the initial vector of the new system. Therefore, the steady state vector of the original system must be found first.

The MGM is again applied to find the steady state vector $\overline{X}$. The stability factor Q has the same definition as for the non-renewal case, however after the appropriate substitutions are made it simplifies as follows:

$$Q = (\beta_1 - p) * \left(1 + \sum_{i=1}^{K2+1}\left(\frac{\theta^i}{\alpha_2}\right)\right) - p\theta^{(K2+1)}\frac{\beta_1}{\beta_2}$$

where

$$\theta = \frac{\beta_1\alpha_2}{\alpha_1\beta_2}$$

Again Q must be positive for a stable system, and $p < \min(\beta_1, \beta_2)$ to meet the arrival rate restrictions.

After applying the MGM, one can either proceed to find the waiting time distribution or calculate the marginal queue length distributions. The derivation of the formulae used to calculate the queue lengths are the same as the non-renewal case. Only the final equations are presented here.

Mean number in queue 1:

$$\mu_{q1} = \overline{X}_1\,(\mathbf{I} - \mathbf{R})^{-2}\,e$$

Mean number in queue 2:

$$\mu_{q2} = \bar{X}_0 \mathbf{V} \mathbf{e} + \bar{X}_1 (\mathbf{I} - \mathbf{R})^{-1} \mathbf{V} \mathbf{e}$$

Mean number in the system:

$$\mu_q = \mu_{q1} + \mu_{q2}$$
$$\mu_q = \bar{X}_0 \mathbf{V} \mathbf{e} + \bar{X}_1 (\mathbf{I} - \mathbf{R})^{-1} [(\mathbf{I} - \mathbf{R})^2 + \mathbf{V}] \mathbf{e}$$

The first step in the solution process for finding the waiting distribution is to setup the absorbing Markov chain. The transition matrix for the system is as follows:

$\hat{P} =$

$$\begin{array}{c c}
L_1 & \begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \ddots \end{array} & \left[\begin{array}{ccccc}
\hat{B}0 & & & & \\
\hat{C}0 & \hat{A}1 & & & \\
 & \hat{A}2 & \hat{A}1 & & \\
 & & \hat{A}2 & \hat{A}1 & \\
 & & & \ddots & \ddots
\end{array}\right]
\end{array}$$

Each of the matrices within $\hat{P}$ are presented in Appendix A.2.2.

Included in the waiting time is a portion of time in which our target customer is receiving service at the first queue. The states which represent the number in each queue ahead of our customer do not distinguish between when queue 1 is truly empty, and when queue 1 is serving our target customer. To account for this auxiliary states, represented with an asterisk, are defined to represent when our target customer is receiving service in the first queue. Otherwise the customer of interest has passed though queue 1 and is waiting for service in the second queue buffer. These auxiliary states are only present in the boundary matrices $\hat{B}0$ and $\hat{C}0$. These are the only matrices which contain the probabilities of queue 1 being empty.

To calculate the waiting time distribution the steady state vector $\bar{Y}$ for the new system must be found:

$$\bar{Y}^{(t+1)} = \bar{Y}^{(t)}\,\hat{P}$$

The auxiliary states are not represented in the steady state vector of the original transition matrix, therefore, it must be modified before it can represent the initial vector of the absorbing Markov chain. If the steady state vector $\bar{X}$ is represented as follows:

$$\bar{X} = (\bar{X}_0, \bar{X}_1, \bar{X}_2, \ldots)$$
$$\bar{X}_i = (X_{i,0}, X_{i,1}, X_{i,2}, \ldots, X_{i,K2}, X_{i,K2+1\,(j=0)}, X_{i,K2+1\,(j=1)})$$
$X_{i,m}$ = the probability of i customers in queue 1, and m customers in queue 2

Only the $\bar{X}_0$ portion is affected by the auxiliary states and the modified vector $\bar{X}^*$ is as follows:

$$\bar{X}^* = (\bar{X}_0^*, \bar{X}_1, \bar{X}_2, \ldots)$$
$$\bar{X}_0^* = (0, X_{0,0}, 0, X_{i,1}, 0, X_{i,2}, \ldots, 0, X_{i,K2}, X_{i,K2+1\,(j=0)}, X_{i,K2+1\,(j=1)})$$

Therefore

$\bar{Y}^{(0)}$ = modified steady state vector $\bar{X}^*$
$W_t = Y_{0,0}^{(t)}$ = probability the waiting time in the system is $\leq t$, departure point
$W_t' = W_t - W_{t-1}$ = probability the waiting time is the system = t

The mean waiting time is as follows:

$$\overline{W} = \sum_{i=0}^{\infty} (1 - W_i)$$

This method utilizes the original steady state vector $\overline{X}$ which is also used in the calculation of all of the other system parameters, including the mean queue length. The calculation of the waiting time distribution only requires the derivation of the absorbing Markov chain and a recursive algorithm. The value of the mean waiting time is only an approximate because of the infinite sum which is terminated at the point of convergence.

3.3.2 Approximate Results

Introduction

The decomposition approach presented here emulates the one presented in Section 3.2.2. The network is decomposed into two isolated queues, and MAP is used to approximate the correlated arrival process into the second queue. The MAP is based on the original transition matrix for the network and is found in the same manner as in the non-renewal case. The isolated queue 2 may now be described as a MAP/Geo/1/K queue. The service probability in the second queue remains the same as in the combined system.

The arrival process into the first queue remains the same, and the modified service rate takes into account customers being blocked from entering queue 2. The new service rate, which depends on the probability of the second queue being full, remains geometric so the isolated queue 1 may be defined as a Geo/Geo/1/∞ queue.

Solution Method of Isolated Queue 2: MAP/Geo/1/K2

The transition matrix for queue 2 utilizes a truncated MAP distribution which has **D0** and **D1** of dimension n by n. The finite departure and no departure matrices are as follows:

D0 =

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \ddots \\ n \end{array} & \left[\begin{array}{ccccc} \mathbf{B0} & \mathbf{B1} & & & \\ & \mathbf{A1''} & \mathbf{A0} & & \\ & & & \ddots & \\ & & & & \mathbf{A1''} + \mathbf{A0} \end{array} \right] \end{array}$$

D1 =

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & \cdots & n\text{-}1 & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \ddots \\ n \end{array} & \left[\begin{array}{ccccc} & & & & \\ \mathbf{A2} & \mathbf{A1'} & & & \\ & & & & \\ & & & \mathbf{A2} & \mathbf{A1'} \end{array} \right] \end{array}$$

The sub-matrices within **D0** and **D1** that appear without a subscript are the same as the sub-matrices within the original transition matrix for the geometric system. The others are presented in Appendix A.2.

The transition matrix for our system is the same as for a standard MAP/Geo/1/K queue. The matrix is as follows:

$\mathbf{P}_{q2} =$

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} 0 & 1 & 2 & 3 & \dots & K2-1 & K2 & K2+1 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \ddots \\ K2 \\ K2+1 \end{array} &
\left[\begin{array}{cccccccc}
D0 & D1 & & & & & & \\
D0\beta_2 & D0\alpha_2 + D1\beta_2 & D1\alpha_2 & & & & & \\
 & D0\beta_2 & D0\alpha_2 + D1\beta_2 & D1\alpha_2 & & & & \\
 & & & & \ddots & & & \\
 & & & & & D0\beta_2 & D0\alpha_2 + D1\beta_2 & D1\alpha_2 \\
 & & & & & & D0\beta_2 & D0\alpha_2 + D1
\end{array} \right]
\end{array}
$$

The steady state vector and the mean number of customers in the system can be found using the same procedure as for the non-renewal case. The formula for the mean number in the queue is presented here for clarity.

$$\mu_{q2} = \sum_{m=0}^{K2+1} m X_m^{(2)}$$

The mean waiting time in the queue, $\overline{W_2}$, not including the service time, can be found from the mean queue length using Little's Law. Little's Law for discrete time states that the mean number in the system is equivalent to the arrival rate multiplied by the mean waiting time in the queue. Therefore, the time in queue 2 is as follows:

$$\overline{W_2} = \frac{\mu_{q2}}{\lambda_2}$$

where

$\lambda_2 = \lambda(1\text{-}BP)$ = modified arrival rate of customers into the second queue and

$\lambda = \pi_D$ **D1** e, and π_D is the steady state vector of **D** = **D0** + **D1**
BP is the steady state probability of queue 2 being full

Solution Method of Isolated Queue 1: Geo/Geo/1/∞

The isolated queue 1 acts as a Geo/Geo/1/∞ queue, and the arrival rate into this queue is the same as the arrival rate into the combined system. The new service rate, Geo*, is less than or equal to the service rate of the first queue in the combined system. The service rate is reduced by the probability of the customer being blocked from entering queue 2 and can be expressed as follows:

$$\beta_1^* = \beta_1 (1 - BP)$$
$$\alpha_1^* = 1 - \beta_1^*$$

where

- BP is the steady state probability of queue 2 being full
- β_1 is the original probability of a service completion in queue 1, $\alpha_1 = 1 - \beta_1$

A phase distribution may have been used to represent the effect of blocking on the service process as well. Although the phase distribution may produce better results, it would also increase the complexity of the solution process. A comparison of the results using a phase service distribution are presented in Section 4.3.2.

Once the new service rate has been established the parameters of the isolated geometric queue are found. The infinite transition matrix for queue 1 is presented below.

$\mathbf{P}_{q1} =$

$$\begin{array}{c|cccccc|}
 & 0 & 1 & 2 & 3 & 4 & \cdots \\
\hline
0 & q & p & & & & \\
1 & q\beta_1^* & q\alpha_1^* + p\beta_1^* & p\alpha_1^* & & & \\
2 & & q\beta_1^* & q\alpha_1^* + p\beta_1^* & p\alpha_1^* & & \\
3 & & & q\beta_1^* & q\alpha_1^* + p\beta_1^* & p\alpha_1^* & \\
\ddots & & & & \ddots & \ddots & \ddots
\end{array}$$

The results of Geo/Geo/1 queue have been well documented and a closed form exists for the mean number in the system. The formula for the mean queue length is as follows:

$$\mu_{q1} = \frac{pq}{\beta_1^* - p}$$

The mean waiting time can also be found using Little's Law and is as follows:

$$\overline{W_1} = \frac{\mu_{q1}}{p}$$

As in the non-renewal case when the probability of blocking, BP, increases the service rate decreases and the number waiting for service in queue 1 increases. When BP = 0, $\beta_1^* = \beta_1$ and the traffic intensity of queue 1, $\rho_1 = p/\beta_1^*$, is a minimum.

3.3.3 Comparison of Combined and Isolated Results

Comparison of Methods Presented in This Section

As before, computer trials were performed on the isolated and combined queues, and the results were compared. Each trial used the same set of inputs: p; β_1; β_2; and K2, and generated the same outputs. The following attributes were observed for each of the trials:

- Q = stability factor
- ρ_2 = traffic intensity into queue 2
- ρ_1 = traffic intensity into queue 1
- μ_q = mean number in system
- μ_{q1} = mean number in queue 1
- μ_{q2} = mean number in queue 2
- % error of $\mu_q = | \mu_q(\text{combo}) - \mu_q(\text{iso}) | / \mu_q(\text{combo})$
- % error of $\mu_{q2} = | \mu_{q2}(\text{combo}) - \mu_{q2}(\text{iso}) | / \mu_{q2}(\text{combo})$
- % error of $\mu_{q1} = | \mu_{q1}(\text{combo}) - \mu_{q1}(\text{iso}) | / \mu_{q1}(\text{combo})$

Where

- mean number in each queue includes customer in service
- (combo) represents combined system solution
- (iso) represents the isolated queue results

The numerical results for the trials are presented in Appendix A.2.4. Reference may be made to section 3.2.3 as the observations for this system are the same as for the case with external MAP arrivals, and they are not presented here.

Observations Related to the Application of the Decomposition Procedure

It would appear that the decomposition method produces values of the mean number in the system within 30% of the actual results when the following is true:

1. Both ρ_1 and ρ_2 are greater than or equal to 0.4 and less than 0.8
2. $\rho_1 > \rho_2$ and $\rho_2 \leq 0.8$
3. The number of blocks in **D0** and **D1** is equal to 2

As in the non-renewal case there appears to be no direct relationship between the value of the percentage error and the values of the traffic intensities, and exceptions to the rule may occur. An additional observation made about the results is that when both ρ_1 and ρ_2 are below the lower bound, the percentage error of queue 2 is high and the percentage error of queue 1 is low. When both traffic intensities are high, both queues have large percentage errors.

A comparison of the results shows that the actual mean waiting times are much less than the approximated values. This was true even when the approximated results for the mean number in the system were comparable. Therefore, the decomposition method presented in this section is not appropriate for approximating the mean waiting time for any values of ρ_1 and ρ_2.

Utilization of an Iterative Procedure

As noted earlier the decomposition method does not always produce acceptable results. In an attempt to increase the values of ρ for which the method was applicable, an iterative procedure was tried. After solving for the isolated queue 1, the new service probability was substituted into the **D0** and **D1** matrices of the second queue and the procedure was repeated. Unfortunately the approximate value of the mean queue length diverged further from the actual with each iteration. This occurred for each trial attempted. Therefore the use of an iterative procedure was abandoned.

Comparison With Yannopoulos & Alfa's Method

The method developed by Yannopoulos and Alfa [55] can also be applied to the geometric system. The following tables present a comparison of all of the methods discussed.

Table 3.3.1: Results of Mean Queue Length for Increasing ρ_1, $\rho_2 = 0.1$

ρ_1	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
0.0556	0.0700	0.1605	129.29%	0.1159	65.57%	0.1153	64.71%
0.0625	0.0886	0.1668	88.26%	0.1237	39.62%	0.1229	38.71%
0.0714	0.1095	0.1754	60.18%	0.1339	22.28%	0.1329	21.37%
0.0833	0.1339	0.1871	39.73%	0.1477	10.31%	0.1465	9.41%
0.1000	0.1640	0.2042	24.51%	0.1678	2.32%	0.1661	1.28%
0.1250	0.2050	0.2315	12.93%	0.1992	2.83%	0.1969	3.95%
0.1667	0.2699	0.2817	4.37%	0.2556	5.30%	0.2522	6.56%
0.2500	0.4070	0.4016	1.33%	0.3864	5.06%	0.3810	6.39%
0.5000	1.0510	1.0197	2.98%	1.0326	1.75%	1.0210	2.85%

Table 3.3.2: Results of Mean Queue Length for Increasing ρ_2, $\rho_1 = 0.1$

ρ_2	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
0.0556	0.1369	0.1587	15.92%	0.1181	13.73%	0.1154	15.70%
0.0625	0.1410	0.1656	17.45%	0.1258	10.78%	0.1230	12.77%
0.0714	0.1463	0.1745	19.28%	0.1358	7.18%	0.1330	9.09%
0.0833	0.1535	0.1867	21.63%	0.1491	2.87%	0.1467	4.43%
0.1000	0.1640	0.2042	24.51%	0.1678	2.32%	0.1661	1.28%
0.1250	0.1804	0.2315	28.33%	0.1963	8.81%	0.1956	8.43%
0.1667	0.2099	0.2802	33.44%	0.2452	16.82%	0.2456	17.01%
0.2500	0.2804	0.3890	38.59%	0.3470	23.75%	0.3483	24.22%
0.5000	0.6914	0.7785	11.86%	0.6784	1.88%	0.6796	1.71%

Table 3.3.3: Results of Mean Queue Length for Increasing K2, $\rho_1 = \rho_2 = 0.2$

K2	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
1	0.3631	0.4205	15.81%	0.3735	2.86%	0.3657	0.72%
2	0.3597	0.4205	16.90%	0.3835	6.62%	0.3692	2.64%
3	0.3592	0.4205	17.07%	0.3851	7.21%	0.3699	2.98%
4	0.3592	0.4205	17.07%	0.3854	7.29%	0.3700	3.01%

Table 3.3.4: Results of Mean Queue Length for Increasing K2, $\rho_1 = \rho_2 = 0.4$

K2	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
1	0.9590	0.9811	2.30%	0.9181	2.23%	0.8896	5.26%
2	0.9485	0.9516	0.33%	0.9746	2.75%	0.9250	2.48%
3	0.9382	0.9509	1.35%	0.9967	6.24%	0.9397	0.16%
4	0.9347	0.9509	1.73%	1.0040	7.41%	0.9446	1.06%
5	0.9337	0.9509	1.84%	1.0062	7.76%	0.9461	1.33%

Unlike the non-renewal case, the approximated results derived in this thesis are similar or inferior to those generated by both methods from [55]. In cases that the decomposition method had lower percentage errors the results are very close to the actual results. This is also true of Yao & Buzacott's and Gelenbe's methods. It can therefore be said that it is not worth the extra effort of utilizing MAP to maintain the relationship between the isolated queues of a tandem network with geometric arrivals.

With independent external arrivals and service completions the dependence between the isolated queues is minimized. When the external arrival process is MAP the departure process is also non-renewal and it is necessary to capture the dependence of the isolated second queue with the first queue.

4. SPLIT CONFIGURATIONS

4.1 Introduction

This section applies the decomposition method developed in Section 3 to a split network. In a split configuration one queue feeds several other queues in parallel. A customer is served at the first queue and then proceeds to one of the second stage queues. The probabilities of proceeding to a particular second stage queue do not have to be equal. Departures from the system occur when service is completed at the second stage queue. The following assumptions hold for the split networks discussed in this section:

- There are only two second stage queues
- Each queue has only one server
- The service times in each of the queues are independent
- Queue 1 has an infinite buffer
- Queues 2 and 3 have finite buffers of size K2 and K3 respectively
- Service follows a first-in, first-out (FIFO) rule

Figure 4.1.1 provides an illustration of the network discussed in this section.

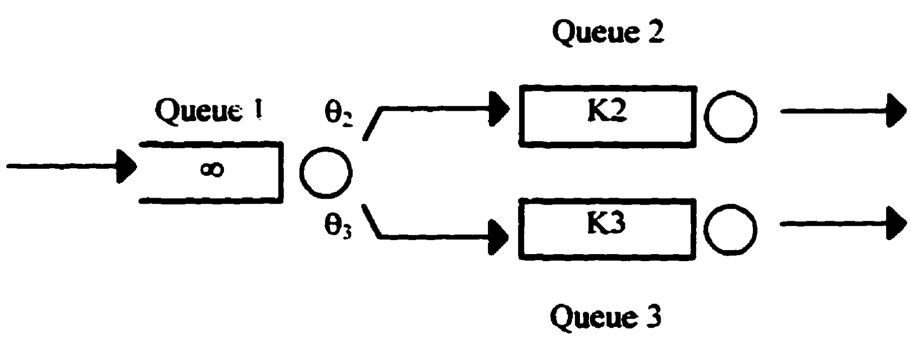

Figure 4.1.1: Split Network, a Single Queue Feeding Two Queues

Two individual cases, *a* and *b*, will be considered. Case *a* will define a system in which the second queue visited is not predetermined. In other words the customer may choose to visit either second stage queue. If a customer becomes blocked they may select the next queue to visit after each time period based on a Bernoulli process. Case *b* will define the system when the second queue is determined after completion of service at the first station and there can be no change. If blocking occurs in this case the customer must wait until the queue they are assigned opens up.

For each case an approximate solution for the split network with correlated arrivals is developed. Existing solution methods do not allow for non-renewal arrival processes, and they do not assume correlation between the isolated queues. Results generated using the decomposition method are compared with those generated using the exact method which is also developed in this section. It may be noticed that many of the formulae and matrices presented in this section are the same as those for the tandem case, when this is true they are repeated for clarity. In addition, when results and formulae apply to both cases *a* and *b* only the differences will be outlined.

This section begins with a presentation of an exact solution method for a split network with a non-renewal arrival process, and continues with the description of the decomposition method. A comparison between the two are then presented, as well as a comparison with the method derived by Yannopoulos and Alfa [55]. This section then continues on with a modification of the methods for a geometric system as a special case of the non-renewal.

4.2 Networks With a Markovian Arrival Process

4.2.1 Exact Results

Introduction

The first split network configuration to be studied is the non-renewal system. As in the tandem case, customers arrive according to the MAP and service follows a phase distribution for each of the queues. As stated in the assumptions of Section 4.1 the first queue is defined as having an infinite buffer, and the second stage queues have finite buffers. The following parameters are used to define both cases of the split system:

- **D0** = No departure matrix of MAP arrival process
- **D1** = Departure matrix of MAP arrival process
- (β_i, S_i) = service distribution of queue i, i = 1, 2, or 3
- K2 = buffer size of queue 2
- K3 = buffer size of queue 3
- θ_2 = probability the customer leaving queue 1 will choose to visit queue 2
- $\theta_3 = 1 - \theta_2 =$ probability the customer leaving queue 1 will choose queue 3
- (j = 0) = state when there are K2 + 1 customers in queue 2 or K3 + 1 customers in queue 3, but there is no blocking
- (j = 2) = state when there are K2 + 1 customers in queue 2 and it blocks an arrival from queue 1
- (j = 3) = state when there are K3 + 1 customers in queue 3 and it blocks an arrival from queue 1

Transition Matrix

The probability matrix which defines this system has three levels. The primary level, L_1 represents the number of customers in queue 1, and sub-levels L_2 and L_3 represent the number of customers in queues 2 and 3 respectively. The following is the transition matrix **P** which defines this system:

P =

$$
\begin{array}{c|c}
L_1 & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \ddots \end{array} &
\left[\begin{array}{cccccc}
\mathbf{B0} & \mathbf{B1} & & & & \\
\mathbf{C0} & \mathbf{A1} & \mathbf{A0} & & & \\
 & \mathbf{A2} & \mathbf{A1} & \mathbf{A0} & & \\
 & & \mathbf{A2} & \mathbf{A1} & \mathbf{A0} & \\
 & & & \ddots & \ddots & \ddots
\end{array}\right]
\end{array}
$$

P has the same tri-diagonal structure for both cases *a* and *b*, however the sub-matrices differ. The sub-matrices for both cases are defined in Appendix A.3.

Solution Method

The exact solution method for the split configuration follows the same steps as the one for a tandem system. The MGM is used to find the steady state vector, which is then used to calculate μ_q, the mean number in the system. Before the system may be solved, the stability factor $Q = \pi_A$ (**A2** - **A0**) e, must be found to be positive to confirm that the system of interest is stable. The value of Q may be verified by a calculation, but for any system $\lambda < \min(\mu_1, \mu_2, \mu_3)$ where:

$\lambda = \pi_D$ **D1** e, where π_D is the steady state vector of **D** = **D0** + **D1**
$\mu_i = [\,\beta_i\,(\mathbf{I} - \mathbf{S_i})^{-1}\, \mathrm{e}\,]^{-1}$, $i = 1, 2,$ or 3

The mean number in the total queue, μ_q, is found as the sum of the mean number in each of the queues, μ_{q1}, μ_{q2} and μ_{q3}. The number in the queue is defined here to include the customer in service, if applicable. The equations are the same for both cases and the derivation of the mean number in queue 1 is as follows:

$$\mu_{q1} = \textstyle\sum_i \sum_m \sum_k i\, X_{i,m,k}$$
$$\mu_{q1} = \textstyle\sum_i i\, \overline{X}_i\, \mathrm{e}$$

given

$$\overline{X}_i = (\overline{X}_{i,0}, \overline{X}_{i,1}, \overline{X}_{i,2}, \ldots, \overline{X}_{i,K2}, \overline{X}_{i,K2+1\,(j=0)}, \overline{X}_{i,K2+1\,(j=2)})$$
$$\overline{X}_{i,m} = (X_{i,m,0}, X_{i,m,1}, X_{i,m,2}, \ldots, X_{i,m,K3}, X_{i,m,K3+1(j=0)}, X_{i,m,K3+1(j=1)}, X_{i,m,K3+1(j=2)}, X_{i,m,K3+1(j=3)})$$

$X_{i,m,k}$ = the probability of i customers in queue 1, m customers in queue 2, and k customers in queue 3

$$\textstyle\sum_m \sum_k X_{i,m,k} = \overline{X}_i\, \mathrm{e}$$

then

$$\mu_{q1} = \textstyle\sum_i i\, (\overline{X}_1\, \mathbf{R}^{i-1})\, \mathrm{e}$$
$$\mu_{q1} = \overline{X}_1 \textstyle\sum_i i\, \mathbf{R}^{i-1}\, \mathrm{e}$$
$$\mu_{q1} = \overline{X}_1\, (\mathbf{I} - \mathbf{R})^{-2}\, \mathrm{e}$$

where

I = Identity matrix
R = rate matrix used in MGM

The mean number in queue 2 is calculated as follows:

$$\mu_{q2} = \textstyle\sum_i \sum_m \sum_k m\, X_{i,m,k}$$
$$\begin{aligned}\mu_{q2} = \textstyle\sum_i [\,& 0\, X_{i,0,0} + 0\, X_{i,0,1} + 0\, X_{i,0,2} + \ldots + 0\, X_{i,0,K3} + \ldots + 0\, X_{i,0,K3+1\,(j=3)} + \\ & 1\, X_{i,1,0} + 1\, X_{i,1,1} + 1\, X_{i,1,2} + \ldots + 1\, X_{i,1,K3} + \ldots + 1\, X_{i,1,K3+1\,(j=3)} + \\ & (K2+1)\, X_{i,K2+1\,(j=2),0} + (K2+1)\, X_{i,K2+1\,(j=2),1} + (K2+1)\, X_{i,K2+1\,(j=2),2} + \ldots + \\ & (K2+1)\, X_{i,K2+1\,(j=2),K3} + \ldots + (K2+1)\, X_{i,K2+1\,(j=2),K3+1\,(j=3)}\,]\end{aligned}$$
$$\mu_{q2} = \textstyle\sum_i \overline{X}_i\, \mathbf{W}\, \mathrm{e}$$

where

$$
W = \begin{bmatrix} I(0) & & & & & \\ & I(1) & & & & \\ & & \ddots & & & \\ & & & I(K2) & & \\ & & & & I(K2+1) & \\ & & & & & I(K2+1) \end{bmatrix}
$$

and

$$\mathbf{I}(n) = n * \mathbf{I}$$

then

$$\mu_{q2} = \overline{X}_0 \, \mathbf{W} \, \mathbf{e} + \Sigma_i \, (\overline{X}_1 \, \mathbf{R}^{i-1}) \, \mathbf{W} \, \mathbf{e}$$
$$\mu_{q2} = \overline{X}_0 \, \mathbf{W} \, \mathbf{e} + \overline{X}_1 \, \Sigma_i \, \mathbf{R}^{i-1} \, \mathbf{W} \, \mathbf{e}$$
$$\mu_{q2} = \overline{X}_0 \, \mathbf{W} \, \mathbf{e} + \overline{X}_1 \, (\mathbf{I} - \mathbf{R})^{-1} \, \mathbf{W} \, \mathbf{e}$$

The derivation of the mean number in queue 3:

$$\mu_{q3} = \Sigma_i \, \Sigma_m \, \Sigma_k \, k \, X_{i,j,k}$$
$$\begin{aligned} \mu_{q3} = \Sigma_i \, [& 0\, X_{i,0,0} + 1\, X_{i,0,1} + 1\, X_{i,0,2} + \ldots + K3\, X_{i,0,K3} + \ldots + (K3+1)\, X_{i,0,K3+1\,(j=3)} + \\ & 0\, X_{i,1,0} + 1\, X_{i,1,1} + 2\, X_{i,1,2} + \ldots + K3\, X_{i,1,K3} + \ldots + (K3+1)\, X_{i,1,K3+1\,(j=3)} + \\ & 0\, X_{i,(K2+1),0} + 1\, X_{i,(K2+1),1} + 2\, X_{i,(K2+1),2} + \ldots + K3\, X_{i,(K2+1),K3} + \ldots + \\ & (K3+1)\, X_{i,(k2+1),K3+1\,(j=3)}] \end{aligned}$$
$$\mu_{q3} = \Sigma_i \, \overline{X}_i \, \mathbf{Y} \, \mathbf{e}$$

where

$$\mathbf{Y} = \mathbf{I} \otimes \mathbf{I}'$$

and

$$
I' = \begin{bmatrix} 0 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & K3 & & & & \\ & & & & K3+1 & & & \\ & & & & & K3+1 & & \\ & & & & & & K3+1 & \\ & & & & & & & K3+1 \end{bmatrix}
$$

then

$$
\begin{aligned}
\mu_{q3} &= \overline{X}_0 \mathbf{Y}\, \mathrm{e} + \Sigma_i (\overline{X}_1 \mathbf{R}^{i-1}) \mathbf{Y}\, \mathrm{e} \\
\mu_{q3} &= \overline{X}_0 \mathbf{Y}\, \mathrm{e} + \overline{X}_1 \Sigma_i \mathbf{R}^{i-1} \mathbf{Y}\, \mathrm{e} \\
\mu_{q3} &= \overline{X}_0 \mathbf{Y}\, \mathrm{e} + \overline{X}_1 (\mathbf{I} - \mathbf{R})^{-1} \mathbf{Y}\, \mathrm{e}
\end{aligned}
$$

The mean number in the system is found as a total of the mean number in each queue and is summarized as follows:

$$
\begin{aligned}
\mu_q &= \mu_{q1} + \mu_{q2} + \mu_{q3} \\
\mu_q &= \overline{X}_1 (\mathbf{I} - \mathbf{R})^{-2} \mathrm{e} + \overline{X}_0 \mathbf{W}\, \mathrm{e} + \overline{X}_1 (\mathbf{I} - \mathbf{R})^{-1} \mathbf{W}\, \mathrm{e} + \overline{X}_0 \mathbf{Y}\, \mathrm{e} + \overline{X}_1 (\mathbf{I} - \mathbf{R})^{-1} \mathbf{Y}\, \mathrm{e} \\
\mu_q &= \overline{X}_0 (\mathbf{W} + \mathbf{Y})\, \mathrm{e} + \overline{X}_1 (\mathbf{I} - \mathbf{R})^{-1} [(\mathbf{I} - \mathbf{R})^2 + \mathbf{W} + \mathbf{Y}]\, \mathrm{e}
\end{aligned}
$$

4.2.2 Approximation Method

Introduction

A network of three queues in a split configuration, with a non-renewal arrival process and finite second stage queues, can be decomposed into three isolated queues. The arrival process into the second stage queues is approximated with a non-renewal arrival process in order to maintain correlation with the departure process from queue 1. Two new MAP^{*i}, where $i = 2$ or 3, are defined with $\mathbf{D0}^{*i}$ and $\mathbf{D1}^{*i}$ based on the transition matrix of the network. For purposes of simplified notation only the '*' will be appended to the new **D0** and **D1** matrix notation when reference is being made to either queues 2 or 3. Queues 2 and 3 may now be described as MAP/PH/1/K queues. The service distributions in the second stage queues are the same as in the combined system.

The arrival process into the first queue remains the same as the arrival process into the network, however the service rate is different. The service rate must be modified to account for customers being blocked from entering either queue 2 or 3. The new service process PH^* depends on the probability of either queue 2 or 3 being full, K2 or K3 in the buffer plus one in service. The isolated queue 1 may be described as a MAP/PH/1/∞ queue.

Although the new MAP^* and PH^* distributions which govern the isolated queues of the split system are different than those for the tandem, the matrices and formulae remain the same. That is, only the end results are different once the appropriate substitutions are made. The generic transition matrices and formulae are presented here in summary form for the sake of completion of the section.

Solution Method of the Isolated Second Stage Queues: MAP/PH/1/K

Queues 2 and 3 are studied in isolation first because of the dependency of the new service rate of queue 1 on the steady state blocking probabilities of queues 2 and 3.

For both cases the arrival processes into queues 2 and 3 are estimated with new MAP distributions based on the departure rate from queue 1. Therefore, the departure and no departure matrices contain an infinite number of blocks. The solution technique uses approximated $\mathbf{D0}^{*}$ and $\mathbf{D1}^{*}$ matrices with a finite number of blocks. The finite $\mathbf{D0}^{*i}$ and $\mathbf{D1}^{*i}$ matrices, $i = 2$ or 3, are provided below:

$\mathbf{D0}^{*2} =$

$$\begin{array}{cc}
 & \begin{array}{cccccc} 1 & 2 & 3 & \cdots & n-1 & n \end{array} \\
\begin{array}{c} 1 \\ 2 \\ \ddots \\ n \end{array} &
\begin{bmatrix}
\mathbf{B0} & \mathbf{B1} & & & & \\
\mathbf{C0}^{*3} & (\mathbf{A1}^{*3})' + \mathbf{A1}'' & \mathbf{A0} & & & \\
 & & & \ddots & & \\
 & & & & \mathbf{A2}^{*3} & (\mathbf{A1}^{*3})' + \mathbf{A1}'' + \mathbf{A0}
\end{bmatrix}
\end{array}$$

$\mathbf{D1}^{*2} =$

$$\begin{array}{cc}
 & \begin{array}{ccccc} 1 & 2 & \cdots & n-1 & n \end{array} \\
\begin{array}{c} 1 \\ 2 \\ \ddots \\ n \end{array} &
\begin{bmatrix}
 & & & & \\
\mathbf{C0}^{*2} & (\mathbf{A1}^{*2})' & & & \\
 & & \ddots & & \\
 & & & \mathbf{A2}^{*2} & (\mathbf{A1}^{*2})'
\end{bmatrix}
\end{array}$$

$\mathbf{D0}^{*3} =$

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & \cdots & n-1 & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \ddots \\ n \end{array} & \begin{bmatrix} \mathbf{B0} & \mathbf{B1} & & & & \\ \mathbf{C0}^{*2} & (\mathbf{A1}^{*2})' + \mathbf{A1}'' & \mathbf{A0} & & & \\ & & & \ddots & & \\ & & & & \mathbf{A2}^{*2} & (\mathbf{A1}^{*2})' + \mathbf{A1}'' + \mathbf{A0} \end{bmatrix} \end{array}$$

$\mathbf{D1}^{*3} =$

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & \cdots & n-1 & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \ddots \\ n \end{array} & \begin{bmatrix} & & & & \\ \mathbf{C0}^{*3} & (\mathbf{A1}^{*3})' & & & \\ & & \ddots & & \\ & & & \mathbf{A2}^{*3} & (\mathbf{A1}^{*3})' \end{bmatrix} \end{array}$$

The definition of the sub-matrices within $\mathbf{D0}^{*}$ and $\mathbf{D1}^{*}$ are dependent on whether we are studying case *a* or *b*. The sub-matrices that appear without a superscript are the same as the sub-matrices within the original transition matrix **P**. The new matrices for both cases are presented in Appendix A.3.

Once the arrival process to each of the second stage queues has been calculated the queues are analyzed as individual MAP/PH/1/K queues. The transition matrices for each of the second stage queues are the same except for the MAP* process. The following matrix is generic for either queue 2 or 3:

$\mathbf{P}_{qi}$ ($i = 2$ or 3) =

$$
\begin{array}{c}
\\ 0 \\ 1 \\ \\ 2 \\ \\ \ddots \\ \mathrm{Ki} \\ \\ \mathrm{Ki}+1 \\ \\
\end{array}
\begin{array}{c}
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & \cdots & \mathrm{Ki}-1 & \mathrm{Ki} & \mathrm{Ki}+1
\end{array} \\
\left[\begin{array}{cccccccc}
D0^{*i} & D1^{*i}\otimes\beta_i & & & & & & \\
D0^{*i}\otimes S_i^0 & D0^{*i}\otimes S_i + & D1^{*i}\otimes S_i & & & & & \\
 & D1^{*i}\otimes S_i^0\beta_i & & & & & & \\
 & D0^{*i}\otimes S_i^0\beta_i & D0^{*i}\otimes S_i + & D1^{*i}\otimes S_i & & & & \\
 & & D1^{*i}\otimes S_i^0\beta_i & & & & & \\
 & & & & \ddots & & & \\
 & & & & & D0^{*i}\otimes S_i^0\beta_i & D0^{*i}\otimes S_i + & D1^{*i}\otimes S_i \\
 & & & & & & D1^{*i}\otimes S_i^0\beta_i & \\
 & & & & & & D0^{*i}\otimes S_i^0\beta_i & D0^{*i}\otimes S_i + \\
 & & & & & & & D1^{*i}
\end{array}\right]
\end{array}
$$

To calculate the mean queue length for these queues the steady state vector of the number in the system must be found. This may be done using several iterative methods including the Power method and Gauss-Seidel. The formula for the mean queue length is then:

$$\mu_{qi} = \sum_{m=0}^{Ki+1} m X_m^{(i)}$$

where

$X_m^{(i)}$ = the steady state probability of m customers in queue i, $i = 2$ or 3

The above equation holds for both case a and b.

Solution Method of Isolated Queue 1: MAP/PH/1/∞

As in the tandem case the isolated queue 1 acts as a MAP/PH/1/∞ queue. The arrival rate into this queue is the same as the arrival rate into the combined system, however the service rate is different. The service rate is altered by the probability of the customer being blocked from entering a second stage queue and can be expressed as $(\beta_1^{\bullet}, S_1^{\bullet})$, where:

$$\beta_1^{\bullet} = \begin{bmatrix} \beta_1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{S}_1^{\bullet} = \begin{bmatrix} S_1 & S_1^0 \beta_2^{\bullet} \theta_2 BP_2 & S_1^0 \beta_3^{\bullet} \theta_3 BP_3 \\ & S_2 & \\ & & S_3 \end{bmatrix}$$

and

BP_i = steady state probabilities of queue i, $i = 2$ or 3, being full
$\beta_i^{\bullet} = \beta_i^{\bullet}(\mathbf{S}_i + \mathrm{S}^0_i\, \beta_i)$, $i = 2$ or 3

Once the new service distribution $\mathrm{PH}^{\bullet}$ is calculated the isolated queue 1 is analyzed as a MAP/PH/1 queue. The infinite transition matrix for this type of queue is as follows:

$\mathbf{P}_{q1} =$

$$\begin{array}{c c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \\ 2 \\ \\ 3 \\ \\ \ddots \end{array} & \left[\begin{array}{cccccc} D0 & D1\otimes\beta_1^{\bullet} & & & & \\ D0\otimes S^0_1{}^{\bullet} & D0\otimes S_1^{\bullet} + & D1\otimes S_1^{\bullet} & & & \\ & D1\otimes S^0_1{}^{\bullet}\beta_1^{\bullet} & & & & \\ & D0\otimes S^0_1{}^{\bullet}\beta_1^{\bullet} & D0\otimes S_1^{\bullet} + & D1\otimes S_1^{\bullet} & & \\ & & D1\otimes S^0_1{}^{\bullet}\beta_1^{\bullet} & & & \\ & & D0\otimes S^0_1{}^{\bullet}\beta_1^{\bullet} & D0\otimes S_1^{\bullet} + & D1\otimes S_1^{\bullet} & \\ & & & D1\otimes S^0_1{}^{\bullet}\beta_1^{\bullet} & & \\ & & & \ddots & \ddots & \ddots \end{array}\right] \end{array}$$

To find the steady state vector of the transition matrix the MGM method may be employed. Once the steady state vector is found, the mean queue length is as follows:

$$\mu_{q1} = X_1^{(1)}\,(\mathbf{I} - \mathbf{R})^{-2}$$

As expected, when the blocking probabilities increase the service rate decreases and the number waiting for service in queue 1 increases. When the blocking probabilities of both queues equal zero, $BP_2 = BP_3 = 0$, then $(\beta_1^*, S_1^*) = (\beta_1, S_1)$, and the traffic intensity of queue 1, $\rho_1 = \lambda/\mu_1^*$, is a minimum.

4.2.3 Comparison of Combined and Isolated Results

Comparison of Methods Presented in This Section

As in tandem case computer trials were performed on the isolated and combined systems, and the results were compared. Each trial used the same set of inputs: **D0** and **D1**; (β_1, S_1); (β_2, S_2); (β_3, S_3); K2; K3; and θ_2, and generated the same set of outputs for each case. The following outputs were compared:

- Q = stability factor
- (β_1^*, S_1^*), modified service process of queue 1
- λ_2 = arrival rate into queue 2
- λ_3 = arrival rate into queue 3
- ρ_1 = traffic intensity into queue 1
- ρ_2 = traffic intensity into queue 2
- ρ_3 = traffic intensity into queue 3
- μ_q = mean number in system
- μ_{q1} = mean number in queue 1
- μ_{q2} = mean number in queue 2
- μ_{q3} = mean number in queue 3
- % error of $\mu_q = | \mu_q(\text{combo}) - \mu_q(\text{iso}) | / \mu_q(\text{combo})$
- % error of $\mu_{q1} = | \mu_{q1}(\text{combo}) - \mu_{q1}(\text{iso}) | / \mu_{q1}(\text{combo})$
- % error of $\mu_{q2} = | \mu_{q2}(\text{combo}) - \mu_{q2}(\text{iso}) | / \mu_{q2}(\text{combo})$
- % error of $\mu_{q3} = | \mu_{q3}(\text{combo}) - \mu_{q3}(\text{iso}) | / \mu_{q3}(\text{combo})$

Where

- mean number in the queue includes the customer in service
- (combo) represents combined system solution
- (iso) represents the isolated queue results

The numerical results of all of the case *a* and case *b* trials are presented in Appendix A.3. From these trials it was confirmed that the overall observations which presented in Section 3.2.3 also hold from split networks. These observations are not presented in this section, however there are differences between the results for case *a* and *b* which are discussed here.

Observations on the Differences Between Case *a* and Case *b*

Due to the differences in the way the customer choose between the second stage queues for each case, the case *b* values of μ_{q1}, μ_{q2}, μ_{q3}, and Q were slightly deviated from the results of case *a*. The following summarizes the differences:

1. The value of Q was higher for case *a* than case *b*. For several trials the inputs resulted in a stable case *a* system, however case *b* was unstable.
2. The value of μ_{q1} was larger for case *b*.
3. As K2 and K3 approach the maximum, the results from case *b* approached the results from case *a*. The method of choosing the next queue becomes unimportant as the probability of blocking decreases.

These results were expected since in case *b* blocking may cause a greater delay of a customer in queue 1 if the other second stage queue is not full. This may lead to a build up of the queue 1 buffer. In case *a* the customer may choose the other queue after one time period, so blocking will have less effect on delaying the customer in the system.

Observations Related to the Application of the Decomposition Procedure for Case *a*

It would appear that the decomposition method will produce values of the mean number in the system within 30% of the actual results for any set of inputs that result in a stable system. In addition, as the values of $\rho_1 = \rho_2 = \rho_3$ decrease, the percentage error of the number in each queue and in the system decrease.

Observations Related to the Application of the Decomposition Procedure for Case *b*

The values of ρ_i, $i = 1, 2$, or 3, for which case *b* decomposition results were within 30% of the actual are much more limited than in case *a*. Results within 30% were produced when both ρ_2 and ρ_3 are less than or equivalent to 0.4, for any values of ρ_1. This is true when the values of ρ are generated with any set of inputs.

The results of case *b* are limited because if either queue 2 or queue 3 builds up queue 1 becomes unstable. Unlike in case *a*, once a second stage destination queue has been chosen no switching is allowed. If the chosen queue is full, queue 1 remains blocked until that queue completes a service. Some additional observations made about the case *b* system results are as follows:

1. As the values of $\rho_1 = \rho_2 = \rho_3$ decrease the percentage error of the number in each queue, and in the total queue decrease.

2. For constant values ρ_1, as the values of $\rho_2 = \rho_3$ decrease the percentage error of the number in queue 1, and the number in the system decrease.

Case *a*: Comparison With Yannopoulos & Alfa's Method

As for the tandem system the validity of the results are supported by comparing them with the results generated using the method by Yannopoulos and Alfa [55]. Not all trials were compared as the split configuration technique in [55] requires several restrictions. The assumptions are listed below:

1. $\theta_2 = \theta_3$

2. $K2 = K3$

3. $(\beta_2, S_2) = (\beta_3, S_3)$

In other words queue 2 must be equivalent to queue 3. The following tables present the results for various sets of inputs which meet all of the specified requirements.

Table 4.2.1: Case *a*: Results of Mean Queue Length for Increasing ρ_1, $\rho_2 = \rho_3 = 0.5$

ρ_1	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
0.1000	1.6357	1.4949	8.61%	1.2623	22.83%	1.2768	21.94%
0.2500	1.8126	1.6112	11.11%	1.3310	26.57%	1.3605	24.94%
0.5000	2.3046	2.0314	11.85%	1.5694	31.90%	1.6063	30.30%
0.7500	3.7879	3.5622	5.96%	2.2081	41.71%	2.2535	40.51%
0.9000	10.3412	11.5300	11.50%	3.9711	61.60%	4.1865	59.52%

Table 4.2.2: Case *a*: Results of Mean Queue Length for Increasing $\rho_2 = \rho_3$, $\rho_1 = 0.25$

$\rho_2 = \rho_3$	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
0.2500	0.8909	0.8092	9.17%	0.6315	29.12%	0.6362	28.59%
0.5000	1.8126	1.6112	11.11%	1.3310	26.57%	1.3605	24.94%
0.7440	3.8590	3.0063	22.10%	2.2123	42.67%	1.9664	49.04%

Table 4.2.3: Case *a*: Results of Mean Queue Length for Increasing $\rho_2 = \rho_3$, $\rho_1 = 0.5$

$\rho_2 = \rho_3$	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
0.2500	1.3268	1.2006	9.51%	0.8051	39.32%	0.7604	42.69%
0.5000	2.3046	2.0314	11.85%	1.5694	31.90%	1.6063	30.30%
0.7440	4.7104	4.0949	13.07%	2.7133	42.40%	2.9116	38.19%

Table 4.2.4: Case *a*: Results of Mean Queue Length for Increasing K2 = K3, $\rho_1 = \rho_2 = \rho_3 = 0.25$

K2 = K3	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
1	0.8909	0.8092	9.17%	0.6315	29.12%	0.6362	28.59%
2	0.9086	0.8117	10.66%	0.6260	31.10%	0.6362	29.98%

To summarize, as in the tandem case the decomposition method produced superior results for all of the trials performed. This was true when values were both comparable and not comparable with the actual results. Therefore, this further supports the notion that a technique which utilizes MAP to maintain the relationship between isolated queues, from a network with a non-renewal external arrival process, produce more accurate results than those which utilize renewal processes.

Case *b*: Comparison With Yannopoulos & Alfa's Method

The values of the results derived from [55] remain the same for both case *a* and case *b*. Although the exact and decomposition results are different, the method in this thesis still produced superior results for all trials performed. The following tables present the results for various sets of inputs which meet all of the specified requirements.

Table 4.2.5: Case *b*: Results of Mean Queue Length for Increasing ρ_1, $\rho_2 = \rho_3 = 0.5$

ρ_1	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
0.1000	2.5107	1.4757	41.22%	1.2623	49.72%	1.2768	49.15%
0.2500	2.6219	1.5325	41.55%	1.3310	49.24%	1.3605	48.11%
0.5000	3.0964	1.9963	35.53%	1.5694	49.32%	1.6063	48.12%
0.7500	5.1693	3.5464	31.39%	2.2081	57.28%	2.2535	56.41%
0.9000	25.3065	11.5255	54.46%	3.9711	84.31%	4.1865	83.46%

Table 4.2.6: Case *b*: Results of Mean Queue Length for Increasing $\rho_2 = \rho_3$, $\rho_1 = 0.25$

$\rho_2 = \rho_3$	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
0.2500	0.9279	0.8071	13.02%	0.6315	31.94%	0.6362	31.44%
0.5000	2.6219	1.5325	41.55%	1.3310	49.24%	1.3605	48.11%
0.7440	unstable	unstable	N/A	2.2123	N/A	1.9664	N/A

Table 4.2.7: Case *b*: Results of Mean Queue Length for Increasing $\rho_2 = \rho_3$, $\rho_1 = 0.5$

$\rho_2 = \rho_3$	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
0.2500	1.3465	1.2005	10.84%	0.8051	40.21%	0.7604	43.53%
0.5000	3.0964	1.9963	35.53%	1.5694	49.32%	1.6063	48.12%
0.7440	unstable	unstable	N/A	2.7133	N/A	2.9116	N/A

Table 4.2.8: Case *b*: Results of Mean Queue Length for Increasing K2 = K3, $\rho_1 = \rho_2 = \rho_3 = 0.25$

K2 = K3	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
1	0.9279	0.8071	13.02%	0.6315	31.94%	0.6362	31.44%
2	0.9279	0.8071	13.02%	0.6260	32.54%	0.6362	31.44%

4.3 Networks With a Geometric Arrival Process

4.3.1 Exact Results

Introduction

In order to determine whether or not the decomposition procedure presented in this thesis is useful for other configurations of geometric networks, the split configuration is examined. The geometric network studied is a special case of the network with MAP arrivals, therefore all of the assumptions hold for both cases. The following parameters are used to define the simplified system:

- p = probability of an arrival to queue 1
- $q = 1 - p$ = probability of no arrival to queue 1
- β_i = probability of a service completion in queue i, i = 1, 2 or 3
- $\alpha_i = 1 - \beta_i$ = probability of no service completion in queue i, i = 1, 2 or 3
- K2 = buffer size of queue 2
- K3 = buffer size of queue 3
- θ_2 = probability the customer leaving queue 1 will choose to visit queue 2
- $\theta_3 = 1 - \theta_2$ = probability the customer leaving queue 1 will choose queue 3
- $(j = 0)$ = state when there are K3 + 1 customers in queue 3 or K2 + 1 in queue 2, but there is no blocking
- $(j = 2)$ = state when there are K2 + 1 customers in queue 2 and it blocks an arrival from queue 1
- $(j = 3)$ = state when there are K3 + 1 customers in queue 3 and it blocks an arrival from queue 1

As before, this network will be looked at in two different ways. Case *a* will define a system in which either second stage queue can be visited, and case *b* will define a system where the second queue is determined after service completion in queue 1 and there can be no change.

Transition Matrix

The discrete time transition matrices $\mathbf{P}_a$ and $\mathbf{P}_b$ which define this system are presented below:

$\mathbf{P}$ =

$$
\begin{array}{c|c}
L_I & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \ddots \end{array} &
\left[\begin{array}{cccccc}
\mathbf{B0} & \mathbf{B1} & & & & \\
\mathbf{A2} & \mathbf{A1} & \mathbf{A0} & & & \\
 & \mathbf{A2} & \mathbf{A1} & \mathbf{A0} & & \\
 & & \mathbf{A2} & \mathbf{A1} & \mathbf{A0} & \\
 & & & \ddots & \ddots & \ddots
\end{array}\right]
\end{array}
$$

Each of sub-matrices within $\mathbf{P}_a$ and $\mathbf{P}_b$ are provided in Appendix A.4.

Solution Method

The stability conditions for either case *a* or *b* are too complicated to be expressed explicitly. However, as usual Q may be expressed as follows:

$$Q = \pi_A (\mathbf{A2} - \mathbf{A0})\, e$$

Once stability has been determined the MGM can be used to find the steady state vector $\overline{X}$. This in turn is used to calculate the marginal queue length distributions and the mean number of customers in the system. The derivations of the mean number in each queue are the same as the non-renewal system. The formulas are presented here for clarity. The mean number in queue 1:

$$\mu_{q1} = \bar{X}_1 (\mathbf{I} - \mathbf{R})^{-2} \mathbf{e}$$

Mean number in queue 2:

$$\mu_{q2} = \bar{X}_0 \mathbf{W} \mathbf{e} + \bar{X}_1 (\mathbf{I} - \mathbf{R})^{-1} \mathbf{W} \mathbf{e}$$

Mean number in queue 3

$$\mu_{q3} = \bar{X}_0 \mathbf{Y} \mathbf{e} + \bar{X}_1 (\mathbf{I} - \mathbf{R})^{-1} \mathbf{Y} \mathbf{e}$$

Mean number in the system

$$\mu_q = \mu_{q1} + \mu_{q2} + \mu_{q3}$$
$$\mu_q = \bar{X}_0 (\mathbf{W} + \mathbf{Y}) \mathbf{e} + \bar{X}_1 (\mathbf{I} - \mathbf{R})^{-1} [(\mathbf{I} - \mathbf{R})^2 + \mathbf{W} + \mathbf{Y}] \mathbf{e}$$

4.3.2 Approximation Method

Introduction

The split configuration system consisting of three Geometric queues can be easily decomposed into a finite queue and 2 infinite queues. A geometric arrival process into the second stage queues does not account for correlation with queue 1, so two new MAP with $\mathbf{D0}^{*i}$ and $\mathbf{D1}^{*i}$, $i = 2$ or 3, based on the transition matrix of the network, are again used to approximate the arrival processes. Queues 2 and 3 may now be described as

MAP/Geo/1/K queues. The service probabilities of the second stage queues remain the same.

The arrival process into the first queue is still geometric, however the service rate must be modified to account for customers being blocked from entering queues 2 or 3.

As in the non-renewal case the matrices and formulae for the isolated queues remain the same as for the tandem system. That is, only the end results are different once the appropriate substitutions are made. The transition matrices and formulae are presented here for clarity.

Solution Method of the Isolated Second Stage Queues: MAP/Geo/1/K

For both cases the arrival processes into queues 2 and 3 are estimated with MAP based on the departure rate from queue 1. Using two truncated MAP, the arrival processes into the second stage queues can be estimated. The $\mathbf{D0}^{*i}$ and $\mathbf{D1}^{*i}$ matrices, i = 2 or 3, are provided below:

$\mathbf{D0}^{*2} =$

$$
\begin{array}{c c}
 & \begin{array}{cccccc} 1 & 2 & 3 & \cdots & n-1 & n \end{array} \\
\begin{array}{c} 1 \\ 2 \\ \\ \vdots \\ n \\ \\ \end{array} &
\left[\begin{array}{cccccc}
\mathbf{B0} & \mathbf{B1} & & & & \\
\mathbf{A2}^{*3} & (\mathbf{A1}^{*3}) + \mathbf{A1}'' & \mathbf{A0} & & & \\
 & & & \ddots & & \\
 & & & & \mathbf{A2}^{*3} & (\mathbf{A1}^{*3})' + \mathbf{A1}'' + \mathbf{A0}
\end{array}\right]
\end{array}
$$

$\mathbf{D1}^{*2} =$

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & \cdots & n-1 & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \ddots \\ n \end{array} & \begin{bmatrix} & & & & \\ \mathbf{A2}^{*2} & (\mathbf{A1}^{*2}) & & & \\ & & \ddots & & \\ & & & \mathbf{A2}^{*2} & (\mathbf{A1}^{*2}) \end{bmatrix} \end{array}$$

$\mathbf{D0}^{*3} =$

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & \cdots & n-1 & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \ddots \\ n \end{array} & \begin{bmatrix} \mathbf{B0} & \mathbf{B1} & & & & \\ \mathbf{A2}^{*2} & \mathbf{A1}'' + (\mathbf{A1}^{*2}) & \mathbf{A0} & & & \\ & & & \ddots & & \\ & & & & \mathbf{A2}^{*2} & (\mathbf{A1}^{*2})' + \mathbf{A1}'' + \mathbf{A0} \end{bmatrix} \end{array}$$

$\mathbf{D1}^{*3} =$

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & \cdots & n-1 & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \ddots \\ n \end{array} & \begin{bmatrix} & & & & \\ \mathbf{A2}^{*3} & (\mathbf{A1}^{*3}) & & & \\ & & \ddots & & \\ & & & \mathbf{A2}^{*3} & (\mathbf{A1}^{*3}) \end{bmatrix} \end{array}$$

The definition of the matrices within $\mathbf{D0}^{*}$ and $\mathbf{D1}^{*}$ are dependent on whether we are studying case *a* or *b*. The sub-matrices for both of the cases are presented in Appendix A.4.

The transition matrix for either of the isolated MAP/Geo/1/K queues is as follows:

$\mathbf{P}_{qi}$ (i = 2 or 3) =

$$
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \ddots \\ Ki \\ Ki+1 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 0 & 1 & 2 & 3 & \ldots & Ki-1 & Ki & Ki+1 \end{array} \\
\left[\begin{array}{cccccccc}
D0^{*i} & D1^{*i} & & & & & & \\
D0^{*i}\beta_i & D0^{*i}\alpha_i + D1^{*i}\beta_i & D1^{*i}\alpha_i & & & & & \\
 & D0^{*i}\beta_i & D0^{*i}\alpha_i + D1^{*i}\beta_i & D1^{*i}\alpha_i & & & & \\
 & & & & \ddots & & & \\
 & & & & & D0^{*i}\beta_i & D0^{*i}\alpha_i + D1^{*i}\beta_i & D1^{*i}\alpha_i \\
 & & & & & & D0^{*i}\beta_i & D0^{*i}\alpha_i + D1^{*i}
\end{array}\right]
\end{array}
$$

The steady state vector of the number in the system can be found using several iterative methods including the Power Method and Gauss-Seidel procedure. The formula for the mean number in the queue is as follows:

$$\mu_{qi} = \sum_{m=0}^{Ki+1} m X_m^{(i)}$$

Solution Method of Isolated Queue 1: Geo/Geo/1/∞

The isolated queue 1 acts as a Geo/Geo/1/∞ queue and the arrival rate into this queue is that same as the arrival rate into the combined system. The service rate, however is less than or equal to the service rate of the first queue in the combined system. The rate is reduced with the probability of the customer being blocked from entering a second stage queue and can be expressed as follows:

$$\beta_1{}^* = \beta_1 (1 - \theta_2 BP_2 - \theta_3 BP_3)$$
$$\alpha_1{}^* = 1 - \beta_1{}^*$$

where

- BP_i is the steady state probability of queue i being full, i = 2 or 3

- β_1 is the original probability of a service completion in queue 1, $\alpha_1 = 1 - \beta_1$
- θ_i is the probability a customer will proceed to queue i upon completing service in queue 1, $i = 2$ or 3

Once the new service rate PH^* has been established the infinite transition matrix for queue 1 is as follows:

$\mathbf{P}_{q1} =$

$$
\begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\
\begin{bmatrix}
q & p & & & & \\
q\beta_1^* & q\alpha_1^* + p\beta_1^* & p\alpha_1^* & & & \\
 & q\beta_1^* & q\alpha_1^* + p\beta_1^* & p\alpha_1^* & & \\
 & & q\beta_1^* & q\alpha_1^* + p\beta_1^* & p\alpha_1^* & \\
 & & & \ddots & \ddots & \ddots
\end{bmatrix}
\end{array}
$$

The closed form formula for the mean queue length is as follows:

$$\mu_{q1} = \frac{pq}{\beta_1^* - p}$$

When the buffer sizes of queues 2 and 3 are greater than or equal to the $K2_{max}$ or $K3_{max}$ the blocking probability is equal to zero. If $BP_2 = BP_3 = 0$ then $\beta_1^* = \beta_1$ and the traffic intensity of queue 1, $\rho_1^* = p/\beta_1^*$, is a minimum.

Utilization of a Phase Service Distribution

As mentioned in Section 3.3.2 a discussion of the results using a phase service distribution instead of a geometric distribution will be presented. Either method will account for blocking probabilities into the second stage queues. The question which remains to be answered is whether or not a phase distribution would produce results closer to the actual. For this to be possible there would have to be significant differences in the values of the service rate calculated with each method.

The service rate utilizing the geometric distribution is:

$$\frac{1}{\mu_g} = \frac{1}{1-\alpha_1^*}$$

For the phase distribution the service rate is more complicated. The formula required can be found below:

$$\frac{1}{\mu_p} = \beta_1^* (I - S_1^*)^{-1} e$$

where

$$\beta_1^* = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$S_1^* = \begin{bmatrix} \alpha_1 & \beta_1\theta_2 BP_2 & \beta_1\theta_3 BP_3 \\ & \alpha_2 & \\ & & \alpha_3 \end{bmatrix}$$

and

BP_i = steady state probabilities of queue i being full, $i = 2$ or 3

The following table presents the service rates using each method for the specified inputs.

Table 4.3.1: Comparison of Service Rates Using Geometric and Phase Distributions

p	β_1	β_2	β_3	μ_g	μ_p	% Error
0.05	0.2	0.2	0.2	5.0258	5.1049	1.57%
0.05	0.3	0.3	0.3	3.3414	3.3521	0.32%
0.05	0.4	0.4	0.4	2.5033	2.5049	0.06%
0.05	0.5	0.5	0.5	2.0016	2.0016	0.00%
0.05	0.6	0.6	0.6	1.6675	1.6672	0.02%
0.05	0.7	0.7	0.7	1.4290	1.4287	0.02%
0.05	0.8	0.8	0.8	1.2502	1.2501	0.01%
0.05	0.9	0.9	0.9	1.1112	1.1111	0.01%
0.1	0.5	0.2	0.5	2.0333	2.0520	0.92%
0.1	0.5	0.3	0.5	2.0151	2.0200	0.24%
0.1	0.5	0.4	0.5	2.0087	2.0098	0.05%
0.1	0.5	0.5	0.2	2.0631	2.1029	1.93%
0.1	0.5	0.5	0.3	2.0246	2.0346	0.49%
0.1	0.5	0.5	0.4	2.0115	2.0138	0.11%
0.1	0.5	0.5	0.5	2.0060	2.0059	0.00%
0.2	0.5	0.5	0.5	2.0209	2.0209	0.00%
0.3	0.5	0.5	0.5	2.0421	2.0421	0.00%
0.4	0.5	0.5	0.5	2.0672	2.0672	0.00%

To summarize the table, all of the service rates found using the two methods were within 2% of each other. Therefore, the results generated with either method would also be approximately the same. In other words, it would appear that utilizing a phase distribution is not worth the extra computational effort required to solve the isolated queue 1.

4.3.3 Comparison of Combined and Isolated Results

Comparison of Methods Presented in This Section

The following trial attributes were compared for the exact and approximated results:

- Q = stability factor
- β_1^* = modified service probability of queue 1
- ρ_1^* = traffic intensity into queue 1
- ρ_1= traffic intensity into queue 1
- ρ_2= traffic intensity into queue 2
- ρ_3= traffic intensity into queue 3
- μ_q = mean number in system
- μ_{q1} = mean number in queue 1
- μ_{q2} = mean number in queue 2
- μ_{q3} = mean number in queue 3
- % error of $\mu_q = | \mu_q(\text{combo}) - \mu_q(\text{iso}) | / \mu_q(\text{combo})$
- % error of $\mu_{q1} = | \mu_{q1}(\text{combo}) - \mu_{q1}(\text{iso}) | / \mu_{q1}(\text{combo})$
- % error of $\mu_{q2} = | \mu_{q2}(\text{combo}) - \mu_{q2}(\text{iso}) | / \mu_{q2}(\text{combo})$
- % error of $\mu_{q3} = | \mu_{q3}(\text{combo}) - \mu_{q3}(\text{iso}) | / \mu_{q3}(\text{combo})$

Where

- mean number in each queue includes the customer in service
- (combo) represents combined system solution
- (iso) represents the isolated queue results

The results of each trial, for each case, are presented in Appendix A.4.3. No new observations were noted for the split geometric system. The observations relating to the individual queues may be found in Section 3.2.3, and those relating to the differences between cases *a* and *b* may be found in Section 4.2.3.

Observations Related to the Application of the Decomposition Procedure for Case *a*

It would appear the decomposition method will produces values of the mean number in the system within 30% of the actual results when:

1. $0.6667 \leq \rho_1 \leq 0.75$, and $0.2 \leq (\rho_2$ and $\rho_3) < 0.5$ for any set of inputs.
2. $0.6667 \leq \rho_1 \leq 0.7$ and $0.1 \leq (\rho_2$ and $\rho_3) \leq 0.6667$
3. $0.4 \leq \rho_1 < 0.6667$ and $0.1 \leq (\rho_2$ and $\rho_3) \leq 0.5$
4. $0.25 < \rho_1 < 0.4$ and $0.2 \leq (\rho_2$ and $\rho_3) \leq 0.5$

An exception to these rules occurs when only one value falls outside or at the boundary. That is the mean number in the system will still be within 30% of the actual results if ρ_2 or ρ_3 is above or equal to the upper bound, and the other is not. An additional observation made about the case *a* results is that as $\rho_2 = \rho_3$ decrease the percentage error for μ_{q1} decreases.

Observations Related to the Application of the Decomposition Procedure for Case *b*

The isolated results for case *b* are quite different than those for case *a*. It would appear that the decomposition method produces acceptable values, μ_q percentage error less than 30%, in fewer instances than case *a*. Acceptable results are produced for:

1. $0.4 \leq \rho_1 \leq 0.86$ when $0.1 \leq (\rho_2$ and $\rho_3) < 0.5$.

2. $0.125 \leq \rho_1 < 0.4$ and $0.156 < (\rho_2$ and $\rho_3) < 0.5$.

The results of case *b* are limited because if either queue 2 or queue 3 builds up queue 1 becomes unstable. Unlike in case *a*, once a second stage destination queue has been chosen no switching is allowed. If the chosen queue is full, queue 1 remains blocked until that queue completes a service.

Case *a*: Comparison of Yannopoulos & Alfa's Method

As for the MAP/PH/1/K system not all of the trials were compared because Yannopoulos and Alfa's [55] technique require that queues 2 and 3 be equivalent. The following tables present the results for various sets of inputs in which the restrictions are met.

Table 4.3.2: Case *a*: Results of Mean Queue Length for Increasing ρ_1, $\rho_2 = \rho_3 = 0.1$

ρ_1	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
0.1250	0.1878	0.3324	77.00%	0.2544	35.46%	0.2510	33.65%
0.2000	0.3530	0.4132	17.05%	0.3584	1.53%	0.3511	0.54%
0.3125	0.5758	0.5789	0.54%	0.5538	3.82%	0.5410	6.04%
0.4000	0.7813	0.7582	2.96%	0.7536	3.55%	0.7365	5.73%
0.5000	1.0279	1.0008	2.64%	0.9866	4.02%	0.9508	7.50%
0.6667	1.8923	1.8348	3.04%	1.8605	1.68%	1.8135	4.16%
0.7500	2.7475	2.6758	2.61%	2.7224	0.91%	2.6700	2.82%
0.8333	4.5842	4.5062	1.70%	4.5698	0.31%	4.5237	1.32%

Table 4.3.3: Case a: Results of Mean Queue Length for Increasing $\rho_2 = \rho_3$, $\rho_1 = 0.125$

$\rho_2 = \rho_3$	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
0.0625	0.1670	0.2518	50.78%	0.1703	1.98%	0.1646	1.44%
0.1000	0.1878	0.3324	77.00%	0.2544	35.46%	0.2510	33.65%
0.2000	0.2471	0.5631	127.88%	0.4913	98.83%	0.4930	99.51%
0.7500	2.7620	1.7022	38.37%	2.1302	22.87%	2.1292	22.91%

Table 4.3.4: Case a: Results of Mean Queue Length for Increasing $\rho_2 = \rho_3$, $\rho_1 = 0.2$

$\rho_2 = \rho_3$	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
0.1000	0.3530	0.4132	17.05%	0.3584	1.53%	0.3511	0.54%
0.6667	2.3467	1.6270	30.67%	1.8967	19.18%	1.8942	19.28%
0.7143	2.3726	1.7382	26.74%	2.0786	12.39%	2.0766	12.48%
0.7500	2.9272	1.7213	41.20%	2.2289	23.86%	2.2282	23.88%

Table 4.3.5: Case a: Results of Mean Queue Length for Increasing K2 = K3, $\rho_1 = 0.4$ & $\rho_2 = \rho_3 = 0.2$

K2 = K3	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
1	0.9995	0.9395	6.00%	0.9953	0.42%	0.9840	1.55%
2	1.0010	0.9347	6.62%	1.0209	1.99%	0.9984	0.26%
3	1.0015	0.9344	6.70%	1.0288	2.73%	1.0034	0.19%
4	1.0016	0.9346	6.69%	1.0309	2.93%	1.0047	0.31%

Like the tandem case, the approximated results for a geometric system generated with the decomposition method presented in this thesis are inferior to those generated by both methods from [55]. In cases that the decomposition method produced lower percentage errors than Yao & Buzacott and Gelenbe the results are very close to the actual results. Therefore, it would appear that the quick and simple method is better suited to geometric systems. That is, maintaining the dependence between the departures from queue 1 and the arrivals into the second stage queues is not significant, in the case where the external arrival process is non-renewal.

Case *b*: Comparison of Yannopoulos & Alfa's Method

Although the values of the mean queue length derived with the exact and decomposition methods changed from case *a* to *b*, the method in this thesis still produced inferior results for most of the trials performed. This further enhances the idea that maintaining the dependence in a geometric system is less significant. The following tables present the results for various sets of inputs which meet all of the specified requirements.

Table 4.3.6: Case *b*: Results of Mean Queue Length for Increasing ρ_1, $\rho_2 = \rho_3 = 0.1$

ρ_1	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
0.1250	0.1881	0.3324	76.71%	0.2544	35.25%	0.2510	33.44%
0.2000	0.3534	0.4132	16.92%	0.3584	1.41%	0.3511	0.65%
0.3125	0.5763	0.5789	0.45%	0.5538	3.90%	0.5410	6.13%
0.4000	0.7819	0.7582	3.03%	0.7536	3.62%	0.7365	5.81%
0.5000	1.0280	1.0008	2.65%	0.9866	4.03%	0.9508	7.51%
0.6667	1.8926	1.8348	3.05%	1.8605	1.70%	1.8135	4.18%
0.7500	2.7479	2.6758	2.62%	2.7224	0.93%	2.6700	2.83%
0.8333	4.5864	4.5062	1.75%	4.5698	0.36%	4.5237	1.37%

Table 4.3.7: Case b: Results of Mean Queue Length for Increasing $\rho_2 = \rho_3$, $\rho_1 = 0.125$

$\rho_2 = \rho_3$	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
0.0625	0.1670	0.2518	50.78%	0.1703	1.98%	0.1646	1.44%
0.1000	0.1881	0.3324	76.71%	0.2544	35.25%	0.2510	33.44%
0.2000	0.5335	0.5440	1.97%	0.4913	7.91%	0.4930	7.59%
0.7500	unstable	unstable	N/A	2.1302	N/A	2.1292	N/A

Table 4.3.8: Case b: Results of Mean Queue Length for Increasing $\rho_2 = \rho_3$, $\rho_1 = 0.2$

$\rho_2 = \rho_3$	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
0.1000	0.3534	0.4132	16.92%	0.3584	1.41%	0.3511	0.65%
0.6667	9.5972	1.4989	84.38%	1.8967	80.24%	1.8942	80.26%
0.7143	88.4355	1.5472	98.25%	2.0786	97.65%	2.0766	97.65%
0.7500	unstable	unstable	N/A	2.2289	N/A	2.2282	N/A

Table 4.3.9: Case b: Results of Mean Queue Length for Increasing K2 = K3, $\rho_1 = 0.4$ & $\rho_2 = \rho_3 = 0.2$

K2 = K3	μ_q exact	μ_q decomp	μ_q % error decomp	μ_q Y & B	μ_q % error Y & B	μ_q Gelenbe	μ_q % error Gelenbe
1	1.0198	0.9394	7.88%	0.9953	2.40%	0.9840	3.51%
2	1.0050	0.9347	7.00%	1.0209	1.58%	0.9984	0.66%
3	1.0023	0.9344	6.77%	1.0288	2.64%	1.0034	0.11%
4	1.0019	0.9344	6.74%	1.0309	2.89%	1.0047	0.28%

5. CONCLUSION

The contribution of this thesis is a decomposition method for obtaining the queue length distributions of open, finite, tandem and split queue networks with a non-renewal external arrival process. The networks explored were two stage systems with MAP arrival processes and phase service distributions. Their geometric equivalents were also studied to determine if maintaining the relationship between the isolated queues improved the approximate results over existing solution methods.

As expected, it can be concluded that for tandem and split systems with non-renewal arrival processes, such as MAP, the method developed in this thesis is superior to existing methods which fail to capture the dependence between the isolated queues. That is, methods such as the one by Yannopoulos and Alfa [55] do not provide good approximate results.

The opposite is true for both configurations of the geometric systems. That is existing methods which do not maintain the relationship between the queues in their decomposition approach produce equal or superior results. When renewal processes are present throughout the system the dependence between the isolated queues is minimized. Therefore, it can be concluded that it is not worth the extra effort to maintain the relationship between isolated geometric queues.

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Appendix A.1

Tandem Networks With a Markovian Arrival Process

A.1.1 Sub-Matrices of the Transition Matrix P

Each of the matrices within the transition matrix P are as follows:

B0 =

$$
\begin{array}{rr}
 & \begin{array}{cccccccc} & & & & & & j=0 & j=1 \\ 0 & 1 & 2 & \dots & K2-1 & K2 & K2+1 & K2+1 \end{array} \\
\begin{array}{rr} & L_2 \\ & 0 \\ & 1 \\ & 2 \\ & \ddots \\ & K2 \\ j=0 & K2+1 \end{array} &
\begin{bmatrix}
D0 & & & & & & & \\
D0 \otimes S^0_2 & D0 \otimes S_2 & & & & & & \\
 & D0 \otimes S^0_2 \beta_2 & D0 \otimes S_2 & & & & & \\
 & & & \ddots & & & & \\
 & & & & D0 \otimes S^0_2 \beta_2 & D0 \otimes S_2 & & \\
 & & & & & D0 \otimes S^0_2 \beta_2 & D0 \otimes S_2 &
\end{bmatrix}
\end{array}
$$

B1 =

$$
\begin{array}{rr}
 & \begin{array}{cccccccc} & & & & & & j=0 & j=1 \\ 0 & 1 & 2 & \dots & K2-1 & K2 & K2+1 & K2+1 \end{array} \\
\begin{array}{rr} & L_2 \\ & 0 \\ & 1 \\ & 2 \\ & \ddots \\ & K2 \\ j=0 & K2+1 \end{array} &
\begin{bmatrix}
D1 \otimes \beta_1 & & & & & & & \\
D1 \otimes \beta_1 \otimes S^0_2 & D1 \otimes \beta_1 \otimes S_2 & & & & & & \\
 & D1 \otimes \beta_1 \otimes S^0_2 \beta_2 & D1 \otimes \beta_1 \otimes S_2 & & & & & \\
 & & & \ddots & & & & \\
 & & & & D1 \otimes \beta_1 \otimes S^0_2 \beta_2 & D1 \otimes \beta_1 \otimes S_2 & & \\
 & & & & & D1 \otimes \beta_1 \otimes S^0_2 \beta_2 & D1 \otimes \beta_1 \otimes S_2 &
\end{bmatrix}
\end{array}
$$

C0 =

$$
\begin{array}{rr}
 & \begin{array}{cccccccc} & & & & & & j=0 & j=1 \\ 0 & 1 & 2 & 3 & \dots & K2 & K2+1 & K2+1 \end{array} \\
\begin{array}{rr} & L_2 \\ & 0 \\ & 1 \\ & 2 \\ & \ddots \\ & K2 \\ j=0 & K2+1 \\ j=1 & K2+1 \end{array} &
\begin{bmatrix}
 & D0 \otimes S^0_1 \otimes \beta_2 & & & & & & \\
 & D0 \otimes S^0_1 \otimes S^0_2 \beta_2 & D0 \otimes S^0_1 \otimes S_2 & & & & & \\
 & & D0 \otimes S^0_1 \otimes S^0_2 \beta_2 & D0 \otimes S^0_1 \otimes S_2 & & & & \\
 & & & & \ddots & & & \\
 & & & & & D0 \otimes S^0_1 \otimes S^0_2 \beta_2 & D0 \otimes S^0_1 \otimes S_2 & \\
 & & & & & & D0 \otimes S^0_1 \otimes S^0_2 \beta_2 & \\
 & & & & & & D0 \otimes S^0_2 \beta_2 &
\end{bmatrix}
\end{array}
$$

A2 =

$$
\begin{array}{cc|cccccccc|}
 & & & & & & & & j=0 & j=1 \\
 & L_2 & 0 & 1 & 2 & 3 & \cdots & K2 & K2+1 & K2+1 \\
\hline
 & 0 & & D0\otimes S^0_1\beta_1\otimes\beta_2 & & & & & & \\
 & 1 & & D0\otimes S^0_1\beta_1\otimes S^0_2\beta_2 & D0\otimes S^0_1\beta_1\otimes S_2 & & & & & \\
 & 2 & & & D0\otimes S^0_1\beta_1\otimes S^0_2\beta_2 & D0\otimes S^0_1\beta_1\otimes S_2 & & & & \\
 & \ddots & & & & & \ddots & & & \\
 & K2 & & & & & & D0\otimes S^0_1\beta_1\otimes S^0_2\beta_2 & D0\otimes S^0_1\beta_1\otimes S_2 & \\
j=0 & K2+1 & & & & & & & D0\otimes S^0_1\beta_1\otimes S^0_2\beta_2 & \\
j=1 & K2+1 & & & & & & & D0\otimes\beta_1\otimes S^0_2\beta_2 & \\
\hline
\end{array}
$$

A1 =

$$
\begin{array}{cc|cccccccc|}
 & & & & & & & & j=0 & j=1 \\
 & L_2 & 0 & 1 & 2 & 3 & \cdots & K2 & K2+1 & K2+1 \\
\hline
 & 0 & D0\otimes S_1 & D1\otimes S^0_1\beta_1\otimes\beta_2 & & & & & & \\
 & 1 & D0\otimes S_1\otimes S^0_2 & D0\otimes S_1\otimes S_2 + D1\otimes S^0_1\beta_1\otimes S^0_2\beta_2 & D1\otimes S^0_1\beta_1\otimes S_2 & & & & & \\
 & 2 & & D0\otimes S_1\otimes S^0_2\beta_2 & D0\otimes S_1\otimes S_2 + D1\otimes S^0_1\beta_1\otimes S^0_2\beta_2 & D1\otimes S^0_1\beta_1\otimes S_2 & & & & \\
 & \ddots & & & & & \ddots & & & \\
j=0 & K2+1 & & & & & & D0\otimes S_1\otimes S^0_2\beta_2 & D0\otimes S_1\otimes S_2 + D1\otimes S^0_1\beta_1\otimes S^0_2\beta_2 & D0\otimes S^0_1\otimes S_2 \\
j=1 & K2+1 & & & & & & & D1\otimes\beta_1\otimes S^0_2\beta_2 & D0\otimes S_2 \\
\hline
\end{array}
$$

A0 =

$$
\begin{array}{cc|cccccccc|}
 & & & & & & & & j=0 & j=1 \\
 & L_2 & 0 & 1 & 2 & \cdots & K2-1 & K2 & K2+1 & K2+1 \\
\hline
 & 0 & D1\otimes S_1 & & & & & & & \\
 & 1 & D1\otimes S_1\otimes S^0_2 & D1\otimes S_1\otimes S_2 & & & & & & \\
 & 2 & & D1\otimes S_1\otimes S^0_2\beta_2 & D1\otimes S_1\otimes S_2 & & & & & \\
 & \ddots & & & & \ddots & & & & \\
 & K2 & & & & & D1\otimes S_1\otimes S^0_2\beta_2 & D1\otimes S_1\otimes S_2 & & \\
j=0 & K2+1 & & & & & & D1\otimes S_1\otimes S^0_2\beta_2 & D1\otimes S_1\otimes S_2 & D1\otimes S^0_1\otimes S_2 \\
j=1 & K2+1 & & & & & & & & D1\otimes S_2 \\
\hline
\end{array}
$$

where:

- **D0** = No departure matrix of MAP arrival process
- **D1** = Departure matrix of MAP arrival process
- **(β_i, S_i)** = service distribution of queue *i*, *i* = 1 or 2
- **K2** = buffer size of queue 2
- (j = 0) = state when there are K2 + 1 customers in queue 2 but there is no blocking
- (j = 1) = state when there are K2 + 1 customers in queue 2 and it blocks an arrival from queue 1

A.1.2 Sub-Matrices of the Modified MAP

Each of the new matrices within **D0*** and **D1*** are as follows:

A1'' =

$$
\begin{array}{cc}
 & \begin{array}{ccccccc} & & & & & j=0 & j=1 \\ 0 & 1 & 2 & \cdots & K2 & K2+1 & K2+1 \end{array} \\
\begin{array}{rr} & L_2 \\ & 0 \\ & 1 \\ & 2 \\ & \ddots \\ j=0 & K2+1 \\ j=1 & K2+1 \end{array} &
\begin{bmatrix}
D0 \otimes S_1 & & & & & & \\
D0 \otimes S_1 \otimes S^0_2 & D0 \otimes S_1 \otimes S_2 & & & & & \\
 & D0 \otimes S_1 \otimes S^0_2 & D0 \otimes S_1 \otimes S_2 & & & & \\
 & & & \ddots & & & \\
 & & & & D0 \otimes S_1 \otimes S^0_2 & D0 \otimes S_1 \otimes S_2 & \\
 & & & & & D1 \otimes \beta_1 \otimes S^0_2 \beta_2 & D0 \otimes S_2
\end{bmatrix}
\end{array}
$$

A1' can be found using the relationship A1' = A1 - A1''

A.1.3 Trial Results

Table A.1.1: Results for Varying Inputs, $n = 2$

	D0	D1	λ	S_1	β_1	μ_1	S_2
1	.45 .45 .45 .45	.05 .05 .05 .05	0.1000	0 0 0 0	.50 .50	1.0000	.375 .375 .375 .375
2	.45 .45 .45 .45	.05 .05 .05 .05	0.1000	0 0 0 0	.50 .50	1.0000	.375 .375 .375 .375
3	.45 .45 .45 .45	.05 .05 .05 .05	0.1000	0 0 0 0	.50 .50	1.0000	.375 .375 .375 .375
4	.45 .45 .45 .45	.05 .05 .05 .05	0.1000	0 0 0 0	.50 .50	1.0000	.375 .375 .375 .375
5	.45 .45 .45 .45	.05 .05 .05 .05	0.1000	.445 .445 .445 .445	.50 .50	0.1100	0 0 0 0
6	.45 .45 .45 .45	.05 .05 .05 .05	0.1000	0 0 0 0	.50 .50	1.0000	.375 .375 .375 .375
7	.45 .45 .45 .45	.05 .05 .05 .05	0.1000	.40 .475 .40 .475	.50 .50	0.1250	0 0 0 0
8	.40 .40 .40 .40	.10 .10 .10 .10	0.2000	0 0 0 0	.50 .50	1.0000	.25 .25 .25 .25
9	.45 .45 .50 .50	.05 .05 0 0	0.0500	.45 .475 .45 .475	.50 .50	0.0750	.25 .25 .25 .25
10	.45 .45 .45 .45	.05 .05 .05 .05	0.1000	0 0 0 0	.50 .50	1.0000	.375 .375 .375 .375
11	.45 .45 .50 .50	.05 .05 0 0	0.0500	.25 .25 .25 .25	.50 .50	0.5000	.25 .25 .25 .25
12	.45 .45 .45 .45	.05 .05 .05 .05	0.1000	.375 .375 .375 .375	.50 .50	0.2500	0 0 0 0
13	.40 .40 .40 .40	.10 .10 .10 .10	0.2000	.05 .05 .05 .05	.50 .50	0.9000	.25 .25 .25 .25
14	.40 .40 .40 .40	.10 .10 .10 .10	0.2000	.25 .25 .25 .25	.50 .50	0.5000	0 0 0 0
15	.45 .45 .45 .45	.05 .05 .05 .05	0.1000	0 0 0 0	.50 .50	1.0000	.375 .375 .375 .375
16	.40 .40 .40 .40	.10 .10 .10 .10	0.2000	.25 .25 .25 .25	.50 .50	0.5000	.05 .05 .05 .05
17	.40 .40 .40 .40	.10 .10 .10 .10	0.2000	.25 .25 .25 .25	.50 .50	0.5000	.10 .10 .10 .10
18	.40 .50 .30 .60	.06 .04 .03 .07	0.1000	.20 .30 .40 .10	.60 .40	0.5000	.20 .30 .40 .10

Table A.1.1: Results for Varying Inputs, $n = 2$

	β_2	μ_2	K2	ρ_1	μ_{q1} combo	μ_{q1} iso	μ_1^*	μ_{q1} % error
1	.50 .50	0.2500	8	0.1000	0.1000	0.1000	0.9998	0.00%
2	.50 .50	0.2500	7	0.1000	0.1001	0.1001	0.9995	0.00%
3	.50 .50	0.2500	6	0.1000	0.1003	0.1002	0.9985	0.10%
4	.50 .50	0.2500	5	0.1000	0.1008	0.1006	0.9955	0.20%
5	.50 .50	1.0000	2	0.9091	9.0000	8.9993	0.1100	0.01%
6	.50 .50	0.2500	4	0.1000	0.1025	0.1020	0.9866	0.49%
7	.50 .50	1.0000	2	0.8000	3.6000	3.6000	0.1250	0.00%
8	.50 .50	0.5000	2	0.2000	0.2083	0.2114	0.9633	1.49%
9	.50 .50	0.5000	2	0.6667	1.9001	1.9000	0.0750	0.01%
10	.50 .50	0.2500	3	0.1000	0.1074	0.1059	0.9605	1.40%
11	.50 .50	0.5000	2	0.1000	0.1056	0.1056	0.4999	0.00%
12	.50 .50	1.0000	2	0.4000	0.6000	0.6000	0.2500	0.00%
13	.50 .50	0.5000	2	0.2222	0.2370	0.2389	0.8740	0.80%
14	.50 .50	1.0000	2	0.4000	0.5333	0.5333	0.5000	0.00%
15	.50 .50	0.2500	2	0.1000	0.1222	0.1182	0.8884	3.27%
16	.50 .50	0.9000	2	0.4000	0.5333	0.5334	0.5000	0.02%
17	.50 .50	0.8000	2	0.4000	0.5334	0.5334	0.4999	0.00%
18	.60 .40	0.5000	2	0.2000	0.2253	0.2252	0.4996	0.04%

Table A.1.1: Results for Varying Inputs, $n = 2$

	ρ_2	μ_{q2} combo	μ_{q2} iso	λ_2	BP	μ_{q2} % error	Q
1	0.4000	0.6000	0.5998	0.1000	0.0000	0.03%	0.1500
2	0.4000	0.5999	0.5995	0.1000	0.0001	0.07%	0.1500
3	0.4000	0.5997	0.5988	0.1000	0.0004	0.15%	0.1500
4	0.4000	0.5992	0.5967	0.1000	0.0011	0.42%	0.1500
5	0.1000	0.1000	0.0553	0.0553	0.0000	44.70%	0.0100
6	0.4000	0.5975	0.5916	0.1001	0.0034	0.99%	0.1500
7	0.1000	0.1000	0.0588	0.0588	0.0000	41.20%	0.0250
8	0.4000	0.5250	0.5107	0.2004	0.0190	2.72%	0.3000
9	0.1000	0.1055	0.0623	0.0309	0.0000	40.95%	0.0250
10	0.4000	0.5926	0.5787	0.1003	0.0103	2.35%	0.1500
11	0.1000	0.1055	0.0986	0.0476	0.0001	6.54%	0.3875
12	0.1000	0.1000	0.0769	0.0769	0.0000	23.10%	0.1500
13	0.4000	0.5249	0.4925	0.1906	0.0165	6.17%	0.2998
14	0.2000	0.2000	0.1667	0.1667	0.0000	16.65%	0.3000
15	0.4000	0.5778	0.5478	0.1009	0.0314	5.19%	0.1500
16	0.2222	0.2286	0.1876	0.1667	0.0000	17.94%	0.2998
17	0.2500	0.2666	0.2149	0.1667	0.0002	19.39%	0.2982
18	0.2000	0.2247	0.1954	0.0909	0.0008	13.04%	0.3375

Table A.1.1: Results for Varying Inputs, $n = 2$

	μ_q combo	μ_q iso	μ_q % error
1	0.7000	0.6998	0.03%
2	0.7000	0.6996	0.06%
3	0.7000	0.6990	0.14%
4	0.7000	0.6973	0.39%
5	9.1000	9.0546	0.50%
6	0.7000	0.6936	0.91%
7	3.7000	3.6588	1.11%
8	0.7333	0.7221	1.53%
9	2.0056	1.9623	2.16%
10	0.7000	0.6846	2.20%
11	0.2111	0.2042	3.27%
12	0.7000	0.6769	3.30%
13	0.7619	0.7314	4.00%
14	0.7333	0.7000	4.54%
15	0.7000	0.6660	4.86%
16	0.7619	0.7210	5.37%
17	0.8000	0.7483	6.46%
18	0.4500	0.4206	6.53%

Table A.1.1: Results for Varying Inputs, $n = 2$

	D0	D1	λ	S_1	β_1	μ_1	S_2
19	.40 .50 .30 .60	.06 .04 .03 .07	0.1000	.20 .30 .40 .10	.60 .40	0.5000	.20 .30 .40 .10
20	.40 .50 .30 .60	.06 .04 .03 .07	0.1000	.20 .30 .40 .10	.60 .40	0.5000	.20 .30 .40 .10
21	.40 .40 .40 .40	.10 .10 .10 .10	0.2000	.10 .10 .10 .10	.50 .50	0.8000	.25 .25 .25 .25
22	.40 .40 .40 .40	.10 .10 .10 .10	0.2000	.25 .25 .25 .25	.50 .50	0.5000	.15 .15 .15 .15
23	.40 .40 .40 .40	.10 .10 .10 .10	0.2000	.15 .15 .15 .15	.50 .50	0.7000	.25 .25 .25 .25
24	.38 .37 .34 .36	.12 .13 .14 .16	0.2275	.08 .17 .11 .19	.49 .51	0.7225	.09 .26 .10 .30
25	.45 .45 .40 .40	.05 .05 .10 .10	0.1500	.25 .25 .25 .25	.50 .50	0.5000	.25 .25 .25 .25
26	.40 .40 .40 .40	.10 .10 .10 .10	0.2000	.25 .25 .25 .25	.50 .50	0.5000	.20 .20 .20 .20
27	.40 .40 .40 .40	.10 .10 .10 .10	0.2000	.35 .35 .35 .35	.50 .50	0.3000	.25 .25 .25 .25
28	.40 .40 .40 .40	.10 .10 .10 .10	0.2000	.20 .20 .20 .20	.50 .50	0.6000	.25 .25 .25 .25
29	.45 .45 .45 .45	.05 .05 .05 .05	0.1000	.40 .475 .40 .475	.50 .50	0.1250	.375 .375 .375 .375
30	.45 .45 .45 .45	.05 .05 .05 .05	0.1000	0 0 0 0	.50 .50	1.0000	.40 .40 .40 .40
31	.45 .45 .20 .20	.05 .05 .30 .30	0.3500	.25 .25 .25 .25	.50 .50	0.5000	.25 .25 .25 .25
32	.30 .50 .40 .40	.10 .10 .05 .15	0.2000	.20 .30 .40 .10	.60 .40	0.5000	.20 .30 .40 .10
33	.40 .40 .40 .40	.10 .10 .10 .10	0.2000	.25 .25 .25 .25	.50 .50	0.5000	.25 .25 .25 .25
34	.30 .50 .40 .40	.10 .10 .05 .15	0.2000	.20 .30 .40 .10	.60 .40	0.5000	.20 .30 .40 .10
35	.30 .50 .40 .40	.10 .10 .05 .15	0.2000	.20 .30 .40 .10	.60 .40	0.5000	.20 .30 .40 .10
36	.30 .50 .40 .40	.10 .10 .05 .15	0.2000	.20 .30 .40 .10	.60 .40	0.5000	.20 .30 .40 .10
37	.40 .40 .40 .40	.10 .10 .10 .10	0.2000	.30 .30 .30 .30	.50 .50	0.4000	.25 .25 .25 .25

Table A.1.1: Results for Varying Inputs, $n = 2$

	β_2	μ_2	K2	ρ_1	μ_{q1} combo	μ_{q1} iso	μ_1^*	μ_{q1} % error
19	.60 .40	0.5000	3	0.2000	0.2250	0.2250	0.5000	0.00%
20	.60 .40	0.5000	4	0.2000	0.2250	0.2250	0.5000	0.00%
21	.50 .50	0.5000	2	0.2500	0.2753	0.2758	0.7825	0.18%
22	.50 .50	0.7000	2	0.4000	0.5339	0.5337	0.4997	0.04%
23	.50 .50	0.5000	2	0.2857	0.3289	0.3279	0.6890	0.30%
24	.46 .54	0.6190	2	0.3813	0.4560	0.4564	0.7139	0.09%
25	.50 .50	0.5000	2	0.3000	0.3666	0.3657	0.4986	0.25%
26	.50 .50	0.6000	2	0.4000	0.5358	0.5349	0.4991	0.17%
27	.50 .50	0.5000	2	0.6667	1.6267	1.6057	0.2996	1.29%
28	.50 .50	0.5000	2	0.3333	0.4094	0.4067	0.5937	0.66%
29	.50 .50	0.2500	2	0.8000	3.8088	3.6173	0.1249	5.03%
30	.50 .50	0.2000	2	0.1000	0.1790	0.1473	0.7690	17.71%
31	.50 .50	0.5000	2	0.7000	2.0094	1.6727	0.4859	16.76%
32	.60 .40	0.5000	2	0.4000	0.5438	0.5390	0.4969	0.88%
33	.50 .50	0.5000	2	0.4000	0.5438	0.5390	0.4969	0.88%
34	.60 .40	0.5000	5	0.4000	0.5335	0.5336	0.5000	0.02%
35	.60 .40	0.5000	3	0.4000	0.5359	0.5342	0.4995	0.32%
36	.60 .40	0.5000	4	0.4000	0.5340	0.5335	0.4999	0.09%
37	.50 .50	0.5000	2	0.5000	0.8134	0.8049	0.3987	1.04%

Table A.1.1: Results for Varying Inputs, $n = 2$

	ρ_2	μ_{q2} combo	μ_{q2} iso	λ_2	BP	μ_{q2} % error	Q
19	0.2000	0.2250	0.1956	0.0909	0.0001	13.07%	0.3500
20	0.2000	0.2250	0.1956	0.0909	0.0000	13.07%	0.3583
21	0.4000	0.5247	0.4718	0.1907	0.0140	10.08%	0.2982
22	0.2857	0.3195	0.2521	0.1667	0.0007	21.10%	0.2916
23	0.4000	0.5245	0.4480	0.1844	0.0114	14.59%	0.2916
24	0.4451	0.5744	0.4801	0.2491	0.0103	16.42%	0.3205
25	0.3000	0.3621	0.2916	0.1305	0.0028	19.47%	0.2875
26	0.3333	0.3978	0.3054	0.1667	0.0022	23.23%	0.2741
27	0.4000	0.5223	0.3004	0.1364	0.0022	42.49%	0.0942
28	0.4000	0.5242	0.4203	0.1766	0.0080	19.82%	0.2741
29	0.4000	0.5710	0.2605	0.0589	0.0021	54.38%	0.0225
30	0.5000	0.8210	0.7329	0.1013	0.0601	10.73%	0.1000
31	0.7000	1.2343	0.6752	0.2586	0.0290	45.30%	0.0875
32	0.4000	0.5237	0.3878	0.1668	0.0063	25.95%	0.2375
33	0.4000	0.5237	0.3878	0.1668	0.0063	25.95%	0.2375
34	0.4000	0.5332	0.3918	0.1667	0.0000	26.52%	0.2643
35	0.4000	0.5308	0.3910	0.1667	0.0009	26.34%	0.2500
36	0.4000	0.5327	0.3917	0.1667	0.0001	26.47%	0.2583
37	0.4000	0.5231	0.3488	0.1539	0.0041	33.32%	0.1766

Table A.1.1: Results for Varying Inputs, $n = 2$

	μ_q combo	μ_q iso	μ_q % error
19	0.4500	0.4206	6.53%
20	0.4500	0.4206	6.53%
21	0.8000	0.7476	6.55%
22	0.8534	0.7858	7.92%
23	0.8534	0.7759	9.08%
24	1.0304	0.9365	9.11%
25	0.7287	0.6573	9.80%
26	0.9336	0.8403	9.99%
27	2.1490	1.9061	11.30%
28	0.9336	0.8270	11.42%
29	4.3798	3.8778	11.46%
30	1.0000	0.8802	11.98%
31	2.6846	2.3479	12.54%
32	1.0675	0.9268	13.18%
33	1.0675	0.9268	13.18%
34	1.0667	0.9254	13.25%
35	1.0667	0.9252	13.27%
36	1.0667	0.9252	13.27%
37	1.3365	1.1537	13.68%

Table A.1.1: Results for Varying Inputs, $n = 2$

	D0	D1	λ	S_1	β_1	μ_1	S_2
38	.275 .275 .275 .275	.225 .225 .225 .225	0.4500	.25 .25 .25 .25	.50 .50	0.5000	.15 .175 .15 .175
39	.45 .45 .30 .30	.05 .05 .20 .20	0.2500	.25 .25 .25 .25	.50 .50	0.5000	.25 .25 .25 .25
40	.40 .40 .40 .40	.10 .10 .10 .10	0.2000	.35 .35 .35 .35	.50 .50	0.3000	.30 .30 .30 .30
41	.40 .40 .40 .40	.10 .10 .10 .10	0.2000	.25 .25 .25 .25	.50 .50	0.5000	.30 .30 .30 .30
42	.30 .30 .30 .30	.20 .20 .20 .20	0.4000	.25 .25 .25 .25	.50 .50	0.5000	.20 .20 .20 .20
43	.35 .35 .35 .35	.15 .15 .15 .15	0.3000	.25 .25 .25 .25	.50 .50	0.5000	.25 .25 .25 .25
44	.40 .40 .40 .40	.10 .10 .10 .10	0.2000	.25 .25 .25 .25	.50 .50	0.5000	.35 .35 .35 .35
45	.40 .40 .40 .40	.10 .10 .10 .10	0.2000	.35 .35 .35 .35	.50 .50	0.3000	.35 .35 .35 .35
46	.30 .30 .30 .30	.20 .20 .20 .20	0.4000	.20 .20 .20 .20	.50 .50	0.6000	.25 .25 .25 .25
47	.45 .45 .50 .50	.05 .05 0 0	0.0500	.25 .25 .25 .25	.50 .50	0.5000	.45 .475 .45 .475
48	.30 .30 .30 .30	.20 .20 .20 .20	0.4000	.25 .25 .25 .25	.50 .50	0.5000	.25 .25 .25 .25
49	.45 .45 .45 .45	.05 .05 .05 .05	0.1000	0 0 0 0	.50 .50	1.0000	.40 .475 .40 .475
50	.275 .275 .275 .275	.225 .225 .225 .225	0.4500	.25 .25 .25 .25	.50 .50	0.5000	.20 .2375 .20 .2375
51	.275 .275 .275 .275	.225 .225 .225 .225	0.4500	.15 .175 .15 .175	.50 .50	0.6750	.25 .25 .25 .25
52	.45 .45 .45 .45	.05 .05 .05 .05	0.1000	.375 .375 .375 .375	.50 .50	0.2500	.40 .475 .40 .475
53	.275 .275 .275 .275	.225 .225 .225 .225	0.4500	.20 .2375 .20 .2375	.50 .50	0.5625	.25 .25 .25 .25

Table A.1.1: Results for Varying Inputs, $n = 2$

	β_2	μ_2	K2	ρ_1	μ_{q1} combo	μ_{q1} iso	μ_1^*	μ_{q1} % error
38	.50 .50	0.6750	2	0.9000	5.9471	5.3350	0.4962	10.29%
39	.50 .50	0.5000	2	0.5000	0.7889	0.7679	0.4942	2.66%
40	.50 .50	0.4000	2	0.6667	1.7150	1.6244	0.2984	5.28%
41	.50 .50	0.4000	2	0.4000	0.5834	0.5556	0.4887	4.77%
42	.50 .50	0.6000	2	0.8000	2.8285	2.5587	0.4935	9.54%
43	.50 .50	0.5000	2	0.6000	1.1841	1.1022	0.4905	6.92%
44	.50 .50	0.3000	2	0.4000	0.8857	0.6374	0.4572	28.03%
45	.50 .50	0.3000	2	0.6667	2.3147	1.7292	0.2925	25.29%
46	.50 .50	0.5000	2	0.6667	2.2805	1.5107	0.5604	33.76%
47	.50 .50	0.0750	2	0.1000	0.6268	0.2450	0.2867	60.91%
48	.50 .50	0.5000	2	0.8000	5.0609	2.9729	0.4806	41.26%
49	.50 .50	0.1250	2	0.1000	1.7938	0.3835	0.4197	78.62%
50	.50 .50	0.5625	2	0.9000	15.9764	6.7071	0.4867	58.02%
51	.50 .50	0.5000	2	0.6667	4.8490	1.7310	0.5974	64.30%
52	.50 .50	0.1250	2	0.4000	2.5298	0.8408	0.2129	66.76%
53	.50 .50	0.5000	2	0.8000	15.4604	3.4357	0.5224	77.78%

Table A.1.1: Results for Varying Inputs, $n = 2$

	ρ_2	μ_{q2} combo	μ_{q2} iso	λ_2	BP	μ_{q2} % error	Q
38	0.6667	0.9980	0.5419	0.3100	0.0102	45.70%	0.0386
39	0.5000	0.7172	0.4842	0.2001	0.0118	32.49%	0.1875
40	0.5000	0.7469	0.4001	0.1365	0.0072	46.43%	0.0824
41	0.5000	0.7532	0.5276	0.1669	0.0185	29.95%	0.1766
42	0.6667	1.0619	0.5830	0.2853	0.0158	45.10%	0.0741
43	0.6000	0.9510	0.5804	0.2307	0.0194	38.97%	0.1375
44	0.6667	1.2634	0.7902	0.1669	0.0562	37.45%	0.0942
45	0.6667	1.2332	0.5934	0.1366	0.0257	51.88%	0.0523
46	0.8000	1.6099	0.8826	0.3122	0.0589	45.18%	0.0741
47	0.6667	1.3788	0.9871	0.0487	0.1116	28.41%	0.0250
48	0.8000	1.5757	0.7674	0.2839	0.0405	51.30%	0.0375
49	0.8000	1.9055	1.2497	0.1000	0.1728	34.42%	0.0250
50	0.8000	1.2798	0.7173	0.3087	0.0308	43.95%	0.0130
51	0.9000	2.0903	1.0797	0.3586	0.0963	48.35%	0.0386
52	0.8000	1.8493	0.9264	0.0777	0.0872	49.91%	0.0225
53	0.9000	1.7200	0.9419	0.3274	0.0682	45.24%	0.0130

Table A.1.1: Results for Varying Inputs, $n = 2$

	μ_q combo	μ_q iso	μ_q % error
38	6.9451	5.8769	15.38%
39	1.5061	1.2521	16.86%
40	2.4619	2.0245	17.77%
41	1.3366	1.0832	18.96%
42	3.8904	3.1417	19.24%
43	2.1351	1.6826	21.19%
44	2.1491	1.4276	33.57%
45	3.5479	2.3226	34.54%
46	3.8904	2.3933	38.48%
47	2.0056	1.2321	38.57%
48	6.6366	3.7403	43.64%
49	3.6993	1.6332	55.85%
50	17.2562	7.4244	56.98%
51	6.9393	2.8107	59.50%
52	4.3791	1.7672	59.64%
53	17.1804	4.3776	74.52%

Appendix A.2

Tandem Networks With A Geometric Arrival Process

A.2.1 Sub-Matrices of the Transition Matrix P

Each of the following matrices are contained within the Geometric system transition matrix **P**:

B0 =

$$\begin{array}{cc}
L_2 & \begin{array}{cccccccc} 0 & 1 & 2 & \cdots & K2-1 & K2 & \substack{j=0\\K2+1} & \substack{j=1\\K2+1} \end{array} \\
\begin{array}{r} 0 \\ 1 \\ 2 \\ \ddots \\ K2 \\ j=0\ \ K2+1 \end{array} &
\left[\begin{array}{cccccccc}
q & & & & & & & \\
q\,\beta_2 & q\,\alpha_2 & & & & & & \\
 & q\,\beta_2 & q\,\alpha_2 & & & & & \\
 & & & \ddots & & & & \\
 & & & & q\,\beta_2 & q\,\alpha_2 & & \\
 & & & & & q\,\beta_2 & q\,\alpha_2 &
\end{array}\right]
\end{array}$$

B1 =

$$\begin{array}{cc}
L_2 & \begin{array}{cccccccc} 0 & 1 & 2 & \cdots & K2-1 & K2 & \substack{j=0\\K2+1} & \substack{j=1\\K2+1} \end{array} \\
\begin{array}{r} 0 \\ 1 \\ 2 \\ \ddots \\ K2 \\ j=0\ \ K2+1 \end{array} &
\left[\begin{array}{cccccccc}
p & & & & & & & \\
p\,\beta_2 & q\,\alpha_2 & & & & & & \\
 & p\,\beta_2 & p\,\alpha_2 & & & & & \\
 & & & \ddots & & & & \\
 & & & & p\,\beta_2 & p\,\alpha_2 & & \\
 & & & & & p\,\beta_2 & p\,\alpha_2 &
\end{array}\right]
\end{array}$$

A2 =

$$\begin{array}{cc}
L_2 & \begin{array}{cccccccc} 0 & 1 & 2 & 3 & \cdots & K2 & \substack{j=0\\K2+1} & \substack{j=1\\K2+1} \end{array} \\
\begin{array}{r} 0 \\ 1 \\ 2 \\ \ddots \\ K2 \\ j=0\ \ K2+1 \\ j=1\ \ K2+1 \end{array} &
\left[\begin{array}{cccccccc}
 & q\,\beta_1 & & & & & & \\
 & q\,\beta_1\beta_2 & q\,\beta_1\alpha_2 & & & & & \\
 & & q\,\beta_1\beta_2 & q\,\beta_1\alpha_2 & & & & \\
 & & & & \ddots & & & \\
 & & & & & q\,\beta_1\beta_2 & q\,\beta_1\alpha_2 & \\
 & & & & & & q\,\beta_1\beta_2 & \\
 & & & & & & q\,\beta_2 &
\end{array}\right]
\end{array}$$

A1 =

$$
\begin{array}{r|cccccccc|}
L_2 & 0 & 1 & 2 & 3 & \dots & K2-1 & K2 & \begin{matrix} j=0 \\ K2+1 \end{matrix} & \begin{matrix} j=1 \\ K2+1 \end{matrix} \\
\hline
0 & q\alpha_1 & p\beta_1 & & & & & & & \\
1 & q\alpha_1\beta_2 & \begin{matrix} q\alpha_1\alpha_2 + \\ p\beta_1\beta_2 \end{matrix} & p\beta_1\alpha_2 & & & & & & \\
2 & & q\alpha_1\beta_2 & \begin{matrix} q\alpha_1\alpha_2 + \\ p\beta_1\beta_2 \end{matrix} & p\beta_1\alpha_2 & & & & & \\
\ddots & & & & & \ddots & & & & \\
K2 & & & & & & q\alpha_1\beta_2 & \begin{matrix} q\alpha_1\alpha_2 + \\ p\beta_1\beta_2 \end{matrix} & p\beta_1\alpha_2 & \\
j=0 \;\; K2+1 & & & & & & & q\alpha_1\beta_2 & \begin{matrix} q\alpha_1\alpha_2 + \\ p\beta_1\beta_2 \end{matrix} & q\beta_1\alpha_2 \\
j=1 \;\; K2+1 & & & & & & & & p\beta_2 & q\alpha_2
\end{array}
$$

A0 =

$$
\begin{array}{r|ccccccc|}
L_2 & 0 & 1 & 2 & \dots & K2-1 & K2 & \begin{matrix} j=0 \\ K2+1 \end{matrix} & \begin{matrix} j=1 \\ K2+1 \end{matrix} \\
\hline
0 & p\alpha_1 & & & & & & & \\
1 & p\alpha_1\beta_2 & p\alpha_1\alpha_2 & & & & & & \\
2 & & p\alpha_1\beta_2 & p\alpha_1\alpha_2 & & & & & \\
\ddots & & & & \ddots & & & & \\
K2 & & & & & p\alpha_1\beta_2 & p\alpha_1\alpha_2 & & \\
j=0 \;\; K2+1 & & & & & & p\alpha_1\beta_2 & p\alpha_1\alpha_2 & p\beta_1\alpha_2 \\
j=1 \;\; K2+1 & & & & & & & & p\alpha_2
\end{array}
$$

where:

- p = probability of an arrival to queue 1
- $q = 1 - p$ = probability of no arrival to queue 1
- β_i = probability of a service completion in queue i, $i = 1$ or 2
- $\alpha_i = 1 - \beta_i$ = probability of no service completion in queue i, $i = 1$ or 2
- K2 = buffer size of queue 2
- ($j = 0$) = state when there are K2 + 1 customers in queue 2 but there is no blocking
- ($j = 1$) = state when there are K2 + 1 customers in queue 2 and a customer leaving queue 1 is blocked from entering queue 2

A.2.2 Sub-Matrices of the Absorbing Markov Chain Transition Matrix

Each of the following matrices are contained within the absorbing Markov chain transition matrix $\hat{P}$:

$\hat{B}0$ =

$$
\begin{array}{c|ccccccccccc|}
L_2 & 0 & 0^* & 1 & 1^* & \ldots & K2-1 & K2-1^* & K2 & K2^* & \begin{matrix} j=0 \\ K2+1 \end{matrix} & \begin{matrix} j=1 \\ K2+1 \end{matrix} \\
\hline
0 & 1 & & & & & & & & & & \\
0^* & \beta_1 & \alpha_1 & & & & & & & & & \\
1 & \beta_2 & & \alpha_2 & & & & & & & & \\
1^* & \beta_2\beta_1 & \beta_2\alpha_1 & \alpha_2\beta_1 & \alpha_2\alpha_1 & & & & & & & \\
\ddots & & & & & \ddots & & & & & & \\
K2 & & & & & & \beta_2 & & \alpha_2 & & & \\
K2^* & & & & & & \beta_2\beta_1 & \beta_2\alpha_1 & \alpha_2\beta_1 & \alpha_2\alpha_1 & & \\
j=0\ K2+1 & & & & & & & & \beta_2\beta_1 & \beta_2\alpha_1 & \alpha_2\alpha_1 & \alpha_2\beta_1 \\
j=1\ K2+1 & & & & & & & & \beta_2 & & & \alpha_2 \\
\hline
\end{array}
$$

$\hat{C}0$ =

$$
\begin{array}{c|ccccccccccccc|}
L_2 & 0 & 0^* & 1 & 1^* & 2 & 2^* & 3 & 3^* & \ldots & K2 & K2^* & \begin{matrix} j=0 \\ K2+1 \end{matrix} & \begin{matrix} j=1 \\ K2+1 \end{matrix} \\
\hline
0 & & & \beta_1 & & & & & & & & & & \\
1 & & & \beta_1\beta_2 & & \beta_1\alpha_2 & & & & & & & & \\
2 & & & & & \beta_1\beta_2 & & \beta_1\alpha_2 & & & & & & \\
\ddots & & & & & & & & & \ddots & & & & \\
K2 & & & & & & & & & & \beta_1\beta_2 & & \beta_1\alpha_2 & \\
j=0\ K2+1 & & & & & & & & & & & & \beta_1\beta_2 & \\
j=1\ K2+1 & & & & & & & & & & & & \beta_2 & \\
\hline
\end{array}
$$

$\hat{A}2$ =

$$
\begin{array}{c|cccccccc|}
L_2 & 0 & 1 & 2 & 3 & \ldots & K2 & \begin{matrix} j=0 \\ K2+1 \end{matrix} & \begin{matrix} j=1 \\ K2+1 \end{matrix} \\
\hline
0 & & \beta_1 & & & & & & \\
1 & & \beta_1\beta_2 & \beta_1\alpha_2 & & & & & \\
2 & & & \beta_1\beta_2 & \beta_1\alpha_2 & & & & \\
. & & & & & & & & \\
K2 & & & & & & \beta_1\beta_2 & \beta_1\alpha_2 & \\
j=0\ K2+1 & & & & & & & \beta_1\beta_2 & \\
j=1\ K2+1 & & & & & & & \beta_2 & \\
\hline
\end{array}
$$

$$\hat{A}1 =$$

$$\begin{array}{rc}
 & \begin{array}{cccccccc} & & & & & & j=0 & j=1 \\ 0 & 1 & 2 & \cdots & K2-1 & K2 & K2+1 & K2+1 \end{array} \\
\begin{array}{r} L_2 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K2 \\ j=0\ \ K2+1 \\ j=1\ \ K2+1 \end{array} &
\begin{array}{c} \\ \left[\begin{array}{cccccccc}
\alpha_1 & & & & & & & \\
\alpha_1\beta_2 & \alpha_1\alpha_2 & & & & & & \\
 & \alpha_1\beta_2 & \alpha_1\alpha_2 & & & & & \\
 & & & \ddots & & & & \\
 & & & & \alpha_1\beta_2 & \alpha_1\alpha_2 & & \\
 & & & & & \alpha_1\beta_2 & \alpha_1\alpha_2 & \beta_1\alpha_2 \\
 & & & & & & & \alpha_2
\end{array}\right] \end{array}
\end{array}$$

A.2.3 Sub-Matrices of the Modified MAP

Each of the new sub-matricies within D0 and D1 are defined as follows:

A1" =

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} 0 & 1 & 2 & \cdots & K2-1 & K2 & \overset{j=0}{K2+1} & \overset{j=1}{K2+1} \end{array} \\
\begin{array}{r} L_2 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K2 \\ j=0 \quad K2+1 \\ j=1 \quad K2+1 \end{array} &
\left[\begin{array}{cccccccc}
q\alpha_1 & & & & & & & \\
q\alpha_1\beta_2 & q\alpha_1\alpha_2 & & & & & & \\
 & q\alpha_1\beta_2 & q\alpha_1\alpha_2 & & & & & \\
 & & & \ddots & & & & \\
 & & & & q\alpha_1\beta_2 & q\alpha_1\alpha_2 & & \\
 & & & & & q\alpha_1\beta_2 & q\alpha_1\alpha_2 & q\beta_1\alpha_2 \\
 & & & & & & & q\alpha_2
\end{array}\right]
\end{array}
$$

A1' = A1 - A1"

A.2.4 Trial Results

Table A.2.1: Results for Varying Inputs, $n = 2$

	ρ	β_1	β_2	K2	ρ_1	μ_{q1} combo	μ_{q1} iso	μ_{q1} % error
1	0.2000	0.5000	0.5000	2	0.4000	0.5619	0.5579	0.71%
2	0.0500	0.2000	0.5000	1	0.2500	0.3170	0.3167	0.09%
3	0.2000	0.5000	0.5000	3	0.4000	0.5542	0.5527	0.27%
4	0.2000	0.5000	0.5000	4	0.4000	0.5523	0.5519	0.07%
5	0.2000	0.5000	0.5000	5	0.4000	0.5519	0.5517	0.04%
6	0.2000	0.5000	0.5000	1	0.4000	0.5711	0.5943	4.06%
7	0.4500	0.5000	0.9000	2	0.9000	4.9597	4.9699	0.21%
8	0.0500	0.1000	0.5000	1	0.5000	0.9508	0.9501	0.07%
9	0.2000	0.3500	0.8500	2	0.5714	1.0667	1.0667	0.00%
10	0.2100	0.4700	0.4700	2	0.4468	0.6572	0.6482	1.37%
11	0.4500	0.5000	0.8000	2	0.9000	5.0657	5.0665	0.02%
12	0.0500	0.3000	0.5000	1	0.1667	0.1903	0.1900	0.16%
13	0.2500	0.5000	0.5000	2	0.5000	0.7835	0.7675	2.04%
14	0.3000	0.4300	0.7000	2	0.6977	1.6263	1.6240	0.14%
15	0.2500	0.4800	0.4800	2	0.5208	0.8640	0.8380	3.01%
16	0.0500	0.5000	0.1000	1	0.1000	0.2330	0.1113	52.23%
17	0.3000	0.7000	0.4100	2	0.4286	0.8340	0.6012	27.91%
18	0.0500	0.4000	0.5000	1	0.1250	0.1360	0.1357	0.22%
19	0.2500	0.4500	0.4500	2	0.5556	1.0258	0.9716	5.28%
20	0.0500	0.5000	1.0000	1	0.1000	0.1056	0.1056	0.00%
21	0.5500	0.7000	0.7000	2	0.7857	2.1321	2.0011	6.14%
22	0.1000	0.5000	0.5000	1	0.2000	0.2274	0.2252	0.97%
23	0.0500	0.5000	0.9000	1	0.1000	0.1056	0.1056	0.00%
24	0.4000	0.8000	0.6250	2	0.5000	0.6492	0.6475	0.26%
25	0.1000	0.5000	0.5000	2	0.2000	0.2252	0.2252	0.00%
26	0.1000	0.5000	0.5000	3	0.2000	0.2250	0.2252	0.09%
27	0.1000	0.5000	0.5000	4	0.2000	0.2250	0.2252	0.09%
28	0.0500	0.5000	0.8000	1	0.1000	0.1056	0.1056	0.00%
29	0.0500	0.5000	0.7000	1	0.1000	0.1056	0.1056	0.00%
30	0.3000	0.5000	0.5000	1	0.6000	1.3807	1.1015	20.22%
31	0.0500	0.5000	0.6000	1	0.1000	0.1056	0.1056	0.00%
32	0.0500	0.5000	0.5000	1	0.1000	0.1058	0.1056	0.19%
33	0.0500	0.5000	0.4000	1	0.1000	0.1062	0.1056	0.56%
34	0.4000	0.7000	0.5000	2	0.5714	1.4661	0.9626	34.34%

Table A.2.1: Results for Varying Inputs, $n = 2$

	$\overline{W}_1$	ρ_2	μ_{q2} combo	μ_{q2} iso	λ	BP	μ_{q2} % error	$\overline{W}_2$
1	2.7895	0.4000	0.3866	0.3937	0.1654	0.0066	1.84%	2.3961
2	6.3340	0.1000	0.0900	0.0849	0.0417	0.0000	5.67%	2.0360
3	2.7635	0.4000	0.3840	0.3982	0.1655	0.0010	3.70%	2.4085
4	2.7595	0.4000	0.3824	0.3990	0.1655	0.0002	4.34%	2.4114
5	2.7585	0.4000	0.3818	0.3992	0.1655	0.0000	4.56%	2.4121
6	2.9715	0.4000	0.3879	0.3868	0.1665	0.0424	0.28%	2.4260
7	11.0442	0.5000	0.5291	0.3568	0.3103	0.0004	32.56%	1.1503
8	19.0020	0.1000	0.1002	0.0696	0.0345	0.0000	30.54%	2.0174
9	5.3335	0.2353	0.2139	0.1741	0.1458	0.0000	18.61%	1.1941
10	3.0867	0.4468	0.4627	0.4282	0.1696	0.0086	7.46%	2.5467
11	11.2589	0.5625	0.6712	0.4208	0.3103	0.0023	37.31%	1.3592
12	3.8000	0.1000	0.0796	0.0917	0.0448	0.0000	15.20%	2.0469
13	3.0700	0.5000	0.5478	0.4822	0.1996	0.0114	11.98%	2.4437
14	5.4133	0.4286	0.4626	0.3311	0.2146	0.0016	28.43%	1.5453
15	3.3520	0.5208	0.6056	0.4989	0.1963	0.0130	17.62%	2.5750
16	2.2260	0.5000	0.4584	0.6621	0.0467	0.0467	44.44%	14.8723
17	2.0040	0.7317	0.9133	0.9206	0.2584	0.0724	0.80%	3.8408
18	2.7140	0.1000	0.0690	0.0958	0.0465	0.0001	38.84%	2.0604
19	3.8864	0.5556	0.7090	0.5259	0.1909	0.0156	25.83%	2.7985
20	2.1120	0.0500	0.0281	0.0476	0.0476	0.0000	69.40%	1.0000
21	3.6384	0.7857	1.2066	0.8342	0.4420	0.0376	30.86%	1.9611
22	2.2520	0.2000	0.1357	0.1953	0.0909	0.0008	43.92%	2.1502
23	2.1120	0.0556	0.0313	0.0531	0.0476	0.0000	69.65%	1.1155
24	1.6188	0.6400	0.5680	0.7686	0.3608	0.0367	35.32%	2.2114
25	2.2520	0.2000	0.1345	0.1953	0.0909	0.0008	45.20%	2.1502
26	2.2520	0.2000	0.1342	0.1953	0.0909	0.0008	45.53%	2.1502
27	2.2520	0.2000	0.1342	0.1953	0.0909	0.0008	45.53%	2.1502
28	2.1120	0.0625	0.0354	0.0600	0.0476	0.0000	69.49%	1.2605
29	2.1120	0.0714	0.0407	0.0689	0.0476	0.0000	69.29%	1.4475
30	3.6717	0.6000	0.7307	0.5771	0.2300	0.0187	21.02%	2.5569
31	2.1120	0.0833	0.0479	0.0811	0.0476	0.0000	69.31%	1.7038
32	2.1120	0.1000	0.0582	0.0986	0.0476	0.0001	69.42%	2.0716
33	2.1120	0.1250	0.0742	0.1259	0.0476	0.0003	69.68%	2.6458
34	2.4065	0.8000	1.2648	0.9703	0.3327	0.0745	23.28%	3.1512

Table A.2.1: Results for Varying Inputs, $n = 2$

	Q	μ_q combo	μ_q iso	μ_q % error	$\overline{W}$ combo	$\overline{W}$ iso	$\overline{W}$ % error
1	0.2278	0.9485	0.9516	0.33%	3.4836	7.1856	106.27%
2	0.1470	0.4070	0.4016	1.33%	5.8853	13.3700	127.18%
3	0.2402	0.9382	0.9509	1.35%	3.4836	7.1720	105.88%
4	0.2485	0.9347	0.9509	1.73%	3.4836	7.1709	105.85%
5	0.2485	0.9337	0.9509	1.84%	3.4836	7.1706	105.84%
6	0.2167	0.9590	0.9811	2.30%	3.4836	7.3975	112.35%
7	0.0498	5.4888	5.3267	2.95%	∞	14.1945	∞
8	0.0498	1.0510	1.0197	2.98%	15.5730	31.0194	99.19%
9	0.1499	1.2806	1.2408	3.11%		9.3847	
10	0.1986	1.1199	1.0764	3.88%		7.7610	
11	0.0482	5.7369	5.4873	4.35%	∞	14.6181	∞
12	0.2358	0.2699	0.2817	4.37%	3.7940	9.1802	141.97%
13	0.1800	1.3313	1.2497	6.13%		7.5137	
14	0.1265	2.0889	1.9551	6.41%		9.2843	
15	0.1600	1.4696	1.3369	9.03%		8.0103	
16	0.0498	0.6914	0.7734	11.86%	7.2307	19.0983	164.13%
17	0.1073	1.7473	1.5218	12.91%	∞	7.2733	∞
18	0.3102	0.2050	0.2315	12.93%	2.8247	7.2744	157.53%
19	0.1396	1.7348	1.4975	13.68%		8.9071	
20	0.4500	0.1337	0.1532	14.58%	2.1111	5.1120	142.15%
21	0.0917	3.3387	2.8353	15.08%		7.0280	
22	0.3167	0.3631	0.4205	15.81%	2.5676	6.4022	149.35%
23	0.4485	0.1369	0.1587	15.92%	2.1245	5.2275	146.06%
24	0.2169	1.2172	1.4161	16.34%		5.0802	
25	0.3375	0.3597	0.4205	16.90%	2.5676	6.4022	149.35%
26	0.3500	0.3592	0.4205	17.07%	2.5676	6.4022	149.35%
27	0.3583	0.3592	0.4205	17.07%	2.5676	6.4022	149.35%
28	0.4425	0.1410	0.1656	17.45%	1.2605	5.3725	326.22%
29	0.4293	0.1463	0.1745	19.28%	2.1662	5.5595	156.65%
30	0.1167	2.1114	1.6786	20.50%	∞	8.2286	∞
31	0.4054	0.1535	0.1867	21.63%	2.2011	5.8158	164.22%
32	0.3667	0.1640	0.2042	24.51%	2.2257	6.1836	177.83%
33	0.3102	0.1804	0.2315	28.33%	2.3508	6.7578	187.47%
34	0.0916	2.7309	1.9329	29.22%		6.9863	

Table A.2.1: Results for Varying Inputs, $n = 2$

	p	β_1	β_2	K2	ρ_1	μ_{q1} combo	μ_{q1} iso	μ_{q1} % error
35	0.2500	0.8000	0.8000	2	0.3125	0.3411	0.3415	0.12%
36	0.0500	0.5000	0.3000	1	0.1000	0.1074	0.1057	1.58%
37	0.0500	0.5000	0.2000	1	0.1000	0.1136	0.1062	6.51%
38	0.0500	0.6000	0.5000	1	0.0833	0.0866	0.0864	0.23%
39	0.4000	0.5000	0.5000	2	0.8000	5.0156	2.9832	40.52%
40	0.6000	0.6500	0.7000	2	0.9231	17.6143	9.9730	43.38%
41	0.1000	0.8000	0.8000	2	0.1250	0.1286	0.1286	0.00%
42	0.3000	0.8500	0.6000	2	0.3529	0.3897	0.3945	1.23%
43	0.0500	0.7000	0.5000	1	0.0714	0.0732	0.0731	0.14%
44	0.1000	0.9000	0.9000	2	0.1111	0.1125	0.1125	0.00%
45	0.4000	0.5000	0.4500	2	0.8000	19.1223	3.4261	82.08%
46	0.0500	0.8000	0.5000	1	0.0625	0.0635	0.0633	0.31%
47	0.0500	0.9000	0.5000	1	0.0556	0.0560	0.0559	0.18%
48	0.0500	1.0000	0.5000	1	0.0500	0.0501	0.0500	0.20%

Table A.2.1: Results for Varying Inputs, $n = 2$

	$\bar{W}_1$	ρ_2	μ_{q2} combo	μ_{q2} iso	λ	BP	μ_{q2} % error	$\bar{W}_2$
35	1.3660	0.3125	0.1542	0.3144	0.2353	0.0012	103.89%	1.3378
36	2.1140	0.1667	0.1025	0.1744	0.0476	0.0012	70.15%	3.6683
37	2.1240	0.2500	0.1668	0.2824	0.0476	0.0057	69.30%	5.9668
38	1.7280	0.1000	0.0473	0.1007	0.0484	0.0001	112.90%	2.0808
39	7.4580	0.8000	1.5069	0.7662	0.2829	0.0391	49.15%	2.8186
40	16.6217	0.8571	1.5958	0.8547	0.4501	0.0399	46.44%	1.9778
41	1.2860	0.1250	0.0388	0.1248	0.0976	0.0001	221.65%	1.2788
42	1.3150	0.5000	0.2634	0.5990	0.2838	0.0208	127.41%	2.1555
43	1.4620	0.1000	0.0363	0.1023	0.0490	0.0002	181.82%	2.0882
44	1.1250	0.1111	0.0226	0.1111	0.0989	0.0000	391.59%	1.1234
45	8.5653	0.8889	1.9701	0.8837	0.2802	0.0599	55.14%	3.3548
46	1.2660	0.1000	0.0251	0.1035	0.0494	0.0002	312.35%	2.0956
47	1.1180	0.1000	0.0140	0.1046	0.0497	0.0002	647.14%	2.1050
48	1.0000	0.1000	0.0127	0.1055	0.5000	0.0003	730.71%	0.2111

Table A.2.1: Results for Varying Inputs, $n = 2$

	Q	μ_q combo	μ_q iso	μ_q % error	$\bar{W}$ combo	$\bar{W}$ iso	$\bar{W}$ % error
35	0.5000	0.4953	0.6559	32.42%		3.9538	
36	0.2358	0.2099	0.2801	33.44%	2.5475	7.7823	205.49%
37	0.1470	0.2804	0.3886	38.59%	3.1078	10.0908	224.69%
38	0.4054	0.1339	0.1871	39.73%	1.8778	5.4755	191.59%
39	0.0375	6.5225	3.7494	42.52%	∞	12.2766	∞
40	0.0115	19.2101	10.8277	43.64%	∞	20.1380	∞
41	0.6500	0.1674	0.2534	51.37%		3.8148	
42	0.2977	0.6531	0.9935	52.12%		4.6469	
43	0.4293	0.1095	0.1754	60.18%	1.6061	4.9787	209.99%
44	0.7700	0.1351	0.2236	65.51%		3.3595	
45	0.0101	21.0924	4.3098	79.57%	∞	13.9200	∞
46	0.4425	0.0886	0.1668	88.26%	1.3998	4.6116	229.44%
47	0.4485	0.0700	0.1605	129.29%	1.2364	4.3342	250.55%
48	0.4500	0.0628	0.1555	147.61%	1.1026	2.2111	100.53%

Appendix A.3

Split Networks With A Markovian Arrival Process

A.3.1 Sub-Matrices of the Transition Matrix

Each of the matrices within **P** for the non-renewal split case are defined as follows:

B0 =

$$\begin{array}{cc}
 & \begin{array}{cccccccc} & & & & & & j=0 & j=2 \\ 0 & 1 & 2 & \cdots & K2-1 & K2 & K2+1 & K2+1 \end{array} \\
\begin{array}{r} L_2 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K2 \\ j=0 \;\; K2+1 \end{array} &
\left[\begin{array}{cccccccc}
\\
\mathbf{b0B0} & & & & & & & \\
\mathbf{c0B0} & \mathbf{a1B0} & & & & & & \\
 & \mathbf{a2B0} & \mathbf{a1B0} & & & & & \\
 & & & \ddots & & & & \\
 & & & & \mathbf{a2B0} & \mathbf{a1B0} & & \\
 & & & & & \mathbf{a2B0} & \mathbf{a1B0} &
\end{array}\right]
\end{array}$$

b0B0 =

$$\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \cdots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0 \;\; K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
\\
D0 & & & & & & & & \\
D0\otimes S^0_3 & D0\otimes S_3 & & & & & & & \\
 & D0\otimes S^0_3\beta_3 & D0\otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & D0\otimes S^0_3\beta_3 & D0\otimes S_3 & & &
\end{array}\right]
\end{array}$$

c0B0 =

$$\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \cdots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0 \;\; K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
\\
D0\otimes S^0_2 & & & & & & & & \\
D0\otimes S^0_2\otimes S^0_3 & D0\otimes S^0_2\otimes S_3 & & & & & & & \\
 & D0\otimes S^0_2\otimes S^0_3\beta_3 & D0\otimes S^0_2\otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & D0\otimes S^0_2\otimes S^0_3\beta_3 & D0\otimes S^0_2\otimes S_3 & & &
\end{array}\right]
\end{array}$$

a2B0 =

$$\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \cdots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0 \;\; K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
\\
D0\otimes S^0_2\beta_2 & & & & & & & & \\
D0\otimes S^0_2\beta_2\otimes S^0_3 & D0\otimes S^0_2\beta_2\otimes S_3 & & & & & & & \\
 & D0\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & D0\otimes S^0_2\beta_2\otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & D0\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & D0\otimes S^0_2\beta_2\otimes S_3 & & &
\end{array}\right]
\end{array}$$

a1B0 =

$$
\begin{array}{r|cccccccccc}
 & & & & & & j=0 & j=1 & j=2 & j=3 \\
L_3 & 0 & 1 & 2 & \cdots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
\hline
0 & D0\otimes S_2 & & & & & & & & \\
1 & D0\otimes S_2\otimes S^0_3 & D0\otimes S_2\otimes S_3 & & & & & & & \\
2 & & D0\otimes S_2\otimes S^0_3\beta_3 & D0\otimes S_2\otimes S_3 & & & & & & \\
\ddots & & & & \ddots & & & & & \\
j=0\;\; K3+1 & & & & & D0\otimes S_2\otimes S^0_3\beta_3 & D0\otimes S_2\otimes S_3 & & &
\end{array}
$$

B1 =

$$
\begin{array}{r|cccccccc}
 & & & & & & & j=0 & j=2 \\
L_2 & 0 & 1 & 2 & \cdots & K2-1 & K2 & K2+1 & K2+1 \\
\hline
0 & b0B1 & & & & & & & \\
1 & c0B1 & a1B1 & & & & & & \\
2 & & a2B1 & a1B1 & & & & & \\
\ddots & & & & \ddots & & & & \\
K2 & & & & & a2B1 & a1B1 & & \\
j=0\;\; K2+1 & & & & & & a2B1 & a1B1 &
\end{array}
$$

b0B1 =

$$
\begin{array}{r|cccccccccc}
 & & & & & & j=0 & j=1 & j=2 & j=3 \\
L_3 & 0 & 1 & 2 & \cdots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
\hline
0 & D1\otimes\beta_1 & & & & & & & & \\
1 & D1\otimes\beta_1\otimes S^0_3 & D1\otimes\beta_1\otimes S_3 & & & & & & & \\
2 & & D1\otimes\beta_1\otimes S^0_3\beta_3 & D1\otimes\beta_1\otimes S_3 & & & & & & \\
\ddots & & & & \ddots & & & & & \\
j=0\;\; K3+1 & & & & & D1\otimes\beta_1\otimes S^0_3\beta_3 & D1\otimes\beta_1\otimes S_3 & & &
\end{array}
$$

c0B1 =

$$
\begin{array}{r|cccccccccc}
 & & & & & & j=0 & j=1 & j=2 & j=3 \\
L_3 & 0 & 1 & 2 & \cdots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
\hline
0 & D1\otimes\beta_1\otimes S^0_2 & & & & & & & & \\
1 & D1\otimes\beta_1\otimes S^0_2\otimes S^0_3 & D1\otimes\beta_1\otimes S^0_2\otimes S_3 & & & & & & & \\
2 & & D1\otimes\beta_1\otimes S^0_2\otimes S^0_3\beta_3 & D1\otimes\beta_1\otimes S^0_2\otimes S_3 & & & & & & \\
\ddots & & & & \cdot & & & & & \\
j=0\;\; K3+1 & & & & & D1\otimes\beta_1\otimes S^0_2\otimes S^0_3\beta_3 & D1\otimes\beta_1\otimes S^0_2\otimes S_3 & & &
\end{array}
$$

a2B1 =

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} 0 & 1 & 2 & \cdots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0\ \ K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
D1\otimes\beta_1\otimes S^0{}_2\beta_2 & & & & & & & & \\
D1\otimes\beta_1\otimes S^0{}_2\beta_2\otimes S^0{}_3 & D1\otimes\beta_1\otimes S^0{}_2\beta_2\otimes S_3 & & & & & & & \\
 & D1\otimes\beta_1\otimes S^0{}_2\beta_2\otimes S^0{}_3\beta_3 & D1\otimes\beta_1\otimes S^0{}_2\beta_2\otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & D1\otimes\beta_1\otimes S^0{}_2\beta_2\otimes S^0{}_3\beta_3 & D1\otimes\beta_1\otimes S^0{}_2\beta_2\otimes S_3 & & &
\end{array}\right]
\end{array}
$$

a1B1 =

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} 0 & 1 & 2 & \cdots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0\ \ K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
D1\otimes\beta_1\otimes S_2 & & & & & & & & \\
D1\otimes\beta_1\otimes S_2\otimes S^0{}_3 & D1\otimes\beta_1\otimes S_2\otimes S_3 & & & & & & & \\
 & D1\otimes\beta_1\otimes S_2\otimes S^0{}_3\beta_3 & D1\otimes\beta_1\otimes S_2\otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & D1\otimes\beta_1\otimes S_2\otimes S^0{}_3\beta_3 & D1\otimes\beta_1\otimes S_2\otimes S_3 & & &
\end{array}\right]
\end{array}
$$

C0 =

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} 0 & 1 & 2 & 3 & \cdots & K2 & \overset{j=0}{K2+1} & \overset{j=2}{K2+1} \end{array} \\
\begin{array}{c} L_2 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0\ \ K2+1 \\ j=2\ \ K2+1 \end{array} &
\left[\begin{array}{cccccccc}
b0C0 & b1C0 & & & & & & \\
c0C0 & a1C0 & a0C0 & & & & & \\
 & a2C0 & a1C0 & a0C0 & & & & \\
 & & & & \ddots & & & \\
 & & & & & a2C0 & a1C0 & \\
 & & & & & b3C0 & b4C0 &
\end{array}\right]
\end{array}
$$

b0C0 =

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & \cdots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ . \\ K3 \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
 & D0\otimes S^0{}_1\theta_3\otimes\beta_3 & & & & & & & & \\
 & D0\otimes S^0{}_1\theta_3\otimes S^0{}_3\beta_3 & D0\otimes S^0{}_1\theta_3\otimes S_3 & & & & & & & \\
 & & D0\otimes S^0{}_1\theta_3\otimes S^0{}_3\beta_3 & D0\otimes S^0{}_1\theta_3\otimes S_3 & & & & & & \\
 & & & & . & & & & & \\
 & & & & & D0\otimes S^0{}_1\theta_3\otimes S^0{}_3\beta_3 & D0\otimes S^0{}_1\theta_3\otimes S_3 & & & \\
 & & & & & & D0\otimes S^0{}_1\theta_3\otimes S^0{}_3\beta_3 & & & \\
 & & & & & & b0C0(1) & & &
\end{array}\right]
\end{array}
$$

$\mathbf{b0C0(1)_a} = \mathbf{D0}\ \theta_3 \otimes S^0{}_3\ \beta_3$

$$b0C0(1)_b = D0 \otimes S^0{}_3\, \beta_3$$

$$b1C0 =$$

$$\begin{array}{c|cccccccc}
 & & & & & & j=0 & j=1 & j=2 & j=3 \\
L_3 & 0 & 1 & 2 & \ldots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
\hline
0 & \begin{array}{c} D0\otimes \\ S^0{}_1\theta_2\otimes\beta_2 \end{array} & & & & & & & & \\
1 & \begin{array}{c} D0\otimes \\ S^0{}_1\theta_2\otimes\beta_2\otimes \\ S^0{}_3 \end{array} & \begin{array}{c} D0\otimes \\ S^0{}_1\theta_2\otimes\beta_2\otimes \\ S_3 \end{array} & & & & & & & \\
2 & & \begin{array}{c} D0\otimes \\ S^0{}_1\theta_2\otimes\beta_2\otimes \\ S^0{}_3\beta_3 \end{array} & \begin{array}{c} D0\otimes \\ S^0{}_1\theta_2\otimes\beta_2\otimes \\ S_3 \end{array} & & & & & & \\
\ddots & & & & \ddots & & & & & \\
j=0\ \ K3+1 & & & & & \begin{array}{c} D0\otimes \\ S^0{}_1\theta_2\otimes\beta_2\otimes \\ S^0{}_3\beta_3 \\ b1C0(1) \end{array} & \begin{array}{c} D0\otimes \\ S^0{}_1\theta_2\otimes\beta_2\otimes \\ S_3 \\ b1C0(2) \end{array} & & & \\
j=3\ \ K3+1 & & & & & & & & &
\end{array}$$

$$b1C0(1)_a = D0\, \theta_2 \otimes \beta_2 \otimes S^0{}_3\beta_3$$
$$b1C0(2)_a = D0\, \theta_2 \otimes \beta_2 \otimes S_3$$

$$b1C0(1)_b = 0$$
$$b1C0(2)_b = 0$$

$$c0C0 =$$

$$\begin{array}{c|cccccccccc}
 & & & & & & & j=0 & j=1 & j=2 & j=3 \\
L_3 & 0 & 1 & 2 & 3 & \ldots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
\hline
0 & & \begin{array}{c} D0\otimes \\ S^0{}_1\theta_3\otimes S^0{}_2\otimes \\ \beta_3 \end{array} & & & & & & & & \\
1 & & \begin{array}{c} D0\otimes \\ S^0{}_1\theta_3\otimes S^0{}_2\otimes \\ S^0{}_3\beta_3 \end{array} & \begin{array}{c} D0\otimes \\ S^0{}_1\theta_3\otimes S^0{}_2\otimes \\ S_3 \end{array} & & & & & & & \\
2 & & & \begin{array}{c} D0\otimes \\ S^0{}_1\theta_3\otimes S^0{}_2\otimes \\ S^0{}_3\beta_3 \end{array} & \begin{array}{c} D0\otimes \\ S^0{}_1\theta_3\otimes S^0{}_2\otimes \\ S_3 \end{array} & & & & & & \\
\ddots & & & & & \ddots & & & & & \\
K3 & & & & & & \begin{array}{c} D0\otimes \\ S^0{}_1\theta_3\otimes S^0{}_2\otimes \\ S^0{}_3\beta_3 \end{array} & \begin{array}{c} D0\otimes \\ S^0{}_1\theta_3\otimes S^0{}_2\otimes \\ S_3 \end{array} & & & \\
j=0\ \ K3+1 & & & & & & & \begin{array}{c} D0\otimes \\ S^0{}_1\theta_3\otimes S^0{}_2\otimes \\ S^0{}_3\beta_3 \\ c0C0(1) \end{array} & & & \\
j=3\ \ K3+1 & & & & & & & & & &
\end{array}$$

$$c0C0(1)_a = D0\, \theta_3 \otimes S^0{}_2 \otimes S^0{}_3\, \beta_3$$

$$c0C0(1)_b = D0 \otimes S^0{}_2 \otimes S^0{}_3\, \beta_3$$

$\mathbf{a2C0} =$

$$
\begin{array}{r l}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & 3 & \cdots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
 & \mathrm{DO}\otimes S^0{}_1\theta_3\otimes S^0{}_2\beta_2\otimes\beta_3 & & & & & & & & \\
 & \mathrm{DO}\otimes S^0{}_1\theta_3\otimes S^0{}_2\beta_2\otimes S^0{}_3\beta_3 & \mathrm{DO}\otimes S^0{}_1\theta_3\otimes S^0{}_2\beta_2\otimes S_3 & & & & & & & \\
 & & \mathrm{DO}\otimes S^0{}_1\theta_3\otimes S^0{}_2\beta_2\otimes S^0{}_3\beta_3 & \mathrm{DO}\otimes S^0{}_1\theta_3\otimes S^0{}_2\beta_2\otimes S_3 & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & \mathrm{DO}\otimes S^0{}_1\theta_3\otimes S^0{}_2\beta_2\otimes S^0{}_3\beta_3 & \mathrm{DO}\otimes S^0{}_1\theta_3\otimes S^0{}_2\beta_2\otimes S_3 & & & \\
 & & & & & & \mathrm{DO}\otimes S^0{}_1\theta_3\otimes S^0{}_2\beta_2\otimes S^0{}_3\beta_3 & & & \\
 & & & & & & \mathrm{a2C0(1)} & & &
\end{array} \right]
\end{array}
$$

$\mathbf{a2C0(1)_a = D0}\ \theta_3 \otimes S^0{}_2\ \beta_2 \otimes S^0{}_3\ \beta_3$

$\mathbf{a2C0(1)_b = D0} \otimes S^0{}_2\ \beta_2 \otimes S^0{}_3\ \beta_3$

$\mathbf{a1C0} =$

$$
\begin{array}{r l}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & 3 & \cdots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
\mathrm{DO}\otimes S^0{}_1\theta_2\otimes S^0{}_2\beta_2 & \mathrm{DO}\otimes S^0{}_1\theta_3\otimes S_2\otimes\beta_3 & & & & & & & & \\
\mathrm{DO}\otimes S^0{}_1\theta_2\otimes S^0{}_2\beta_2\otimes S^0{}_3 & \mathrm{DO}\otimes S^0{}_1\theta_2\otimes S^0{}_2\beta_2\otimes S_3 + \mathrm{DO}\otimes S^0{}_1\theta_3\otimes S_2\otimes S^0{}_3\beta_3 & \mathrm{DO}\otimes S^0{}_1\theta_3\otimes S_2\otimes S_3 & & & & & & & \\
 & \mathrm{DO}\otimes S^0{}_1\theta_2\otimes S^0{}_2\beta_2\otimes S^0{}_3\beta_3 & \mathrm{DO}\otimes S^0{}_1\theta_2\otimes S^0{}_2\beta_2\otimes S_3 + \mathrm{DO}\otimes S^0{}_1\theta_3\otimes S_2\otimes S^0{}_3\beta_3 & \mathrm{DO}\otimes S^0{}_1\theta_3\otimes S_2\otimes S_3 & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & \mathrm{DO}\otimes S^0{}_1\theta_2\otimes S^0{}_2\beta_2\otimes S^0{}_3\beta_3 & \mathrm{DO}\otimes S^0{}_1\theta_2\otimes S^0{}_2\beta_2\otimes S_3 + \mathrm{DO}\otimes S^0{}_1\theta_3\otimes S_2\otimes S^0{}_3\beta_3 & & & \\
 & & & & & \mathrm{a1C0(1)} & \mathrm{a1C0(2)} & & &
\end{array} \right]
\end{array}
$$

$\mathbf{a1C0(1)_a = D0}\ \theta_2 \otimes S^0{}_2\ \beta_2 \otimes S^0{}_3\ \beta_3$

$\mathbf{a1C0(2)_a = D0}\ \theta_2 \otimes S^0{}_2\ \beta_2 \otimes \mathbf{S_3} + \mathbf{D0}\ \theta_3 \otimes \mathbf{S_2} \otimes S^0{}_3\ \beta_3$

$\mathbf{a1C0(1)_b = 0}$

$\mathbf{a1C0(2)_b = D0 \otimes S_2 \otimes S^0{}_3\, \beta_3}$

$\mathbf{a0C0 =}$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} 0 & 1 & 2 & \dots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0 \quad K3+1 \\ j=3 \quad K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
\begin{matrix} D0\otimes \\ S^0{}_1\theta_2\otimes S_2 \end{matrix} & & & & & & & & \\
\begin{matrix} D0\otimes \\ S^0{}_1\theta_2\otimes \\ S_2\otimes S^0{}_3 \end{matrix} & \begin{matrix} D0\otimes \\ S^0{}_1\theta_2\otimes \\ S_2\otimes S_3 \end{matrix} & & & & & & & \\
 & \begin{matrix} D0\otimes \\ S^0{}_1\theta_2\otimes \\ S_2\otimes S^0{}_3\beta_3 \end{matrix} & \begin{matrix} D0\otimes \\ S^0{}_1\theta_2\otimes \\ S_2\otimes S_3 \end{matrix} & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & \begin{matrix} D0\otimes \\ S^0{}_1\theta_2\otimes \\ S_2\otimes S^0{}_3\beta_3 \\ a0C0(1) \end{matrix} & \begin{matrix} D0\otimes \\ S^0{}_1\theta_2\otimes \\ S_2\otimes S_3 \\ a0C0(2) \end{matrix} & & & \\
 & & & & & & & &
\end{array}\right]
\end{array}
$$

$\mathbf{a0C0(1)_a = D0\, \theta_2 \otimes S_2 \otimes S^0{}_3\, \beta_3}$

$\mathbf{a0C0(2)_a = D0\, \theta_2 \otimes S_2 \otimes S_3}$

$\mathbf{a0C0(1)_b = 0}$

$\mathbf{a0C0(2)_a = 0}$

$\mathbf{b3C0_a =}$

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & \dots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=2 \quad K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
 & \begin{matrix} D0\theta_3\otimes \\ S^0{}_2\beta_2\otimes\beta_3 \end{matrix} & & & & & & & & \\
 & \begin{matrix} D0\theta_3\otimes \\ S^0{}_2\beta_2\otimes S^0{}_3 \end{matrix} & \begin{matrix} D0\theta_3\otimes \\ S^0{}_2\beta_2\otimes S_3 \end{matrix} & & & & & & & \\
 & & \begin{matrix} D0\theta_3\otimes \\ S^0{}_2\beta_2\otimes S^0{}_3\beta_3 \end{matrix} & \begin{matrix} D0\theta_3\otimes \\ S^0{}_2\beta_2\otimes S_3 \end{matrix} & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & \begin{matrix} D0\theta_3\otimes \\ S^0{}_2\beta_2\otimes S^0{}_3\beta_3 \end{matrix} & \begin{matrix} D0\theta_3\otimes \\ S^0{}_2\beta_2\otimes S_3 \end{matrix} & & & \\
 & & & & & & \begin{matrix} D0\theta_3\otimes \\ S^0{}_2\beta_2\otimes S^0{}_3\beta_3 \end{matrix} & & &
\end{array}\right]
\end{array}
$$

$\mathbf{b3C0_b = 0}$

$$\mathbf{b4C0_a} =$$

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & 3 & \cdots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=2\;\; K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
\begin{array}{c} DO\theta_2\otimes \\ S^0_2\beta_2 \end{array} & \begin{array}{c} DO\theta_3\otimes \\ S_2\otimes\beta_3 \end{array} & & & & & & & & \\
\begin{array}{c} DO\theta_2\otimes \\ S^0_2\beta_2\otimes S^0_3 \end{array} & \begin{array}{c} DO\theta_2\otimes \\ S^0_2\beta_2\otimes S_3 \\ +\, DO\theta_3\otimes \\ S_2\otimes S^0_3\beta_3 \end{array} & \begin{array}{c} DO\theta_3\otimes \\ S_2\otimes S_3 \end{array} & & & & & & & \\
 & \begin{array}{c} DO\theta_2\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3 \end{array} & \begin{array}{c} DO\theta_2\otimes \\ S^0_2\beta_2\otimes S_3 \\ +\, DO\theta_3\otimes \\ S_2\otimes S^0_3\beta_3 \end{array} & \begin{array}{c} DO\theta_3\otimes \\ S_2\otimes S_3 \end{array} & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & \begin{array}{c} DO\theta_2\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3 \end{array} & \begin{array}{c} DO\theta_2\otimes \\ S^0_2\beta_2\otimes S_3 \\ +\, DO\theta_3\otimes \\ S_2\otimes S^0_3\beta_3 \end{array} & & &
\end{array}\right]
\end{array}
$$

$$\mathbf{b4C0_b} =$$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \cdots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=2\;\; K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
DO\otimes S^0_2\beta_2 & & & & & & & & \\
\begin{array}{c} DO\otimes \\ S^0_2\beta_2\otimes S^0_3 \end{array} & \begin{array}{c} DO\otimes \\ S^0_2\beta_2\otimes S_3 \end{array} & & & & & & & \\
 & \begin{array}{c} DO\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3 \end{array} & \begin{array}{c} DO\otimes \\ S^0_2\beta_2\otimes S_3 \end{array} & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & \begin{array}{c} DO\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3 \end{array} & \begin{array}{c} DO\otimes \\ S^0_2\beta_2\otimes S_3 \end{array} & & &
\end{array}\right]
\end{array}
$$

$$\mathbf{A2} =$$

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} & & & & & & j=0 & j=2 \\ 0 & 1 & 2 & 3 & \cdots & K2 & K2+1 & K2+1 \end{array} \\
\begin{array}{c} L_2 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0\;\; K2+1 \\ j=2\;\; K2+1 \end{array} &
\left[\begin{array}{cccccccc}
b0A2 & b1A2 & & & & & & \\
c0A2 & a1A2 & a0A2 & & & & & \\
 & a2A2 & a1A2 & a0A2 & & & & \\
 & & & & \ddots & & & \\
 & & & & & a2A2 & a1A2 & \\
 & & & & & b3A2 & b4A2 &
\end{array}\right]
\end{array}
$$

b0A2 =

$$\begin{array}{c|cccccccccc|}
L_3 & 0 & 1 & 2 & 3 & \cdots & K3 & \begin{matrix} j=0 \\ K3+1 \end{matrix} & \begin{matrix} j=1 \\ K3+1 \end{matrix} & \begin{matrix} j=2 \\ K3+1 \end{matrix} & \begin{matrix} j=3 \\ K3+1 \end{matrix} \\
\hline
0 & & \begin{matrix} D0\otimes \\ S^0_1\beta_1\theta_3\otimes\beta_3 \end{matrix} & & & & & & & & \\
1 & & \begin{matrix} D0\otimes \\ S^0_1\beta_1\theta_3\otimes \\ S^0_3\beta_3 \end{matrix} & \begin{matrix} D0\otimes \\ S^0_1\beta_1\theta_3\otimes S_3 \end{matrix} & & & & & & & \\
2 & & & \begin{matrix} D0\otimes \\ S^0_1\beta_1\theta_3\otimes \\ S^0_3\beta_3 \end{matrix} & \begin{matrix} D0\otimes \\ S^0_1\beta_1\theta_3\otimes S_3 \end{matrix} & & & & & & \\
\ddots & & & & & \ddots & & & & & \\
K3 & & & & & & \begin{matrix} D0\otimes \\ S^0_1\beta_1\theta_3\otimes \\ S^0_3\beta_3 \end{matrix} & \begin{matrix} D0\otimes \\ S^0_1\beta_1\theta_3\otimes S_3 \end{matrix} & & & \\
\begin{matrix} j=0 \;\; K3+1 \\ \\ j=3 \;\; K3+1 \end{matrix} & & & & & & & \begin{matrix} D0\otimes \\ S^0_1\beta_1\theta_3\otimes \\ S^0_3\beta_3 \\ b0A2(1) \end{matrix} & & & \\
\hline
\end{array}$$

$$\mathbf{b0A2(1)_a = D0 \otimes \beta_1\, \theta_3 \otimes S^0{}_3\, \beta_3}$$

$$\mathbf{b0A2(1)_b = D0 \otimes \beta_1 \otimes S^0{}_3\, \beta_3}$$

b1A2 =

$$\begin{array}{c|ccccccccc|}
L_3 & 0 & 1 & 2 & \cdots & K3 & \begin{matrix} j=0 \\ K3+1 \end{matrix} & \begin{matrix} j=1 \\ K3+1 \end{matrix} & \begin{matrix} j=2 \\ K3+1 \end{matrix} & \begin{matrix} j=3 \\ K3+1 \end{matrix} \\
\hline
0 & \begin{matrix} D0\otimes \\ S^0_1\beta_1\theta_2\otimes\beta_2 \end{matrix} & & & & & & & & \\
1 & \begin{matrix} D0\otimes \\ S^0_1\beta_1\theta_2\otimes \\ \beta_2\otimes S^0_3 \end{matrix} & \begin{matrix} D0\otimes \\ S^0_1\beta_1\theta_2\otimes \\ \beta_2\otimes S_3 \end{matrix} & & & & & & & \\
2 & & \begin{matrix} D0\otimes \\ S^0_1\beta_1\theta_2\otimes \\ \beta_2\otimes S^0_3\beta_3 \end{matrix} & \begin{matrix} D0\otimes \\ S^0_1\beta_1\theta_2\otimes \\ \beta_2\otimes S_3 \end{matrix} & & & & & & \\
\ddots & & & & \ddots & & & & & \\
\begin{matrix} j=0 \;\; K3+1 \\ \\ j=3 \;\; K3+1 \end{matrix} & & & & & \begin{matrix} D0\otimes \\ S^0_1\beta_1\theta_2\otimes \\ \beta_2\otimes S^0_3\beta_3 \\ b1A2(1) \end{matrix} & \begin{matrix} D0\otimes \\ S^0_1\beta_1\theta_2\otimes \\ \beta_2\otimes S_3 \\ b1A2(2) \end{matrix} & & & \\
\hline
\end{array}$$

$$\mathbf{b1A2(1)_a = D0 \otimes\beta_1\, \theta_2 \otimes \beta_2 \otimes S^0{}_3\, \beta_3}$$
$$\mathbf{b1A2(2)_a = D0 \otimes\beta_1\, \theta_2 \otimes \beta_2 \otimes S_3}$$

$$\mathbf{b1A2(1)_b = 0}$$
$$\mathbf{b1A2(2)_a = 0}$$

c0A2 =

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & \cdots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
 & D0\otimes S^0_1\beta_1\theta_3\otimes S^0_2\otimes\beta_3 & & & & & & & & \\
 & D0\otimes S^0_1\beta_1\theta_3\otimes S^0_2\otimes S^0_3\beta_3 & D0\otimes S^0_1\beta_1\theta_3\otimes S^0_2\otimes S_3 & & & & & & & \\
 & & D0\otimes S^0_1\beta_1\theta_3\otimes S^0_2\otimes S^0_3\beta_3 & D0\otimes S^0_1\beta_1\theta_3\otimes S^0_2\otimes S_3 & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & D0\otimes S^0_1\beta_1\theta_3\otimes S^0_2\otimes S^0_3\beta_3 & D0\otimes S^0_1\beta_1\theta_3\otimes S^0_2\otimes S_3 & & & \\
 & & & & & & D0\otimes S^0_1\beta_1\theta_3\otimes S^0_2\otimes S^0_3\beta_3 & & & \\
 & & & & & & c0A2(1) & & &
\end{array}\right]
\end{array}
$$

$$\mathbf{c0A2(1)_a = D0 \otimes \beta_1\,\theta_3 \otimes S^0{}_2 \otimes S^0{}_3\,\beta_3}$$

$$\mathbf{c0A2(1)_b = D0 \otimes \beta_1 \otimes S^0{}_2 \otimes S^0{}_3\,\beta_3}$$

a2A2 =

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & \cdots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
 & D0\otimes S^0_1\beta_1\theta_3\otimes S^0_2\beta_2\otimes\beta_3 & & & & & & & & \\
 & D0\otimes S^0_1\beta_1\theta_3\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & D0\otimes S^0_1\beta_1\theta_3\otimes S^0_2\beta_2\otimes S_3 & & & & & & & \\
 & & D0\otimes S^0_1\beta_1\theta_3\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & D0\otimes S^0_1\beta_1\theta_3\otimes S^0_2\beta_2\otimes S_3 & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & D0\otimes S^0_1\beta_1\theta_3\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & D0\otimes S^0_1\beta_1\theta_3\otimes S^0_2\beta_2\otimes S_3 & & & \\
 & & & & & & D0\otimes S^0_1\beta_1\theta_3\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & & & \\
 & & & & & & a2A2(1) & & &
\end{array}\right]
\end{array}
$$

$$\mathbf{a2A2(1)_a = D0 \otimes \beta_1\,\theta_3 \otimes S^0{}_2\,\beta_2 \otimes S^0{}_3\,\beta_3}$$

$$\mathbf{a2A2(1)_b = D0 \otimes \beta_1 \otimes S^0{}_2\,\beta_2 \otimes S^0{}_3\,\beta_3}$$

a1A2 =

$$
\begin{array}{c|cccccccccc|}
 & & & & & & & j=0 & j=1 & j=2 & j=3 \\
L_3 & 0 & 1 & 2 & 3 & \dots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
0 & \substack{D0\otimes \\ S^0_1\beta_1\theta_2\otimes \\ S^0_2\beta_2} & \substack{D0\otimes \\ S^0_1\beta_1\theta_3\otimes \\ S_2\otimes\beta_3} & & & & & & & & \\
1 & \substack{D0\otimes \\ S^0_1\beta_1\theta_2\otimes \\ S^0_2\beta_2\otimes S^0_3} & \substack{D0\otimes \\ S^0_1\beta_1\theta_2\otimes \\ S^0_2\beta_2\otimes S_3 \\ +\ D0\otimes \\ S^0_1\beta_1\theta_3\otimes \\ S_2\otimes S^0_3\beta_3} & \substack{D0\otimes \\ S^0_1\beta_1\theta_3\otimes \\ S_2\otimes S_3} & & & & & & & \\
2 & & \substack{D0\otimes \\ S^0_1\beta_1\theta_2\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3} & \substack{D0\otimes \\ S^0_1\beta_1\theta_2\otimes \\ S^0_2\beta_2\otimes S_3 \\ +\ D0\otimes \\ S^0_1\beta_1\theta_3\otimes \\ S_2\otimes S^0_3\beta_3} & \substack{D0\otimes \\ S^0_1\beta_1\theta_3\otimes \\ S_2\otimes S_3} & & & & & & \\
\ddots & & & & & \ddots & & & & & \\
j=0\quad K3+1 & & & & & & \substack{D0\otimes \\ S^0_1\beta_1\theta_2\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3} & \substack{D0\otimes \\ S^0_1\beta_1\theta_2\otimes \\ S^0_2\beta_2\otimes S_3 \\ +\ D0\otimes \\ S^0_1\beta_1\theta_3\otimes \\ S_2\otimes S^0_3\beta_3} & & & \\
j=3\quad K3+1 & & & & & & a1A2(1) & a1A2(2) & & &
\end{array}
$$

$$\mathbf{a1A2(1)_a} = \mathbf{D0} \otimes \beta_1\, \theta_2 \otimes S^0{}_2\, \beta_2 \otimes S^0{}_3\, \beta_3$$
$$\mathbf{a1A2(2)_a} = \mathbf{D0} \otimes \beta_1\, \theta_2 \otimes S^0{}_2\, \beta_2 \otimes \mathbf{S_3} + \mathbf{D0} \otimes \beta_1\, \theta_3 \otimes \mathbf{S_2} \otimes S^0{}_3\, \beta_3$$

$$\mathbf{a1A2(1)_b} = 0$$
$$\mathbf{a1A2(2)_b} = \mathbf{D0} \otimes \beta_1 \otimes \mathbf{S_2} \otimes S^0{}_3\, \beta_3$$

a0A2 =

$$
\begin{array}{c|ccccccccc|}
 & & & & & & j=0 & j=1 & j=2 & j=3 \\
L_3 & 0 & 1 & 2 & \dots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
0 & \substack{D0\otimes \\ S^0_1\beta_1\theta_2\otimes S_2} & & & & & & & & \\
1 & \substack{D0\otimes \\ S^0_1\beta_1\theta_2\otimes \\ S_2\otimes S^0_3} & \substack{D0\otimes \\ S^0_1\beta_1\theta_2\otimes \\ S_2\otimes S_3} & & & & & & & \\
2 & & \substack{D0\otimes \\ S^0_1\beta_1\theta_2\otimes \\ S_2\otimes S^0_3\beta_3} & \substack{D0\otimes \\ S^0_1\beta_1\theta_2\otimes \\ S_2\otimes S_3} & & & & & & \\
. & & & & . & & & & & \\
j=0\quad K3+1 & & & & & \substack{D0\otimes \\ S^0_1\beta_1\theta_2\otimes \\ S_2\otimes S^0_3\beta_3} & \substack{D0\otimes \\ S^0_1\beta_1\theta_2\otimes \\ S_2\otimes S_3} & & & \\
j=3\quad K3+1 & & & & & \substack{D0\otimes\beta_1\theta_2\otimes \\ S_2\otimes S^0_3\beta_3} & \substack{D0\otimes\beta_1\theta_2\otimes \\ S_2\otimes S_3} & & &
\end{array}
$$

$$\mathbf{a0A2(1)_a} = D0 \otimes \beta_1\, \theta_2 \otimes \mathbf{S_2} \otimes S^0{}_3\, \beta_3$$
$$\mathbf{a0A2(2)_a} = \mathbf{D0} \otimes \beta_1\, \theta_2 \otimes \mathbf{S_2} \otimes \mathbf{S_3}$$

$$\mathbf{a0A2(1)_b} = 0$$

$$\mathbf{a0A2(2)_b} = 0$$

$$\mathbf{b3A2_a} =$$

$$\begin{array}{cc}
L_3 & \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & \dots & K3 & \substack{j=0 \\ K3+1} & \substack{j=1 \\ K3+1} & \substack{j=2 \\ K3+1} & \substack{j=3 \\ K3+1} \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=2\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
 & \substack{\mathrm{DO}\otimes\beta_1\theta_3\otimes \\ S^0_2\beta_2\otimes\beta_3} & & & & & & & & \\
 & \substack{\mathrm{DO}\otimes\beta_1\theta_3\otimes \\ S^0_2\beta_2\otimes S^0_3} & \substack{\mathrm{DO}\otimes\beta_1\theta_3\otimes \\ S^0_2\beta_2\otimes S_3} & & & & & & & \\
 & & \substack{\mathrm{DO}\otimes\beta_1\theta_3\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3} & \substack{\mathrm{DO}\otimes\beta_1\theta_3\otimes \\ S^0_2\beta_2\otimes S_3} & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & \substack{\mathrm{DO}\otimes\beta_1\theta_3\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3} & \substack{\mathrm{DO}\otimes\beta_1\theta_3\otimes \\ S^0_2\beta_2\otimes S_3} & & & \\
 & & & & & & \substack{\mathrm{DO}\otimes\beta_1\theta_3\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3} & & &
\end{array}\right]
\end{array}$$

$$\mathbf{b3A2_b} = 0$$

$$\mathbf{b4A2_a} =$$

$$\begin{array}{cc}
L_3 & \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & \dots & K3 & \substack{j=0 \\ K3+1} & \substack{j=1 \\ K3+1} & \substack{j=2 \\ K3+1} & \substack{j=3 \\ K3+1} \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \ddots \\ j=2\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
\substack{\mathrm{DO}\otimes\beta_1\theta_2\otimes \\ S^0_2\beta_2} & \substack{\mathrm{DO}\otimes\beta_1\theta_3\otimes \\ S_2\otimes\beta_3} & & & & & & & & \\
\substack{\mathrm{DO}\otimes\beta_1\theta_2\otimes \\ S^0_2\beta_2\otimes S^0_3} & \substack{\mathrm{DO}\otimes\beta_1\theta_2\otimes \\ S^0_2\beta_2\otimes S_3 \\ + \\ \mathrm{DO}\otimes\beta_1\theta_3\otimes \\ S_2\otimes S^0_3\beta_3} & \substack{\mathrm{DO}\otimes\beta_1\theta_3\otimes \\ S_2\otimes S_3} & & & & & & & \\
 & \substack{\mathrm{DO}\otimes\beta_1\theta_2\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3} & \substack{\mathrm{DO}\otimes\beta_1\theta_2\otimes \\ S^0_2\beta_2\otimes S_3 \\ + \\ \mathrm{DO}\otimes\beta_1\theta_3\otimes \\ S_2\otimes S^0_3\beta_3} & \substack{\mathrm{DO}\otimes\beta_1\theta_3\otimes \\ S_2\otimes S_3} & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & \substack{\mathrm{DO}\otimes\beta_1\theta_2\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3} & \substack{\mathrm{DO}\otimes\beta_1\theta_2\otimes \\ S^0_2\beta_2\otimes S_3 \\ + \\ \mathrm{DO}\otimes\beta_1\theta_3\otimes \\ S_2\otimes S^0_3\beta_3} & & &
\end{array}\right]
\end{array}$$

$\mathbf{b4A2_b} =$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & \ldots & K3 & \begin{array}{c} j=0 \\ K3+1 \end{array} & \begin{array}{c} j=1 \\ K3+1 \end{array} & \begin{array}{c} j=2 \\ K3+1 \end{array} & \begin{array}{c} j=3 \\ K3+1 \end{array} \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=2 \quad K3+1 \end{array} &
\left[\begin{array}{ccccccccccc}
DO\otimes\beta_1\otimes S^0{}_2\beta_2 & & & & & & & & & \\
DO\otimes\beta_1\otimes S^0{}_2\beta_2\otimes S^0{}_3 & DO\otimes\beta_1\otimes S^0{}_2\beta_2\otimes S_3 & & & & & & & & \\
 & DO\otimes\beta_1\otimes S^0{}_2\beta_2\otimes S^0{}_3\beta_3 & DO\otimes\beta_1\otimes S^0{}_2\beta_2\otimes S_3 & & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & DO\otimes\beta_1\otimes S^0{}_2\beta_2\otimes S^0{}_3\beta_3 & DO\otimes\beta_1\otimes S^0{}_2\beta_2\otimes S_3 & & &
\end{array}\right]
\end{array}
$$

$\mathbf{A1} =$

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} 0 & 1 & 2 & 3 & \ldots & K2 & \begin{array}{c} j=0 \\ K2+1 \end{array} & \begin{array}{c} j=2 \\ K2+1 \end{array} \end{array} \\
\begin{array}{c} L_2 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0 \quad K2+1 \\ j=2 \quad K2+1 \end{array} &
\left[\begin{array}{cccccccc}
\mathbf{b0A1} & \mathbf{b1A1} & & & & & & \\
\mathbf{c0A1} & \mathbf{a1A1} & \mathbf{a0A1} & & & & & \\
 & \mathbf{a2A1} & \mathbf{a1A1} & \mathbf{a0A1} & & & & \\
 & & & & \ddots & & & \\
 & & & & & \mathbf{a2A1} & \mathbf{a1A1} & \mathbf{b2A1} \\
 & & & & & \mathbf{b3A1} & \mathbf{b4A1} & \mathbf{b5A1}
\end{array}\right]
\end{array}
$$

$\mathbf{b0A1} =$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & \ldots & K3 & \begin{array}{c} j=0 \\ K3+1 \end{array} & \begin{array}{c} j=1 \\ K3+1 \end{array} & \begin{array}{c} j=2 \\ K3+1 \end{array} & \begin{array}{c} j=3 \\ K3+1 \end{array} \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0 \quad K3+1 \\ j=3 \quad K3+1 \end{array} &
\left[\begin{array}{ccccccccccc}
D0\otimes S_1 & D1\otimes S^0{}_1\beta_1\theta_3\otimes\beta_3 & & & & & & & & \\
D0\otimes S_1\otimes S^0{}_3 & D0\otimes S_1\otimes S_3 + D1\otimes S^0{}_1\beta_1\theta_3\otimes S^0{}_3\beta_3 & D1\otimes S^0{}_1\beta_1\theta_3\otimes S_3 & & & & & & & \\
 & D0\otimes S_1\otimes S^0{}_3\beta_3 & D0\otimes S_1\otimes S_3 + D1\otimes S^0{}_1\beta_1\theta_3\otimes S^0{}_3\beta_3 & D1\otimes S^0{}_1\beta_1\theta_3\otimes S_3 & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & D0\otimes S_1\otimes S^0{}_3\beta_3 & D0\otimes S_1\otimes S_3 + D1\otimes S^0{}_1\beta_1\theta_3\otimes S^0{}_3\beta_3 & & & D0\otimes S^0{}_1\theta_3\otimes S_3 \\
 & & & & & & \mathrm{b0A1(1)} & & & \mathrm{b0A1(2)}
\end{array}\right]
\end{array}
$$

$\mathbf{b0A1(1)_a = D1 \otimes \beta_1\, \theta_3 \otimes S^0{}_3\, \beta_3}$
$\mathbf{b0A1(2)_a = D0\, \theta_3 \otimes S_3}$

$\mathbf{b0A1(1)_b = D1 \otimes \beta_1 \otimes S^0{}_3\, \beta_3}$
$\mathbf{b0A1(2)_b = D0 \otimes S_3}$

b1A1 =

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \cdots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
D1\otimes S^0{}_1\beta_1\theta_2\otimes\beta_2 & & & & & & & & \\
D1\otimes S^0{}_1\beta_1\theta_2\otimes\beta_2\otimes S^0{}_3 & D1\otimes S^0{}_1\beta_1\theta_2\otimes\beta_2\otimes S_3 & & & & & & & \\
 & D1\otimes S^0{}_1\beta_1\theta_2\otimes\beta_2\otimes S^0{}_3\beta_3 & D1\otimes S^0{}_1\beta_1\theta_2\otimes\beta_2\otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & D1\otimes S^0{}_1\beta_1\theta_2\otimes\beta_2\otimes S^0{}_3\beta_3 & D1\otimes S^0{}_1\beta_1\theta_2\otimes\beta_2\otimes S_3 & & & \\
 & & & & b1A1(1) & b1A1(2) & & &
\end{array}\right]
\end{array}
$$

$$b1A1(1)_a = D1 \otimes \beta_1\,\theta_2 \otimes \beta_2 \otimes S^0{}_3\,\beta_3$$
$$b1A1(2)_a = D1 \otimes \beta_1\,\theta_2 \otimes \beta_2 \otimes S_3$$

$$b1A1(1)_b = 0$$
$$b1A1(2)_b = 0$$

c0A1 =

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & 3 & \cdots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
D0\otimes S_1\otimes S^0{}_2 & D1\otimes S^0{}_1\beta_1\theta_3\otimes S^0{}_2\otimes\beta_3 & & & & & & & & \\
D0\otimes S_1\otimes S^0{}_2\otimes S^0{}_3 & D0\otimes S_1\otimes S^0{}_2\otimes S_3 + D1\otimes S^0{}_1\beta_1\theta_3\otimes S^0{}_2\otimes S^0{}_3\beta_3 & D1\otimes S^0{}_1\beta_1\theta_3\otimes S^0{}_2\otimes S_3 & & & & & & & \\
 & D0\otimes S_1\otimes S^0{}_2\otimes S^0{}_3\beta_3 & D0\otimes S_1\otimes S^0{}_2\otimes S_3 + D1\otimes S^0{}_1\beta_1\theta_3\otimes S^0{}_2\otimes S^0{}_3\beta_3 & D1\otimes S^0{}_1\beta_1\theta_3\otimes S^0{}_2\otimes S_3 & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & D0\otimes S_1\otimes S^0{}_2\otimes S^0{}_3\beta_3 & D0\otimes S_1\otimes S^0{}_2\otimes S_3 + D1\otimes S^0{}_1\beta_1\theta_3\otimes S^0{}_2\otimes S^0{}_3\beta_3 & & & D0\otimes S^0{}_1\theta_3\otimes S^0{}_2\otimes S_3 \\
 & & & & & & c0A1(1) & & & c0A1(2)
\end{array}\right]
\end{array}
$$

$$c0A1(1)_a = D1 \otimes \beta_1\,\theta_3 \otimes S^0{}_2 \otimes S^0{}_3\,\beta_3$$
$$c0A1(2)_a = D0\,\theta_3 \otimes S^0{}_2 \otimes S_3$$

$$c0A1(1)_b = D1 \otimes \beta_1 \otimes S^0{}_2 \otimes S^0{}_3\,\beta_3$$
$$c0A1(2)_b = D0 \otimes S^0{}_2 \otimes S_3$$

a2A1 =

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc}
 & & & & & & j=0 & j=1 & j=2 & j=3 \\
0 & 1 & 2 & 3 & \ldots & K3 & K3+1 & K3+1 & K3+1 & K3+1
\end{array} \\
\begin{array}{c}
L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0 \;\; K3+1 \\ j=3 \;\; K3+1
\end{array}
&
\left[\begin{array}{cccccccccc}
\begin{array}{c}D0\otimes S_1\otimes\\ S^0{}_2\beta_2\end{array} & \begin{array}{c}D1\otimes\\ S^0{}_1\beta_1\theta_3\otimes\\ S^0{}_2\beta_2\otimes\beta_3\end{array} & & & & & & & & \\
\begin{array}{c}D0\otimes S_1\otimes\\ S^0{}_2\beta_2\otimes S^0{}_3\end{array} & \begin{array}{c}D0\otimes S_1\otimes\\ S^0{}_2\beta_2\otimes S_3\\ +D1\otimes\\ S^0{}_1\beta_1\theta_3\otimes\\ S^0{}_2\beta_2\otimes S^0{}_3\beta_3\end{array} & \begin{array}{c}D1\otimes\\ S^0{}_1\beta_1\theta_3\otimes\\ S^0{}_2\beta_2\otimes S_3\end{array} & & & & & & & \\
 & \begin{array}{c}D0\otimes S_1\otimes\\ S^0{}_2\beta_2\otimes S^0{}_3\beta_3\end{array} & \begin{array}{c}D0\otimes S_1\otimes\\ S^0{}_2\beta_2\otimes S_3\\ +D1\otimes\\ S^0{}_1\beta_1\theta_3\otimes\\ S^0{}_2\beta_2\otimes S^0{}_3\beta_3\end{array} & \begin{array}{c}D1\otimes\\ S^0{}_1\beta_1\theta_3\otimes\\ S^0{}_2\beta_2\otimes S_3\end{array} & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & \begin{array}{c}D0\otimes S_1\otimes\\ S^0{}_2\beta_2\otimes S^0{}_3\beta_3\end{array} & \begin{array}{c}D0\otimes S_1\otimes\\ S^0{}_2\beta_2\otimes S_3\\ +D1\otimes\\ S^0{}_1\beta_1\theta_3\otimes\\ S^0{}_2\beta_2\otimes S^0{}_3\beta_3\end{array} & & & \begin{array}{c}D0\otimes\\ S^0{}_1\theta_3\otimes\\ S^0{}_2\beta_2\otimes S_3\end{array} \\
 & & & & & & a2A1(1) & & & a2A1(2)
\end{array}\right]
\end{array}
$$

$$\mathbf{a2A1(1)_a = D1 \otimes \beta_1\, \theta_3 \otimes S^0{}_2\, \beta_2 \otimes S^0{}_3\, \beta_3}$$
$$\mathbf{a2A1(2)_a = D0\; \theta_3 \otimes S^0{}_2\, \beta_2 \otimes S_3}$$

$$\mathbf{a2A1(1)_b = D1 \otimes \beta_1 \otimes S^0{}_2\, \beta_2 \otimes S^0{}_3\, \beta_3}$$
$$\mathbf{a2A1(2)_b = D0 \otimes S^0{}_2\, \beta_2 \otimes S_3}$$

a1A1 =

$$
\begin{array}{r|c|c|c|c|c|c|c|c|c|c}
 & & & & & & & j=0 & j=1 & j=2 & j=3 \\
L_3 & 0 & 1 & 2 & 3 & \ldots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
\hline
0 & \begin{matrix} D0\otimes S_1\otimes \\ S_2 + \\ D1\otimes \\ S^0{}_1\beta_1\theta_2\otimes \\ S^0{}_2\beta_2 \end{matrix} & \begin{matrix} D1\otimes \\ S^0{}_1\beta_1\theta_3\otimes \\ S_2\otimes\beta_3 \end{matrix} & & & & & & & & \\
1 & \begin{matrix} D0\otimes S_1\otimes \\ S_2\otimes S^0{}_3 \\ + D1\otimes \\ S^0{}_1\beta_1\theta_2\otimes \\ S^0{}_2\beta_2\otimes \\ S^0{}_3 \end{matrix} & \begin{matrix} D0\otimes S_1\otimes \\ S\otimes S_3 \\ + D1\otimes \\ S^0{}_1\beta_1\theta_2\otimes \\ S^0{}_2\beta_2\otimes S_3 \\ + D1\otimes \\ S^0{}_1\beta_1\theta_3\otimes \\ S_2\otimes S^0{}_3\beta_3 \end{matrix} & \begin{matrix} D1\otimes \\ S^0{}_1\beta_1\theta_3\otimes \\ S_2\otimes S_3 \end{matrix} & & & & & & & \\
2 & & \begin{matrix} D0\otimes S_1\otimes \\ S_2\otimes S^0{}_3\beta_3 \\ + D1\otimes \\ S^0{}_1\beta_1\theta_2\otimes \\ S^0{}_2\beta_2\otimes S^0{}_3\beta_3 \end{matrix} & \begin{matrix} D0\otimes S_1\otimes \\ S\otimes S_3 \\ + D1\otimes \\ S^0{}_1\beta_1\theta_2\otimes \\ S^0{}_2\beta_2\otimes S_3 \\ + D1\otimes \\ S^0{}_1\beta_1\theta_3\otimes \\ S_2\otimes S^0{}_3\beta_3 \end{matrix} & \begin{matrix} D1\otimes \\ S^0{}_1\beta_1\theta_3\otimes \\ S_2\otimes S_3 \end{matrix} & & & & & & \\
\ddots & & & & & \ddots & & & & & \\
j=0\ \ K3+1 & & & & & & \begin{matrix} D0\otimes S_1\otimes \\ S_2\otimes S^0{}_3\beta_3 \\ + D1\otimes \\ S^0{}_1\beta_1\theta_2\otimes \\ S^0{}_2\beta_2\otimes S^0{}_3\beta_3 \end{matrix} & \begin{matrix} D0\otimes S_1\otimes \\ S\otimes S_3 \\ + D1\otimes \\ S^0{}_1\beta_1\theta_2\otimes \\ S^0{}_2\beta_2\otimes S_3 \\ + D1\otimes \\ S^0{}_1\beta_1\theta_3\otimes \\ S_2\otimes S^0{}_3\beta_3 \end{matrix} & & & \begin{matrix} D0\otimes \\ S^0{}_1\theta_3\otimes \\ S_2\otimes S_3 \end{matrix} \\
j=3\ \ K3+1 & & & & & & a1A1(1) & a1A1(2) & & & a1A1(3)
\end{array}
$$

$$\mathbf{a1A1(1)_a = D1 \otimes \beta_1\, \theta_2 \otimes S^0{}_2\, \beta_2 \otimes S^0{}_3\, \beta_3}$$
$$\mathbf{a1A1(2)_a = D1 \otimes \beta_1\, \theta_2 \otimes S^0{}_2\, \beta_2 \otimes S_3 + D1 \otimes \beta_1\, \theta_3 \otimes S_2 \otimes S^0{}_3\, \beta_3}$$
$$\mathbf{a1A1(3)_a = D0\, \theta_3 \otimes S_2 \otimes S_3}$$

$$\mathbf{a1A1(1)_b = 0}$$
$$\mathbf{a1A1(2)_b = D1 \otimes \beta_1 \otimes S_2 \otimes S^0{}_3\, \beta_3}$$
$$\mathbf{a1A1(3)_b = D0 \otimes S_2 \otimes S_3}$$

$\mathbf{a0A1} =$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \cdots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
\begin{matrix} D1\otimes \\ S^0_1\beta_1\theta_2\otimes S_2 \end{matrix} & & & & & & & & \\
\begin{matrix} D1\otimes \\ S^0_1\beta_1\theta_2\otimes \\ S_2\otimes S^0_3 \end{matrix} & \begin{matrix} D1\otimes \\ S^0_1\beta_1\theta_2\otimes \\ S_2\otimes S_3 \end{matrix} & & & & & & & \\
 & \begin{matrix} D1\otimes \\ S^0_1\beta_1\theta_2\otimes \\ S_2\otimes S^0_3\beta_3 \end{matrix} & \begin{matrix} D1\otimes \\ S^0_1\beta_1\theta_2\otimes \\ S_2\otimes S_3 \end{matrix} & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & \begin{matrix} D1\otimes \\ S^0_1\beta_1\theta_2\otimes \\ S_2\otimes S^0_3\beta_3 \\ a0A1(1) \end{matrix} & \begin{matrix} D1\otimes \\ S^0_1\beta_1\theta_2\otimes \\ S_2\otimes S_3 \\ a0A1(2) \end{matrix} & & &
\end{array}\right]
\end{array}
$$

$$\mathbf{a0A1(1)_a = D1 \otimes \beta_1\, \theta_2 \otimes S_2 \otimes S^0{}_3\, \beta_3}$$
$$\mathbf{a0A1(2)_a = D1 \otimes \beta_1\, \theta_2 \otimes S_2 \otimes S_3}$$

$$\mathbf{a0A1(1)_b = 0}$$
$$\mathbf{a0A1(2)_b = 0}$$

$\mathbf{b2A1} =$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \cdots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
\begin{matrix} D0\otimes \\ S^0_1\theta_2\otimes S_2 \end{matrix} & & & & & & & & \\
\begin{matrix} D0\otimes \\ S^0_1\theta_2\otimes \\ S_2\otimes S^0_3 \end{matrix} & \begin{matrix} D0\otimes \\ S^0_1\theta_2\otimes \\ S_2\otimes S_3 \end{matrix} & & & & & & & \\
 & \begin{matrix} D0\otimes \\ S^0_1\theta_2\otimes \\ S_2\otimes S^0_3\beta_3 \end{matrix} & \begin{matrix} D0\otimes \\ S^0_1\theta_2\otimes \\ S_2\otimes S_3 \end{matrix} & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & \begin{matrix} D0\otimes \\ S^0_1\theta_2\otimes \\ S_2\otimes S^0_3\beta_3 \\ b2A1(1) \end{matrix} & & & \begin{matrix} D0\otimes \\ S^0_1\theta_2\otimes \\ S_2\otimes S_3 \\ b2A1(2) \end{matrix} &
\end{array}\right]
\end{array}
$$

$$\mathbf{b2A1(1)_a = D0\, \theta_2 \otimes S_2 \otimes S^0{}_3\, \beta_3}$$
$$\mathbf{b2A1(2)_a = D0\, \theta_2 \otimes S_2 \otimes S_3}$$

$$\mathbf{b2A1(1)_b = 0}$$
$$\mathbf{b2A1(2)_b = 0}$$

$\mathbf{b3A1_a} =$

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & \cdots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=2\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
 & \begin{array}{c} D1\otimes\beta_1\theta_3\otimes \\ S^0_2\beta_2\otimes\beta_3 \end{array} & & & & & & & & \\
 & \begin{array}{c} D1\otimes\beta_1\theta_3\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3 \end{array} & \begin{array}{c} D1\otimes\beta_1\theta_3\otimes \\ S^0_2\beta_2\otimes S_3 \end{array} & & & & & & & \\
 & & \begin{array}{c} D1\otimes\beta_1\theta_3\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3 \end{array} & \begin{array}{c} D1\otimes\beta_1\theta_3\otimes \\ S^0_2\beta_2\otimes S_3 \end{array} & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & & \begin{array}{c} D1\otimes\beta_1\theta_3\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3 \end{array} & & & \begin{array}{c} D0\theta_3\otimes \\ S^0_2\beta_2\otimes S_3 \end{array}
\end{array} \right]
\end{array}
$$

$\mathbf{b3A1_b} = 0$

$\mathbf{b4A1_a} =$

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & \cdots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=2\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
\begin{array}{c} D1\otimes \\ \beta_1\theta_2\otimes \\ S^0_2\beta_2 \end{array} & \begin{array}{c} D1\otimes\beta_1\theta_3\otimes \\ S_2\otimes\beta_3 \end{array} & & & & & & & & \\
\begin{array}{c} D1\otimes \\ \beta_1\theta_2\otimes \\ S^0_2\beta_2\otimes S^0_3 \end{array} & \begin{array}{c} D1\otimes\beta_1\theta_2\otimes \\ S^0_2\beta_2\otimes S_3 \\ + \\ D1\otimes\beta_1\theta_3\otimes \\ S_2\otimes S^0_3\beta_3 \end{array} & \begin{array}{c} D1\otimes\beta_1\theta_3\otimes \\ S_2\otimes S_3 \end{array} & & & & & & & \\
 & \begin{array}{c} D1\otimes \\ \beta_1\theta_2\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3 \end{array} & \begin{array}{c} D1\otimes\beta_1\theta_2\otimes \\ S^0_2\beta_2\otimes S_3 \\ + \\ D1\otimes\beta_1\theta_3\otimes \\ S_2\otimes S^0_3\beta_3 \end{array} & \begin{array}{c} D1\otimes\beta_1\theta_3\otimes \\ S_2\otimes S_3 \end{array} & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & \begin{array}{c} D1\otimes \\ \beta_1\theta_2\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3 \end{array} & \begin{array}{c} D1\otimes\beta_1\theta_2\otimes \\ S^0_2\beta_2\otimes S_3 \\ + \\ D1\otimes\beta_1\theta_3\otimes \\ S_2\otimes S^0_3\beta_3 \end{array} & & & \begin{array}{c} D0\theta_3\otimes \\ S_2\otimes S_3 \end{array}
\end{array} \right]
\end{array}
$$

$\mathbf{b4A1_b} =$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} 0 & 1 & 2 & \cdots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \cdot \\ j=2\ \ K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
\begin{array}{c} D1\otimes\beta_1\otimes \\ S^0_2\beta_2 \end{array} & & & & & & & & \\
\begin{array}{c} D1\otimes\beta_1\otimes \\ S^0_2\beta_2\otimes S^0_3 \end{array} & \begin{array}{c} D1\otimes\beta_1\otimes \\ S^0_2\beta_2\otimes S_3 \end{array} & & & & & & & \\
 & \begin{array}{c} D1\otimes\beta_1\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3 \end{array} & \begin{array}{c} D1\otimes\beta_1\otimes \\ S^0_2\beta_2\otimes S_3 \end{array} & & & & & & \\
 & & & \cdot & & & & & \\
 & & & & \begin{array}{c} D1\otimes\beta_1\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3 \end{array} & \begin{array}{c} D1\otimes\beta_1\otimes \\ S^0_2\beta_2\otimes S_3 \end{array} & & &
\end{array} \right]
\end{array}
$$

$b5A1_a$ =

$$
\begin{array}{r c}
 & \begin{array}{ccccccccc} 0 & 1 & 2 & \dots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=2\ \ K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
D0\theta_2\otimes S_2 & & & & & & & & \\
D0\theta_2\otimes S_2\otimes S^0_3 & D0\theta_2\otimes S_2\otimes S_3 & & & & & & & \\
 & D0\theta_2\otimes S_2\otimes S^0_3\beta_3 & D0\theta_2\otimes S_2\otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & D0\theta_2\otimes S_2\otimes S^0_3\beta_3 & & & D0\theta_2\otimes S_2\otimes S_3 &
\end{array}\right]
\end{array}
$$

$b5A1_b$ =

$$
\begin{array}{r c}
 & \begin{array}{ccccccccc} 0 & 1 & 2 & \dots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=2\ \ K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
D0\otimes S_2 & & & & & & & & \\
D0\otimes S_2\otimes S^0_3 & D0\otimes S_2\otimes S_3 & & & & & & & \\
 & D0\otimes S_2\otimes S^0_3\beta_3 & D0\otimes S_2\otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & D0\otimes S_2\otimes S^0_3\beta_3 & & & D0\otimes S_2\otimes S_3 &
\end{array}\right]
\end{array}
$$

A0 =

$$
\begin{array}{r c}
 & \begin{array}{cccccccc} 0 & 1 & 2 & \dots & K2-1 & K2 & \overset{j=0}{K2+1} & \overset{j=2}{K2+1} \end{array} \\
\begin{array}{r} L_2 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K2 \\ j=0\ \ K2+1 \\ j=2\ \ K2+1 \end{array} &
\left[\begin{array}{cccccccc}
b0A0 & & & & & & & \\
c0A0 & a1A0 & & & & & & \\
 & a2A0 & a1A0 & & & & & \\
 & & & \ddots & & & & \\
 & & & & a2A0 & a1A0 & & \\
 & & & & & a2A0 & a1A0 & b2A0 \\
 & & & & & b3A0 & b4A0 & b5A0
\end{array}\right]
\end{array}
$$

b0A0 =

$$
\begin{array}{r c}
 & \begin{array}{ccccccccc} 0 & 1 & 2 & \dots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ . \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
D1\otimes S_1 & & & & & & & & \\
D1\otimes S_1\otimes S^0_3 & D1\otimes S_1\otimes S_3 & & & & & & & \\
 & D1\otimes S_1\otimes S^0_3\beta_3 & D1\otimes S_1\otimes S_3 & & & & & & \\
 & & & . & & & & & \\
 & & & & D1\otimes S_1\otimes S^0_3\beta_3 & D1\otimes S_1\otimes S_3 & & & D1\otimes S^0_3\theta_3\otimes S_3 \\
 & & & & & & & & b0A0(1)
\end{array}\right]
\end{array}
$$

$b0A0(1)_a = D1\ \theta_3 \otimes S_3$

$b0A0(1)_b = D1 \otimes S_3$

c0A0 =

$$\begin{array}{r c}
 & \begin{array}{ccccccccc} & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \ldots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
D1\otimes S_1\otimes S^0_2 & & & & & & & & \\
D1\otimes S_1\otimes S^0_2\otimes S^0_3 & D1\otimes S_1\otimes S^0_2\otimes S_3 & & & & & & & \\
 & D1\otimes S_1\otimes S^0_2\otimes S^0_3\beta_3 & D1\otimes S_1\otimes S^0_2\otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & D1\otimes S_1\otimes S^0_2\otimes S^0_3\beta_3 & D1\otimes S_1\otimes S^0_2\otimes S_3 & & & D1\otimes S^0_1\theta_3\otimes S^0_2\otimes S_3 \\
 & & & & & & & & c0A0(1)
\end{array}\right]
\end{array}$$

$$\mathbf{c0A0(1)_a = D1\ \theta_3 \otimes S^0{}_2 \otimes S_3}$$

$$\mathbf{c0A0(1)_b = D1 \otimes S^0{}_2 \otimes S_3}$$

a2A0 =

$$\begin{array}{r c}
 & \begin{array}{ccccccccc} & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \ldots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
D1\otimes S_1\otimes S^0_2\beta_2 & & & & & & & & \\
D1\otimes S_1\otimes S^0_2\beta_2\otimes S^0_3 & D1\otimes S_1\otimes S^0_2\beta_2\otimes S_3 & & & & & & & \\
 & D1\otimes S_1\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & D1\otimes S_1\otimes S^0_2\beta_2\otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & D1\otimes S_1\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & D1\otimes S_1\otimes S^0_2\beta_2\otimes S_3 & & & D1\otimes S^0_1\theta_3\otimes S^0_2\beta_2\otimes S_3 \\
 & & & & & & & & a2A0(1)
\end{array}\right]
\end{array}$$

$$\mathbf{a2A0(1)_a = D1\ \theta_3 \otimes S^0{}_2\ \beta_2 \otimes S_3}$$

$$\mathbf{a2A0(1)_b = D1 \otimes S^0{}_2\ \beta_2 \otimes S_3}$$

a1A0 =

$$\begin{array}{r c}
 & \begin{array}{ccccccccc} & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \ldots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \cdot \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
D1\otimes S_1\otimes S_2 & & & & & & & & \\
D1\otimes S_1\otimes S_2\otimes S^0_3 & D1\otimes S_1\otimes S_2\otimes S_3 & & & & & & & \\
 & D1\otimes S_1\otimes S_2\otimes S^0_3\beta_3 & D1\otimes S_1\otimes S_2\otimes S_3 & & & & & & \\
 & & & \cdot & & & & & \\
 & & & & D1\otimes S_1\otimes S_2\otimes S^0_3\beta_3 & D1\otimes S_1\otimes S_2\otimes S_3 & & & D1\otimes S^0_1\theta_3\otimes S_2\otimes S_3 \\
 & & & & & & & & a1A0(1)
\end{array}\right]
\end{array}$$

$$\mathbf{a1A0(1)_a = D1\ \theta_3 \otimes S_2 \otimes S_3}$$

$$\mathbf{a1A0(1)_b = D1 \otimes S_2 \otimes S_3}$$

$$
\mathbf{b2A0} =
$$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \cdots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0 \quad K3+1 \\ j=3 \quad K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
\begin{matrix} D1\otimes S^0_1\theta_2\otimes \\ S_2 \end{matrix} & & & & & & & & \\
\begin{matrix} D1\otimes S^0_1\theta_2\otimes \\ S_2\otimes S^0_3 \end{matrix} & \begin{matrix} D1\otimes S^0_1\theta_2\otimes \\ S_2\otimes S_3 \end{matrix} & & & & & & & \\
 & \begin{matrix} D1\otimes S^0_1\theta_2\otimes \\ S_2\otimes S^0_3\beta_3 \end{matrix} & \begin{matrix} D1\otimes S^0_1\theta_2\otimes \\ S_2\otimes S_3 \end{matrix} & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & \begin{matrix} D1\otimes S^0_1\theta_2\otimes \\ S_2\otimes S^0_3\beta_3 \\ b2A0(1) \end{matrix} & & & \begin{matrix} D1\otimes S^0_1\theta_2\otimes \\ S_2\otimes S_3 \\ b2A0(2) \end{matrix} & \\
 & & & & & & & &
\end{array}\right]
\end{array}
$$

$$
\mathbf{b2A0(1)_a = D1\ \theta_2 \otimes S_2 \otimes S^0{}_3\ \beta_3}
$$
$$
\mathbf{b2A0(2)_a = D1\ \theta_2 \otimes S_2 \otimes S_3}
$$

$$
\mathbf{b2A0(1)_b = 0}
$$
$$
\mathbf{b2A0(2)_b = 0}
$$

$$
\mathbf{b3A0_a} =
$$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \cdots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{c} L_3 \\ j=2 \quad K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
 & & & & & & & & D1\theta_3\otimes S^0_2\beta_2\otimes S_3
\end{array}\right]
\end{array}
$$

$$
\mathbf{b3A0_b = 0}
$$

$$
\mathbf{b4A0_a} =
$$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \cdots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{c} L_3 \\ j=2 \quad K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
 & & & & & & & & D1\theta_3\otimes S_2\otimes S_3
\end{array}\right]
\end{array}
$$

$$
\mathbf{b4A0_b = 0}
$$

$$
\mathbf{b5A0_a} =
$$

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \cdot \\ j=2 \quad K3+1 \end{array} &
\left[\begin{array}{cccccccc}
D1\theta_2\otimes S_2 & & & & & & & \\
\begin{matrix} D1\theta_2\otimes \\ S_2\otimes S^0_3 \end{matrix} & \begin{matrix} D1\theta_2\otimes \\ S_2\otimes S_3 \end{matrix} & & & & & & \\
 & \begin{matrix} D1\theta_2\otimes \\ S_2\otimes S^0_3\beta_3 \end{matrix} & \begin{matrix} D1\theta_2\otimes \\ S_2\otimes S_3 \end{matrix} & & & & & \\
 & & & & & & & \\
 & & & \begin{matrix} D1\theta_2\otimes \\ S_2\otimes S^0_3\beta_3 \end{matrix} & & & \begin{matrix} D1\theta_2\otimes \\ S_2\otimes S_3 \end{matrix} &
\end{array}\right]
\end{array}
$$

$b5A0_b$ =

$$
\begin{array}{cc}
L_3 & \begin{array}{ccccccccc} 0 & 1 & 2 & \cdots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \ddots \\ j=2 \quad K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
D1\otimes S_2 & & & & & & & & \\
D1\otimes S_2\otimes S^0_3 & D1\otimes S_2\otimes S_3 & & & & & & & \\
 & D1\otimes S_2\otimes S^0_3\beta_3 & D1\otimes S_2\otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & D1\otimes S_2\otimes S^0_3\beta_3 & & & D1\otimes S_2\otimes S_3 &
\end{array}\right]
\end{array}
$$

where:

- **D0** = No departure matrix of MAP arrival process
- **D1** = Departure matrix of MAP arrival process
- (β_i, S_i) = service distribution of queue i, i = 1, 2, or 3
- Ki = buffer size of queue i, i = 2 or 3
- θ_2 = probability the customer leaving queue 1 will choose to visit queue 2
- $\theta_3 = 1 - \theta_2$ = probability the customer leaving queue 1 will choose queue 3
- (j = 0) = state when there are K2 + 1 customers in queue 2 or K3 + 1 customers in queue 3, but there is no blocking
- (j = 2) = state when there are K2 + 1 customers in queue 2 and it blocks an arrival from queue 1
- (j = 3) = state when there are K3 + 1 customers in queue 3 and it blocks an arrival from queue 1

A.3.2 Sub-Matrices of the Modified MAP

Each of the new matrices within $\mathbf{D0}^{*i}$ and $\mathbf{D1}^{*i}$, $i = 2$ or 3, are defined as follows:

$\mathbf{A1''} =$

$$
\begin{array}{cc}
 & \begin{array}{ccccccc} 0 & 1 & 2 & \ldots & K2 & \overset{j=0}{K2+1} & \overset{j=2}{K2+1} \end{array} \\
\begin{array}{c} L_2 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0\;\; K2+1 \\ j=2\;\; K2+1 \end{array} &
\left[\begin{array}{ccccccc}
\mathbf{b0A1''} & & & & & & \\
\mathbf{c0A1''} & \mathbf{a1A1''} & & & & & \\
 & \mathbf{a2A1''} & \mathbf{a1A1''} & & & & \\
 & & & \ddots & & & \\
 & & & & \mathbf{a2A1''} & \mathbf{a1A1''} & \mathbf{b2A1''} \\
 & & & & \mathbf{b3A1''} & \mathbf{b4A1''} & \mathbf{b5A1''}
\end{array}\right]
\end{array}
$$

$\mathbf{b0A1''} =$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} 0 & 1 & 2 & \ldots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0\;\; K3+1 \\ j=3\;\; K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
D0\otimes S_1 & & & & & & & & \\
D0\otimes S_1\otimes S^0_3 & D0\otimes S_1\otimes S_3 & & & & & & & \\
 & D0\otimes S_1\otimes S^0_3 & D0\otimes S_1\otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & D0\otimes S_1\otimes S^0_3 & D0\otimes S_1\otimes S_3 & & & D0\otimes S^0_1\theta_3\otimes S_3 \\
 & & & & & & & & b0A1''(1)
\end{array}\right]
\end{array}
$$

$\mathbf{b0A1''(1)_a = D0}\ \theta_3 \otimes \mathbf{S_3}$

$\mathbf{b0A1''(1)_b = D0 \otimes S_3}$

$\mathbf{c0A1''} =$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} 0 & 1 & 2 & \ldots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0\;\; K3+1 \\ j=3\;\; K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
D0\otimes S_1\otimes S^0_2 & & & & & & & & \\
D0\otimes S_1\otimes S^0_2\otimes S^0_3 & D0\otimes S_1\otimes S^0_2\otimes S_3 & & & & & & & \\
 & D0\otimes S_1\otimes S^0_2\otimes S^0_3\beta_3 & D0\otimes S_1\otimes S^0_2\otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & D0\otimes S_1\otimes S^0_2\otimes S^0_3\beta_3 & D0\otimes S_1\otimes S^0_2\otimes S_3 & & & D0\otimes S^0_1\theta_3\otimes S^0_2\otimes S_3 \\
 & & & & & & & & c0A1''(1)
\end{array}\right]
\end{array}
$$

$\mathbf{c0A1''(1)_a = D0}\ \theta_3 \otimes \mathbf{S^0_2 \otimes S_3}$

$\mathbf{c0A1''(1)_b = D0 \otimes S^0_2 \otimes S_3}$

a2A1" =

$$
\begin{array}{rr}
 & \begin{array}{ccccccccc} & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \ldots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
L_3 & \\
\begin{array}{r} 0 \\ 1 \\ 2 \\ \ddots \\ j=0 \quad K3+1 \\ \\ j=3 \quad K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
D0\otimes S_1\otimes S^0_2\beta_2 & & & & & & & & \\
D0\otimes S_1\otimes S^0_2\beta_2\otimes S^0_3 & D0\otimes S_1\otimes S^0_2\beta_2\otimes S_3 & & & & & & & \\
 & D0\otimes S_1\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & D0\otimes S_1\otimes S^0_2\beta_2\otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & D0\otimes S_1\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & D0\otimes S_1\otimes S^0_2\beta_2\otimes S_3 & & & D0\otimes S^0_1\theta_3\otimes S^0_2\beta_2\otimes S_3 \\
 & & & & & & & & \\
 & & & & & & & & a2A1''(1)
\end{array}\right]
\end{array}
$$

$$\mathbf{a2A1''(1)_a = D0\,\theta_3 \otimes S^0_2\,\beta_2 \otimes S_3}$$

$$\mathbf{a2A1''(1)_b = D0 \otimes S^0_2\,\beta_2 \otimes S_3}$$

a1A1" =

$$
\begin{array}{rr}
 & \begin{array}{ccccccccc} & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \ldots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
L_3 & \\
\begin{array}{r} 0 \\ 1 \\ 2 \\ \ddots \\ j=0 \quad K3+1 \\ \\ j=3 \quad K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
D0\otimes S_1\otimes S_2 & & & & & & & & \\
D0\otimes S_1\otimes S_2\otimes S^0_3 & D0\otimes S_1\otimes S\otimes S_3 & & & & & & & \\
 & D0\otimes S_1\otimes S_2\otimes S^0_3\beta_3 & D0\otimes S_1\otimes S\otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & D0\otimes S_1\otimes S_2\otimes S^0_3\beta_3 & D0\otimes S_1\otimes S\otimes S_3 & & & D0\otimes S^0_1\theta_3\otimes S_2\otimes S_3 \\
 & & & & & & & & \\
 & & & & & & & & a1A1''(1)
\end{array}\right]
\end{array}
$$

$$\mathbf{a1A1''(1)_a = D0\,\theta_3 \otimes S_2 \otimes S_3}$$

$$\mathbf{a1A1''(1)_b = D0 \otimes S_2 \otimes S_3}$$

b2A1" =

$$
\begin{array}{c|cccccccccc}
L_3 & 0 & 1 & 2 & \cdots & K3 & \begin{array}{c} j=0 \\ K3+1 \end{array} & \begin{array}{c} j=1 \\ K3+1 \end{array} & \begin{array}{c} j=2 \\ K3+1 \end{array} & \begin{array}{c} j=3 \\ K3+1 \end{array} \\
\hline
0 & \begin{array}{c} D0\otimes \\ S^0_1\theta_2\otimes S_2 \end{array} & & & & & & & & \\
1 & \begin{array}{c} D0\otimes \\ S^0_1\theta_2\otimes \\ S_2\otimes S^0_3 \end{array} & \begin{array}{c} D0\otimes \\ S^0_1\theta_2\otimes \\ S_2\otimes S_3 \end{array} & & & & & & & \\
2 & & \begin{array}{c} D0\otimes \\ S^0_1\theta_2\otimes \\ S_2\otimes S^0_3\beta_3 \end{array} & \begin{array}{c} D0\otimes \\ S^0_1\theta_2\otimes \\ S_2\otimes S_3 \end{array} & & & & & & \\
\ddots & & & & \ddots & & & & & \\
j=0\ \ K3+1 & & & & & \begin{array}{c} D0\otimes \\ S^0_1\theta_2\otimes \\ S_2\otimes S^0_3\beta_3 \\ b2A1"(1) \end{array} & & & \begin{array}{c} D0\otimes \\ S^0_1\theta_2\otimes \\ S_2\otimes S_3 \\ b2A1"(2) \end{array} & \\
j=3\ \ K3+1 & & & & & & & & &
\end{array}
$$

$$b2A1"(1)_a = D0\,\theta_2 \otimes S_2 \otimes S^0_3\,\beta_3$$
$$b2A1"(2)_a = D0\,\theta_2 \otimes S_2 \otimes S_3$$

$$b2A1"(1)_b = 0$$
$$b2A1"(2)_b = 0$$

b3A1"$_a$ =

$$
\begin{array}{c|cccccccccc}
L_3 & 0 & 1 & 2 & \cdots & K3 & \begin{array}{c} j=0 \\ K3+1 \end{array} & \begin{array}{c} j=1 \\ K3+1 \end{array} & \begin{array}{c} j=2 \\ K3+1 \end{array} & \begin{array}{c} j=3 \\ K3+1 \end{array} \\
\hline
j=2\ \ K3+1 & & & & & & & & & \begin{array}{c} D0\theta_1\otimes \\ S^0_2\beta_2\otimes S_3 \end{array}
\end{array}
$$

$$b3A1"_b = 0$$

b4A1"$_a$ =

$$
\begin{array}{c|cccccccccc}
L_3 & 0 & 1 & 2 & \cdots & K3 & \begin{array}{c} j=0 \\ K3+1 \end{array} & \begin{array}{c} j=1 \\ K3+1 \end{array} & \begin{array}{c} j=2 \\ K3+1 \end{array} & \begin{array}{c} j=3 \\ K3+1 \end{array} \\
\hline
j=2\ \ K3+1 & & & & & & & & & \begin{array}{c} D0\theta_1\otimes \\ S_2\otimes S_3 \end{array}
\end{array}
$$

$$b4A1"_b = 0$$

b5A1"$_a$ =

$$
\begin{array}{c|cccccccccc}
L_3 & 0 & 1 & 2 & \cdots & K3 & \begin{array}{c} j=0 \\ K3+1 \end{array} & \begin{array}{c} j=1 \\ K3+1 \end{array} & \begin{array}{c} j=2 \\ K3+1 \end{array} & \begin{array}{c} j=3 \\ K3+1 \end{array} \\
\hline
0 & D0\theta_2\otimes S_2 & & & & & & & & \\
1 & \begin{array}{c} D0\theta_2\otimes \\ S_2\otimes S^0_3 \end{array} & \begin{array}{c} D0\theta_2\otimes \\ S_2\otimes S_3 \end{array} & & & & & & & \\
2 & & \begin{array}{c} D0\theta_2\otimes \\ S_2\otimes S^0_3\beta_3 \end{array} & \begin{array}{c} D0\theta_2\otimes \\ S_2\otimes S_3 \end{array} & & & & & & \\
\cdot & & & & \cdot & & & & & \\
j=2\ \ K3+1 & & & & & D0\theta_2\otimes & & & D0\theta_2\otimes &
\end{array}
$$

$S_2 \otimes S^0{}_3\beta_3$ $S_2 \otimes S_3$

$b5A1''_b =$

$$\begin{array}{cc}
 & \begin{array}{cccccccccc} L_3 & 0 & 1 & 2 & \cdots & K3 & \substack{j=0\\ K3+1} & \substack{j=1\\ K3+1} & \substack{j=2\\ K3+1} & \substack{j=3\\ K3+1} \end{array} \\
\begin{array}{r} 0 \\ 1 \\ 2 \\ \ddots \\ j=2 \quad K3+1 \end{array} &
\begin{bmatrix}
D0 \otimes S_2 & & & & & & & & \\
D0 \otimes S_2 \otimes S^0_3 & D0 \otimes S_2 \otimes S_3 & & & & & & & \\
 & \substack{D0 \otimes S_2 \otimes \\ S^0_3 \beta_3} & D0 \otimes S_2 \otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & \substack{D0 \otimes S_2 \otimes \\ S^0_3 \beta_3} & & & D0 \otimes S_2 \otimes S_3 &
\end{bmatrix}
\end{array}$$

$(A1^{*2})' =$

$$\begin{array}{cc}
 & \begin{array}{cccccccc} L_2 & 0 & 1 & 2 & 3 & \cdots & \substack{j=0\\ K2+1} & \substack{j=2\\ K2+1} \end{array} \\
\begin{array}{r} 0 \\ 1 \\ 2 \\ \ddots \\ j=0 \quad K2+1 \\ j=2 \quad K2+1 \end{array} &
\begin{bmatrix}
 & (b1A1^{*2})' & & & & & \\
 & (a1A1^{*2})' & (a0A1^{*2})' & & & & \\
 & & (a1A1^{*2})' & (a0A1^{*2})' & & & \\
 & & & & \ddots & & \\
 & & & & & (a1A1^{*2})' & \\
 & & & & & (b4A1^{*2})' &
\end{bmatrix}
\end{array}$$

$(b1A1^{*2}) =$

$$\begin{array}{cc}
 & \begin{array}{cccccccccc} L_3 & 0 & 1 & 2 & \cdots & K3 & \substack{j=0\\ K3+1} & \substack{j=1\\ K3+1} & \substack{j=2\\ K3+1} & \substack{j=3\\ K3+1} \end{array} \\
\begin{array}{r} 0 \\ 1 \\ 2 \\ \ddots \\ j=0 \quad K3+1 \\ j=3 \quad K3+1 \end{array} &
\begin{bmatrix}
\substack{D1 \otimes \\ S^0_1 \beta_1 \theta_2 \otimes \beta_2} & & & & & & & & \\
\substack{D1 \otimes \\ S^0_1 \beta_1 \theta_2 \otimes \\ \beta_2 \otimes S^0_3} & \substack{D1 \otimes \\ S^0_1 \beta_1 \theta_2 \otimes \\ \beta_2 \otimes S_3} & & & & & & & \\
 & \substack{D1 \otimes \\ S^0_1 \beta_1 \theta_2 \otimes \\ \beta_2 \otimes S^0_3 \beta_3} & \substack{D1 \otimes \\ S^0_1 \beta_1 \theta_2 \otimes \\ \beta_2 \otimes S_3} & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & \substack{D1 \otimes \\ S^0_1 \beta_1 \theta_2 \otimes \\ \beta_2 \otimes S^0_3 \beta_3} & \substack{D1 \otimes \\ S^0_1 \beta_1 \theta_2 \otimes \\ \beta_2 \otimes S_3} & & & \\
 & & & & (b1A1^{*2})'(1) & (b1A1^{*2})'(2) & & &
\end{bmatrix}
\end{array}$$

$(b1A1^{*2})'(1)_a = D1 \otimes \beta_1 \theta_2 \otimes \beta_2 \otimes S^0_3 \beta_3$

$(b1A1^{*2})'(2)_a = D1 \otimes \beta_1 \theta_2 \otimes \beta_2 \otimes S_3$

$(b1A1^{*2})'(1)_b = 0$

$(b1A1^{*2})'(2)_b = 0$

$(\mathbf{a1A1}^{*2})' =$

$$
\begin{array}{c|c}
 & \begin{array}{cccccccccc} L_3 & 0 & 1 & 2 & \ldots & K3 & \substack{j=0\\ K3+1} & \substack{j=1\\ K3+1} & \substack{j=2\\ K3+1} & \substack{j=3\\ K3+1} \end{array} \\
\begin{array}{c} 0 \\ \\ 1 \\ \\ 2 \\ \\ \ddots \\ j=0\ \ K3+1 \\ \vdots \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccc}
\substack{D1\otimes\\ S^0{}_1\beta_1\theta_2\otimes\\ S^0{}_2\beta_2} & & & & & & & \\
\substack{D1\otimes\\ S^0{}_1\beta_1\theta_2\otimes\\ S^0{}_2\beta_2\otimes S^0{}_3} & \substack{D1\otimes\\ S^0{}_1\beta_1\theta_2\otimes\\ S^0{}_2\beta_2\otimes S_3} & & & & & & \\
 & \substack{D1\otimes\\ S^0{}_1\beta_1\theta_2\otimes\\ S^0{}_2\beta_2\otimes S^0{}_3\beta_3} & \substack{D1\otimes\\ S^0{}_1\beta_1\theta_2\otimes\\ S^0{}_2\beta_2\otimes S_3} & & & & & \\
 & & & \ddots & & & & \\
 & & & & \substack{D1\otimes\\ S^0{}_1\beta_1\theta_2\otimes\\ S^0{}_2\beta_2\otimes S^0{}_3\beta_3\\ (a1A1^{*2})'(1)} & \substack{D1\otimes\\ S^0{}_1\beta_1\theta_2\otimes\\ S^0{}_2\beta_2\otimes S_3\\ (a1A1^{*2})'(2)} & &
\end{array}\right]
\end{array}
$$

$(\mathbf{a1A1}^{*2})'(1)_a = \mathbf{D1} \otimes \beta_1\theta_2 \otimes S^0{}_2\beta_2 \otimes S^0{}_3\beta_3$
$(\mathbf{a1A1}^{*2})'(2)_a = \mathbf{D1} \otimes \beta_1\, \theta_2 \otimes S^0{}_2\, \beta_2 \otimes S_3$

$(\mathbf{a1A1}^{*2})'(1)_b = 0$
$(\mathbf{a1A1}^{*2})'(2)_b = 0$

$(\mathbf{a0A1}^{*2})' =$

$$
\begin{array}{c|c}
 & \begin{array}{cccccccccc} L_3 & 0 & 1 & 2 & \ldots & K3 & \substack{j=0\\ K3+1} & \substack{j=1\\ K3+1} & \substack{j=2\\ K3+1} & \substack{j=3\\ K3+1} \end{array} \\
\begin{array}{c} 0 \\ 1 \\ \\ 2 \\ \\ \ddots \\ j=0\ \ K3+1 \\ \vdots \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccc}
\substack{D1\otimes\\ S^0{}_1\beta_1\theta_2\otimes S_2} & & & & & & & \\
\substack{D1\otimes\\ S^0{}_1\beta_1\theta_2\otimes\\ S_2\otimes S^0{}_3} & \substack{D1\otimes\\ S^0{}_1\beta_1\theta_2\otimes\\ S_2\otimes S_3} & & & & & & \\
 & \substack{D1\otimes\\ S^0{}_1\beta_1\theta_2\otimes\\ S_2\otimes S^0{}_3\beta_3} & \substack{D1\otimes\\ S^0{}_1\beta_1\theta_2\otimes\\ S_2\otimes S_3} & & & & & \\
 & & & \ddots & & & & \\
 & & & & \substack{D1\otimes\\ S^0{}_1\beta_1\theta_2\otimes\\ S_2\otimes S^0{}_3\beta_3\\ (a0A1^{*2})'(1)} & \substack{D1\otimes\\ S^0{}_1\beta_1\theta_2\otimes\\ S_2\otimes S_3\\ (a0A1^{*2})'(2)} & &
\end{array}\right]
\end{array}
$$

$(\mathbf{a0A1}^{*2})'(1)_a = \mathbf{D1} \otimes \beta_1\, \theta_2 \otimes \mathbf{S_2} \otimes S^0{}_3\, \beta_3$
$(\mathbf{a0A1}^{*2})'(2)_a = \mathbf{D1} \otimes \beta_1\, \theta_2 \otimes \mathbf{S_2} \otimes \mathbf{S_3}$

$(\mathbf{a0A1}^{*2})'(1)_b = 0$
$(\mathbf{a0A1}^{*2})'(2)_b = 0$

$(b4A1^{*2})'_a =$

$$
\begin{array}{cc|ccccccccc}
 & & & & & & & j=0 & j=1 & j=2 & j=3 \\
 & L_3 & 0 & 1 & 2 & \ldots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
\hline
 & 0 & D1\otimes\beta_1\theta_2\otimes S^0_2\beta_2 & & & & & & & & \\
 & 1 & D1\otimes\beta_1\theta_2\otimes S^0_2\beta_2\otimes S^0_3 & D1\otimes\beta_1\theta_2\otimes S^0_2\beta_2\otimes S_3 & & & & & & & \\
 & 2 & & D1\otimes\beta_1\theta_2\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & D1\otimes\beta_1\theta_2\otimes S^0_2\beta_2\otimes S_3 & & & & & & \\
 & \ddots & & & & \ddots & & & & & \\
j=2 & K3+1 & & & & & D1\otimes\beta_1\theta_2\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & D1\otimes\beta_1\theta_2\otimes S^0_2\beta_2\otimes S_3 & & &
\end{array}
$$

$(b4A1^{*2})'_b =$

$$
\begin{array}{cc|ccccccccc}
 & & & & & & & j=0 & j=1 & j=2 & j=3 \\
 & L_3 & 0 & 1 & 2 & \ldots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
\hline
 & 0 & D1\otimes\beta_1\otimes S^0_2\beta_2 & & & & & & & & \\
 & 1 & D1\otimes\beta_1\otimes S^0_2\beta_2\otimes S^0_3 & D1\otimes\beta_1\otimes S^0_2\beta_2\otimes S_3 & & & & & & & \\
 & 2 & & D1\otimes\beta_1\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & D1\otimes\beta_1\otimes S^0_2\beta_2\otimes S_3 & & & & & & \\
 & \ddots & & & & \ddots & & & & & \\
j=2 & K3+1 & & & & & D1\otimes\beta_1\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & D1\otimes\beta_1\otimes S^0_2\beta_2\otimes S_3 & & &
\end{array}
$$

$(A1^{*3})'$ can be found using $(A1^{*3})' = A1 - (A1'' + (A1^{*2})')$

$C0^{*2} =$

$$
\begin{array}{cc|cccccccc}
 & & & & & & & & j=0 & j=2 \\
 & L_2 & 0 & 1 & 2 & 3 & \ldots & K2 & K2+1 & K2+1 \\
\hline
 & 0 & & b1C0^{*2} & & & & & & \\
 & 1 & & a1C0^{*2} & a0C0^{*2} & & & & & \\
 & 2 & & & a1C0^{*2} & a0C0^{*2} & & & & \\
 & \ddots & & & & & \ddots & & & \\
 & K2 & & & & & & a1C0^{*2} & a0C0^{*2} & \\
j=0 & K2+1 & & & & & & & a1C0^{*2} & \\
j=2 & K2+1 & & & & & & & b4C0^{*2} &
\end{array}
$$

$\mathbf{b1C0^{*2}} =$

$$
\begin{array}{c|c c c c c c c c c}
L_3 & 0 & 1 & 2 & \cdots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \\
\hline
0 & \begin{smallmatrix} DO\otimes \\ S^0_1\theta_2\otimes\beta_2 \end{smallmatrix} & & & & & & & & \\
1 & \begin{smallmatrix} DO\otimes \\ S^0_1\theta_2\otimes\beta_2\otimes \\ S^0_3 \end{smallmatrix} & \begin{smallmatrix} DO\otimes \\ S^0_1\theta_2\otimes\beta_2\otimes \\ S_3 \end{smallmatrix} & & & & & & & \\
2 & & \begin{smallmatrix} DO\otimes \\ S^0_1\theta_2\otimes\beta_2\otimes \\ S^0_3\beta_3 \end{smallmatrix} & \begin{smallmatrix} DO\otimes \\ S^0_1\theta_2\otimes\beta_2\otimes \\ S_3 \end{smallmatrix} & & & & & & \\
\ddots & & & & \ddots & & & & & \\
j=0\ \ K3+1 & & & & & & & & & \\
j=3\ \ K3+1 & & & & & \begin{smallmatrix} DO\otimes \\ S^0_1\theta_2\otimes\beta_2\otimes \\ S^0_3\beta_3 \\ b1C0^{*2}(1) \end{smallmatrix} & \begin{smallmatrix} DO\otimes \\ S^0_1\theta_2\otimes\beta_2\otimes \\ S_3 \\ b1C0^{*2}(2) \end{smallmatrix} & & &
\end{array}
$$

$\mathbf{b1C0^{*2}(1)_a = D0\ \theta_2 \otimes \beta_2 \otimes S^0_3\, \beta_3}$
$\mathbf{b1C0^{*2}(2)_a = D0\ \theta_2 \otimes \beta_2 \otimes S_3}$

$\mathbf{b1C0^{*2}(1)_b = 0}$
$\mathbf{b1C0^{*2}(2)_b = 0}$

$\mathbf{a1C0^{*2}} =$

$$
\begin{array}{c|c c c c c c c c c}
L_3 & 0 & 1 & 2 & \cdots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \\
\hline
0 & \begin{smallmatrix} DO\otimes \\ S^0_1\theta_2\otimes S^0_2\beta_2 \end{smallmatrix} & & & & & & & & \\
1 & \begin{smallmatrix} DO\otimes \\ S^0_1\theta_2\otimes \\ S^0_2\beta_2\otimes S^0_3 \end{smallmatrix} & \begin{smallmatrix} DO\otimes \\ S^0_1\theta_2\otimes \\ S^0_2\beta_2\otimes S_3 \end{smallmatrix} & & & & & & & \\
2 & & \begin{smallmatrix} DO\otimes \\ S^0_1\theta_2\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3 \end{smallmatrix} & \begin{smallmatrix} DO\otimes \\ S^0_1\theta_2\otimes \\ S^0_2\beta_2\otimes S_3 \end{smallmatrix} & & & & & & \\
\ddots & & & & \ddots & & & & & \\
j=0\ \ K3+1 & & & & & & & & & \\
j=3\ \ K3+1 & & & & & \begin{smallmatrix} DO\otimes \\ S^0_1\theta_2\otimes \\ S^0_2\beta_2\otimes S^0_3\beta_3 \\ a1C0^{*2}(1) \end{smallmatrix} & \begin{smallmatrix} DO\otimes \\ S^0_1\theta_2\otimes \\ S^0_2\beta_2\otimes S_3 \\ a1C0^{*2}(2) \end{smallmatrix} & & &
\end{array}
$$

$\mathbf{a1C0^{*2}(1)_a = D0\ \theta_2 \otimes S^0_2\, \beta_2 \otimes S^0_3\, \beta_3}$
$\mathbf{a1C0^{*2}(2)_a = D0\ \theta_2 \otimes S^0_2\, \beta_2 \otimes S_3}$

$\mathbf{a1C0^{*2}(1)_b = 0}$
$\mathbf{a1C0^{*2}(2)_b = 0}$

$$\mathbf{a0C0^{*2}} =$$

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} L_3 & 0 & 1 & 2 & \cdots & K3 & \begin{array}{c} j=0 \\ K3+1 \end{array} & \begin{array}{c} j=1 \\ K3+1 \end{array} & \begin{array}{c} j=2 \\ K3+1 \end{array} & \begin{array}{c} j=3 \\ K3+1 \end{array} \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \ddots \\ j=0 \;\; K3+1 \\ \vdots \\ j=3 \;\; K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
D0\otimes S^0_1\theta_2\otimes S_2 & & & & & & & & \\
D0\otimes S^0_1\theta_2\otimes S_2\otimes S^0_3 & D0\otimes S^0_1\theta_2\otimes S_2\otimes S_3 & & & & & & & \\
 & D0\otimes S^0_1\theta_2\otimes S_2\otimes S^0_3\beta_3 & D0\otimes S^0_1\theta_2\otimes S_2\otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & D0\otimes S^0_1\theta_2\otimes S_2\otimes S^0_3\beta_3 \;\; a0C0^{*2}(1) & D0\otimes S^0_1\theta_2\otimes S_2\otimes S_3 \;\; a0C0^{*2}(2) & & &
\end{array} \right]
\end{array}
$$

$$\mathbf{a0C0^{*2}(1)_a = D0\ \theta_2 \otimes S_2 \otimes S^0{}_3\ \beta_3}$$
$$\mathbf{a0C0^{*2}(2)_a = D0\ \theta_2 \otimes S_2 \otimes S_3}$$

$$\mathbf{a0C0^{*2}(1)_b = 0}$$
$$\mathbf{a0C0^{*2}(2)_b = 0}$$

$$\mathbf{b4C0^{*2}{}_a} =$$

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} L_3 & 0 & 1 & 2 & \cdots & K3 & \begin{array}{c} j=0 \\ K3+1 \end{array} & \begin{array}{c} j=1 \\ K3+1 \end{array} & \begin{array}{c} j=2 \\ K3+1 \end{array} & \begin{array}{c} j=3 \\ K3+1 \end{array} \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \ddots \\ j=2 \;\; K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
D0\theta_2\otimes S^0_2\beta_2 & & & & & & & & \\
D0\theta_2\otimes S^0_2\beta_2\otimes S^0_3 & D0\theta_2\otimes S^0_2\beta_2\otimes S_3 & & & & & & & \\
 & D0\theta_2\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & D0\theta_2\otimes S^0_2\beta_2\otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & D0\theta_2\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & D0\theta_2\otimes S^0_2\beta_2\otimes S_3 & & &
\end{array} \right]
\end{array}
$$

$$\mathbf{b4C0^{*2}{}_b} =$$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} L_3 & 0 & 1 & 2 & K3 & \begin{array}{c} j=0 \\ K3+1 \end{array} & \begin{array}{c} j=1 \\ K3+1 \end{array} & \begin{array}{c} j=2 \\ K3+1 \end{array} & \begin{array}{c} j=3 \\ K3+1 \end{array} \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \cdot \\ j=2 \;\; K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
D0\otimes S^0_2\beta_2 & & & & & & & & \\
D0\otimes S^0_2\beta_2\otimes S^0_3 & D0\otimes S^0_2\beta_2\otimes S_3 & & & & & & & \\
 & D0\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & D0\otimes S^0_2\beta_2\otimes S_3 & & & & & & \\
 & & & \cdot & & & & & \\
 & & & & D0\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & D0\otimes S^0_2\beta_2\otimes S_3 & & &
\end{array} \right]
\end{array}
$$

$$\mathbf{C0^{*3} = C0 - C0^{*2}}$$

$\mathbf{A2^{*2}} =$

$$
\begin{array}{c|c}
 & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & \cdots & \overset{j=0}{K2+1} & \overset{j=2}{K2+1} \end{array} \\
\begin{array}{c} L_2 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0\ \ K2+1 \\ j=2\ \ K2+1 \end{array} &
\left[\begin{array}{ccccccc}
 & \mathbf{b1A2^{*2}} & & & & & \\
 & \mathbf{a1A2^{*2}} & \mathbf{a0A2^{*2}} & & & & \\
 & & \mathbf{a1A2^{*2}} & \mathbf{a0A2^{*2}} & & & \\
 & & & & \ddots & & \\
 & & & & & \mathbf{a1A2^{*2}} & \\
 & & & & & \mathbf{b4A2^{*2}} &
\end{array}\right]
\end{array}
$$

$\mathbf{b1A2^{*2}} =$

$$
\begin{array}{c|c}
 & \begin{array}{ccccccccc} 0 & 1 & 2 & \cdots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0\ \ K3+1 \\ \vdots \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
D0\otimes S^0_1\beta_1\theta_2\otimes\beta_2 & & & & & & & & \\
D0\otimes S^0_1\beta_1\theta_2\otimes \beta_2\otimes S^0_3 & D0\otimes S^0_1\beta_1\theta_2\otimes \beta_2\otimes S_3 & & & & & & & \\
 & D0\otimes S^0_1\beta_1\theta_2\otimes \beta_2\otimes S^0_3\beta_3 & D0\otimes S^0_1\beta_1\theta_2\otimes \beta_2\otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & D0\otimes S^0_1\beta_1\theta_2\otimes \beta_2\otimes S^0_3\beta_3 & D0\otimes S^0_1\beta_1\theta_2\otimes \beta_2\otimes S_3 & & & \\
 & & & & b1A2^{*2}(1) & b1A2^{*2}(2) & & &
\end{array}\right]
\end{array}
$$

$$\mathbf{b1A2^{*2}(1)_a = D0 \otimes \beta_1\, \theta_2 \otimes \beta_2 \otimes S^0{}_3\, \beta_3}$$
$$\mathbf{b1A2^{*2}(2)_a = D0 \otimes \beta_1\, \theta_2 \otimes \beta_2 \otimes S_3}$$

$$\mathbf{b1A2^{*2}(1)_b} = 0$$
$$\mathbf{b1A2^{*2}(2)_b} = 0$$

$\mathbf{a1A2^{*2}} =$

$$
\begin{array}{c|c}
 & \begin{array}{ccccccccc} 0 & 1 & 2 & \cdots & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \cdot \\ j=0\ \ K3+1 \\ \vdots \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
D0\otimes S^0_1\beta_1\theta_2\otimes S^0_2\beta_2 & & & & & & & & \\
D0\otimes S^0_1\beta_1\theta_2\otimes S^0_2\beta_2\otimes S^0_3 & D0\otimes S^0_1\beta_1\theta_2\otimes S^0_2\beta_2\otimes S_3 & & & & & & & \\
 & D0\otimes S^0_1\beta_1\theta_2\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & D0\otimes S^0_1\beta_1\theta_2\otimes S^0_2\beta_2\otimes S_3 & & & & & & \\
 & & & \cdot & & & & & \\
 & & & & D0\otimes S^0_1\beta_1\theta_2\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & D0\otimes S^0_1\beta_1\theta_2\otimes S^0_2\beta_2\otimes S_3 & & & \\
 & & & & a1A2^{*2}(1) & a1A2^{*2}(2) & & &
\end{array}\right]
\end{array}
$$

$$a1A2^{*2}(1)_a = D0 \otimes \beta_1\theta_2 \otimes S^0{}_2\beta_2 \otimes S^0{}_3\beta_3$$
$$a1A2^{*2}(2)_a = D0 \otimes \beta_1\, \theta_2 \otimes S^0{}_2\, \beta_2 \otimes S_3$$

$$a1A2^{*2}(1)_b = 0$$
$$a1A2^{*2}(2)_b = 0$$

$$a0A2^{*2} =$$

$$\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \dots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=0 \quad K3+1 \\ j=3 \quad K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
\begin{array}{c} D0\otimes \\ S^0{}_1\beta_1\theta_2\otimes S_2 \end{array} & & & & & & & & \\
\begin{array}{c} D0\otimes \\ S^0{}_1\beta_1\theta_2\otimes \\ S_2\otimes S^0{}_3 \end{array} & \begin{array}{c} D0\otimes \\ S^0{}_1\beta_1\theta_2\otimes \\ S_2\otimes S_3 \end{array} & & & & & & & \\
 & \begin{array}{c} D0\otimes \\ S^0{}_1\beta_1\theta_2\otimes \\ S_2\otimes S^0{}_3\beta_3 \end{array} & \begin{array}{c} D0\otimes \\ S^0{}_1\beta_1\theta_2\otimes \\ S_2\otimes S_3 \end{array} & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & \begin{array}{c} D0\otimes \\ S^0{}_1\beta_1\theta_2\otimes \\ S_2\otimes S^0{}_3\beta_3 \end{array} & \begin{array}{c} D0\otimes \\ S^0{}_1\beta_1\theta_2\otimes \\ S_2\otimes S_3 \end{array} & & & \\
 & & & & a0A2^{*2}(1) & a0A2^{*2}(2) & & &
\end{array}\right]
\end{array}$$

$$a0A2^{*2}(1)_a = D0 \otimes \beta_1\theta_2 \otimes S_2 \otimes S^0{}_3\beta_3$$
$$a0A2^{*2}(2)_a = D0 \otimes \beta_1\, \theta_2 \otimes S_2 \otimes S_3$$

$$a0A2^{*2}(1)_b = 0$$
$$a0A2^{*2}(2)_b = 0$$

$$b4A2^{*2}{}_a =$$

$$\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \dots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ j=2 \quad K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
\begin{array}{c} D0\otimes\beta_1\theta_2\otimes \\ S^0{}_2\beta_2 \end{array} & & & & & & & & \\
\begin{array}{c} D0\otimes\beta_1\theta_2\otimes \\ S^0{}_2\beta_2\otimes S^0{}_3 \end{array} & \begin{array}{c} D0\otimes\beta_1\theta_2\otimes \\ S^0{}_2\beta_2\otimes S_3 \end{array} & & & & & & & \\
 & \begin{array}{c} D0\otimes\beta_1\theta_2\otimes \\ S^0{}_2\beta_2\otimes S^0{}_3\beta_3 \end{array} & \begin{array}{c} D0\otimes\beta_1\theta_2\otimes \\ S^0{}_2\beta_2\otimes S_3 \end{array} & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & \begin{array}{c} D0\otimes\beta_1\theta_2\otimes \\ S^0{}_2\beta_2\otimes S^0{}_3\beta_3 \end{array} & \begin{array}{c} D0\otimes\beta_1\theta_2\otimes \\ S^0{}_2\beta_2\otimes S_3 \end{array} & & &
\end{array}\right]
\end{array}$$

$$b4A2^{*2}_{b} =$$

$$\begin{array}{cc}
L_3 & \begin{array}{ccccccccc} 0 & 1 & 2 & \cdots & K3 & \substack{j=0\\ K3+1} & \substack{j=1\\ K3+1} & \substack{j=2\\ K3+1} & \substack{j=3\\ K3+1} \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \ddots \\ j=2\ \ K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
DO\otimes\beta_1\otimes S^0_2\beta_2 & & & & & & & & \\
DO\otimes\beta_1\otimes S^0_2\beta_2\otimes S^0_3 & DO\otimes\beta_1\otimes S^0_2\beta_2\otimes S_3 & & & & & & & \\
 & DO\otimes\beta_1\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & DO\otimes\beta_1\otimes S^0_2\beta_2\otimes S_3 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & DO\otimes\beta_1\otimes S^0_2\beta_2\otimes S^0_3\beta_3 & DO\otimes\beta_1\otimes S^0_2\beta_2\otimes S_3 & & &
\end{array}\right]
\end{array}$$

$$A2^{*3} = A2 - A2^{*2}$$

A.3.3 Trial Results

Table A.3.1: Case *a*, Results for Varying Inputs, $n = 2$

	D0	D1	λ	S_1	β_1	μ_1	S_2
1	.325 .30 .325 .30	.175 .20 .175 .20	0.3750	.125 .125 .125 .125	.50 .50	0.7500	.375 .375 .375 .375
2	.45 .45 .50 .50	.05 .05 0 0	0.0500	.25 .25 .25 .25	.50 .50	0.5000	.325 .30 .325 .30
3	.45 .45 .50 .50	.05 .05 0 0	0.0500	.25 .25 .25 .25	.50 .50	0.5000	.375 .375 .375 .375
4	.40 .475 .40 .475	.10 .025 .10 .025	0.1250	.25 .25 .25 .25	.50 .50	0.5000	.458 .458 .458 .458
5	.325 .30 .325 .30	.175 .20 .175 .20	0.3750	.25 .25 .25 .25	.50 .50	0.5000	.325 .30 .325 .30
6	.325 .30 .325 .30	.175 .20 .175 .20	0.3750	.25 .25 .25 .25	.50 .50	0.5000	.125 .125 .125 .125
7	.40 .475 .40 .475	.10 .025 .10 .025	0.1250	.25 .25 .25 .25	.50 .50	0.5000	.40 .475 .40 .475
8	.45 .45 .50 .50	.05 .05 0 0	0.0500	.45 .45 .45 .45	.50 .50	0.1000	.375 .375 .375 .375
9	.45 .45 .50 .50	.05 .05 0 0	0.0500	.25 .25 .25 .25	.50 .50	0.5000	.45 .45 .50 .50
10	.40 .475 .40 .475	.10 .025 .10 .025	0.1250	.25 .25 .25 .25	.50 .50	0.5000	.40 .475 .40 .475
11	.40 .475 .40 .475	.10 .025 .10 .025	0.1250	.25 .25 .25 .25	.50 .50	0.5000	.375 .375 .375 .375
12	.375 .375 .375 .375	.125 .125 .125 .125	0.2500	.25 .25 .25 .25	.50 .50	0.5000	.25 .25 .25 .25
13	.325 .30 .325 .30	.175 .20 .175 .20	0.3750	.25 .25 .25 .25	.50 .50	0.5000	.125 .125 .125 .125
14	.40 .475 .40 .475	.10 .025 .10 .025	0.1250	.25 .25 .25 .25	.50 .50	0.5000	.375 .375 .375 .375
15	.40 .475 .40 .475	.10 .025	0.1250	.25 .25 .25 .25	.50 .50	0.5000	.40 .475 .40 .475
16	.275 .275 .275 .275	.225 .225 .225 .225	0.4500	.25 .25 .25 .25	.50 .50	0.5000	.275 .275 .275 .275
17	.40 .475 .40 .475	.10 .025 .10 .025	0.1250	.375 .375 .375 .375	.50 .50	0.2500	.375 .375 .375 .375
18	.375 .375 .375 .375	.125 .125 .125 .125	0.2500	.25 .25 .25 .25	.50 .50	0.5000	.375 .375 .375 .375

Table A.3.1: Case a, Results for Varying Inputs, $n = 2$

	β_2	μ_2	S_3	β_3	μ_3	K2	K3	θ_2
1	.50 .50	0.2500	.325 .30 .325 .30	.50 .50	0.3750	1	1	0.5
2	.50 .50	0.3750	.40 .475 .40 .475	.50 .50	0.1250	1	1	0.75
3	.50 .50	0.2500	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5
4	.50 .50	0.0840	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5
5	.50 .50	0.3750	.325 .30 .325 .30	.50 .50	0.3750	1	1	0.5
6	.50 .50	0.7500	.125 .125 .125 .125	.50 .50	0.7500	1	1	0.5
7	.50 .50	0.1250	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5
8	.50 .50	0.2500	.45 .45 .50 .50	.50 .50	0.0500	1	1	0.5
9	.50 .50	0.0500	.45 .45 .50 .50	.50 .50	0.0500	1	1	0.5
10	.50 .50	0.1250	.325 .30 .325 .30	.50 .50	0.3750	1	1	0.25
11	.50 .50	0.2500	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5
12	.50 .50	0.5000	.25 .25 .25 .25	.50 .50	0.5000	1	1	0.5
13	.50 .50	0.7500	.375 .375 .375 .375	.50 .50	0.2500	2	2	0.5
14	.50 .50	0.2500	.375 .375 .375 .375	.50 .50	0.2500	2	2	0.5
15	.50 .50	0.1250	.40 .475 .40 .475	.50 .50	0.1250	1	1	0.5
16	.50 .50	0.4500	.275 .275 .275 .275	.50 .50	0.4500	1	1	0.5
17	.50 .50	0.2500	.40 .475 .40 .475	.50 .50	0.1250	1	1	0.5
18	.50 .50	0.2500	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5

Table A.3.1: Case a, Results for Varying Inputs, $n = 2$

	ρ_1	μ_{q1} combo	μ_{q1} iso	μ_1^*	μ_{q1} % error	ρ_2	μ_{q2} combo
1	0.5000	1.0871	1.6718	0.5551	53.79%	0.7500	1.0035
2	0.1000	0.1060	0.1070	0.4947	0.94%	0.1000	0.1064
3	0.1000	0.1061	0.1069	0.4948	0.75%	0.1000	0.1073
4	0.2500	0.3610	0.7497	0.3301	107.67%	0.7440	0.8725
5	0.7500	2.4751	2.7428	0.4624	10.82%	0.5000	0.6564
6	0.7500	1.8915	1.9051	0.4978	0.72%	0.2500	0.2683
7	0.2500	0.3273	0.4371	0.4082	33.55%	0.5000	0.6118
8	0.5000	0.9714	1.1083	0.0941	14.09%	0.1000	0.1219
9	0.1000	0.2155	0.4011	0.2221	86.13%	0.5000	0.7101
10	0.2500	0.3011	0.3218	0.4741	6.87%	0.2500	0.2958
11	0.2500	0.3021	0.3178	0.4740	5.20%	0.2500	0.2944
12	0.5000	0.7614	0.7730	0.4925	1.52%	0.2500	0.2827
13	0.7500	2.2740	2.4781	0.4750	8.98%	0.2500	0.2964
14	0.2500	0.2936	0.2955	0.4959	0.65%	0.2500	0.3075
15	0.2500	0.4182	0.5904	0.3509	41.18%	0.5000	0.6972
16	0.9000	9.1792	10.7596	0.4731	17.22%	0.5000	0.5810
17	0.5000	0.9396	1.0530	0.2323	12.07%	0.2500	0.3246
18	0.5000	0.9514	1.1194	0.4305	17.66%	0.5000	0.6766

Table A.3.1: Case a, Results for Varying Inputs, $n = 2$

	μ_{a2} iso	λ_2	BP_2	μ_{a2} % error	ρ_3	μ_{a3} combo	μ_{a3} iso
1	0.6966	0.1466	0.1690	30.58%	0.5000	0.7481	0.5270
2	0.0993	0.0357	0.0042	6.67%	0.1000	0.1081	0.1011
3	0.1002	0.0238	0.0053	6.62%	0.1000	0.1073	0.1002
4	0.6521	0.0464	0.1616	25.26%	0.2500	0.3656	0.2789
5	0.4097	0.1345	0.0610	37.58%	0.5000	0.6564	0.4097
6	0.1881	0.1363	0.0066	29.89%	0.2500	0.2683	0.1881
7	0.4832	0.0515	0.0974	21.02%	0.2500	0.3266	0.2602
8	0.0737	0.0180	0.0016	39.54%	0.5000	0.5841	0.3776
9	0.5469	0.0232	0.1251	22.98%	0.5000	0.7101	0.5469
10	0.2468	0.0274	0.0319	16.57%	0.2500	0.2911	0.2429
11	0.2457	0.0554	0.0275	16.54%	0.2500	0.2944	0.2457
12	0.2138	0.0998	0.0152	24.37%	0.2500	0.2827	0.2138
13	0.1932	0.1394	0.0003	34.82%	0.7500	1.2385	0.7467
14	0.2581	0.0555	0.0041	16.07%	0.2500	0.3075	0.2581
15	0.5104	0.0543	0.1062	26.79%	0.5000	0.6972	0.5104
16	0.3852	0.1536	0.0511	33.70%	0.5000	0.5810	0.3852
17	0.2035	0.0470	0.0167	37.31%	0.5000	0.6124	0.4014
18	0.4560	0.0981	0.0807	32.60%	0.5000	0.6766	0.4560

Table A.3.1: Case *a*, Results for Varying Inputs, $n = 2$

	λ_3	BP_3	μ_{a3} % error	Q	μ_q combo	μ_q iso	μ_q % error
1	0.1691	0.0975	29.55%	0.1481	2.8387	2.8954	2.00%
2	0.0119	0.0065	6.48%	0.3479	0.3205	0.3074	4.09%
3	0.0238	0.0053	6.62%	0.3416	0.3207	0.3073	4.18%
4	0.0624	0.0339	23.71%	0.1671	1.5991	1.6807	5.10%
5	0.1345	0.0610	37.58%	0.0811	3.7879	3.5622	5.96%
6	0.1363	0.0066	29.89%	0.1227	2.4281	2.2813	6.05%
7	0.0584	0.0302	20.33%	0.1980	1.2657	1.1805	6.73%
8	0.0164	0.0624	35.35%	0.0465	1.6774	1.5596	7.02%
9	0.0232	0.1251	22.98%	0.0495	1.6357	1.4949	8.61%
10	0.0834	0.0227	16.56%	0.2729	0.888	0.8115	8.61%
11	0.0554	0.0275	16.54%	0.2666	0.8909	0.8092	9.17%
12	0.0998	0.0152	24.37%	0.2319	1.3268	1.2006	9.51%
13	0.1311	0.0526	39.71%	0.0922	3.8089	3.418	10.26%
14	0.0555	0.0041	16.07%	0.2952	0.9086	0.8117	10.66%
15	0.0543	0.1062	26.79%	0.1132	1.8126	1.6112	11.11%
16	0.1536	0.0511	33.70%	0.0242	10.3412	11.53	11.50%
17	0.0434	0.0679	34.45%	0.0993	1.8766	1.6579	11.65%
18	0.0981	0.0807	32.60%	0.1416	2.3046	2.0314	11.85%

Table A.3.1: Case *a*, Results for Varying Inputs, $n = 2$

	D0	D1	λ	S_1	β_1	μ_1	S_2
19	.325 .30 .325 .30	.175 .20 .175 .20	0.3750	.25 .25 .25 .25	.50 .50	0.5000	.325 .30 .325 .30
20	.375 .375 .375 .375	.125 .125 .125 .125	0.2500	.25 .25 .25 .25	.50 .50	0.5000	.416 .416 .416 .416
21	.40 .475 .40 .475	.10 .025 .10 .025	0.1250	.25 .25 .25 .25	.50 .50	0.5000	.4 .4125 .4 .4125
22	.325 .30 .325 .30	.175 .20 .175 .20	0.3750	.25 .25 .25 .25	.50 .50	0.5000	.125 .125 .125 .125
23	.325 .30 .325 .30	.175 .20 .175 .20	0.3750	.25 .25 .25 .25	.50 .50	0.5000	.375 .375 .375 .375
24	.40 .475 .40 .475	.10 .025 .10 .025	0.1250	.25 .25 .25 .25	.50 .50	0.5000	.458 .458 .458 .458
25	.275 .275 .275 .275	.225 .225 .225 .225	0.4500	.05 .05 .05 .05	.50 .50	0.9000	.375 .375 .375 .375
26	.275 .275 .275 .275	.225 .225 .225 .225	0.4500	.25 .25 .25 .25	.50 .50	0.5000	.375 .375 .375 .375

Table A.3.1: Case *a*, Results for Varying Inputs, $n = 2$

	β_2	μ_2	S_3	β_3	μ_3	K2	K3	θ_2
19	.50 .50	0.3750	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5
20	.50 .50	0.1680	.416 .416 .416 .416	.50 .50	0.1680	1	1	0.5
21	.50 .50	0.1875	.40 .475 .40 .475	.50 .50	0.1250	1	1	0.75
22	.50 .50	0.7500	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5
23	.50 .50	0.2500	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5
24	.50 .50	0.0840	.458 .458 .458 .458	.50 .50	0.0840	1	1	0.5
25	.50 .50	0.2500	.275 .275 .275 .275	.50 .50	0.4500	1	1	0.5
26	.50 .50	0.2500	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5

Table A.3.1: Case a, Results for Varying Inputs, $n = 2$

	ρ_1	μ_{q1} combo	μ_{q1} iso	μ_1^*	μ_{q1} % error	ρ_2	μ_{q2} combo
19	0.7500	3.6230	5.0044	0.4272	38.13%	0.5000	0.7359
20	0.5000	2.3678	2.7657	0.3396	16.80%	0.7440	1.1713
21	0.2500	0.3633	0.4227	0.4067	16.35%	0.5000	0.6412
22	0.7500	2.6230	3.7979	0.4438	44.79%	0.2500	0.3104
23	0.7500	11.1735	14.4792	0.3942	29.59%	0.7500	0.9677
24	0.2500	1.4300	1.5647	0.2354	9.42%	0.7440	1.2145
25	0.5000	1.2529	2.5552	0.5985	103.94%	0.9000	1.1766
26	0.9000					0.9000	

Table A.3.1: Case a, Results for Varying Inputs, $n = 2$

	μ_{q2} iso	λ_2	BP_2	μ_{q2} % error	ρ_3	μ_{q3} combo	μ_{q3} iso
19	0.4251	0.1393	0.0644	42.23%	0.7500	1.0008	0.5940
20	0.6646	0.0943	0.1587	43.26%	0.7440	1.1713	0.6646
21	0.4923	0.0789	0.0958	23.22%	0.2500	0.3643	0.2744
22	0.1982	0.1435	0.0070	36.15%	0.7500	0.9255	0.5841
23	0.6152	0.1300	0.1342	36.43%	0.7500	0.9677	0.6152
24	0.7208	0.0516	0.1889	40.65%	0.7440	1.2145	0.7208
25	0.8132	0.1718	0.2196	30.89%	0.5000	0.7896	0.5738
26					0.9000		

Table A.3.1: Case a, Results for Varying Inputs, $n = 2$

	λ_3	BP_3	μ_{q3} % error	Q	μ_q combo	μ_q iso	μ_q % error
19	0.1257	0.1274	40.65%	0.0522	5.3597	6.0235	12.39%
20	0.0943	0.1587	43.26%	0.0537	4.7104	4.0949	13.07%
21	0.0303	0.0380	24.68%	0.1554	1.3688	1.1894	13.11%
22	0.1237	0.1244	36.89%	0.0794	3.8589	4.5802	18.69%
23	0.1300	0.1342	36.43%	0.0166	13.1089	15.7096	19.84%
24	0.0516	0.1889	40.65%	0.0401	3.859	3.0063	22.10%
25	0.2201	0.2201	27.33%	0.1422	3.2191	3.9422	22.46%
26				unstable			

Table A.3.2: Case *b*, Results for Varying Inputs, $n = 2$

	D0	D1	λ	S_1	β_1	μ_1	S_2
1	.41 .41 .41 .41	.09 .09 .09 .09	0.1800	.40 .40 .40 .40	.50 .50	0.2000	.05 .05 .05 .05
2	.275 .275 .275 .275	.225 .225 .225 .225	0.4500	.25 .25 .25 .25	.50 .50	0.5000	.05 .05 .05 .05
3	.425 .425 .425 .425	.075 .075 .075 .075	0.1500	.40 .40 .40 .40	.50 .50	0.2000	.125 .125 .125 .125
4	.45 .45 .50 .50	.05 .05 0 0	0.0500	.25 .25 .25 .25	.50 .50	0.5000	.325 .30 .325 .30
5	.45 .45 .50 .50	.05 .05 0 0	0.0500	.25 .25 .25 .25	.50 .50	0.5000	.375 .375 .375 .375
6	.40 .475 .40 .475	.10 .025 .10 .025	0.1250	.375 .375 .375 .375	.50 .50	0.2500	.188 .188 .188 .188
7	.325 .30 .325 .30	.175 .20 .175 .20	0.3750	.25 .25 .25 .25	.50 .50	0.5000	.125 .125 .125 .125
8	.275 .275 .275 .275	.225 .225 .225 .225	0.4500	.25 .25 .25 .25	.50 .50	0.5000	.2 .2375 .2 .2375
9	.375 .375 .375 .375	.125 .125 .125 .125	0.2500	.25 .25 .25 .25	.50 .50	0.5000	.25 .25 .25 .25
10	.40 .475 .40 .475	.10 .025 .10 .025	0.1250	.25 .25 .25 .25	.50 .50	0.5000	.375 .375 .375 .375
11	.40 .475 .40 .475	.10 .025 .10 .025	0.1250	.25 .25 .25 .25	.50 .50	0.5000	.375 .375 .375 .375
12	.45 .45 .50 .50	.05 .05 0 0	0.0500	.25 .25 .25 .25	.50 .50	0.5000	.45 .45 .45 .45
13	.40 .475 .40 .475	.10 .025 .10 .025	0.1250	.25 .25 .25 .25	.50 .50	0.5000	.40 .475 .40 .475
14	.325 .30 .325 .30	.175 .20 .175 .20	0.3750	.25 .25 .25 .25	.50 .50	0.5000	.25 .2812 .25 .2812
15	.40 .40 .40 .40	.10 .10 .10 .10	0.2000	.25 .25 .25 .25	.50 .50	0.5000	.30 .30 .30 .30
16	.40 .475 .40 .475	.10 .025 .10 .025	0 1250	.25 .25 .25 .25	.50 .50	0.5000	.375 .375 .375 .375
17	.375 .375 .375 .375	.125 .125 .125 .125	0.2500	.25 .25 .25 .25	.50 .50	0.5000	.35 .3375 .35 .3375
18	.45 .45 .45 .45	.05 .05 .05 .05	0.1000	.375 .375 .375 .375	.50 .50	0.2500	.25 .25 .25 .25

Table A.3.2: Case b, Results for Varying Inputs, $n = 2$

	β_2	μ_2	S_3	β_3	μ_3	K2	K3	θ_2
1	.50 .50	0.9000	.05 .05 .05 .05	.50 .50	0.9000	1	1	0.5
2	.50 .50	0.9000	.05 .05 .05 .05	.50 .50	0.9000	1	1	0.5
3	.50 .50	0.7500	.125 .125 .125 .125	.50 .50	0.7500	1	1	0.5
4	.50 .50	0.3750	.40 .475 .40 .475	.50 .50	0.1250	1	1	0.75
5	.50 .50	0.2500	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5
6	.50 .50	0.6250	.188 .188 .188 .188	.50 .50	0.6250	1	1	0.5
7	.50 .50	0.7500	.125 .125 .125 .125	.50 .50	0.7500	1	1	0.5
8	.50 .50	0.5625	.2 .2375 .2 .2375	.50 .50	0.5625	1	1	0.5
9	.50 .50	0.5000	.25 .25 .25 .25	.50 .50	0.5000	1	1	0.5
10	.50 .50	0.2500	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5
11	.50 .50	0.2500	.375 .375 .375 .375	.50 .50	0.2500	2	2	0.5
12	.50 .50	0.1	.45 .45 .45 .45	.50 .50	0.1	1	1	0.5
13	.50 .50	0.1250	.325 .30 .325 .30	.50 .50	0.3750	1	1	0.25
14	.50 .50	0.4688	.25 .2812 .25 .2812	.50 .50	0.4688	1	1	0.5
15	.50 .50	0.4000	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5
16	.50 .50	0.2500	.4 .44375 .4 .44375	.50 .50	0.1563	1	1	0.5
17	.50 .50	0.3125	.35 .3375 .35 .3375	.50 .50	0.3125	1	1	0.5
18	.50 .50	0.5000	.40 .475 .40 .475	.50 .50	0.1250	1	1	0.5

Table A.3.2: Case b, Results for Varying Inputs, $n = 2$

	ρ_1	μ_{q1} combo	μ_{q1} iso	μ_1^*	μ_{q1} % error	ρ_2	μ_{q2} combo
1	0.9000	7.3811	7.3819	0.2000	0.01%	0.1000	0.0995
2	0.9000	4.9697	5.0138	0.4993	0.89%	0.2500	0.2575
3	0.7500	2.5515	2.5509	0.2000	0.02%	0.1000	0.1027
4	0.1000	0.1072	0.1071	0.4947	0.09%	0.1000	0.1065
5	0.1000	0.1070	0.1069	0.4947	0.09%	0.1000	0.1077
6	0.5000	0.8758	0.8755	0.2499	0.03%	0.1000	0.1040
7	0.7500	1.8957	1.9051	0.4978	0.50%	0.2500	0.2690
8	0.9000	6.9394	6.4746	0.4880	6.70%	0.4000	0.4811
9	0.5000	0.7759	0.7731	0.2425	0.36%	0.2500	0.2853
10	0.2500	0.3281	0.3179	0.4739	3.11%	0.2500	0.2999
11	0.2500	0.3281	0.3179	0.4739	3.11%	0.2500	0.2999
12	0.1000	0.1513	0.1360	0.4214	10.11%	0.2500	0.3080
13	0.2500	0.3376	0.3221	0.4739	4.59%	0.2500	0.3062
14	0.7500	2.3498	2.1961	0.4819	6.54%	0.4000	0.5010
15	0.4000	0.6732	0.6149	0.4671	8.66%	0.2500	0.2927
16	0.2500	0.4406	0.3771	0.4386	14.41%	0.2500	0.3019
17	0.5000	1.0269	0.9050	0.4626	11.87%	0.4000	0.5240
18	0.4000	0.7683	0.6692	0.2373	12.90%	0.1000	0.1056

Table A.3.2: Case *b*, Results for Varying Inputs, $n = 2$

	μ_{a2} iso	λ_2	BP_2	μ_{a2} % error	ρ_3	μ_{a3} combo	μ_{a3} iso
1	0.0583	0.0523	0.0001	41.41%	0.1000	0.0995	0.0583
2	0.1748	0.1552	0.0025	32.12%	0.2500	0.2575	0.1748
3	0.0629	0.0469	0.0004	38.75%	0.1000	0.1027	0.0629
4	0.0992	0.0357	0.0042	6.85%	0.1000	0.1087	0.1014
5	0.1002	0.0238	0.0053	6.96%	0.1000	0.1077	0.1002
6	0.0736	0.0454	0.0009	29.23%	0.1000	0.1040	0.0736
7	0.1881	0.1363	0.0066	30.07%	0.2500	0.2690	0.1881
8	0.2994	0.1543	0.0276	37.77%	0.4000	0.4811	0.2994
9	0.2137	0.0998	0.0153	25.10%	0.2500	0.2853	0.2137
10	0.2446	0.0551	0.0275	18.44%	0.2500	0.2999	0.2446
11	0.2446	0.0551	0.0275	18.44%	0.2500	0.2999	0.2446
12	0.2656	0.0234	0.0373	13.77%	0.2500	0.3080	0.2656
13	0.2480	0.0275	0.0325	19.01%	0.2500	0.2935	0.2407
14	0.3195	0.1351	0.0353	36.23%	0.4000	0.5010	0.3195
15	0.2229	0.0821	0.0194	23.85%	0.4000	0.5302	0.3760
16	0.2407	0.0542	0.0270	20.27%	0.4000	0.5425	0.4020
17	0.3562	0.0981	0.0506	32.02%	0.4000	0.5240	0.3562
18	0.0774	0.0380	0.0014	26.70%	0.4000	0.5430	0.3473

Table A.3.2: Case *b*, Results for Varying Inputs, $n = 2$

	λ_3	BP_3	μ_{q3} % error	Q	μ_q combo	μ_q iso	μ_q % error
1	0.0523	0.0001	41.41%	0.0200	7.5801	7.4985	1.08%
2	0.1552	0.0025	32.12%	0.0497	5.4847	5.3634	2.21%
3	0.0469	0.0004	38.75%	0.0499	2.7569	2.6767	2.91%
4	0.0119	0.0066	6.72%	0.2862	0.3224	0.3077	4.56%
5	0.0238	0.0053	6.96%	0.2853	0.3224	0.3073	4.68%
6	0.0454	0.0009	29.23%	0.1244	1.0838	1.0227	5.64%
7	0.1363	0.0066	30.07%	0.1223	2.4337	2.2813	6.26%
8	0.1543	0.0276	37.77%	0.0335	7.9016	7.0734	10.48%
9	0.0998	0.0153	25.10%	0.2225	1.3465	1.2005	10.84%
10	0.0551	0.0275	18.44%	0.2103	0.9279	0.8071	13.02%
11	0.0551	0.0275	18.44%	0.2103	0.9279	0.8071	13.02%
12	0.0234	0.0373	13.77%	0.0976	0.7673	0.6672	13.05%
13	0.0826	0.0225	17.99%	0.2112	0.9373	0.8108	13.50%
14	0.1351	0.0353	36.23%	0.0898	3.3518	2.8351	15.42%
15	0.0821	0.0584	29.08%	0.1767	1.4961	1.2138	18.87%
16	0.0542	0.0706	25.90%	0.1384	1.2850	1.0198	20.64%
17	0.0981	0.0506	32.02%	0.1395	2.0749	1.6174	22.05%
18	0.0380	0.0530	36.04%	0.0913	1.4169	1.0939	22.80%

Table A.3.2: Case *b*, Results for Varying Inputs, $n = 2$

	D0	D1	λ	S_1	β_1	μ_1	S_2
19	.40 .475 .40 .475	.10 .025 .10 .025	0.1250	.25 .25 .25 .25	.50 .50	0.5000	.4 .44375 .4 .44375
20	.40 .475 .40 .475	.10 .025 .10 .025	0.1250	.25 .25 .25 .25	.50 .50	0.5000	.4 .4125 .4 .4125
21	.45 .45 .50 .50	.05 .05 0 0	0.0500	.25 .25 .25 .25	.50 .50	0.5000	.45 .4875 .45 .4875
22	.40 .475 .40 .475	.10 .025 .10 .025	0.1250	.25 .25 .25 .25	.50 .50	0.5000	.40 .475 .40 .475
23	.325 .30 .325 .30	.175 .20 .175 .20	0.3750	.25 .25 .25 .25	.50 .50	0.5000	.325 .30 .325 .30
24	.40 .475 .40 .475	.10 .025 .10 .025	0.1250	.375 .375 .375 .375	.50 .50	0.25	.375 .375 .375 .375
25	.45 .45 .50 .50	.05 .05 0 0	0.0500	.25 .25 .25 .25	.50 .50	0.5000	.45 .45 .45 .45
26	.45 .45 .50 .50	.05 .05 0 0	0.0500	.45 .45 .45 .45	.50 .50	0.1	.375 .375 .375 .375
27	.375 .375 .375 .375	.125 .125 .125 .125	0.2500	.25 .25 .25 .25	.50 .50	0.5000	.375 .375 .375 .375
28	.45 .45 .50 .50	.05 .05 0 0	0.0500	.25 .25 .25 .25	.50 .50	0.5000	.45 .45 .50 .50
29	.40 .475 .40 .475	.10 .025	0.1250	.25 .25 .25 .25	.50 .50	0.5000	.40 .475 .40 .475
30	.275 .275 .275 .275	.225 .225 .225 .225	0.4500	.25 .25 .25 .25	.50 .50	0.5000	.275 .275 .275 .275
31	.325 .30 .325 .30	.175 .20 .175 .20	0.3750	.25 .25 .25 .25	.50 .50	0.5000	.125 .125 .125 .125
32	.325 .30 .325 .30	.175 .20 .175 .20	0.3750	.125 .125 .125 .125	.50 .50	0.75	.375 .375 .375 .375
33	.40 .475 .40 .475	.10 .025 .10 .025	0.1250	.25 .25 .25 .25	.50 .50	0.5000	.458 .458 .458 .458
34	.325 .30 .325 .30	.175 .20 .175 .20	0.3750	.25 .25 .25 .25	.50 .50	0.5000	.125 .125 .125 .125
35	.40 .475 .40 .475	.10 .025 .10 .025	0.1250	.25 .25 .25 .25	.50 .50	0.5000	.458 .458 .458 .458
36	.275 .275 .275 .275	.225 .225 .225 .225	0.4500	.05 .05 .05 .05	.50 .50	0.9000	.375 .375 .375 .375
37	.275 .275 .275 .275	.225 .225 .225 .225	0.4500	.25 .25 .05 .05	.50 .50 .50 .50	0.5000	.375 .375 .375 .375

Table A.3.2: Case b, Results for Varying Inputs, $n = 2$

	β_2	μ_2	S_3	β_3	μ_3	K2	K3	θ_2
19	.50 .50	0.1563	.4 .44375 .4 .44375	.50 .50	0.1563	1	1	0.5
20	.50 .50	0.1875	.40 .475 .40 .475	.50 .50	0.1250	1	1	0.75
21	.50 .50	0.0625	.45 .4875 .45 .4875	.50 .50	0.0625	1	1	0.5
22	.50 .50	0.1250	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5
23	.50 .50	0.3750	.325 .30 .325 .30	.50 .50	0.3750	1	1	0.5
24	.50 .50	0.2500	.40 .475 .40 .475	.50 .50	0.1250	1	1	0.5
25	.50 .50	0.1	.45 .45 .50 .50	.50 .50	0.0500	1	1	0.5
26	.50 .50	0.2500	.45 .45 .50 .50	.50 .50	0.0500	1	1	0.5
27	.50 .50	0.2500	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5
28	.50 .50	0.0500	.45 .45 .50 .50	.50 .50	0.0500	1	1	0.5
29	.50 .50	0.1250	.40 .475 .40 .475	.50 .50	0.1250	1	1	0.5
30	.50 .50	0.4500	.275 .275 .275 .275	.50 .50	0.4500	1	1	0.5
31	.50 .50	0.7500	.375 .375 .375 .375	.50 .50	0.2500	2	2	0.5
32	.50 .50	0.2500	.325 .30 .325 .30	.50 .50	0.3750	1	1	0.5
33	.50 .50	0.0840	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5
34	.50 .50	0.7500	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5
35	.50 .50	0.0840	.40 .475 .40 .475	.50 .50	0.1250	1	1	0.5
36	.50 .50	0.2500	.275 .275 .275 .275	.50 .50	0.4500	1	1	0.5
37	.50 .50	0.2500	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5

Table A.3.2: Case b, Results for Varying Inputs, $n = 2$

	ρ_1	μ_{q1} combo	μ_{q1} iso	μ_1^*	μ_{q1} % error	ρ_2	μ_{q2} combo
19	0.2500	0.5713	0.4242	0.4093	25.75%	0.4000	0.5457
20	0.2500	0.5677	0.4208	0.4072	25.88%	0.5000	0.7219
21	0.1000	0.4130	0.2382	0.3022	42.32%	0.4000	0.5575
22	0.2500	0.6598	0.4425	0.4059	32.93%	0.5000	0.7297
23	0.7500	3.8007	2.7362	0.4626	28.01%	0.5000	0.6843
24	0.5000	1.4326	1.0599	0.2317	26.02%	0.2500	0.3028
25	0.1000	0.5012	0.2529	0.2988	49.54%	0.2500	0.3149
26	0.5000	1.5542	1.1186	0.0938	28.03%	0.1000	0.1086
27	0.5000	1.6712	1.1079	0.4321	33.71%	0.5000	0.7126
28	0.1000	1.0027	0.3648	0.2371	63.62%	0.5000	0.7540
29	0.2500	1.1425	0.5667	0.3586	50.40%	0.5000	0.7397
30	0.9000	24.2203	10.7601	0.4731	55.57%	0.5000	0.5431
31	0.7500	6.1006	2.5132	0.4738	58.80%	0.2500	0.2711
32	0.5000	5.5656	1.6355	0.5596	70.61%	0.7500	1.2277
33	0.2500	3.1860	0.7880	0.3231	75.27%	0.7440	1.2649
34	0.7500	15.9787	4.0064	0.4405	74.93%	0.2500	0.2339
35	0.2500	5.4120	0.9158	0.2965	83.08%	0.7440	1.2751
36	0.5000	296.9764	2.4995	0.6036	99.16%	0.9000	0.1685
37	0.9000					0.9000	

Table A.3.2: Case *b*, Results for Varying Inputs, $n = 2$

	μ_{a2} iso	λ_2	BP_2	μ_{a2} % error	ρ_3	μ_{a3} combo	μ_{a3} iso
19	0.3957	0.0534	0.0693	27.49%	0.4000	0.5457	0.3957
20	0.5002	0.0802	0.0983	30.71%	0.2500	0.3082	0.2403
21	0.4191	0.0224	0.0819	24.83%	0.4000	0.5575	0.4191
22	0.4992	0.0532	0.1027	31.59%	0.2500	0.3051	0.2361
23	0.4051	0.1330	0.0606	40.80%	0.5000	0.6843	0.4051
24	0.1923	0.0445	0.0155	36.49%	0.5000	0.7223	0.4117
25	0.2533	0.0224	0.0353	19.56%	0.5000	0.7428	0.5258
26	0.0691	0.0169	0.0014	36.37%	0.5000	0.7313	0.3907
27	0.4442	0.0957	0.0785	37.66%	0.5000	0.7126	0.4442
28	0.5039	0.0215	0.1109	33.17%	0.5000	0.7540	0.5039
29	0.4829	0.0515	0.0985	34.72%	0.5000	0.7397	0.4829
30	0.3827	0.1525	0.0512	29.53%	0.5000	0.5431	0.3827
31	0.1848	0.1333	0.0003	31.83%	0.7500	1.5075	0.7627
32	0.7033	0.1483	0.1719	42.71%	0.5000	0.7169	0.4604
33	0.6918	0.0495	0.1757	45.31%	0.2500	0.3232	0.2194
34	0.1784	0.1292	0.0063	23.73%	0.7500	1.0136	0.6104
35	0.6729	0.0482	0.1696	47.23%	0.5000	0.7747	0.4507
36	0.8336	0.1766	0.2271	394.72%	0.5000	0.0814	0.4575
37					0.9000		

Table A.3.2: Case *b*, Results for Varying Inputs, $n = 2$

	λ_3	BP_3	μ_{q3} % error	Q	μ_q combo	μ_q iso	μ_q % error
19	0.0534	0.0693	27.49%	0.1005	1.6627	1.2156	26.89%
20	0.0267	0.0312	22.03%	0.1006	1.5978	1.1613	27.32%
21	0.0224	0.0819	24.83%	0.0431	1.5280	1.0764	29.55%
22	0.0532	0.0264	22.62%	0.0998	1.6946	1.1778	30.50%
23	0.1330	0.0606	40.80%	0.0536	5.1693	3.5464	31.39%
24	0.0445	0.0711	43.00%	0.0613	2.4577	1.6639	32.30%
25	0.0224	0.1170	29.21%	0.0428	1.5589	1.0320	33.80%
26	0.0169	0.0663	46.57%	0.0255	2.3941	1.5784	34.07%
27	0.0957	0.0785	37.66%	0.0853	3.0964	1.9963	35.53%
28	0.0215	0.1109	33.17%	0.0246	2.5107	#REF!	#REF!
29	0.0515	0.0985	34.72%	0.0580	2.6219	1.5325	41.55%
30	0.1525	0.0512	29.53%	0.0092	25.3065	11.5255	54.46%
31	0.1333	0.0552	49.41%	0.0416	7.8792	3.4607	56.08%
32	0.1483	0.0826	35.78%	0.0469	7.5102	2.7992	62.73%
33	0.0495	0.0243	32.12%	0.0362	4.7741	1.6992	64.41%
34	0.1292	0.1329	39.78%	0.0172	17.2262	4.7952	72.16%
35	0.0482	0.0908	41.82%	0.0198	7.4618	2.0394	72.67%
36	0.1766	0.0824	462.04%	0.0010	297.2263	3.7906	98.72%
37				unstable			

Table A.3.2: Case *b*, Results for Varying Inputs, $n = 2$

	D0	D1	λ	S_1	β_1	μ_1	S_2
38	.275 .275 .275 .275	.225 .225 .225 .225	0.4500	.05 .05 .05 .05	.50 .50	0.9000	.375 .375 .375 .375
39	.325 .30 .325 .30	.175 .20 .175 .20	0.3750	.25 .25 .25 .25	.50 .50	0.5000	.375 .375 .375 .375
40	.325 .30 .325 .30	.175 .20 .175 .20	0.3750	.25 .25 .25 .25	.50 .50	0.5000	.325 .30 .325 .30
41	.375 .375 .375 .375	.125 .125 .125 .125	0.2500	.25 .25 .25 .25	.50 .50	0.5000	.416 .416 .416 .416
42	.40 .475 .40 .475	.10 .025 .10 .025	0.1250	.25 .25 .25 .25	.50 .50	0.5000	.458 .458 .458 .458
43	.45 .45 .50 .50	.05 .05 0 0	0.0500	.25 .25 .25 .25	.50 .50	0.5000	.48 .4864 .48 .4864
44	.45 .45 .50 .50	.05 .05 0 0	0.0500	.25 .25 .25 .25	.50 .50	0.5000	.486 .486 .486 .486

Table A.3.2: Case *b*, Results for Varying Inputs, $n = 2$

	β_2	μ_2	S_3	β_3	μ_3	K2	K3	θ_2
38	.50 .50	0.2500	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5
39	.50 .50	0.2500	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5
40	.50 .50	0.3750	.375 .375 .375 .375	.50 .50	0.2500	1	1	0.5
41	.50 .50	0.1680	.416 .416 .416 .416	.50 .50	0.1680	1	1	0.5
42	.50 .50	0.0840	.458 .458 .458 .458	50 .50	0.0840	1	1	0.5
43	.50 .50	0.0336	.48 .4864 .48 .4864	.50 .50	0.0336	1	1	0.5
44	.50 .50	0.0280	.486 .486 .486 .486	.50 .50	0.0280	1	1	0.5

Table A.3.2: Case *b*, Results for Varying Inputs, $n = 2$

	ρ_1	ρ_2	ρ_3	Q
38	0.5000	0.9000	0.9000	unstable
39	0.7500	0.7500	0.7500	unstable
40	0.7500	0.5000	0.7500	unstable
41	0.5000	0.7440	0.7440	unstable
42	0.2500	0.7440	0.7440	unstable
43	0.1000	0.7440	0.7440	unstable
44	0.1000	0.8929	0.8929	unstable

Appendix A.4

Split Networks With a Geometric Arrival Process

A.4.1 Sub-Matrices of the Transition Matrix P

Each of the matrices within **P** for both case *a* and case *b* are defined as follows:

B0 =

$$
\begin{array}{c|cccccccc}
L_2 & 0 & 1 & 2 & \cdots & K2-1 & K2 & \overset{j=0}{K2+1} & \overset{j=2}{K2+1} \\
\hline
0 & \mathbf{b0B0} & & & & & & & \\
1 & \mathbf{a2B0} & \mathbf{a1B0} & & & & & & \\
2 & & \mathbf{a2B0} & \mathbf{a1B0} & & & & & \\
\ddots & & & & \ddots & & & & \\
K2 & & & & & \mathbf{a2B0} & \mathbf{a1B0} & & \\
j=0\;\; K2+1 & & & & & & \mathbf{a2B0} & \mathbf{a1B0} &
\end{array}
$$

b0B0 =

$$
\begin{array}{c|cccccccccc}
L_3 & 0 & 1 & 2 & \cdots & K3-1 & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \\
\hline
0 & q & & & & & & & & & \\
1 & q\beta_3 & q\alpha_3 & & & & & & & & \\
2 & & q\beta_3 & q\alpha_3 & & & & & & & \\
\ddots & & & & \ddots & & & & & & \\
K3 & & & & & q\beta_3 & q\alpha_3 & & & & \\
j=0\;\; K3+1 & & & & & & q\beta_3 & q\alpha_3 & & &
\end{array}
$$

a2B0 =

$$
\begin{array}{c|cccccccccc}
L_3 & 0 & 1 & 2 & \cdots & K3-1 & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \\
\hline
0 & q\beta_2 & & & & & & & & & \\
1 & q\beta_2\beta_3 & q\beta_2\alpha_3 & & & & & & & & \\
2 & & q\beta_2\beta_3 & q\beta_2\alpha_3 & & & & & & & \\
\ddots & & & & \ddots & & & & & & \\
K3 & & & & & q\beta_2\beta_3 & q\beta_2\alpha_3 & & & & \\
j=0\;\; K3+1 & & & & & & q\beta_2\beta_3 & q\beta_2\alpha_3 & & &
\end{array}
$$

a1B0 =

$$
\begin{array}{c|cccccccccc}
L_3 & 0 & 1 & 2 & & K3-1 & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \\
\hline
0 & q\alpha_2 & & & & & & & & & \\
1 & q\alpha_2\beta_3 & q\alpha_2\alpha_3 & & & & & & & & \\
2 & & q\alpha_2\beta_3 & q\alpha_2\alpha_3 & & & & & & & \\
. & & & & . & & & & & & \\
K3 & & & & & q\alpha_2\beta_3 & q\alpha_2\alpha_3 & & & & \\
j=0\;\; K3+1 & & & & & & q\alpha_2\beta_3 & q\alpha_2\alpha_3 & & &
\end{array}
$$

B1 =

$$
\begin{array}{rr}
 & \begin{array}{cccccccc} & & & & & & j=0 & j=2 \\ 0 & 1 & 2 & \dots & K2-1 & K2 & K2+1 & K2+1 \end{array} \\
\begin{array}{r} L_2 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K2 \\ j=0 \quad K2+1 \end{array} &
\left[\begin{array}{cccccccc}
\mathbf{b0B1} & & & & & & & \\
\mathbf{a2B1} & \mathbf{a1B1} & & & & & & \\
 & \mathbf{a2B1} & \mathbf{a1B1} & & & & & \\
 & & & \ddots & & & & \\
 & & & & \mathbf{a2B1} & \mathbf{a1B1} & & \\
 & & & & & \mathbf{a2B1} & \mathbf{a1B1} &
\end{array}\right]
\end{array}
$$

b0B1 =

$$
\begin{array}{rr}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \dots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=0 \quad K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
p & & & & & & & & & \\
p\beta_3 & p\alpha_3 & & & & & & & & \\
 & p\beta_3 & p\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & p\beta_3 & p\alpha_3 & & & & \\
 & & & & & p\beta_3 & p\alpha_3 & & &
\end{array}\right]
\end{array}
$$

a2B1 =

$$
\begin{array}{rr}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \dots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=0 \quad K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
p\beta_2 & & & & & & & & & \\
p\beta_2\beta_3 & p\beta_2\alpha_3 & & & & & & & & \\
 & p\beta_2\beta_3 & p\beta_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & p\beta_2\beta_3 & p\beta_2\alpha_3 & & & & \\
 & & & & & p\beta_2\beta_3 & p\beta_2\alpha_3 & & &
\end{array}\right]
\end{array}
$$

a1B1 =

$$
\begin{array}{rr}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \dots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=0 \quad K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
p\alpha_2 & & & & & & & & & \\
p\alpha_2\beta_3 & p\alpha_2\alpha_3 & & & & & & & & \\
 & p\alpha_2\beta_3 & p\alpha_2\alpha_3 & & & & & & & \\
 & & & & & & & & & \\
 & & & & p\alpha_2\beta_3 & p\alpha_2\alpha_3 & & & & \\
 & & & & & p\alpha_2\beta_3 & p\alpha_2\alpha_3 & & &
\end{array}\right]
\end{array}
$$

A2 =

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & & & & & j=0 & j=2 \\ 0 & 1 & 2 & 3 & \dots & K2-1 & K2 & K2+1 & K2+1 \end{array} \\
\begin{array}{c} L_2 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K2 \\ j=0\quad K2+1 \\ j=2\quad K2+1 \end{array} &
\left[\begin{array}{ccccccccc}
\mathbf{b0A2} & \mathbf{b1A2} & & & & & & & \\
\mathbf{a2A2} & \mathbf{a1A2} & \mathbf{a0A2} & & & & & & \\
 & \mathbf{a2A2} & \mathbf{a1A2} & \mathbf{a0A2} & & & & & \\
 & & & & \ddots & & & & \\
 & & & & & \mathbf{a2A2} & \mathbf{a1A2} & \mathbf{a0A2} & \\
 & & & & & & \mathbf{a2A2} & \mathbf{a1A2} & \\
 & & & & & & \mathbf{b3A2} & \mathbf{b4A2} &
\end{array}\right]
\end{array}
$$

b0A2 =

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & 3 & \dots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=0\quad K3+1 \\ j=3\quad K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
 & q\beta_1\theta_3 & & & & & & & & \\
 & q\beta_1\theta_3\beta_3 & q\beta_1\theta_3\alpha_3 & & & & & & & \\
 & & q\beta_1\theta_3\beta_3 & q\beta_1\theta_3\alpha_3 & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & q\beta_1\theta_3\beta_3 & q\beta_1\theta_3\alpha_3 & & & \\
 & & & & & & q\beta_1\theta_3\beta_3 & & & \\
 & & & & & & \mathrm{b0A2(1)} & & &
\end{array}\right]
\end{array}
$$

b0A2(1)$_a$ = q θ_3 β_3

b0A2(1)$_b$ = q β_3

b1A2 =

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \dots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=0\quad K3+1 \\ j=3\quad K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
q\beta_1\theta_2 & & & & & & & & & \\
q\beta_1\theta_2\beta_3 & q\beta_1\theta_2\alpha_3 & & & & & & & & \\
 & q\beta_1\theta_2\beta_3 & q\beta_1\theta_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & q\beta_1\theta_2\beta_3 & q\beta_1\theta_2\alpha_3 & & & & \\
 & & & & & q\beta_1\theta_2\beta_3 & q\beta_1\theta_2\alpha_3 & & & \\
 & & & & & \mathrm{b1A2(1)} & \mathrm{b1A2(2)} & & &
\end{array}\right]
\end{array}
$$

b1A2(1)$_a$ = q θ_2 β_3
b1A2(2)$_a$ = q θ_2 α_3

b1A2(1)$_b$ = 0
b1A2(2)$_b$ = 0

a2A2 =

$$
\begin{array}{ll|cccccccccc|}
 & & & & & & & & j=0 & j=1 & j=2 & j=3 \\
 & L_3 & 0 & 1 & 2 & 3 & \ldots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
\hline
 & 0 & & q\beta_1\theta_3\beta_2 & & & & & & & & \\
 & 1 & & q\beta_1\theta_3\beta_2\beta_3 & q\beta_1\theta_3\beta_2\alpha_3 & & & & & & & \\
 & 2 & & & q\beta_1\theta_3\beta_2\beta_3 & q\beta_1\theta_3\beta_2\alpha_3 & & & & & & \\
 & \ddots & & & & & \ddots & & & & & \\
 & K3 & & & & & & q\beta_1\theta_3\beta_2\beta_3 & q\beta_1\theta_3\beta_2\alpha_3 & & & \\
j=0 & K3+1 & & & & & & & q\beta_1\theta_3\beta_2\beta_3 & & & \\
j=3 & K3+1 & & & & & & & a2A2(1) & & & \\
\hline
\end{array}
$$

$$\mathbf{a2A2(1)_a = q\,\theta_3\,\beta_2\,\beta_3}$$

$$\mathbf{a2A2(1)_b = q\,\beta_2\,\beta_3}$$

a1A2 =

$$
\begin{array}{ll|cccccccccc|}
 & & & & & & & & j=0 & j=1 & j=2 & j=3 \\
 & L_3 & 0 & 1 & 2 & \ldots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
\hline
 & 0 & q\beta_1\theta_2\beta_2 & q\beta_1\theta_3\alpha_2 & & & & & & & & \\
 & 1 & q\beta_1\theta_2\beta_2\beta_3 & \begin{array}{c} q\beta_1\theta_2\beta_2\alpha_3 + \\ q\beta_1\theta_3\alpha_2\beta_3 \end{array} & q\beta_1\theta_3\alpha_2\alpha_3 & & & & & & & \\
 & \ddots & & & & \ddots & & & & & & \\
 & K3 & & & & & q\beta_1\theta_2\beta_2\beta_3 & \begin{array}{c} q\beta_1\theta_2\beta_2\alpha_3 \\ + \\ q\beta_1\theta_3\alpha_2\beta_3 \end{array} & q\beta_1\theta_3\alpha_2\alpha_3 & & & \\
j=0 & K3+1 & & & & & & q\beta_1\theta_2\beta_2\beta_3 & \begin{array}{c} q\beta_1\theta_2\beta_2\alpha_3 + \\ q\beta_1\theta_3\alpha_2\beta_3 \end{array} & & & \\
j=3 & K3+1 & & & & & & a1A2(1) & a1A2(2) & & & \\
\hline
\end{array}
$$

$$\mathbf{a1A2(1)_a = q\,\theta_2\,\beta_2\,\beta_3}$$
$$\mathbf{a1A2(2)_a = q\,\theta_2\,\beta_2\,\alpha_3 + q\,\theta_3\,\alpha_2\,\beta_3}$$

$$\mathbf{a1A2(1)_b = 0}$$
$$\mathbf{a1A2(2)_b = q\,\alpha_2\,\beta_3}$$

a0A2 =

$$
\begin{array}{ll|ccccccccc|}
 & & & & & & & j=0 & j=1 & j=2 & j=3 \\
 & L_3 & 0 & 1 & 2 & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
\hline
 & 0 & q\beta_1\theta_2\alpha_2 & & & & & & & & \\
 & 1 & q\beta_1\theta_2\alpha_2\beta_3 & q\beta_1\theta_2\alpha_2\alpha_3 & & & & & & & \\
 & 2 & & q\beta_1\theta_2\alpha_2\beta_3 & q\beta_1\theta_2\alpha_2\alpha_3 & & & & & & \\
 & . & & & & . & & & & & \\
 & K3 & & & & q\beta_1\theta_2\alpha_2\beta_3 & q\beta_1\theta_2\alpha_2\alpha_3 & & & & \\
j=0 & K3+1 & & & & & q\beta_1\theta_2\alpha_2\beta_3 & q\beta_1\theta_2\alpha_2\alpha_3 & & & \\
j=3 & K3+1 & & & & & a0A2(1) & a0A2(2) & & & \\
\hline
\end{array}
$$

$$\mathbf{a0A2(1)_a = q\,\theta_2\,\alpha_2\,\beta_3}$$
$$\mathbf{a0A2(2)_a = q\,\theta_2\,\alpha_2\,\alpha_3}$$

$$\mathbf{a0A2(1)_b = 0}$$
$$\mathbf{a0A2(2)_b = 0}$$

$$\mathbf{b3A2_a =}$$

$$\begin{array}{rl}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & 3 & \ldots & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=2 \quad K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
 & q\theta_3\beta_2 & & & & & & & & \\
 & q\theta_3\beta_2\beta_3 & q\theta_3\beta_2\alpha_3 & & & & & & & \\
 & & q\theta_3\beta_2\beta_3 & q\theta_3\beta_2\alpha_3 & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & q\theta_3\beta_2\beta_3 & q\theta_3\beta_2\alpha_3 & & & \\
 & & & & & & q\theta_3\beta_2\beta_3 & & &
\end{array}\right]
\end{array}$$

$$\mathbf{b3A2_b = 0}$$

$$\mathbf{b4A2_a =}$$

$$\begin{array}{rl}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \ldots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ \ddots \\ K3 \\ j=2 \quad K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
q\theta_2\beta_2 & q\theta_3\alpha_2 & & & & & & & & \\
q\theta_2\beta_2\beta_3 & q\theta_2\beta_2\alpha_3 + q\theta_3\alpha_2\beta_3 & q\theta_3\alpha_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & q\theta_2\beta_2\beta_3 & q\theta_2\beta_2\alpha_3 + q\theta_3\alpha_2\beta_3 & q\theta_3\alpha_2\alpha_3 & & & \\
 & & & & & q\theta_2\beta_2\beta_3 & q\theta_2\beta_2\alpha_3 + q\theta_3\alpha_2\beta_3 & & &
\end{array}\right]
\end{array}$$

$$\mathbf{b4A2_b =}$$

$$\begin{array}{rl}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \ldots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=2 \quad K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
q\beta_2 & & & & & & & & & \\
q\beta_2\beta_3 & q\beta_2\alpha_3 & & & & & & & & \\
 & q\beta_2\beta_3 & q\beta_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & q\beta_2\beta_3 & q\beta_2\alpha_3 & & & & \\
 & & & & & q\beta_2\beta_3 & q\beta_2\alpha_3 & & &
\end{array}\right]
\end{array}$$

$$\mathbf{A1 =}$$

$$\begin{array}{rl}
 & \begin{array}{cccccccc} & & & & & & j=0 & j=2 \\ 0 & 1 & 2 & \ldots & K2-1 & K2 & K2+1 & K2+1 \end{array} \\
\begin{array}{r} L_2 \\ 0 \\ 1 \\ . \\ K2 \\ j=0 \quad K2+1 \\ j=2 \quad K2+1 \end{array} &
\left[\begin{array}{cccccccc}
\mathbf{b0A1} & \mathbf{b1A1} & & & & & & \\
\mathbf{a2A1} & \mathbf{a1A1} & \mathbf{a0A1} & & & & & \\
 & & & . & & & & \\
 & & & & \mathbf{a2A1} & \mathbf{a1A1} & \mathbf{a0A1} & \\
 & & & & & \mathbf{a2A1} & \mathbf{a1A1} & \mathbf{b2A1} \\
 & & & & & \mathbf{b3A1} & \mathbf{b4A1} & \mathbf{b5A1}
\end{array}\right]
\end{array}$$

b0A1 =

$$\begin{array}{r|cccccccccc|}
 & & & & & & & j=0 & j=1 & j=2 & j=3 \\
L_3 & 0 & 1 & 2 & \cdots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
\hline
0 & q\alpha_1 & p\beta_1\theta_3 & & & & & & & & \\
1 & q\alpha_1\beta_3 & \begin{array}{c} q\alpha_1\alpha_3 + \\ p\beta_1\theta_3\beta_3 \end{array} & p\beta_1\theta_3\alpha_3 & & & & & & & \\
\ddots & & & & \ddots & & & & & & \\
K3 & & & & & q\alpha_1\beta_3 & \begin{array}{c} q\alpha_1\alpha_3 + \\ p\beta_1\theta_3\beta_3 \end{array} & p\beta_1\theta_3\alpha_3 & & & \\
j=0\ \ K3+1 & & & & & & q\alpha_1\beta_3 & \begin{array}{c} q\alpha_1\alpha_3 + \\ p\beta_1\theta_3\beta_3 \end{array} & & & q\beta_1\theta_3\alpha_3 \\
j=3\ \ K3+1 & & & & & & & b0A1(1) & & & b0A1(2) \\
\hline
\end{array}$$

$b0A1(1)_a = p\,\theta_3\,\beta_3$
$b0A1(2)_a = p\,\theta_3\,\alpha_3$

$b0A1(1)_b = p\,\beta_3$
$b0A1(2)_b = p\,\alpha_3$

b1A1 =

$$\begin{array}{r|cccccccccc|}
 & & & & & & & j=0 & j=1 & j=2 & j=3 \\
L_3 & 0 & 1 & 2 & \cdots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
\hline
0 & p\beta_1\theta_2 & & & & & & & & & \\
1 & p\beta_1\theta_2\beta_3 & p\beta_1\theta_2\alpha_3 & & & & & & & & \\
2 & & p\beta_1\theta_2\beta_3 & p\beta_1\theta_2\alpha_3 & & & & & & & \\
\ddots & & & & \ddots & & & & & & \\
K3 & & & & & p\beta_1\theta_2\beta_3 & p\beta_1\theta_2\alpha_3 & & & & \\
j=0\ \ K3+1 & & & & & & p\beta_1\theta_2\beta_3 & p\beta_1\theta_2\alpha_3 & & & \\
j=3\ \ K3+1 & & & & & & b1A1(1) & b1A1(2) & & & \\
\hline
\end{array}$$

$b1A1(1)_a = p\,\theta_2\,\beta_3$
$b1A1(2)_a = p\,\theta_2\,\alpha_3$

$b1A1(1)_b = 0$
$b1A1(2)_b = 0$

a2A1 =

$$\begin{array}{r|cccccccccc|}
 & & & & & & & j=0 & j=1 & j=2 & j=3 \\
L_3 & 0 & 1 & 2 & & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
\hline
0 & q\alpha_1\beta_2 & p\beta_1\theta_3\beta_2 & & & & & & & & \\
1 & q\alpha_1\beta_2\beta_3 & \begin{array}{c} q\alpha_1\beta_2\alpha_3 + \\ p\beta_1\theta_3\beta_2\beta_3 \end{array} & p\beta_1\theta_3\beta_2\alpha_3 & & & & & & & \\
. & & & & . & & & & & & \\
K3 & & & & & q\alpha_1\beta_2\beta_3 & \begin{array}{c} q\alpha_1\beta_2\alpha_3 + \\ p\beta_1\theta_3\beta_2\beta_3 \end{array} & p\beta_1\theta_3\beta_2\alpha_3 & & & \\
j=0\ \ K3+1 & & & & & & q\alpha_1\beta_2\beta_3 & \begin{array}{c} q\alpha_1\beta_2\alpha_3 + \\ p\beta_1\theta_3\beta_2\beta_3 \end{array} & & & q\beta_1\theta_3\beta_2\alpha_3 \\
j=3\ \ K3+1 & & & & & & & a2A1(1) & & & a2A1(2) \\
\hline
\end{array}$$

$$\mathbf{a2A1(1)_a = p\,\theta_3\,\beta_2\,\beta_3}$$
$$\mathbf{a2A1(2)_a = p\,\theta_3\,\beta_2\,\alpha_3}$$

$$\mathbf{a2A1(1)_b = p\,\beta_2\,\beta_3}$$
$$\mathbf{a2A1(2)_b = p\,\beta_2\,\alpha_3}$$

a1A1 =

$$
\begin{array}{rr|cccccccccc|}
 & & & & & & & & j=0 & j=1 & j=2 & j=3 \\
 & L_3 & 0 & 1 & 2 & \dots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
\hline
 & 0 & \begin{array}{c} q\alpha_1\alpha_2 + \\ p\beta_1\theta_2\beta_2 \end{array} & p\beta_1\theta_3\alpha_2 & & & & & & & & \\
 & 1 & \begin{array}{c} q\alpha_1\alpha_2\beta_3 + \\ p\beta_1\theta_2\beta_2\beta_3 \end{array} & \begin{array}{c} q\alpha_1\alpha_2\alpha_3 + \\ p\beta_1\theta_2\beta_2\alpha_3 + \\ p\beta_1\theta_3\alpha_2\beta_3 \end{array} & p\beta_1\theta_3\alpha_2\alpha_3 & & & & & & & \\
 & \ddots & & & & \ddots & & & & & & \\
 & K3 & & & & & \begin{array}{c} q\alpha_1\alpha_2\beta_3 + \\ p\beta_1\theta_2\beta_2\beta_3 \end{array} & \begin{array}{c} q\alpha_1\alpha_2\alpha_3 + \\ p\beta_1\theta_2\beta_2\alpha_3 \\ + \\ p\beta_1\theta_3\alpha_2\beta_3 \end{array} & p\beta_1\theta_3\alpha_2\alpha_3 & & & \\
j=0 & K3+1 & & & & & & \begin{array}{c} q\alpha_1\alpha_2\beta_3 + \\ p\beta_1\theta_2\beta_2\beta_3 \end{array} & \begin{array}{c} q\alpha_1\alpha_2\alpha_3 + \\ p\beta_1\theta_2\beta_2\alpha_3 + \\ p\beta_1\theta_3\alpha_2\beta_3 \end{array} & & & q\beta_1\theta_3\alpha_2\alpha_3 \\
j=3 & K3+1 & & & & & & \mathrm{a1A1(1)} & \mathrm{a1A1(2)} & & & \mathrm{a1A1(3)} \\
\hline
\end{array}
$$

$$\mathbf{a1A1(1)_a = p\,\theta_2\,\beta_2\,\beta_3}$$
$$\mathbf{a1A1(2)_a = p\,\theta_2\,\beta_2\,\alpha_3 + p\,\theta_3\,\alpha_2\,\beta_3}$$
$$\mathbf{a1A1(3)_a = q\,\theta_3\,\alpha_2\,\alpha_3}$$

$$\mathbf{a1A1(1)_b = 0}$$
$$\mathbf{a1A1(2)_b = p\,\alpha_2\,\beta_3}$$
$$\mathbf{a1A1(3)_b = q\,\alpha_2\,\alpha_3}$$

a0A1 =

$$
\begin{array}{rr|cccccccccc|}
 & & & & & & & & j=0 & j=1 & j=2 & j=3 \\
 & L_3 & 0 & 1 & 2 & \dots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
\hline
 & 0 & p\beta_1\theta_2\alpha_2 & & & & & & & & & \\
 & 1 & p\beta_1\theta_2\alpha_2\beta_3 & p\beta_1\theta_2\alpha_2\alpha_3 & & & & & & & & \\
 & 2 & & p\beta_1\theta_2\alpha_2\beta_3 & p\beta_1\theta_2\alpha_2\alpha_3 & & & & & & & \\
 & . & & & & \ddots & & & & & & \\
 & K3 & & & & & p\beta_1\theta_2\alpha_2\beta_3 & p\beta_1\theta_2\alpha_2\alpha_3 & & & & \\
j=0 & K3+1 & & & & & & p\beta_1\theta_2\alpha_2\beta_3 & p\beta_1\theta_2\alpha_2\alpha_3 & & & \\
j=3 & K3+1 & & & & & & \mathrm{a0A1(1)} & \mathrm{a0A1(2)} & & & \\
\hline
\end{array}
$$

$$\mathbf{a0A1(1)_a = p\,\theta_2\,\alpha_2\,\beta_3}$$
$$\mathbf{a0A1(2)_a = p\,\theta_2\,\alpha_2\,\alpha_3}$$

$$\mathbf{a0A1(1)_b = 0}$$
$$\mathbf{a0A1(2)_b = 0}$$

b2A1 =

$$
\begin{array}{r l}
 & \begin{array}{cccccccccc} 0 & 1 & 2 & \ldots & K3-1 & K3 & \begin{matrix} j=0 \\ K3+1 \end{matrix} & \begin{matrix} j=1 \\ K3+1 \end{matrix} & \begin{matrix} j=2 \\ K3+1 \end{matrix} & \begin{matrix} j=3 \\ K3+1 \end{matrix} \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=0 \quad K3+1 \\ j=3 \quad K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
q\beta_1\theta_2\alpha_2 & & & & & & & & & \\
q\beta_1\theta_2\alpha_2\beta_3 & q\beta_1\theta_2\alpha_2\alpha_3 & & & & & & & & \\
 & q\beta_1\theta_2\alpha_2\beta_3 & q\beta_1\theta_2\alpha_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & q\beta_1\theta_2\alpha_2\beta_3 & q\beta_1\theta_2\alpha_2\alpha_3 & & & & \\
 & & & & & q\beta_1\theta_2\alpha_2\beta_3 & & & q\beta_1\theta_2\alpha_2\alpha_3 & \\
 & & & & & q\theta_2\alpha_2\beta_3 & & & q\theta_2\alpha_2\alpha_3 &
\end{array} \right]
\end{array}
$$

b2A1(1)$_a$ = $q\,\theta_2\,\alpha_2\,\beta_3$
b2A1(2)$_a$ = $q\,\theta_2\,\alpha_2\,\alpha_3$

b2A1(1)$_b$ = 0
b2A1(2)$_b$ = 0

b3A1$_a$ =

$$
\begin{array}{r l}
 & \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & \ldots & K3 & \begin{matrix} j=0 \\ K3+1 \end{matrix} & \begin{matrix} j=1 \\ K3+1 \end{matrix} & \begin{matrix} j=2 \\ K3+1 \end{matrix} & \begin{matrix} j=3 \\ K3+1 \end{matrix} \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=2 \quad K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
 & p\theta_3\beta_2 & & & & & & & & \\
 & p\theta_3\beta_2\beta_3 & p\theta_3\beta_2\alpha_3 & & & & & & & \\
 & & p\theta_3\beta_2\beta_3 & p\theta_3\beta_2\alpha_3 & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & p\theta_3\beta_2\beta_3 & p\theta_3\beta_2\alpha_3 & & & \\
 & & & & & & p\theta_3\beta_2\beta_3 & & & q\theta_3\beta_2\alpha_3
\end{array} \right]
\end{array}
$$

b3A1$_b$ = 0

b4A1$_a$ =

$$
\begin{array}{r l}
 & \begin{array}{cccccccccc} 0 & 1 & 2 & \ldots & K3-1 & K3 & \begin{matrix} j=0 \\ K3+1 \end{matrix} & \begin{matrix} j=1 \\ K3+1 \end{matrix} & \begin{matrix} j=2 \\ K3+1 \end{matrix} & \begin{matrix} j=3 \\ K3+1 \end{matrix} \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ . \\ K3 \\ j=2 \quad K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
p\theta_2\beta_2 & p\theta_3\alpha_2 & & & & & & & & \\
p\theta_2\beta_2\beta_3 & \begin{matrix} p\theta_2\beta_2\alpha_3 + \\ p\theta_3\alpha_2\beta_3 \end{matrix} & p\theta_3\alpha_2\alpha_3 & & & & & & & \\
 & & & & & & & & & \\
 & & & & p\theta_2\beta_2\beta_3 & \begin{matrix} p\theta_2\beta_2\alpha_3 + \\ p\theta_3\alpha_2\beta_3 \end{matrix} & p\theta_3\alpha_2\alpha_3 & & & \\
 & & & & & p\theta_2\beta_2\beta_3 & \begin{matrix} p\theta_2\beta_2\alpha_3 + \\ p\theta_3\alpha_2\beta_3 \end{matrix} & & & q\theta_3\alpha_2\alpha_3
\end{array} \right]
\end{array}
$$

$$b4A1_b =$$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccc} 0 & 1 & 2 & \cdots & K3-1 & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=2\ \ K3+1 \end{array} &
\begin{bmatrix}
p\beta_2 & & & & & & & & & \\
p\beta_2\beta_3 & p\beta_2\alpha_3 & & & & & & & & \\
 & p\beta_2\beta_3 & p\beta_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & p\beta_2\beta_3 & p\beta_2\alpha_3 & & & & \\
 & & & & & p\beta_2\beta_3 & p\beta_2\alpha_3 & & &
\end{bmatrix}
\end{array}
$$

$$b5A1_a =$$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccc} 0 & 1 & 2 & \cdots & K3-1 & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=2\ \ K3+1 \end{array} &
\begin{bmatrix}
q\theta_2\alpha_2 & & & & & & & & & \\
q\theta_2\alpha_2\beta_3 & q\theta_2\alpha_2\alpha_3 & & & & & & & & \\
 & q\theta_2\alpha_2\beta_3 & q\theta_2\alpha_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & q\theta_2\alpha_2\beta_3 & q\theta_2\alpha_2\alpha_3 & & & & \\
 & & & & & q\theta_2\alpha_2\beta_3 & & & q\theta_2\alpha_2\alpha_3 &
\end{bmatrix}
\end{array}
$$

$$b5A1_b =$$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccc} 0 & 1 & 2 & \cdots & K3-1 & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=2\ \ K3+1 \end{array} &
\begin{bmatrix}
q\alpha_2 & & & & & & & & & \\
q\alpha_2\beta_3 & q\alpha_2\alpha_3 & & & & & & & & \\
 & q\alpha_2\beta_3 & q\alpha_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & q\alpha_2\beta_3 & q\alpha_2\alpha_3 & & & & \\
 & & & & & q\alpha_2\beta_3 & q\alpha_2\alpha_3 & & &
\end{bmatrix}
\end{array}
$$

$$A0 =$$

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} 0 & 1 & 2 & \cdots & K2-1 & K2 & \overset{j=0}{K2+1} & \overset{j=2}{K2+1} \end{array} \\
\begin{array}{r} L_2 \\ 0 \\ 1 \\ 2 \\ . \\ K2 \\ j=0\ \ K2+1 \\ j=2\ \ K2+1 \end{array} &
\begin{bmatrix}
b0A0 & & & & & & & \\
a2A0 & a1A0 & & & & & & \\
 & a2A0 & a1A0 & & & & & \\
 & & & & & & & \\
 & & & & a2A0 & a1A0 & & \\
 & & & & & a2A0 & a1A0 & b2A0 \\
 & & & & & b3A0 & b5A0 & b5A0
\end{bmatrix}
\end{array}
$$

b0A0 =

$$
\begin{array}{r|cccccccccc|}
 & & & & & & & j=0 & j=1 & j=2 & j=3 \\
L_3 & 0 & 1 & 2 & \cdots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
\hline
0 & p\alpha_1 & & & & & & & & & \\
1 & p\alpha_1\beta_3 & p\alpha_1\alpha_3 & & & & & & & & \\
2 & & p\alpha_1\beta_3 & p\alpha_1\alpha_3 & & & & & & & \\
\ddots & & & & \ddots & & & & & & \\
K3 & & & & & p\alpha_1\beta_3 & p\alpha_1\alpha_3 & & & & \\
j=0\ \ K3+1 & & & & & & p\alpha_1\beta_3 & p\alpha_1\alpha_3 & & & p\beta_1\theta_3\alpha_3 \\
j=3\ \ K3+1 & & & & & & & & & & b0A0(1) \\
\hline
\end{array}
$$

$b0A0(1)_a = p\,\theta_3\,\alpha_3$

$b0A0(1)_b = p\,\alpha_3$

a2A0 =

$$
\begin{array}{r|cccccccccc|}
 & & & & & & & j=0 & j=1 & j=2 & j=3 \\
L_3 & 0 & 1 & 2 & \cdots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
\hline
0 & p\alpha_1\beta_2 & & & & & & & & & \\
1 & p\alpha_1\beta_2\beta_3 & p\alpha_1\beta_2\alpha_3 & & & & & & & & \\
2 & & p\alpha_1\beta_2\beta_3 & p\alpha_1\beta_2\alpha_3 & & & & & & & \\
\ddots & & & & \ddots & & & & & & \\
K3 & & & & & p\alpha_1\beta_2\beta_3 & p\alpha_1\beta_2\alpha_3 & & & & \\
j=0\ \ K3+1 & & & & & & p\alpha_1\beta_2\beta_3 & p\alpha_1\beta_2\alpha_3 & & & p\beta_1\theta_3\beta_2\alpha_3 \\
j=3\ \ K3+1 & & & & & & & & & & a2A0(1) \\
\hline
\end{array}
$$

$a2A0(1)_a = p\,\theta_3\,\beta_2\,\alpha_3$

$a2A0(1)_b = p\,\beta_2\,\alpha_3$

$a1A0_a$ =

$$
\begin{array}{r|cccccccccc|}
 & & & & & & & j=0 & j=1 & j=2 & j=3 \\
L_3 & 0 & 1 & 2 & \cdot & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \\
\hline
0 & p\alpha_1\alpha_2 & & & & & & & & & \\
1 & p\alpha_1\alpha_2\beta_3 & p\alpha_1\alpha_2\alpha_3 & & & & & & & & \\
2 & & p\alpha_1\alpha_2\beta_3 & p\alpha_1\alpha_2\alpha_3 & & & & & & & \\
\cdot & & & & & & & & & & \\
K3 & & & & & p\alpha_1\alpha_2\beta_3 & p\alpha_1\alpha_2\alpha_3 & & & & \\
j=0\ \ K3+1 & & & & & & p\alpha_1\alpha_2\beta_3 & p\alpha_1\alpha_2\alpha_3 & & & p\beta_1\theta_3\alpha_2\alpha_3 \\
j=3\ \ K3+1 & & & & & & & & & & a1A0(1) \\
\hline
\end{array}
$$

$a1A0(1)_a = p\,\theta_3\,\alpha_2\,\alpha_3$

$a1A0(1)_b = p\,\alpha_2\,\alpha_3$

b2A0 =

$$
\begin{array}{c|ccccccccccc|}
L_3 & 0 & 1 & 2 & \dots & K3-1 & K3 & \begin{smallmatrix} j=0 \\ K3+1 \end{smallmatrix} & \begin{smallmatrix} j=1 \\ K3+1 \end{smallmatrix} & \begin{smallmatrix} j=2 \\ K3+1 \end{smallmatrix} & \begin{smallmatrix} j=3 \\ K3+1 \end{smallmatrix} \\
\hline
0 & p\beta_1\theta_2\alpha_2 & & & & & & & & & \\
1 & p\beta_1\theta_2\alpha_2\beta_3 & p\beta_1\theta_2\alpha_2\alpha_3 & & & & & & & & \\
2 & & p\beta_1\theta_2\alpha_2\beta_3 & p\beta_1\theta_2\alpha_2\alpha_3 & & & & & & & \\
\ddots & & & & \ddots & & & & & & \\
K3 & & & & & p\beta_1\theta_2\alpha_2\beta_3 & p\beta_1\theta_2\alpha_2\alpha_3 & & & & \\
j=0\ \ K3+1 & & & & & & p\beta_1\theta_2\alpha_2\beta_3 & & & p\beta_1\theta_2\alpha_2\alpha_3 & \\
j=3\ \ K3+1 & & & & & & b2A0(1) & & & b2A0(2) & \\
\hline
\end{array}
$$

$b2A0(1)_a = p\,\theta_2\,\alpha_2\,\beta_3$
$b2A0(2)_a = p\,\theta_2\,\alpha_2\,\alpha_3$

$b2A0(1)_b = 0$
$b2A0(2)_b = 0$

$b3A0_a =$

$$
\begin{array}{c|ccccccccc|}
L_3 & 0 & 1 & 2 & \dots & K3 & \begin{smallmatrix} j=0 \\ K3+1 \end{smallmatrix} & \begin{smallmatrix} j=1 \\ K3+1 \end{smallmatrix} & \begin{smallmatrix} j=2 \\ K3+1 \end{smallmatrix} & \begin{smallmatrix} j=3 \\ K3+1 \end{smallmatrix} \\
\hline
j=2\ \ K3+1 & & & & & & & & & p\theta_3\beta_2\alpha_3 \\
\hline
\end{array}
$$

$b3A0_b = 0$

$b4A0_a =$

$$
\begin{array}{c|ccccccccc|}
L_3 & 0 & 1 & 2 & \dots & K3 & \begin{smallmatrix} j=0 \\ K3+1 \end{smallmatrix} & \begin{smallmatrix} j=1 \\ K3+1 \end{smallmatrix} & \begin{smallmatrix} j=2 \\ K3+1 \end{smallmatrix} & \begin{smallmatrix} j=3 \\ K3+1 \end{smallmatrix} \\
\hline
j=2\ \ K3+1 & & & & & & & & & p\theta_3\alpha_2\alpha_3 \\
\hline
\end{array}
$$

$b4A0_b = 0$

$b5A0_a =$

$$
\begin{array}{c|cccccccccc|}
L_3 & 0 & 1 & 2 & & K3-1 & K3 & \begin{smallmatrix} j=0 \\ K3+1 \end{smallmatrix} & \begin{smallmatrix} j=1 \\ K3+1 \end{smallmatrix} & \begin{smallmatrix} j=2 \\ K3+1 \end{smallmatrix} & \begin{smallmatrix} j=3 \\ K3+1 \end{smallmatrix} \\
\hline
0 & p\theta_2\alpha_2 & & & & & & & & & \\
1 & p\theta_2\alpha_2\beta_3 & p\theta_2\alpha_2\alpha_3 & & & & & & & & \\
2 & & p\theta_2\alpha_2\beta_3 & p\theta_2\alpha_2\alpha_3 & & & & & & & \\
. & & & & & & & & & & \\
K3 & & & & & p\theta_2\alpha_2\beta_3 & p\theta_2\alpha_2\alpha_3 & & & & \\
j=2\ \ K3+1 & & & & & & p\theta_2\alpha_2\beta_3 & & & p\theta_2\alpha_2\alpha_3 & \\
\hline
\end{array}
$$

b5A0$_b$ =

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \ldots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{rr} & L_3 \\ & 0 \\ & 1 \\ & 2 \\ & \ddots \\ & K3 \\ j=2 & K3+1 \end{array} &
\begin{bmatrix}
p\alpha_2 & & & & & & & & & \\
p\alpha_2\beta_3 & p\alpha_2\alpha_3 & & & & & & & & \\
 & p\alpha_2\beta_3 & p\alpha_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & p\alpha_2\beta_3 & p\alpha_2\alpha_3 & & & & \\
 & & & & & p\alpha_2\beta_3 & p\alpha_2\alpha_3 & & &
\end{bmatrix}
\end{array}
$$

where:

- p = probability of an arrival to queue 1
- $q = 1 - p$ = probability of no arrival to queue 1
- β_i = probability of a service completion in queue i, i = 1, 2 or 3
- $\alpha_i = 1 - \beta_i$ = probability of no service completion in queue i, i = 1, 2 or 3
- Ki = buffer size of queue 1, i = 2 or 3
- θ_2 = probability the customer leaving queue 1 will choose to visit queue 2
- $\theta_3 = 1 - \theta_2$ = probability the customer leaving queue 1 will choose queue 3
- (j = 0) = state when there are K3 + 1 customers in queue 3 or K2 + 1 in queue 2, but there is no blocking
- (j = 2) = state when there are K2 + 1 customers in queue 2 and it blocks an arrival from queue 1
- (j = 3) = state when there are K3 + 1 customers in queue 3 and it blocks an arrival from queue 1

A.4.2 Sub-Matrices of the Modified MAP

Each of the matrices within $\mathbf{D0}^{*i}$ and $\mathbf{D1}^{*i}$, $i = 2$ or 3 are defined as follows:

A1" =

$$
\begin{array}{rl}
 & \begin{array}{ccccccccc} 0 & 1 & 2 & \ldots & K2-1 & K2 & \overset{j=0}{K2+1} & \overset{j=2}{K2+1} \end{array} \\
\begin{array}{r} L_2 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K2 \\ j=0\ \ K2+1 \\ j=2\ \ K2+1 \end{array} &
\left[\begin{array}{cccccccc}
\mathbf{b0A1''} & & & & & & & \\
\mathbf{a2A1''} & \mathbf{a1A1''} & & & & & & \\
 & \mathbf{a2A1''} & \mathbf{a1A1''} & & & & & \\
 & & & \ddots & & & & \\
 & & & & \mathbf{a2A1''} & \mathbf{a1A1''} & & \\
 & & & & & \mathbf{a2A1''} & \mathbf{a1A1''} & \mathbf{b2A1''} \\
 & & & & & \mathbf{b3A1''} & \mathbf{b4A1''} & \mathbf{b5A1''}
\end{array}\right]
\end{array}
$$

b0A1" =

$$
\begin{array}{rl}
 & \begin{array}{cccccccccc} 0 & 1 & 2 & \ldots & K3-1 & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
q\alpha_1 & & & & & & & & & \\
q\alpha_1\beta_3 & q\alpha_1\alpha_3 & & & & & & & & \\
 & q\alpha_1\beta_3 & q\alpha_1\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & q\alpha_1\beta_3 & q\alpha_1\alpha_3 & & & & \\
 & & & & & q\alpha_1\beta_3 & q\alpha_1\alpha_3 & & & q\beta_1\theta_3\alpha_3 \\
 & & & & & & & & & \mathrm{b0A1''(1)}
\end{array}\right]
\end{array}
$$

$\mathbf{b0A1''(1)_a} = q\ \theta_3\ \alpha_3$

$\mathbf{b0A1''(1)_b} = q\ \alpha_3$

a2A1" =

$$
\begin{array}{rl}
 & \begin{array}{cccccccccc} 0 & 1 & 2 & \ldots & K3-1 & K3 & \overset{j=0}{K3+1} & \overset{j=1}{K3+1} & \overset{j=2}{K3+1} & \overset{j=3}{K3+1} \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \cdot \\ K3 \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
q\alpha_1\beta_2 & & & & & & & & & \\
q\alpha_1\beta_2\beta_3 & q\alpha_1\beta_2\alpha_3 & & & & & & & & \\
 & q\alpha_1\beta_2\beta_3 & q\alpha_1\beta_2\alpha_3 & & & & & & & \\
 & & & \cdot & & & & & & \\
 & & & & q\alpha_1\beta_2\beta_3 & q\alpha_1\beta_2\alpha_3 & & & & \\
 & & & & & q\alpha_1\beta_2\beta_3 & q\alpha_1\beta_2\alpha_3 & & & q\beta_1\theta_3\beta_2\alpha_3 \\
 & & & & & & & & & \mathrm{a2A1''(1)}
\end{array}\right]
\end{array}
$$

$\mathbf{a2A1''(1)_a} = q\ \theta_3\ \beta_2\ \alpha_3$

$\mathbf{a2A1''(1)_b} = q\ \beta_2\ \alpha_3$

a1A1" =

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} 0 & 1 & 2 & \dots & K3-1 & K3 & \substack{j=0\\ K3+1} & \substack{j=1\\ K3+1} & \substack{j=2\\ K3+1} & \substack{j=3\\ K3+1} \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
q\alpha_1\alpha_2 & & & & & & & & & \\
q\alpha_1\alpha_2\beta_3 & q\alpha_1\alpha_2\alpha_3 & & & & & & & & \\
 & q\alpha_1\alpha_2\beta_3 & q\alpha_1\alpha_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & q\alpha_1\alpha_2\beta_3 & q\alpha_1\alpha_2\alpha_3 & & & & \\
 & & & & & q\alpha_1\alpha_2\beta_3 & q\alpha_1\alpha_2\alpha_3 & & & q\beta_1\theta_3\alpha_2\alpha_3 \\
 & & & & & & & & & a1A1''(1)
\end{array}\right]
\end{array}
$$

a1A1"(1)$_a$ = $q\ \theta_3\ \alpha_2\ \alpha_3$

a1A1"(1)$_b$ = $q\ \alpha_2\ \alpha_3$

b2A1" =

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} 0 & 1 & 2 & \dots & K3-1 & K3 & \substack{j=0\\ K3+1} & \substack{j=1\\ K3+1} & \substack{j=2\\ K3+1} & \substack{j=3\\ K3+1} \end{array} \\
\begin{array}{c} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
q\beta_1\theta_2\alpha_2 & & & & & & & & & \\
q\beta_1\theta_2\alpha_2\beta_3 & q\beta_1\theta_2\alpha_2\alpha_3 & & & & & & & & \\
 & q\beta_1\theta_2\alpha_2\beta_3 & q\beta_1\theta_2\alpha_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & q\beta_1\theta_2\alpha_2\beta_3 & q\beta_1\theta_2\alpha_2\alpha_3 & & & & \\
 & & & & & q\beta_1\theta_2\alpha_2\beta_3 & & & q\beta_1\theta_2\alpha_2\alpha_3 & \\
 & & & & & b2A1''(1) & & & b2A1''(2) &
\end{array}\right]
\end{array}
$$

b2A1"(1)$_a$ = $q\ \theta_2\ \alpha_2\ \beta_3$
b2A1"(2)$_a$ = $q\ \theta_2\ \alpha_2\ \alpha_3$

b2A1"(1)$_b$ = 0
b2A1"(2)$_b$ = 0

b3A1"$_a$ =

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} 0 & 1 & 2 & & K3 & \substack{j=0\\ K3+1} & \substack{j=1\\ K3+1} & \substack{j=2\\ K3+1} & \substack{j=3\\ K3+1} \end{array} \\
\begin{array}{c} L_3 \\ j=2\ \ K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
 & & & & & & & & q\theta_3\beta_2\alpha_3
\end{array}\right]
\end{array}
$$

b3A1"$_b$ = 0

b4A1"$_a$ =

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} 0 & 1 & 2 & \dots & K3 & \substack{j=0\\ K3+1} & \substack{j=1\\ K3+1} & \substack{j=2\\ K3+1} & \substack{j=3\\ K3+1} \end{array} \\
\begin{array}{c} L_3 \\ j=2\ \ K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
 & & & & & & & & q\theta_3\alpha_2\alpha_3
\end{array}\right]
\end{array}
$$

$\mathbf{b4A1''_b = 0}$

$\mathbf{b5A1''_a =}$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccc} & & & & & & & j=0 & j=1 & j=2 & j=3 \\ L_3 & 0 & 1 & 2 & \dots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=2 \quad K3+1 \end{array} &
\begin{bmatrix}
q\theta_2\alpha_2 & & & & & & & & & \\
q\theta_2\alpha_2\beta_3 & q\theta_2\alpha_2\alpha_3 & & & & & & & & \\
 & q\theta_2\alpha_2\beta_3 & q\theta_2\alpha_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & q\theta_2\alpha_2\beta_3 & q\theta_2\alpha_2\alpha_3 & & & & \\
 & & & & & q\theta_2\alpha_2\beta_3 & & & q\theta_2\alpha_2\alpha_3 &
\end{bmatrix}
\end{array}
$$

$\mathbf{b5A1''_b =}$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccc} & & & & & & & j=0 & j=1 & j=2 & j=3 \\ L_3 & 0 & 1 & 2 & \dots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=2 \quad K3+1 \end{array} &
\begin{bmatrix}
q\alpha_2 & & & & & & & & & \\
q\alpha_2\beta_3 & q\alpha_2\alpha_3 & & & & & & & & \\
 & q\alpha_2\beta_3 & q\alpha_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & q\alpha_2\beta_3 & q\alpha_2\alpha_3 & & & & \\
 & & & & & q\alpha_2\beta_3 & & & q\alpha_2\alpha_3 &
\end{bmatrix}
\end{array}
$$

$\mathbf{(A1^{*2})' =}$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & & & & & j=0 & j=2 \\ L_2 & 0 & 1 & 2 & 3 & \dots & K2 & K2+1 & K2+1 \end{array} \\
\begin{array}{r} 0 \\ 1 \\ 2 \\ \ddots \\ K2 \\ j=0 \quad K2+1 \\ j=2 \quad K2+1 \end{array} &
\begin{bmatrix}
 & (b1A1^{*2})' & & & & & & \\
 & (a1A1^{*2})' & (a0A1^{*2})' & & & & & \\
 & & (a1A1^{*2})' & (a0A1^{*2})' & & & & \\
 & & & & \ddots & & & \\
 & & & & & (a1A1^{*2})' & (a0A1^{*2})' & \\
 & & & & & & (a1A1^{*2})' & \\
 & & & & & & (b4A1^{*2})' &
\end{bmatrix}
\end{array}
$$

$\mathbf{(b1A1^{*2})' =}$

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ L_3 & 0 & 1 & 2 & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} 0 \\ 1 \\ 2 \\ . \\ K3 \\ j=0 \quad K3+1 \\ j=3 \quad K3+1 \end{array} &
\begin{bmatrix}
p\beta_1\theta_2 & & & & & & & & \\
p\beta_1\theta_2\beta_3 & p\beta_1\theta_2\alpha_3 & & & & & & & \\
 & p\beta_1\theta_2\beta_3 & p\beta_1\theta_2\alpha_3 & & & & & & \\
 & & . & & & & & & \\
 & & & p\beta_1\theta_2\beta_3 & p\beta_1\theta_2\alpha_3 & & & & \\
 & & & & p\beta_1\theta_2\beta_3 & p\beta_1\theta_2\alpha_3 & & & \\
 & & & & (b1A1^{*2})'(1) & (b1A1^{*2})'(2) & & &
\end{bmatrix}
\end{array}
$$

$\mathbf{(b1A1^{*2})'(1)_a = p\,\theta_2\,\beta_3}$
$\mathbf{(b1A1^{*2})'(2)_a = p\,\theta_2\,\alpha_3}$

$(\mathbf{b1A1}^{*2})'(1)_b = 0$
$(\mathbf{b1A1}^{*2})'(2)_a = 0$

$(\mathbf{a1A1}^{*2})' =$

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \ldots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
p\beta_1\theta_2\beta_2 & & & & & & & & & \\
p\beta_1\theta_2\beta_2\beta_3 & p\beta_1\theta_2\beta_2\alpha_3 & & & & & & & & \\
 & p\beta_1\theta_2\beta_2\beta_3 & p\beta_1\theta_2\beta_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & p\beta_1\theta_2\beta_2\beta_3 & p\beta_1\theta_2\beta_2\alpha_3 & & & & \\
 & & & & & p\beta_1\theta_2\beta_2\beta_3 & p\beta_1\theta_2\beta_2\alpha_3 & & & \\
 & & & & & (a1A1^{*2})'(1) & (a1A1^{*2})'(1) & & &
\end{array}\right]
\end{array}
$$

$(\mathbf{a1A1}^{*2})'(1)_a = p\,\theta_2\,\beta_2\,\beta_3$
$(\mathbf{a1A1}^{*2})'(2)_a = p\,\theta_2\,\beta_2\,\alpha_3$

$(\mathbf{a1A1}^{*2})'(1)_b = 0$
$(\mathbf{a1A1}^{*2})'(2)_b = 0$

$(\mathbf{a0A1}^{*2})' =$

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \ldots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
p\beta_1\theta_2\alpha_2 & & & & & & & & & \\
p\beta_1\theta_2\alpha_2\beta_3 & p\beta_1\theta_2\alpha_2\alpha_3 & & & & & & & & \\
 & p\beta_1\theta_2\alpha_2\beta_3 & p\beta_1\theta_2\alpha_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & p\beta_1\theta_2\alpha_2\beta_3 & p\beta_1\theta_2\alpha_2\alpha_3 & & & & \\
 & & & & & p\beta_1\theta_2\alpha_2\beta_3 & p\beta_1\theta_2\alpha_2\alpha_3 & & & \\
 & & & & & (a0A1^{*2})'(1) & (a0A1^{*2})'(2) & & &
\end{array}\right]
\end{array}
$$

$(\mathbf{a0A1}^{*2})'(1)_a = p\,\theta_2\,\alpha_2\,\beta_3$
$(\mathbf{a0A1}^{*2})'(2)_a = p\,\theta_2\,\alpha_2\,\alpha_3$

$(\mathbf{a0A1}^{*2})'(1)_b = 0$
$(\mathbf{a0A1}^{*2})'(2)_b = 0$

$(\mathbf{b4A1}^{*2})'_a =$

$$\begin{array}{r c}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \cdots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=2\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
 & & & & & & & & & \\
p\theta_2\beta_2 & & & & & & & & & \\
p\theta_2\beta_2\beta_3 & p\theta_2\beta_2\alpha_3 & & & & & & & & \\
 & p\theta_2\beta_2\beta_3 & p\theta_2\beta_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & p\theta_2\beta_2\beta_3 & p\theta_2\beta_2\alpha_3 & & & & \\
 & & & & & p\theta_2\beta_2\beta_3 & p\theta_2\beta_2\alpha_3 & & &
\end{array}\right]
\end{array}$$

$(\mathbf{b4A1}^{*2})'_b =$

$$\begin{array}{r c}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \cdots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=2\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
 & & & & & & & & & \\
p\beta_2 & & & & & & & & & \\
p\beta_2\beta_3 & p\beta_2\alpha_3 & & & & & & & & \\
 & p\beta_2\beta_3 & p\beta_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & p\beta_2\beta_3 & p\beta_2\alpha_3 & & & & \\
 & & & & & p\beta_2\beta_3 & p\beta_2\alpha_3 & & &
\end{array}\right]
\end{array}$$

$(\mathbf{A1}^{*3})'$ **can be found from** $(\mathbf{A1}^{*3})' = \mathbf{A1} - ((\mathbf{A1}^{*2})' + \mathbf{A1}'')$

$\mathbf{A2}^{*2} =$

$$\begin{array}{r c}
 & \begin{array}{cccccccc} & & & & & & j=0 & j=2 \\ 0 & 1 & 2 & 3 & \cdots & K2 & K2+1 & K2+1 \end{array} \\
\begin{array}{r} L_2 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K2 \\ j=0\ \ K2+1 \\ j=2\ \ K2+1 \end{array} &
\left[\begin{array}{cccccccc}
 & & & & & & & \\
 & \mathbf{b1A2}^{*2} & & & & & & \\
 & \mathbf{a1A2}^{*2} & \mathbf{a0A2}^{*2} & & & & & \\
 & & \mathbf{a1A2}^{*2} & \mathbf{a0A2}^{*2} & & & & \\
 & & & & \ddots & & & \\
 & & & & & \mathbf{a1A2}^{*2} & \mathbf{a0A2}^{*2} & \\
 & & & & & & \mathbf{a1A2}^{*2} & \\
 & & & & & & \mathbf{b4A2}^{*2} &
\end{array}\right]
\end{array}$$

$\mathbf{b1A2}^{*2} =$

$$\begin{array}{r c}
 & \begin{array}{ccccccccc} & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ . \\ K3 \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{ccccccccc}
 & & & & & & & & \\
q\beta_1\theta_2 & & & & & & & & \\
q\beta_1\theta_2\beta_3 & q\beta_1\theta_2\alpha_3 & & & & & & & \\
 & q\beta_1\theta_2\beta_3 & q\beta_1\theta_2\alpha_3 & & & & & & \\
 & & & . & & & & & \\
 & & & q\beta_1\theta_2\beta_3 & q\beta_1\theta_2\alpha_3 & & & & \\
 & & & & q\beta_1\theta_2\beta_3 & q\beta_1\theta_2\alpha_3 & & & \\
 & & & & \mathrm{b1A2}^{*2}(1) & \mathrm{b1A2}^{*2}(2) & & &
\end{array}\right]
\end{array}$$

$\mathbf{b1A2}^{*2}(1)_a = q\,\theta_2\,\beta_3$
$\mathbf{b1A2}^{*2}(2)_a = q\,\theta_2\,\alpha_3$

$$b1A2^{*2}(1)_b = 0$$
$$b1A2^{*2}(2)_b = 0$$

$$a1A2^{*2} =$$

$$\begin{array}{cc}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \cdots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
q\beta_1\theta_2\beta_2 & & & & & & & & & \\
q\beta_1\theta_2\beta_2\beta_3 & q\beta_1\theta_2\beta_2\alpha_3 & & & & & & & & \\
 & q\beta_1\theta_2\beta_2\beta_3 & q\beta_1\theta_2\beta_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & q\beta_1\theta_2\beta_2\beta_3 & q\beta_1\theta_2\beta_2\alpha_3 & & & & \\
 & & & & & q\beta_1\theta_2\beta_2\beta_3 & q\beta_1\theta_2\beta_2\alpha_3 & & & \\
 & & & & & a1A2^{*2}(1) & a1A2^{*2}(2) & & &
\end{array}\right]
\end{array}$$

$$a1A2^{*2}(1)_a = q\,\theta_2\,\beta_2\,\beta_3$$
$$a1A2^{*2}(2)_a = q\,\theta_2\,\beta_2\,\alpha_3$$

$$a1A2^{*2}(1)_b = 0$$
$$a1A2^{*2}(2)_b = 0$$

$$a0A2^{*2} =$$

$$\begin{array}{cc}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \cdots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=0\ \ K3+1 \\ j=3\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
q\beta_1\theta_2\alpha_2 & & & & & & & & & \\
q\beta_1\theta_2\alpha_2\beta_3 & q\beta_1\theta_2\alpha_2\alpha_3 & & & & & & & & \\
 & q\beta_1\theta_2\alpha_2\beta_3 & q\beta_1\theta_2\alpha_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & q\beta_1\theta_2\alpha_2\beta_3 & q\beta_1\theta_2\alpha_2\alpha_3 & & & & \\
 & & & & & q\beta_1\theta_2\alpha_2\beta_3 & q\beta_1\theta_2\alpha_2\alpha_3 & & & \\
 & & & & & a0A2^{*2}(1) & a0A2^{*2}(2) & & &
\end{array}\right]
\end{array}$$

$$a0A2^{*2}(1)_a = q\,\theta_2\,\alpha_2\,\beta_3$$
$$a0A2^{*2}(2)_a = q\,\theta_2\,\alpha_2\,\alpha_3$$

$$a0A2^{*2}(1)_b = 0$$
$$a0A2^{*2}(2)_b = 0$$

$$b4A2^{*2}{}_a =$$

$$\begin{array}{cc}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ . \\ K3 \\ j=2\ \ K3+1 \end{array} &
\left[\begin{array}{cccccccccc}
q\theta_2\beta_2 & & & & & & & & & \\
q\theta_2\beta_2\beta_3 & q\theta_2\beta_2\alpha_3 & & & & & & & & \\
 & q\theta_2\beta_2\beta_3 & q\theta_2\beta_2\alpha_3 & & & & & & & \\
 & & & . & & & & & & \\
 & & & & q\theta_2\beta_2\beta_3 & q\theta_2\beta_2\alpha_3 & & & & \\
 & & & & & q\theta_2\beta_2\beta_3 & q\theta_2\beta_2\alpha_3 & & &
\end{array}\right]
\end{array}$$

$b4A2^{*2}_{b} =$

$$
\begin{array}{rc}
 & \begin{array}{cccccccccc} & & & & & & j=0 & j=1 & j=2 & j=3 \\ 0 & 1 & 2 & \ldots & K3-1 & K3 & K3+1 & K3+1 & K3+1 & K3+1 \end{array} \\
\begin{array}{r} L_3 \\ 0 \\ 1 \\ 2 \\ \ddots \\ K3 \\ j=2 \quad K3+1 \end{array} &
\begin{bmatrix}
q\beta_2 & & & & & & & & & \\
q\beta_2\beta_3 & q\beta_2\alpha_3 & & & & & & & & \\
 & q\beta_2\beta_3 & q\beta_2\alpha_3 & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & q\beta_2\beta_3 & q\beta_2\alpha_3 & & & & \\
 & & & & & q\beta_2\beta_3 & q\beta_2\alpha_3 & & &
\end{bmatrix}
\end{array}
$$

$A2^{*3}$ can be found using the relationship $A2^{*3} = A2 - A2^{*2}$

A.4.3 Trial Results

Table A.4.1: Case a, Results for Varying Inputs, n = 2

	p	β_1	β_2	β_3	K2	K3	θ_2	ρ_1	μ_{ul} combo
1	0.6000	0.8000	0.8000	0.8000	4	3	0.75	0.7500	1.2005
2	0.6000	0.8000	0.8000	0.8000	5	3	0.75	0.7500	1.2001
3	0.6000	0.8000	0.8000	0.8000	3	3	0.75	0.7500	1.2025
4	0.1000	0.3200	0.5000	0.5000	1	1	0.5	0.3125	0.4096
5	0.6000	0.7000	0.9000	0.9000	1	1	0.5	0.8571	2.4171
6	0.6000	0.8000	0.8000	0.8000	3	3	0.5	0.7500	1.2003
7	0.6000	0.8000	0.8000	0.8000	4	4	0.5	0.7500	1.2000
8	0.6000	0.8000	0.8000	0.8000	2	2	0.75	0.7500	1.2126
9	0.4000	0.4500	0.9000	0.9000	1	1	0.5	0.8889	4.8111
10	0.6000	0.8000	0.8000	0.8000	2	2	0.5	0.7500	1.2029
11	0.1250	0.1500	0.6250	0.6250	1	1	0.5	0.8333	4.3810
12	0.1500	0.2000	0.7500	0.7500	1	1	0.5	0.7500	2.5513
13	0.1500	0.3000	0.7500	0.7500	1	1	0.5	0.5000	0.8503
14	0.1000	0.2500	0.5000	0.5000	1	1	0.5	0.4000	0.6007
15	0.1500	0.2250	0.7500	0.7500	1	1	0.5	0.6667	1.7007
16	0.4000	0.4800	0.3600	0.8000	1	1	0.5	0.8333	3.7150
17	0.6000	0.7000	0.9000	0.9000	1	1	0.75	0.8571	2.4380
18	0.2500	0.3000	0.2500	0.5000	1	1	0.5	0.8333	4.4124
19	0.6000	0.8000	0.8000	0.8000	1	1	0.5	0.7500	1.2287
20	0.1000	0.2500	0.2500	0.2500	1	1	0.5	0.4000	0.6067
21	0.1000	0.2500	0.2500	0.2500	3	3	0.5	0.4000	0.6001
22	0.1000	0.2500	0.2500	0.2500	2	2	0.5	0.4000	0.6010
23	0.1000	0.2500	0.2500	0.2500	4	4	0.5	0.4000	0.6000
24	0.3000	0.4500	0.2143	0.2143	1	1	0.5	0.6667	3.5233
25	0.1500	0.8000	0.1250	0.1000	1	1	0.75	0.1875	0.588
26	0.5000	0.6000	0.4000	0.4000	1	1	0.5	0.8333	6.6506
27	0.2000	0.4000	0.1425	0.1425	1	1	0.5	0.5000	1.8486
28	0.6000	0.8000	0.4000	0.6000	1	1	0.5	0.7500	2.2138
29	0.2000	0.5000	0.2000	0.2000	1	1	0.5	0.4000	0.6616
30	0.6000	0.9000	0.6000	0.4500	1	1	0.5	0.6667	1.1295
31	0.6000	0.8000	0.8000	0.8000	1	1	0.75	0.7500	1.2678
32	0.1000	0.2000	0.2000	0.2000	1	1	0.5	0.5000	0.9182
33	0.6000	0.8000	0.4500	0.4500	1	1	0.5	0.7500	2.7793
34	0.2000	0.4000	0.2000	0.2000	1	1	0.5	0.5000	0.9860

Table A.4.1: Case *a*, Results for Varying Inputs, $n = 2$

	μ_{q1} iso	μ_{q1} % error	ρ_2	μ_{q2} combo	μ_{q2} iso	BP_2	μ_{q2} % error	ρ_3
1	1.2000	0.04%	0.5625	0.5712	0.5728	0.0000	0.28%	0.1875
2	1.2000	0.01%	0.5625	0.5711	0.5730	0.0000	0.33%	0.1875
3	1.2047	0.18%	0.5625	0.5711	0.5717	0.0013	0.11%	0.1875
4	0.4103	0.17%	0.1000	0.0831	0.0843	0.0020	1.44%	0.1000
5	2.5424	5.18%	0.3333	0.3238	0.2726	0.0080	15.81%	0.3333
6	1.2000	0.02%	0.3750	0.3457	0.3548	0.0000	2.63%	0.3750
7	1.2000	0.00%	0.3750	0.3457	0.3549	0.0000	2.66%	0.3750
8	1.2319	1.59%	0.5625	0.5691	0.5660	0.0087	0.54%	0.1875
9	4.8880	1.60%	0.2222	0.2198	0.1509	0.0020	31.35%	0.2222
10	1.2097	0.57%	0.3750	0.3456	0.3541	0.0020	2.46%	0.3750
11	4.3882	0.16%	0.1000	0.1016	0.0590	0.0005	41.93%	0.1000
12	2.5500	0.05%	0.1000	0.0981	0.0629	0.0004	35.88%	0.1000
13	0.8514	0.13%	0.1000	0.0888	0.0747	0.0008	15.88%	0.1000
14	0.6014	0.12%	0.1000	0.0903	0.0784	0.0014	13.18%	0.1000
15	1.7020	0.08%	0.1000	0.0958	0.0664	0.0004	30.69%	0.1000
16	3.8339	3.20%	0.5556	0.6557	0.4293	0.0676	34.53%	0.2500
17	2.6826	10.03%	0.5000	0.5041	0.4161	0.0195	17.46%	0.1667
18	4.5203	2.45%	0.5000	0.5954	0.3486	0.0485	41.45%	0.2500
19	1.3274	8.03%	0.3750	0.3438	0.3476	0.0240	1.11%	0.3750
20	0.6113	0.76%	0.2000	0.1964	0.1641	0.0111	16.45%	0.2000
21	0.6000	0.02%	0.2000	0.2007	0.1672	0.0001	16.69%	0.2000
22	0.6009	0.02%	0.2000	0.2000	0.1669	0.0009	16.55%	0.2000
23	0.6000	0.00%	0.2000	0.2008	0.1673	0.0000	16.68%	0.2000
24	2.2727	35.50%	0.7001	1.0099	0.5818	0.1280	42.39%	0.7001
25	0.2558	56.50%	0.9000	0.7386	0.7972	0.2139	7.93%	0.3750
26	5.9552	10.46%	0.6250	0.8438	0.5309	0.0967	37.08%	0.6250
27	1.1211	39.35%	0.7018	1.0097	0.6123	0.1432	39.36%	0.7018
28	2.2489	1.59%	0.7500	0.8961	0.7001	0.1597	21.87%	0.5000
29	0.6276	5.14%	0.5000	0.5308	0.4767	0.0901	10.19%	0.5000
30	1.2352	9.36%	0.5000	0.4887	0.5370	0.0853	9.88%	0.6667
31	1.4659	15.63%	0.5625	0.5555	0.5387	0.0587	3.02%	0.1875
32	0.9292	1.20%	0.2500	0.2697	0.1928	0.0157	28.51%	0.2500
33	2.6201	5.73%	0.6667	0.8582	0.6498	0.1355	24.28%	0.6667
34	0.9456	4.10%	0.5000	0.5901	0.4394	0.0770	25.54%	0.5000

Table A.4.1: Case *a*, Results for Varying Inputs, $n = 2$

	μ_{a3} combo	μ_{a3} iso	BP_3	μ_{a3} % error	Q	μ_q combo	μ_q iso	μ_q % error
1	0.1628	0.1689	0.0000	3.75%	0.1990	1.9345	1.9417	0.37%
2	0.1628	0.1689	0.0000	3.75%	0.1996	1.9340	1.9419	0.41%
3	0.1630	0.1689	0.0000	3.62%	0.1974	1.9366	1.9453	0.45%
4	0.0831	0.0843	0.0020	1.44%	0.2167	0.5758	0.5789	0.54%
5	0.3238	0.2726	0.0080	15.81%	0.0989	3.0647	3.0876	0.75%
6	0.3457	0.3548	0.0000	2.63%	0.1997	1.8917	1.9096	0.95%
7	0.3457	0.3549	0.0000	2.66%	0.2000	1.8914	1.9098	0.97%
8	0.1636	0.1690	0.0002	3.30%	0.1929	1.9453	1.9669	1.11%
9	0.2198	0.1509	0.0020	31.35%	0.0498	5.2507	5.1898	1.16%
10	0.3456	0.3541	0.0020	2.46%	0.1985	1.8941	1.9179	1.26%
11	0.1016	0.0590	0.0005	41.93%	0.0249	4.5842	4.5062	1.70%
12	0.0981	0.0629	0.0004	35.88%	0.0499	2.7475	2.6758	2.61%
13	0.0888	0.0747	0.0008	15.88%	0.1497	1.0279	1.0008	2.64%
14	0.0903	0.0784	0.0014	13.18%	0.1488	0.7813	0.7582	2.96%
15	0.0958	0.0664	0.0004	30.69%	0.0749	1.8923	1.8348	3.04%
16	0.2698	0.1827	0.0049	32.28%	0.0600	4.6405	4.4459	4.19%
17	0.1584	0.1344	0.0017	15.15%	0.0975	3.1005	3.2331	4.28%
18	0.2933	0.1686	0.0083	42.52%	0.0397	5.3011	5.0375	4.97%
19	0.3438	0.3476	0.0240	1.11%	0.1907	1.9163	2.0226	5.55%
20	0.1964	0.1641	0.0111	16.45%	0.1392	0.9995	0.9395	6.00%
21	0.2007	0.1672	0.0001	16.69%	0.1484	1.0015	0.9344	6.70%
22	0.2000	0.1669	0.0009	16.55%	0.1459	1.0010	0.9347	6.70%
23	0.2008	0.1673	0.0000	16.68%	0.1493	1.0016	0.9346	6.70%
24	1.0099	0.5818	0.1280	42.39%	0.0437	4.2925	3.4363	6.70%
25	0.5118	0.5380	0.1166	5.12%	0.0668	1.5062	1.5910	6.70%
26	0.8438	0.5309	0.0967	37.08%	0.0313	7.6428	7.0170	6.70%
27	1.0097	0.6123	0.1432	39.36%	0.0546	3.1405	2.3457	6.70%
28	0.6090	0.5005	0.0735	17.82%	0.0913	3.7189	3.4495	7.24%
29	0.5308	0.4767	0.0901	10.19%	0.1437	1.7232	1.5810	8.25%
30	0.6476	0.6828	0.1496	5.44%	0.1674	2.2658	2.4550	8.35%
31	0.1667	0.1689	0.0053	1.32%	0.1793	1.9900	2.1735	9.22%
32	0.2697	0.1928	0.0157	28.51%	0.0915	1.4576	1.3148	9.80%
33	0.8582	0.6498	0.1355	24.28%	0.0704	4.4957	3.9197	12.81%
34	0.5901	0.4394	0.0770	25.54%	0.1137	2.1662	1.8244	15.78%

Table A.4.1: Case a, Results for Varying Inputs, $n = 2$

	p	β_1	β_2	β_3	K2	K3	θ_2	ρ_1	μ_{q1} combo
35	0.2000	0.6500	0.2000	0.2000	1	1	0.5	0.3077	0.4390
36	0.1000	0.5000	0.5000	0.5000	1	1	0.5	0.2000	0.2254
37	0.6000	0.9000	0.4000	0.4500	1	1	0.5	0.6667	2.0385
38	0.2000	0.6000	0.3000	0.3000	1	1	0.5	0.3333	0.4210
39	0.1500	0.8000	0.1250	0.2000	1	1	0.75	0.1875	0.3294
40	0.1500	0.7500	0.1050	0.1050	1	1	0.5	0.2000	0.7720
41	0.3000	0.6000	0.2500	0.4000	1	1	0.75	0.5000	1.4124
42	0.4000	0.6000	0.3000	0.3000	1	1	0.5	0.6667	2.5345
43	0.1000	0.5000	0.0750	0.0750	1	1	0.5	0.2000	0.6543
44	0.4500	0.6500	0.3250	0.3250	1	1	0.5	0.6923	3.1479
45	0.2500	0.8000	0.8000	0.8000	1	1	0.5	0.3125	0.3413
46	0.6000	0.9000	0.4000	0.4000	1	1	0.5	0.6667	2.9449
47	0.5000	0.6000	0.4500	0.4000	1	1	0.75	0.8333	12.6834
48	0.5000	0.6000	0.3750	0.3750	1	1	0.5	0.8333	10.6685
49	0.1500	0.3000	0.1125	0.1125	1	1	0.5	0.5000	1.6432
50	0.1500	0.3750	0.1125	0.1125	1	1	0.5	0.4000	1.1715
51	0.6000	0.9000	0.3500	0.4500	1	1	0.5	0.6667	3.1843
52	0.3000	0.7500	0.2000	0.2000	1	1	0.5	0.4000	1.5310
53	0.1000	0.8000	0.0667	0.0667	1	1	0.5	0.1250	0.9548
54	0.4500	0.6000	0.3000	0.3375	1	1	0.5	0.7500	5.7246
55	0.1500	0.7500	0.1000	0.1000	1	1	0.5	0.2000	1.0678
56	0.4500	0.5400	0.9000	0.3938	1	1	0.25	0.8333	14.2968
57	0.3000	0.4000	0.2143	0.2143	1	1	0.5	0.7500	6.6826
58	0.3000	0.6000	0.2000	0.2000	1	1	0.5	0.5000	2.2156
59	0.1000	0.8000	0.8000	0.8000	1	1	0.5	0.1250	0.1286
60	0.1000	0.3200	0.0667	0.0667	1	1	0.5	0.3125	1.6069
61	0.2000	0.6400	0.1250	0.1250	1	1	0.5	0.3125	2.1283
62	0.3000	0.6000	0.1875	0.1875	1	1	0.5	0.5000	3.3504
63	0.2000	0.5000	0.1250	0.1250	1	1	0.5	0.4000	2.6659
64	0.5000	0.6000	0.3571	0.3571	1	1	0.5	0.8333	21.5938
65	0.4500	0.6000	0.3000	0.3000	1	1	0.5	0.7500	9.8646
66	0.2000	0.8000	0.2000	0.2000	1	1	0.5	0.2500	0.3150
67	0.2500	0.6500	0.1500	0.1500	1	1	0.5	0.3846	3.3822
68	0.2000	0.6400	0.1200	0.1200	1	1	0.5	0.3125	2.9871
69	0.2500	0.5000	0.1500	0.1500	1	1	0.5	0.5000	4.9497

Table A.4.1: Case *a*, Results for Varying Inputs, $n = 2$

	μ_{a1} iso	μ_{a1} % error	ρ_2	μ_{a2} combo	μ_{a2} iso	BP_2	μ_{a2} % error	ρ_3
35	0.4190	4.56%	0.5000	0.4205	0.5155	0.1048	22.59%	0.5000
36	0.2258	0.18%	0.1000	0.0638	0.0937	0.0030	46.87%	0.1000
37	1.6139	20.83%	0.7500	0.8987	0.7539	0.1812	16.11%	0.6667
38	0.4292	1.95%	0.3333	0.2500	0.3303	0.0453	32.12%	0.3333
39	0.2463	25.23%	0.9000	0.5842	0.7862	0.2083	34.58%	0.1875
40	0.2756	64.30%	0.7143	0.8003	0.7313	0.1832	8.62%	0.7143
41	0.9947	29.57%	0.9000	1.0400	0.7602	0.1923	26.90%	0.1875
42	1.9154	24.43%	0.6667	0.8935	0.5945	0.1245	33.46%	0.6667
43	0.2798	57.24%	0.6667	0.8462	0.6736	0.1566	20.40%	0.6667
44	2.2011	30.08%	0.6923	0.9466	0.6259	0.1347	33.88%	0.6923
45	0.3429	0.47%	0.1563	0.0745	0.1510	0.0040	102.68%	0.1563
46	1.7798	39.56%	0.7500	0.9871	0.7606	0.1835	22.95%	0.7500
47	8.5310	32.74%	0.8333	1.1732	0.6930	0.1473	40.93%	0.3125
48	6.9832	34.54%	0.6667	0.9345	0.5658	0.1070	39.45%	0.6667
49	1.1740	28.55%	0.6667	0.9605	0.5660	0.1276	41.07%	0.6667
50	0.7291	37.76%	0.6667	0.9272	0.6065	0.1451	34.59%	0.6667
51	1.7941	43.66%	0.8571	1.1003	0.8153	0.2117	25.90%	0.6667
52	0.6786	55.68%	0.7500	0.9585	0.7352	0.1874	23.30%	0.7500
53	0.1696	82.24%	0.7500	0.9036	0.7663	0.2118	15.19%	0.7500
54	3.4119	40.40%	0.7500	1.0499	0.6382	0.1406	39.21%	0.6667
55	0.2085	80.47%	0.7500	0.9297	0.7564	0.2045	18.64%	0.7500
56	8.3727	41.44%	0.1250	0.1609	0.0981	0.0007	39.03%	0.8570
57	3.9238	41.28%	0.7001	1.0302	0.5504	0.1162	46.57%	0.7001
58	1.0526	52.49%	0.7500	1.0625	0.6901	0.1675	35.05%	0.7500
59	0.1286	0.00%	0.0625	0.0192	0.0616	0.0005	220.83%	0.0625
60	0.5500	65.77%	0.7500	1.1570	0.6886	0.1761	40.48%	0.7500
61	0.5079	76.14%	0.8000	1.1747	0.7643	0.1953	34.94%	0.8000
62	1.0927	67.39%	0.8000	1.2064	0.7257	0.1797	39.85%	0.8000
63	0.7585	71.55%	0.8000	1.2326	0.7267	0.1781	41.04%	0.8000
64	8.9670	58.47%	0.7001	1.0110	0.5928	0.1202	41.36%	0.7001
65	3.8624	60.85%	0.7500	1.1152	0.6457	0.1432	42.10%	0.7500
66	0.3151	0.03%	0.5000	0.2770	0.5419	0.1153	95.63%	0.5000
67	0.7069	79.10%	0.8333	1.2850	0.7753	0.2073	39.67%	0.8333
68	0.5265	82.37%	0.8333	1.2890	0.7842	0.2127	39.16%	0.8333
69	1.1916	75.93%	0.8333	1.3327	0.7272	0.1853	45.43%	0.8333

Table A.4.1: Case a, Results for Varying Inputs, n = 2

	μ_{q3} combo	μ_{q3} iso	BP_3	μ_{q3} % error	Q	μ_q combo	μ_q iso	μ_q % error
35	0.4205	0.5515	0.1048	31.15%	0.1672	1.2800	1.4860	16.09%
36	0.0638	0.0937	0.0030	46.87%	0.3819	0.3530	0.4132	17.05%
37	0.8182	0.7005	0.1550	14.39%	0.0985	3.7554	3.0683	18.30%
38	0.2500	0.3303	0.0453	32.12%	0.2701	0.9210	1.0898	18.33%
39	0.1818	0.2826	0.0371	55.45%	0.1291	1.0954	1.3151	20.06%
40	0.8003	0.7313	0.1832	8.62%	0.0569	2.3726	1.7382	26.74%
41	0.2669	0.1991	0.0156	25.40%	0.1056	2.7193	1.9540	28.14%
42	0.8935	0.5945	0.1245	33.46%	0.0701	4.3215	3.1044	28.16%
43	0.8462	0.6736	0.1566	20.40%	0.0481	2.3467	1.6270	30.67%
44	0.9466	0.6259	0.1347	33.88%	0.0598	5.0411	3.4529	31.51%
45	0.0745	0.1510	0.0040	102.68%	0.5407	0.4903	0.6449	31.53%
46	0.9871	0.7606	0.1835	22.95%	0.0646	4.9191	3.3010	32.89%
47	0.4438	0.2711	0.0294	38.91%	0.0168	14.3004	9.4951	33.60%
48	0.9345	0.5658	0.1070	39.45%	0.0193	12.5375	8.1148	35.28%
49	0.9605	0.5660	0.1276	41.07%	0.0487	3.5642	2.3060	35.30%
50	0.9272	0.6065	0.1451	34.59%	0.0593	3.0259	1.9421	35.82%
51	0.9213	0.7089	0.1577	23.05%	0.0610	5.2059	3.3183	36.26%
52	0.9585	0.7352	0.1874	23.30%	0.0753	3.4480	2.1490	37.67%
53	0.9036	0.7663	0.2118	15.19%	0.0326	2.7620	1.7022	38.37%
54	0.9691	0.5870	0.1176	39.43%	0.0341	7.7436	4.6371	40.12%
55	0.9297	0.7564	0.2045	18.64%	0.0473	2.9272	1.7213	41.20%
56	1.1831	0.6798	0.1490	42.54%	0.0150	15.6408	9.1506	41.50%
57	1.0302	0.5504	0.1162	46.57%	0.0242	8.7430	5.0246	42.53%
58	1.0625	0.6901	0.1675	35.05%	0.0614	4.3406	2.4328	43.95%
59	0.0192	0.0616	0.0005	220.83%	0.6907	0.1670	0.2518	50.78%
60	1.1570	0.6886	0.1761	40.48%	0.0299	3.9209	1.9272	50.85%
61	1.1747	0.7643	0.1953	34.94%	0.0429	4.4777	2.0365	54.52%
62	1.2064	0.7257	0.1797	39.85%	0.0436	5.7632	2.5441	55.86%
63	1.2326	0.7267	0.1781	41.04%	0.0382	5.1311	2.2119	56.89%
64	1.0110	0.5928	0.1202	41.36%	0.0096	23.6158	10.1526	57.01%
65	1.1152	0.6457	0.1432	42.10%	0.0201	12.0950	5.1538	57.39%
66	0.2770	0.5419	0.1153	95.63%	0.1781	0.8690	1.3989	60.98%
67	1.2850	0.7753	0.2073	39.67%	0.0373	5.9522	2.2575	62.07%
68	1.2890	0.7842	0.2127	39.16%	0.0388	5.5651	2.0949	62.36%
69	1.3327	0.7272	0.1853	45.43%	0.0279	7.6151	2.6460	65.25%

Table A.4.1: Case a, Results for Varying Inputs, n = 2

	p	β_1	β_2	β_3	K2	K3	θ_2	ρ_1	μ_{q1} combo
70	0.1000	0.4500	0.0600	0.0600	1	1	0.5	0.2222	2.8625
71	0.3000	0.4500	0.1875	0.1875	1	1	0.5	0.6667	9.0462
72	0.4000	0.5250	0.2600	0.2600	1	1	0.5	0.7619	21.1413
73	0.1000	0.8000	0.5000	0.5000	1	1	0.5	0.1250	0.1288
74	0.3000	0.4500	0.1800	0.1800	1	1	0.5	0.6667	17.3363
75	0.1000	0.8000	0.1500	0.2000	1	1	0.75	0.1250	0.1396
76	0.1000	0.8000	0.2500	0.2500	1	1	0.5	0.1250	0.1297
77	0.2000	0.9000	0.2000	0.2000	1	1	0.5	0.2222	0.2547

Table A.4.1: Case a, Results for Varying Inputs, n = 2

	μ_{q1} iso	μ_{q1} % error	ρ_2	μ_{q2} combo	μ_{q2} iso	BP_2	μ_{q2} % error	ρ_3
70	0.3561	87.56%	0.8333	1.3643	0.7804	0.2162	42.80%	0.8333
71	2.5468	71.85%	0.8000	1.2577	0.6558	0.1501	47.86%	0.8000
72	4.3382	79.48%	0.7692	1.1771	0.6344	0.1409	46.10%	0.7692
73	0.1292	0.31%	0.1000	0.0295	0.1016	0.0042	244.41%	0.1000
74	2.7349	84.22%	0.8333	1.3469	0.6787	0.1627	49.61%	0.8333
75	0.1429	2.36%	0.5000	0.1784	0.5236		193.50%	0.1250
76	0.1321	1.85%	0.2000	0.0587	0.2155		267.12%	0.2000
77	0.2706	6.24%	0.5000	0.1592	0.5551	0.1125	248.68%	0.5000

Table A.4.1: Case a, Results for Varying Inputs, n = 2

	μ_{q3} combo	μ_{q3} iso	BP_3	μ_{q3} % error	Q	μ_q combo	μ_q iso	μ_q % error
70	1.3643	0.7804	0.2162	42.80%	0.0189	5.5911	1.9169	65.72%
71	1.2577	0.6558	0.1501	47.86%	0.0180	11.5616	3.8584	66.63%
72	1.1771	0.6344	0.1409	46.10%	0.0091	23.4955	5.6070	76.14%
73	0.0295	0.1016	0.0042	244.41%	0.6023	0.1878	0.3324	77.00%
74	1.3469	0.6787	0.1627	49.61%	0.0097	20.0301	4.0923	79.57%
75	0.0456	0.1568	0.0140	243.86%	0.2032	0.3636	0.8233	126.43%
76	0.0587	0.2155		267.12%	0.3563	0.2471	0.5631	127.88%
77	0.1592	0.5551	0.1125	248.68%	0.1821	0.5731	1.3808	140.94%

Table A.4.2: Case b, Results for Varying Inputs, n = 2

	p	β_1	β_2	β_3	K2	K3	θ_2	ρ_1	μ_{q1} combo
1	0.6000	0.8000	0.8000	0.8000	4	3	0.75	0.7500	1.2006
2	0.6000	0.8000	0.8000	0.8000	3	3	0.75	0.7500	1.2029
3	0.6000	0.8000	0.8000	0.8000	5	3	0.75	0.7500	1.2001
4	0.1000	0.3200	0.5000	0.5000	1	1	0.5	0.3125	0.4099
5	0.6000	0.7000	0.9000	0.9000	1	1	0.5	0.8571	2.4188
6	0.6000	0.8000	0.8000	0.8000	2	2	0.75	0.7500	1.2150
7	0.6000	0.8000	0.8000	0.8000	3	3	0.5	0.7500	1.2004
8	0.6000	0.8000	0.8000	0.8000	4	4	0.5	0.7500	1.2000
9	0.6000	0.8000	0.8000	0.8000	2	2	0.5	0.7500	1.2038
10	0.4000	0.4500	0.9000	0.9000	1	1	0.5	0.8889	4.8120
11	0.1250	0.1500	0.6250	0.6250	1	1	0.5	0.8333	4.3830
12	0.0300	0.2400	0.0750	0.0750	1	1	0.5	0.1250	0.1561
13	0.1500	0.2000	0.7500	0.7500	1	1	0.5	0.7500	2.5515
14	0.1500	0.3000	0.7500	0.7500	1	1	0.5	0.5000	0.8504
15	0.1000	0.2500	0.5000	0.5000	1	1	0.5	0.4000	0.6011
16	0.1500	0.2250	0.7500	0.7500	1	1	0.5	0.6667	1.7008
17	0.6000	0.7000	0.9000	0.9000	1	1	0.75	0.8571	2.4406
18	0.6000	0.8000	0.8000	0.8000	1	1	0.5	0.7500	1.2368
19	0.1000	0.2500	0.2500	0.2500	4	4	0.5	0.4000	0.6001
20	0.1000	0.2500	0.2500	0.2500	3	3	0.5	0.4000	0.6005
21	0.1000	0.2500	0.2500	0.2500	2	2	0.5	0.4000	0.6032
22	0.1000	0.2500	0.2500	0.2500	1	1	0.5	0.4000	0.6210
23	0.6000	0.8000	0.8000	0.8000	1	1	0.75	0.7500	1.2812
24	0.2000	0.6000	0.3000	0.3000	1	1	0.5	0.3333	0.4702
25	0.4000	0.8000	0.8000	0.8000	1	1	0.5	0.5000	0.6033
26	0.1000	0.5000	0.5000	0.5000	1	1	0.5	0.2000	0.2256
27	0.2000	0.6500	0.2000	0.2000	1	1	0.5	0.3077	0.8848
28	0.6000	0.9000	0.6000	0.4500	1	1	0.5	0.6667	2.1601
29	0.2500	0.8000	0.8000	0.8000	1	1	0.5	0.3125	0.3414
30	0.4000	0.4800	0.3600	0.8000	1	1	0.5	0.8333	5.6153
31	0.2500	0.3000	0.2500	0.5000	1	1	0.5	0.8333	6.6255
32	0.2000	0.5000	0.2000	0.2000	1	1	0.5	0.4000	1.2875
33	0.1000	0.4500	0.1000	0.1000	1	1	0.5	0.2222	0.9891
34	0.2000	0.4000	0.2000	0.2000	1	1	0.5	0.5000	1.8416

Table A.4.2: Case *b*, Results for Varying Inputs, $n = 2$

	μ_{q1} iso	μ_{q1} % error	ρ_2	μ_{q2} combo	μ_{q2} iso	BP_2	μ_{q2} % error	ρ_3
1	1.2000	0.05%	0.5625	0.5714	0.5728	0.0000	0.25%	0.1875
2	1.2047	0.15%	0.5625	0.5719	0.5717	0.0013	0.03%	0.1875
3	1.2000	0.01%	0.5625	0.5712	0.5730	0.0000	0.32%	0.1875
4	0.4103	0.10%	0.1000	0.0832	0.0843	0.0020	1.32%	0.1000
5	2.5424	5.11%	0.3333	0.3246	0.2726	0.0080	16.02%	0.3333
6	1.2319	1.39%	0.5625	0.5721	0.5663	0.0087	1.01%	0.1875
7	1.2000	0.03%	0.3750	0.3458	0.3548	0.0000	2.60%	0.3750
8	1.2000	0.00%	0.3750	0.3458	0.3549	0.0000	2.63%	0.3750
9	1.2097	0.49%	0.3750	0.3460	0.3541	0.0020	2.34%	0.3750
10	4.8880	1.58%	0.2222	0.2199	0.1509	0.0020	31.38%	0.2222
11	4.3882	0.12%	0.1000	0.1017	0.0590	0.0005	41.99%	0.1000
12	0.1422	8.90%	0.2000	0.1883	0.2009	0.0221	6.69%	0.2000
13	2.5500	0.06%	0.1000	0.0982	0.0629	0.0004	35.95%	0.1000
14	0.8514	0.12%	0.1000	0.0888	0.0747	0.0008	15.88%	0.1000
15	0.6014	0.05%	0.1000	0.0904	0.0784	0.0014	13.27%	0.1000
16	1.7020	0.07%	0.1000	0.0959	0.0664	0.0004	30.76%	0.1000
17	2.6826	9.92%	0.5000	0.5056	0.4163	0.0195	17.66%	0.1667
18	1.3274	7.33%	0.3750	0.3460	0.3475	0.0240	0.43%	0.3750
19	0.6000	0.02%	0.2000	0.2009	0.1672	0.0000	16.77%	0.2000
20	0.6000	0.08%	0.2000	0.2009	0.1672	0.0001	16.77%	0.2000
21	0.6009	0.38%	0.2000	0.2009	0.1669	0.0009	16.92%	0.2000
22	0.6114	1.55%	0.2000	0.1992	0.1640	0.0112	17.67%	0.2000
23	1.4677	14.56%	0.5625	0.5649	0.5401	0.0590	4.39%	0.1875
24	0.4292	8.72%	0.3333	0.2612	0.3273	0.0451	25.31%	0.3333
25	0.6129	1.59%	0.2500	0.1653	0.2372	0.0105	43.50%	0.2500
26	0.2258	0.09%	0.1000	0.0639	0.0937	0.0030	46.64%	0.1000
27	0.4147	53.13%	0.5000	0.4742	0.4921	0.0988	3.77%	0.5000
28	1.2284	43.13%	0.5000	0.5177	0.5017	0.0793	3.09%	0.6667
29	0.3429	0.44%	0.1563	0.0746	0.1510	0.0040	102.41%	0.1563
30	3.8710	31.06%	0.5556	0.7532	0.4837	0.0702	35.78%	0.2500
31	4.5433	31.43%	0.5000	0.6816	0.3546	0.0502	47.98%	0.2500
32	0.6229	51.62%	0.5000	0.5835	0.4598	0.0863	21.20%	0.5000
33	0.2948	70.20%	0.5000	0.5705	0.4822	0.0993	15.48%	0.5000
34	0.9407	48.92%	0.5000	0.6396	0.4273	0.0748	33.19%	0.5000

Table A.4.2: Case *b*, Results for Varying Inputs, *n* = 2

	μ_{a3} combo	μ_{a3} iso	BP_3	μ_{a3} % error	Q	μ_q combo	μ_q iso	μ_q % error
1	0.1628	0.1689	0.0000	3.75%	0.1989	1.9348	1.9417	0.36%
2	0.1629	0.1688	0.0000	3.62%	0.1970	1.9377	1.9452	0.39%
3	0.1628	0.1689	0.0000	3.75%	0.1996	1.9341	1.9419	0.40%
4	0.0832	0.0843	0.0020	1.32%	0.2150	0.5763	0.5789	0.45%
5	0.3246	0.2726	0.0080	16.02%	0.0988	3.0680	3.0876	0.64%
6	0.1631	0.1668	0.0002	2.27%	0.1918	1.9502	1.9650	0.76%
7	0.3458	0.3548	0.0000	2.60%	0.1997	1.8920	1.9096	0.93%
8	0.3458	0.3549	0.0000	2.63%	0.1999	1.8916	1.9098	0.96%
9	0.3460	0.3541	0.0020	2.34%	0.1891	1.8958	1.9179	1.17%
10	0.2199	0.1509	0.0020	31.38%	0.0498	5.2518	5.1898	1.18%
11	0.1017	0.0590	0.0005	41.99%	0.0249	4.5864	4.5062	1.75%
12	0.1891	0.2009	0.0221	6.24%	0.0737	0.5335	0.5440	1.97%
13	0.0982	0.0629	0.0004	35.95%	0.0499	2.7479	2.6758	2.62%
14	0.0888	0.0747	0.0008	15.88%	0.1497	1.0280	1.0008	2.65%
15	0.0904	0.0784	0.0014	13.27%	0.1481	0.7819	0.7582	3.03%
16	0.0959	0.0664	0.0004	30.76%	0.0749	1.8926	1.8348	3.05%
17	0.1580	0.1342	0.0017	15.06%	0.0974	3.1042	3.2331	4.15%
18	0.3460	0.3475	0.0240	0.43%	0.1889	1.9288	2.0224	4.85%
19	0.2009	0.1672	0.0000	16.77%	0.1482	1.0019	0.9344	6.74%
20	0.2009	0.1672	0.0001	16.77%	0.1458	1.0023	0.9344	6.77%
21	0.2009	0.1669	0.0009	16.92%	0.1402	1.0050	0.9347	7.00%
22	0.1996	0.1640	0.0112	17.84%	0.1273	1.0198	0.9394	7.88%
23	0.1643	0.1674	0.0054	1.89%	0.1765	2.0104	2.1752	8.20%
24	0.2631	0.3272	0.0451	24.36%	0.2049	0.9945	1.0837	8.97%
25	0.1653	0.2372	0.0105	43.50%	0.3889	0.9339	1.0873	16.43%
26	0.0639	0.0937	0.0030	46.64%	0.3723	0.3534	0.4132	16.92%
27	0.4923	0.4921	0.0988	0.04%	0.0863	1.8513	1.3989	24.44%
28	0.7949	0.6915	0.1532	13.01%	0.1030	3.4727	2.4216	30.27%
29	0.0746	0.1510	0.0040	102.41%	0.5389	0.4906	0.6449	31.45%
30	0.2566	0.1763	0.0048	31.29%	0.0404	6.6251	4.5310	31.61%
31	0.2804	0.1639	0.0080	41.55%	0.0260	7.5875	5.0618	33.29%
32	0.6035	0.4597	0.0863	23.83%	0.0766	2.4745	1.5424	37.67%
33	0.6026	0.4821	0.0993	20.00%	0.0425	2.1622	1.2591	41.77%
34	0.6592	0.4273	0.0748	35.18%	0.0621	3.1404	1.7953	42.83%

Table A.4.2: Case *b*, Results for Varying Inputs, *n* = 2

	p	β_1	β_2	β_3	K2	K3	θ_2	ρ_1	μ_{q1} combo
35	0.1000	0.4000	0.1000	0.1000	1	1	0.5	0.2500	1.0981
36	0.1000	0.8000	0.8000	0.8000	1	1	0.5	0.1250	0.1286
37	0.7500	0.9000	0.5627	0.5627	1	1	0.5	0.8333	27.5459
38	0.6000	0.8000	0.4000	0.6000	1	1	0.5	0.7500	10.4505
39	0.1000	0.8000	0.5000	0.5000	1	1	0.5	0.1250	0.1289
40	0.1000	0.8000	0.1500	0.2000	1	1	0.75	0.1250	0.1938
41	0.6000	0.8000	0.4500	0.4500	1	1	0.5	0.7500	20.4735
42	0.1500	0.7500	0.1125	0.1125	1	1	0.5	0.2000	7.7097
43	0.1500	0.8000	0.1250	0.2000	1	1	0.75	0.1875	7.2811
44	0.7125	0.9500	0.5088	0.5088	1	1	0.5	0.7500	26.0116
45	0.3000	0.6000	0.2500	0.4000	1	1	0.75	0.5000	13.0563
46	0.6000	0.9000	0.4285	0.4285	1	1	0.5	0.6667	20.4065
47	0.1500	0.3750	0.1125	0.1125	1	1	0.5	0.4000	15.4786
48	0.4000	0.8000	0.2850	0.2850	1	1	0.5	0.5000	21.8919
49	0.4000	0.6000	0.3000	0.3000	1	1	0.5	0.6667	44.2036
50	0.6000	0.9000	0.4000	0.4500	1	1	0.5	0.6667	43.2480
51	0.1500	0.3000	0.1125	0.1125	1	1	0.5	0.5000	34.6288
52	0.1500	0.8000	0.1250	0.1000	1	1	0.75	0.1875	28.9817
53	0.1500	0.7500	0.1050	0.1050	1	1	0.5	0.2000	85.8373
54	0.7916	0.9500	0.7125	0.6333	1	1	0.75	0.8333	3.7423
55	0.1750	0.3500	0.1094	0.1094	1	1	0.5	0.5000	
56	0.3000	0.4500	0.1800	0.2250	1	1	0.5	0.6667	
57	0.1250	0.3250	0.0750	0.0750	1	1	0.5	0.3846	
58	0.3000	0.4500	0.2000	0.2000	1	1	0.5	0.6667	
59	0.5000	0.6000	0.4000	0.4000	1	1	0.5	0.8333	
60	0.3000	0.4500	0.1875	0.1875	1	1	0.5	0.6667	
61	0.2000	0.4000	0.1333	0.1333	1	1	0.5	0.5000	
62	0.1000	0.3200	0.0667	0.0667	1	1	0.5	0.3125	
63	0.4500	0.5400	0.9000	0.3938	1	1	0.25	0.8333	
64	0.3000	0.4500	0.1800	0.1800	1	1	0.5	0.6667	
65	0.4500	0.6000	0.3000	0.3375	1	1	0.5	0.7500	
66	0.1500	0.3750	0.0938	0.0938	1	1	0.5	0.4000	
67	0.3000	0.7500	0.2000	0.2000	1	1	0.5	0.4000	
68	0.1500	0.7500	0.1000	0.1000	1	1	0.5	0.2000	
69	0.1000	0.8000	0.0667	0.0667	1	1	0.5	0.1250	

Table A.4.2: Case b, Results for Varying Inputs, n = 2

	μ_{q1} iso	μ_{q1} % error	ρ_2	μ_{q2} combo	μ_{q2} iso	BP_2	μ_{q2} % error	ρ_3
35	0.3439	68.68%	0.5000	0.6016	0.4730	0.0957	21.38%	0.5000
36	0.1286	0.00%	0.0625	0.0192	0.0616	0.0005	220.83%	0.0625
37	7.5301	72.66%	0.6665	0.9518	0.6732	0.1390	29.27%	0.6665
38	2.2214	78.74%	0.7500	1.1355	0.7160	0.1655	36.94%	0.5000
39	0.1292	0.23%	0.1000	0.0296	0.1016	0.0042	243.24%	0.1000
40	0.1437	25.85%	0.5000	0.2200	0.5447	0.1192	147.59%	0.1250
41	2.5147	87.72%	0.6667	0.9854	0.6275	0.1307	36.32%	0.6667
42	0.2633	96.58%	0.6667	0.8914	0.6179	0.6190	30.68%	0.6667
43	0.2486	96.59%	0.9000	1.3203	0.8254	0.2228	37.48%	0.1875
44	2.3957	90.79%	0.7002	1.0402	0.7145	0.1606	31.31%	0.7002
45	1.0124	92.25%	0.9000	1.5552	0.7889	0.2022	49.27%	0.1875
46	1.5046	92.63%	0.7001	1.0574	0.6852	0.1561	35.20%	0.7001
47	0.6975	95.49%	0.6667	1.0768	0.5582	0.1242	48.16%	0.6667
48	0.8621	96.06%	0.7018	1.0990	0.6487	0.1520	40.97%	0.7018
49	1.8576	95.80%	0.6667	1.3097	0.5678	0.1180	56.65%	0.6667
50	1.5231	96.48%	0.7500	1.1931	0.7305	0.1752	38.77%	0.6667
51	1.0954	96.84%	0.6667	1.0887	0.5282	0.1120	51.48%	0.6667
52	0.2514	99.13%	0.9000	1.5637	0.8107	0.2178	48.16%	0.3750
53	0.2659	99.69%	0.7143	1.2170	0.6408	0.1607	47.35%	0.7143
54	189.6000	4966.40%	0.8333	1.1743	0.9224	0.2126	21.45%	0.3125
55			0.8000					0.8000
56			0.8333					0.6667
57			0.8333					0.8333
58			0.7500					0.7500
59			0.6250					0.6250
60			0.8000					0.8000
61			0.7502					0.7502
62			0.7500					0.7500
63			0.1250					0.8570
64			0.8333					0.8333
65			0.7500					0.6667
66			0.8000					0.8000
67			0.7500					0.7500
68			0.7500					0.7500
69			0.7500					0.7500

Table A.4.2: Case *b*, Results for Varying Inputs, $n = 2$

	μ_{q3} combo	μ_{q3} iso	BP_3	μ_{q3} % error	Q	μ_q combo	μ_q iso	μ_q % error
35	0.6345	0.4729	0.0957	25.47%	0.0417	2.3342	1.2898	44.74%
36	0.0192	0.0616	0.0005	220.83%	0.6889	0.1670	0.2518	50.78%
37	0.9550	0.6732	0.1390	29.51%	0.0083	29.4527	8.8765	69.86%
38	0.6346	0.4568	0.0644	28.02%	0.0277	12.2206	3.3942	72.23%
39	0.0296	0.1016	0.0042	243.24%	0.5581	0.1881	0.3324	76.71%
40	0.0396	0.1247	0.0098	214.90%	0.0960	0.4534	0.8131	79.33%
41	0.9969	0.6274	0.1307	37.06%	0.0131	22.4558	3.7696	83.21%
42	0.9961	0.6177	0.6190	37.99%	0.0125	9.5972	1.4989	84.38%
43	0.2127	0.1629	0.0174	23.41%	0.0147	8.8141	1.2369	85.97%
44	1.0481	0.7144	0.1606	31.84%	0.0107	28.0999	3.8246	86.39%
45	0.2214	0.1484	0.0106	32.97%	0.0173	14.8329	1.9497	86.86%
46	1.0759	0.6852	0.1561	36.31%	0.0145	22.5398	2.8750	87.24%
47	1.1770	0.5582	0.1242	52.57%	0.0080	17.7324	1.8139	89.77%
48	1.1550	0.6486	0.1520	43.84%	0.0122	24.1459	2.1594	91.06%
49	1.0743	0.5678	0.1180	47.15%	0.0057	46.5876	2.9932	93.58%
50	1.0310	0.6466	0.1413	37.28%	0.0073	45.4721	2.9002	93.62%
51	1.1800	0.5281	0.1120	55.25%	0.0037	36.8975	2.1517	94.17%
52	0.7232	0.3361	0.0611	53.53%	0.0043	31.2686	1.3982	95.53%
53	1.3812	0.6405	0.1607	53.63%	0.0016	88.4355	1.5472	98.25%
54	0.3334	0.3111	0.0322	6.69%	0.0468	5.2500	190.8335	3534.92%
55					unstable			
56					unstable			
57					unstable			
58					unstable			
59					unstable			
60					unstable			
61					unstable			
62					unstable			
63					unstable			
64					unstable			
65					unstable			
66					unstable			
67					unstable			
68					unstable			
69					unstable			

Table A.4.2: Case *b*, Results for Varying Inputs, $n = 2$

	p	β_1	β_2	β_3	K2	K3	θ_2	ρ_1
70	0.4000	0.5250	0.2600	0.2600	1	1	0.5	0.7619
71	0.5000	0.6000	0.3571	0.3571	1	1	0.5	0.8333
72	0.2000	0.6400	0.1250	0.1250	1	1	0.5	0.3125
73	0.2500	0.5000	0.1500	0.1500	1	1	0.5	0.5000
74	0.3000	0.9600	0.1800	0.1800	1	1	0.5	0.3125
75	0.2000	0.9000	0.1200	0.1200	1	1	0.5	0.2222

Table A.4.2: Case *b*, Results for Varying Inputs, $n = 2$

	ρ_2	ρ_3	Q
70	0.7692	0.7692	unstable
71	0.7001	0.7001	unstable
72	0.8000	0.8000	unstable
73	0.8333	0.8333	unstable
74	0.8333	0.8333	unstable
75	0.8333	0.8333	unstable