

NON-LEFT-ORDERABLE SURGERIES OF TWISTED TORUS KNOTS

by

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Abstract

The topic of study of this thesis belongs both to knot theory and to group theory. A knot is a smooth embedding of a circle in \mathbb{R}^3 or $S^3 = \mathbb{R}^3 \cup \{+\infty\}$. With any knot K we can do an operation which depends on two integer coefficients p and q , called $\frac{p}{q}$ Dehn surgery, resulting in a 3-manifold M denoted by $M := S^3(K, \frac{p}{q})$. A group is left-orderable if it can be given a total strict ordering which is invariant under multiplication from the left. It is hard to understand Dehn surgery geometrically, but algebraically it is clear - the fundamental group $\pi_1(M)$ equals to the fundamental group of the knot complement of K with one relation added. Although the fundamental group of the knot complement is always left-orderable, $\pi_1(M)$ may not be left-orderable. We study left-orderability of $\pi_1(M)$ in case where K is a twisted torus knot.

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Chapter 1

Motivation

In this chapter we discuss the purpose of this research along with its goals. We give some necessary definitions here which will be considered in greater detail later on.

Firstly, a *knot* is a smooth embedding of a circle into S^3 . We require smoothness to avoid so-called wild knots (they have bizarre pathologies), which we exclude from our consideration.

One can consider an open tubular neighborhood of a knot. Then a complement of a tubular neighborhood is called *the knot complement*. We have that the knot complement of a knot in S^3 is a compact, connected, orientable, irreducible 3-manifold. One can calculate the fundamental group of that manifold, it is called *the knot group*.

A strict ordering of a set X is a binary relation $<$ which is transitive and such that $x < y$ and $y < x$ cannot both hold. A group G is called *left-orderable* if its elements can be given a strict total ordering which is left invariant, meaning $g < h$ implies $fg < fh$ for all $f, g, h \in G$.

Let M be a closed, orientable, irreducible 3-manifold. The *Heegaard-Floer homology group* of M , denoted $\widehat{HF}(M)$, is a finitely generated abelian group. We won't go further than this sentence. An irreducible manifold M with finite first homology group, $|H_1(M)| < \infty$, is called an *L-space* if $\text{rank} \widehat{HF}(M) = |H_1(M)|$.

The following conjecture from [14] is a focus of current research in low dimensional topology:

Conjecture 1.1. *Let M be closed, orientable, irreducible 3-manifold. Then the following are equivalent:*

- 1) M is not an L -space;
- 2) $\pi_1(M)$ is left-orderable;
- 3) M admits a C^0 co-orientable taut foliation.

It seems that this conjecture is far from being proved. What is known so far is that 1) implies 3), see [19]. Checking this conjecture in some specific cases may give some insight on why this is true or not in general. This thesis is concerned with the implication from 1) to 2), i.e. if it is known that a manifold M is an L -space, we will show in some particular cases that its fundamental group is not left-orderable.

Dehn surgery is a method of constructing new manifolds using knots [5]. The result of doing Dehn surgery with coefficient $\frac{p}{q}$, where p and q are relatively prime, on a knot K is a manifold denoted $M := S^3(K, \frac{p}{q})$. It is known that the homeomorphism type of the manifold M depends only on the knot K and $\frac{p}{q}$.

A Dehn surgery is called an *L -space surgery* if the surgered 3-manifold is an L -space, and a knot admitting a non-trivial L -space surgery is called an *L -space knot*.

Heegaard-Floer homology behaves very well with respect to Dehn surgery, so there are many theorems about L -spaces and Dehn surgery which could be translated into statements about left-orderable fundamental groups, if conjecture is true. For example, the conjecture below is just a restatement of the following theorem [15]:

Theorem 1.2. *Let K be a knot in S^3 , and suppose there exists $\frac{p}{q} \in \mathbb{Q}$ such that $S^3(K, \frac{p}{q})$ is an L -space. Then $S^3(K, \frac{p'}{q'})$ is an L -space if and only if $\frac{p'}{q'} \geq 2g(K) - 1$, where $g(K)$ is knot genus, see Definition 2.13.*

Conjecture 1.3. *Let K be a knot in S^3 , and suppose there exists $\frac{p}{q} \in \mathbb{Q}$ such that $\pi_1(S^3(K, \frac{p}{q}))$ is not left-orderable. Then $\pi_1(S^3(K, \frac{p'}{q'}))$ is not left-orderable if and only if $\frac{p'}{q'} \geq 2g(K) - 1$.*

It is also reasonable to work from the perspective of non-left-orderability, i.e. perform Dehn surgery with coefficient $r \in \mathbb{Q}$ on a knot, show that the fundamental

group of the resulting manifold is not left-orderable, and then show that the same is true for any rational r' such that $r' > r$. See [4, 6, 7].

In this thesis we will consider twisted torus knots $T_{p,pk-1}^{2,1}$ and $T_{p,pk+1}^{2,1}$, see the definition before Figure 6.4 on page 35. Then we will aim to prove that there exists $r \in \mathbb{Q}$ such that $\pi_1(S^3(T_{p,pk-1}^{2,1}, r))$ is not left-orderable and that for any rational r' such that $r' > r$, $\pi_1(S^3(T_{p,pk-1}^{2,1}, r'))$ is also not left-orderable. And the same for $T_{p,pk+1}^{2,1}$.

Proving that a group is not left-orderable may have different level of complexity for different presentations of the group, that is why a choice of presentation is extremely important. Thus, we spend some time reviewing methods of computing knot groups in Chapters 3 and 4, see [20] and [11] for details. Although the presentations of the knots groups we work with in Chapter 6 is obtained by using Seifert-van Kampen theorem, we use Wirtinger presentation to prove Theorem 5, and provide Dehn presentation in order to demonstrate different approaches and because it is very easy to compute.

Chapter 2

Introduction

This chapter serves as a brief introduction to a branch of mathematics called knot theory. Knot theory is the study of mathematical knots, which are an abstraction of knots that we see in real world. As we will soon see, mathematical knots differ from physically existing knots.

This chapter is supposed to be self-contained. The information given here should be enough for understanding the subsequent chapters. Some facts provided here will not be used later and are given to tie together different aspects of the theory. Many of the facts and definitions in this chapter (and the following chapters) are not obvious at all, the reader is referred to [8, 17, 20] for further information.

As in number theory, many problems in knot theory are easy to state, but often hard to answer. One of the other advantages of the theory is that we can depict our object of study—knots.

Definition 2.1. *A (tame) knot K is a smooth (or a piecewise linear) embedding of a circle S^1 into 3-dimensional sphere S^3 , or \mathbb{R}^3 .*

Remark 2.2. In knot theory we are interested in an image of a smooth (or piecewise linear) embedding, not in the function itself. It will be made explicit when we wish to speak of the embedding itself instead of its image.

The spaces S^3 and \mathbb{R}^3 are interchangeable through this thesis unless otherwise stated. The reason for that is that S^3 is a one-point compactification of \mathbb{R}^3 , $S^3 =$

$\mathbb{R}^3 \cup \{\infty\}$, and we can always consider our knots to be away from the point at infinity.

From the definition it is understood that a knot is a closed 1-manifold, though we will primarily be concerned with how the knot sits in \mathbb{R}^3 .

Below one can see in Figure 2.1 the easiest possible knot—the unknot.

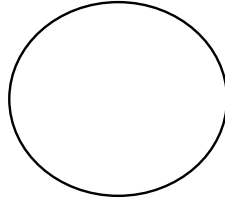


Figure 2.1: The unknot.

Definition 2.3. *Suppose a smooth curve is embedded in S^3 (or \mathbb{R}^3). A tubular neighbourhood to such a smooth curve is a submanifold of S^3 (or \mathbb{R}^3) defined as the union of all open discs such that: 1) all the discs have the same fixed radius; 2) the center of each disc lies on the curve; 3) each disc lies in a plane normal to the curve where the curve passes through that disc's center.*

A tame knot K is a smooth curve so it has a tubular neighbourhood (see [20, Chapter 2] for details), which we will denote by $N(K)$. Note that $\overline{N(K)} \cong D^2 \times S^1$.

In the definition of a tame knot we ask for our embedding to be smooth (or piecewise linear), the reason for that is that we want to exclude from our consideration another (pathological) type of knots called wild knots. One way to say it informally is that wild knots have infinitely many kinks, as in Figure 2.2 below. Wild knots differ from tame knots, for example, they may not have a tubular neighbourhood, and a knot having a tubular neighbourhood is essential for much of knot theory. We don't consider wild knots, and that is why we omit the word tame from our work.

Definition 2.4. *A link L of m components is a subset of S^3 , or \mathbb{R}^3 , that consists of m disjoint, smooth (or piecewise linear), simple closed curves.*

Below one can see the easiest possible link—two linked unknots.

So a link L of m components consists of m knots. The objects of study in Chapter 6 are knots, so we focus more on knots rather than links. The majority of facts about

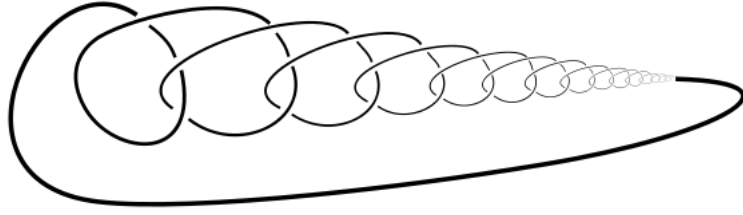


Figure 2.2: A wild knot.

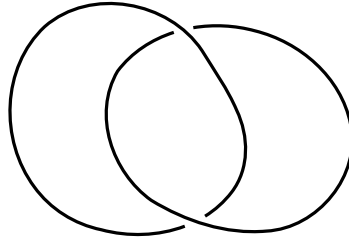


Figure 2.3: Two linked unknots.

knots can be translated into the language of links, on the other hand every fact about links holds for knots, since a knot is a link with one component.

So far we have seen pictures of two knots and a link. One may ask whether every knot admits such a picture. The answer to this question is affirmative, and it is guaranteed by the following [8, Theorem 3.2.1]:

Theorem 2.5. *Every tame link has a regular projection.*

By a regular projection of a knot we mean a projection of the knot to a plane with no multiple points which are more than double, and at each double point we know which branch of the knot is uppermost. Every knot in \mathbb{R}^3 has a regular projection. The double points are called *crossings*. We have seen crossings on diagrams of the wild knot and two linked unknots before. Such regular projections are called *diagrams*.

Below one can see in Figure 2.4 another trivial embedding of a circle. Some portion of the unknot in Figure 2.1 is changed as shown in Figure 2.4. The diagram in Figure 2.4 also represents the unknot.

In general the following holds:

Proposition 2.6. *Every tame link has infinitely many regular projections.*

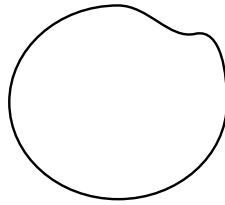


Figure 2.4: Another diagram of the unknot.

But in knot theory we don't want to distinguish two embeddings if they represent the same knot. We need a type of a proper deformation, i.e. an equivalence relation, such that any two diagrams of the same knot are related by that kind of deformation. We have several candidates for such deformation.

First of all, homotopy is useless for deforming knots, because it allows a curve to pass through itself, thus all knots are homotopic (equivalent to the unknot).

Secondly, isotopy also doesn't work, because it allows us to pull a knot until it becomes a point and disappears. This continuous deformation is called “bachelors' unknotting”, see the example in Figure 2.5.

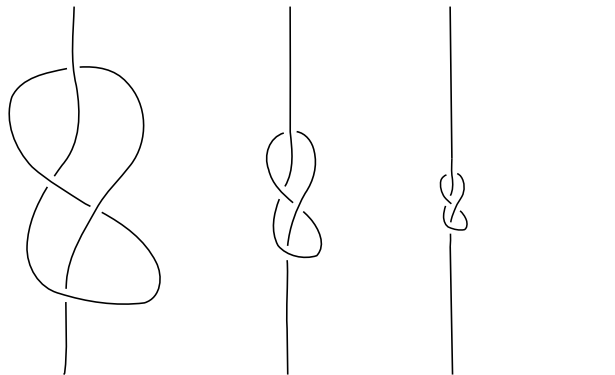


Figure 2.5: Bachelors' unknotting.

Finally, we come to a candidate that works: we need an ambient isotopy to model the topological deformation of knots.

Definition 2.7. *Two knots K_1 and K_2 are ambient isotopic (equivalent) if there is an isotopy $h : \mathbb{R}^3 \times [0, 1] \rightarrow \mathbb{R}^3$ such that $h(K_1, 0) = h_0(K_1) = K_1$ and $h(K_1, 1) = h_1(K_1) = K_2$.*

As we mentioned before, we may take links to be smooth or piecewise linear. The reason why one can switch between smooth and piecewise linear embeddings is given in [8, Theorem 1.11.7 and Exercise 1.12.11]:

Theorem 2.8. *An image of a piecewise linear embedding is ambient isotopic to the image of a smooth function.*

Take two equivalent knots, choose a diagram of each knot. Then these two diagrams are related by a sequence of *Reidemeister moves*. These moves are local changes meant to reflect an ambient isotopy of the knot. They are named after the German mathematician Kurt W. F. Reidemeister (1893-1971). The moves are given in Figure 2.6 (Types 1 to 3 appear from the top to the bottom), equivalence of a local change (before and after the move) is indicated by two-sided arrow. Finding a sequence of Reidemeister moves that changes one diagram of a knot to another equivalent diagram is not simple even in case of knots with few crossings, but its existence is frequently used in proofs.

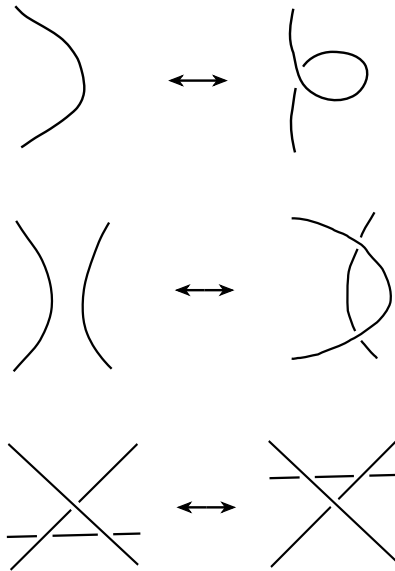


Figure 2.6: Reidemeister moves.

Every knot (a 1-manifold) is orientable. Obviously there are only two orientations for each knot. Every link with m components has 2^m orientations.

Let K be an oriented knot. Denote by $-K$ the knot with reversed orientation of K , it is called the *reverse* of K . It is possible that K and $-K$ are not equivalent (if we require our ambient isotopy to preserve orientations).

Let's get back to crossings. There are two types of crossings, positive and negative. Which is which is just a matter of a convention, ours is shown below:

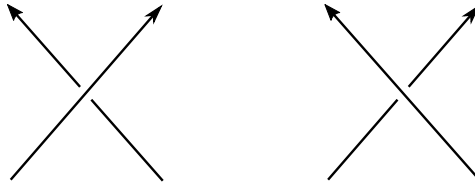


Figure 2.7: Positive (left) and negative (right) crossings. This choice of positive and negative crossings is sometimes called the right-hand screw convention.

Definition 2.9. *The crossing number of a link L is the minimum number of crossings in any diagram D of L , and is denoted by $c(L)$:*

$$c(L) = \min\{c(D) : D \text{ is a diagram of } L\}.$$

Note 2.10. The crossing number of a knot is often extremely hard to compute, but there exist inequalities on the crossing number with bounds that are relatively easy to calculate. The crossing number of the unknot is 0, there are no knots with the crossing number 1 or 2, the only two knots with crossing number 3 are left- and right-handed trefoils, shown below in Figure 2.8. There are 253293 (unoriented) knots with the crossing number 15.

Note 2.11. Let K be the left-handed trefoil (conventionally called the trefoil). Switching all crossings of K from undermost to uppermost, and vice versa, one gets the knot called *mirror image*, *reflection* or *obverse* of K and it is denoted by K^* . In our case K^* is the right-handed trefoil, we have that K and K^* are not equivalent. Also K and $-K$, as well as K^* and $-K^*$ are equivalent.

Let's now talk about the linking number of two knots.

Let D be an oriented diagram, and let c be a crossing in D and define

$$\varepsilon(c) = \begin{cases} +1 & \text{if } c \text{ is a positive crossing,} \\ -1 & \text{if } c \text{ is a negative crossing.} \end{cases}$$

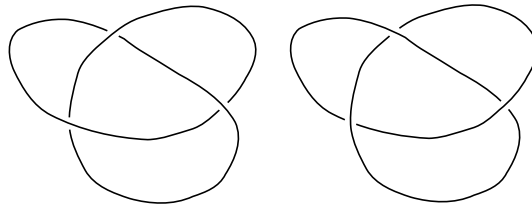


Figure 2.8: Left- and right-handed trefoils respectively.

Definition 2.12. Let D be a diagram of a 2-component link $K_1 \cup K_2$, and let D_i denote the component of D corresponding to K_i . The crossings of D are of three types: D_i with itself, for $i = 1, 2$, and D_1 with D_2 . We shall concentrate on the last group which we will denote by $D_1 \cap D_2$. The linking number of D_1 with D_2 is defined to be

$$lk(D_1, D_2) = \frac{1}{2} \sum_{c \in D_1 \cap D_2} \varepsilon(c).$$

One can find seven other equivalent ways to define the linking number in [20]. This number is independent of the diagram D [8, Theorem 3.8.2], and therefore is an invariant of the link $K_1 \cup K_2$. So we can write $lk(K_1, K_2)$.

A surface F spans a link L if its boundary ∂F is ambient isotopic to L .

Definition 2.13. The genus of an oriented link L is the minimum genus of any connected orientable surface that spans L . The genus of an unoriented link is the minimum taken over all possible choices of orientation. We denote the genus of L by $g(L)$.

One may ask whether the genus is well-defined. It is answered both affirmatively and constructively in [20]:

Theorem 2.14 (Seifert's algorithm). *Every link bounds an orientable surface.*

Proof. Seifert's construction goes as follows. If the link L is unoriented then choose an orientation for each component. Choose a diagram for the link in the xy -plane in \mathbb{R}^3 . Make the following local change to the diagram in a small neighbourhood of each crossing: delete the crossing and reconnect the loose ends in the only way compatible with the orientation. When this is done at every crossing, the diagram becomes a set

of disjoint simple loops in the plane, it is a diagram with no crossings. These loops are called *Seifert circles*. Seifert circles may be nested. We will assign an index to each one: for each circle γ , let $h(\gamma)$ be the number of Seifert circles that contain γ . We will use this index as a height function. We shall now construct a surface. For each Seifert circle γ , take a disc D in the plane $z = h(\gamma)$ such that under projection $(x, y, z) \mapsto (x, y)$ the image of ∂D is γ . Our collection of discs sits in the upper half-space \mathbb{R}_+^3 and the discs are stacked in such a way that when viewed from above, the boundary of each disc is visible. The discs inherit an orientation from the diagram. To complete the required surface, we need to insert a small half-twisted rectangle at the site of each crossing, the sense of the half-twist being chosen to produce the correct kind of crossing needed to recover the original knot. This produces a surface with boundary L . This surface is called a *Seifert surface*. Note that we have not proved here that this surface is orientable. \square

We have the following theorem and a corollary (which we will use later) of it for the Seifert's construction [8]:

Theorem 2.15. *The Euler characteristic of a Seifert surface F constructed from a diagram D with $s(D)$ Seifert circles and $c(D)$ crossings is $\chi(F) = s(D) - c(D)$.*

Corollary 2.16. *The genus of a Seifert surface F constructed from a connected diagram D satisfies $2g(F) = [1 - \chi(F)] + [1 - \mu(D)] = [1 - s(D) + c(D)] + [1 - \mu(D)]$, where $\mu(D)$ is the number of components of the link L .*

Chapter 3

The knot group

In this chapter we introduce historically the first and second techniques of computing the knot group, which is $\pi_1(S^3 \setminus N(K))$. We need the Wirtinger presentation in our discussion of Dehn surgery in Chapter 4, to define the longitude.

3.1 Wirtinger presentation

This method is named after Austrian mathematician Wilhelm Wirtinger(1865-1945). Historically it is the first technique of computing the knot group, it was presented for the first time during a lecture in 1904 and published by H. Tietze in 1908 [22].

Let K be a knot in S^3 . As was said before (in Chapter 2) K has a regular projection. First, draw some regular projection of the knot. The crossings break our projection into a set of arcs, an arc starts when the knot comes under a crossing and ends with the knot going under a crossing. Name the arcs, suppose we have n of them, $\alpha_1, \dots, \alpha_n$. Secondly, chose some orientation for our knot. Let the base point for the knot group be above the page. Then the knot group is generated by loops $x_i, 1 \leq i \leq n$, which pass under these arcs. Orient the generators x_i by the right-hand screw convention (see Figure 2.7 on page 12). Every crossing contributes one relation between the generators to the presentation, the presentation consists of all such relations. They all are of the form $x_i = x_j x_k x_j^{-1}$, we will see it in details in Example 3.2.

Finally, $\pi_1(S^3 \setminus N(K)) = \langle x_1, \dots, x_n \mid \text{all relations} \rangle$. Actually, one may omit one relation because of the following proposition, see [20, Theorem 3.D.2].

Proposition 3.1. *Any one Wirtinger relation is a consequence of the others.*

Example 3.2. We are ready to find the knot group of 4_1 . It is depicted in Figure 3.1, it has four crossings.

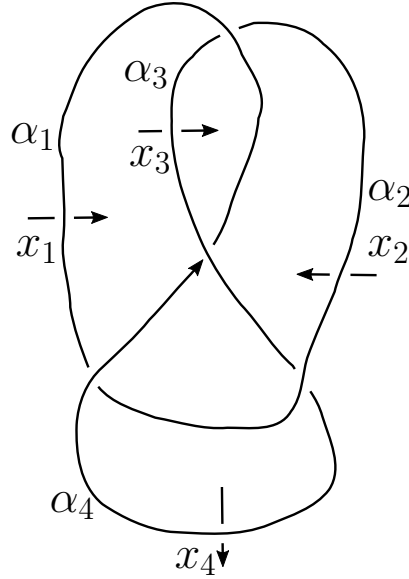


Figure 3.1: 4_1 knot, also called figure-eight knot, with generators x_i indicated by small arrows.

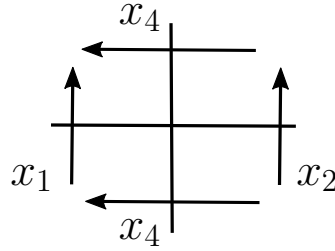


Figure 3.2: 1st crossing of 4_1 with generators x_1 , x_2 and x_4 .

The first crossing gives us $x_2 = x_4 x_1 x_4^{-1}$.

From the second crossing we obtain $x_4 = x_2 x_3 x_2^{-1}$.

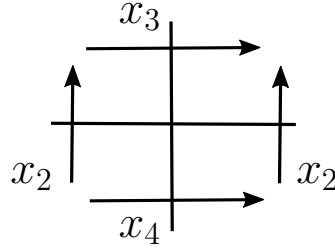


Figure 3.3: 2nd crossing of 4_1 with generators x_2 , x_3 and x_4 .

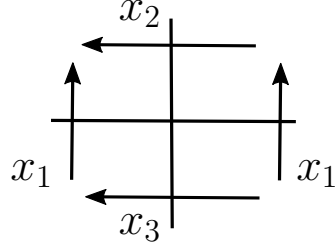


Figure 3.4: 3rd crossing of 4_1 with generators x_1 , x_2 and x_3 .

The third tells us that $x_2 = x_1 x_3 x_1^{-1}$.

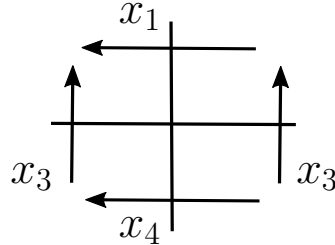


Figure 3.5: 4th crossing of 4_1 with generators x_1 , x_3 and x_4 .

From the last one, $x_4 = x_3 x_1 x_3^{-1}$. So

$$\begin{aligned} \pi_1(S^3 \setminus N(4_1)) &= \langle x_1, x_2, x_3, x_4 \mid x_2 = x_4 x_1 x_4^{-1}, x_4 = x_2 x_3 x_2^{-1}, x_2 = x_1 x_3 x_1^{-1}, x_4 = x_3 x_1 x_3^{-1} \rangle \\ &= \{\text{see Proposition 3.1}\} = \langle x_1, x_2, x_3, x_4 \mid x_2 = x_4 x_1 x_4^{-1}, x_4 = x_2 x_3 x_2^{-1}, \\ &\quad x_2 = x_1 x_3 x_1^{-1} \rangle. \end{aligned}$$

3.2 Dehn presentation

This method is named after a German-born American mathematician Max Dehn (1878-1952).

Let K be a knot. Firstly, we choose an orientation of K . The diagram of K divides the plane into faces. Let's name these faces f_1, \dots, f_n . These are our generators.

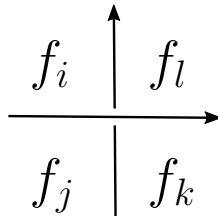


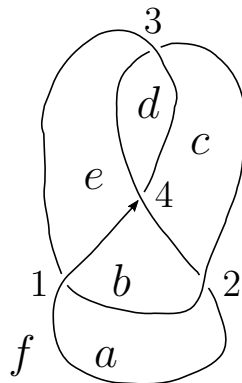
Figure 3.6: Faces near a crossing.

As in the Wirtinger presentation we get a relation from each crossing. Suppose we have a crossing as in Figure 3.6. Look at the label that lies to the left from the under arc that leaves the crossing, in our case it is f_i . In the relation f_i has power 1. Starting with f_i go anticlockwise and give alternating signs (-1 and +1) to the rest of the labels. So here we get $f_i f_j^{-1} f_k f_l^{-1}$. Do this for each crossing to produce $c(K)$ relations. At the end set one of the generators to be 1, for example $f_i = 1$.

The Dehn presentation of the knot group is

$$\pi_1(S^3 \setminus N(K)) = \langle f_1, \dots, f_n \mid \text{relations from crossings, } f_i = 1 \rangle.$$

Example 3.3. Let's find Dehn presentation of 4_1 . The knot is drawn in Figure 3.7.

Figure 3.7: 4_1 knot with faces labeled.

It has four crossings. The first crossing gives us $be^{-1}fa^{-1}$. From the second crossing we obtain $fc^{-1}ba^{-1}$. The third tells us that $dc^{-1}fe^{-1}$. From the last one, $de^{-1}bc^{-1}$. So

$$\begin{aligned} \pi_1(S^3 \setminus N(4_1)) &= \langle a, b, c, d, e, f \mid be^{-1}fa^{-1}, fc^{-1}ba^{-1}, dc^{-1}fe^{-1}, de^{-1}bc^{-1}, f \rangle \\ &= \langle a, b, c, d, e \mid be^{-1}a^{-1}, c^{-1}ba^{-1}, dc^{-1}e^{-1}, de^{-1}bc^{-1} \rangle. \end{aligned}$$

Chapter 4

Dehn surgery

In this chapter we will see a method of constructing 3-manifolds from knots called Dehn surgery. We will see how the fundamental groups of the constructed manifold and the knot group are related. The chapter concludes with a unifying theorem from low dimensional topology, that illustrates the significance of Dehn surgery.

Let K be an oriented knot in S^3 .

Remark 4.1. We consider S^3 , because the knot complement in S^3 is compact, while the knot complement in \mathbb{R}^3 is unbounded, hence not compact.

The operation of removing of an open tubular neighbourhood of the knot $N(K)$ and “gluing in” (“sewing”) a solid torus in a specific way is called Dehn surgery.

Observe that $\overline{N(K)} \cong S^1 \times D^2$. Let the curve $S^1 \times \{0\}$ correspond to K . The curve $\lambda = S^1 \times \{1\}$ is called a longitude, it is disjoint from K and parallel to it, meaning that λ and K have the same orientation. Notice that λ may wrap around the knot K an arbitrary number of times, depending on the embedding of $S^1 \times D^2$ in S^3 .

We want to choose λ to be homologically trivial in $S^3 \setminus N(K)$, this is called the *preferred* longitude. Let’s explain this requirement. Because of the form of our relations in the Wirtinger presentation, $x_i x_j x_i^{-1} = x_k$, we have a well-defined map $\pi(S^3 \setminus N(K)) \rightarrow \mathbb{Z}$, where all generators map to 1. Also we have an inclusion of a torus $T \hookrightarrow \partial(S^3 \setminus N(K))$ which induces a map $\pi_1(T) \simeq \mathbb{Z} \oplus \mathbb{Z} \hookrightarrow \pi(S^3 \setminus N(K))$. Let $\pi_1(T) = \langle \lambda, \mu \rangle$ and so we have a map $m : \pi_1(T) \rightarrow \mathbb{Z}$ such that $m(\lambda) = 0$ and $m(\mu)$ generates \mathbb{Z} .

To satisfy the requirement that $m(\lambda) = 0$, we specify that the linking number of λ with K is zero, and λ and K are oriented in the same direction as was said before. If $lk(\lambda, K) = n$ then λ is called *n-framed* longitude. We will see *v*-framed longitude in Theorem 6.18 on page 39.

A meridian is the curve $\mu \cong \{1\} \times S^1$. Then λ and μ meet at a single point $\{1\} \times \{1\}$. Since λ and μ lie on the torus $\partial\overline{N(K)}$, their classes in $\pi_1(S^3 \setminus N(K))$, which we also denote by λ and μ , commute and serve as generators of the subgroup $\pi_1(\partial\overline{N(K)})$ of $\pi_1(S^3 \setminus N(K))$. We orient μ such that the linking number with K is 1.

We will see how to choose λ and μ in Example 4.8 on page 21.

A subgroup of $\pi_1(S^3 \setminus N(K))$ generated by such λ and μ is called a *peripheral subgroup*.

If J is any homotopically nontrivial simple closed curve on the torus $\partial\overline{N(K)}$, we may form the space $M := (S^3 \setminus N(K)) \cup_{\varphi} (S^1 \times D^2)$, where $\varphi : S^1 \times S^1 \rightarrow \partial\overline{N(K)}$ is a homeomorphism that sends the meridian $\{1\} \times S^1$ of $S^1 \times D^2$ to J . The existence of φ is guaranteed by the following proposition.

Proposition 4.2. *Suppose that J is oriented and $[J] = \mu^p \lambda^q$ in $\pi_1(\partial\overline{N(K)})$ for some relatively prime integers p and q . Then there exists a homeomorphism $\varphi : S^1 \times S^1 \rightarrow \partial\overline{N(K)}$ such that $\varphi(\mu) = [J]$.*

Proof. Consider the mapping class group of a torus T

$$MCG(T) := \{h : T \rightarrow T \mid h \text{ is a homeomorphism}\} / \text{isotopy}.$$

Every $\varphi \in MCG(T)$ induces the homomorphism $\varphi_* : \pi_1(T) \rightarrow \pi_1(T)$, recall $\pi_1(T) \simeq \mathbb{Z} \oplus \mathbb{Z}$. One can conclude that $MCG(T) = SL_2(\mathbb{Z})$ [10, Theorem 2.5]. Let A be a matrix in $SL_2(\mathbb{Z})$. The equation $A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$ has a solution when p and q are relatively prime, since then $pr + qs = 1$ and $A = \begin{pmatrix} p & -s \\ q & r \end{pmatrix}$. In this case $A^{-1} \in SL_2(\mathbb{Z})$ and φ , such that $\varphi_* = A$, sends the meridian to J . \square

Definition 4.3. *The result of doing Dehn surgery with coefficient $\frac{p}{q}$, where p and q are relatively prime, on a knot K is a manifold denoted $M := S^3(K, \frac{p}{q})$. The manifold M is called $\frac{p}{q}$ (Dehn) surgered manifold of K .*

The following theorem tells us that it is enough to consider only relatively prime p and q [12, Proposition 6.2].

Proposition 4.4. *The homeomorphism type of $\frac{p}{q}$ surgered manifold of K depends only on K and $\frac{p}{q}$.*

Remark 4.5. The fraction $\frac{p}{q}$ in Proposition 4.4 is called the slope of the Dehn surgery.

Definition 4.6. *A curve J in S^3 bounds a disk, if there exists a smooth map $h : D^2 \hookrightarrow S^3$ such that $h(\partial D^2) = J$.*

In M the curve J bounds a disk, so the class of J in the knot group becomes trivial in $\pi_1(M)$. The main result of this chapter is formulated in the following proposition, see [20] for full details.

Proposition 4.7. *The fundamental group $\pi_1(M)$ is equal to $\pi_1(S^3 \setminus N(K))$ with the relation $[J] = 1$ added.*

Proof. Let's compute $\pi_1(M)$. Let's divide the process of obtaining M into two steps. First adjoin a meridian disk $\{0\} \times D^2$ of $S^1 \times D^2$ to the knot exterior $S^3 \setminus N(K)$. Then add a collar neighbourhood $(-\varepsilon, \varepsilon) \times D^2$ of that disk and obtain 3-manifold M' . We have that $\partial M' = S^2$, and also that $(S^1 \times D^2) \setminus ((-\varepsilon, \varepsilon) \times D^2)$ is a 3-ball B^3 . Finally sew that 3-ball B^3 with M' . We have that $\pi_1(M)$ can be obtained by applying Seifert-van Kampen theorem to $M = M' \cup_{S^2} B^3$. Since $\pi_1(B^3)$ is trivial, the only thing that amalgamation contributes to $\pi_1(M)$ is the image of the meridian, which is J , equals to 1, so add $[J] = 1$ to the knot group. \square

Example 4.8. Let's demonstrate the previous proposition with $\frac{p}{q}$ Dehn surgery on the figure-eight knot 4_1 . We will use Example 3.1 here. Note that a meridian μ can be represented by each of the arcs x_1, x_2, x_3, x_4 , say x_1 in our example. The preferred longitude λ is given in Figure 4.1. We find $\lambda = x_4^{-1} x_1 x_2^{-1} x_3$ and $m(\lambda) = 0$, where $m : \pi_1(S^3 \setminus N(4_1)) \rightarrow \mathbb{Z}$, that is why λ doesn't wrap around the knot. Then $\pi_1(S^3(4_1, \frac{p}{q})) = \langle x_1, x_2, x_3, x_4 \mid x_2 = x_4 x_1 x_4^{-1}, x_4 = x_2 x_3 x_2^{-1}, x_2 = x_1 x_3 x_1^{-1}, x_1^p (x_4^{-1} x_1 x_2^{-1} x_3)^q \rangle$.

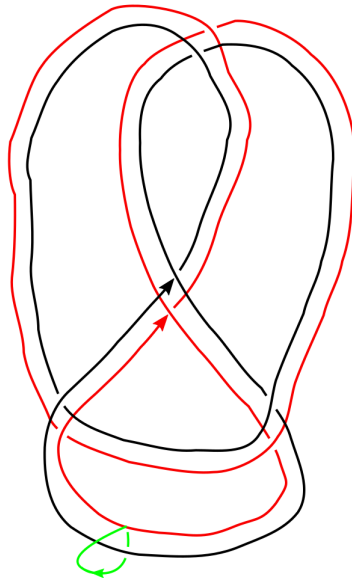


Figure 4.1: 4_1 (black), the longitude λ (red) and the meridian μ .

We finish this chapter with a significant result in low dimensional topology that highlights the importance of Dehn surgery.

We can do Dehn surgery on a link. To specify a surgery about a link, one simply specifies the surgery slope for each component. Here is a more detailed view of this structure. Suppose a link L has l components (knots, which are enumerated) and l irreducible fractions $\frac{p_1}{q_1}, \dots, \frac{p_l}{q_l}$ are fixed. Orient all components somehow. Consider non-intersecting tubular neighbourhoods of all the knots. We can choose meridians μ_1, \dots, μ_l and longitudes $\lambda_1, \dots, \lambda_l$ on the boundaries of the respective neighbourhoods as in case of one knot. Then we have l sewing homeomorphisms as in Proposition 4.2. Finally we obtain a 3-manifold which we denote as $M := S^3(L, \frac{p_1}{q_1}, \dots, \frac{p_l}{q_l})$.

Then we have the following theorem, see [24] or [16]:

Theorem 4.9. (*Lickorish-Wallace*) *Every closed, orientable connected 3-manifold may be obtained by Dehn surgery on a link in S^3 .*

Manifolds play important role in mathematics nowadays. Especially three- and four-manifolds, since 0-,1- and 2-manifolds have been studied extensively already.

Mathematicians are interested in manifolds for various reasons. To begin with, often attempts to answer a big question in manifold theory lead to discovery of new concepts, which become interesting by themselves. For example, R.S. Hamilton has discovered the Ricci flows in order to solve the Poincaré conjecture, which was used by Grigori Perelman in his proof of the Thurston geometrization conjecture, and the Poincaré conjecture that follows from it. Also manifolds are used in modern physics. For instance, Lagrangian mechanics and Hamiltonian mechanics, when considered geometrically, are naturally manifold theories. Whereas Calabi-Yau manifolds appear in superstring theory. And finally, since self-motivation is essential to mathematics, manifolds are studied as objects in their own right. In conclusion, a subfamily of closed, orientable connected 3-manifolds is considered to be one of the most important subfamilies of 3-manifolds, and Theorem 4.9 tells us that Dehn surgery upon links provides an exhaustive source for all closed, orientable connected 3-manifolds.

Chapter 5

Orderability of groups

In this chapter we show that knot groups are left-orderable.

An ordering $<$ of a set X is *strict* if it is:

1. Irreflexive, i.e. $x < x$ never holds for any $x \in X$.
2. Asymmetric, i.e. $x < y$ and $y < x$ can not hold at the same time for any $x, y \in X$.
3. Transitive, meaning $x < y$ and $y < z \Rightarrow x < z$.

It is a strict *total* ordering if for all $x, y \in X$ exactly one of $x < y, y < x$ or $x = y$ holds.

Definition 5.1. A group G is called *left-orderable* if its elements can be given a strict total ordering $<$ which is left invariant, meaning $f < g \Rightarrow hf < hg$ for all $f, g, h \in G$.

Elements that are bigger or smaller than an identity of a group are called *positive* and *negative* respectively.

Another way to define left-orderability is as follows:

Theorem 5.2. A group G is left-orderable if and only if there exists a subset $P \subset G$ such that

1. $P \cdot P \subset P$;
2. for every $g \neq 1$ in G , exactly one of g or g^{-1} belongs to P .

Proof. If G is left-orderable, set $P = \{g \in G \mid g > 1\}$. Then $P \cdot P \subset P$, because a product of positive elements is positive. Also if $g \in P$ then $g^{-1} < 1$, so g^{-1} is not in P .

Given such a P , the recipe $g < h$ if and only if $g^{-1}h \in P$ defines a left-invariant strict total order. Let's prove this. If $f < g$ and $g < h$, then we have that $f^{-1}g, g^{-1}h \in P$, and by condition 1, $f^{-1}g \cdot g^{-1}h = f^{-1}h \in P$, so $f < h$, which means $<$ is transitive. Also $f < h$ and $h < f$ can not both hold, because $h^{-1}f = (f^{-1}h)^{-1}$, and condition 2 says that exactly one of g or g^{-1} belongs to P , so $<$ is strict ordering. Note that $G = P \cup P^{-1} \cup \{1\}$, so $<$ is total. Finally, if $g < h$, then $fg < fh$, because $(fg)^{-1}fh = g^{-1}f^{-1}fh = g^{-1}h \in P$, so $<$ is left-invariant. \square

Such a subset P is called a *positive cone*.

A group that admits an ordering which is both left and right invariant is called *bi-orderable*.

Proposition 5.3. *A group G is bi-orderable if and only if it admits a subset P satisfying both conditions in Theorem 5.2, and in addition $gPg^{-1} \subset P$ for all $g \in G$.*

Example 5.4. The groups $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$ are both bi-orderable with their usual orderings of integers, reals.

Remark 5.5. Left-orderable groups are torsion-free and therefore infinite. Suppose G is left-orderable and $g \in G, g \neq 1$, and suppose $g > 1$. Then $g^2 > g$ by left-invariance. So $g^2 > 1$ by transitivity. This implies $g^n > 1$ for all $n > 0$, so $g^n \neq 1$. If G is non-trivial and torsion-free, then it is infinite.

We need the following lemma [5, Problem 1.8]:

Lemma 5.6. *Suppose G is a group with normal subgroup K and quotient group $H \cong G/K$. Let $p : G \rightarrow H$ denote the quotient map. In other words, suppose there is a short exact sequence $1 \rightarrow K \hookrightarrow G \rightarrow H \rightarrow 1$. Further suppose $(H, <_H)$*

and $(K, <_K)$ are left-orderable groups. We can then give G a left-ordering in a lexicographic way: declare that $g < g'$ if and only if either $p(g) < p(g')$ or else $p(g) = p(g')$ (so $g^{-1}g' \in K$) and $1 <_K g^{-1}g'$.

Proof. We need to show that $<$ is left-invariant and transitive.

Suppose $g < g'$. Let's consider cases:

Case 1 : $p(g) <_H p(g')$. Then $p(hg) = p(h)p(g) <_H p(h)p(g') = p(hg')$, so $hg < hg'$.

Case 2 : $p(g) = p(g')$ and $1 <_K g^{-1}g'$. Then $p(hg) = p(hg')$ and $(hg)^{-1}hg' = g^{-1}h^{-1}hg' = g^{-1}g' >_K 1$, so $hg < hg'$. Hence $<$ is left-invariant.

Suppose $g < g'$ and $g' < h$. Let's consider cases:

Case 1 : $p(g) <_H p(g')$ and $p(g') <_H p(h)$. Then $p(g) <_H p(h)$, so $g <_H h$.

Case 2 : $p(g) <_H p(g')$ and $p(g') = p(h)$. Then $p(g) <_H p(h)$, so $g <_H h$.

Case 3 : $p(g) = p(g')$ and $p(g') <_H p(h)$. Then $p(g) <_H p(h)$, so $g <_H h$.

Case 4 : $p(g) = p(g')$, $1 <_K g^{-1}g'$, $p(g') = p(h)$, $1 <_K g'^{-1}h$. Multiplying both sides of $g'^{-1}h >_K 1$ by $g^{-1}g'$ which is positive, we get $g^{-1}h = (g^{-1}g')g'^{-1}h >_K g^{-1}g' >_K 1$, so $g <_H h$. Hence $<$ is transitive. □

A subgroup C of an ordered group G which is a convex subset (meaning if $a < b < c$ and $a, c \in C$, then $b \in C$) of G with respect to the given order is called a *convex subgroup*. Normal convex subgroups are exactly the kernels of homomorphisms from an ordered group onto an ordered image which preserve the order. A subgroup of an orderable group which is convex for any order is called an *absolutely convex* subgroup; if it is convex only for a certain order, it is called a *relatively convex* subgroup.

Proposition 5.7. *The family of left-orderable groups is closed under: taking subgroups, direct products, free products, quotients by convex normal subgroups, extensions from Lemma 5.6.*

For a bi-orderable group we can say much more [5, Problem 1.20-1.22]:

Proposition 5.8. *If G is bi-orderable then:*

1) G has no generalized torsion, i.e. a product of conjugates of positive powers of a

non-trivial element is non-trivial;

2) *G has unique roots: $g^n = h^n \Rightarrow g = h$;*

3) *if $[g^n, h] = 1$ in G then $[g, h] = 1$.*

Example 5.9. Let K be the fundamental group of the Klein bottle, $K \cong \langle x, y \mid xyx^{-1} = y^{-1} \rangle$. Then K is left-orderable by Lemma 5.6, because we have a short exact sequence $1 \rightarrow \langle\langle y \rangle\rangle \rightarrow K \rightarrow \langle x \rangle \rightarrow 1$, and $\langle x \rangle, \langle\langle y \rangle\rangle$ are isomorphic to \mathbb{Z} so left-orderable. However K is not bi-orderable. Notice that x^2 and y commute: $yx^2 = xy^{-1}x^{-1}x^2 = xy^{-1}x = xxyx^{-1}x = x^2y$. Also $x \neq yx$ and $(yx)^2 = yx^2x^{-1}yx = x^2yy^{-1} = x^2$, so K doesn't have unique roots, so K is not bi-orderable by Proposition 5.8.

Remark 5.10. Note that $x < y$ and $z < w$ imply $xz < yw$ in bi-orderable groups, but it is not true for left-orderable groups. Let's show it using the Klein bottle. Suppose we choose the signs of x, y so that $x < yx$. Then we would have $x \cdot x < yx \cdot yx$, but we know that they are equal, so we get a contradiction.

Remark 5.11. Note that $x < y$ implies $x^{-1} > y^{-1}$ in bi-orderable groups, but it is not true for left-orderable groups.

Let's consider a specific type of orderings now [5].

Definition 5.12. *A left-ordering $<$ of a group G is called Archimedean if for every pair of positive elements $g, h \in G$ there exists $n > 0$ such that $h < g^n$.*

Example 5.13. The standard orderings of $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$ are Archimedean.

Here are three lemmas [5, Lemmas 2.2, 2.4 and 2.5] which are used to prove the surprising result of Hölder [13].

Lemma 5.14. *Every Archimedean left-ordering is a bi-ordering.*

Proof. Let P denote the positive cone of an Archimedean left-ordering $<$ of a group G . To prove that $<$ is a bi-ordering, by Proposition 5.3 it suffices to show that $g^{-1}Pg \subset P$ for all $g \in G$.

We need to consider two cases, i.e. g being positive and negative. We will prove one of them. Suppose g is a positive element in the ordering $<$, and let $h \in P$.

Since $<$ is Archimedean there exists $n > 0$ such that $g < h^n$. But then we have that $1 < g^{-1}h^n$. Therefore $1 < g^{-1}h^ng$ as it is a product of positive elements. Now $1 < g^{-1}hg$, since its n -th power is positive, so $g^{-1}hg \in P$ for all $h \in P$. \square

Lemma 5.15. *Every Archimedean left-ordered group G is abelian.*

We need the following Lemma 5.16 to prove Lemma 5.15, [5]:

Lemma 5.16. *If G is bi-ordered and does not have a least positive element, then given $p > 1$ there exists $g > 1$ in G such that $1 < g^2 < p$.*

Proof. Let $p > r > s > 1$ and consider $rs^{-1} > 1$. If $(rs^{-1})^2 \geq r$, then $s^{-1}rs^{-1} \geq 1$ and $r \geq s^2$, so we can choose $q = s$. Otherwise, let $q = rs^{-1}$. \square

Proof of Lemma 5.15. By Lemma 5.14 our ordering is bi-invariant. We consider two cases.

Case 1 : The positive cone P of our ordering has a least element p . Then $G = \langle p \rangle$. If $g \in G \setminus \langle p \rangle$, then there exists n such that $p^n < g < p^{n+1}$ and hence $1 < p^{-n}g < p$, giving a contradiction to p being least element. So $G \cong \mathbb{Z}$, and the theorem follows.

Case 2 : Suppose P does not have a least element. We prove by contraposition. Suppose $g, h \in G$ do not commute. Without loss of generality, we may assume that g, h and their commutator $ghg^{-1}h^{-1}$ are all positive. Then there exists $x > 1$ in G such that $1 < x^2 < ghg^{-1}h^{-1}$ by Lemma 5.16. Our ordering is Archimedean, so there exist m and n such that $x^m \leq g < x^{m+1}$ and $x^n \leq h < x^{n+1}$. Then $g^{-1} \leq x^{-m}$ and $h^{-1} \leq x^{-n}$, because $<$ is a bi-ordering, recall Remark 5.11. By multiplying the appropriate inequalities (recall Remark 5.10) we get $ghg^{-1}h^{-1} < x^{m+1+n+1-m-n} = x^2$, a contradiction. \square

Theorem 5.17 (Hölder). *If G is a group with an Archimedean left-ordering, then G is isomorphic with a subgroup of additive reals, by an isomorphism under which the ordering of G corresponds to the usual order of \mathbb{R} .*

Remark 5.18. By Lemmas 5.15 and 5.14, we have that G is an abelian bi-orderable group in theorem above.

Note 5.19. The groups we work with in Chapter 6 are not abelian, so the orderings we consider are not Archimedean. For arbitrary positive elements of these groups we can not take a power of one of them and say that it is greater than the other element. This shows us that ordered groups already behave much differently than \mathbb{R} , \mathbb{Z} .

We consider a first example of an ordering of a group.

Example 5.20. The group $(\mathbb{Z}^2, +)$ is left-orderable. Actually it is bi-orderable, since it is abelian. It can be ordered lexicographically, i.e. $(m_1, m_2) < (n_1, n_2)$ if and only if $m_1 < n_1$ or $m_1 = n_1$ and $m_2 < n_2$. Let's provide one more ordering. Imagine \mathbb{Z}^2 sitting in \mathbb{R}^2 in the usual way, and choose a vector $v = (v_1, v_2) \in \mathbb{R}^2$ which has an irrational slope. We can order $m = (m_1, m_2), n = (n_1, n_2)$ according to their dot product with v , i.e. $(m_1, m_2) < (n_1, n_2)$ if and only if $m_1 v_1 + m_2 v_2 < n_1 v_1 + n_2 v_2$. We get uncountably many orderings of \mathbb{Z}^2 in this way. The same trick works for $\mathbb{Z}^n, n > 2$. Consider a hyperplane H given by $x_1 - kx_2 + 0 \cdot x_3 + \dots + 0 \cdot x_n = 0$ in \mathbb{R}^n , where k is irrational. It contains only one point of \mathbb{Z}^n , $\bar{0}$. Let $s = (1, -k, 0, \dots, 0) \in \mathbb{Z}^n$. Take any $m_1 = (m_{11}, \dots, m_{1n}), m_2 = (m_{21}, \dots, m_{2n}) \in \mathbb{Z}^n$. Then $m_1 < m_2$ if and only if $m_1 \cdot s < m_2 \cdot s$. As before it also gives an ordering.

Let's consider one important example of a left-orderable group. Let $\text{Homeo}_+(\mathbb{R})$ denote the group of all order-preserving homeomorphisms of the real line, i.e. continuous functions with continuous inverses that preserve the usual order of the reals. Let x_1, x_2, \dots be a countable dense set of real numbers. For two functions $f, g \in \text{Homeo}_+(\mathbb{R})$, let $m = m(f, g)$ denote the minimum i such that $f(x_i) \neq g(x_i)$. Declare $f < g$ if $f(x_m) < g(x_m)$. Let's prove that it is an ordering. The number $m(f, f)$ doesn't exist, so it is irreflexive. The inequalities $f(x_i) < g(x_i)$ and $g(x_i) < f(x_i)$ don't hold at the same time, so it is asymmetric. Suppose that $f < g$ and $g < h$. Suppose $m(f, g) = i$ and $m(g, h) = j$. Then $m(f, h) = \min\{i, j\}$, and we have transitivity. It is total because the set $\{x_i\}$ is dense so $f = g$ iff $f(x_i) = g(x_i)$ for all i .

This group serves as an ambient group for all countable left-orderable groups in the following sense [5]:

Theorem 5.21. *A countable group G is left-orderable if and only if it embeds in $\text{Homeo}_+(\mathbb{R})$.*

Few more examples of orderable groups are given here as corollaries of Theorem 5.21:

Corollary 5.22. *Free groups and torsion-free abelian groups are bi-orderable.*

Proof. We have already discussed torsion-free abelian groups in Example 5.20.

Note that a non-abelian free group of rank 2 has subgroups of all countable ranks. See [5] for the proof of a non-abelian free group of rank 2 being bi-orderable. □

Corollary 5.23. *Braid groups are left-orderable, but not bi-orderable.*

Proof. Several proofs can be found in [9]. □

We need the following theorem [3]:

Theorem 5.24. *If M is a compact, connected, orientable, irreducible 3-manifold ($M \neq S^3$) then $\pi_1(M)$ is left-orderable if and only if there exists a surjection $\pi_1(M) \rightarrow L$ onto a nontrivial left-orderable group.*

We will use the following to prove Theorem 5.24:

Theorem 5.25 (Burns-Hale criterion). *A non-trivial group G is left-orderable if and only if every non-trivial finitely generated subgroup of G has a left-orderable quotient.*

Proof. See [1] for the proof. □

Proof. One direction is immediate: if $\pi_1(M)$ is left-orderable then its quotient by the trivial subgroup is left-orderable.

For the other direction, suppose that $\pi_1(M)$ surjects onto a left-orderable group G . By the Burns-Hale criterion it suffices to prove that every finitely generated subgroup of $\pi_1(M)$ surjects onto a left-orderable group. Let H be such a subgroup. We shall distinguish two cases to whether H has finite or infinite index in $\pi_1(M)$.

If H has finite index in $\pi_1(M)$ then its image under the projection onto G has finite index as well. Since G is infinite (being left-orderable) the surjection $\pi_1(M) \rightarrow G$ restricts to a surjection of H onto a non trivial subgroup of G . Thus H surjects onto a left-orderable group and we are done.

The case when H has infinite index in $\pi_1(M)$ involves more background that is not given in this thesis. So we will discuss just a sketch of the proof here. See [1] for details. If H has infinite index consider the infinite-index covering associated to H , i.e. the unique connected covering space $p : M' \rightarrow M$ such that $p_*(\pi_1(M')) = H$. Manifold M' is non-compact, but by the Compact Core Theorem (see [21]) there exists a compact, connected, three-dimensional submanifold $M_c \subset M'$ such that $i_*(\pi_1(M_c)) = \pi_1(M')$, where $i : M_c \hookrightarrow M'$ is the inclusion. Notice that $\partial M_c \neq \emptyset$, otherwise $M_c = M'$, which is impossible since M_c is compact whereas M' is not.

Furthermore, it can be shown that M_c has no boundary components with positive Euler characteristic. Having that it is possible to show that M_c has positive first Betti number b_1 . Now a surjection of $H \cong \pi_1(M') \cong \pi_1(M_c)$ onto a left-orderable group is given by the composition

$$\pi_1(M_c) \rightarrow H_1(M_c, \mathbb{Z}) = \mathbb{Z}_1^{b_1} \oplus \text{torsion} \rightarrow \mathbb{Z}_1^{b_1},$$

where the first map is the abelianization, while the second one is the projection onto the first factor. □

We conclude this chapter with an important result [3]:

Theorem 5.26. *Every knot group is left-orderable.*

Proof. Let K be a knot. It was mentioned in Chapter 1 that the knot complement is a compact, connected, orientable, irreducible 3-manifold. Let $M := S^3 \setminus N(K)$. If we can construct a surjection $h : \pi_1(M) \rightarrow \mathbb{Z}$, then we are done, since \mathbb{Z} is left-orderable. Suppose we computed $\pi_1(M)$ using Wirtinger's method. Then we can send each generator to 1 in \mathbb{Z} , since this rule respects any relation of the kind $x_i = x_j x_k x_j^{-1}$, as $h(x_i) = 1 = 1 + 1 - 1 = h(x_j x_k x_j^{-1})$. Also h is surjective. □

Chapter 6

Non-left-orderable surgeries of twisted torus knots

In this chapter, we introduce twisted torus knots $T_{c,d}^{a,b}$, and investigate orderability of $\pi_1(S^3(T_{c,d}^{a,b}, r))$. Our goal is to show that there exist $r_1, r_2 \in \mathbb{Q}$ such that $\forall r'_1, r'_2 \in \mathbb{Q}, r'_1 > r_1, r'_2 > r_2$ the fundamental groups of $S^3(T_{p,pk+1}^{2,1}, r'_1)$ and $S^3(T_{p,pk-1}^{2,1}, r'_2)$ are not left-orderable.

This chapter consists of three sections. First, we introduce twisted torus knots. In the second section we will see known results on the connection between being an L-space knot and having non-left-orderable surgeries for twisted torus knots. This chapter concludes with calculations for two families of twisted torus knots mentioned above.

6.1 Twisted-torus knots

Let's first introduce torus knots. Torus knots are called “torus” because they lie on the surface of a standard torus. More precisely, let's take a circle of radius R in the yz -plane centred on the y -axis at distance $R + r$ from the origin, then rotate it about the z -axis. This rotation generates a torus. Parameterise the circle by the angle

$\varphi \in [0, 2\pi]$, and the rotation by $\theta \in [0, 2\pi]$, so we can express the torus as

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ r + R \cos(\varphi) \\ R \sin(\varphi) \end{pmatrix} = \begin{pmatrix} -\sin(\theta)(r + R \cos(\varphi)) \\ \cos(\theta)(r + R \cos(\varphi)) \\ R \sin(\varphi) \end{pmatrix}.$$

There, R is the radius of the tube and r is the radius of the hole, any point on the torus can be labelled by a pair of the form (φ, θ) . See Figure 6.1.

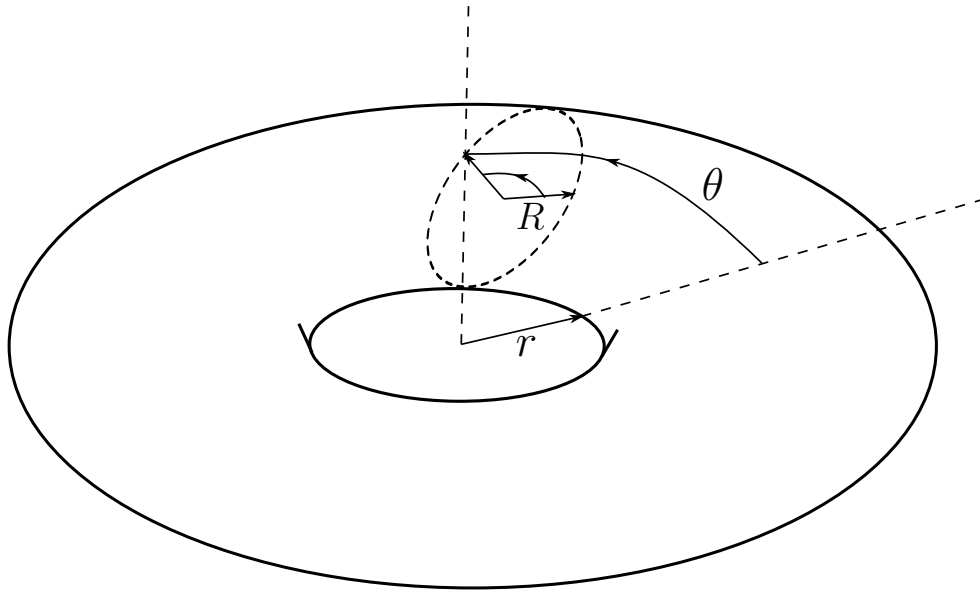


Figure 6.1: Coordinates on a torus, angle φ is the unnamed angle.

For coprime p and q the equation $p\theta = q\varphi$ defines the subset of points which lie on the torus and form a knot. We solve the equation modulo 2π , or we can look at the image of the function $t \mapsto (-\sin(qt)(r + R \cos(pt)), \cos(qt)(r + R \cos(pt)), R \sin(pt))$, $t \in [0, 2\pi]$.

This function covers the curve more than once, but as always we are interested in the image set, not the function. This knot is called the (p, q) torus knot and is denoted by $T(p, q)$.

Example 6.1. A left-handed trefoil $3_1 = T(2, 3)$. See the figure below.

Let's mention some properties of torus knots:

Proposition 6.2. *Every torus knot satisfies the following:*

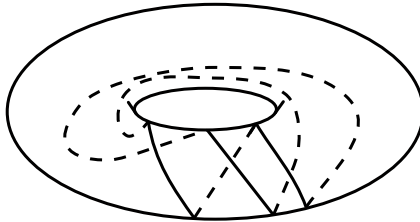
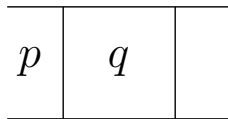


Figure 6.2: A left-handed trefoil on a torus.

1. $T(p, q) = T(q, p)$;
2. $T(-p, q)$ and $T(p, q)$ are mirror images.

We will draw p strings that are half-twisted q times as shown below. In Figure 6.6 we see 4 strings doing 2 twists at the top part, and 4 strings doing 5 half-twists at the bottom part:

Figure 6.3: p strings half-twisted q times.

Now we are ready to introduce twisted torus knots. Suppose we have the knot $T(p, q)$. Take l strands of this torus knot and additionally twist them m full times and one will get the (p, q, l, m) -twisted torus knot denoted $T_{p,q}^{l,m}$. We have that the (p, q, l, m) -twisted torus knot lies on a closed genus 2 surface as in Figure 6.4. We denote the knot group of $T_{p,q}^{l,m}$ as $G_{p,q}^{l,m}$, and the 3-manifold produced by r -surgery on $T_{p,q}^{l,m}$ as $M_{p,q}^{l,m}(r)$.

Throughout, we will assume $p \geq 2, q > 0, p > l > 0, m > 0, k \geq 1, r > 0, r \in \mathbb{Q}$.

6.2 Known results

We start with the following definition:

Definition 6.3. *We say that a knot K in S^3 which admits a positive (i.e. with positive slope) Dehn surgery that yields an L -space is an L -space knot.*

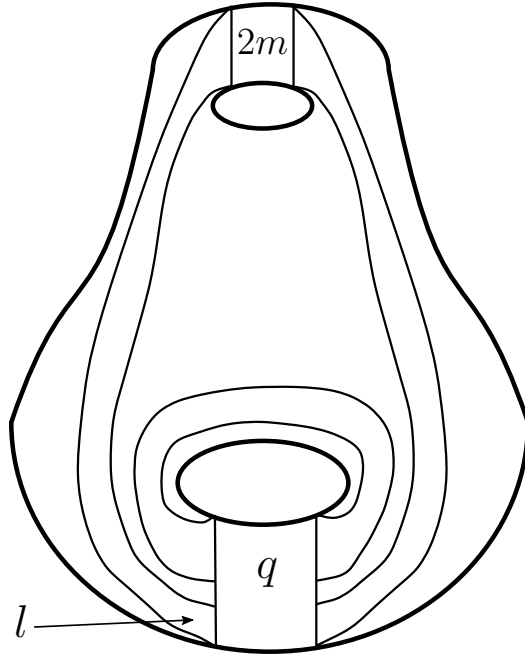


Figure 6.4: (p, q, l, m) -twisted torus knot on closed genus 2 surface.

Example 6.4. The unknot and torus knots are L-space knots [2].

Example 6.5. It was proved in [18, Theorem 1] that $(-2, 3, 1)$ -pretzel knot (see Figure 6.5) is an L-space knot. The full statement is as follows:

Theorem 6.6. *Let K be a pretzel knot. Then, K admits an L-space surgery if and only if K is ambient isotopic to a $\pm(-2, 3, q)$ -pretzel knot for odd $q \geq 1$ or is a $T(2, 2n + 1)$ torus knot for some n .*

For the case of twisted torus knots, the following was proved in [23]:

Theorem 6.7. *The knot $T_{p, pk \pm 1}^{l, m}$ is an L-space knot if and only if one of the following holds:*

- 1) $l = p - 1$;
- 2) $l = p - 2$ and $m = 1$;
- 3) $l = 2$ and $m = 1$.

Note 6.8. The question of what happens when the respective parameter is not $pk \pm 1$ remains unanswered.

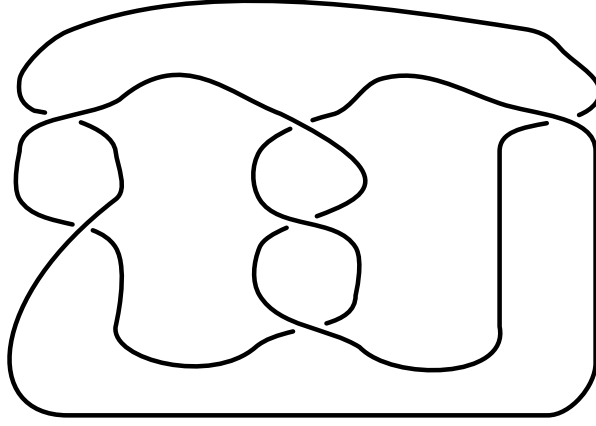


Figure 6.5: $(-2,3,1)$ pretzel knot.

Note 6.9. Case 3 with $p = 3$ is contained in Case 1, and Case 3 with $p = 4$ is contained in Case 2. We will consider these cases in Proposition 6.27.

Representatives of the respective families from Theorem 6.7 are shown in Figures 6.6, 6.7 and 6.8:

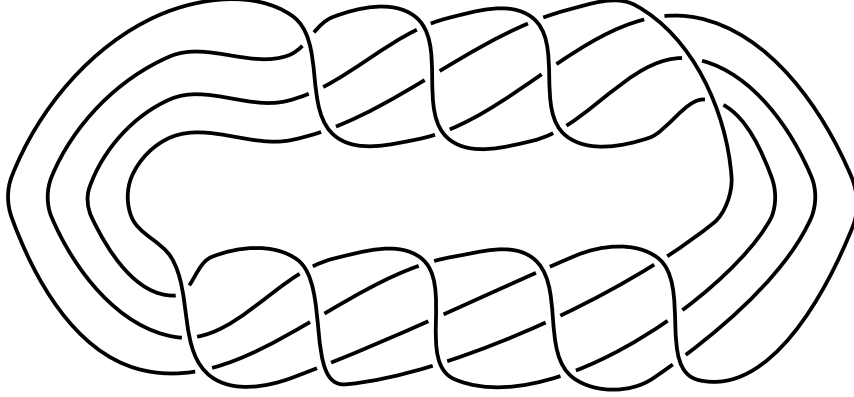


Figure 6.6: $T_{p,pk-1}^{p-1,m} = T_{5,4}^{4,2}$, when $p = 5, k = 1, m = 2$.

In order to satisfy Conjecture 1.1 all these L-space knots must have surgeries with non-left-orderable knot groups. The L-space surgeries correspond to specific slopes as given by the following result from [19]:

Theorem 6.10. *Let $K \subset S^3$, and suppose that there exists $\frac{p}{q} \in \mathbb{Q}$ such that $S^3(K, \frac{p}{q})$ is an L-space. Then $S^3(K, \frac{p'}{q'})$ is an L-space if and only if $\frac{p'}{q'} \geq 2g(K) - 1$.*

Correspondingly, we should have

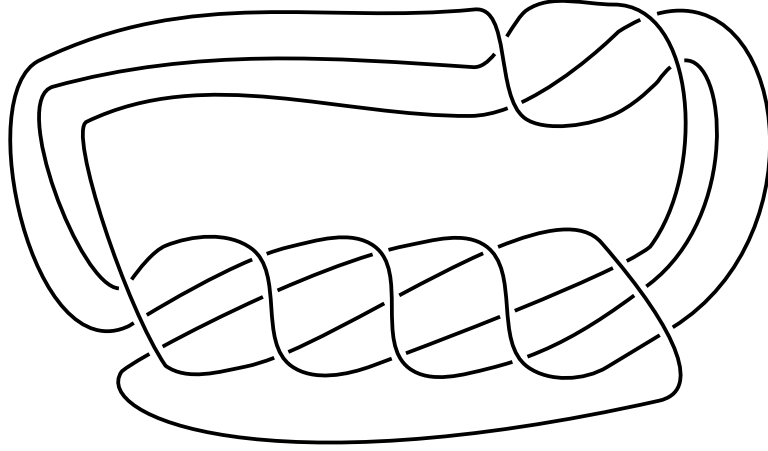


Figure 6.7: $T_{p,pk-1}^{p-2,1} = T_{5,4}^{3,1}$, when $p = 5, k = 1$.

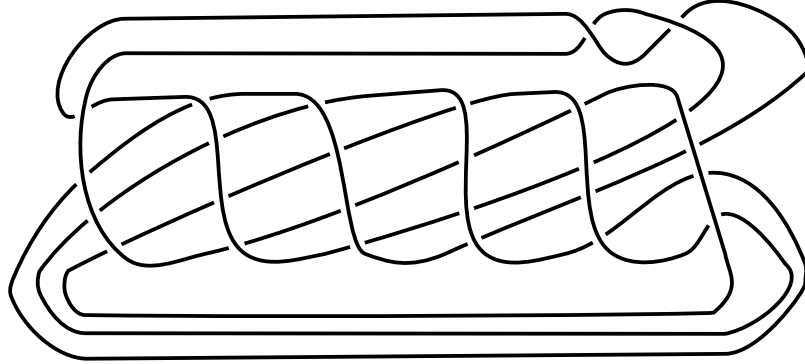


Figure 6.8: $T_{p,pk+1}^{2,1} = T_{5,6}^{2,1}$, when $p = 5, k = 1$.

Conjecture 6.11. *Let $K \subset S^3$, and suppose that there exists $\frac{p}{q} \in \mathbb{Q}$ such that $\pi_1(S^3(K, \frac{p}{q}))$ is not left-orderable. Then $\pi_1(S^3(K, \frac{p'}{q'}))$ is not left-orderable if and only if $\frac{p'}{q'} \geq 2g(K) - 1$.*

For L-space knots of Types 1) and 2) in Theorem 6.7, the above was shown to hold in [4] as follows:

Proposition 6.12. *For each type of knots in Cases 1 and 2 in Theorem 6.7 there exists $r' \in \mathbb{Q}$ such that for any $r \geq r'$ the surgered manifold $M_{c,d}^{a,b}(r)$ has a non-left-orderable fundamental group as follows:*

1. *The manifold $M_{p,pk-1}^{p-1,m}(r)$ has a non-left-orderable fundamental group for $r \geq p(pk-1) + (p-1)^2m$.*

2. The manifold $M_{p,pk-1}^{p-2,1}(r)$ has a non-left-orderable fundamental group for $r \geq p(pk-1) + (p-2)^2$.
3. The manifold $M_{p,pk+1}^{p-1,m}(r)$ has a non-left-orderable fundamental group for $r \geq p(pk+1) + (p-1)^2m$.
4. The manifold $M_{p,pk+1}^{p-2,1}(r)$ has a non-left-orderable fundamental group for $r \geq p(pk+1) + (p-2)^2$.

Remark 6.13. The lower bounds of slopes in Proposition 6.12 are bigger than the corresponding values $2g(T_{k,l}^{i,j}) - 1$ in Conjecture 6.11 for all $p \geq p_0$, $k \geq k_0$, where p_0 , k_0 depends on the case. To see this, let's calculate the genus of a Seifert surface F of $T_{p,pk+1}^{p-2,1}$. From the definition of a twisted torus knot it follows that $p \geq 4$ and $k \geq 1$ here. Consider a diagram D of $T_{p,pk+1}^{p-2,1}$ similar to the one in Figure 6.7. Then D has $s(D) = p$ Seifert circles, and $c(D) = (pk+1)(p-1) + (p-2)(2-1)$ crossings. By Theorem 2.15 the Euler characteristic of the Seifert surface is $\chi(F) = s(D) - c(D) = p - (pk+1)(p-1) - (p-2)$. Then by Corollary 2.16 we have $2g(F) = [1 - p + (pk+1)(p-1) + (p-2)] + [1 - \mu(D)] = (T_{p,pk+1}^{2,1}$ is a knot, so $\mu(D) = 1) = p^2k - pk + p - 2$. Consider respective r from Proposition 6.12. We have that $r - (2g(F) - 1) = p(pk-1) + (p-2)^2 - (p^2k - pk + p - 3) = p^2 + p(k-4) + 7 > 0$, whenever $p \geq 4$ and $k \geq 1$, so for all p and k under consideration. And since $g(F) \geq g(T_{p,pk+1}^{p-2,1})$, we have $r > 2g(T_{p,pk+1}^{p-2,1}) - 1$, so the lower bound of slopes in Proposition 6.12 is bigger than $2g(T_{k,l}^{i,j}) - 1$ in this case. We have similar situations in all remaining cases.

Remark 6.14. A presentation of the fundamental group of the manifold $M_{c,d}^{a,b}(r)$ is obtained by the following method in [4]. The sphere S^3 has a Heegaard splitting into two handlebodies both having the same 2-tori as a boundary. The twisted torus knot $T_{c,d}^{a,b}$ lies on this 2-tori. By applying the Seifert-van Kampen theorem to this construction we may compute the knot group $G_{c,d}^{a,b}$. Then apply Proposition 4.7 to get $\pi_1(M_{c,d}^{a,b}(r))$.

Since Cases 1 and 2 are resolved in in [4], we will focus on Case 3 from Theorem 6.7.

Using similar arguments from [4, Propositions 15 and 16] it can be shown that:

Proposition 6.15. *For the $(p, pk - 1, 2, 1)$ -twisted torus knot,*

- 1) *The knot group is $G_{p, pk-1}^{2,1} = \langle a, b \mid a^{p-2}(a(b^{1-k(p-2)}a^{p-2}))a = b^k a^{p-2} b^k \rangle$;*
- 2) *The peripheral subgroup is generated by the meridian $\mu = a^{-1}b^k$ and the surface framing $s = \mu^{p(pk-1)+4}\lambda = a^{p-3}(ab^{1-k(p-2)}a^{p-2})^2a$.*

Note 6.16. Recall that on page 20 we defined a subgroup $\pi_1(\overline{\partial N(K)})$ of $\pi_1(S^3 \setminus N(K))$ that is called the peripheral subgroup. It is generated by a meridian μ and a v -framed longitude s , where $v \in \mathbb{Z}$. If v is fixed then choosing a framing is equivalent to choosing appropriate power of μ in s , because s is determined by the surface $\overline{\partial N(K)}$.

Proposition 6.17. *For the $(p, pk + 1, 2, 1)$ -twisted torus knot,*

- 1) *The knot group is $G_{p, pk+1}^{2,1} = \langle a, b \mid a((b^{k(p-2)+1}a^{2-p})a)a^{p-2} = b^{k(p-1)+1}a^{2-p}b^{k(p-1)+1} \rangle$;*
- 2) *The peripheral subgroup is generated by the meridian $\mu = b^{-k}a$ and the surface framing $s = \mu^{p(pk+1)+4}\lambda = ((b^{k(p-2)+1}a^{2-p})a)^2a^{p-2}$.*

To prove the main results of [4] the authors use the following theorem:

Theorem 6.18. *Let K be a nontrivial knot in S^3 . Let G denote the knot group of K , and let $G(p/q)$ be the quotient of G resulting from p/q -surgery. Let μ be the meridian of K and s be a v -framed longitude with $v > 0$. Suppose that G has two generators, x and y , such that $x = \mu$ and s is a word which excludes x^{-1} and y^{-1} and contains at least one x . Suppose further that every homomorphism $\Phi : G(p/q) \rightarrow \text{Homeo}_+(\mathbb{R})$ satisfies $\Phi(x)(t) > t$ for all $t \Rightarrow \Phi(y)(t) \geq t$ for all t . If $p, q > 0$, then, for $p/q > v$, $G(p/q)$ is not left-orderable.*

Let's show how Theorem 6.18 is applied to prove Case 1 of Proposition 6.12 in [4]. It is sufficient to check that respective s can be written with only positive powers of x and y , and at least one x . In this case x equals the meridian μ , so $x = \mu = a^{-1}b^k$, they make the following choice for y , $y = b^{1-k}a$, and then $s = (a(b^{1-k}a)^m)^{p-1}a = ((yx)^{k-1}y^{m+1})^{p-1}(yx)^{k-1}y$, so clearly the above condition is satisfied.

6.3 A new approach to the problem

Let's use Theorem 6.18 in Case 3 of Theorem 6.7 and proceed as in [4] in order to witness the obstruction to applying it here. First we do it for the $(p, pk - 1, 2, 1)$ -twisted torus knot. Let $x = \mu = a^{-1}b^k$. Suppose then $y = b^{1-k}a$. Then $b = yx$ and $a = (yx)^{k-1}y$, which means that x and y generate $G_{p, pk-1}^{2,1}$. Now it is enough to show that s can be written using only positive powers of x and y with at least one x . We have

$$\begin{aligned}
s &= a^{p-3}(ab^{1-k(p-2)}a^{p-2})^2a = a^{p-2}b^{1-k(p-2)}a^{p-2}b^{1-k(p-2)}a^{p-1} \\
&= ((yx)^{k-1}y)^{p-2}(yx)^{1-k(p-2)}((yx)^{k-1}y)^{p-2}(yx)^{1-k(p-2)}((yx)^{k-1}y)^{p-1} \\
&= ((yx)^{k-1}y)^{p-2}(x^{-1}y^{-1})^{k(p-2)-1}((yx)^{k-1}y)^{p-2}(x^{-1}y^{-1})^{k(p-2)-1}((yx)^{k-1}y)^{p-1} \\
&= ((yx)^{k-1}y)^{p-2}(x^{-1}y^{-1})^{k(p-2)-1}(yx)^{k-1}y((yx)^{k-1}y)^{p-3}(x^{-1}y^{-1})^{k(p-2)-1}(yx)^{k-1}y \\
&\quad \cdot ((yx)^{k-1}y)^{p-2} \\
&= ((yx)^{k-1}y)^{p-2}(x^{-1}y^{-1})^{k(p-2)-1-(k-1)-1}\underline{x^{-1}}((yx)^{k-1}y)^{p-3}(x^{-1}y^{-1})^{k(p-2)-1-(k-1)-1}\underline{x^{-1}} \\
&\quad \cdot ((yx)^{k-1}y)^{p-3}.
\end{aligned}$$

We can see that this y doesn't work, because of the underlined x^{-1} above.

Here we will see calculations for $(p, pk + 1, 2, 1)$ -twisted torus knot. So $x = \mu = b^{-k}a$. Suppose then $y = a^{-1}b^{k+1}$. Then $b = xy$ and $a = (xy)^kx$, which means that x and y generate $G_{p, pk+1}^{2,1}$. Now it is enough to show that s can be written using only positive powers of x and y with at least one x . We have

$$\begin{aligned}
s &= ((b^{k(p-2)+1}a^{2-p})a)^2a^{p-2} = ((xy)^{k(p-2)+1}((xy)^kx)^{3-p})^2((xy)^kx)^{p-2} \\
&= ((xy)^{k(p-2)+1}(x^{-1}(xy)^{-k})^{p-3})^2((xy)^kx)^{p-2} \\
&= (xy)^{k(p-2)+1}(x^{-1}(xy)^{-k})^{p-3}(xy)^{k(p-2)+1}(x^{-1}(xy)^{-k})^{p-3}((xy)^kx)^{p-2} \\
&= (xy)^{k(p-2)+1}(x^{-1}(xy)^{-k})^{p-4}x^{-1}(xy)^{-k}(xy)^{k(p-2)+1}(xy)^kx \\
&= (xy)^{k(p-2)+1}(x^{-1}(xy)^{-k})^{p-4}x^{-1}(xy)^{k(p-2)+1-k}(xy)^kx \\
&= (xy)^{k(p-2)+1}(x^{-1}(xy)^{-k})^{p-4}x^{-1}(xy)^{k(p-2)+1}x \\
&= (xy)^{k(p-2)+1}(x^{-1}(xy)^{-k})^{p-4}x^{-1}xy(xy)^{k(p-2)}x
\end{aligned}$$

$$\begin{aligned}
&= (xy)^{k(p-2)+1} (x^{-1}(xy)^{-k})^{p-5} x^{-1} (xy)^{-k} y (xy)^{k(p-2)} x \\
&= (xy)^{k(p-2)+1} (x^{-1}(xy)^{-k})^{p-5} x^{-1} \underline{y^{-k} x^{-k} y} (xy)^{k(p-2)} x.
\end{aligned}$$

We see that this y doesn't work either, because of the underlined $y^{-k}x^{-k}$ above.

We need to rewrite some powers of a and b in terms of x and y . For example, in the $(p, pk-1, 2, 1)$ case, $x = \mu = a^{-1}b^k$, so $a = b^k x^{-1}$ and we need to rewrite only some powers of b now. Rewriting only b^k is not enough, since we encounter $b^{1-k(p-2)} = b^{-k(p-2)}b$ in s , which means we need to rewrite b itself in order to rewrite s . This limits the number of y 's that could possibly work. The author tried different y 's in both cases, but did not succeed.

We will try another method from [7]. This method of studying left-orderability of the fundamental group of surgered manifolds works in general, contrary to Theorem 6.18, which works only in special cases.

We need Corollary 6.26 of Theorem 6.19, [7, Corollary 11 of Theorem 9]:

Theorem 6.19. *Let $\frac{p}{q}, \frac{p_0}{q_0}, \frac{p_1}{q_1} \in \mathbb{Q}^+$ be given, with $\frac{p}{q} \in (\frac{p_0}{q_0}, \frac{p_1}{q_1})$ and $p, q, p_i, q_i > 0$, for $i = 0, 1$. Suppose that M is compact, connected, orientable 3-manifold with incompressible torus boundary, and suppose that $\langle \mu, \lambda \rangle \cong \pi_1(\partial M)$ is not sent to 1 under the quotient map $\pi_1(M) \rightarrow \pi_1(M(\frac{p}{q}))$. If $\pi_1(M(\frac{p}{q}))$ is left-orderable, then there exists a left-ordering of $\pi_1(M)$ relative to which the elements $\mu^{p_0}\lambda^{q_0}$ and $\mu^{p_1}\lambda^{q_1}$ have opposite signs.*

To prove Theorem 6.19 we need a sequence of results all from [7].

Recall the definition of a convex subgroup on page 26.

Lemma 6.20. *Let H be a nontrivial subgroup of the left-ordered group $(G, >)$. If C is convex relative to the left-ordering $>$, then $C \cap H$ is convex relative to the restriction ordering $>_H$.*

Proposition 6.21. *Suppose that $1 \rightarrow K \rightarrow G \rightarrow H \rightarrow 1$ is a short exact sequence of nontrivial groups, and $\phi : G \rightarrow H$ is the quotient map. Then K is convex relative to the left-ordering $<$ of G if and only if K and H are left-orderable. Moreover, the left-ordering $<$ of G is related to the left-orderings $>_K$ and $>_H$ by the following rule:*

given $g \in G$, if $\phi(g) \neq 1$, then $g > 1$ if and only if $\phi(g) >_H 1$; otherwise, $\phi(g) = 1$ and $g > 1$ if and only if $g >_K 1$.

Proof. A proof is similar to the proof of Lemma 5.6. \square

Proposition 6.22. *Suppose that M is compact, connected, orientable 3-manifold with incompressible torus boundary, and let α be a slope in ∂M , and suppose $\pi_1(\partial M)$ is not sent to 1 under a quotient map $\pi_1(M) \rightarrow \pi_1(M(\alpha))$. If $\pi_1(M(\alpha))$ is left-orderable, then we may define a left-ordering $>$ of $\pi_1(M)$ such that $\langle \alpha \rangle$ is convex relative to restriction of $>_{\pi_1(\partial M)}$ of $\pi_1(\partial M) \cong \mathbb{Z} \times \mathbb{Z}$.*

Note 6.23. We call $\beta \in \mathbb{Z} \times \mathbb{Z}$ a slope, because if generators μ and λ of $\mathbb{Z} \times \mathbb{Z}$ are fixed, then $\beta = \mu^p \lambda^q$, which defines a slope $\frac{p}{q}$. We require β to be primitive, i.e. not a proper power.

Proof. Suppose $\pi_1(M(\alpha))$ is left-orderable. Let $\langle\langle \alpha \rangle\rangle$ denote the normal closure of α in $\pi_1(M)$. By Theorem 5.24, the group $\pi_1(M)$ is left-orderable. Hence $\langle\langle \alpha \rangle\rangle$ is left-orderable as a subgroup of a left-orderable group by Proposition 5.7. Proposition 6.21 says that the short exact sequence

$$1 \rightarrow \langle\langle \alpha \rangle\rangle \rightarrow \pi_1(M) \rightarrow \pi_1(M(\alpha)) \rightarrow 1$$

gives a left-ordering of $\pi_1(M)$ relative to which $\langle\langle \alpha \rangle\rangle$ is a convex subgroup.

If C is a nontrivial subgroup of $\mathbb{Z} \times \mathbb{Z}$ then it is either isomorphic to \mathbb{Z} or it is of rank two. Let's show that if C is convex and of rank two then $C \cong \mathbb{Z} \times \mathbb{Z}$. We know that in convex groups, if $g^k \in C$ for some k then $g \in C$. Thus, if u and v generate C and g is any element of $\mathbb{Z} \times \mathbb{Z}$, then $au + bv = g^k$ for some integers a and b , hence $g \in C$ and $C = \mathbb{Z} \times \mathbb{Z}$.

By Lemma 6.20, the group $\langle\langle \alpha \rangle\rangle \cap \pi_1(\partial M)$ is convex in $\pi_1(\partial M) \cong \mathbb{Z} \times \mathbb{Z}$ relative to the left-ordering $<_{\pi_1(\partial M)}$. So we can apply the previous argument to it. Since $\pi_1(\partial M)$ is not sent to 1 under a quotient map $\pi_1(M) \rightarrow \pi_1(M(\alpha))$, we have that $\langle\langle \alpha \rangle\rangle \cap \pi_1(\partial M) \cong \mathbb{Z}$. Since α is primitive, we have that $\langle\langle \alpha \rangle\rangle \cap \pi_1(\partial M) = \langle \alpha \rangle$ and this subgroup is convex relative to the left-ordering $<_{\pi_1(\partial M)}$ of $\pi_1(\partial M)$. \square

Lemma 6.24. *Let $\frac{p}{q}, \frac{p_0}{q_0}, \frac{p_1}{q_1} \in \mathbb{Q}^+$ be given, $p, q, p_i, q_i > 0$, for $i = 0, 1$. Let $>$ be any ordering of $\mathbb{Z} \times \mathbb{Z}$ relative to which the subgroup $\langle(p, q)\rangle$ is convex. If $\frac{p}{q} \in (\frac{p_0}{q_0}, \frac{p_1}{q_1})$ then (p_0, q_0) and (p_1, q_1) have opposite signs in the ordering of $\mathbb{Z} \times \mathbb{Z}$.*

Sketch of the proof of Lemma 6.24. Let $\mathbb{Z} \times \mathbb{Z}$ sit in \mathbb{R}^2 . Then the $\frac{p}{q}$ defines a line l in \mathbb{R}^2 , and an ordering $>$ of $\mathbb{Z} \times \mathbb{Z}$. Clearly points (p_0, q_0) and (p_1, q_1) belong to the different half-planes defined by the line l , therefore they have opposite signs in the ordering $>$ of $\mathbb{Z} \times \mathbb{Z}$. See [7, Proposition 18] for details. \square

Proof of Theorem 6.19. We have that M is compact, connected, orientable 3-manifold with incompressible torus boundary, so by Proposition 6.22, we may create a left-ordering of $\pi_1(M)$ such that $\langle\mu^p\lambda^q\rangle$ is convex relative to restriction ordering $>_{\pi_1(\partial M)}$ of $\pi_1(\partial M)$. Then by Lemma 6.24, we have that $\langle\mu^{p_0}\lambda^{q_0}\rangle$ and $\langle\mu^{p_1}\lambda^{q_1}\rangle$ have opposite signs in the restriction ordering $>_{\pi_1(\partial M)}$, and therefore they have opposite signs in the ordering $>$ of $\pi_1(M)$. And we are done. \square

Here are two corollaries of Theorem 6.19:

Corollary 6.25. *Let $\frac{p}{q}, \frac{p_0}{q_0}, \frac{p_1}{q_1} \in \mathbb{Q}^+$ be given, with $\frac{p}{q} \in (\frac{p_0}{q_0}, \frac{p_1}{q_1})$ and $p, q, p_i, q_i > 0$, for $i = 0, 1$. If $\mu^{p_0}\lambda^{q_0} > 1$ implies $\mu^{p_1}\lambda^{q_1} > 1$ for every left-ordering $>$ of $\pi_1(M)$, then $\pi_1(M(\frac{p}{q}))$ is not left-orderable.*

Proof. Let's prove it by contraposition. Suppose $\pi_1(M(\frac{p}{q}))$ is left-orderable. If $\pi_1(M(\frac{p}{q}))$ is trivial then we are done, since finite groups are not left-orderable. Let $\pi_1(M(\frac{p}{q}))$ be non-trivial. Then $\pi_1(\partial M)$ is not sent to 1 under the quotient map $\pi_1(M) \rightarrow \pi_1(M(\frac{p}{q}))$. But then by Theorem 6.19 there exists a left-ordering $>$ of $\pi_1(M)$ such that $\mu^{p_0}\lambda^{q_0}$ and $\mu^{p_1}\lambda^{q_1}$ have opposite signs, a contradiction. \square

Corollary 6.26. *Let $\frac{p}{q}, r \in \mathbb{Q}^+$ be given, with $p, q > 0$ and $r > \frac{p}{q}$. If $s = \mu^p\lambda^q > 1$ implies $\mu^N s > 1$ for all $N \geq 0$, then $\pi_1(M(r))$ is not left-orderable.*

Proof. Take N such that $r \in (\frac{p}{q}, \frac{p+N}{q})$ and apply Corollary 6.25. \square

We will paraphrase this corollary using Theorem 6.18 as follows:

Proposition 6.27. *Suppose the peripheral subgroups of $G_{p,pk-1}^{2,1}$ (respectively $G_{p,pk+1}^{2,1}$) are generated by respective s and μ . Then $s > 1 \Rightarrow s\mu^N > 1 \forall N \geq 1$, therefore:*

- 1) *The manifold $M_{p,pk-1}^{2,1}(r)$ has a non-left-orderable fundamental group for $r \geq p(pk-1) + 4$;*
- 2) *The manifold $M_{p,pk+1}^{2,1}(r)$ has a non-left-orderable fundamental group for $r \geq p(pk+1) + 4$.*

Note that this proposition is not proved completely, see Remark 6.28 below.

Remark 6.28. Recall that $p \geq 2$. In both Cases 1 and 2 above we prove Proposition 6.27 when $p = 2$ and $p = 3$. When $p > 3$ we prove two subcases out of four for both families of knots. All other subcases remain unsolved.

Proof of Proposition 6.27. We start with some observations about Proposition 6.27, which we will use throughout this proof:

1. If the hypothesis of our subcase imply that $s > 1$ is not possible then our case is not possible.
2. If the hypothesis of our subcase imply that $\mu > 1$ then the implication in Proposition 6.27 is straightforward.
3. If the hypothesis of our subcase imply that $\mu < 1 \Rightarrow s < 1$ then our case is not possible.
4. If the hypothesis of our subcase leads to a contradiction in the relations of $G_{p,pk-1}^{2,1}$ (or $G_{p,pk+1}^{2,1}$) then our case is not possible.

We tried to use Theorem 6.18 (which works in some special cases) and we did not succeed. We will use Corollary 6.26 which works in general case. Recall Note 6.9 on page 36. We will do these already solved cases with new approach for two reasons. Firstly, solving the cases with $p = 2$ and $p = 3$ may give us a hint for solving the case $p > 3$. Secondly, we are going to consider cases $p = 2$, $p = 3$ and $p > 3$ separately, because some powers of our generators have different signs then, and it changes the argumentation of the proof. In each case we use the observations above to eliminate

cases which are not possible. After that we rewrite μ using the relation in a way we can see that $\mu^N s$ is positive. Note that since $s \in \langle \mu, \lambda | \mu\lambda = \lambda\mu \rangle$ then s and μ commute, so $\mu^N s = s\mu^N$.

We work with $G_{p,pk-1}^{2,1}$ first. Recall that $s = a^{p-3}(ab^{1-k(p-2)}a^{p-2})^2a$, $\mu = a^{-1}b^k$ and the relation $a^{p-1}b^{1-k(p-2)}a^{p-1} = b^k a^{p-2} b^k$ holds in this case.

Case 1 : $p = 2$. Then we compute $aba = b^{2k}$, $\mu = a^{-1}b^k = bab^{-k}$, $s = a^{-1}(ab)^2a = babab = b^{2k+1}$. We organize our argument into subcases as follows:

Subcase 1.1 : $a > 1, b < 1$. Then as $s = b^{2k+1} < 1$, we are done by Observation 1.

Subcase 1.2 : $a < 1, b > 1$. We have $s = b^{2k+1} > 1$ and $\mu = a^{-1}b^k > 1$, so $s\mu^N > 1$ for all $N \geq 0$, as it is a product of only positive elements.

Subcase 1.3 : $a < 1, b < 1$. Since $s = b^{2k+1} < 1$, we are done by Observation 1.

Subcase 1.4 : $a > 1, b > 1$. We have the implication $\mu = a^{-1}b^k < 1 \Rightarrow b^{-k}a > 1$. Then

$$\begin{aligned} s\mu^N &= \mu^N s = (a^{-1}b^k)^N \cdot b^{2k+1} = (bab^{-k})^N b^{2k+1} = bab(b^{-k}ab)^{N-1} b^{-k-1} b^{2k+1} \\ &= bab(b^{-k}ab)^{N-1} b^k > 1, \text{ since } b^{-k}a > 1. \text{ So } s\mu^N > 1, \forall N \geq 0. \end{aligned}$$

Remark 6.29. When $p > 2, a < 1, b > 1$ we have that $s > 1$ is not possible, since a occurs in s with only non-negative powers, and b only with non-positive powers.

Case 2 : $p = 3$. We compute that $a^2 b^{1-k} a^2 = b^k a b^k$, $s = (ab^{1-k}a)^2 a = ab^{1-k} \cdot a^2 b^{1-k} a^2 = abab^k$.

Subcase 2.1 : $a < 1, b < 1$. Then $s = abab^k < 1$, and we are done by Observation 1.

Subcase 2.2 : $a > 1, b > 1$. We have that $\mu = a^{-1}b^k = ab^{1-k}a^2 b^{-k}a^{-1}$ and $\mu^{-1} = b^{-k}a > 1$. Thus

$$\begin{aligned} s\mu^N &= \mu^N s = (ab^{1-k}a^2 b^{-k}a^{-1})^N \cdot abab^k = a(b^{1-k}a^2 b^{-k}a^{-1})^N a^{-1} \cdot abab^k \\ &= a(b \cdot b^{-k}a \cdot ab^{-k})^N bab^k > 1, \end{aligned}$$

as it is a product of only positive elements when $ab^{-k} > 1$.

Suppose $ab^{-k} < 1$, so $b^k a^{-1} > 1$. Then $\mu^N = (a^{-1}b^k)^N = a^{-1}(b^k a^{-1})^N a$ and thus

$$s\mu^N = abab^k \cdot \mu^N = abab^k \cdot a^{-1}(b^k a^{-1})^N a = aba(b^k a^{-1})^{N+1} a > 1,$$

as it is a product of only positive elements.

Subcase 2.3 : $a > 1, b < 1$. Now $\mu = a^{-1}b^k = a^{-2}b^{-k}a^2b^{1-k}a^2$, and so

$$\begin{aligned} s\mu^N &= abab^k \cdot (a^{-2}b^{-k}a^2b^{1-k}a^2)^N = ab \cdot ab^k \cdot a^{-2}(b^{-k}a^2b^{1-k})^N a^2 \\ &= ab \cdot b^{-k}a^2b^{1-k}a^2 \cdot a^{-2}(b^{-k}a^2b^{1-k})^N a^2 = ab^{1-k}a^2b^{1-k}(b^{-k}a^2b^{1-k})^N a^2 > 1, \end{aligned}$$

as it is a product of only positive elements, since $b < 1$ occurs only with non-positive powers, because $k \geq 1$.

Case 3 : $p > 3$. We have that $a^{p-1}b^{1-k(p-2)}a^{p-1} = b^k a^{p-2}b^k$, $s = a^{p-3}(ab^{1-k(p-2)}a^{p-2})^2 a$.

Subcase 3.1 : $a > 1, b < 1$. Write b^k from $a^{p-1}b^{1-k(p-2)}a^{p-1} = b^k a^{p-2}b^k$ as $b^k = a^{2-p}b^{-k}a^{p-1}b^{1-k(p-2)}a^{p-1}$. Then $\mu = a^{-1}b^k = a^{-1} \cdot a^{2-p}b^{-k}a^{p-1}b^{1-k(p-2)}a^{p-1} = a^{1-p}b^{-k}a^{p-1}b^{1-k(p-2)}a^{p-1}$. Thus

$$\begin{aligned} s\mu^N &= a^{p-2}b^{1-k(p-2)}a^{p-1}b^{1-k(p-2)}a^{p-1} \cdot (a^{1-p}b^{-k}a^{p-1}b^{1-k(p-2)}a^{p-1})^N \\ &= a^{p-2}b^{1-k(p-2)}a^{p-1}b^{1-k(p-2)} \cdot (b^{-k}a^{p-1}b^{1-k(p-2)})^N a^{p-1} > 1, \end{aligned}$$

as it is a product of positive elements.

Subcase 3.2 : $a < 1, b < 1$. This case remains unsolved.

Subcase 3.3 : $a > 1, b > 1$. This case remains unsolved.

Let's now work with $G_{p,pk+1}^{2,1}$. Recall that $s = ((b^{k(p-2)+1}a^{2-p})a)^2 a^{p-2}$, $\mu = b^{-k}a$ and the relation is $ab^{k(p-2)+1}a = b^{k(p-1)+1}a^{2-p}b^{k(p-1)+1}$ in this case.

Case 1 : $p = 2$. We compute $aba = b^{2k+2}$, $s = (ba)^2$.

Subcase 1.1 : $a > 1, b < 1$. Since $\mu = b^{-k}a > 1$ and it doesn't depend on p so by Observation 2 we have shown that the claim is true for all p whenever $a > 1, b < 1$.

Subcase 1.2 : $a < 1, b < 1$. As $s = (ba)^2 < 1$, and so we are done by Observation 1.

Subcase 1.3 : $a < 1, b > 1$. We have the implication $s = (ba)^2 > 1 \Rightarrow ba > 1$. Also $\mu = b^{-k}a = b^{-k} \cdot b^{2k+2}a^{-1}b^{-1} = b^{k+2}a^{-1}b^{-1}$. So

$$\begin{aligned} \mu^N s &= (b^{k+2}a^{-1}b^{-1})^N \cdot s = b(b^{k+1}a^{-1})^N b^{-1} \cdot baba = b(b^{k+1}a^{-1})^N aba \\ &= b(b^{k+1}a^{-1})^{N-1}b^{k+1}a^{-1}aba = b(b^{k+1} \cdot a^{-1})^{N-1}b^{k+1} \cdot ba > 1, \end{aligned}$$

as it is a product of positive elements.

Subcase 1.4 : $a > 1, b > 1$. Observe that $\mu = b^{-k}a = b^{-k-1}a^{-1}b^{2k+2}$, $\mu^{-1} =$

$a^{-1}b^k > 1$. And then $s\mu^N = baba \cdot (b^{-k-1}a^{-1}b^{2k+2})^N = b^{2k+3} \cdot b^{-k-1}(a^{-1}b^{k+1})^N b^{k+1} = b^{k+2} \cdot (a^{-1}b^k \cdot b)^N b^{k+1} > 1$, as it is a product of positive elements.

Case 2 : $p = 3$. We calculate $ab^{k+1}a = b^{2k+1}a^{-1}b^{2k+1}$, $s = (b^{k+1})^2a = b^{2k+2}a$.

Subcase 2.1 : $a < 1, b < 1$. Whereas $s = b^{2k+2}a < 1$, we are done by Observation 1.

Subcase 2.2 : $a > 1, b > 1$. Rewrite μ using the relation as $\mu = b^{-k}a = b^{-k} \cdot b^{2k+1}a^{-1}b^{2k+1}a^{-1}b^{-k-1} = b^{k+1}a^{-1}b^{2k+1}a^{-1}b^{-k-1}$. Then

$$\begin{aligned}\mu^N s &= (b^{k+1}a^{-1}b^{2k+1}a^{-1}b^{-k-1})^N \cdot b^{2k+2}a = b^{k+1}(a^{-1}b^{2k+1}a^{-1})^N b^{-k-1} \cdot b^{2k+2}a \\ &= b^{k+1}(a^{-1}b^k \cdot b^{k+1}a^{-1})^N b^{k+1}a.\end{aligned}$$

If $b^{k+1}a^{-1} > 1$, then $\mu^N s > 1$, as it is a product of only positive elements. If $b^{k+1}a^{-1} < 1 \Rightarrow a > b^{k+1} \Rightarrow \mu = b^{-k}a > b^{-k}b^{k+1} = b > 1$, get a contradiction.

Subcase 2.3 : $a < 1, b > 1$. Rewrite $\mu = b^{-k} \cdot a = b^{-k} \cdot b^{2k+1}a^{-1}b^{2k+1}a^{-1}b^{-k-1} = b^{k+1}a^{-1}b^{2k+1}a^{-1}b^{-k-1}$. Then we have

$$\begin{aligned}\mu^N s &= (b^{k+1}a^{-1}b^{2k+1}a^{-1}b^{-k-1})^N \cdot b^{2k+2}a = b^{k+1}(a^{-1}b^{2k+1}a^{-1})^N b^{-k-1} \cdot b^{2k+2}a \\ &= b^{k+1}(a^{-1}b^k \cdot b^{k+1} \cdot a^{-1})^N b^{k+1}a > 1,\end{aligned}$$

as it is a product of only positive elements, since $b^{k+1}a = a^{-1}b^{2k+1}a^{-1}b^{2k+1} > 1$.

Case 3 : $p > 3$. We compute that $ab^{k(p-2)+1}a = b^{k(p-1)+1}a^{2-p}b^{k(p-1)+1}$, $s = ((b^{k(p-2)+1}a^{2-p})a)^2a^{p-2} = b^{k(p-2)+1}a^{3-p}b^{k(p-2)+1}a$.

Subcase 3.1 : $a < 1, b > 1$. Rewrite $a = b^{-k(p-2)-1}a^{-1}b^{k(p-1)+1}a^{2-p}b^{k(p-1)+1}$ from the relation. Then

$$\begin{aligned}s\mu^N &= b^{k(p-2)+1}a^{3-p}b^{k(p-2)+1}a \cdot (b^{-k}(a))^N \\ &= b^{k(p-2)+1}a^{2-p} \cdot ab^{k(p-2)+1}a \cdot (b^{-k(p-1)-1}a^{-1}b^{k(p-1)+1}a^{2-p}b^{k(p-1)+1})^N \\ &= b^{k(p-2)+1}a^{2-p}b^{k(p-1)+1}a^{2-p}b^{k(p-1)+1} \cdot b^{-k(p-1)-1}a^{-1}(b^{k(p-1)+1}a^{1-p})^N ab^{k(p-1)+1} \\ &= b^{k(p-2)+1}a^{2-p}(b^{k(p-1)+1}a^{1-p})^{N+1} \cdot ab^{k(p-1)+1} > 1,\end{aligned}$$

if $ab^{k(p-1)+1} > 1$. Suppose $ab^{k(p-1)+1} < 1$. Then $ab^{k(p-2)+1}a = ab^{k(p-1)+1} \cdot b^{-k}a = (\text{negative} \cdot \text{negative}) = b^{k(p-1)+1} \cdot a^{2-p} \cdot b^{k(p-1)+1} = (\text{positive} \cdot \text{positive} \cdot \text{positive})$, we get a contradiction.

Subcase 3.2 : $a < 1, b < 1$. This case remains unsolved.

Subcase 3.3 : $a > 1, b > 1$. This case remains unsolved.

□

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