IMPROVING THE PERFORMANCE OF BEAM ELEMENTS UNDERGOING FORCED VIBRATIONS

by

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A thesis
presented to the University of Manitoba
in partial fulfillment of the
requirements for the degree of
Master of Science
in
Mechanical Engineering

Winnipeg, Manitoba

ISBN 0-315-33582-3

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A thesis submitted to the Faculty of Graduate Studies of the University of Manitoba in partial fulfillment of the requirements of the degree of

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ABSTRACT

Condensation procedures are presented for reducing the degrees of freedom in a finite element analysis. Specifically, the Guyan-Irons and Downs-Fricker procedures are investigated in beam vibrations. The traditional Guyan-Irons procedure involves the reduction of the stiffness and mass matrices, separately. This results in approximations because the inertial forces (frequency dependent terms) are neglected in the condensation. The Downs-Fricker procedure involves the reduction of dynamic element stiffness matrices which combine the stiffness and mass matrices, and the inertial forces. Thus, there are no approximations involved. The objective of this thesis is to use the Downs-Fricker condensation to improve the performance of beam elements under the conditions of forced vibrations.

The advantages of eliminating internal degrees of freedom are discussed. Also, the condensation procedures are illustrated from a mathematical and computational point of view. All results are taken from the simply supported beam problem for which the exact solution is known. As expected, results show that the accuracies of solutions using the Downs-Fricker condensation are the same as accuracies from solutions without condensation, when the same original shape function

is used. The ability of the condensation to retain the inertial characteristics is evident by the large error encountered at resonant frequencies.

On the other hand, the accuracies of solutions employing the Guyan- Irons condensation is never as high as those obtained from the Downs-Fricker condensation. Also, the error does not get very large at resonant frequencies indicating very little recognition of the inertial characteristics.

ACKNOWLEDGEMENTS

The author would like to thank Dr.W.L.Cleghorn for his guidance throughout this research. His financial support in the form of research assistantships is also appreciated.

Discussions with Dr.B.Tabarrok and Mr.C.J.Konzelman, both from the University of Toronto, proved very valuable.

The author would like to thank his parents for their patience and understanding.

CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

A	cross-sectional area of beam finite element
$\{C\}$	vector of constants from polynomial shape function
E	modulus of elasticity
F	external forcing function on simply supported beam
F_0	magnitude of external force
g	gravitational constant
$[I]$	Identity matrix
I	second moment of area of beam element about its neutral axis , symbol used to represent functional
$[K_e]$	element stiffness matrix
$[K]$	global stiffness matrix
$[K_r]$	reduced element stiffness matrix
L	length of beam, Lagrangian functional
M	bending moment
$[M_e]$	element mass matrix
$[M]$	global mass matrix
$[\overset{1}{M_e}],[\overset{2}{M_e}]$	dynamic correction matrices
m_1,m_2	curvature nodal variables
N_e	number of elements
P_e	total energy of beam
$\{q\}$	element load vector
$\{Q\}$	global load vector
$\{\bar{Q}\}$	time independent global load vector
$\{q_r\}$	reduced load vector
T_e^*	complementary kinetic energy of beam element
t	time variable

U_{ye}	assumed deflection distribution throughout beam element (shape function)
v_1, v_2, v_c	translational displacement (deflection) nodal variables
V_e	elastic strain energy of beam element
W_e	virtual work due to external forcing function
$\{X\}$	vector of spacial variables from polynomial shape function
x	length (space) variable
$y(x)$	exact deflection distribution throughout beam
$[Z_e]$	dynamic element stiffness matrix

Greek

γ	specific weight
δ	variational operator, deflection at any point along beam
ε	deflection error of finite element solutions
ε^s	stress error of finite element solutions
ρ	mass density
σ	flexural stress
$\{\phi_e\}$	vector of element nodal variables
$\{\phi\}$	vector of global nodal variables
$\{\bar{\phi}\}, \{\bar{\phi}_e\}$	time independent nodal variables
ψ_1, ψ_2	rotational displacement nodal variables
ω	frequency of vibration

Overdots indicate differentiation with respect to time, and primes denote differentiation with respect to the space variable.

Chapter I

INTRODUCTION

1.1 BACKGROUND

A finite element analysis of continuous media is usually non-exact because of its inability to represent infinite degrees of freedom. Consequently, increasing the degrees of freedom often leads to increased accuracy.

The two ways of increasing degrees of freedom in a finite element analysis are to increase the number of elements in the discretization and to increase the complexity of the shape functions used to formulate the element equations. This thesis is primarily concerned with the latter technique.

In the analysis of beam vibrations, polynomials are usually chosen as the shape functions for beam elements. By increasing the degree of the polynomial, more degrees of freedom may be accommodated resulting in a more accurate solution.

On a global scale, the number of degrees of freedom in an analysis corresponds to the number of equations to be solved. Increasing the number of equations increases the computational time and memory. Therefore, there is a trade-off between accuracy and computational effort.

Condensation of equations reduces the number of equations to be solved with the objective of maintaining the accuracy corresponding to the unreduced equations.

1.2 LITERATURE SURVEY

Since its inception, finite element users have incorporated condensation in one form or another. In 1956, Turner et al. [1] used condensation in statical structural analysis in a pioneering paper on finite elements. In order to generate the element stiffness matrix for a quadrilateral two dimensional element, they first subdivided the element into four triangles (Figure 1). Since the element stiffness matrix had already been established for a triangle, the quadrilateral element stiffness matrix is derived by assembling the element matrices for the triangles and condensing out the interior node or degree of freedom. Since no forces are applied to the interior degree of freedom, they are no approximations involved in the condensation.

Guyan [2] and Irons [3,4] were the first to extend the condensation procedure used by Turner et al. to vibration problems. Guyan used the static system (strain energy) to obtain a constraint or transformation equation before proceeding to use it to reduce both the strain and kinetic energies, separately. The transformation procedure corresponds to eliminating degrees of freedom at which no forces are applied. The kinetic energy is reduced by simply differentiating the statical constraint equation with respect to time. Irons suggested the same condensation procedure as Guyan, but rationalized it from a mathematical point of view. He referred to degrees of freedom that are eliminated as "slaves" and those that are retained as "masters". He proposed that slaves take values giving least strain energy regardless of their effect on the kinetic energy.

In the Guyan-Irons condensation procedure, no account is made for the inertial forces in the constraint equation. Therefore, the reduction of the kinetic energy equations (mass condensation) is only approximate. Wright and Miles [5], and Geradin [6] have shown that the Guyan- Irons condensation corresponds to a first order approximation by considering the complete eigenvalue problem. However, Wright and Miles conclude that including second (and higher) order approximations to obtain more accuracy is not economical.

The Guyan-Irons condensation is used on the nondiagonal mass matrix obtained from the consistent mass matrix formulation [7]. When condensation is used on the inconsistent (diagonal) mass matrix in a similar manner, no approximations are involved because the zero diagonal entries can be partitioned to form a null submatrix [8,9]. Nevertheless, there are numerous advantages to using the consistent mass matrix formulation [7,8,10,11].

Notwithstanding the approximations in the Guyan-Irons condensation, fairly accurate representations of the lower modes of vibration in complex structures can still be obtained when suitable choices for the location, number and types of master degrees of freedom are made [12,10,13,14]. Popplewell et al. [14] generalized the criteria for master selection in free and forced vibration.

Vysloukh et al. [15] proposed a modification of the Guyan-Irons condensation procedure. This technique is based on using independent functions to derive the stiffness and mass matrix. The mass matrix is formulated using a truncated function (hence, truncated mass matrix). The stiffness matrix is reduced to the level of the mass matrix by the usual static condensation. Although the computational effort may be less for this technique [15] than the Guyan-Irons technique for certain problems, the latter is more general in scope [16].

The inadequacy of the standard finite element formulation for dynamic analysis has been recognized by several authors. For beam elements, which have received considerable attention, Paz and Dung [17], and Downs [18], and others have expressed the element matrices as a power series of ω^2 for harmonic motion, i.e.,

$$[Z_e] = [K_e] - \omega^2[M_e] - \omega^4[\overset{1}{M}_e] - \omega^6[\overset{2}{M}_e] \qquad (1.1)$$

where

$[Z_e]$ = dynamic element stiffness matrix,

$[K_e]$ = static element stiffness matrix,

$[M_e]$ = consistent mass matrix,

$[M_e^1], [M_e^2]$ = additional dynamic correction matrices,

and ω = frequency of vibration.

The standard formulation is seen as a first order approximation to $[Z_e]$. More importantly, the condensation of $[Z_e]$ will not suffer from the approximations of the Guyan-Irons condensation, even if only the standard formulation is used. This is because the inertial forces are represented in equation (1.1) through ω. Downs [19] and Fricker [20] were the first to include frequency dependent terms in condensation of dynamic equations. For free vibration problems, ω in equation (1.1) corresponds to the natural frequency of vibration, and is unknown. This complicates the reduction procedure. For example, Fricker used fixed frequency analysis to synthesize the dynamic element stiffness matrix before proceeding to solve the eigenvalue problem.

1.3 OBJECTIVE

In this thesis, condensation of dynamic element stiffness matrices is investigated as a means of improving the performance of beam elements. The standard formulation, where only the first order terms in equation (1.1) is included, is used. The proposed condensation procedure is applied to a forced vibration problem, where ω in equation (1.1) corresponds to the known external forcing frequency.

The validity of the proposed condensation procedure is demonstrated by comparing solutions with and without condensation. Also, the proposed condensation is compared to the Guyan-Irons condensation.

Chapter II

FINITE ELEMENT THEORY

The derivation of the governing elemental equations for a typical beam element will be presented. The method of solution of the global equations for forced and free vibrations will also be illustrated.

The derivation of element equations is divided into three parts. First, an element displacement function is assumed and expressed in terms of the element nodal variables. This is used in the derivations of expressions for the discretized form of the energies associated with the element. Finally, Hamilton's Principle, which utilizes these energies, will be employed to generate the governing equations.

2.1 DEFLECTION DISTRIBUTION

Consider the beam element shown in Figure 2. The conventional cubic polynomial is chosen to represent the transverse deflection distribution throughout the length of the element,

$$U_{ye}(x,t) = C_0(t) + C_1(t)x + C_2(t)x^2 + C_3(t)x^3 \qquad (2.1.1)$$

where x and t represent the length and time variable of the element and U_{ye} is the deflection distribution. This degree of approximation requires four degrees of freedom (d.o.f.) or nodal variables which are chosen as deflections, v_1 and v_2 ,and rotations, ψ_1 and ψ_2, at the ends of the element. Equation (2.1.1) in matrix form is

$$U_{ye}(x,t) = \{C\}^T\{X\} \tag{2.1.2}$$

where

$$\{C\}^T = [C_0(t)\ \ C_1(t)\ \ C_2(t)\ \ C_3(t)] \tag{2.1.3}$$

and

$$\{X\} = [1\ x\ x^2\ x^3]^T. \tag{2.1.4}$$

The constants in equation (2.1.3) can be written in terms of the degrees of freedom in Figure 2 as follows,

$$v_1 = U_{ye}(0,t) = C_0(t) \tag{2.1.5}$$

$$\psi_1 = U'_{ye}(0,t) = C_1(t) \tag{2.1.6}$$

$$v_2 = U_{ye}(L,t) = C_0(t) + C_1(t)L + C_2(t)L^2 + C_3(t)L^3 \tag{2.1.7}$$

$$\psi_2 = U'_{ye}(L,t) = C_1(t) + 2C_2(t)L + 3C_3(t)L^2. \tag{2.1.8}$$

Rewriting equations (2.1.5)-(2.1.8) in matrix form,

$$\{\phi_e\} = [D]\{C\} \tag{2.1.9}$$

where

$$\{\phi_e\} = [v_1 \ \psi_1 \ v_2 \ \psi_2]^T \tag{2.1.10}$$

is the element nodal vector and

$$[D] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix}. \tag{2.1.11}$$

Solving equation (2.1.9) for {C} gives

$$\{C\} = [D]^{-1}\{\phi_e\}. \tag{2.1.12}$$

Substituting equation (2.1.12) in (2.1.2) yields

$$U_{ye}(x,t) = \{\phi_e\}^T([D]^{-1})^T\{X\}. \tag{2.1.13}$$

Equation (2.1.13) expresses the deflection distribution in terms of the element nodal variables.

2.2 ENERGIES OF A BEAM ELEMENT

2.2.1 Strain energy

The elastic strain energy of a uniform beam element, V_e, is given in analytical form as

$$V_e = \frac{EI}{2}\int_0^L \left(\frac{\partial^2 U_{ye}}{\partial x^2}\right)^2 dx \tag{2.2.1}$$

where E is the modulus of elasticity and I is the second moment of area of the element about its neutral axis. In order to discretize V_e, let

$$\frac{\partial^2 U_{ye}}{\partial x^2}(x,t) = \{\phi_e\}^T\left([D]^{-1}\right)^T\{X''\} \tag{2.2.2}$$

where $\{X\}$ represents the only space dependent variables in equation (2.1.13). Substituting equation (2.2.2) in (2.2.1) gives

$$V_e = \frac{EI}{2}\int_0^L\left(\{\phi_e\}^T\left([D]^{-1}\right)^T\{X''\}\right)^2 dx \tag{2.2.3}$$

which can also be expressed as

$$V_e = \frac{EI}{2}\int_0^L \{\phi_e\}^T\left([D]^{-1}\right)^T\{X''\}\{X''\}^T[D]^{-1}\{\phi_e\}dx. \tag{2.2.4}$$

Again, recognizing that $\{X''\}$ represents the only space dependent variables, equation (2.2.4) can be rewritten as

$$V_e = \frac{EI}{2}\{\phi_e\}^T\left([D]^{-1}\right)^T\int_0^L \{X''\}\{X''\}^T dx\,[D]^{-1}\{\phi_e\}. \tag{2.2.5}$$

Carrying out the multiplications and integration of equation (2.2.5) give

$$V_e = \frac{1}{2}\{\phi_e\}^T[K_e]\{\phi_e\} \tag{2.2.6}$$

in which $[K_e]$ is known as the element stiffness matrix and is defined as

$$[K_e] = EI \begin{bmatrix} 12L^{-3} & 6L^{-2} & -12L^{-3} & 6L^{-2} \\ & 4L^{-1} & -6L^{-2} & 2L^{-1} \\ \text{SYMMETRIC} & & 12L^{-3} & -6L^{-2} \\ & & & 4L^{-1} \end{bmatrix} \tag{2.2.7}$$

2.2.2 Complementary kinetic energy

The complementary kinetic energy, T_e^* , for a beam element vibrating about a stationary configuration is

$$T_e^* = \frac{\rho A}{2} \int_0^L \left(\frac{\partial U_{ye}}{\partial t} \right)^2 dx \tag{2.2.8}$$

where ρ is the mass density and A is the cross-sectional area of the element. In order to discretize T_e^* let

$$\frac{\partial U_{ye}}{\partial t}(x,t) = \{\dot{\phi}_e\}^T \left([D]^{-1} \right)^T \{X\} \tag{2.2.9}$$

where $\{\phi_e\}$ contains the only time dependent quantity in equation (2.1.13). Substituting equation (2.2.9) in (2.2.8), and expanding, gives

$$T_e^* = \frac{\rho A}{2}\{\dot{\phi}_e\}^T\left([D]^{-1}\right)^T \int_0^L \{X\}\{X\}^T dx\,[D]^{-1}\{\dot{\phi}_e\}. \qquad (2.2.10)$$

The multiplications and integration of equation (2.2.10) lead to

$$T_e^* = \frac{1}{2}\{\dot{\phi}_e\}^T[M_e]\{\dot{\phi}_e\} \qquad (2.2.11)$$

where $[M_e]$ is the consistent element mass matrix and is defined as

$$[M_e] = \frac{\rho A}{35}\begin{bmatrix} 13L & \frac{11L^2}{6} & \frac{9L}{2} & -\frac{13L^2}{12} \\ & \frac{L^3}{3} & \frac{13L^2}{12} & \frac{-L^3}{4} \\ \text{SYMMETRIC} & & & \\ & & 13L & -\frac{11L^2}{6} \\ & & & \frac{L^3}{3} \end{bmatrix} \qquad (2.2.12)$$

2.2.3 Virtual work

Figure 3 shows a simply supported beam with a sinusoidally time varying load which is uniformly distributed. The virtual work associated with the external forcing function is given by

$$W_e = \int_0^L F\left(U_{ye}(x,t)\right)dx \tag{2.2.13}$$

where F contains the magnitude of the external force. The discretized form of W_e is obtained by substituting equation (2.1.13) in (2.2.13),

$$W_e = \int_0^L F\{\phi_e\}^T\left([D]^{-1}\right)^T\{X\}dx. \tag{2.2.14}$$

Carrying out the integration and multiplication of equation (2.2.14) give

$$W_e = \{\phi_e\}^T\{q\} \tag{2.2.15}$$

where {q} is the element load vector, and is defined as

$$\{q\}^T = F\left(\frac{L}{2} \quad \frac{L}{12} \quad \frac{L}{2} \quad \frac{-L}{12}\right). \tag{2.2.16}$$

2.3 HAMILTON'S PRINCIPLE

Hamilton's Principle [11,pp.60] is used to generate equations of motion for dynamic problems. This principle could also be used for static problems by neglecting the kinetic energy. In this case, it reverts to the well known Principle of Minimum Potential Energy [21].

Hamilton's Principle states that

$$\delta \int_{t_1}^{t_2} L \, dt = 0 \tag{2.3.1}$$

where L is a functional called the Lagrangian and is defined as

$$L = T_e^* + W_e - V_e \tag{2.3.2}$$

and δ is the symbol used in variational calculus to denote the variation of some quantity. The variational principle which governs the behaviour is the one which makes the Lagrangian functional a minimum.

In order to generate the equations of motion for a beam element vibrating about a stationary configuration let

$$\delta I = \delta \int_{t_1}^{t_2} L \, dt = \delta \int_{t_1}^{t_2} \left(T_e^* + W_e - V_e \right) dt \tag{2.3.3}$$

where δI is used to represent the overall functional. Since the operation of variation is commutative with integration,

$$\delta I = \int_{t_1}^{t_2} \left(\delta T_e^* + \delta W_e - \delta V_e \right) dt. \qquad (2.3.4)$$

The discretized form of the energies from equations (2.2.6), (2.2.11) and (2.2.16) give

$$\delta T_e^* = \delta \left\{ \frac{1}{2} \{\dot{\phi}_e\}^T [M_e] \{\dot{\phi}_e\} \right\} \qquad (2.3.5)$$

$$\delta V_e = \delta \left\{ \frac{1}{2} \{\phi_e\}^T [K_e] \{\phi_e\} \right\} \qquad (2.3.6)$$

$$\delta W_e = \delta \left\{ \{\phi_e\}^T \{\phi_e\} \right\} \qquad (2.3.7)$$

from which it can be seen that

$$I = I \left(\{\dot{\phi}_e\}, \{\phi_e\} \right). \qquad (2.3.8)$$

The variation of a function of several variables or of a functional is defined in a manner similiar to the calculus definition of a total differential,

$$\delta I = \frac{\partial I}{\partial \{\dot{\phi}_e\}} \delta\{\dot{\phi}_e\} + \frac{\partial I}{\partial \{\phi_e\}} \delta\{\phi_e\} \qquad (2.3.9)$$

Therefore,

$$\delta T_e^* = [M_e]\{\dot{\phi}_e\}\delta\{\dot{\phi}_e\} \tag{2.3.10}$$

$$\delta W_e = \{q\}\delta\{\phi_e\} \tag{2.3.11}$$

$$\delta V_e = [K_e]\{\phi_e\}\delta\{\phi_e\} \tag{2.3.12}$$

and

$$\delta\, I = \int_{t_1}^{t_2} \left([M_e]\{\dot{\phi}_e\}\delta\{\dot{\phi}_e\} + \{q\}\delta\{\phi_e\} - [K_e]\{\phi_e\}\delta\{\phi_e\} \right) dt. \tag{2.3.13}$$

Integration by parts on the first term of equation (2.3.13) gives

$$\int_{t_1}^{t_2} [M_e]\{\dot{\phi}_e\}\delta\{\dot{\phi}_e\}dt = [M_e]\{\dot{\phi}_e\}\delta\{\phi_e\}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} [M_e]\{\ddot{\phi}_e\}\delta\{\phi_e\}\, dt. \tag{2.3.14}$$

Since

$$\delta\{\phi_e\}\Big|_{t_1} = \delta\{\phi_e\}\Big|_{t_2}\ , \tag{2.3.15}$$

$$\int_{t_1}^{t_2} [M_e]\{\dot{\phi}_e\}\delta\{\dot{\phi}_e\}dt = -\int_{t_1}^{t_2} [M_e]\{\ddot{\phi}_e\}\delta\{\phi_e\}\, dt. \tag{2.3.16}$$

Substituting equation (2.3.16) in (2.3.13) gives

$$\delta I = \int_{t_1}^{t_2} \left(-[M_e]\{\ddot{\phi}_e\} - [K_e]\{\phi_e\} + \{q\} \right) \delta\{\phi_e\}\, dt. \tag{2.3.17}$$

$\delta \bar{I}$ must equal zero in order to satisfy Hamilton's Principle. Since the variation of the element nodal variables is completely arbitrary, the expression in brackets must vanish. This leads to the equations of motion of a beam element as

$$[M_e]\{\ddot{\phi}_e\} + [K_e]\{\phi_e\} = \{q\}. \tag{2.3.18}$$

The equations of motion for free vibration can be obtained similarly by neglecting the external force,

$$[M_e]\{\ddot{\phi}_e\} + [K_e]\{\phi_e\} = \{0\}. \tag{2.3.19}$$

Also for static problems, where the kinetic energy of the beam is not considered,

$$[K_e]\{\phi_e\} = \{q\}. \tag{2.3.20}$$

2.4 BEAM ELEMENTS

Figure 4 shows the range of beam elements investigated in this thesis besides the cubic element. All elements in this figure are more refined than the cubic element. To accommodate higher degree elements, curvatures, m_1 and m_2, as well as deflections and rotations are used on some elements.

Appendix A gives the element stiffness and mass matrices for beam elements ranging from quartic to sextic. The element load vectors for the same range of beam elements are also included.

The degree of freedom associated with a particular row or column of the element matrices and load vectors can be found by examining their respective beam elements. The order of degrees of freedom moving from left to right or top to bottom along the matrices or vectors are the same as moving from left to right along the beam element. Additionally, where there is more than one degree of freedom at any point on the element, the order is deflection, rotation and curvature (when applicable).

For example, consider the cubic element matrices and vector given in equations (2.2.7), (2.2.12) and (2.2.16), respectively. The degree of freedom correspondence is v_1, ψ_1, v_2 and ψ_2 from the beam element shown in Figure 2.

2.5 METHOD OF SOLUTION

Let the global equations representing the entire length of the beam for forced and free vibration be represented by

$$[M]\{\ddot{\phi}\} + [K]\{\phi\} = \{Q\} \tag{2.5.1}$$

and

$$[M]\{\ddot{\phi}\} + [K]\{\phi\} = \{0\} \tag{2.5.2}$$

respectively, where [M] and [K] are global mass and stiffness matrices respectively, {Q} is the global load vector and $\{\phi\}$ is the global nodal vector. To obtain the steady-state solution of equation (2.5.1), the following solution is assumed

$$\{\phi\} = \{\bar{\phi}\} \sin(\omega t) \tag{2.5.3}$$

where $\{\bar{\phi}\}$ is the time independent global nodal vector. Substituting equation (2.5.3) and its second derivative with respect to time in equation (2.5.1) yields

$$\left[-[M]\omega^2\{\bar{\phi}\} + [K]\{\bar{\phi}\} \right] \sin(\omega t) = \{Q\} \tag{2.5.4}$$

Since the global load vector has the same sinusoidal time dependence as the left side in equation (2.5.4), (see Figure 3),

$$\left[[K] - \omega^2 [M] \right] \{\bar{\phi}\} = \{\bar{Q}\} \tag{2.5.5}$$

where $\{\bar{Q}\}$ is the time independent global load vector. Equation (2.5.5) is a simultaneous linear set of equations which can be solved for the elements of the global nodal vector.

For free vibrations the right side of equation (2.5.5) is zero,

$$\left[[K] - \omega^2 [M] \right] \{\bar{\phi}\} = \{0\} \tag{2.5.6}$$

Multiplying equation (2.5.6) by $[M]^{-1}$ gives

$$\left[[M]^{-1} [K] - \omega^2 [I] \right] \{\bar{\phi}\} = \{0\} \tag{2.5.7}$$

where [I] is the identity matrix. Since

$$\{\bar{\phi}\} = \{0\} \tag{2.5.8}$$

and

$$[M]^{-1} [K] - \omega^2 [I] = \{0\} \tag{2.5.9}$$

a unique solution only exists when

$$\left| [M]^{-1} [K] - \omega^2 [I] \right| = 0. \tag{2.5.10}$$

This corresponds to an eigenvalue/eigenvector problem. The solution will yield n eigenvalues and eigenvectors, where n is the number of global degrees of freedom of the beam. The square roots of each eigenvalue represents natural frequencies of vibration, ω_n, of the beam and each eigenvector gives the corresponding nodal values.

Chapter III

CONDENSATION OF ELEMENT EQUATIONS

3.1 INTERNAL DEGREES OF FREEDOM

Recall from Chapter I that by increasing the degree of the polynomial shape function for an element, more degrees of freedom may be accommodated. When condensation is to be performed on the resulting element equations, it is preferable to introduce internal nodes, which may subsequently be treated as slaves, to increase the number of degrees of freedom. By keeping the end nodes unaltered, the process of element assembly will remain intact and the internal nodes remain independent. This permits energy minimization to be carried out with respect to the internal nodes at the elemental level. Figure 4(a),(b) and (d) show beam elements with internal degrees of freedom which are all located at the midpoint of the beam.

3.2 CONDENSATION PROCEDURES

Condensation procedures for static and dynamic analyses will be illustrated. The quartic to cubic reduction is used as an example for all the procedures. The quartic beam element is shown in Figure 4(a).

3.2.1 Static condensation

In the static analysis of beams, a constraint equation is obtained from the total potential energy by minimizing it with respect to the internal degree of freedom,

$$\frac{\delta P_e}{\delta v_c} = 0 \tag{3.2.1}$$

where

$$P_e = V_e - W_e \tag{3.2.2}$$

is the total potential energy. The resulting constraint equation can be expressed in the form

$$v_c = v_c(v_1, v_2, \psi_1, \psi_2, F_0). \tag{3.2.3}$$

To complete the reduction, this constraint equation is substituted back into P_e to eliminate v_c. The elimination of v_c corresponds to reducing the beam element from a quartic to a cubic level (Figure 2).

Equation (3.2.1) corresponds to establishing equilibrium with respect to the slave variable, v_c, on an elemental basis. This is permitted because v_c remains independent in the assembly procedure.

WHen an internal variable, at which no external forces are applied (Figure 1), is to be eliminated, only the elastic strain energy is minimized, producing a constraint equation of the form

$$v_c = v_c(v_1, v_2, \psi_1, \psi_2). \qquad (3.2.4)$$

3.2.2 Guyan-Irons condensation

This is basically static condensation applied to dynamic problems without the inclusion of inertial forces. Consequently, the reduction of the elastic strain energy and virtual work is accomplished in the same manner as the static case. To reduce the complementary kinetic energy, T_e^* , the constraint equation (3.2.2) is differentiated with respect to time to obtain

$$\dot{v}_c = \dot{v}_c(\dot{v}_1, \dot{v}_2, \dot{\psi}_1, \dot{\psi}_2). \qquad (3.2.5)$$

This equation is substituted back in T_e^* to eliminate $\dot{v}_c$.

3.2.3 Downs-Fricker condensation

In this procedure, the dynamic element stiffness matrix, $[Z_e]$ (Section 1.2), is first formulated. To accomplish this, a sinusoidal time dependence for the element nodal vector is assumed,

$$\{\phi_e\} = \{\bar{\phi}_e\} \sin(\omega t) \qquad (3.2.6)$$

where $\{\bar{\phi}\}$ is the time independent element nodal vector. Equation (3.2.6) along with its second derivative,

$$\{\ddot{\phi}_e\} = -\omega^2 \{\bar{\phi}_e\} \sin(\omega t) \qquad (3.2.7)$$

is substituted in the elemental governing equations (2.3.18) to obtain

$$[Z_e]\{\phi_e\} = \{q\} \tag{3.2.8}$$

where

$$[Z_e] = [K_e] - \omega^2[M_e]. \tag{3.2.9}$$

The Downs-Fricker condensation is static condensation applied to the steady-state equations of dynamic problems with the inclusion of inertial forces. To condense the steady-state equations represented in $[Z_e]$, the total energy must again be minimized with respect to the internal degree of freedom. For this case the total energy is

$$P_e = (V_e + T_e^*) - W_e \tag{3.2.10}$$

where $(V_e + T_e^*)$ is the energy associated with the dynamic stiffness matrix. The minimization produces a constraint equation of the form

$$v_c = v_c(v_1, v_c, \psi_1, \psi_2, \omega, F_0) \tag{3.2.11}$$

which is substituted back in equation (3.2.10) to complete the reduction.

3.3 IMPLEMENTATION

In order to offer additional insights into the condensation procedures, it is necessary to illustrate the reduction procedures from a computational point of view. Consider the simultaneous linear system of equations obtained from a static analysis

$$[K]\{\phi\} = \{q\} \tag{2.3.20}$$

where it is understood that equation (2.3.20) represent elemental equations. These equations may be partitioned in the following form

$$\left[\begin{array}{c|c} [K_{mm}] & [K_{ms}] \\ \hline [K_{sm}] & [K_{ss}] \end{array}\right] \left\{\begin{array}{c} \{\phi_m\} \\ \hline \{\phi_s\} \end{array}\right\} = \left\{\begin{array}{c} \{q_m\} \\ \hline \{q_s\} \end{array}\right\} \tag{3.3.1}$$

where subscript m refers to masters and s to slaves. Equation (2.3.20), decoupled as shown in equation (3.3.1), can be separated as

$$[K_{mm}]\{\phi_m\} + [K_{ms}]\{\phi_s\} = \{q_m\} \tag{3.3.2}$$

and

$$[K_{sm}]\{\phi_m\} + [K_{ss}]\{\phi_s\} = \{q_s\} \tag{3.3.3}$$

Solving equation (3.3.3) for $\{\phi_s\}$,

$$\{\phi_s\} = -[K_{ss}]^{-1}[K_{sm}]\{\phi_m\} + [K_{ss}]^{-1}\{q_s\} \tag{3.3.4}$$

which is substituted back in equation (3.3.2) to complete the reduction. Equation (3.3.4) is a partitioned form of the constraint equation in static condensation. After simplifying, the condensed version of equation (2.3.20) is

$$[K_r]\{\phi_m\} = \{q_r\} \tag{3.3.5}$$

where

$$[K_r] = [K_{mm}] - [K_{ms}][K_{ss}]^{-1}[K_{sm}] \tag{3.3.6}$$

and

$$\{q_r\} = \{q_m\} - [K_{ms}][K_{ss}]^{-1}\{q_s\} \tag{3.3.7}$$

are the reduced element stiffness matrix and load vector, respectively.

When the number of slaves per element is small, computation of equations (3.3.4),(3.3.6) and (3.3.7) is straightforward if condensation is performed on an elemental level. For example, if there is only one slave per element, as in the quartic element, $[K_{ss}]$ becomes a scalar quantity and is easily inverted. If, however, condensation is performed on the global level, the accumulation of slaves can become quite large, in which case the inversion of $[K_{ss}]$ becomes complicated.

In the Guyan-Irons condensation, the reduced stiffness matrix and load vector are obtained in the same manner as the static condensation. The reduced mass matrix is also obtained from an equation of the form (3.3.6) since the strain energy constraint equation is simply differentiated with respect to time to obtain one that is applicable to the complementary kinetic energy.

In the Downs-Fricker condensation, equation (2.3.20) is replaced by equation (3.2.8). The reduced dynamic element stiffness matrix and load vector are again given in the form of equations (3.3.6) and (3.3.7), respectively.

3.4 HIGHER LEVEL REDUCTIONS

The quartic to cubic reduction is one of a variety of combinations of reductions that may be used depending upon the desired accuracy. In this thesis, sextic to quintic and quintic to cubic reductions have also been studied.

For the sextic to quintic reduction (Figures 4(d) and (c)), there are six masters and one slave degree of freedom. Similarly, for the quintic to cubic case (Figures 4(b) and 2), elimination of two slaves permit the quintic (polynomial) beam element to be reduced to a cubic element. As the number of slaves eliminated increases for a particular element, more substantial reductions in degrees of freedom are achieved on a global scale.

3.5 REDUCED MATRICES

In the static analysis of beams subjected to a continuously distributed load of constant magnitude, F, the exact analytical solution can be obtained from the equilibrium equation of the beam,

$$EI\, y^{iv}(x) = F \tag{3.5.1}$$

where EI is the flexural rigidity of the beam, and $y^{iv}(x)$ is the fourth derivative of the deflection distribution. In this type of analysis, the general solution of equation (3.5.1) will be a quartic polynomial.

In view of the nature of the analytical solution, it is expected that a quartic polynomial shape function will provide the exact solution throughout the beam. A cubic polynomial will be only able to satisfy the homogenous solution of equation (3.5.1) and consequently provide exact values at the nodes of the beam. The extra degree of freedom provided by the quartic polynomial may be viewed as a correction term to obtain exact values within the element [22]. Also, degrees of freedom introduced by shape functions that are more complex than the quartic polynomial will be redundant.

In the static analysis, condensation of quartic and higher degree element stiffness matrices and load vectors to the cubic level result in reduced matrices that are the same as those obtained by the conventional cubic polynomial shape function. This is due to two reasons. The first is that there are no approximations involved in using static condensation. The second explanation lies in the nature of the analytical solution. Since every element in the cubic stiffness matrices and load vectors contribute to giving exact nodal values, they will not be altered when higher degree matrices or vectors are reduced to the cubic level. On the other hand, the additional elements in the quintic and sextic stiffness matrices and load vectors introduced by the higher degree polynomials are redundant. Consequently, sextic to quintic reduction of element stiffness matrices and load vectors will not yield matrices and vectors that are the same as their regular counterparts. Figure A9 shows an element stiffness matrix obtained from the sextic to quintic reduction which is different from the regular quintic element stiffness matrix in Figure A2.

In the Guyan-Irons condensation, the dynamic equations are treated statically to apply condensation. This results in the same pattern of observations for regular and reduced matrices as in the static analysis. This includes the mass matrix. Figures A10 and A3 show the reduced and regular element mass matrices. The reduced matrix is also obtained from the sextic to quintic reduction.

In the Downs-Fricker condensation, reduced $[Z_e]$ matrices will not yield the same matrices as the corresponding regular matrices. This is also due to two reasons. The first is that they are no approximations involved in using the condensation. The second is that the analytical solution requires infinite degrees of freedom for the exact solution. This makes every additional degree of freedom introduced through the shape functions significant.

Chapter IV

RESULTS AND DISCUSSION

To determine the accuracy of the numerical solutions, a sample problem is solved for which the exact analytical solution is known, both for the static and dynamic cases.

In order to test the validity and effectiveness of the condensation procedures, comparisons have been made between solutions with (modified) and without (regular) condensation.

4.1 SAMPLE PROBLEM

Figure 3 shows a simply supported beam with a uniformly distributed load. In the static analysis, the load is constant with magnitude, $F=F_0$. In this case, the exact solution can be derived from the equilibrium equation of the beam, (3.5.1), as

$$y(x) = \frac{F}{24EI}\left(L^3x - 2Lx^3 + x^4\right) \tag{4.1.1}$$

where $y(x)$ is the transverse deflection distribution throughout the length of the beam.

In the dynamic case, the steady-state solution of the beam in Figure 3, [23], is

$$y(x) = \frac{F_0 g}{A\gamma\omega^2}\left\{\frac{\cos\left(\sqrt{\frac{\omega}{a}}\left(\frac{L}{2} - x\right)\right)}{2\cos\left(\sqrt{\frac{\omega}{a}}\,\frac{L}{2}\right)} + \frac{\cosh\left(\sqrt{\frac{\omega}{a}}\left(\frac{L}{2} - x\right)\right)}{2\cosh\left(\sqrt{\frac{\omega}{a}}\,\frac{L}{2}\right)} - 1\right\} \tag{4.1.2}$$

where

$a^2 = EIg/A\gamma$,

γ = specific weight,

L = total length of beam,

A = cross-sectional area,

and g = gravitational constant.

All finite element solutions in this chapter have been obtained using equal length elements. In addition, all figures illustrating the results have been obtained using non-dimensionalized parameters. The actual dimensions of parameters used in generating the results are included in Appendix B for reference.

4.2 REGULAR AND MODIFIED FINITE ELEMENT PROGRAMS

In conventional finite element programs, an assembly procedure is setup whereby the element equations and boundary conditions are used in generating the global equations representing the entire domain. These equations may then be solved for the primary unknowns in the problem, for example, nodal displacements. This is usually followed by calculations of secondary quantities, such as deflections and stresses.

When condensation is incorporated in the finite element program, the element equations are reduced before proceeding to assemble the global equations. Also, when the primary unknowns are obtained, constraint equations from the reduction are used to restore the original order of element equations. In this way, the number of global equations to be solved will be less than in the conventional procedure.

Figure 5 lists a flowchart illustrating the incorporation of condensation in a finite element program.

4.3 STATIC ANALYSIS

Figure 6 shows the variation of the error of the deflection distribution throughout the length of the beam. The conventional cubic polynomial is employed in the finite element solutions. The error is defined as

$$\varepsilon = \frac{(\delta_{exact} - \delta_{f.e.})}{\delta_{max}} \times 100 \qquad (4.3.1)$$

where $(\delta_{exact} - \delta_{f.e.})$ is the difference between the exact and finite element deflections and δ_{max} is the maximum exact deflection. Error plots are obtained from finite element solutions using one, two and four elements. The maximum error for each case is designated as $(\varepsilon_{max})_i$ with the subscript i denoting the number of elements used in each case.

When quartic and other higher degree shape functions are used in the finite element analysis, the exact deflection distribution is obtained throughout the beam. This is independent of the number of elements used in the discretization.

The modified finite element solution using the quartic to cubic reduction (modified cubic) provides the same results as those obtained from the regular quartic solution. Figure 7 illustrates the difference between the modified and regular solutions, using two degrees of freedom, through a plot of the deflected beam configuration.

4.4 DYNAMIC ANALYSIS

Figures 8-11 show plots of the maximum error of regular and modified finite element solutions versus the number of elements, N_e , and degrees of freedom, d.o.f. The legends in the figures indicate the order of the effective shape function used in solving the problem. Consequently, a regular solution using a cubic beam element is designated a 'cubic' solution and a modified solution using the sextic to quintic reduction is designated a 'modified quintic' solution. The modified cubic (1) and (2) solutions correspond to the quartic to cubic and quintic to cubic reductions, respectively. The maximum error is defined as

$$\varepsilon_{max} = \frac{|\delta_{exact} - {}_{f.e.}|_{max}}{\delta_{max}} \times 100 \qquad (4.4.1)$$

where δ_{max} is the maximum deflection determined from the exact solution, and $|\delta_{exact} - \delta_{f.e.}|_{max}$ is the maximum absolute difference between the exact and finite element solutions over the beam length. In Figures 8 and 9, the modified solutions are obtained using the Guyan-Irons condensation and in Figures 10 and 11, the Downs- Fricker condensation.

Figure 12 gives the maximum stress error of regular and modified solutions with respect to the number of elements. The stress error is defined analogously to the deflection error,

$$\varepsilon^{s}_{max} = \frac{|\sigma_{exact} - \sigma_{f.e.}|_{max}}{\sigma_{max}} \times 100. \qquad (4.4.2)$$

The flexural stress is computed from the well-known flexure formula,

$$\sigma_{max} = \frac{Mc}{I} \qquad (4.4.3)$$

where c is equal to half the height of the rectangular beam and M is the bending moment obtained from

$$M = EI\, y''(x) \tag{4.4.4}$$

The modified solutions are obtained from the Downs-Fricker condensation in this figure.

In Figures 8-12, the error plots are obtained by a search along the length of the beam for the maximum error of the finite element solutions. Figures 13-15 gives the variation of the absolute value of the error of deflections and stresses throughout the length of the beam. Two elements are used in all cases. The error distributions are symmetrical about the midpoint of the beam and, therefore, only the first half of the distribution is plotted. The absolute error of the deflection is defined as

$$\varepsilon = \frac{|\delta_{exact} - \delta_{f.e.}|}{\delta_{max}} \times 100 \tag{4.4.5}$$

and similarly for the stress,

$$\varepsilon^s = \frac{|\sigma_{exact} - \sigma_{f.e.}|}{\sigma_{max}} \times 100. \tag{4.4.6}$$

In Figure 13, the modified solutions are obtained from the Guyan- Irons condensation, and in Figures 14 and 15, the Downs-Fricker condensation.

In Figures 8-15, all error plots are obtained at one, constant, frequency, $\omega/\omega_n = 0.124$. Figures 16 and 17 show the variation of the maximum deflection error with the normalized external forcing frequency for the Guyan-Irons and Downs-Fricker condensation, respectively. In both figures two elements are used. The normalizing factor is the fundamental natural frequency of vibration of the beam obtained using the same finite element model. For example, the normalizing factor in Figure 16 is obtained using a model consisting of two elements and the quartic polynomial shape function.

4.5 DISCUSSION

As expected, the results of Figure 6 show that exact nodal values are obtained from the static analysis when the conventional cubic polynomial shape function is used, irrespective of the number of elements used. Figure 6 also shows that the maximum error occurs at the midpoint of beam elements. It has been confirmed that the extra degree of freedom provided by the quartic polynomial shape function is enough to provide the exact solution throughout the beam in the static analysis. Condensation may be used to provide that extra degree of freedom. Figure 7 is an illustration of this. Rimrott and Tabarrok [22] have shown the improvement in stress distribution obtained from the modified cubic solution. Errors associated with a given (polynomial) function are accentuated by the process of differentiation.

Thus the conventional cubic shape function can be expected to give quite poor results for stress. On the other hand, the modified cubic solution is able to provide more accurate results because of its ability to provide higher degree solutions. Consequently, the exact stress distribution is obtained in the static analysis starting with only one element.

As discussed in Section 3.5, the quartic to cubic reduction of element stiffness matrices results in matrices that are the same as those obtained from the regular cubic shape function. This translates into a reduction of computational effort when condensation is incorporated in finite element programs because there is no need to perform a reduction of element equations (Figure 5). Only the constraint equations need to be applied in the post processing phase of the program.

Figures 8 and 9 show that the modified cubic and quintic solutions from the Guyan-Irons condensation are more accurate than the regular cubic solution, but not as accurate as the quartic or higher degree solutions. Moreover, on a degree of freedom basis, the modified cubic solution is more accurate than the modified quintic. Similar results in Figures 10 and 11 show the superiority of the Downs-Fricker condensation. In Figure 10, the results indicate that the modified cubic (1) and modified quintic results coincide exactly with the regular quartic and sextic results, respectively. This clearly indicates that there are no approximations involved in using the Downs-Fricker condensation.

Figure 11 shows that the modified solutions give consistently and significantly more accurate results than their regular counterparts when compared on a degree of freedom basis. For example, the modified cubic (1) solution is more accurate than the cubic solution. Additionally, the modified solutions are even more accurate than their regular unreduced counterparts. For example, the modified cubic (1) solution is more accurate than the quartic solution. This is in contrast to the Guyan-Irons condensation. For this type of condensation, the modified cubic solution, although better than the regular cubic solution, is not as good as the quartic solution. The modified quintic solution is so low in accuracy that it must immediately be discounted as a valid means of improving beam element performance.

In the Downs-Fricker condensation, increasing the number of slaves within an element increases the improvement in accuracy of the modified solutions over their regular counterparts. Compare, for example, the error plots obtained from the cubic, modified cubic (1) and modified cubic (2) solutions.

From Figure 10, the modified cubic (2) solution does not coincide exactly with the regular quintic solution, in contrast to the other levels of modified solutions. In fact, the modified solution is slightly more accurate than its quintic counterpart. This can be attributed to the nodal variables used in the respective beam elements. For the modified cubic (2) beam element, shown in Figure 4(b), only translational and rotational displacements are considered at the ends of the beam. In the regular quintic beam element, shown in Figure 4(c), translational, rotational and curvature boundary conditions are taken into account. Translational and rotational boundary conditions are considered kinematic while curvature is a force boundary condition. Kinematic boundary conditions arise from compatibility requirements and curvature from equilibrium requirements. Setting the curvatures at the ends of the element equal to zero results in the force constraints restricting the energy minimization and the accuracy of the regular quintic solution is therefore not as good as the modified cubic (2) solution. This is in agreement with the results obtained by Popplewell and McDonald [24] on vibration of plate bending elements. They found that the absence of force constraints reduces the plates overall stiffness and, hence, the natural frequencies tend to decrease. An alternative to imposing the force constraints is to float the curvature variables. However, this has the disadvantage of increasing the degrees of freedom in the analysis.

The results for the stress calculations from the Downs-Fricker condensation show the same trends as the results for the deflections. Since flexural stress is directly related to the second derivative of the deflection distribution, it is obvious that the stress distribution obtained from the modified cubic (1) and (2) solutions will be closer to the exact distribution than the one obtained from the regular cubic solution.

In Figures 13-15, the variation of the error along the length of the beam appears discontinuous because the absolute error is plotted. The true nature of the polynomial is, of course, continuous. The relative positions of the modified and regular solutions is maintained in Figures 13-15 as in Figures 8,10 and 12. In Figure 13, the shape of the error distribution curve for the modified solutions are not the same as their regular unreduced counterparts. For example, the modified cubic solution does not yield the same error variation as the regular quartic solution. This is in contrast to the modified solutions obtained from the Downs-Fricker condensation in Figure 14. This is another indication that there are inaccuracies in the Guyan- Irons condensation. The error variation of the modified cubic curve, for example, indicates that the effective shape function provided by the constraint equation is closer to a cubic polynomial rather than a quartic polynomial.

Figure 14 shows that the deflections obtained by the modified cubic (2) solution is, indeed, more accurate than the regular quintic solution, throughout the length of the beam.

However, for the stress distribution in Figure 15, the modified cubic (2) solution is not more accurate than the regular quintic solution at <u>every</u> point along the beam. In view of this, the criss-cross pattern exhibited by the modified cubic (2) and regular quintic solutions in Figure 12 is not surprising.

The only error variation in Figure 15 that starts out with zero error is the regular quintic solution. All other solutions show appreciable error at the ends of the beam. This is because only in the regular quintic solution are curvature boundary conditions (force constraints) enforced.

Figure 16 and 17 reveal that the error variation with external frequency is approximately the same for both the Guyan-Irons and Downs-Fricker condensation when the modified cubic solutions are employed. However, the fundamental difference between the two types of condensation is again evident in these two figures. When the external frequency approaches the natural frequency of vibration, resonance is encountered. At the first resonance ($\omega/\omega_n = 1$), the error is finite and relatively small for the Guyan-Irons condensation and infinite for the Downs-Fricker condensation. As well, at the second resonance ($\omega/\omega_n = 4$), a significant jump in error is still observed in the Downs-Fricker case. The error will not be infinite due to computational error. This is in contrast to the Guyan-Irons case where the error variation is continuous through that point. The discontinuity at the resonance points in Figure 17 illustrates the ability of the Downs-Fricker condensation to retain the inertial characteristics of the dynamic problem.

The error does not get large at resonance for the Guyan-Irons condensation because the inertial forces are completely neglected. The error variation in the figure is due to the influence of the inertial forces in the formulation, instead of in the condensation. This trend is also evident in the Downs-Fricker condensation. At a frequency of zero, the static case, the solution is exact in both cases because there are no inertial forces.

The Guyan-Irons condensation is immediately applicable to the free vibration problem because it is frequency independent. However, recall that the quartic to cubic reductions results in matrices that are the same as those associated with the regular cubic shape function. This leads to the result that natural frequencies obtained from the reduced Guyan-Irons matrices are the same as those provided by the regular cubic function.

Comparisons of CPU time between regular and modified solutions show that they are approximately equal. This is due to the small number of degrees of freedom used in the solutions. In complex problems where the degrees of freedom are large, the Downs-Fricker condensation should be just as effective as the Guyan-Irons condensation in reducing degrees of freedom. As a quantitative example, consider a hypothetical case where fifty elements is used to discretize the beam. Using the quintic to cubic reduction, the degrees of freedom will be reduced from one hundred and fifty-one to one hundred, a reduction of approximately thirty three percent.

Chapter V

CONCLUSIONS AND RECOMMENDATIONS

5.1 CONCLUSIONS

Condensation procedures, which previously have been based almost entirely on the Guyan-Irons technique, have concentrated mainly on reducing degrees of freedom. However, these procedures have never been able to maintain the accuracy corresponding to the unreduced system. An improved condensation procedure, named the Downs-Fricker procedure, is presented in this thesis; it overcomes the approximations inherent in the traditional Guyan-Irons approach under certain conditions. The approximations are eliminated upon inclusion of the inertial forces in the condensation.

In view of the results obtained from steady-state forced vibrations of a simply supported beam, the following conclusions can be reached:

1. The Downs-Fricker condensation achieves the dual objectives of reducing degrees of freedom as well as maintaining the accuracy corresponding to the unreduced system. The Guyan-Irons condensation reduces degrees of freedom but cannot maintain the accuracy corresponding to the unreduced system.

2. The accuracy of the solution from the Down-Fricker condensation is independent of the level of the element equation reduction whereas only the quartic to cubic reduction is valid for the Guyan- Irons condensation.
3. Error variation with external frequency shows that the Downs-Fricker condensation in much better at recognizing resonance than the Guyan- Irons condensation.
4. The attractiveness of having corresponding regular and reduced matrices the same, as in the quartic to cubic reduction in the static [22] and Guyan-Irons [25] condensation, is not available for the Downs-Fricker case. This requires the implementation of a routine to reduce matrices in programs incorporating the condensation procedure.
5. Solutions employing beam elements with purely kinematic degrees of freedom give more accurate solutions than those with a mixture of kinematic and force (curvature) degrees of freedom. This is in agreement with the conclusions of Popplewell and McDonald [24].

5.2 RECOMMENDATIONS FOR FURTHER WORK

While performing the research for this thesis, the following topics were identified for possible, further study.

1. Since only internal degrees of freedom are eliminated and the assembly procedure remains unchanged, it is recommended that the Downs-Fricker condensation procedure be used on beam problems with other types of end conditions.
2. It is suggested that the Downs-Fricker condensation procedure be applied to problems employing two and three dimensional finite elements by incorporating internal degrees of freedom.
3. It is suggested that the Downs-Fricker condensation used in this thesis be applied to more complicated problems where the degrees of freedom are greater. Quantitative results on the reduction of computational effort is needed. The CPU time is a popular indicator of computational effort.
4. In this thesis only the standard formulation is used. The effect of including higher order terms in the dynamic element stiffness matrix (equation 1.1) could be investigated. As mentioned previously, some research has already been done in the context of free vibration.
5. It is suggested that the applicability of the Downs-Fricker condensation be investigated in other forced

vibration problems, for example, the analysis of flexible mechanisms [26]. In this case, the external force corresponds to inertial loads which can be linearly varying instead of constant.

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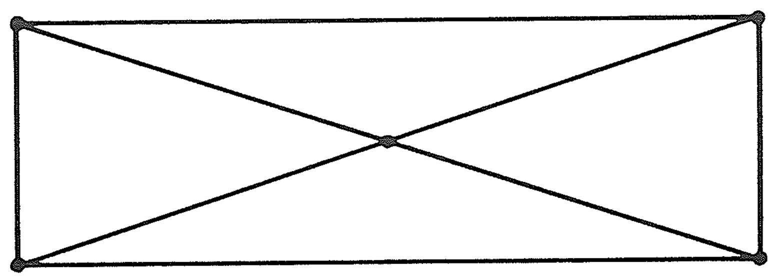

Figure 1: Quadrilateral finite element

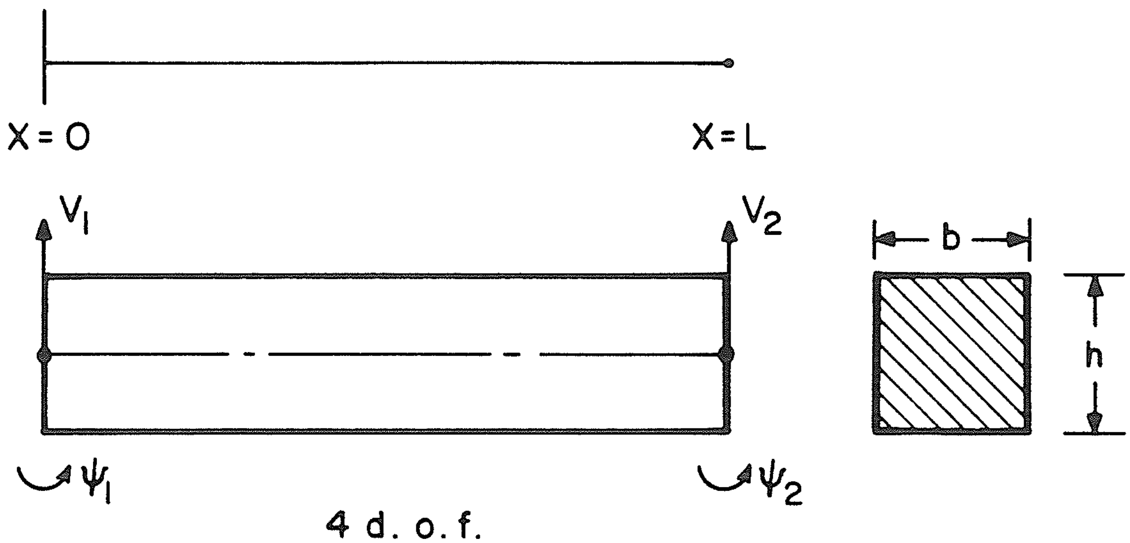

Figure 2: Cubic beam element

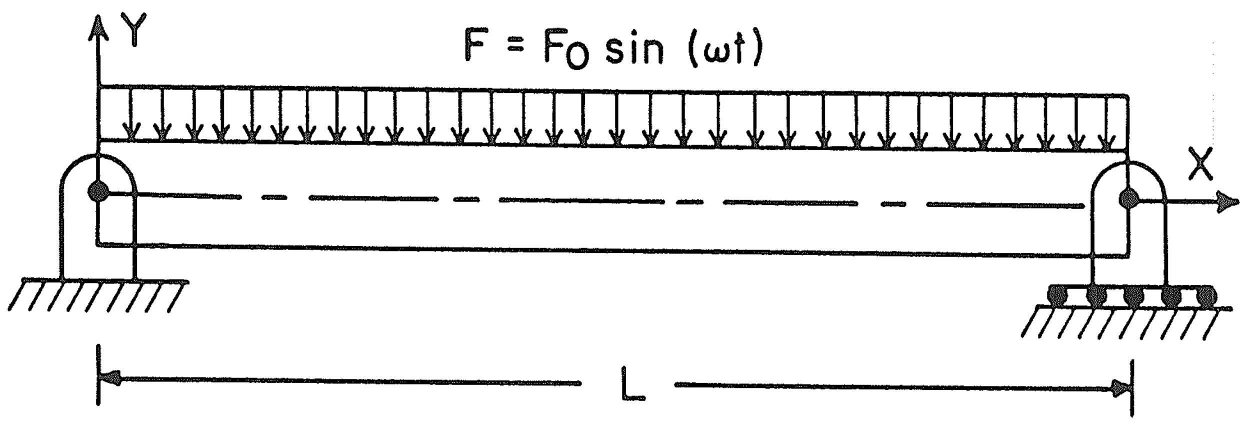

Figure 3: Simply supported beam with uniformly distributed forcing function

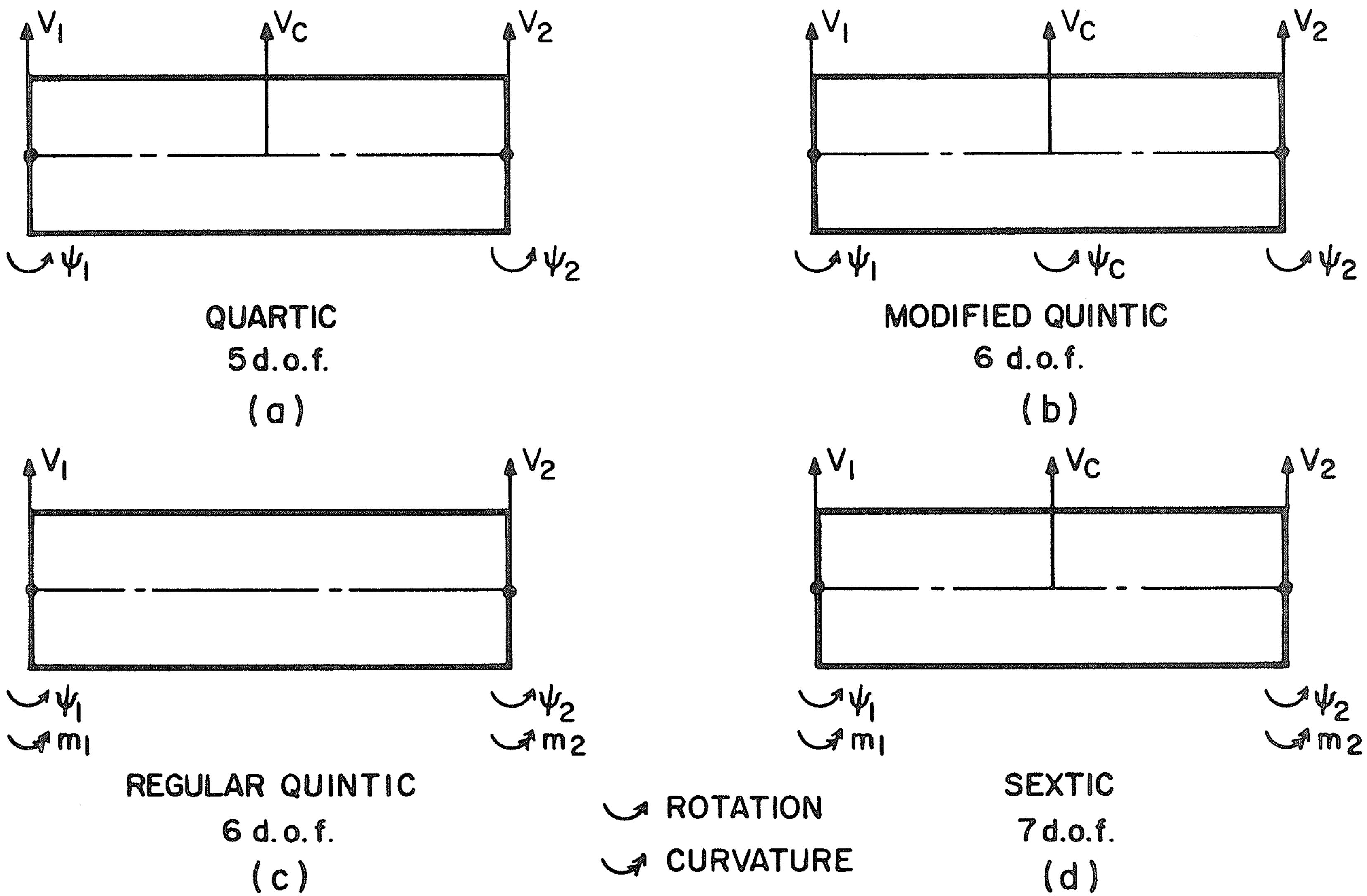

Figure 4: Higher degree beam elements

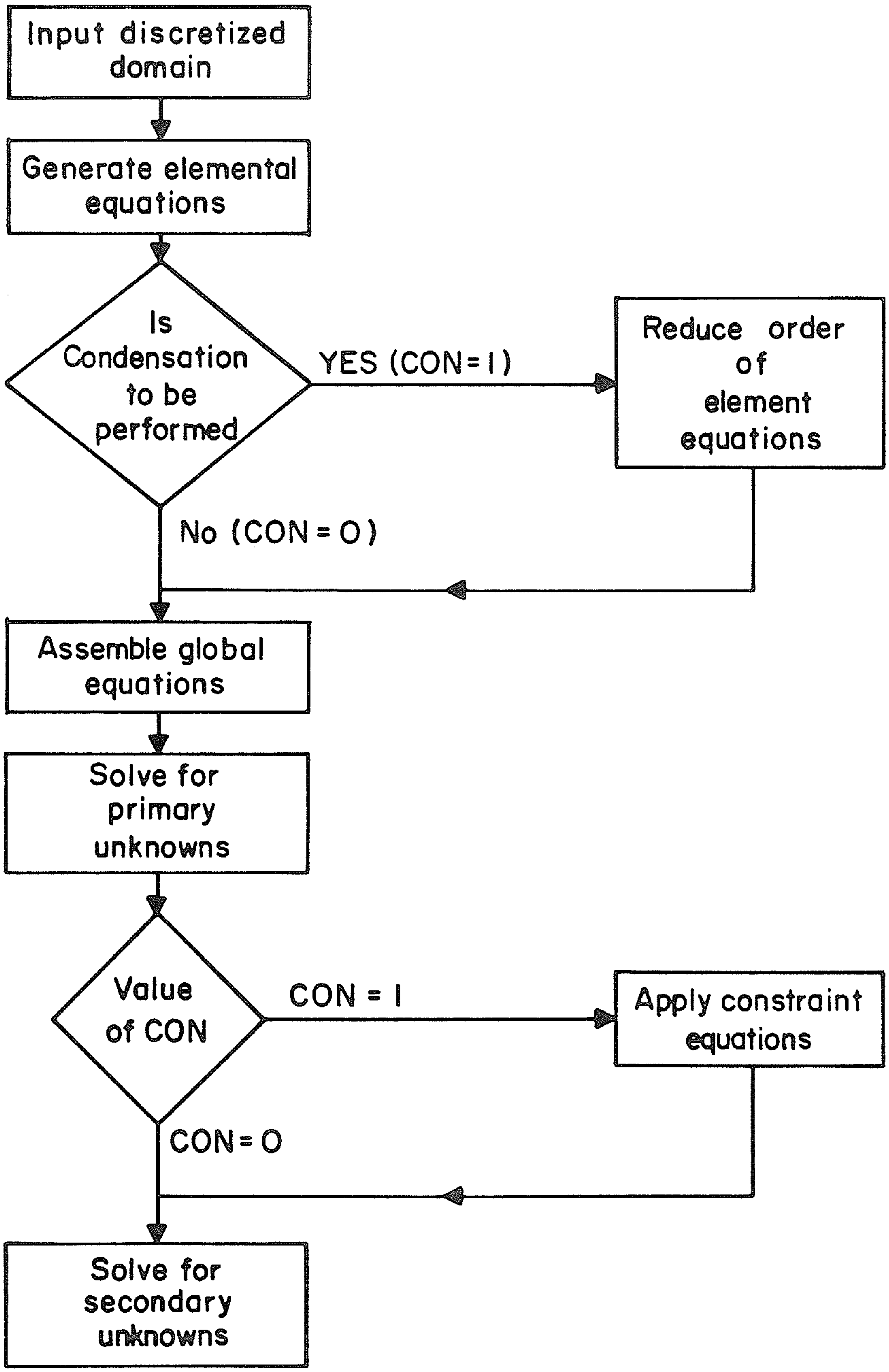

Figure 5: Illustration of the incorporation of condensation in conventional finite element programs

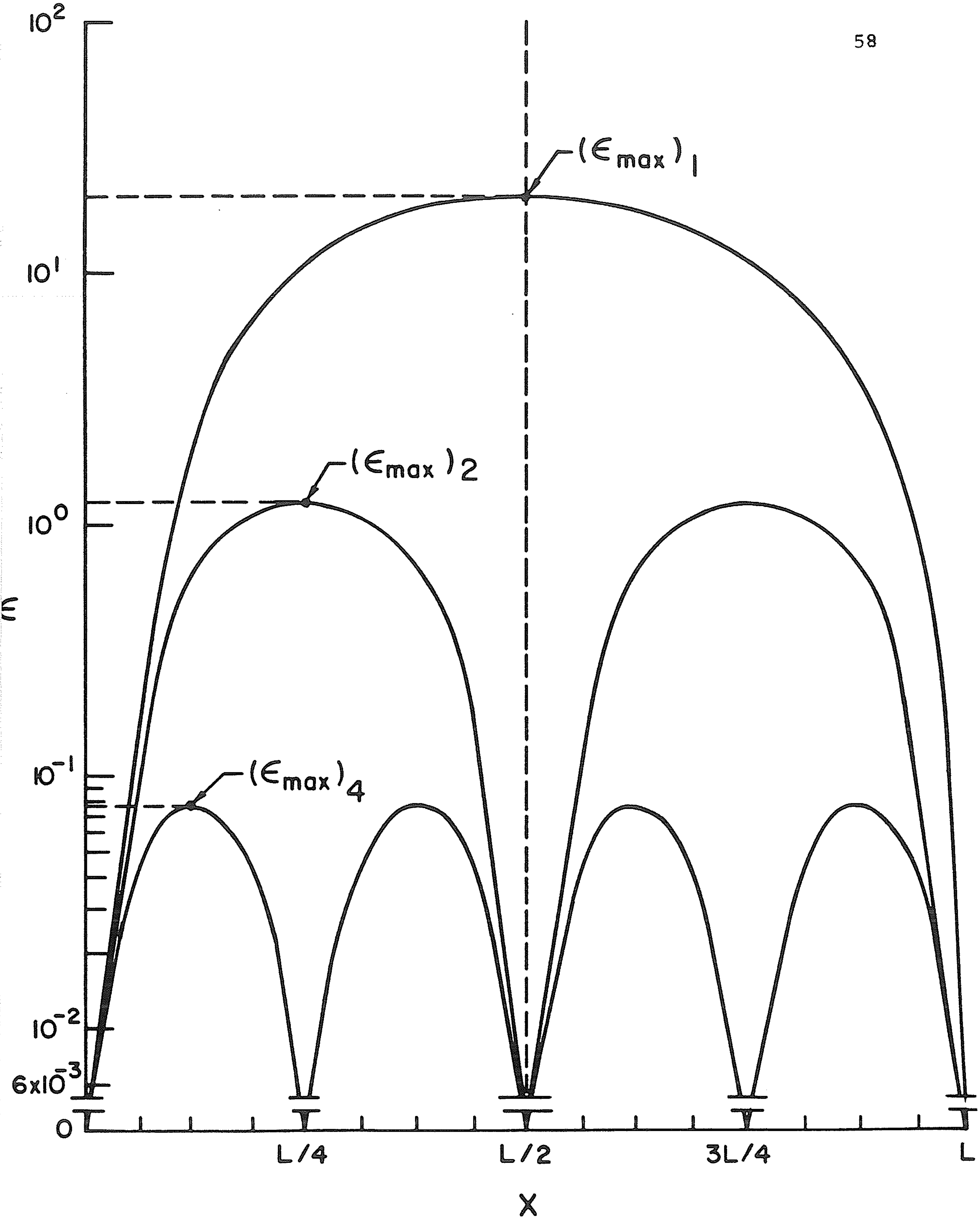

Figure 6 : Error distribution of deflections throughout the beam for the static case using the cubic shape function

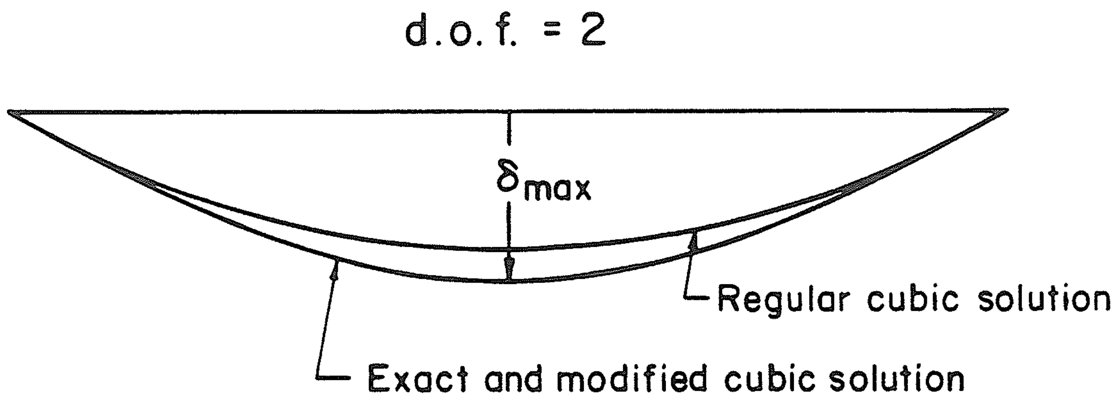

Figure 7: Deflected beam configuration from the exact and finite element solutions for the static case

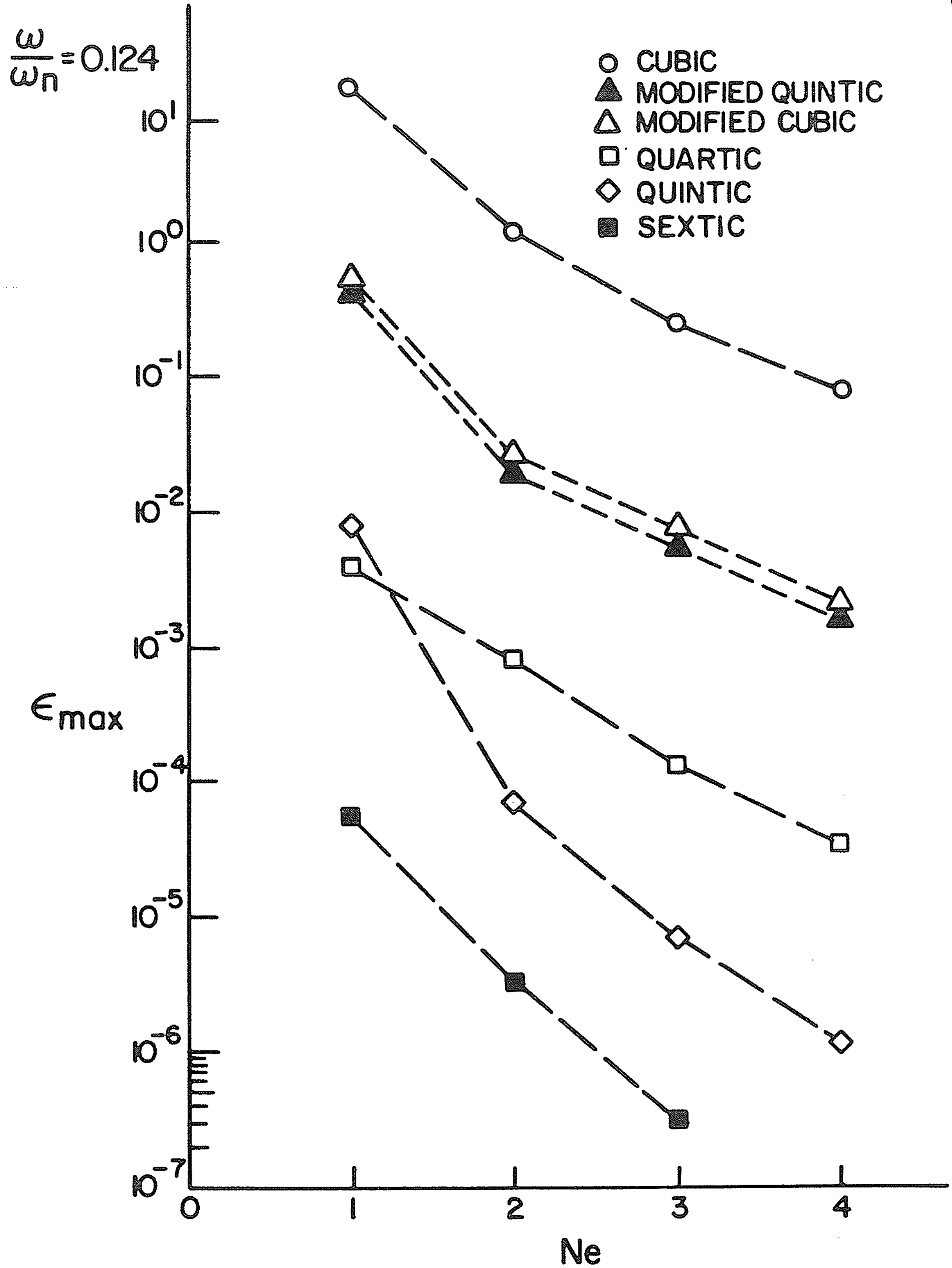

Figure 8: Maximum deflection error of regular and modified (Guyan-Irons) finite element solutions versus the number of elements

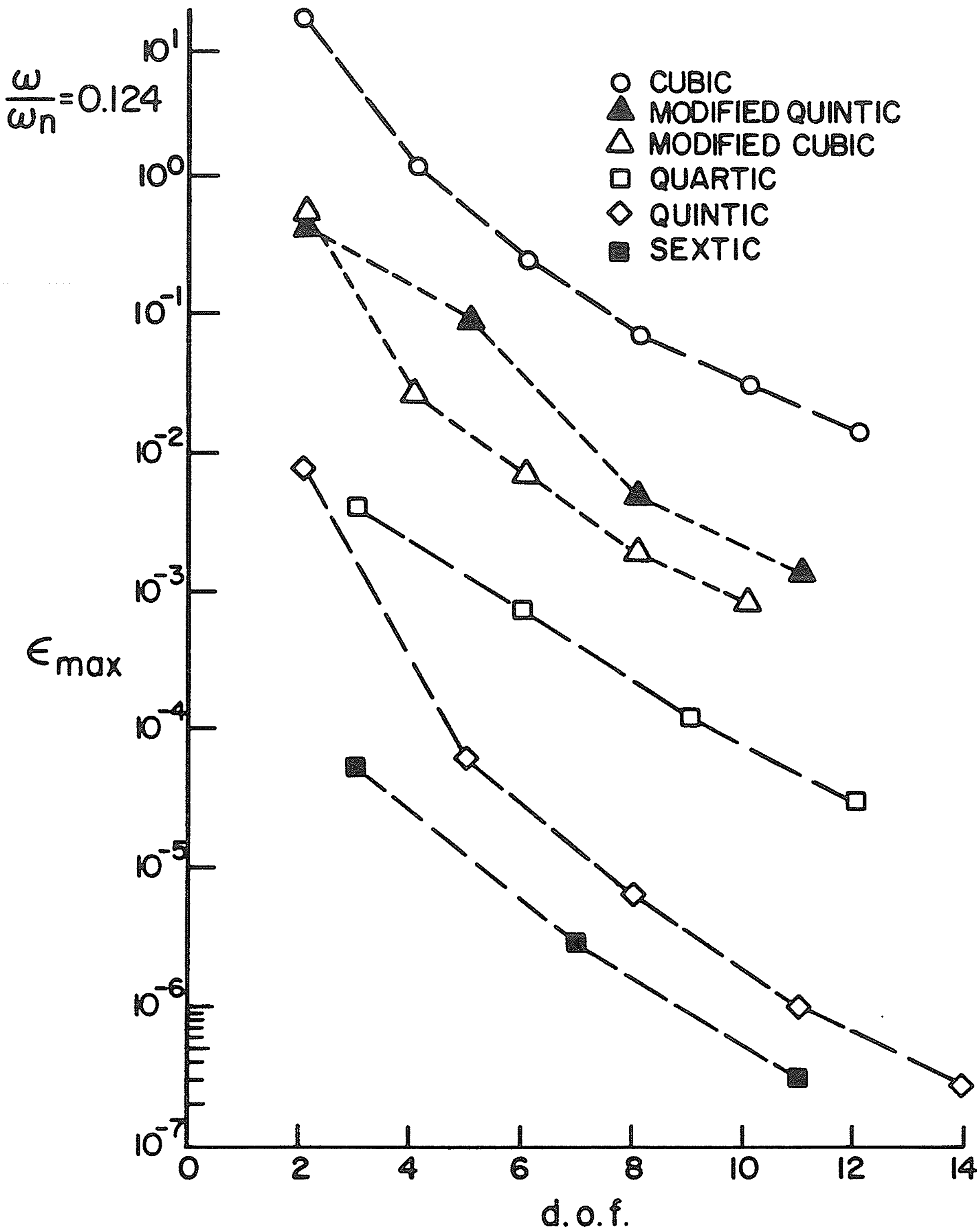

Figure 9 : Maximum deflection error of regular and modified (Guyan-Irons) finite element solutions versus the degrees of freedom

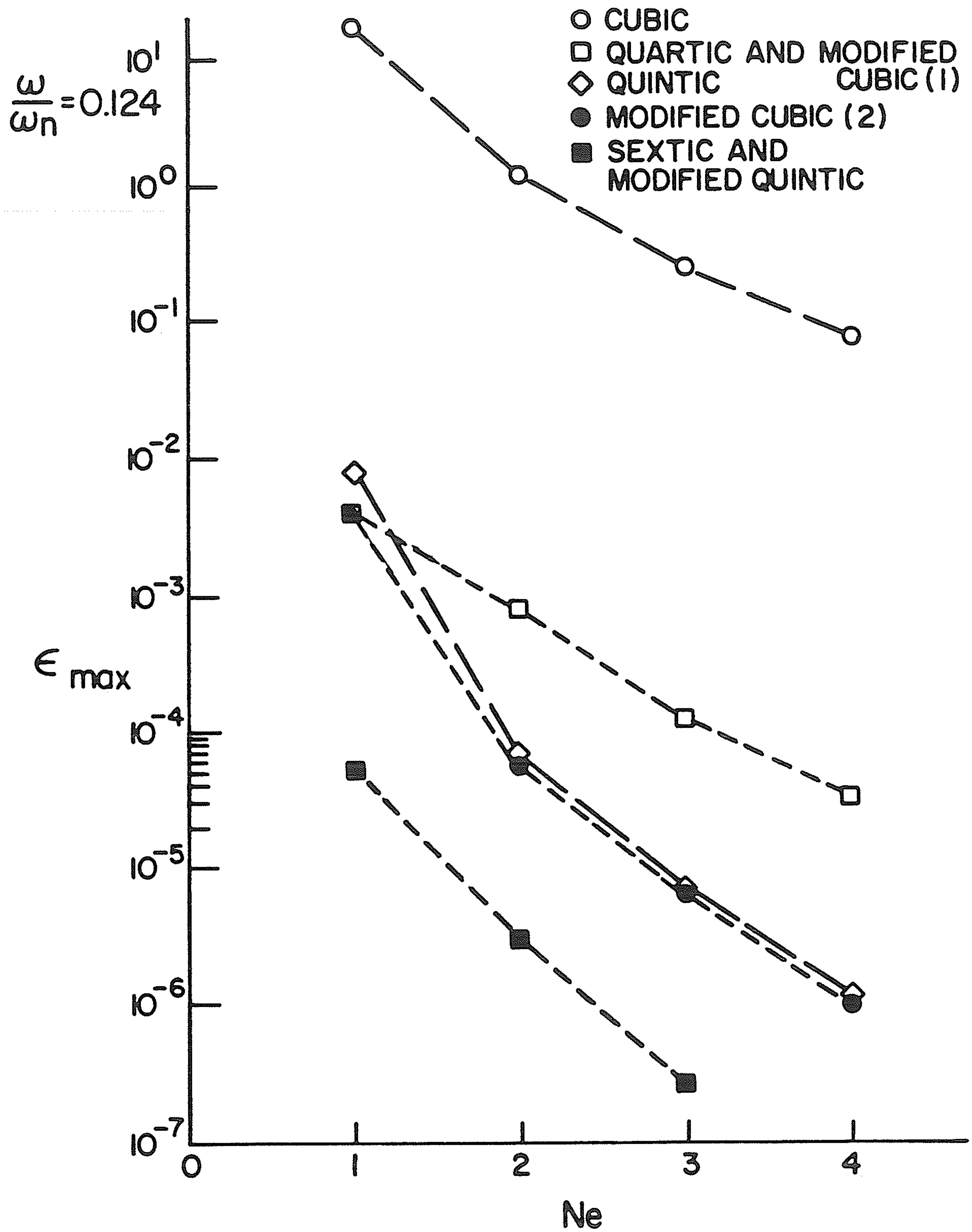

Figure 10 : Maximum deflection error of regular and modified (Downs-Fricker) finite element solutions versus the number of elements

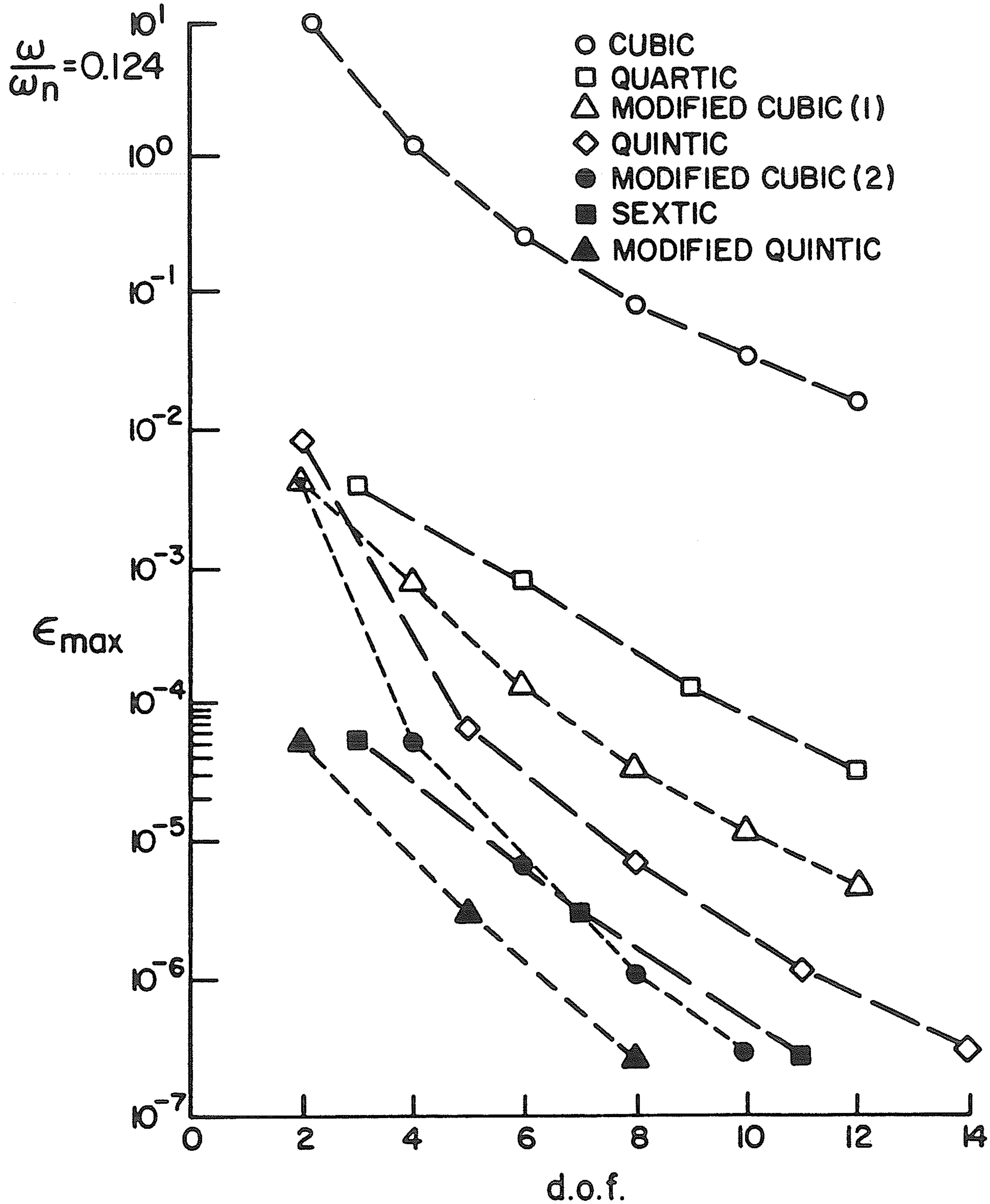

Figure 11: Maximum deflection error of regular and modified (Downs-Fricker) finite element solutions versus the degrees of freedom

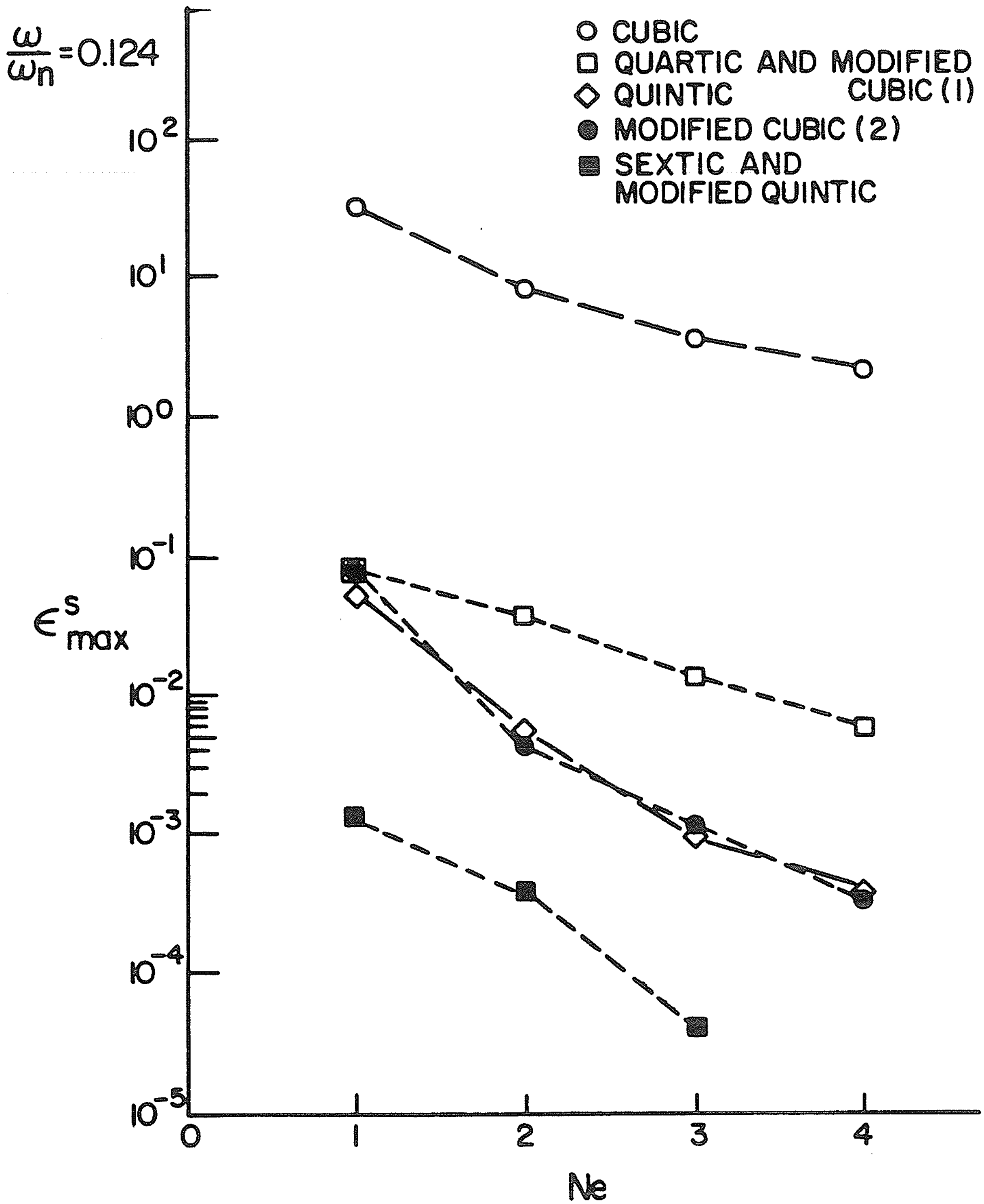

Figure 12: Maximum stress error of regular and modified (Downs-Fricker) finite element solutions versus the number of elements

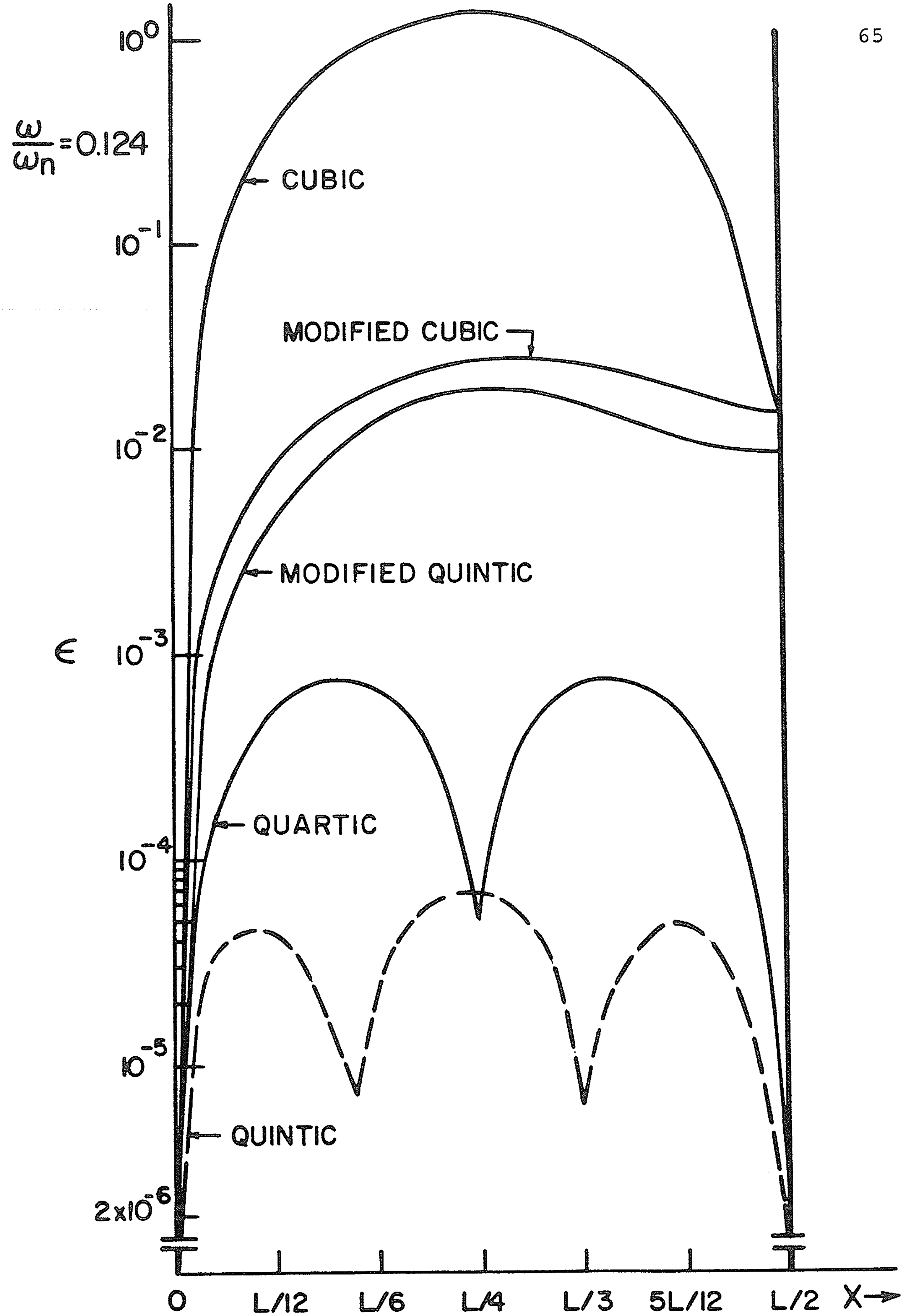

Figure 13: Error distribution of deflections of regular and modified (Guyan-Irons) finite element solutions over the beam length using two elements

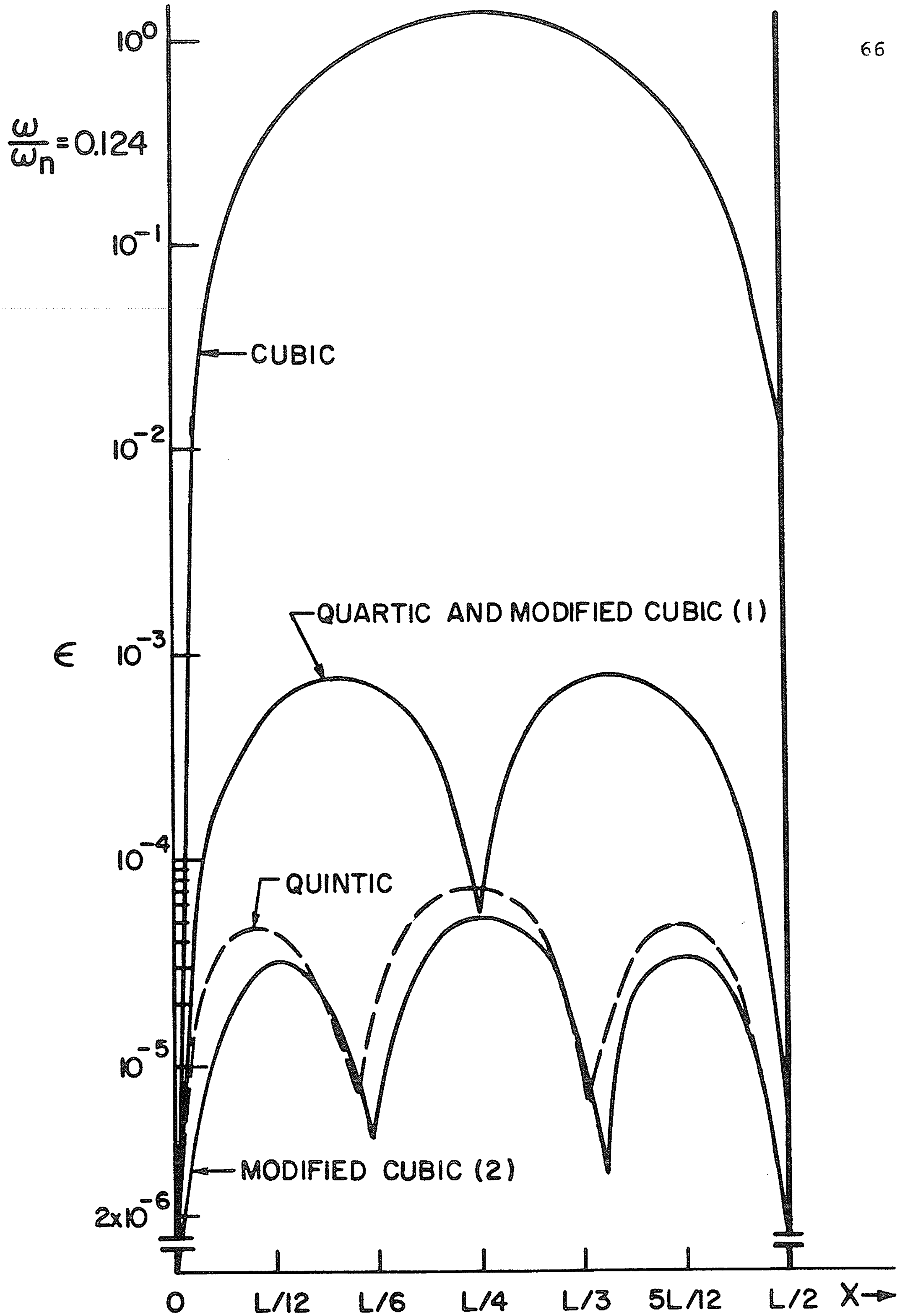

Figure 14 : Error distribution of deflections of regular and modified (Downs-Fricker) finite element solutions over the beam length using two elements

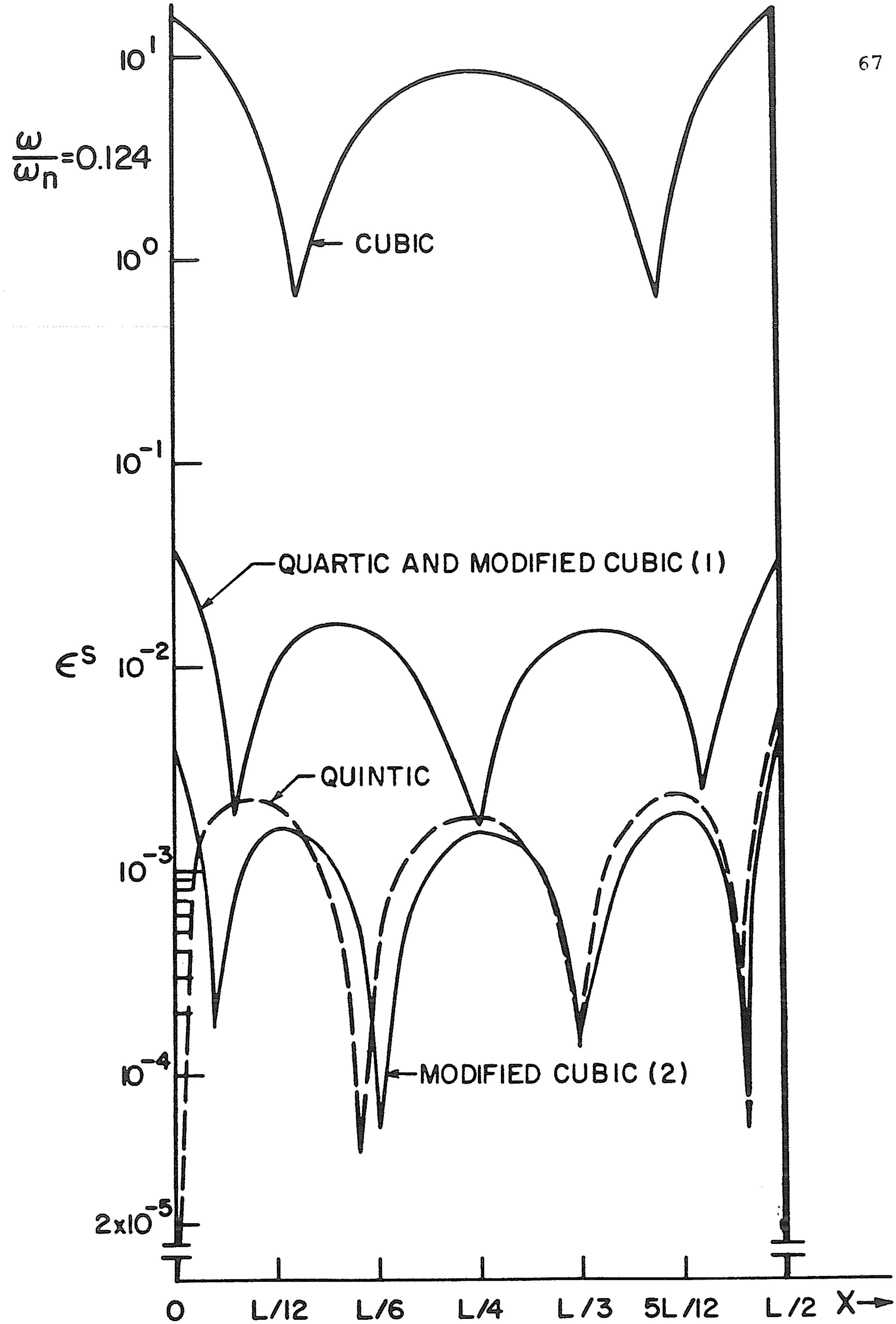

Figure 15: Error distribution of stress of regular and modified (Downs-Fricker) finite element solutions over the beam length using two elements

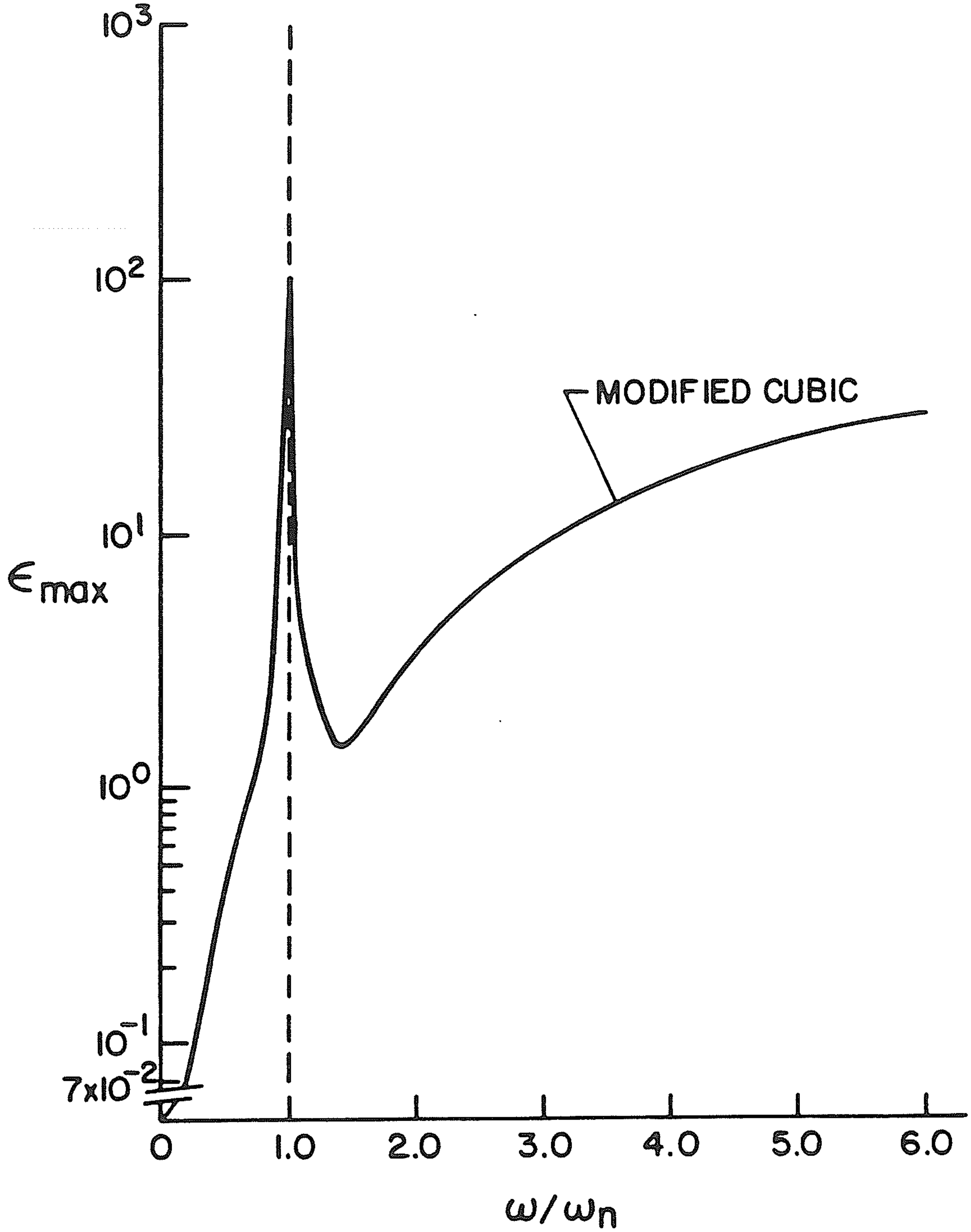

Figure 16 : Maximum deflection error versus the normalized external frequency using the Guyan-Irons condensation with two elements

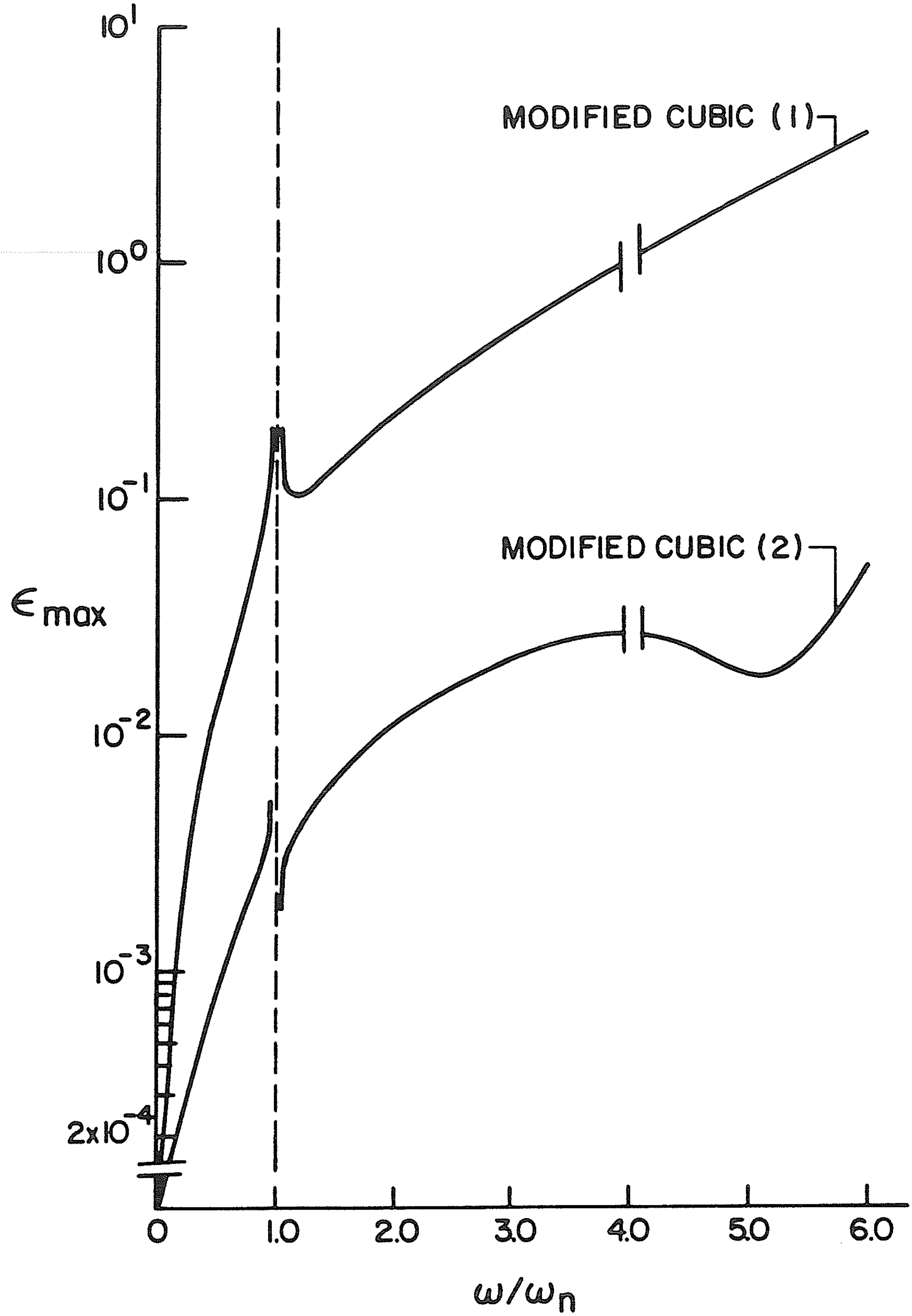

Figure 17 : Maximum deflection error versus the normalized external frequency using the Downs-Fricker condensation with two elements

Appendix A

ELEMENT MATRICES AND VECTORS

The matrices and vectors presented in this appendix have all been derived similarly to the cubic element matrices and vector, in Chapter 2.

$$
\{K_e\} = EI \begin{bmatrix}
\frac{316}{5}\frac{1}{L^3} & \frac{94}{5}\frac{1}{L^2} & \frac{-512}{5}\frac{1}{L^3} & \frac{196}{5}\frac{1}{L^3} & \frac{-34}{5}\frac{1}{L^2} \\
 & \frac{36}{5}\frac{1}{L} & \frac{-128}{5}\frac{1}{L^2} & \frac{34}{5}\frac{1}{L^2} & \frac{-6}{5}\frac{1}{L} \\
 & & \frac{1024}{5}\frac{1}{L^3} & \frac{-512}{5}\frac{1}{L^3} & \frac{128}{5}\frac{1}{L^2} \\
 & \text{SYMMETRIC} & & \frac{316}{5}\frac{1}{L^3} & \frac{-94}{5}\frac{1}{L^2} \\
 & & & & \frac{36}{5}\frac{1}{L}
\end{bmatrix}
$$

Figure A1(a) Stiffness matrix for the quartic element

$$
\{M_e\} = \rho A \begin{bmatrix}
\frac{13}{63} L & \frac{1}{63} L^2 & \frac{4}{63} L & \frac{-23}{630} L & \frac{1}{180} L^2 \\
 & \frac{1}{630} L^3 & \frac{4}{630} L^2 & \frac{-1}{180} L^2 & \frac{1}{1260} L^3 \\
 & & \frac{128}{315} L & \frac{4}{63} L & \frac{-4}{630} L^2 \\
 & \text{SYMMETRIC} & & \frac{13}{63} L & \frac{-1}{63} L^2 \\
 & & & & \frac{1}{630} L^3
\end{bmatrix}
$$

Figure A1(b) Mass matrix for the quartic element

$$
\left[K_e\right] = EI \begin{bmatrix}
\frac{120}{7}\frac{1}{L^3} & \frac{60}{7}\frac{1}{L^2} & \frac{3}{7}\frac{1}{L} & -\frac{120}{7}\frac{1}{L^3} & \frac{60}{7}\frac{1}{L^2} & -\frac{3}{7}\frac{1}{L} \\
 & \frac{192}{35}\frac{1}{L} & \frac{22}{70} & -\frac{60}{7}\frac{1}{L^2} & \frac{108}{35}\frac{1}{L} & -\frac{4}{35} \\
 & & \frac{6}{70}L & -\frac{3}{7}\frac{1}{L} & \frac{4}{35} & \frac{1}{70}L \\
 & & & \frac{120}{7}\frac{1}{L^3} & -\frac{60}{7}\frac{1}{L^2} & \frac{3}{7}\frac{1}{L} \\
 & \text{SYMMETRIC} & & & \frac{192}{35}\frac{1}{L} & -\frac{22}{70} \\
 & & & & & \frac{6}{70}L
\end{bmatrix}
$$

Figure A2: Stiffness matrix for the regular quintic element

$$
\left[M_e\right] = \rho A \begin{bmatrix}
\frac{181}{462} L & \frac{311}{4620} L^2 & \frac{281}{55440} L^3 & \frac{25}{231} L & \frac{-151}{4620} L^2 & \frac{181}{55440} L^3 \\
 & \frac{52}{3465} L^3 & \frac{23}{18480} L^4 & \frac{151}{4620} L^2 & \frac{-19}{1980} L^3 & \frac{13}{13860} L^4 \\
 & & \frac{1}{9240} L^5 & \frac{181}{55440} L^3 & \frac{-13}{13860} L^4 & \frac{1}{11088} L^5 \\
 & & & \frac{181}{462} L & \frac{-311}{4620} L^2 & \frac{281}{55440} L^3 \\
 & \text{SYMMETRIC} & & & \frac{52}{3465} L^3 & \frac{-23}{18480} L^4 \\
 & & & & & \frac{1}{9240} L^5
\end{bmatrix}
$$

Figure A3: Mass matrix for the regular quintic element

$$
\left[K_e\right] = EI \begin{bmatrix}
\frac{5092}{35}\frac{1}{L^3} & \frac{1138}{35}\frac{1}{L^2} & -\frac{512}{5}\frac{1}{L^3} & \frac{384}{7}\frac{1}{L^2} & -\frac{1508}{35}\frac{1}{L^3} & \frac{242}{35}\frac{1}{L^2} \\
 & \frac{332}{35}\frac{1}{L} & -\frac{128}{5}\frac{1}{L^2} & \frac{64}{7}\frac{1}{L} & -\frac{242}{35}\frac{1}{L^2} & \frac{38}{35}\frac{1}{L} \\
 & & \frac{1024}{5}\frac{1}{L^3} & 0 & -\frac{512}{5}\frac{1}{L^3} & 128\,\frac{1}{L^2} \\
 & & & \frac{256}{7}\frac{1}{L} & -\frac{384}{7}\frac{1}{L^2} & \frac{64}{7}\frac{1}{L} \\
 & \text{SYMMETRIC} & & & \frac{5092}{35}\frac{1}{L^3} & -\frac{1138}{35}\frac{1}{L^2} \\
 & & & & & \frac{332}{35}\frac{1}{L}
\end{bmatrix}
$$

Figure A4: Stiffness matrix for the modified quintic element

$$\left[M_e\right] = \rho A \begin{bmatrix} \frac{523}{3465} L & \frac{19}{2310} L^2 & \frac{4}{63} L & \frac{-8}{693} L^2 & \frac{131}{6930} L & \frac{-29}{13860} L^2 \\ & \frac{2}{3465} L^3 & \frac{2}{315} L^2 & \frac{-1}{1155} L^3 & \frac{29}{13860} L^2 & \frac{-1}{4620} L^3 \\ & & \frac{128}{315} L & 0 & \frac{4}{63} L & \frac{-2}{315} L^2 \\ & & & \frac{32}{3465} L^3 & \frac{8}{693} L^2 & \frac{-1}{1155} L^3 \\ & \text{SYMMETRIC} & & & \frac{523}{3465} L & \frac{-19}{2310} L^2 \\ & & & & & \frac{2}{3465} L^3 \end{bmatrix}$$

Figure A5: Mass matrix for the modified quintic element

$$
\left[K_e\right] = EI \begin{bmatrix}
\frac{2648}{35}\frac{1}{L^3} & \frac{844}{35}\frac{1}{L^2} & \frac{31}{35}\frac{1}{L} & \frac{-4096}{35}\frac{1}{L^3} & \frac{1448}{35}\frac{1}{L^3} & \frac{-244}{35}\frac{1}{L^2} & \frac{1}{35}\frac{1}{L} \\
 & \frac{332}{35}\frac{1}{L} & \frac{13}{35} & \frac{-1088}{35}\frac{1}{L^2} & \frac{244}{35}\frac{1}{L^2} & \frac{-32}{35}\frac{1}{L} & \frac{-2}{35} \\
 & & \frac{2}{35}L & \frac{-32}{35}\frac{1}{L} & \frac{1}{35}\frac{1}{L} & \frac{2}{35} & \frac{-1}{70}L \\
 & & & \frac{8192}{35}\frac{1}{L^3} & \frac{-4096}{35}\frac{1}{L^3} & \frac{1088}{35}\frac{1}{L^2} & \frac{-32}{35}\frac{1}{L} \\
 & \text{SYMMETRIC} & & & \frac{2648}{35}\frac{1}{L^3} & \frac{-844}{35}\frac{1}{L^2} & \frac{31}{35}\frac{1}{L} \\
 & & & & & \frac{332}{35}\frac{1}{L} & \frac{-13}{35} \\
 & & & & & & \frac{2}{35}L
\end{bmatrix}
$$

Figure A6: Stiffness matrix for the sextic element

$$
\left[M_e\right] = \rho A \begin{bmatrix}
\frac{2487}{10010} L & \frac{271}{10010} L^2 & \frac{919}{720720} L^3 & \frac{872}{15015} L & \frac{-527}{15015} L & \frac{227}{30030} L^2 & \frac{-127}{240240} L^3 \\
 & \frac{347}{90090} L^3 & \frac{29}{144144} L^4 & \frac{136}{15015} L^2 & \frac{-227}{30030} L^2 & \frac{281}{180180} L^3 & \frac{-19}{180180} L^4 \\
 & & \frac{1}{90090} L^5 & \frac{4}{9009} L^3 & \frac{-127}{240240} L^3 & \frac{19}{180180} L^4 & \frac{-1}{144144} L^5 \\
 & & & \frac{1024}{3003} L & \frac{872}{15015} L & \frac{-136}{15015} L^2 & \frac{4}{9009} L^3 \\
 & \text{SYMMETRIC} & & & \frac{2487}{10010} L & \frac{-271}{10010} L^2 & \frac{919}{720720} L^3 \\
 & & & & & \frac{347}{90090} L^3 & \frac{-29}{144144} L^4 \\
 & & & & & & \frac{1}{90090} L^5
\end{bmatrix}
$$

Figure A7: Mass matrix for the sextic element

$$\{q\}^T = F \left(\frac{7L}{30} \quad \frac{L^2}{60} \quad \frac{8L}{15} \quad \frac{7L}{30} \quad \frac{-L^2}{60} \right)$$

(a) Quartic element

$$\{q\}^T = F \left(\frac{L}{2} \quad \frac{L^2}{10} \quad \frac{L^3}{120} \quad \frac{L}{2} \quad \frac{-L^2}{10} \quad \frac{L^3}{120} \right)$$

(b) Regular quintic element

$$\{q\}^T = F \left(\frac{7L}{30} \quad \frac{L^2}{60} \quad \frac{8L}{15} \quad 0 \quad \frac{7L}{30} \quad \frac{-L^2}{60} \right)$$

(c) Modified quintic element

$$\{q\}^T = F \left(\frac{19L}{70} \quad \frac{L^2}{35} \quad \frac{L^3}{840} \quad \frac{16L}{35} \quad \frac{19L}{70} \quad \frac{-L^2}{35} \quad \frac{L^3}{840} \right)$$

(d) Sextic element

Figure A8: Load vectors for higher degree beam elements

$$
\left[K_e\right] = EI \begin{bmatrix}
\frac{120}{7}\frac{1}{L^3} & \frac{60}{7}\frac{1}{L^2} & \frac{3}{7}\frac{1}{L} & \frac{-120}{7}\frac{1}{L^3} & \frac{60}{7}\frac{1}{L^2} & \frac{-3}{7}\frac{1}{L} \\
 & \frac{75}{14}\frac{1}{L} & \frac{1}{4} & \frac{-60}{7}\frac{1}{L^2} & \frac{45}{14}\frac{1}{L} & \frac{-5}{28} \\
 & & \frac{3}{56}L & \frac{-3}{7}\frac{1}{L} & \frac{5}{28} & \frac{-1}{56}L \\
 & & & \frac{120}{7}\frac{1}{L^3} & \frac{-60}{7}\frac{1}{L^2} & \frac{3}{7}\frac{1}{L} \\
 & \text{SYMMETRIC} & & & \frac{75}{14}\frac{1}{L} & \frac{-1}{4} \\
 & & & & & \frac{3}{56}L
\end{bmatrix}
$$

Figure A9: Stiffness matrix obtained from the sextic to quintic reduction

$$\left[M_e\right] = \rho A \begin{bmatrix} \frac{181}{462} L & \frac{229}{3696} L^2 & \frac{53}{22176} L^3 & \frac{25}{231} L & \frac{-101}{3696} L^2 & \frac{13}{22176} L^3 \\ & \frac{1769}{144144} L^3 & \frac{227}{480480} L^4 & \frac{101}{3696} L^2 & \frac{-989}{144144} L^3 & \frac{239}{1441440} L^4 \\ & & \frac{19}{960960} L^5 & \frac{13}{22176} L^3 & \frac{-239}{1441440} L^4 & \frac{1}{576576} L^5 \\ & & & \frac{181}{462} L & \frac{-229}{3696} L^2 & \frac{53}{22176} L^3 \\ & \text{SYMMETRIC} & & & \frac{1769}{144144} L^3 & \frac{-227}{480480} L^4 \\ & & & & & \frac{19}{960960} L^5 \end{bmatrix}$$

Figure A10: Mass matrix obtained from the sextic to quintic reduction

Appendix B

DIMENSIONS OF PARAMETERS

$\rho = 0.28 \text{ lbm/in}^3$

$b = 0.25 \text{ ins}$

$h = 0.50 \text{ ins}$

$L = 60.0 \text{ ins}$

$F_0 = 5 \text{ lbs}$

$E = 30.0 \times 10^6 \text{ lb/in}^2$

$\delta_{max} = 10.97 \text{ ins}$

$\omega_n = 80.47 \text{ rad/s}$

Appendix C

THE COMPUTER PROGRAMS

The computer programs developed and used to obtain the results of Chapter V are written in the WATFIV programming language using double precision arithmetic. The AMDAHL 580 mainframe computer at the University of Manitoba is used to perform all computations.

Figure C1 shows a chart that gives a breakdown of the different versions of finite element programs used to generate the results. In each version of the programs, solutions corresponding to different degree of shape functions can be obtained by inserting the appropriate element matrices and redimensioning the element matrices and loop counters. The range of shape functions used are from cubic to sextic.

Examples of all versions of the programs are given in the following listings. The quartic polynomial shape function is used in all examples. This means that in the modified versions, the quartic beam element will be reduced to the cubic level before the global equations are assembled. However, for the static and Guyan-Irons condensations, the cubic element equations are substituted directly because the reduction produces equations which are the same as the cubic equations.

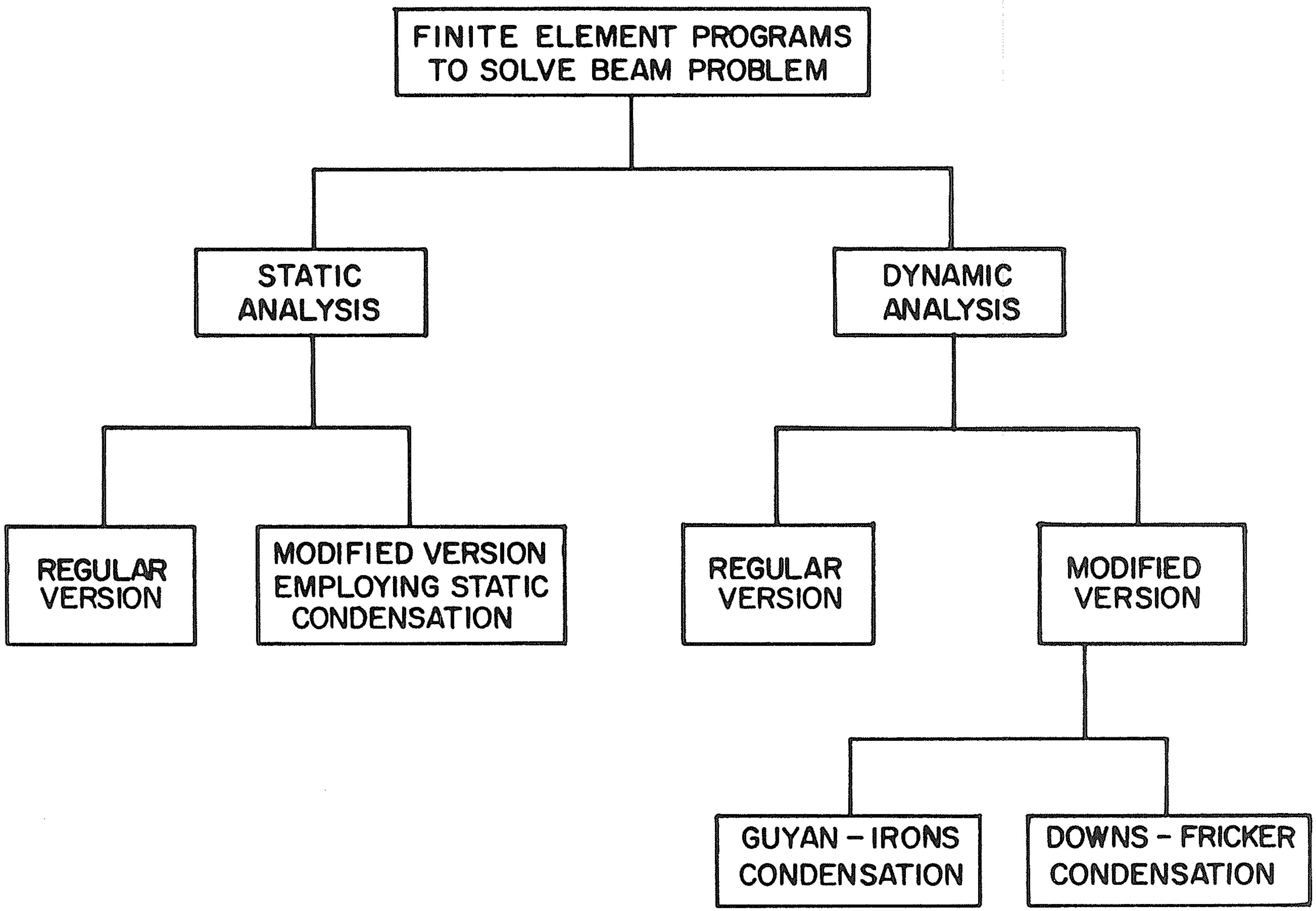

Figure C1: Chart breakdown of the versions of finite element programs used

```
$JOB WATFIV SURENDRA-RAJPAL,NOEXT,NOWARN
C ***********************************************************************
C *                                                                     *
C *         FINITE ELEMENT SOLUTION OF STATIC BEAM PROBLEM              *
C *                  FOURTH ORDER SOLUTION                              *
C *                                                                     *
C ***********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION WKAREA(200),NCON(10,7)
      DIMENSION GK(2,2),EK(7,7),GQ(2,1),EQ(7),E(10),B(10),H(10),
     &          XL(10),QBAR(10),WBAR(10),RHOV(10),EM(7,7)
      INTEGER NCON/70*0/,NOP/0/
      COMMON Y1,Y2,Y3,Y4,Y5,Y6,Y7,CC1,CC2,CC3,CC4,CC5,CC6,CC7
      PI=.3141592653589793D+01
C
C     READ DIMENSION OF ELEMENT EQUATIONS
C
      READ, N
C
C     READ OPTION FOR REGULAR (0) OR MODIFIED (1) SOLUTION
C
      READ, NOP
C
C     READ NUMBER OF ELEMENTS AND NUMBER OF DEGREES OF FREEDOM
C
      READ, NE,NTF
C
C     READ CONNECTION MATRIX (BOUNDARY CONSTRAINTS) AND ELEMENT
C     PROPERTIES
C
      TL=0.0D0
      DO 20 I=1,NE
        READ, (NCON(I,J),J=1,N),E(I),RHOV(I),B(I),H(I),XL(I),WBAR(I)
     &           ,QBAR(I)
        RHOV(I)=RHOV(I)/386.0D0
        TL=TL+XL(I)
   20 E(I)=E(I)*1.0D6
C
C     PRINT CONNECTION MATRIX
C
      PRINT 602
      PRINT 698
      PRINT 684
      DO 81 I=1,NE
   81 PRINT 644,I,(NCON(I,J),J=1,N)
C
C     INITIALIZE GLOBAL STIFFNESS MATRIX AND LOAD VECTOR
C
      DO 26 I=1,NTF
        DO 27 J=1,NTF
   27   GK(I,J)=0.0D0
   26 GQ(I,1)=0.0D0
C
C     COMBINE ELEMENT MATRICES INTO GLOBAL MATRICES
C
      DO 36 K=1,NE
        PRINT 609,K
        PRINT 610,K,E(K)
        PRINT 611,K,B(K)
        PRINT 612,K,H(K)
        PRINT 613,K,XL(K)
        PRINT 614,K,QBAR(K)
C
C       SUBROUTINE ELMT SUPPLIES VALUES OF ELEMENT MATRICES
```

```
C
      CALL ELMT(EM,EK,EQ,E(K),RHOV(K),B(K),H(K),XL(K),QBAR(K),
     &          WBAR(K))
C
C     PRINT ELEMENT STIFFNESS MATRIX AND LOAD VECTOR
C
      PRINT 604
      PRINT 620
      PRINT 604
      DO 9 I=1,N
    9 PRINT 681,(EK(I,J),J=1,N),EQ(I)
      PRINT 602
      DO 36 I=1,N
        LX=NCON(K,I)
        IF(LX.NE.0)GQ(LX,1)=GQ(LX,1)+EQ(I)
        DO 36 J=1,N
          LY=NCON(K,J)
          IF(LX.EQ.0.OR.LY.EQ.0)GO TO 36
          GK(LX,LY)=GK(LX,LY)+EK(I,J)
   36 CONTINUE
C
C    PRINT GLOBAL STIFFNESS MATRIX AND LOAD VECTOR
C
     PRINT 602
     PRINT 621
     DO 192 I=1,NTF
       PRINT 604
       PRINT 603,(GK(I,J),J=1,NTF),GQ(I,1)
  192 CONTINUE
C
C    IMSL SUBROUTINE TO SOLVE LINEAR SYSTEM OF EQUATIONS
C
     IDGT=14
     PRINT 701
     CALL LEQT2F(GK,1,NTF,NTF,GQ,IDGT,WKAREA,IER)
     DO 12 I=1,NTF
       PRINT 700,GQ(I,1)
   12 CONTINUE
     YMAX=QBAR(1)/E(1)/B(1)/H(1)**3/2.0D0
     YMAX=YMAX*TL**4*5.0D0/16.0D0
     X1=X2=DIFF=TLC=0.0D0
     DO 13 I=1,NE
       PRINT 701
       PRINT 706,I
       PRINT 705
       Y1=Y2=Y3=Y4=Y5=Y6=Y7=0.0D0
       CC1=CC2=CC3=CC4=CC5=CC6=CC7=0.0D0
C
C      CONVERT GLOBAL NODAL VALUES TO ELEMENT NODAL VALUES
C
       DO 14 K=1,NTF
         DO 15 J=1,7
           IF(NCON(I,J).EQ.K)THEN DO
             IF(J.EQ.1) Y1=GQ(K,1)
             IF(J.EQ.2) Y2=GQ(K,1)
             IF(J.EQ.3) Y3=GQ(K,1)
             IF(J.EQ.4) Y4=GQ(K,1)
             IF(J.EQ.5) Y5=GQ(K,1)
             IF(J.EQ.6) Y6=GQ(K,1)
             IF(J.EQ.7) Y7=GQ(K,1)
           END IF
   15    CONTINUE
   14  CONTINUE
C
```

```
C       SUBROUTINE CON COMPUTES THE VALUES OF THE CONSTANTS OF THE
C       SHAPE FUNCTIONS
C
        CALL CON(XL,B,H,E,QBAR,I,NOP)
        PRINT 702,CC1,CC2,CC3,CC4,CC5,CC6,CC7
        PRINT 701
        X=0.0D0
        PRINT 709
C
C       DEFLECTIONS FROM THE FINITE ELEMENT AND EXACT SOLUTIONS AND
C       THE RESULTING ERROR ARE COMPUTED AND PRINTED
C
        WHILE(X.LE.XL(I))DO
          Y=CC1+CC2*X+CC3*X**2+CC4*X**3+CC5*X**4+CC6*X**5+CC7*X**6
          Y1=QBAR(I)/2.0D0/E(I)/B(I)/H(I)**3*(TL**3*X1-2.0D0*TL
     &       *X1**3+X1**4)
          Y2=(Y1-Y)/YMAX*100.0D0
          X=X+TL/60.0D0
          X1=X1+TL/60.0D0
          PRINT 703,Y,Y1,Y2
        END WHILE
        TLC=TLC+XL(I)
        X1=TLC
        X=ST01=0.0D0
C
C       THE MAXIMUM ERROR OF THE DEFLECTION IS CALCULATED, STORED AND
C       PRINTED
C
        WHILE(X.LE.XL(I))DO
          Y=CC1+CC2*X+CC3*X**2+CC4*X**3+CC5*X**4+CC6*X**5+CC7*X**6
          Y1=QBAR(I)/2.0D0/E(I)/B(I)/H(I)**3*(TL**3*X2-2.0D0*TL
     &       *X2**3+X2**4)
          ST01=DABS(Y1)-DABS(Y)
          IF(ST01.GT.DIFF) DIFF=ST01
          X=X+TL/1000.0D0
          X2=X2+TL/1000.0D0
        END WHILE
        X2=TLC
   13 CONTINUE
      PRINT 604
      DIFF=DIFF/YMAX*100.0D0
      PRINT 707,DIFF
  709 FORMAT(T9,'NUM',14X,'EXACT',14X,'ERROR'/)
  707 FORMAT(T2,'DIFF =',D16.7)
  700 FORMAT('0',T2,D16.7)
  701 FORMAT('-')
  702 FORMAT(6(D16.7))
  703 FORMAT(3(D18.9))
  705 FORMAT(T3,'CONSTANTS FOR SHAPE FUNCTIONS'/)
  706 FORMAT(T3,'ELEMENT NO.',I2/)
      PRINT 602
  602 FORMAT('1')
  603 FORMAT('0',T2,11D11.3)
  604 FORMAT('0')
  609 FORMAT('0',T23,'ELEMENT NUMBER ',I2)
  610 FORMAT('0',T20,'E(',I2,') =',D16.7)
  611 FORMAT(' ',T20,'B(',I2,') =',D16.7)
  612 FORMAT(' ',T20,'H(',I2,') =',D16.7)
  613 FORMAT(' ',T20,'XL(',I2,') =',D16.7)
  614 FORMAT(' ',T20,'QBAR(',I2,') =',D16.7)
  620 FORMAT('0',T13,'ELEMENT STIFFNESS MATRIX AND ELEMENT LOAD VECTOR',
     &       ' (RIGHT COLUMN) ORIENTED IN GLOBAL COORDINATES')
  621 FORMAT('0',T10,'GLOBAL STIFFNESS MATRIX AND GLOBAL LOAD VECTOR ',
     &       '(RIGHT COLUMN)')
```

```
  641 FORMAT('0',10X,7D12.3)
  644 FORMAT('0',T26,'I=',I2,1X,7I4)
  677 FORMAT(' ',T5,5D18.8)
  681 FORMAT('0',10X,7D12.3,10X,D12.3)
  698 FORMAT('0',T31,'CONNECTION MATRIX, NCON(I,J)')
  684 FORMAT(' ',T33,'J=1',1X,'J=2',1X,'J=3',1X,'J=4',1X,'J=5')
      STOP
      END
C
C     *****************************************************************
      SUBROUTINE ELMT(EMG,EKG,EQG,E,RHOV,B,H,XL,Q,W)
C
C     *****************************************************************
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION EKG(7,7),EQG(7),EMG(7,7)
      EQG(1)=Q*7.0D0*XL/30.0D0
      EQG(2)=Q*XL*XL/60.0D0
      EQG(3)=Q*8.0D0*XL/15.0D0
      EQG(4)=EQG(1)
      EQG(5)=-EQG(2)
      EQG(6)=0.0D0
      EQG(7)=0.0D0
      EI=E/12.0D0*B*H**3
      AREA=B*H
      RHO=RHOV*AREA
      DO 33 I=1,7
        DO 33 J=1,7
          EKG(I,J)=0.0D0
          EMG(I,J)=0.0D0
   33 CONTINUE
      EI=EI/5.0D0
      EKG(1,1)=EI*316.0D0/XL**3
      EKG(1,2)=EI*94.0D0/XL**2
      EKG(1,3)=-EI*512.0D0/XL**3
      EKG(1,4)=EI*196.0D0/XL**3
      EKG(1,5)=-EI*34.0D0/XL**2
      EKG(2,2)=EI*36.0D0/XL
      EKG(2,3)=-EI*128.0D0/XL**2
      EKG(2,4)=EI*34.0D0/XL**2
      EKG(2,5)=-EI*6.0D0/XL
      EKG(3,3)=EI*1024.0D0/XL**3
      EKG(3,4)=-EI*512.0D0/XL**3
      EKG(3,5)=EI*128.0D0/XL**2
      EKG(4,4)=EI*316.0D0/XL**3
      EKG(4,5)=-EI*94.0D0/XL**2
      EKG(5,5)=EI*36.0D0/XL
      EMG(1,1)=RHO*XL*13.0D0/63.0D0
      EMG(1,2)=RHO*XL**2/63.0D0
      EMG(1,3)=RHO*XL*4.0D0/63.0D0
      EMG(1,4)=-RHO*XL*23.0D0/630.0D0
      EMG(1,5)=RHO*XL**2/180.0D0
      EMG(2,2)=RHO*XL**3/630.0D0
      EMG(2,3)=RHO*XL**2*4.0D0/630.0D0
      EMG(2,4)=-RHO*XL**2/180.0D0
      EMG(2,5)=RHO*XL**3/1260.0D0
      EMG(3,3)=RHO*XL*128.0D0/315.0D0
      EMG(3,4)=EMG(1,3)
      EMG(3,5)=-EMG(2,3)
      EMG(4,4)=EMG(1,1)
      EMG(4,5)=-EMG(1,2)
      EMG(5,5)=EMG(2,2)
      DO 3 I=1,5
        DO 3 J=1,5
```

```
         EKG(J,I)=EKG(I,J)
         EMG(J,I)=EMG(I,J)
    3 CONTINUE
      RETURN
      END
C
C     *******************************************************************
      SUBROUTINE CON(XL,B,H,E,QBAR,I,NOP)
C
C     *******************************************************************
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION XL(10),B(10),H(10),QBAR(10),E(10)
      COMMON Y1,Y2,Y3,Y4,Y5,Y6,Y7,CC1,CC2,CC3,CC4,CC5,CC6,CC7
      CC1=Y1
      CC2=Y2
      CC3=-11.0D0*Y1/XL(I)**2-4.0D0*Y2/XL(I)+16.0D0*Y3/XL(I)**2
     &   -5.0D0*Y4/XL(I)**2+Y5/XL(I)
      CC4=18.0D0*Y1/XL(I)**3+5.0D0*Y2/XL(I)**2-32.0D0*Y3/XL(I)**3
     &   +14.0D0*Y4/XL(I)**3-3.0D0*Y5/XL(I)**2
      CC5=-8.0D0*Y1/XL(I)**4-2.0D0*Y2/XL(I)**3+16.0D0*Y3/XL(I)**4
     &   -8.0D0*Y4/XL(I)**4+2.0D0*Y5/XL(I)**3
      RETURN
      END
```

```
$JOB WATFIV SURENDRA-RAJPAL,NOEXT,NOWARN
C ***********************************************************************
C *                                                                     *
C *   STATIC CONDENSATION--FOURTH TO THIRD ORDER REDUCTION--            *
C *           SOLUTION OF STATIC BEAM PROBLEM                           *
C *                                                                     *
C ***********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION WKAREA(200),NCON(10,7)
      DIMENSION GK(2,2),EK(7,7),GQ(2,1),EQ(7),E(10),B(10),H(10),
     &          XL(10),QBAR(10),WBAR(10),RHOV(10),EM(7,7)
      INTEGER NCON/70*0/,NOP/0/
      COMMON Y1,Y2,Y3,Y4,Y5,Y6,Y7,CC1,CC2,CC3,CC4,CC5,CC6,CC7
      PI=.3141592653589793D+01
C
C     READ DIMENSION OF ELEMENT EQUATIONS
C
      READ, N
C
C     READ OPTION FOR REGULAR (0) OR MODIFIED (1) SOLUTION
C
      READ, NOP
C
C     READ NUMBER OF ELEMENTS AND NUMBER OF DEGREES OF FREEDOM
C
      READ, NE,NTF
C
C     READ CONNECTION MATRIX (BOUNDARY CONSTRAINTS) AND ELEMENT
C     PROPERTIES
C
      TL=0.0D0
      DO 20 I=1,NE
        READ, (NCON(I,J),J=1,N),E(I),RHOV(I),B(I),H(I),XL(I),WBAR(I)
     &        ,QBAR(I)
        RHOV(I)=RHOV(I)/386.0D0
        TL=TL+XL(I)
   20 E(I)=E(I)*1.0D6
C
C     PRINT CONNECTION MATRIX
C
      PRINT 602
      PRINT 698
      PRINT 684
      DO 81 I=1,NE
   81 PRINT 644,I,(NCON(I,J),J=1,N)
C
C     INITIALIZE GLOBAL STIFFNESS MATRIX AND LOAD VECTOR
C
      DO 26 I=1,NTF
        DO 27 J=1,NTF
   27   GK(I,J)=0.0D0
   26 GQ(I,1)=0.0D0
C
C     COMBINE ELEMENT MATRICES INTO GLOBAL MATRICES
C
      DO 36 K=1,NE
        PRINT 609,K
        PRINT 610,K,E(K)
        PRINT 611,K,B(K)
        PRINT 612,K,H(K)
        PRINT 613,K,XL(K)
        PRINT 614,K,QBAR(K)
C
C       SUBROUTINE ELMT SUPPLIES VALUES OF ELEMENT MATRICES
```

```
C
      CALL ELMT(EM,EK,EQ,E(K),RHOV(K),B(K),H(K),XL(K),QBAR(K),
     &          WBAR(K))
C
C     PRINT ELEMENT STIFFNESS MATRIX AND LOAD VECTOR
C
      PRINT 604
      PRINT 620
      PRINT 604
      DO 9 I=1,N
    9 PRINT 681,(EK(I,J),J=1,N),EQ(I)
      PRINT 602
      DO 36 I=1,N
        LX=NCON(K,I)
        IF(LX.NE.0)GQ(LX,1)=GQ(LX,1)+EQ(I)
        DO 36 J=1,N
          LY=NCON(K,J)
          IF(LX.EQ.0.OR.LY.EQ.0)GO TO 36
          GK(LX,LY)=GK(LX,LY)+EK(I,J)
   36 CONTINUE
C
C     PRINT GLOBAL STIFFNESS MATRIX AND LOAD VECTOR
C
      PRINT 602
      PRINT 621
      DO 192 I=1,NTF
        PRINT 604
        PRINT 603,(GK(I,J),J=1,NTF),GQ(I,1)
  192 CONTINUE
C
C     IMSL SUBROUTINE TO SOLVE LINEAR SYSTEM OF EQUATIONS
C
      IDGT=14
      PRINT 701
      CALL LEQT2F(GK,1,NTF,NTF,GQ,IDGT,WKAREA,IER)
      DO 12 I=1,NTF
        PRINT 700,GQ(I,1)
   12 CONTINUE
      YMAX=QBAR(1)/E(1)/B(1)/H(1)**3/2.0D0
      YMAX=YMAX*TL**4*5.0D0/16.0D0
      X1=X2=DIFF=TLC=0.0D0
      DO 13 I=1,NE
        PRINT 701
        PRINT 706,I
        PRINT 705
        Y1=Y2=Y3=Y4=Y5=Y6=Y7=0.0D0
        CC1=CC2=CC3=CC4=CC5=CC6=CC7=0.0D0
C
C       CONVERT GLOBAL NODAL VALUES TO ELEMENT NODAL VALUES
C
        DO 14 K=1,NTF
          DO 15 J=1,7
            IF(NCON(I,J).EQ.K)THEN DO
              IF(J.EQ.1)  Y1=GQ(K,1)
              IF(J.EQ.2)  Y2=GQ(K,1)
              IF(J.EQ.3)  Y3=GQ(K,1)
              IF(J.EQ.4)  Y4=GQ(K,1)
              IF(J.EQ.5)  Y5=GQ(K,1)
              IF(J.EQ.6)  Y6=GQ(K,1)
              IF(J.EQ.7)  Y7=GQ(K,1)
            END IF
   15     CONTINUE
   14   CONTINUE
C
```

```
C       SUBROUTINE CON COMPUTES THE VALUES OF THE CONSTANTS OF THE
C       SHAPE FUNCTIONS
C
        CALL CON(XL,B,H,E,QBAR,I,NOP)
        PRINT 702,CC1,CC2,CC3,CC4,CC5,CC6,CC7
        PRINT 701
        X=0.0D0
        PRINT 709
C
C       DEFLECTIONS FROM THE FINITE ELEMENT AND EXACT SOLUTIONS AND
C       THE RESULTING ERROR ARE COMPUTED AND PRINTED
C
        WHILE(X.LE.XL(I))DO
          Y=CC1+CC2*X+CC3*X**2+CC4*X**3+CC5*X**4+CC6*X**5+CC7*X**6
          Y1=QBAR(I)/2.0D0/E(I)/B(I)/H(I)**3*(TL**3*X1-2.0D0*TL
     &        *X1**3+X1**4)
          Y2=(Y1-Y)/YMAX*100.0D0
          X=X+TL/60.0D0
          X1=X1+TL/60.0D0
          PRINT 703,Y,Y1,Y2
        END WHILE
        TLC=TLC+XL(I)
        X1=TLC
        X=ST01=0.0D0
C
C       THE MAXIMUM ERROR OF THE DEFLECTION IS CALCULATED, STORED AND
C       PRINTED
C
        WHILE(X.LE.XL(I))DO
          Y=CC1+CC2*X+CC3*X**2+CC4*X**3+CC5*X**4+CC6*X**5+CC7*X**6
          Y1=QBAR(I)/2.0D0/E(I)/B(I)/H(I)**3*(TL**3*X2-2.0D0*TL
     &        *X2**3+X2**4)
          ST01=DABS(Y1)-DABS(Y)
          IF(ST01.GT.DIFF) DIFF=ST01
          X=X+TL/1000.0D0
          X2=X2+TL/1000.0D0
        END WHILE
        X2=TLC
   13 CONTINUE
      PRINT 604
      DIFF=DIFF/YMAX*100.0D0
      PRINT 707,DIFF
  709 FORMAT(T9,'NUM',14X,'EXACT',14X,'ERROR'/)
  707 FORMAT(T2,'DIFF =',D16.7)
  700 FORMAT('0',T2,D16.7)
  701 FORMAT('-')
  702 FORMAT(6(D16.7))
  703 FORMAT(3(D18.9))
  705 FORMAT(T3,'CONSTANTS FOR SHAPE FUNCTIONS'/)
  706 FORMAT(T3,'ELEMENT NO.',I2/)
      PRINT 602
  602 FORMAT('1')
  603 FORMAT('0',T2,11D11.3)
  604 FORMAT('0')
  609 FORMAT('0',T23,'ELEMENT NUMBER ',I2)
  610 FORMAT('0',T20,'E(',I2,') =',D16.7)
  611 FORMAT(' ',T20,'B(',I2,') =',D16.7)
  612 FORMAT(' ',T20,'H(',I2,') =',D16.7)
  613 FORMAT(' ',T20,'XL(',I2,') =',D16.7)
  614 FORMAT(' ',T20,'QBAR(',I2,') =',D16.7)
  620 FORMAT('0',T13,'ELEMENT STIFFNESS MATRIX AND ELEMENT LOAD VECTOR',
     &       ' (RIGHT COLUMN) ORIENTED IN GLOBAL COORDINATES')
  621 FORMAT('0',T10,'GLOBAL STIFFNESS MATRIX AND GLOBAL LOAD VECTOR ',
     &       '(RIGHT COLUMN)')
```

```
  641 FORMAT('0',10X,7D12.3)
  644 FORMAT('0',T26,'I=',I2,1X,7I4)
  677 FORMAT(' ',T5,5D18.8)
  681 FORMAT('0',10X,7D12.3,10X,D12.3)
  698 FORMAT('0',T31,'CONNECTION MATRIX, NCON(I,J)')
  684 FORMAT(' ',T33,'J=1',1X,'J=2',1X,'J=3',1X,'J=4')
      STOP
      END
C
C     ******************************************************************
      SUBROUTINE ELMT(EMG,EKG,EQG,E,RHOV,B,H,XL,Q,W)
C
C     ******************************************************************
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION EMG(7,7),EKG(7,7),EQG(7)
      EQG(1)=Q*XL/2.0D0
      EQG(2)=Q*XL*XL/12.0D0
      EQG(3)=Q*XL/2.0D0
      EQG(4)=-Q*XL*XL/12.0D0
      EQG(5)=0.0D0
      EQG(6)=0.0D0
      EQG(7)=0.0D0
      EI=E/12.0D0*B*H**3
      AREA=B*H
      RHO=RHOV*B*H
      DO 33 I=1,7
        DO 33 J=1,7
          EMG(I,J)=0.0D0
          EKG(I,J)=0.0D0
   33 CONTINUE
      EKG(1,1)=EI*12.0D0/XL**3
      EKG(1,2)=EI*6.0D0/XL**2
      EKG(1,3)=-EI*12.0D0/XL**3
      EKG(1,4)=EKG(1,2)
      EKG(2,2)=EI*4.0D0/XL
      EKG(2,3)=-EKG(1,2)
      EKG(2,4)=EI*2.0D0/XL
      EKG(3,3)=EKG(1,1)
      EKG(3,4)=-EKG(1,2)
      EKG(4,4)=EKG(2,2)
      EMG(1,1)=RHO*XL*13.0D0/35.0D0
      EMG(1,2)=RHO*XL**2*11.0D0/210.0D0
      EMG(1,3)=RHO*XL*9./70.0D0
      EMG(1,4)=-RHO*XL**2*13.0D0/420.0D0
      EMG(2,2)=RHO*XL**3/105.0D0
      EMG(2,3)=-EMG(1,4)
      EMG(2,4)=-RHO*XL**3/140.0D0
      EMG(3,3)=EMG(1,1)
      EMG(3,4)=-EMG(1,2)
      EMG(4,4)=EMG(2,2)
      DO 3 I=1,4
        DO 3 J=1,4
          EMG(J,I)=EMG(I,J)
          EKG(J,I)=EKG(I,J)
    3 CONTINUE
      RETURN
      END
C
C     ******************************************************************
      SUBROUTINE CON(XL,B,H,E,QBAR,I,NOP)
C
C     ******************************************************************
C
```

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION XL(10),B(10),H(10),QBAR(10),E(10)
      COMMON Y1,Y2,Y3,Y4,Y5,Y6,Y7,CC1,CC2,CC3,CC4,CC5,CC6,CC7
      IF(NOP.EQ.1)THEN DO
C
C     THE CONSTRAINT EQUATION IS USED TO RAISE THE ORDER OF THE
C     EQUATIONS BACK TO THEIR ORIGINAL LEVEL
C
        TIN=B(I)*H(I)**3/12.0D0
        YC=(Y1/2.D0)+(Y2*XL(I)/8.D0)+(Y3/2.D0)-(Y4*XL(I)/8.D0)
     &      +QBAR(I)*XL(I)**4/384.0D0/E(I)/TIN
        Y5=Y4
        Y4=Y3
        Y3=YC
      END IF
      CC1=Y1
      CC2=Y2
      CC3=-11.0D0*Y1/XL(I)**2-4.0D0*Y2/XL(I)+16.0D0*Y3/XL(I)**2
     &   -5.0D0*Y4/XL(I)**2+Y5/XL(I)
      CC4=18.0D0*Y1/XL(I)**3+5.0D0*Y2/XL(I)**2-32.0D0*Y3/XL(I)**3
     &   +14.0D0*Y4/XL(I)**3-3.0D0*Y5/XL(I)**2
      CC5=-8.0D0*Y1/XL(I)**4-2.0D0*Y2/XL(I)**3+16.0D0*Y3/XL(I)**4
     &   -8.0D0*Y4/XL(I)**4+2.0D0*Y5/XL(I)**3
      RETURN
      END
```

```
$JOB WATFIV SURENDRA-RAJPAL,NOEXT,NOWARN
C ******************************************************************
C *                                                                *
C *      FINITE ELEMENT SOLUTION OF BEAM VIBRATION PROBLEM         *
C *                  FOURTH ORDER SOLUTION                         *
C *                                                                *
C ******************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION WKAREA(500),NCON(10,7)
      DIMENSION GM(4,4),GK(4,4),EM(7,7),EK(7,7),GQ(4,1),EQ(7),
C     DIMENSION GM(11,11),GK(11,11),EM(7,7),EK(7,7),GQ(11,1),EQ(7),
     &          E(10),RHOV(10),B(10),H(10),XL(10),FBAR(10),WBAR(10),
     &          DK(7,7)
      INTEGER NCON/70*0/,NOP/0/
      COMMON Y1,Y2,Y3,Y4,Y5,Y6,Y7,CC1,CC2,CC3,CC4,CC5,CC6,CC7
      PI=.3141592653589793D+01
C
C     READ DIMENSION OF ELEMENT EQUATIONS
C
      READ, N
C
C     READ OPTION FOR REGULAR (0) OR MODIFIED (1) SOLUTION
C
      READ, NOP
C
C     READ NUMBER OF ELEMENTS AND NUMBER OF DEGREES OF FREEDOM
C
      READ, NE,NTF
C
C     READ CONNECTION MATRIX (BOUNDARY CONSTRAINTS) AND ELEMENT
C     PROPERTIES
C
      TL=0.0D0
      DO 20 I=1,NE
        READ, (NCON(I,J),J=1,N),E(I),RHOV(I),B(I),H(I),XL(I),WBAR(I),
     &        FBAR(I)
        RHOV(I)=RHOV(I)/386.0D0
        TL=TL+XL(I)
   20 E(I)=E(I)*1.0D6
      DO 22 I=1,NE
      WBAR(I)=WBAR(I)*(PI/TL)**2*DSQRT(E(I)*H(I)**2/12.0D0/RHOV(I))
   22 CONTINUE
C
C     PRINT CONNECTION MATRIX
C
      PRINT 602
      PRINT 698
      PRINT 684
      DO 81 I=1,NE
   81 PRINT 644,I,(NCON(I,J),J=1,N)
C
C     INITIALIZE GLOBAL STIFFNESS AND MASS MATRICES AND LOAD VECTOR
C
      DO 26 I=1,NTF
        DO 27 J=1,NTF
          GM(I,J)=0.0D0
   27   GK(I,J)=0.0D0
   26 GQ(I,1)=0.0D0
C
C     COMBINE ELEMENT MATRICES INTO GLOBAL MATRICES
C
      DO 36 K=1,NE
        PRINT 609,K
        PRINT 610,K,E(K)
```

```
          PRINT 630,K,RHOV(K)
          PRINT 611,K,B(K)
          PRINT 612,K,H(K)
          PRINT 613,K,XL(K)
          PRINT 614,K,FBAR(K)
          PRINT 631,K,WBAR(K)
C
C         SUBROUTINE ELMT SUPPLIES VALUES OF THE ELEMENT MATRICES
C
          CALL ELMT(EM,EK,EQ,E(K),RHOV(K),B(K),H(K),XL(K),FBAR(K),
     &               WBAR(K))
C
C         PRINT ELEMENT MASS AND STIFFNESS MATRICES AND LOAD VECTOR
C
          PRINT 604
          PRINT 640
          PRINT 604
          DO 79 I=1,N
   79     PRINT 641,(EM(I,J),J=1,N)
          PRINT 604
          PRINT 620
          PRINT 604
          DO 9 I=1,N
    9     PRINT 681,(EK(I,J),J=1,N),EQ(I)
          PRINT 602
          DO 36 I=1,N
            LX=NCON(K,I)
            IF(LX.NE.0)GQ(LX,1)=GQ(LX,1)+EQ(I)
            DO 36 J=1,N
              LY=NCON(K,J)
              IF(LX.EQ.0.OR.LY.EQ.0)GO TO 36
              GM(LX,LY)=GM(LX,LY)+EM(I,J)
              GK(LX,LY)=GK(LX,LY)+EK(I,J)
   36 CONTINUE
C
C     PRINT GLOBAL MASS AND STIFFNESS MATRICES AND LOAD VECTOR
C
      PRINT 661
      DO 29 I=1,NTF
        PRINT 604
        PRINT 603,(GM(I,J),J=1,NTF)
   29 CONTINUE
      PRINT 602
      PRINT 621
      DO 192 I=1,NTF
        PRINT 604
        PRINT 603,(GK(I,J),J=1,NTF)
  192 CONTINUE
C
C     CALCULATE AND PRINT GLOBAL DYNAMIC STIFFNESS MATRIX
C
      DO 10 I=1,NTF
        DO 11 J=1,NTF
          GK(I,J)=GK(I,J)-(WBAR(1)**2*GM(I,J))
   11   CONTINUE
   10 CONTINUE
      PRINT 602
      PRINT 622
      DO 193 I=1,NTF
        PRINT 604
        PRINT 605,(GK(I,J),J=1,NTF)
  193 CONTINUE
C
C     IMSL SUBROUTINE TO SOLVE LINEAR SYSTEM OF EQUATIONS
```

```
C
      IDGT=12
      PRINT 701
      CALL LEQT2F(GK,1,NTF,NTF,GQ,IDGT,WKAREA,IER)
      DO 12 I=1,NTF
        PRINT 700,GQ(I,1)
   12 CONTINUE
      A1=DSQRT(E(1)*H(1)**2/RHOV(1)/12.0D0)
      T1=FBAR(1)/RHOV(1)/B(1)/H(1)/WBAR(1)**2
      T2=1.0D0/(2.0D0*DCOS(DSQRT(WBAR(1)/A1)*TL/2.0D0))
      T3=1.0D0/(2.0D0*DCOSH(DSQRT(WBAR(1)/A1)*TL/2.0D0))
      YMAX=T1*(T2+T3-1.0D0)
      SMAX=T1*WBAR(1)/A1*(T3-T2)*E(1)*H(1)/2.0D0
      X1=X2=0.0D0
      DIFF=DIFFS=TLC=0.0D0
      DO 13 I=1,NE
        PRINT 701
        PRINT 705
        Y1=Y2=Y3=Y4=Y5=Y6=Y7=0.0D0
        CC1=CC2=CC3=CC4=CC5=CC6=CC7=0.0D0
C
C       CONVERT GLOBAL NODAL VALUES TO ELEMENT NODAL VALUES
C
        DO 14 K=1,NTF
          DO 15 J=1,7
            IF(NCON(I,J).EQ.K)THEN DO
              IF(J.EQ.1)  Y1=GQ(K,1)
              IF(J.EQ.2)  Y2=GQ(K,1)
              IF(J.EQ.3)  Y3=GQ(K,1)
              IF(J.EQ.4)  Y4=GQ(K,1)
              IF(J.EQ.5)  Y5=GQ(K,1)
              IF(J.EQ.6)  Y6=GQ(K,1)
              IF(J.EQ.7)  Y7=GQ(K,1)
            END IF
   15     CONTINUE
   14   CONTINUE
C
C       SUBROUTINE CON SUPPLIES THE VALUES OF THE CONSTANTS OF THE
C       SHAPE FUNCTIONS
C
        CALL CON(XL,B,H,E,FBAR,I,NOP)
        PRINT 706,I
        PRINT 702,CC1,CC2,CC3,CC4,CC5,CC6,CC7
        PRINT 701
        X=ST01=ST02=0.0D0
        PRINT 709
C
C       DEFLECTIONS AND STRESSES AND THEIR CORRESPONDING ERRORS ARE
C       COMPUTED AND PRINTED AT EQUAL INTERVALS ALONG THE LENGTH OF
C       THE BEAM.
C
        WHILE(X.LE.XL(I))DO
          Y1=CC1+CC2*X+CC3*X**2+CC4*X**3+CC5*X**4+CC6*X**5+CC7*X**6
          Y2=T1*(DCOS(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X1))*T2
     &        +DCOSH(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X1))*T3-1.0D0)
          Y3=2.0D0*CC3+6.0D0*CC4*X+12.0D0*CC5*X**2+20.0D0*CC6*X**3
     &        +30.0D0*CC7*X**4
          Y3=Y3*E(I)*H(I)/2.0D0
          Y4=T1*WBAR(I)/A1*(DCOSH(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X1))
     &          *T3-DCOS(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X1))*T2)
          Y4=Y4*E(I)*H(I)/2.0D0
          ST01=DABS(Y4)-DABS(Y3)
          ST01=ST01/SMAX*100.0D0
          ST02=(Y2-Y1)/YMAX*100.0D0
```

```
          X=X+TL/10.0D0
          X1=X1+TL/10.0D0
          PRINT 703,Y1,Y2,ST02,Y3,Y4,ST01
        END WHILE
        TLC=TLC+XL(I)
        X1=TLC
        X=ST01=ST02=0.0D0
C
C       THE MAXIMUM ERROR OF THE DEFLECTIONS AND STRESSES ARE
C       DETERMINED, STORED AND PRINTED
C
        WHILE(X.LE.XL(I))DO
          Y1=CC1+CC2*X+CC3*X**2+CC4*X**3+CC5*X**4+CC6*X**5+CC7*X**6
          Y2=T1*(DCOS(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X2))*T2
     &       +DCOSH(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X2))*T3-1.0D0)
          Y3=2.0D0*CC3+6.0D0*CC4*X+12.0D0*CC5*X**2+20.0D0*CC6*X**3
     &       +30.0D0*CC7*X**4
          Y3=Y3*E(I)*H(I)/2.0D0
          Y4=T1*WBAR(I)/A1*(DCOSH(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X2))
     &        *T3-DCOS(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X2))*T2)
          Y4=Y4*E(I)*H(I)/2.0D0
          ST01=DABS(Y4)-DABS(Y3)
          ST02=DABS(Y2-Y1)
          IF(ST02.GT.DIFF) DIFF=ST02
          IF(X2.EQ.0.0D0.OR.X2.GE.TL) GOTO 85
          IF(ST01.GT.DIFFS) DIFFS=ST01
   85     X=X+TL/500.0D0
          X2=X2+TL/500.0D0
        END WHILE
        X2=TLC
   13 CONTINUE
      PRINT 604
      PRINT, SMAX
      DIFFS=DIFFS/SMAX*100.0D0
      DIFF=DIFF/YMAX*100.0D0
      PRINT 707,DIFF
      PRINT 708,DIFFS
  709 FORMAT('0',T9,'DEFL(N)',11X,'DEFL(E)',10X,'ERROR(D)',10X,
     &        'STRESS(N)',9X,'STRESS(E)',9X,'ERROR(S)')
  707 FORMAT(T2,'DIFF =',D16.7)
  708 FORMAT(T2,'DIFFS =',D16.7)
  700 FORMAT('0',T2,D16.7)
  701 FORMAT('-')
  702 FORMAT(6(D16.7))
  703 FORMAT(6(D18.9))
  705 FORMAT(T3,'CONSTANTS FOR SHAPE FUNCTIONS')
  706 FORMAT(T3,'ELEMENT NO.',I2///)
      PRINT 602
  602 FORMAT('1')
  603 FORMAT('0',T2,11D11.3)
  604 FORMAT('0')
  605 FORMAT('0',T2,11D16.8)
  609 FORMAT('0',T23,'ELEMENT NUMBER ',I2)
  610 FORMAT('0',T20,'E(',I2,') =',D16.7)
  611 FORMAT(' ',T20,'B(',I2,') =',D16.7)
  612 FORMAT(' ',T20,'H(',I2,') =',D16.7)
  613 FORMAT(' ',T20,'XL(',I2,') =',D16.7)
  614 FORMAT(' ',T20,'FBAR(',I2,') =',D16.7)
  620 FORMAT('0',T13,'ELEMENT STIFFNESS MATRIX AND ELEMENT LOAD VECTOR',
     &        ' (RIGHT COLUMN) ORIENTED IN GLOBAL COORDINATES')
  621 FORMAT('0',T10,'GLOBAL STIFFNESS MATRIX AND GLOBAL LOAD VECTOR ',
     &        '(RIGHT COLUMN)')
  622 FORMAT('0',T10,'GLOBAL DYNAMIC STIFFNESS MATRIX')
  624 FORMAT('0',T13,'DYNAMIC ELEMENT STIFFNESS MATRIX')
```

```
  630 FORMAT(' ',T20,'RHOV(',I2,') =',D16.7)
  631 FORMAT(' ',T20,'WBAR(',I2,') =',D16.7)
  640 FORMAT('0',T13,'ELEMENT MASS MATRIX ORIENTATED IN GLOBAL ',
     &        'COORDINATES')
  641 FORMAT('0',10X,7D12.3)
  644 FORMAT('0',T26,'I=',I2,1X,7I4)
  661 FORMAT('0',T10,'GLOBAL MASS MATRIX')
  677 FORMAT(' ',T5,5D18.8)
  681 FORMAT('0',10X,7D12.3,10X,D12.3)
  698 FORMAT('0',T31,'CONNECTION MATRIX, NCON(I,J)')
  684 FORMAT(' ',T33,'J=1',1X,'J=2',1X,'J=3',1X,'J=4',1X,'J=5')
      STOP
      END
C
C     ******************************************************************
      SUBROUTINE ELMT(EMG,EKG,EQG,E,RHOV,B,H,XL,Q,W)
C
C     ******************************************************************
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION EKG(7,7),EQG(7),EMG(7,7)
      EQG(1)=Q*7.0D0*XL/30.0D0
      EQG(2)=Q*XL*XL/60.0D0
      EQG(3)=Q*8.0D0*XL/15.0D0
      EQG(4)=EQG(1)
      EQG(5)=-EQG(2)
      EQG(6)=0.0D0
      EQG(7)=0.0D0
      EI=E/12.0D0*B*H**3
      AREA=B*H
      RHO=RHOV*AREA
      DO 33 I=1,7
        DO 33 J=1,7
          EKG(I,J)=0.0D0
          EMG(I,J)=0.0D0
   33 CONTINUE
      EI=EI/5.0D0
      EKG(1,1)=EI*316.0D0/XL**3
      EKG(1,2)=EI*94.0D0/XL**2
      EKG(1,3)=-EI*512.0D0/XL**3
      EKG(1,4)=EI*196.0D0/XL**3
      EKG(1,5)=-EI*34.0D0/XL**2
      EKG(2,2)=EI*36.0D0/XL
      EKG(2,3)=-EI*128.0D0/XL**2
      EKG(2,4)=EI*34.0D0/XL**2
      EKG(2,5)=-EI*6.0D0/XL
      EKG(3,3)=EI*1024.0D0/XL**3
      EKG(3,4)=-EI*512.0D0/XL**3
      EKG(3,5)=EI*128.0D0/XL**2
      EKG(4,4)=EI*316.0D0/XL**3
      EKG(4,5)=-EI*94.0D0/XL**2
      EKG(5,5)=EI*36.0D0/XL
      EMG(1,1)=RHO*XL*13.0D0/63.0D0
      EMG(1,2)=RHO*XL**2/63.0D0
      EMG(1,3)=RHO*XL*4.0D0/63.0D0
      EMG(1,4)=-RHO*XL*23.0D0/630.0D0
      EMG(1,5)=RHO*XL**2/180.0D0
      EMG(2,2)=RHO*XL**3/630.0D0
      EMG(2,3)=RHO*XL**2*4.0D0/630.0D0
      EMG(2,4)=-RHO*XL**2/180.0D0
      EMG(2,5)=RHO*XL**3/1260.0D0
      EMG(3,3)=RHO*XL*128.0D0/315.0D0
      EMG(3,4)=EMG(1,3)
      EMG(3,5)=-EMG(2,3)
```

```
      EMG(4,4)=EMG(1,1)
      EMG(4,5)=-EMG(1,2)
      EMG(5,5)=EMG(2,2)
      DO 3 I=1,5
        DO 3 J=1,5
          EKG(J,I)=EKG(I,J)
          EMG(J,I)=EMG(I,J)
    3 CONTINUE
      RETURN
      END
C
C     ****************************************************************
      SUBROUTINE CON(XL,B,H,E,QBAR,I,NOP)
C
C     ****************************************************************
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION XL(10),B(10),H(10),QBAR(10),E(10)
      COMMON Y1,Y2,Y3,Y4,Y5,Y6,Y7,CC1,CC2,CC3,CC4,CC5,CC6,CC7
      CC1=Y1
      CC2=Y2
      CC3=-11.0D0*Y1/XL(I)**2-4.0D0*Y2/XL(I)+16.0D0*Y3/XL(I)**2
     &    -5.0D0*Y4/XL(I)**2+Y5/XL(I)
      CC4=18.0D0*Y1/XL(I)**3+5.0D0*Y2/XL(I)**2-32.0D0*Y3/XL(I)**3
     &    +14.0D0*Y4/XL(I)**3-3.0D0*Y5/XL(I)**2
      CC5=-8.0D0*Y1/XL(I)**4-2.0D0*Y2/XL(I)**3+16.0D0*Y3/XL(I)**4
     &    -8.0D0*Y4/XL(I)**4+2.0D0*Y5/XL(I)**3
      RETURN
      END
```

```
$JOB WATFIV SURENDRA-RAJPAL,NOEXT,NOWARN
C ***********************************************************************
C *                                                                     *
C *   GUYAN-IRONS CONDENSATION--FOURTH TO THIRD ORDER REDUCTION--       *
C *             SOLUTION OF BEAM VIBRATION PROBLEM                      *
C *                                                                     *
C ***********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION WKAREA(500),NCON(10,7)
      DIMENSION GM(4,4),GK(4,4),EM(7,7),EK(7,7),GQ(4,1),EQ(7),
C     DIMENSION GM(11,11),GK(11,11),EM(7,7),EK(7,7),GQ(11,1),EQ(7),
     &          E(10),RHOV(10),B(10),H(10),XL(10),FBAR(10),WBAR(10),
     &          DK(7,7)
      INTEGER NCON/70*0/,NOP/0/
      COMMON Y1,Y2,Y3,Y4,Y5,Y6,Y7,CC1,CC2,CC3,CC4,CC5,CC6,CC7
      PI=.3141592653589793D+01
C
C     READ DIMENSION OF ELEMENT EQUATIONS
C
      READ, N
C
C     READ OPTION FOR REGULAR (0) OR MODIFIED (1) SOLUTION
C
      READ, NOP
C
C     READ NUMBER OF ELEMENTS AND NUMBER OF DEGREES OF FREEDOM
C
      READ, NE,NTF
C
C     READ CONNECTION MATRIX (BOUNDARY CONSTRAINTS) AND ELEMENT
C     PROPERTIES
C
      TL=0.0D0
      DO 20 I=1,NE
        READ, (NCON(I,J),J=1,N),E(I),RHOV(I),B(I),H(I),XL(I),WBAR(I),
     &        FBAR(I)
        RHOV(I)=RHOV(I)/386.0D0
        TL=TL+XL(I)
   20 E(I)=E(I)*1.0D6
      DO 22 I=1,NE
      WBAR(I)=WBAR(I)*(PI/TL)**2*DSQRT(E(I)*H(I)**2/12.0D0/RHOV(I))
   22 CONTINUE
C
C     PRINT CONNECTION MATRIX
C
      PRINT 602
      PRINT 698
      PRINT 684
      DO 81 I=1,NE
   81 PRINT 644,I,(NCON(I,J),J=1,N)
C
C     INITIALIZE GLOBAL STIFFNESS AND MASS MATRICES AND LOAD VECTOR
C
      DO 26 I=1,NTF
        DO 27 J=1,NTF
          GM(I,J)=0.0D0
   27   GK(I,J)=0.0D0
   26 GQ(I,1)=0.0D0
C
C     COMBINE ELEMENT MATRICES INTO GLOBAL MATRICES
C
      DO 36 K=1,NE
        PRINT 609,K
        PRINT 610,K,E(K)
```

```
      PRINT 630,K,RHOV(K)
      PRINT 611,K,B(K)
      PRINT 612,K,H(K)
      PRINT 613,K,XL(K)
      PRINT 614,K,FBAR(K)
      PRINT 631,K,WBAR(K)
C
C     SUBROUTINE ELMT SUPPLIES VALUES OF THE ELEMENT MATRICES
C
      CALL ELMT(EM,EK,EQ,E(K),RHOV(K),B(K),H(K),XL(K),FBAR(K),
     &          WBAR(K))
C
C     PRINT ELEMENT MASS AND STIFFNESS MATRICES AND LOAD VECTOR
C
      PRINT 604
      PRINT 640
      PRINT 604
      DO 79 I=1,N
   79 PRINT 641,(EM(I,J),J=1,N)
      PRINT 604
      PRINT 620
      PRINT 604
      DO 9 I=1,N
    9 PRINT 681,(EK(I,J),J=1,N),EQ(I)
      PRINT 602
      DO 36 I=1,N
        LX=NCON(K,I)
        IF(LX.NE.0)GQ(LX,1)=GQ(LX,1)+EQ(I)
        DO 36 J=1,N
          LY=NCON(K,J)
          IF(LX.EQ.0.OR.LY.EQ.0)GO TO 36
          GM(LX,LY)=GM(LX,LY)+EM(I,J)
          GK(LX,LY)=GK(LX,LY)+EK(I,J)
   36 CONTINUE
C
C     PRINT GLOBAL MASS AND STIFFNESS MATRICES AND LOAD VECTOR
C
      PRINT 661
      DO 29 I=1,NTF
        PRINT 604
        PRINT 603,(GM(I,J),J=1,NTF)
   29 CONTINUE
      PRINT 602
      PRINT 621
      DO 192 I=1,NTF
        PRINT 604
        PRINT 603,(GK(I,J),J=1,NTF)
  192 CONTINUE
C
C     CALCULATE AND PRINT GLOBAL DYNAMIC STIFFNESS MATRIX
C
      DO 10 I=1,NTF
        DO 11 J=1,NTF
          GK(I,J)=GK(I,J)-(WBAR(1)**2*GM(I,J))
   11   CONTINUE
   10 CONTINUE
      PRINT 602
      PRINT 622
      DO 193 I=1,NTF
        PRINT 604
        PRINT 605,(GK(I,J),J=1,NTF)
  193 CONTINUE
C
C     IMSL SUBROUTINE TO SOLVE LINEAR SYSTEM OF EQUATIONS
```

```
C
      IDGT=12
      PRINT 701
      CALL LEQT2F(GK,1,NTF,NTF,GQ,IDGT,WKAREA,IER)
      DO 12 I=1,NTF
        PRINT 700,GQ(I,1)
   12 CONTINUE
      A1=DSQRT(E(1)*H(1)**2/RHOV(1)/12.0D0)
      T1=FBAR(1)/RHOV(1)/B(1)/H(1)/WBAR(1)**2
      T2=1.0D0/(2.0D0*DCOS(DSQRT(WBAR(1)/A1)*TL/2.0D0))
      T3=1.0D0/(2.0D0*DCOSH(DSQRT(WBAR(1)/A1)*TL/2.0D0))
      YMAX=T1*(T2+T3-1.0D0)
      SMAX=T1*WBAR(1)/A1*(T3-T2)*E(1)*H(1)/2.0D0
      X1=X2=0.0D0
      DIFF=DIFFS=TLC=0.0D0
      DO 13 I=1,NE
        PRINT 701
        PRINT 705
        Y1=Y2=Y3=Y4=Y5=Y6=Y7=0.0D0
        CC1=CC2=CC3=CC4=CC5=CC6=CC7=0.0D0
C
C       CONVERT GLOBAL NODAL VALUES TO ELEMENT NODAL VALUES
C
        DO 14 K=1,NTF
          DO 15 J=1,7
            IF(NCON(I,J).EQ.K)THEN DO
              IF(J.EQ.1)  Y1=GQ(K,1)
              IF(J.EQ.2)  Y2=GQ(K,1)
              IF(J.EQ.3)  Y3=GQ(K,1)
              IF(J.EQ.4)  Y4=GQ(K,1)
              IF(J.EQ.5)  Y5=GQ(K,1)
              IF(J.EQ.6)  Y6=GQ(K,1)
              IF(J.EQ.7)  Y7=GQ(K,1)
            END IF
   15     CONTINUE
   14   CONTINUE
C
C       SUBROUTINE CON SUPPLIES THE VALUES OF THE CONSTANTS OF THE
C       SHAPE FUNCTIONS
C
        CALL CON(XL,B,H,E,FBAR,I,NOP)
        PRINT 706,I
        PRINT 702,CC1,CC2,CC3,CC4,CC5,CC6,CC7
        PRINT 701
        X=STO1=STO2=0.0D0
        PRINT 709
C
C       DEFLECTIONS AND STRESSES AND THEIR CORRESPONDING ERRORS ARE
C       COMPUTED AND PRINTED AT EQUAL INTERVALS ALONG THE LENGTH OF
C       THE BEAM.
C
        WHILE(X.LE.XL(I))DO
          Y1=CC1+CC2*X+CC3*X**2+CC4*X**3+CC5*X**4+CC6*X**5+CC7*X**6
          Y2=T1*(DCOS(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X1))*T2
     &       +DCOSH(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X1))*T3-1.0D0)
          Y3=2.0D0*CC3+6.0D0*CC4*X+12.0D0*CC5*X**2+20.0D0*CC6*X**3
     &       +30.0D0*CC7*X**4
          Y3=Y3*E(I)*H(I)/2.0D0
          Y4=T1*WBAR(I)/A1*(DCOSH(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X1))
     &        *T3-DCOS(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X1))*T2)
          Y4=Y4*E(I)*H(I)/2.0D0
          STO1=DABS(Y4)-DABS(Y3)
          STO1=STO1/SMAX*100.0D0
          STO2=(Y2-Y1)/YMAX*100.0D0
```

```
           X=X+TL/10.0D0
           X1=X1+TL/10.0D0
           PRINT 703,Y1,Y2,ST02,Y3,Y4,ST01
         END WHILE
         TLC=TLC+XL(I)
         X1=TLC
         X=ST01=ST02=0.0D0
C
C        THE MAXIMUM ERROR OF THE DEFLECTIONS AND STRESSES ARE
C        DETERMINED, STORED AND PRINTED
C
         WHILE(X.LE.XL(I))DO
           Y1=CC1+CC2*X+CC3*X**2+CC4*X**3+CC5*X**4+CC6*X**5+CC7*X**6
           Y2=T1*(DCOS(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X2))*T2
     &        +DCOSH(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X2))*T3-1.0D0)
           Y3=2.0D0*CC3+6.0D0*CC4*X+12.0D0*CC5*X**2+20.0D0*CC6*X**3
     &        +30.0D0*CC7*X**4
           Y3=Y3*E(I)*H(I)/2.0D0
           Y4=T1*WBAR(I)/A1*(DCOSH(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X2))
     &         *T3-DCOS(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X2))*T2)
           Y4=Y4*E(I)*H(I)/2.0D0
           ST01=DABS(Y4)-DABS(Y3)
           ST02=DABS(Y2-Y1)
           IF(ST02.GT.DIFF) DIFF=ST02
           IF(X2.EQ.0.0D0.OR.X2.GE.TL) GOTO 85
           IF(ST01.GT.DIFFS) DIFFS=ST01
   85      X=X+TL/500.0D0
           X2=X2+TL/500.0D0
         END WHILE
         X2=TLC
   13 CONTINUE
       PRINT 604
       PRINT, SMAX
       DIFFS=DIFFS/SMAX*100.0D0
       DIFF=DIFF/YMAX*100.0D0
       PRINT 707,DIFF
       PRINT 708,DIFFS
  709 FORMAT('0',T9,'DEFL(N)',11X,'DEFL(E)',10X,'ERROR(D)',10X,
     &        'STRESS(N)',9X,'STRESS(E)',9X,'ERROR(S)')
  707 FORMAT(T2,'DIFF =',D16.7)
  708 FORMAT(T2,'DIFFS =',D16.7)
  700 FORMAT('0',T2,D16.7)
  701 FORMAT('-')
  702 FORMAT(6(D16.7))
  703 FORMAT(6(D18.9))
  705 FORMAT(T3,'CONSTANTS FOR SHAPE FUNCTIONS')
  706 FORMAT(T3,'ELEMENT NO.',I2///)
      PRINT 602
  602 FORMAT('1')
  603 FORMAT('0',T2,11D11.3)
  604 FORMAT('0')
  605 FORMAT('0',T2,11D16.8)
  609 FORMAT('0',T23,'ELEMENT NUMBER ',I2)
  610 FORMAT('0',T20,'E(',I2,') =',D16.7)
  611 FORMAT(' ',T20,'B(',I2,') =',D16.7)
  612 FORMAT(' ',T20,'H(',I2,') =',D16.7)
  613 FORMAT(' ',T20,'XL(',I2,') =',D16.7)
  614 FORMAT(' ',T20,'FBAR(',I2,') =',D16.7)
  620 FORMAT('0',T13,'ELEMENT STIFFNESS MATRIX AND ELEMENT LOAD VECTOR',
     &        ' (RIGHT COLUMN) ORIENTED IN GLOBAL COORDINATES')
  621 FORMAT('0',T10,'GLOBAL STIFFNESS MATRIX AND GLOBAL LOAD VECTOR ',
     &        '(RIGHT COLUMN)')
  622 FORMAT('0',T10,'GLOBAL DYNAMIC STIFFNESS MATRIX')
  624 FORMAT('0',T13,'DYNAMIC ELEMENT STIFFNESS MATRIX')
```

```
  630 FORMAT(' ',T20,'RHOV(',I2,') =',D16.7)
  631 FORMAT(' ',T20,'WBAR(',I2,') =',D16.7)
  640 FORMAT('0',T13,'ELEMENT MASS MATRIX ORIENTATED IN GLOBAL ',
     &        'COORDINATES')
  641 FORMAT('0',10X,7D12.3)
  644 FORMAT('0',T26,'I=',I2,1X,7I4)
  661 FORMAT('0',T10,'GLOBAL MASS MATRIX')
  677 FORMAT(' ',T5,5D18.8)
  681 FORMAT('0',10X,7D12.3,10X,D12.3)
  698 FORMAT('0',T31,'CONNECTION MATRIX, NCON(I,J)')
  684 FORMAT(' ',T33,'J=1',1X,'J=2',1X,'J=3',1X,'J=4')
      STOP
      END
C
C     *****************************************************************
      SUBROUTINE ELMT(EMG,EKG,EQG,E,RHOV,B,H,XL,Q,W)
C
C     *****************************************************************
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION EMG(7,7),EKG(7,7),EQG(7)
      EQG(1)=Q*XL/2.0D0
      EQG(2)=Q*XL*XL/12.0D0
      EQG(3)=Q*XL/2.0D0
      EQG(4)=-Q*XL*XL/12.0D0
      EQG(5)=0.0D0
      EQG(6)=0.0D0
      EQG(7)=0.0D0
      EI=E/12.0D0*B*H**3
      AREA=B*H
      RHO=RHOV*B*H
      DO 33 I=1,7
        DO 33 J=1,7
          EMG(I,J)=0.0D0
          EKG(I,J)=0.0D0
   33 CONTINUE
      EKG(1,1)=EI*12.0D0/XL**3
      EKG(1,2)=EI*6.0D0/XL**2
      EKG(1,3)=-EI*12.0D0/XL**3
      EKG(1,4)=EKG(1,2)
      EKG(2,2)=EI*4.0D0/XL
      EKG(2,3)=-EKG(1,2)
      EKG(2,4)=EI*2.0D0/XL
      EKG(3,3)=EKG(1,1)
      EKG(3,4)=-EKG(1,2)
      EKG(4,4)=EKG(2,2)
      EMG(1,1)=RHO*XL*13.0D0/35.0D0
      EMG(1,2)=RHO*XL**2*11.0D0/210.0D0
      EMG(1,3)=RHO*XL*9./70.0D0
      EMG(1,4)=-RHO*XL**2*13.0D0/420.0D0
      EMG(2,2)=RHO*XL**3/105.0D0
      EMG(2,3)=-EMG(1,4)
      EMG(2,4)=-RHO*XL**3/140.0D0
      EMG(3,3)=EMG(1,1)
      EMG(3,4)=-EMG(1,2)
      EMG(4,4)=EMG(2,2)
      DO 3 I=1,4
        DO 3 J=1,4
          EMG(J,I)=EMG(I,J)
          EKG(J,I)=EKG(I,J)
    3 CONTINUE
      RETURN
      END
C
```

```
C       ****************************************************************
        SUBROUTINE CON(XL,B,H,E,QBAR,I,NOP)
C
C       ****************************************************************
C
        IMPLICIT REAL*8 (A-H,O-Z)
        DIMENSION XL(10),B(10),H(10),QBAR(10),E(10)
        COMMON Y1,Y2,Y3,Y4,Y5,Y6,Y7,CC1,CC2,CC3,CC4,CC5,CC6,CC7
        IF(NOP.EQ.1)THEN DO
C
C       THE CONSTRAINT EQUATION IS USED TO RAISE THE ORDER OF THE
C       EQUATIONS BACK TO THEIR ORIGINAL LEVEL
C
          TIN=B(I)*H(I)**3/12.0D0
          YC=(Y1/2.D0)+(Y2*XL(I)/8.D0)+(Y3/2.D0)-(Y4*XL(I)/8.D0)
     &       +QBAR(I)*XL(I)**4/384.0D0/E(I)/TIN
          Y5=Y4
          Y4=Y3
          Y3=YC
        END IF
        CC1=Y1
        CC2=Y2
        CC3=-11.0D0*Y1/XL(I)**2-4.0D0*Y2/XL(I)+16.0D0*Y3/XL(I)**2
     &    -5.0D0*Y4/XL(I)**2+Y5/XL(I)
        CC4=18.0D0*Y1/XL(I)**3+5.0D0*Y2/XL(I)**2-32.0D0*Y3/XL(I)**3
     &    +14.0D0*Y4/XL(I)**3-3.0D0*Y5/XL(I)**2
        CC5=-8.0D0*Y1/XL(I)**4-2.0D0*Y2/XL(I)**3+16.0D0*Y3/XL(I)**4
     &    -8.0D0*Y4/XL(I)**4+2.0D0*Y5/XL(I)**3
        RETURN
        END
```

```
C ********************************************************************
C *                                                                  *
C * DOWNS-FRICKER CONDENSATION--FOURTH TO THIRD ORDER REDUCTION--    *
C * SOLUTION OF THE BEAM VIBRATION PROBLEM                           *
C *                                                                  *
C ********************************************************************
$JOB WATFIV SURENDRA-RAJPAL,NOEXT,NOWARN
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION FBAR(10),WKAREA(500),NCON(10,4)
      DIMENSION E(10),B(10),H(10),XL(10),EKN(4,4),EQN(4)
      DIMENSION GK(4,4),EK(5,5),GQ(4,1),EQ(5),DK(5,5)
C     DIMENSION GK(12,12),EK(5,5),GQ(12,1),EQ(5),DK(5,5)
      DIMENSION EM(5,5),RHOV(10),WBAR(10)
      PI=.3141592653589793D+01
C
C     READ NUMBER OF ELEMENTS AND DEGREES OF FREEDOM
C
      READ, NE,NTF
C
C     READ CONNECTION MATRIX (BOUNDARY CONSTRAINTS) AND ELEMENT
C     PROPERTIES
C
      TL=0.0D0
      DO 20 I=1,NE
        READ, (NCON(I,J),J=1,4),E(I),RHOV(I),B(I),H(I),XL(I),
     &         WBAR(I),FBAR(I)
        RHOV(I)=RHOV(I)/386.0D0
        TL=TL+XL(I)
   20 E(I)=E(I)*1.0D6
      DO 22 I=1,NE
      WBAR(I)=WBAR(I)*(PI/TL)**2*DSQRT(E(I)*H(I)**2/12.0D0/RHOV(I))
   22 CONTINUE
C
C     INITIALIZE GLOBAL STIFFNESS MATRIX AND LOAD VECTOR
C
      DO 26 I=1,NTF
        DO 27 J=1,NTF
   27   GK(I,J)=0.0D0
   26 GQ(I,1)=0.0D0
C
C     PRINT CONNECTION MATRIX
C
      PRINT 602
      PRINT 698
      PRINT 684
      DO 81 I=1,NE
   81 PRINT 644,I,(NCON(I,J),J=1,4)
C
C     ASSEMBLE ELEMENT MATRICES INTO GLOBAL MATRICES
C
      DO 36 K=1,NE
        PRINT 609,K
        PRINT 610,K,E(K)
        PRINT 611,K,B(K)
        PRINT 612,K,H(K)
        PRINT 613,K,XL(K)
        PRINT 614,K,FBAR(K)
        PRINT 631,K,WBAR(K)
C
C       SUBROUTINE ELMT SUPPLIES THE VALUES OF THE ELEMENT MATRICES
C
        CALL ELMT(EM,EK,DK,EQ,E(K),RHOV(K),B(K),H(K),XL(K),FBAR(K),
     &            WBAR(K))
C
```

```
C       PRINT THE ELEMENT MASS AND STIFFNESS MATRICES AND LOAD VECTOR
C
        PRINT 604
        PRINT 640
        PRINT 604
        DO 79 I=1,5
   79   PRINT 641,(EM(I,J),J=1,5)
        PRINT 604
        PRINT 620
        PRINT 604
        DO 9 I=1,5
    9   PRINT 681,(EK(I,J),J=1,5),EQ(I)
C
C       PRINT THE ELEMENT DYNAMIC STIFFNESS MATRIX
C
        PRINT 604
        PRINT 624
        PRINT 604
        DO 14 I=1,5
   14   PRINT 641,(DK(I,J),J=1,5)
C
C       SUBROUTINE REDUCE CONDENSES THE ELEMENTAL EQUATIONS BY ONE
C       ORDER
C
        CALL REDUCE(DK,EQ,EKN,EQN)
C
C       PRINT THE REDUCED ELEMENT DYNAMIC STIFFNESS MATRIX AND LOAD
C       VECTOR
C
        PRINT 604
        PRINT 622
        PRINT 604
        DO 10 I=1,4
   10   PRINT 682,(EKN(I,J),J=1,4),EQN(I)
        DO 36 I=1,4
          LX=NCON(K,I)
          IF(LX.NE.0)GQ(LX,1)=GQ(LX,1)+EQN(I)
          DO 36 J=1,4
            LY=NCON(K,J)
            IF(LX.EQ.0.OR.LY.EQ.0)GO TO 36
            GK(LX,LY)=GK(LX,LY)+EKN(I,J)
   36 CONTINUE
C
C     PRINT GLOBAL DYNAMIC STIFFNESS MATRIX AND LOAD VECTOR
C
      PRINT 604
      PRINT 621
      DO 192 I=1,NTF
        PRINT 604
        PRINT 603,(GK(I,J),J=1,NTF),GQ(I,1)
  192 CONTINUE
C
C     IMSL ROUTINE TO SOLVE LINEAR SYSTEM OF EQUATIONS
C
      PRINT 701
      IDGT=14
      CALL LEQT2F(GK,1,NTF,NTF,GQ,IDGT,WKAREA,IER)
      DO 5 I=1,NTF
        PRINT 700, GQ(I,1)
    5 CONTINUE
      PRINT 602
      A1=DSQRT(E(1)*H(1)**2/RHOV(1)/12.0D0)
      T1=FBAR(1)/RHOV(1)/B(1)/H(1)/WBAR(1)**2
      T2=1.0D0/(2.0D0*DCOS(DSQRT(WBAR(1)/A1)*TL/2.0))
```

```
      T3=1.0D0/(2.0D0*DCOSH(DSQRT(WBAR(1)/A1)*TL/2.0))
      YMAX=T1*(T2+T3-1.0D0)
      SMAX=T1*WBAR(1)/A1*(T3-T2)*E(1)*H(1)/2.0D0
      X1=X2=TLC=0.0D0
      DIFFB=DIFFC=DIFFS=0.0D0
      DO 16 I=1,NE
      Y1=Y2=YC=Y4=Y5=0.0D0
C
C     CONVERT GLOBAL NODAL VALUES TO ELEMENT NODAL VALUES
C
        DO 17 K=1,NTF
          DO 18 J=1,4
            IF(NCON(I,J).EQ.K)THEN DO
              IF(J.EQ.1) Y1=GQ(K,1)
              IF(J.EQ.2) Y2=GQ(K,1)
              IF(J.EQ.3) Y4=GQ(K,1)
              IF(J.EQ.4) Y5=GQ(K,1)
            END IF
 18       CONTINUE
 17     CONTINUE
C
C       THE CONSTRAINT EQUATION IS USED TO RAISE THE ORDER OF THE
C       EQUATIONS BACK TO THEIR ORIGINAL LEVEL
C
        CALL ELMT(EM,EK,DK,EQ,E(I),RHOV(I),B(I),H(I),XL(I),FBAR(I),
     &            WBAR(I))
        YC=(EQ(3)-(DK(3,1)*Y1+DK(3,2)*Y2+DK(3,4)*Y4+DK(3,5)*Y5))
     &     /DK(3,3)
        PRINT 701
        PRINT 711,YC
        PRINT 701
        PRINT 705
C
C       THE VALUES OF THE CONSTANTS OF THE SHAPE FUNCTIONS ARE
C       CALCULATED
C
        C1=Y1
        C2=Y2
        C3=-11.0D0*Y1/XL(I)**2-4.0D0*Y2/XL(I)+16.0D0*YC/XL(I)**2
     &     -5.0D0*Y4/XL(I)**2+Y5/XL(I)
        C4=18.0D0*Y1/XL(I)**3+5.0D0*Y2/XL(I)**2-32.0D0*YC/XL(I)**3
     &     +14.0D0*Y4/XL(I)**3-3.0D0*Y5/XL(I)**2
        C5=-8.0D0*Y1/XL(I)**4-2.0D0*Y2/XL(I)**3+16.0D0*YC/XL(I)**4
     &     -8.0D0*Y4/XL(I)**4+2.0D0*Y5/XL(I)**3
        PRINT 706,I
        PRINT 702,C1,C2,C3,C4,C5
        PRINT 701
        X=ST01=ST03=0.0D0
C
C       DEFLECTIONS, STRESSES AND THEIR CORRESPONDING ERRORS ARE
C       CALCULATED AND PRINTED
C
        PRINT 604
        PRINT 710
        PRINT 604
        WHILE(X.LE.XL(I))DO
          Y=C1+C2*X+C3*X**2+C4*X**3+C5*X**4
          Y2=T1*(DCOS(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X1))*T2
     &       +DCOSH(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X1))*T3-1.0D0)
          Y3=2.0D0*C3+6.0D0*C4*X+12.0D0*C5*X**2
          Y3=Y3*E(I)*H(I)/2.0D0
          Y4=T1*WBAR(I)/A1*(DCOSH(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X1))
     &       *T3-DCOS(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X1))*T2)
          Y4=Y4*E(I)*H(I)/2.0D0
```

```
          ST01=(Y2-Y)/YMAX*100.0D0
          ST03=DABS(Y4)-DABS(Y3)
          ST03=ST03/SMAX*100.0D0
          X=X+TL/60.0D0
          X1=X1+TL/60.0D0
          PRINT 703,Y,Y2,ST01,Y3,Y4,ST03
        END WHILE
        TLC=TLC+XL(I)
        X1=TLC
        X=ST01=ST02=ST03=0.0D0
C
C       THE MAXIMUM ERROR OF THE DEFLECTIONS AND STRESSES ARE
C       DETERMINED, STORED AND PRINTED
C
        WHILE(X.LE.XL(I))DO
          Y=C1+C2*X+C3*X**2+C4*X**3+C5*X**4
          Y2=T1*(DCOS(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X2))*T2
     &       +DCOSH(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X2))*T3-1.0D0)
          Y3=2.0D0*C3+6.0D0*C4*X+12.0D0*C5*X**2
          Y3=Y3*E(I)*H(I)/2.0D0
          Y4=T1*WBAR(I)/A1*(DCOSH(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X2))
     &        *T3-DCOS(DSQRT(WBAR(I)/A1)*(TL/2.0D0-X2))*T2)
          Y4=Y4*E(I)*H(I)/2.0D0
          ST01=DABS(Y2-Y)
          ST03=DABS(Y4)-DABS(Y3)
          IF(ST01.GT.DIFFB) DIFFB=ST01
          IF(X2.EQ.0.0D0.OR.X2.GE.TL) GOTO 85
          IF(ST03.GT.DIFFS) DIFFS=ST03
   85     X=X+TL/500.0D0
          X2=X2+TL/500.0D0
        END WHILE
        X2=TLC
   16 CONTINUE
      PRINT 604
      PRINT, SMAX
      DIFFB=DIFFB/YMAX*100.0D0
      DIFFS=DIFFS/SMAX*100.0D0
      PRINT 707,DIFFB
      PRINT 709,DIFFS
  710 FORMAT('0',T9,'DEFL(N)',11X,'DEFL(E)',10X,'ERROR(D)',10X,
     &          'STRESS(N)',9X,'STRESS(E)',9X,'ERROR(S)')
  707 FORMAT(T2,'DIFFB =',D25.16)
  709 FORMAT(T2,'DIFFS =',D25.16)
  700 FORMAT('0',T2,D16.7)
  701 FORMAT('-')
  702 FORMAT(5(D16.7))
  703 FORMAT(6(D18.9))
  704 FORMAT(4(D16.7))
  705 FORMAT(T3,'CONSTANTS FOR SHAPE FUNCTIONS')
  706 FORMAT(T3,'ELEMENT NO.',I2///)
  711 FORMAT(T2,'THE VALUE OF THE CENTRAL NODE IS',D15.8)
      PRINT 602
  602 FORMAT('1')
  603 FORMAT('0',T2,11D15.7)
  604 FORMAT('0')
  609 FORMAT('0',T23,'ELEMENT NUMBER ',I2)
  610 FORMAT('0',T20,'E(',I2,') =',D16.7)
  611 FORMAT(' ',T20,'B(',I2,') =',D16.7)
  612 FORMAT(' ',T20,'H(',I2,') =',D16.7)
  613 FORMAT(' ',T20,'XL(',I2,') =',D16.7)
  614 FORMAT(' ',T20,'FBAR(',I2,') =',D16.7)
  620 FORMAT('0',T13,'ELEMENT STIFFNESS MATRIX AND ELEMENT LOAD VECTOR',
     &        ' (RIGHT COLUMN)')
  621 FORMAT('0',T10,'GLOBAL STIFFNESS MATRIX AND GLOBAL LOAD VECTOR ',
```

```
      &        '(RIGHT COLUMN)')
  622 FORMAT('0',T13,'REDUCED ELEMENT DYNAMIC STIFFNESS MATRIX',
      &        ' AND LOAD VECTOR (RIGHT) COLUMN')
  624 FORMAT('0',T13,'DYNAMIC ELEMENT STIFFNESS MATRIX')
  630 FORMAT(' ',T20,'RHOV(',I2,') =',D16.7)
  631 FORMAT(' ',T20,'WBAR(',I2,') =',D16.7)
  640 FORMAT('0',T13,'ELEMENT MASS MATRIX')
  641 FORMAT('0',10X,5D12.3)
  644 FORMAT('0',T26,'I=',I2,1X,4I4)
  677 FORMAT(' ',T5,5D18.8)
  678 FORMAT(' ',T5,4D18.8)
  681 FORMAT('0',10X,5D12.3,10X,D12.3)
  682 FORMAT('0',10X,4D21.13,10X,D21.13)
  684 FORMAT(' ',T33,'J=1',1X,'J=2',1X,'J=3',1X,'J=4')
  698 FORMAT('0',T31,'CONNECTION MATRIX, NCON(I,J)')
      STOP
      END
C
C     ******************************************************************
C
      SUBROUTINE ELMT(EMG,EKG,DK,EQG,E,RHOV,B,H,XL,F,W)
C
C     ******************************************************************
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION EKG(5,5),EQG(5),EMG(5,5),DK(5,5)
      EQG(1)=F*7.0D0*XL/30.0D0
      EQG(2)=F*XL*XL/60.0D0
      EQG(3)=F*8.0D0*XL/15.0D0
      EQG(4)=EQG(1)
      EQG(5)=-EQG(2)
      EI=E/12.0D0*B*H**3
      AREA=B*H
      RHO=RHOV*AREA
      DO 33 I=1,5
        DO 33 J=1,5
          EKG(I,J)=0.0D0
          EMG(I,J)=0.0D0
          DK(I,J)=0.0D0
   33 CONTINUE
      EI=EI/5.0D0
      EKG(1,1)=EI*316.0D0/XL**3
      EKG(1,2)=EI*94.0D0/XL**2
      EKG(1,3)=-EI*512.0D0/XL**3
      EKG(1,4)=EI*196.0D0/XL**3
      EKG(1,5)=-EI*34.0D0/XL**2
      EKG(2,2)=EI*36.0D0/XL
      EKG(2,3)=-EI*128.0D0/XL**2
      EKG(2,4)=EI*34.0D0/XL**2
      EKG(2,5)=-EI*6.0D0/XL
      EKG(3,3)=EI*1024.0D0/XL**3
      EKG(3,4)=-EI*512.0D0/XL**3
      EKG(3,5)=EI*128.0D0/XL**2
      EKG(4,4)=EI*316.0D0/XL**3
      EKG(4,5)=-EI*94.0D0/XL**2
      EKG(5,5)=EI*36.0D0/XL
      EMG(1,1)=RHO*XL*13.0D0/63.0D0
      EMG(1,2)=RHO*XL**2/63.0D0
      EMG(1,3)=RHO*XL*4.0D0/63.0D0
      EMG(1,4)=-RHO*XL*23.0D0/630.0D0
      EMG(1,5)=RHO*XL**2/180.0D0
      EMG(2,2)=RHO*XL**3/630.0D0
      EMG(2,3)=RHO*XL**2*4.0D0/630.0D0
      EMG(2,4)=-RHO*XL**2/180.0D0
```

```
      EMG(2,5)=RHO*XL**3/1260.0D0
      EMG(3,3)=RHO*XL*128.0D0/315.0D0
      EMG(3,4)=EMG(1,3)
      EMG(3,5)=-EMG(2,3)
      EMG(4,4)=EMG(1,1)
      EMG(4,5)=-EMG(1,2)
      EMG(5,5)=EMG(2,2)
C
C     CALCULATE THE ELEMENT DYNAMIC STIFFNESS MATRIX
C
      DO 3 I=1,5
        DO 3 J=I,5
          EKG(J,I)=EKG(I,J)
          EMG(J,I)=EMG(I,J)
          DK(I,J)=EKG(I,J)-(W**2*EMG(I,J))
          DK(J,I)=DK(I,J)
    3 CONTINUE
      RETURN
      END
C
C ************************************************************************
C
      SUBROUTINE REDUCE(DK,EQ,EKN,EQN)
C
C ************************************************************************
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION DK(5,5),EKN(4,4),EQ(5),EQN(4)
      II=1
      DO 10 I=1,4
        IF(II.EQ.3) II=II+1
        JJ=1
        DO 11 J=1,4
          IF(JJ.EQ.3) JJ=JJ+1
          EKN(I,J)=DK(II,JJ)-DK(3,JJ)*DK(II,3)/DK(3,3)
          JJ=JJ+1
   11   CONTINUE
        EQN(I)=EQ(II)-EQ(3)*DK(3,II)/DK(3,3)
        II=II+1
   10 CONTINUE
      RETURN
      END
$ENTRY
```