

TIME OPTIMAL CONTROL OF LINEAR
SYSTEMS WITH DELAY

A Thesis

Presented to
the Faculty of Graduate Studies and Research
University of Manitoba

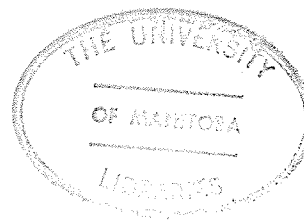
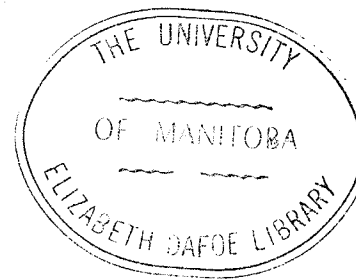
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ABSTRACT

This thesis considers linear time invariant differential-difference systems. Theorems on Controllability, Stability, and Null Controllability of such systems are presented.

The major part of the thesis investigates the problem of determining which control, out of a set of admissible controls, will reduce the state of the system to zero in minimum time. Necessary conditions for such a time optimal control are presented. Two examples are given that demonstrate the technique of finding the time optimal control.

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CHAPTER I

INTRODUCTION

I. GENERAL INTRODUCTION

In recent years, much of the study in the field of control systems engineering has been directed towards the optimization of control systems or processes. Some of these industrial processes have an inherent transport lag and the dynamic behaviour of the perturbed system can be modelled adequately by linear differential difference equations. See for example the mathematical model of a Bipropellant Gas-pressurized Liquid Rocket System given by Day and Hsia¹. Examples of delays are also found in such diversified fields as the study of communications between space vehicles, the study of traffic flow, and in economic theory.

Consider also systems where any time lags may be neglected. Many such plants are of high order and the solution of Pontryagin's Maximum Principle² offers considerable difficulty. In many cases the high order plant may be approximated by a low order plant and a pure delay. This approximation can be quite good (See for example the discussion by Fuller³), and it may offer a more realistic mathematical model of plants with distributed parameters.

¹K.S. Day and T.C. Hsia, "Optimal Control of Linear Time-Lag Systems", 1968 Joint Automatic Control Conference (Ann Harbor, Michigan, June 1968), pp. 1046-1055.

²L.S. Pontryagin et al. The Mathematical Theory of Optimal Processes, (New York: Interscience Publishers, 1962).

³A.T. Fuller, "Optimal Nonlinear Control of Systems with Pure Delay", International Journal of Control, Vol. 8, No. 2, 1968, pp. 145-168.

II. THE SYSTEM

In this thesis systems are considered that are characterized by the linear time-invariant differential-difference system

$$\dot{\underline{x}}(t) = \sum_{i=0}^k A_i \underline{x}(t-\tau_i) + B \underline{u}(t) \quad \text{--- I-1}$$

where

$$0 = \tau_0 < \tau_1 < \dots < \tau_k = \tau$$

$$A_i = n \times n \text{ constant matrix, } i = 0, \dots, k$$

$$B = n \times r \text{ constant matrix}$$

$$\underline{x}(t) = n\text{-dimensional vector}$$

$$\underline{u}(t) = r\text{-dimensional control vector}$$

and all the τ_i are constant.

The state space is the Banach space of continuous functions over a time interval of length τ . From Lee and Markus⁴, Repin⁵, and Reeve⁶ the state of the system at any time t is denoted by the function

$$\underline{x}_t(\sigma) = \underline{x}(t+\sigma), \quad -\tau \leq \sigma \leq 0 \quad \text{--- I-2}$$

If the state, as defined above, of the system in I-1, is given at time t_0 , the output of the system at time t is uniquely determined by the state at time t_0 and the input $\underline{u}(t)$ of the system in (t_0, t) .

⁴E.B. Lee and L. Markus, Foundations of Optimal Control Theory, (New York: John Wiley and Sons, Inc., 1967), pp. 521-522.

⁵Iu.M. Repin, "On the Approximate Replacement of Systems with Lag by Ordinary Dynamical Systems", Journal of Applied Mathematics and Mechanics, Vol. 29, No. 2, 1966 (Russian 1965), pp. 254-264.

⁶P.J. Reeve, "A Method of Approximating to Pure Time Delay", International Journal of Control, Vol. 8, No. 1, 1968, pp. 53-63.

The initial state is

$$\underline{x}_{t_0}(\sigma) = \underline{\phi}(\sigma) \quad , \quad -\tau \leq \sigma \leq 0 \quad \text{--- I-3}$$

where $\underline{\phi}(\sigma)$ is the specified continuous initial function. This definition of state is in agreement with the discussion by Johnson⁷ about the misuse of the term 'state'.

III. A BRIEF SUMMARY AND PREVIOUS RELATED WORK

The first part of this paper is concerned with Controllability, Stability, and Null Controllability of the system. Some previous results on these topics are not always easy to apply. In an article by Weiss⁸ for example it is necessary to solve a differential difference equation to determine controllability. Some results that are easier to apply, but less complete, are given in Chapter II.

The second part of this paper investigates the time optimal regulator problem. That is the problem of determining which control, out of a set of admissible controls, will reduce the error of the system given by I-1 to zero in minimum time and maintain it at zero.

The system given by I-1 has been investigated before. Kharatishvili⁹,

⁷R.A. Johnson, "State Space and Systems Incorporating Delay", Electronics Letters, Vol. 2, No. 7, 1966, pp. 277-278.

⁸Leonard Weiss, "On the Controllability of Delay-Differential Systems", SIAM Journal on Control, Vol. 5, No. 4, 1967, pp. 575-587.

⁹Pontryagin, op. cit., pp. 213-226.

Banks¹⁰, and Oguztoreli¹¹, to name a few, have also considered the time optimal regulator problem. However, they only considered the problem of reducing the error to zero in minimum time, but not the problem of keeping it there afterwards. Chyung and Lee¹², Seierstand¹³, and Halanay¹⁴ use quite a general cost index but they specify the final time.

Fuller¹⁵ considers the time optimal problem of reducing the error to zero and maintaining it at zero in minimum time. For certain examples, he is able to represent the optimal control as a function of the state variables, however, he only considers a lag in the control. For a very large reference list of articles on delay systems, see M.N. Oguztoreli¹⁶.

At the end of this thesis, two examples are given. They were chosen because they had been used before in a thesis by J.D. Stebbing¹⁷ on the time optimal regulator problem where he considered reducing the error to zero in minimum time but not the problem of keeping it at zero. A comparison is given between Stebbing's results and the results of this thesis.

¹⁰H.T. Banks, "Necessary Conditions for Control Problems with Variable Time Lags", SIAM Journal on Control, Vol. 6, No. 1, 1968, pp. 9-47.

¹¹M.N. Oguztoreli, Time-Lag Control Systems. (New York: Academic Press Inc., 1966).

¹²D.H. Chyung and E.B. Lee, "Linear Optimal Systems with Time Delays", SIAM Journal on Control, Vol. 4, 1966, pp. 548-575.

¹³Atle Seierstand, "A Pontryagin Maximum Principle in Banach Space", IEEE Transactions on Automatic Control, Vol. 13, No. 3, 1968, p. 299.

¹⁴A. Halanay, "Optimal Control for Systems with Time Lag", SIAM Journal on Control, Vol. 6, No. 2, 1968, pp. 215-234.

¹⁵Fuller, loc. cit.

¹⁶Oguztoreli, loc. cit.

¹⁷James D. Stebbing, "An Investigation into the Time Optimal Control of Linear Systems with Delay" (unpublished Master's thesis, The University of Manitoba, 1967).

CHAPTER II

CONTROLLABILITY

I. CONTROLLABILITY

Consider the system characterized by

$$\dot{\underline{x}}(t) = \sum_{i=0}^k A_i \underline{x}(t-\tau_i) + B \underline{u}(t) \quad \text{--- II-1}$$

where

$$0 = \tau_0 < \tau_1 < \dots < \tau_k = \tau$$

$$A_i = n \times n \text{ constant matrix, } i = 0, \dots, k$$

$$B = n \times r \text{ constant matrix}$$

$$\underline{x} = n\text{-dimensional vector}$$

$$\underline{u} = r\text{-dimensional control vector}$$

$$\underline{x}_{t_0}(\sigma) = \underline{\phi}(\sigma) \quad -\tau \leq \sigma \leq 0$$

where $\underline{\phi}(\sigma)$ is the continuous initial function.

The state space is the Banach space of continuous functions over a time interval of length τ . Complete controllability to the origin of the state space (function space) on $[t_0, t_f + \tau]$ means that for all given continuous initial functions on $[t_0 - \tau, t_0]$ there exists a piecewise continuous control $\underline{u}(t)$ on $[t_0, t_f + \tau]$ such that $\underline{x}(t) = 0$ for $t \in [t_f, t_f + \tau]$.

Theorem 1

The system II-1 is completely controllable to the origin of the state space only if

$$(sI - \sum_{i=0}^k A_i e^{-s\tau_i})^{-1} B \quad \text{--- II-2}$$

has n independent rows.

Proof. The system II-1 can be approximated to any degree of accuracy by a linear ordinary differential system. See Repin¹ for a good discussion of this approximation method.

Let

$$\left. \begin{aligned} \underline{x}_1(t) &= \underline{x}(t - \frac{\tau}{m}) \\ \underline{x}_2(t) &= \underline{x}(t - \frac{2\tau}{m}) \\ &\vdots \\ \underline{x}_{\ell_i}(t) &= \underline{x}(t - \frac{\ell_i \tau}{m}) \\ &\vdots \\ \underline{x}_m(t) &= \underline{x}(t - \tau) \end{aligned} \right\} \quad \text{--- II-3}$$

$$\text{such that } \tau_i \approx \frac{\ell_i \tau}{m} \quad i = 1, \dots, k \quad \text{--- II-4}$$

$$\ell_i = \text{integer} \leq m \quad \text{--- II-5}$$

$$\ell_k = m \quad \text{--- II-6}$$

Then II-1 can be replaced by

$$\left. \begin{aligned} \dot{\underline{x}}(t) &= \sum_{i=0}^k A_i \underline{x}_{\ell_i}(t) + B \underline{u}(t) \\ \dot{\underline{x}}_1(t) &= \frac{m}{\tau} (\underline{x}(t) - \underline{x}_1(t)) \\ \dot{\underline{x}}_2(t) &= \frac{m}{\tau} (\underline{x}_1(t) - \underline{x}_2(t)) \\ &\vdots \\ \dot{\underline{x}}_m(t) &= \frac{m}{\tau} (\underline{x}_{m-1}(t) - \underline{x}_m(t)) \end{aligned} \right\} \quad \text{--- II-7}$$

¹Iu.M. Repin, "On the Approximate Replacement of Systems with Lag by Ordinary Dynamical Systems", Journal of Applied Mathematics and Mechanics, Vol. 29, No. 2, 1966 (Russian 1965), pp. 254-264.

for m large. Repin has shown that the accuracy of the approximation given by II-7 increases as m increases, and in the limit as $m \rightarrow \infty$, II-7 becomes equal to II-1.

Let

$$\underline{X}(t) = \begin{bmatrix} \underline{x}(t) \\ \underline{x}_1(t) \\ \vdots \\ \underline{x}_m(t) \end{bmatrix}, \text{ a } (m+1)n \text{ vector,} \quad \text{--- II-8}$$

$$C = \begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ a } (m+1)n \times r \text{ matrix,} \quad \text{--- II-9}$$

$$A = \begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 & \cdots & 0 & A_k \\ \frac{m}{\tau} I_n & -\frac{m}{\tau} I_n & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \frac{m}{\tau} I_n & -\frac{m}{\tau} I_n & \cdots & 0 & \cdots & 0 & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \frac{m}{\tau} I_n \\ & & & & & & & -\frac{m}{\tau} I_n \end{bmatrix}$$

$\ell_i + 1$
 \downarrow
 A_i

--- II-10

a $(m+1)n \times (m+1)n$ matrix, where I_n is the $n \times n$ identity matrix. Therefore, from II-7 --- II-10 we have

$$\dot{\underline{X}}(t) = A \underline{X}(t) + C \underline{u}(t) \quad \text{--- II-11}$$

the $(m+1)n$ order approximation to the system II-1.

A necessary condition that the system II-11 be completely controllable is that

$$(sI - A)^{-1}C \quad \text{--- II-12}$$

have $(m+1)n$ independent rows. By repeated row operations on

$$sI_{(m+1)n} - A \quad \left[\begin{array}{cccc} I_n & 0 & \dots & 0 \\ 0 & I_n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & I_n \end{array} \right] \quad \text{--- II-13}$$

the left half can be reduced to the $(m+1)n$ identity matrix and the right half becomes the matrix $(sI - A)^{-1}$, and

$$(sI - A)^{-1} = \left[\begin{array}{ccc} F(s)^{-1} & \dots & \\ \left(\frac{m}{\tau}\right) \left(s + \frac{m}{\tau}\right)^{-1} F(s)^{-1} & \dots & \\ \vdots & & \vdots \\ \left(\frac{m}{\tau}\right)^m \left(s + \frac{m}{\tau}\right)^{-m} F(s)^{-1} & \dots & \end{array} \right] \quad \text{--- II-14}$$

where

$$F(s) = sI_n - \sum_{i=0}^k \left(\frac{m}{\tau}\right)^{\ell_i} \left(s + \frac{m}{\tau}\right)^{-\ell_i} A_i \quad \text{--- II-15}$$

Only the first n columns of $(sI-A)^{-1}$ are given in II-14 because the interest is in $(sI-A)^{-1}C$, and C has at most the first n rows non zero. It is easily verified that II-14 is correct by multiplying $(sI-A)$ times the first n columns of $(sI-A)^{-1}$ as given by II-14.

From II-9 and II-14

$$(sI - A)^{-1}C = \begin{bmatrix} F(s)^{-1} \\ (\frac{m}{\tau})(s + \frac{m}{\tau})^{-1}F(s)^{-1} \\ \vdots \\ (\frac{m}{\tau})^m(s + \frac{m}{\tau})^{-m}F(s)^{-1} \end{bmatrix} B \quad \text{--- II-16}$$

If the system II-1 is controllable to the origin of the state space, then the approximating system II-11 is controllable, to the origin $\underline{X} = 0$ in the limit $m \rightarrow \infty$. If the approximating system II-11 is controllable then from II-12 and II-16, $F(s)^{-1}B$ must have n independent rows. From II-4

$$\ell_i = m \frac{\tau_i}{\tau} \quad \text{--- II-17}$$

and therefore

$$\left(\frac{m}{\tau}\right)^{\ell_i} \left(s + \frac{m}{\tau}\right)^{-\ell_i} = \left(\frac{m}{s\tau + m}\right)^{\ell_i} = \left(\frac{m}{s\tau + m}\right)^{\frac{\tau_i}{m}} \quad \text{--- II-18}$$

From the binomial expansion it is easily verified that

$$\lim_{m \rightarrow \infty} \left(\frac{m}{s\tau + m}\right)^m = e^{-s\tau} \quad \text{--- II-19}$$

Therefore, from II-15 and II-19

$$\lim_{m \rightarrow \infty} F(s) = sI_n - \sum_{i=0}^k e^{-s\tau_i} A_i \quad \text{--- II-20}$$

and $F(s)^{-1}B$ must have n independent rows in the limit as $m \rightarrow \infty$, i.e. if system II-1 is controllable

$$\left(sI - \sum_{i=0}^k A_i e^{-s\tau_i}\right)^{-1}B \quad \text{--- II-21}$$

must have n -independent rows.

Q.E.D.

Consider the special case where

$$A_i \underline{v} = B \underline{u} \quad \text{--- II-22}$$

yields a solution for \underline{u} for any \underline{v} and all $i = 1, 2, \dots, k$. In this case a \underline{u} can be found that exactly cancels out the delay terms.

Theorem 2

If a r -vector \underline{u} can be found to satisfy the equation

$$A_i \underline{v} = B \underline{u} \quad \text{--- II-23}$$

for arbitrary \underline{v} and for all $i = 1, 2, \dots, k$ where the A_i and B are given by the system equation II-1

$$\dot{\underline{x}}(t) = \sum_{i=0}^k A_i \underline{x}(t-\tau_i) + B \underline{u}(t)$$

then the system is completely controllable to the origin of the state space if and only if the matrix

$$[B \quad A_0 B \quad \dots \quad A_0^{n-1} B] \quad \text{--- II-24}$$

has rank n .

Proof. From II-23 we can pick a vector $\underline{\omega}(t)$ such that

$$\sum_{i=1}^k A_i \underline{x}(t-\tau_i) = -B \underline{\omega}(t) \quad \text{--- II-25}$$

Let

$$\underline{u}(t) = \underline{\omega}(t) + \underline{z}(t) \quad \text{--- II-26}$$

The system II-1 then becomes

$$\dot{\underline{x}}(t) = A_0 \underline{x}(t) + B \underline{z}(t) \quad \text{--- II-27}$$

It is well known that the system given in II-27 is completely controllable

to the origin if and only if the matrix²

$$[B \quad A_0 B \quad \dots \quad A_0^{n-1} B] \quad \text{--- II-28}$$

has rank n . If there exists a $\underline{z}(t)$ that brings $\underline{x}(t)$ to the origin in II-27 then the $\underline{u}(t)$ given by II-26 will bring the system in II-1 to the origin. On the other hand, if there exists a $\underline{u}(t)$ that brings $\underline{x}(t)$ in II-1 to the origin, then the $\underline{z}(t)$ given by II-26 will bring $\underline{x}(t)$ in II-27 to the origin. Hence the system given by II-1 is completely controllable to the origin if and only if the matrix given by II-28 has rank n .

Q.E.D.

A more complicated proof where the A_i ($i = 0, 1, \dots, k$) are functions of time is given by Buckalo³.

II. STABILITY

Consider the system II-1 with the control $\underline{u}(t) = 0$

$$\dot{\underline{x}}(t) = \sum_{i=0}^k A_i \underline{x}(t-\tau_i) \quad \text{--- II-29}$$

For

$$\underline{\phi}(\sigma) = \underline{0} \quad -\tau \leq \sigma < 0 \quad \text{--- II-30}$$

$$\underline{\phi}(0) = \underline{x}_0 \quad \text{--- II-31}$$

the Laplace Transform of II-29 is

$$\underline{x}(s) = (sI - \sum_{i=0}^k A_i e^{-\tau_i s})^{-1} \underline{x}_0 \quad \text{--- II-32}$$

²M. Athans and P.L. Falb, Optimal Control, (New York: McGraw-Hill Book Company, 1966), p. 205.

³Andrew F. Buckalo, "Explicit Conditions for Controllability of Linear Systems with Time Lag", IEEE Transactions on Automatic Control, Vol. 13, No. 2, 1968, pp. 193-195.

The Nyquist criterion is easily applied to II-32 to determine if the roots of

$$\Phi(s) = \left| sI - \sum_{i=0}^k A_i e^{-\tau_i s} \right| = 0 \quad \text{--- II-33}$$

all lie in the left hand complex plane⁴. The system is stable if and only if the plot of $\left| sI - \sum_{i=0}^k A_i e^{-\tau_i s} \right|$, as s encircles the right half complex plane, makes no net encirclements of the origin of the complex plane.

The values of the roots of the system II-29 can be determined graphically, to get a better idea of stability using the method by Huang and Li⁵. The method consists of letting

$$s = p_0 + j\omega \quad \text{--- II-34}$$

then

$$\left| sI - \sum_{i=0}^k A_i e^{-\tau_i s} \right| = 0 \quad \text{--- II-35}$$

becomes

$$\text{Re} \left| (p_0 + j\omega)I - \sum_{i=0}^k A_i e^{-p_0 \tau_i} (\cos \tau_i \omega - j \sin \tau_i \omega) \right| = 0 \quad \text{--- II-36}$$

$$\text{Im} \left| (p_0 + j\omega)I - \sum_{i=0}^k A_i e^{-p_0 \tau_i} (\cos \tau_i \omega - j \sin \tau_i \omega) \right| = 0 \quad \text{--- II-37}$$

Equations II-36 and II-37 each can be used to give a plot of ω versus p_0 . The intersection points are the characteristic roots of the system.

⁴N.H. Choksy, "Time Lag Systems", Progress in Control Engineering - 1, (London: Heywood and Co. Ltd., 1962), pp. 17-38.

⁵I. Huang and L.L. Li, "Root Locus Determination of Linear Systems with Transport Lag", IEEE Transactions on Automatic Control, Vol. 12, No. 5, Oct. 1967, pp. 632-634.

III. DOMAIN OF NULL CONTROLLABILITY

For the control process II-1

$$\dot{\underline{x}}(t) = \sum_{i=0}^k A_i \underline{x}(t-\tau_i) + B \underline{u}(t)$$

the domain of null controllability consists of all these continuous functions $\underline{x}_{t_0}(\sigma) = \underline{\phi}(\sigma)$ on $[-\tau, 0]$ that can be steered to the origin of the state space by the r -dimensional control vector $\underline{u}(t)$ on some finite time interval $t \in [t_0, t_f + \tau]$, where each component $u_i(t)$, $i = 1, \dots, r$, of $\underline{u}(t)$ must satisfy the constraint $|u_i(t)| \leq \epsilon$ for a fixed constant $\epsilon > 0$. This definition is analogous to the one for systems without time delay. See, for example, the definition by Lee and Markus⁶.

Theorem 3

Consider the system II-1

$$\dot{\underline{x}}(t) = \sum_{i=0}^k A_i \underline{x}(t-\tau_i) + B \underline{u}(t)$$

with control constraint

$$|u_i(t)| \leq \epsilon \quad i = 1, \dots, r \quad \text{--- II-38}$$

If the system is

- a) Completely Controllable, and
- b) Asymptotically Stable, i.e. every eigenvalue λ has $\text{Re } \lambda < 0$,

⁶E.B. Lee and L. Markus, Foundations of Optimal Control Theory (New York: John Wiley and Sons, Inc., 1967), p. 32.

then the domain of null controllability is $\underline{x}_t(\sigma) \in C^0[-\tau, 0]^*$ where \underline{x} is an n -vector.

Proof. The system II-1 can be approximated to any degree of accuracy by the linear autonomous system without time delays

$$\dot{\underline{X}}(t) = A \underline{X}(t) + C \underline{u}(t) \quad \text{--- II-39}$$

which is defined by equations II-3 to II-11. Controllability of system II-1 implies controllability of system II-39 as $m \rightarrow \infty$. Stability of system II-1 implies stability of system II-39 for m large enough. Therefore, system II-39 satisfies conditions a) and b) of the theorem, and system II-39 has domain of null controllability⁷ $R^{(m+1)n}$. Since

$$\underline{X}(t) = \begin{bmatrix} \underline{x}(t) \\ \underline{x}(t - \frac{\tau}{m}) \\ \underline{x}(t - \frac{2\tau}{m}) \\ \vdots \\ \underline{x}(t - \tau) \end{bmatrix}; \text{ a } (m+1)n \text{ vector,} \quad \text{--- II-40}$$

in the limit as $m \rightarrow \infty$

defines $\underline{x}_t(\sigma) = \underline{x}(t+\sigma)$, $\sigma \in [-\tau, 0]$, it is clear that $\underline{X}(t) = \underline{0}$ is the same as $\underline{x}_t(\sigma) = \underline{0}$. Because the controls are bounded, the control that takes $\underline{X}(t)$ to the origin will take $\underline{x}(t)$ in II-1 to the origin of state space. Therefore, the domain of null controllability is $\underline{x}_t(\sigma) \in C^0[-\tau, 0]$, where \underline{x} is an n -vector.

* $C^0[-\tau, 0]$ denotes the set of real functions continuous for $\sigma \in [-\tau, 0]$.

⁷Ibid., p. 85.

CHAPTER III

CONDITIONS FOR OPTIMALITY

I. THE ADJOINT

A problem. Given the system

$$\dot{\underline{x}}(t) = \sum_{i=0}^k A_i \underline{x}(t-\tau_i) + B \underline{u}(t) \quad \text{--- III-1}$$

with continuous initial function

$$\underline{x}(t_0 + \sigma) = \underline{x}_{t_0}(\sigma) = \underline{\phi}(\sigma) \quad -\tau \leq \sigma \leq 0 \quad \text{--- III-2}$$

where

$$0 = \tau_0 < \tau_1 < \dots < \tau_k = \tau$$

$$A_i = nxn \text{ constant matrix, } i = 0, \dots, k$$

$$B = nxr \text{ constant matrix}$$

$$\underline{x}(t) = n\text{-vector}$$

$$\underline{u}(t) = r \text{ dimensional control vector}$$

$$|u_j(t)| \leq 1 \quad j = 1, 2, \dots, r \quad \text{--- III-3}$$

where the u_j are the components of \underline{u} . Find the control $\underline{u}(t)$ that transfers the state from the initial state, $\underline{x}_{t_0}(\sigma) = \underline{\phi}(\sigma)$, to the zero state, $\underline{x}_{t_f+\tau}(\sigma) = \underline{0}$, $\sigma \in [-\tau, 0]$, in minimum time. Such a $\underline{u}(t)$ will be called a time optimal control.

Theorem 4

Let $\underline{u}(t)$ be a solution to the above problem. Let $\underline{x}_t(\sigma)$ be the state of system III-1 corresponding to $\underline{u}(t)$ and the given initial state $\underline{x}_{t_0}(\sigma)$. Let t_f denote the minimum time such that

$$\underline{x}_{t+\tau}(\sigma) = \underline{x}(t+\tau+\sigma) = \underline{0}, \quad -\tau \leq \sigma \leq 0 \quad \text{--- III-4}$$

Then there exists a corresponding adjoint vector $\underline{p}(t)$ such that

$$a) \quad \dot{\underline{p}}(t) = - \sum_{i=0}^k A_i^T \underline{p}(t+\tau_i), \quad t \in [t_0, t_f] \quad \text{--- III-5}$$

and

$$b) \quad \underline{u}(t) = - \operatorname{sgn} \{ B^T \underline{p}(t) \}, \quad t \in [t_0, t_f] \quad \text{--- III-6}$$

In this thesis $u_j = - \operatorname{sgn}\{0\}$ implies that u_j is not defined.

Proof. The system III-1

$$\dot{\underline{x}}(t) = \sum_{i=0}^k A_i \underline{x}(t-\tau_i) + B \underline{u}(t)$$

can be approximated to any degree of accuracy by the linear time-invariant system without time delays

$$\dot{\underline{X}}(t) = A \underline{X}(t) + C \underline{u}(t) \quad \text{--- III-7}$$

which is defined by equations II-3 to II-11. The accuracy improves as the dimension of the system III-7 is increased. Equation II-3 shows that the initial state $\underline{x}_{t_0}(\sigma) = \underline{\phi}(\sigma)$ implies an initial condition $\underline{X}(t_0) = \underline{X}_0$ for system III-7, and the target $\underline{x}_{t_f+\tau}(\sigma) = \underline{0}$ implies the target set $\underline{X}(t_f+\tau) = \underline{0}$.

Let $\underline{v}(t)$ be the time optimal control for system III-7, steering $\underline{X}(t)$ from \underline{X}_0 to $\underline{0}$. From Pontryagin's Maximum Principle¹ $\underline{v}(t)$ must satisfy

$$\underline{v}(t) = - \operatorname{sgn} \{ C^T \underline{p}(t) \}, \quad t \in [t_0, t_f+\tau] \quad \text{--- III-8}$$

¹E.B. Lee and L. Markus, Foundations of Optimal Control Theory (New York: John Wiley and Sons, Inc., 1967), pp. 129-135.

where $\underline{P}(t)$ is a non-trivial solution of the adjoint equation

$$\dot{\underline{P}}(t) = -A^T \underline{P}(t), \quad t \in [t_0, t_f + \tau] \quad \text{--- III-9}$$

Let

$$\underline{P}(t) = \begin{bmatrix} \underline{p}(t) \\ \underline{p}_1(t) \\ \vdots \\ \underline{p}_m(t) \end{bmatrix} \quad \text{--- III-10}$$

where the $\underline{p}(t)$ and $\underline{p}_i(t)$, $i = 1, \dots, m$; are n -vectors. Equations

II-10 and III-9 give

$$\begin{bmatrix} \dot{\underline{p}} \\ \dot{\underline{p}}_1 \\ \dot{\underline{p}}_2 \\ \vdots \\ \dot{\underline{p}}_{\ell_i} \\ \vdots \\ \dot{\underline{p}}_m \end{bmatrix} = \begin{bmatrix} -A_0^T & -\frac{m}{\tau} I_n & 0 & \dots & 0 \\ 0 & \frac{m}{\tau} I_n & -\frac{m}{\tau} I_n & \dots & 0 \\ 0 & 0 & \frac{m}{\tau} I_n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -A_i^T & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ -A_k^T & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \underline{p} \\ \underline{p}_1 \\ \underline{p}_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \underline{p}_m \end{bmatrix} \quad \text{--- III-11}$$

Reversing time and letting

$$\underline{Z}(t) = \underline{P}(t_f + \tau - t) = \begin{bmatrix} \underline{z}_0(t) \\ \underline{z}_1(t) \\ \vdots \\ \underline{z}_m(t) \end{bmatrix} \quad \text{--- III-12}$$

III-9 becomes

$$\frac{d\underline{P}(t_f + \tau - t)}{d(t_f + \tau - t)} = - \frac{d\underline{P}(t_f + \tau - t)}{dt} = - A^T \underline{P}(t_f + \tau - t) \quad \text{--- III-13}$$

or

$$\dot{\underline{Z}}(t) = A^T \underline{Z}(t) \quad \text{--- III-14}$$

Since the system is time-invariant let

$$\underline{Z}(0) = \underline{P}(t_f + \tau) \quad \text{--- III-15}$$

The Laplace Transform of III-14 yields

$$\underline{Z}(s) = (sI_{n(m+1)} - A^T)^{-1} \underline{Z}(0) \quad \text{--- III-16}$$

The inverse of $(sI_{n(m+1)} - A^T)$ can be found by the method of row transformations. By repeated row operations on

$$\left[\begin{array}{c|ccc} sI_{n(m+1)} - A^T & I_n & 0 & 0 \\ & 0 & I_n & \vdots \\ & \vdots & & \vdots \\ & \vdots & & 0 \\ & 0 & 0 & I_n \end{array} \right] \quad \text{--- III-17}$$

the left side can be reduced to the identity matrix and III-17 becomes

$$\left[\begin{array}{c|ccc} I_n & 0 & 0 & \\ 0 & I_n & \vdots & \\ \vdots & & \vdots & \\ \vdots & & 0 & \\ 0 & 0 & I_n & \end{array} \right]$$

$$\begin{bmatrix}
 F^{-1} & F^{-1} \left(\frac{m}{m+s\tau} \right) \dots \dots \dots F^{-1} \left(\frac{m}{m+s\tau} \right)^m \\
 \vdots & \vdots \\
 (s + \frac{m}{\tau})^{-1} \left(\sum_{i=h}^k A_i^T \left(\frac{m}{m+s\tau} \right)^{\frac{m}{\tau} (\tau_i - \tau_h)} F^{-1} \right) & \vdots \\
 \vdots & \vdots \\
 (s + \frac{m}{\tau})^{-1} A_k^T F^{-1} & (s + \frac{m}{\tau})^{-1} A_k^T F^{-1} \left(\frac{m}{m+s\tau} \right) \dots (s + \frac{m}{\tau})^{-1} A_k^T F^{-1} \left(\frac{m}{m+s\tau} \right)^m + I_n
 \end{bmatrix}$$

--- III-18

where

$$F = sI_n - \sum_{i=0}^k A_i^T \left(\frac{m}{m+s\tau} \right)^m \frac{\tau_i}{\tau}$$

--- III-19

and therefore

$$(sI_{n(m+1)} - A^T)^{-1} = \begin{bmatrix}
 F^{-1} & F^{-1} \left(\frac{m}{m+s\tau} \right) \dots \dots \dots F^{-1} \left(\frac{m}{m+s\tau} \right)^m \\
 \vdots & \vdots \\
 \vdots & \vdots \\
 \vdots & \vdots
 \end{bmatrix}$$

--- III-20

From III-12 and III-16

$$\underline{z}_0(s) = F^{-1} \sum_{\ell=0}^m \left(\frac{m}{m+s\tau} \right)^\ell \underline{z}_\ell(0)$$

--- III-21

Let

$$\underline{z}_0(0) = \underline{c} \quad \underline{z}_\ell(0) = \underline{0}, \ell = 1, \dots, m$$

--- III-22

then

$$\underline{z}_0(s) = F^{-1} \underline{z}_0(0)$$

--- III-23

But

$$\lim_{m \rightarrow \infty} \left(\frac{m}{m+s\tau} \right)^m \frac{\tau_i}{\tau} = e^{-s\tau_i}$$

--- III-24

and from III-19

$$\lim_{m \rightarrow \infty} F = sI_n - \sum_{i=0}^k A_i^T e^{-s\tau_i} \quad \text{--- III-25}$$

and therefore

$$\underline{z}_0(s) = (sI_n - \sum_{i=0}^k A_i^T e^{-s\tau_i})^{-1} \underline{z}_0(0) \quad \text{--- III-26}$$

which is the Laplace transform of the equation

$$\left. \begin{aligned} \dot{\underline{z}}_0(t) &= \sum_{i=0}^k A_i^T \underline{z}_0(t-\tau_i) \\ \underline{z}_0(0) &= \underline{C} \quad \underline{z}_0(t) = \underline{0} \quad t < 0 \end{aligned} \right\} \quad \text{--- III-27}$$

Let

$$\left. \begin{aligned} \underline{z}_\ell(0) &= \underline{0} \quad \ell = 0, 1, \dots, r-1, r+1, \dots, m \\ \underline{z}_r(0) &= \underline{C}_r \end{aligned} \right\} \quad \text{--- III-28}$$

then using the same method, III-21 is the Laplace transform of

$$\left. \begin{aligned} \dot{\underline{z}}_0(t) &= \sum_{i=0}^k A_i^T \underline{z}_0(t-\tau_i) \\ \underline{z}_0(\tau \frac{r}{m}) &= \underline{C}_r \quad \underline{z}_0(t) = 0 \quad t < \tau \frac{r}{m} \end{aligned} \right\} \quad \text{--- III-29}$$

Since equations III-27 and III-29 are linear their combined solutions must also satisfy

$$\dot{\underline{z}}_0(t) = \sum_{i=0}^k A_i^T \underline{z}_0(t-\tau_i) \quad \text{on } t > \tau \quad \text{--- III-30}$$

with the initial function on $[0, \tau]$ equal to the sum of the solutions of III-27 and III-29 on $[0, \tau]$. Therefore $\underline{z}_0(t)$ must satisfy

$$\dot{\underline{z}}_0(t) = \sum_{i=0}^k A_i^T \underline{z}_0(t-\tau_i) \quad \text{on } t > \tau \quad \text{--- III-31}$$

From III-12 and III-15

$$\frac{d\underline{z}_0(t_f + \tau - t)}{d(t_f + \tau - t)} = - \frac{d\underline{p}(t)}{dt} = \sum_{i=0}^k A_i^T \underline{p}(t + \tau_i) \quad \text{--- III-32}$$

or

$$\dot{\underline{p}}(t) = - \sum_{i=0}^k A_i^T \underline{p}(t + \tau_i) \quad \text{on } [t_0, t_f] \quad \text{--- III-33}$$

From II-9 and III-8

$$C^T \underline{p}(t) = B^T \underline{p}(t) \quad \text{--- III-34}$$

and

$$\underline{v}(t) = - \operatorname{sgn}\{B^T \underline{p}(t)\} \quad \text{--- III-35}$$

If $\underline{u}(t)$ is the time optimal control for system III-1, then in the limit as $m \rightarrow \infty$ it is also the time optimal control for system III-7.

Therefore, from III-35 and III-33, $\underline{u}(t)$ must satisfy

$$\underline{u}(t) = - \operatorname{sgn} B^T \underline{p}(t) \quad \text{on } [t_0, t_f] \quad \text{--- III-36}$$

where

$$\dot{\underline{p}}(t) = - \sum_{i=0}^k A_i^T \underline{p}(t + \tau_i) \quad \text{on } [t_0, t_f] \quad \text{--- III-37}$$

Q.E.D.

II. TRANSVERSALITY

Consider the system described by

$$\dot{\underline{x}}(t) = A_0 \underline{x}(t) + A_1 \underline{x}(t - \tau) + B \underline{u}(t) \quad \text{--- III-38}$$

where

$\underline{x}(t)$ is an n -vector

$\underline{u}(t)$ is the r -dimensional control

A_0 , A_1 , and B are constant matrices

$\tau > 0$ is a constant

Suppose B is of rank $q < r$; then there are $r-q$ dependent columns in B . i.e.

$$B = (\underline{b}_1, \dots, \underline{b}_i, \dots, \underline{b}_r) \quad \text{--- III-39}$$

with \underline{b}_i a dependent column at $i = d_1, \dots, d_{r-q}$ and then

$$\underline{b}_i = \alpha_{1i} \underline{b}_1 + \dots + \alpha_{ji} \underline{b}_j + \dots + \alpha_{ri} \underline{b}_r$$

$$\text{for } i = d_1, \dots, d_{r-q} \quad \text{--- III-40}$$

$$j \neq d_1, \dots, d_{r-q}$$

$$\underline{B}\underline{u} = (\underline{b}_1, \dots, \underline{b}_i, \dots, \underline{b}_r) \begin{bmatrix} u_1 \\ \vdots \\ u_i \\ \vdots \\ u_r \end{bmatrix}$$

$$= \underline{b}_1 u_1 + \dots + \underline{b}_i u_i + \dots + \underline{b}_r u_r$$

$$= \underline{b}_1 u_1 + \dots + (\alpha_{1i} \underline{b}_1 + \dots + \alpha_{ji} \underline{b}_j + \dots + \alpha_{ri} \underline{b}_r) u_i + \dots + \underline{b}_r u_r$$

$$= \underline{b}_1 (u_1 + \sum_i \alpha_{1i} u_i) + \dots + \underline{b}_r (u_r + \sum_i \alpha_{ri} u_i)$$

$$= (\underline{b}_1, \dots, \underline{b}_j, \dots, \underline{b}_r) \begin{bmatrix} u_1 + \sum_i \alpha_{1i} u_i \\ \vdots \\ u_j + \sum_i \alpha_{ji} u_i \\ \vdots \\ u_r + \sum_i \alpha_{ri} u_i \end{bmatrix}$$

$$= B_q \underline{u}_q \quad \text{--- III-41}$$

$$\begin{aligned} \text{for } i &= d_1, \dots, d_{r-q} \\ j &\neq d_1, \dots, d_{r-q} \end{aligned}$$

Hence

Theorem 5

If B , the $n \times r$ matrix given in III-38, has rank $q < r$, then the forcing term $B\mathbf{u}$ may be replaced by a forcing term of the form $B_q \mathbf{u}_q$ without any loss of control; where B_q is a $n \times q$ matrix of rank q , and \mathbf{u}_q is a q vector.

It will be assumed, without loss of generality, for the rest of this section that the $n \times r$ matrix in III-38 is of rank r , with the control $\mathbf{u}(t)$ an r -vector. Since B is of rank r it can be reduced to its normal form by elementary row transformations². Therefore, there exists a nonsingular $n \times n$ matrix H such that

$$HB = \begin{bmatrix} I_r \\ 0 \end{bmatrix} \text{ an } n \times r \text{ matrix} \quad \text{--- III-42}$$

where I_r is the $r \times r$ identity matrix.

The target for our problem is the zero state

$$\mathbf{x}_{t_f+\tau}(\sigma) = \mathbf{x}(t_f+\tau+\sigma) = \mathbf{0}, \quad \sigma \in [-\tau, 0] \quad \text{--- III-43}$$

$$\text{or} \quad \mathbf{x}(t) = \mathbf{0} \quad \text{for } t \in [t_f, t_f+\tau] \quad \text{--- III-44}$$

This implies by substitution in III-38 that

$$A_1 \mathbf{x}(t-\tau) + B \mathbf{u}(t) = \mathbf{0}, \quad t \in [t_f, t_f+\tau] \quad \text{--- III-45}$$

$$\text{or} \quad A_1 \mathbf{x}(t) + B \mathbf{u}(t+\tau) = \mathbf{0}, \quad t \in [t_f-\tau, t_f] \quad \text{--- III-46}$$

²Frank Ayres, Theory and Problems of Matrices (New York: Schaum Publishing Co., 1962), pp. 41-42.

and $\underline{x}(t_f) = 0$

--- III-47

Multiplying III-46 by H , from III-42

$$HA_1 \underline{x}(t) + \begin{bmatrix} I_r \\ 0 \end{bmatrix} \underline{u}(t+\tau) = 0, \quad t \in [t_f - \tau, t_f] \quad \text{--- III-48}$$

The last $n-r$ equations of III-48 define the conditions that $\underline{x}(t)$ must satisfy on $[t_f - \tau, t_f]$, and with the condition $\underline{x}(t_f) = 0$ these define a new target.

If $\underline{u}(t)$ is bounded by

$$|u_i(t)| \leq 1, \quad i = 1, \dots, r \quad \text{--- III-49}$$

then there is the additional restriction

$$\begin{aligned} -1 &\leq \text{1st row of } HA_1 \underline{x}(t) < 1 \\ &\vdots \\ -1 &\leq \text{rth row of } HA_1 \underline{x}(t) < 1 \end{aligned} \quad t \in [t_f - \tau, t_f] \quad \text{--- III-50}$$

If III-38 is approximated by the $n(m+1)$ dimensional equation given in II-11

$$\dot{\underline{X}}(t) = A \underline{X}(t) + C \underline{u}(t) \quad \text{--- III-51}$$

then

$$\underline{X}(t) = \begin{bmatrix} \underline{x}(t) \\ \underline{x}_1(t) \\ \vdots \\ \underline{x}_m(t) \end{bmatrix} = \begin{bmatrix} \underline{x}(t) \\ \underline{x}(t - \frac{\tau}{m}) \\ \vdots \\ \underline{x}(t - \tau) \end{bmatrix} \quad \text{--- III-52}$$

approximates the state of III-38 for m large. From III-47 to III-50, the target set for $\underline{X}(t)$ is defined by

$$\underline{x}(t_f) = 0 \quad \text{--- III-53}$$

with the last $n-r$ rows of $HA_1 \underline{x}_i(t_f) = 0$ for $i = 1, \dots, m$. If the

components of \underline{u} are bounded as in III-49 then there is the restriction

$$\left. \begin{array}{l} -1 \leq \text{1st row of } HA_{1-i} \underline{x}_i(t_f) \leq 1 \\ -1 \leq \text{rth row of } HA_{1-i} \underline{x}_i(t_f) \leq 1 \end{array} \right\} \quad i = 1, \dots, m \quad \text{--- III-54}$$

Let the last $n-r$ rows of $HA_{1-i} \underline{x}_i = 0$ be denoted by

$$f_j(\underline{x}_i) = 0, \quad j = 1, \dots, n-r \quad \text{--- III-55}$$

Let $S_i(f_1, \dots, f_{n-r})$ denote the set of points \underline{x}_i in R_n at which all the $f_j(\underline{x}_i) = 0, j = 1, \dots, n-r$. Since the $f_j(\underline{x}_i) = 0$ are linear, they define hyper planes in R_n , and since $\underline{x}_i = 0$ is a solution to III-55

S_i is not empty. Therefore, $S_i(f_1, \dots, f_{n-r})$ is a smooth r -fold in R_n and a vector \underline{p}_i is transversal to S_i at some $\underline{x}_{i0} \in S_i$ if and only if \underline{p}_i is a linear combination of the $n-r$ vectors

$$\frac{\partial f_1}{\partial \underline{x}_i}, \frac{\partial f_2}{\partial \underline{x}_i}, \dots, \frac{\partial f_{n-r}}{\partial \underline{x}_i} \quad \text{--- III-56}$$

at \underline{x}_{i0} ³. If S_i is a smooth r -fold in R_n then the target set is a smooth mr -fold in $R_{n(m+1)}$ with boundaries defined by the equality signs of III-54. Therefore, a vector \underline{P} is transversal to the target if and only if \underline{P} is a linear combination of the $n(m+1)-mr$ vectors

$$\frac{\partial f_j(\underline{x}_i)}{\partial \underline{x}} = \begin{bmatrix} \frac{\partial}{\partial \underline{x}} \\ \vdots \\ \frac{\partial}{\partial \underline{x}_i} \\ \vdots \\ \frac{\partial}{\partial \underline{x}_m} \end{bmatrix} f_j(\underline{x}_i) = \begin{bmatrix} 0 \\ \vdots \\ \frac{\partial f_j(\underline{x}_i)}{\partial \underline{x}_i} \\ \vdots \\ 0 \end{bmatrix} \quad \text{for } i = 1, \dots, m; \quad j = 1, \dots, n-r;$$

and

$$\frac{\partial \underline{x}_\ell}{\partial \underline{x}} \quad \text{for } \ell = 1, \dots, n \quad \text{--- III-57}$$

³M. Athans and P.L. Falb, Optimal Control (New York: McGraw Hill Book Company, 1966), pp. 94-95.

where x_ℓ is a member of $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

If $\underline{p}(t)$ is the non-trivial solution to the adjoint equation of Pontryagin's Maximum Principle as given by III-9, then it must be transversal to the target set at $t = t_f^4$. i.e. If

$$\underline{p}(t) = \begin{bmatrix} p(t) \\ p_1(t) \\ \vdots \\ p_m(t) \end{bmatrix}, \quad p(t) = n\text{-vector}, p_i(t) = n\text{-vector};$$

i = 1, ..., m

--- III-58

then from III-55, III-56, and III-57, the $p_i(t_f)$ are linear combinations of

$$\frac{\partial f_j(\underline{x}_i)}{\partial \underline{x}_i}, \quad j = 1, \dots, n-r;$$

--- III-59

and since there is no tangent at $\underline{x}(t_f) = 0$, $\underline{p}(t_f)$ is arbitrary.

In the case where $r = n$, i.e. B is of rank n , then $p_i(t_f) = 0$, $i = 1, \dots, m$. In the case where the last $n-r$ rows of $HA_1\underline{x}$ are identically zero⁵, the $\frac{\partial f_j(\underline{x}_i)}{\partial \underline{x}_i} = 0$ for all $j = 1, \dots, n-r$, and $p_i(t_f) = 0$, $i = 1, \dots, m$. In reverse time this corresponds to equations III-22 to III-27, which in forward time corresponds to the condition

$$\underline{p}(t_f) \neq 0, \quad \underline{p}(t) = 0 \quad t > t_f$$

--- III-60

⁴Ibid., p. 280.

⁵This corresponds to the condition given in II-23

in the $\lim \rightarrow \infty$. This acts as the final function for the adjoint equation

$$\dot{\underline{p}}(t) = -A_0^T \underline{p}(t) - A_1^T \underline{p}(t+\tau) \quad \text{--- III-61}$$

This is equivalent to the condition that $\underline{p}(t)$ must satisfy

$$\dot{\underline{p}}(t) = -A_0^T \underline{p}(t) \quad , \quad t \in [t_f - \tau, t_f] \quad \text{--- III-62}$$

with $\underline{p}(t_f)$ unspecified. If the optimal trajectory hits the target at some point $\underline{x}_b(t_f)$ on the boundary defined by the equalities of III-54, the transversal may not be defined, i.e. there may be a $\underline{p}_b(t_f) \neq 0$. From III-28 to III-30 we see that we must add to the final function defined by III-62, the function that satisfies

$$\dot{\underline{p}}(t) = -A_0^T \underline{p}(t) \quad \text{--- III-63}$$

with $\underline{p}(t_f - \tau \frac{b}{m}) \neq 0$, $\underline{p}(t) = 0$, $t > t_f - \tau \frac{b}{m}$. That is if $\underline{x}(t)$ satisfies the equalities of III-50 at $t = t_f - \tau \frac{b}{m}$ then the final function is the sum of the solution of III-62 and the solution of III-63.

CHAPTER IV

EXAMPLES

I. FIRST ORDER SYSTEM

The control system considered in this example is shown in Fig. 1.

The difference between the desired mixture and the actual mixture at the pump outlet will be denoted by x . When the motor is turning at constant speed, the mixture will be constant. The motor has a constant field current and its speed is controlled by the armature voltage. The motor's transfer function contains only one time constant, due to the moment of inertia of the pump and the motor and the constant of proportionality relating the motor speed to its back emf. The gains of the motor and amplifier are assumed to be linear and are lumped together. The resulting block diagram of the system error is shown in Fig. 2.

The delay time τ is given by the quotient, D/v , where v is the velocity of the fluid in the pipe and D is the distance from the additive input port to the point where the concentration of the mixture is measured. The concentration cannot be measured at the input port as time must be allowed for the additive to mix with the fluid. It is assumed that there is no change in velocity so that the time delay is constant.

The problem to be solved is to determine the control signal, $u(t)$, so that the error, $x(t)$, is reduced to zero, in minimum time, and stays there. There is a constraint on the magnitude of the control signal. That is the inequality

$$|u(t)| \leq 1$$

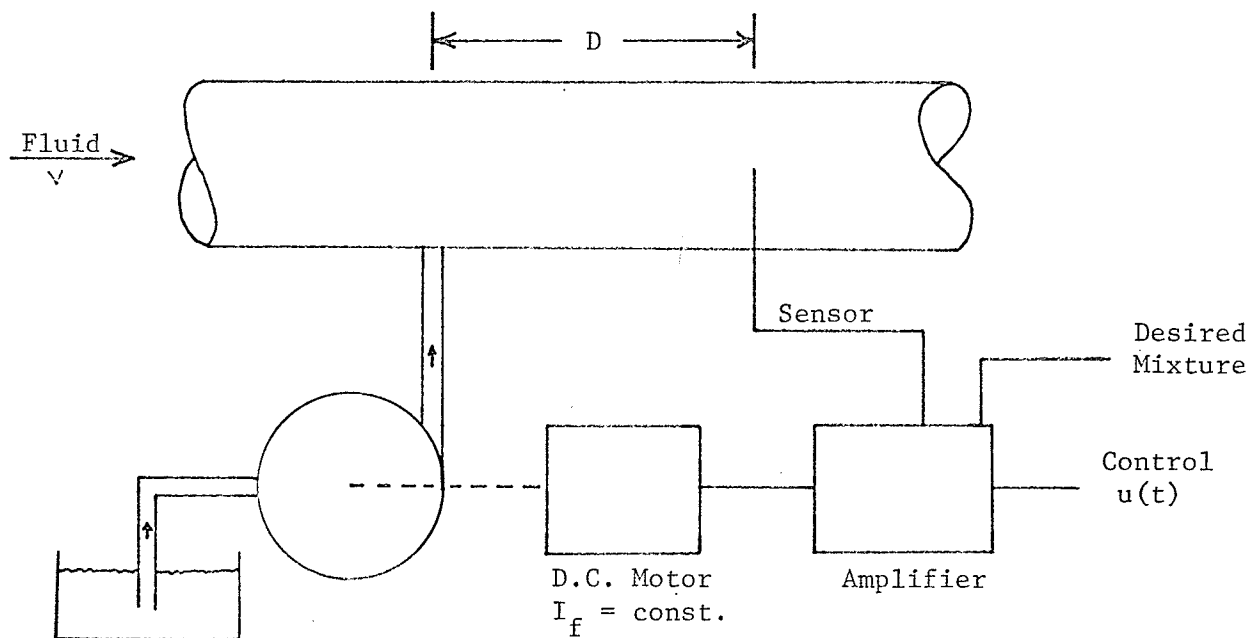


Fig. 1

Mixture Control System

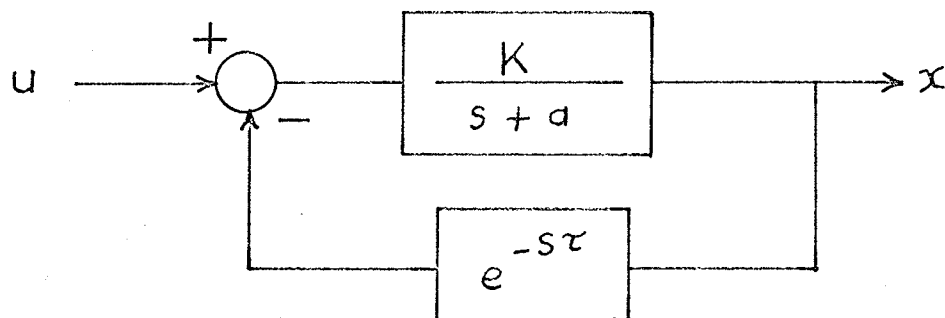


Fig. 2

Mixture Control System

Block Diagram

must be satisfied at all times.

This system, then, is described by the differential-difference equation

$$\dot{x}(t) = -a x(t) - K x(t-\tau) + K u(t) \quad \text{--- IV-2}$$

For a unique solution, an initial function,

$$x_{t_0}(\sigma) = \phi(\sigma) \quad \text{on} \quad [-\tau, 0] \quad \text{--- IV-3}$$

must be specified, where t_0 is the initial time and $\phi(\sigma)$ is a given function.

For this problem the gain K was chosen equal to 2.0, the time delay $\tau = 1.0$, $a = 1.0$. Then IV-2 becomes

$$\dot{x}(t) = -x(t) - 2x(t-1) + 2u(t) \quad \text{--- IV-4}$$

The optimal control for this problem may be found by using Theorem 4 and the transversality condition given in Chapter III. From Theorem 4, the optimal control must satisfy

$$u(t) = -\operatorname{sgn} \left\{ B^T p(t) \right\} \quad t \in [t_0, t_f] \quad \text{--- IV-5}$$

where the adjoint $p(t)$ must satisfy

$$\dot{p}(t) = - \sum_{i=0}^k A_i^T p(t+\tau_i) \quad t \in [t_0, t_f] \quad \text{--- IV-6}$$

In our problem

$$B = K = 2.0, \quad k = 1, \quad \tau_0 = 0, \quad \tau_1 = 1.0 \quad \text{--- IV-7}$$

$$A_0 = -a = -1.0, \quad A_1 = -K = -2.0 \quad \text{--- IV-8}$$

and equations IV-5 and IV-6 become

$$u(t) = \operatorname{sgn} \{ 2p(t) \} \quad t \in [t_0, t_f] \quad \text{--- IV-9}$$

$$\dot{p}(t) = p(t) + 2p(t+1) \quad t \in [t_0, t_f] \quad \text{--- IV-10}$$

From Theorem 2 the system IV-4 is completely controllable to the origin of the state space since

a) $u = -v$ satisfies $A_1 v = Bu$ and

b) the matrix $[B] = [2]$ has rank $n = 1$.

The roots of the system IV-4 can be determined using the method of Section II, Chapter II. Since in this example, the solution of the adjoint equation IV-10 is just the reverse time solution of the system equation IV-4 with $u = 0$, the roots will be the same but of opposite sign for IV-10 and IV-4. The Laplace transform of IV-4 with $u = 0$ is

$$x(s) = \frac{x(0) - 2 \int_{-1}^0 x(t) e^{-s(t+1)} dt}{s + 1 + 2e^{-s}} \quad \text{--- IV-11}$$

The characteristic roots of equation IV-4 are the solutions of

$$s + 1 + 2e^{-s} = 0 \quad \text{--- IV-12}$$

Let $s = p_0 + j\omega$ --- IV-13

and IV-12 becomes

$$p_0 + j\omega + 1 + 2e^{-p_0}(\cos \omega - j \sin \omega) = 0 \quad \text{--- IV-14}$$

or

$$p_0 + 1 + 2e^{-p_0} \cos \omega = 0 \quad \text{--- IV-15}$$

$$\omega - 2e^{-p_0} \sin \omega = 0 \quad \text{--- IV-16}$$

These two equations are plotted in Fig. 3, the intersection points give the roots of the equation. Fig. 3 shows that the dominant root has

$$s = -0.09 + j 2.0 \quad \text{--- IV-17}$$

Therefore $p(t)$ will have a frequency of

$$f \approx \frac{\omega}{2\pi} = \frac{2}{2\pi} = 0.32 \quad \text{--- IV-18}$$

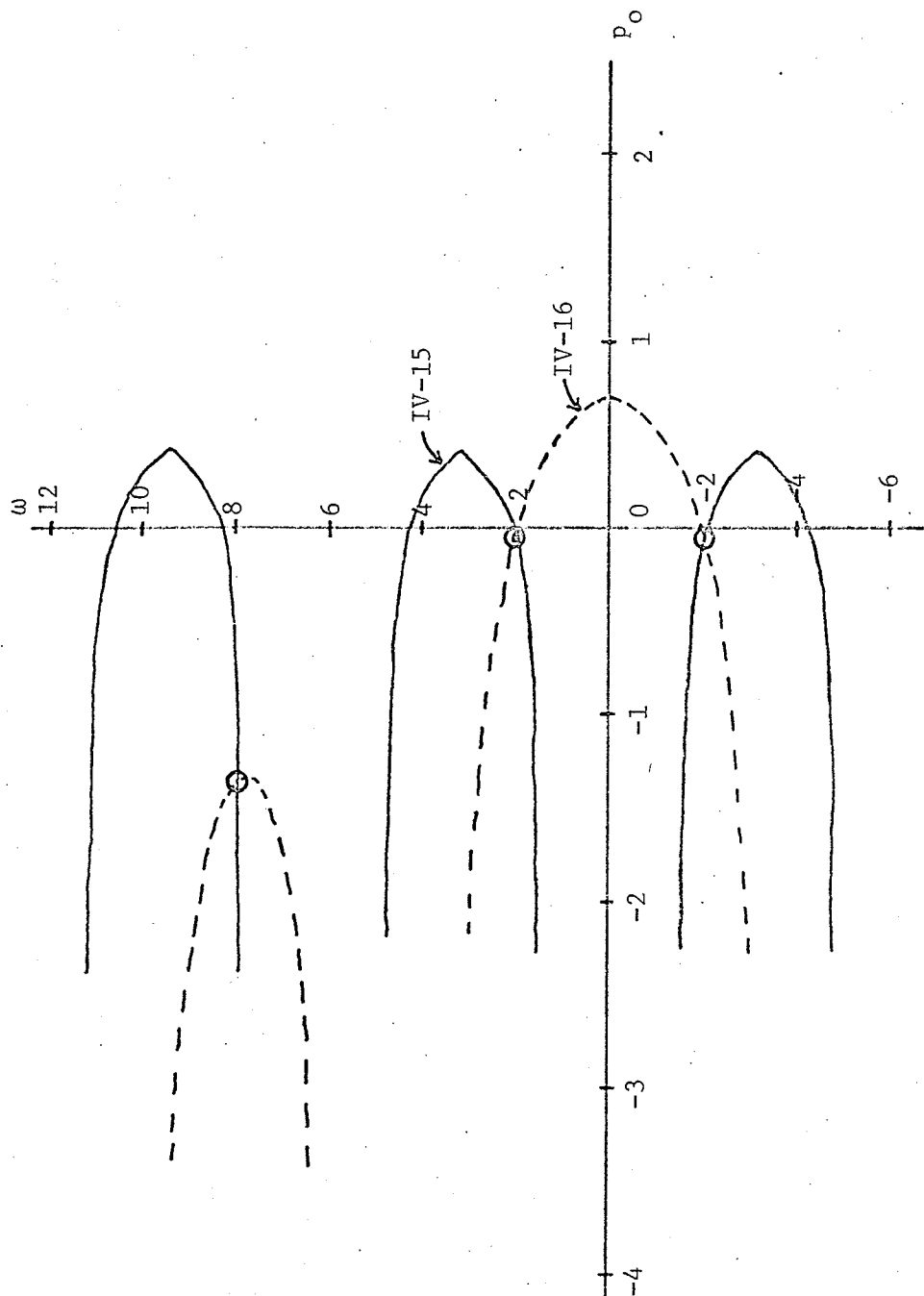


Fig. 3

Poles of System Equation in Example No. 1

and the zero crossings will be approximately

$$\frac{T}{2} = \frac{1}{2f} = 1.6 \text{ seconds} \quad \text{--- IV-19}$$

apart.

Therefore

$$p(t) \approx D \cos(2t+\theta) \quad \text{--- IV-20}$$

and from IV-9

$$u(t) \approx -\operatorname{sgn} \{2D \cos(2t+\theta)\} \quad \text{--- IV-21}$$

for some θ .

Fig. 3 shows that the roots of the system all have $\operatorname{Re} s < 0$, and therefore the system is stable. The system has been shown to be controllable, and therefore by Theorem 3 the domain of null controllability is $x_t(\sigma) \in C^0[-1, 0]$; i.e. any continuous initial function can be steered to the origin by a $u(t)$ with $|u| \leq 1$.

In this example, as in all 1st order systems, the rank of $B = K$ in IV-2 is the same as the order of the system. Therefore from the transversality condition given by III-62, $p(t)$ must satisfy

$$\dot{p}(t) = -A_0^T p(t) = p(t) \quad t \in [t_f - \tau, t_f] \quad \text{--- IV-22}$$

$$p(t_f) = \pi \quad \text{--- IV-23}$$

Equations IV-4 and III-50 show that the boundary of the target set is defined by

$$|x(t)| = 1 \quad t \in [t_f - 1, t_f] \quad \text{--- IV-24}$$

or

$$|x_{t_f}(\sigma)| = 1 \quad \sigma \in [-1, 0] \quad \text{--- IV-25}$$

If $x(t)$ hits this boundary at $t_f - \tau \frac{b}{m}$ then from the discussion in

Chapter III and III-63, the function that satisfies

$$\dot{p}(t) = p(t) \quad t \in [t_f - 1, t_f] \quad \text{--- IV-26}$$

$$p(t_f - \tau \frac{b}{m}) = \pi_b, \quad p(t) = 0 \quad t > \tau \frac{b}{m} \quad \text{--- IV-27}$$

must be added to the function given by IV-22 and IV-23. The IBM 360/65 digital computer was used to solve for $p(t)$ in the case where

$$|x(t_f - 1)| = 1 \quad \text{--- IV-28}$$

$$|x(t)| < 1 \quad t \in [t_f - 1, t_f] \quad \text{--- IV-29}$$

In this case $p(t)$ is the sum of solutions to

$$\dot{p}(t) = p(t) \quad t \in [t_f - 1, t_f] \quad \text{--- IV-30}$$

$$\text{with } p(t_f) = \pi \quad \text{--- IV-31}$$

$$\text{and with } p(t_f - 1) = \pi, \quad p(t) = 0 \quad t > t_f - 1 \quad \text{--- IV-32}$$

From IV-10 $p(t)$ is the solution to

$$\dot{p}(t) = p(t) + 2p(t+1) \quad t \in [t_0, t_f - 1] \quad \text{--- IV-33}$$

$p(t)$ is shown in Fig. 4 for various values of π and π_1 . If $|x(t)| < 1$ for $t \in [t_f - 1, t_f]$ i.e. $x(t)$ doesn't hit the boundary, then $p(t)$ corresponds to the solution given in Fig. 4 with $\pi_1 = 0$.

Fig. 4 shows that $p(t)$ has zero crossings approximately 1.55 seconds apart which agrees well with IV-19. The system equation IV-4 was solved on the IBM 360/65 digital computer using the Continuous System Modeling Program. Since $u(t) = -\text{sgn}\{2p(t)\}$, the switchings were made 1.55 seconds apart except for the last switching which may vary slightly as seen in Fig. 4. Different $u(t)$ were tried by varying the time of the first switching and the last switching, until the time optimal trajectory

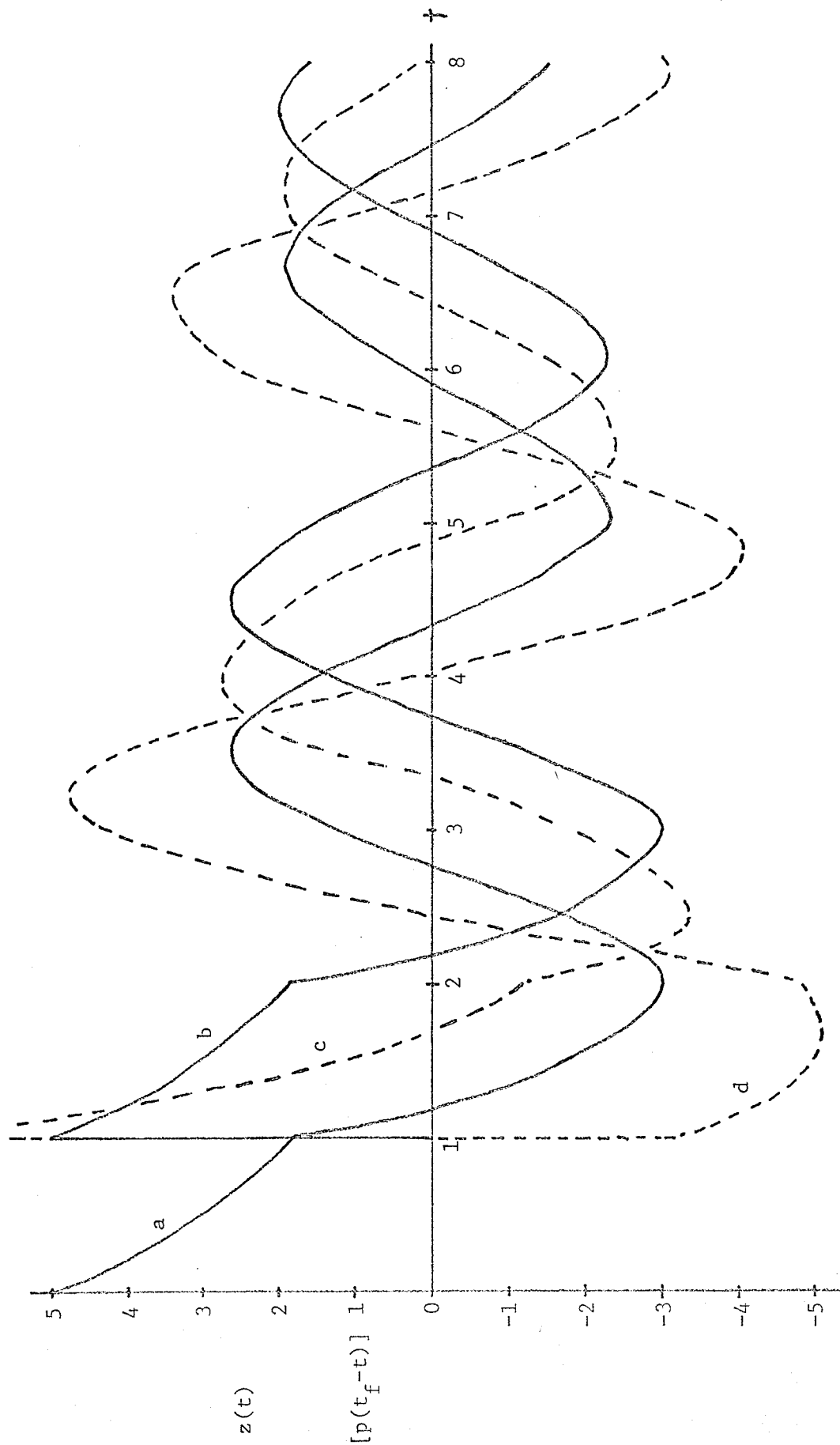


Fig. 4

Possible Values of $z(t) = p(t_f - t)$

- (a) Initial condition $z(0) = 5$, $z(t) = 0$, $t < 0$
- (b) Initial condition $z(1) = 5$, $z(t) = 0$, $t < 1$
- (c) Sum of (a) and (b)
- (d) Difference between (a) and (b)

was found. Fig. 5 shows the computer input statements for the initial function $x(t) = 5$ on $t \in [-1, 0]$.

The time optimal trajectory, the trajectory for $u = 0$, and the time optimal control for the initial function $x(t) = 5$ on $t \in [-1, 0]$ are shown in Fig. 6. Time optimal controls were also found, using the above method for the initial states $x_0(\sigma) = 1.1, 2.0, 3.0$. The time optimal trajectories and controls are shown in Fig. 7.

The system equation IV-4 has been used previously in an example of time optimal controls¹. In this example, however, the object was to get the vector $x(t)$ to zero in minimum time without any consideration to keeping it at zero afterwards. The optimum $u(t)$ found for an initial function $x(t) = 5$ on $t \in [-1, 0]$ brought $x(t)$ to zero in 0.35 seconds, but $x(t)$ could not be kept there after this time. Fig. 6 shows that a minimum time of 4.4 seconds is necessary to bring $x(t)$ to zero and keep it there after this time.

The system equation IV-4 was approximated by the method given by equations II-3 to II-11 in order to get a feel for the method. For $m = 10$

$$\left. \begin{aligned} x_1(t) &= x(t - \frac{1}{10}) \\ x_2(t) &= x(t - \frac{2}{10}) \\ &\vdots \\ x_{10}(t) &= x(t-1) \end{aligned} \right\} \quad \text{--- IV-34}$$

¹James D. Stebbing, "An Investigation into the Time Optimal Control of Linear Systems with Delay" (unpublished Master's Thesis, The University of Manitoba, 1967).

****CONTINUOUS SYSTEM MODELING PROGRAM****

PROBLEM INPUT STATEMENTS

```

IN=2.*VERR
X=REALPL(5.0,1.0,IN)
VERR=U-XD
DELOUT=DELAY(100,1.,X)
XIC=COMPAR(.999,TIME)
XD=5.*XIC-DELOUT
UA=COMPAR(TIME,TA)
UB=COMPAR(TIME,TB)
UC=COMPAR(TIME,TC)
U=-1.0+2.*UA-2.*UB+2.*UC
TIMER DELT=0.01,FINTIM=5.0,PRDEL=0.01,OUTDEL=0.01
PRTPLT X(U,XD)
PARAM TA=0.08,TB=1.65,TC=3.23
END
PARAM TA=0.11,TB=1.68,TC=3.26
END
PARAM TA=0.13,TB=1.70,TC=3.29
END
STOP

```

Fig. 5

Computer Input Statements Example No. 1

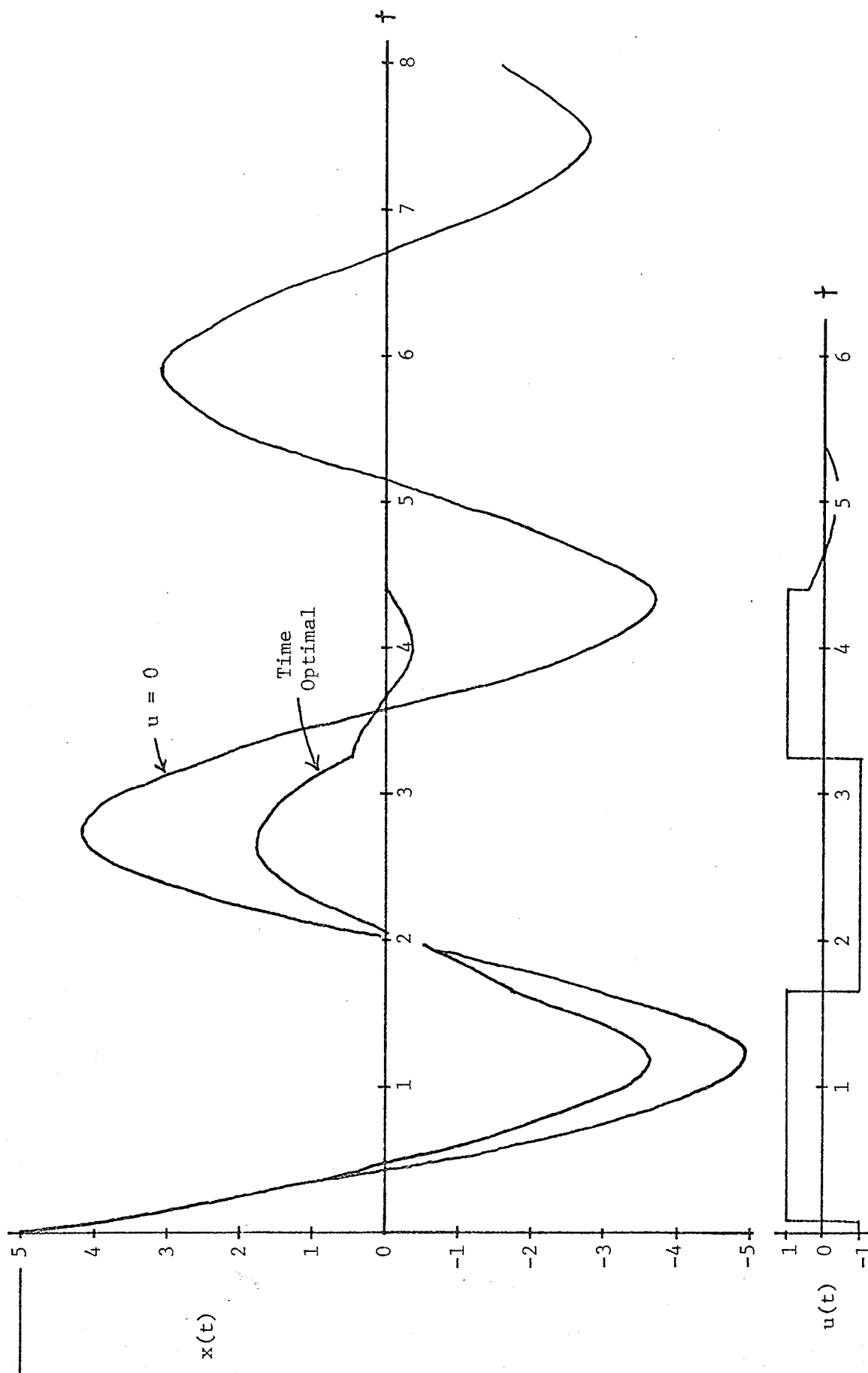


Fig. 6

Solutions For Example 1

 $x(t) = 5, -1 < t < 0$

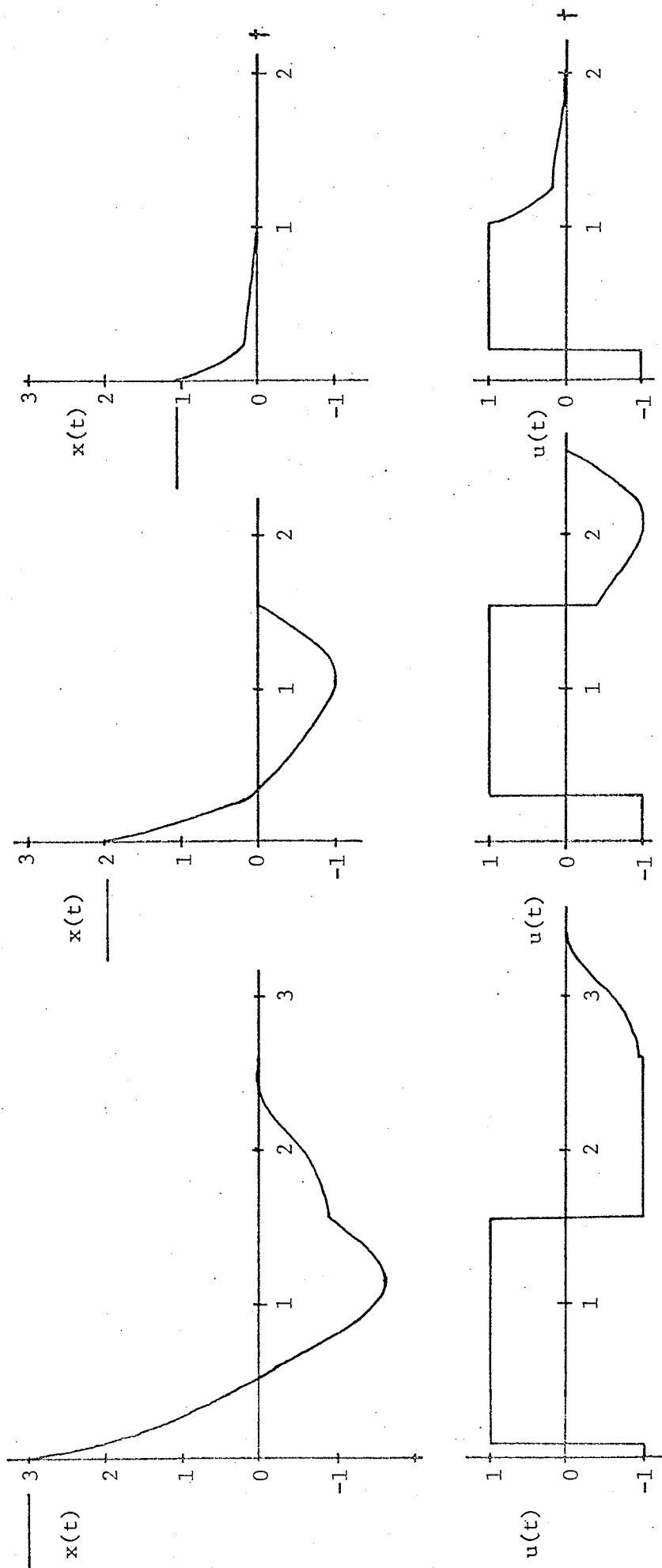


Fig. 7

Time Optimal Solutions For Example 1

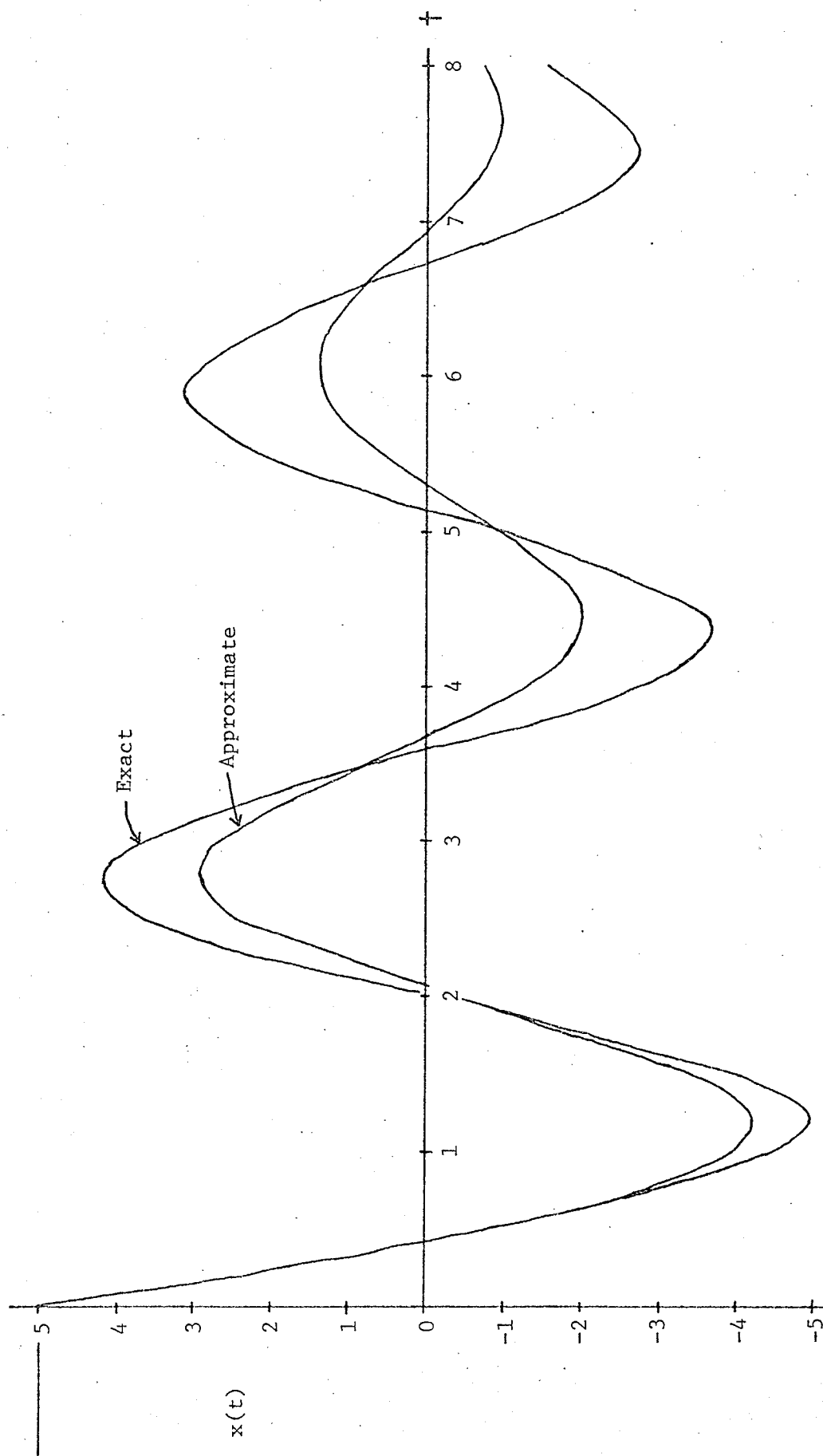


Fig. 8

Solutions to the System Equations of Example 1

For $u(t) = 0$

and for $u = 0$

$$\left. \begin{aligned} \dot{x}(t) &= -x(t) \\ \dot{x}_1(t) &= 10(x(t) - x_1(t)) \\ &\vdots \\ \dot{x}_{10}(t) &= 10(x_9(t) - x_{10}(t)) \end{aligned} \right\} \text{--- IV-35}$$

with

$$x(0) = x_1(0) = \dots = x_{10}(0) = 5.0 \quad \text{--- IV-36}$$

The solution to the approximating equations IV-34 and IV-35, and the exact solution are shown in Fig. 8. In this case although a fairly large order approximation was used, the accuracy was poor. Caution should be exercised in use of such finite approximations.

II. SECOND ORDER SYSTEM

In the mixture control system of the previous example, if the motor and pump are replaced by a valve whose position is determined by a motor, then the system will take on the form of Figure 9.

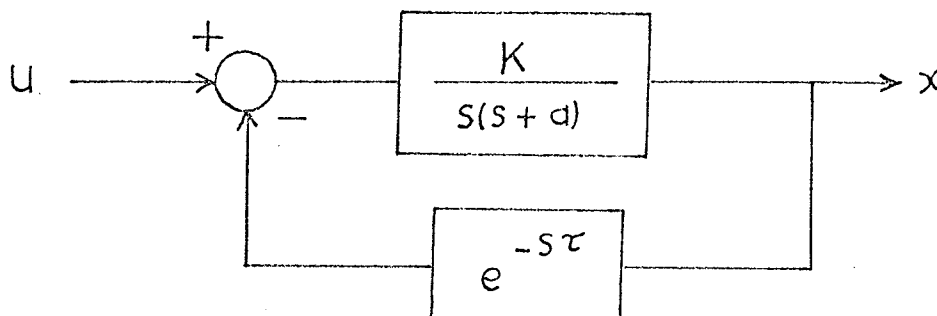


Figure 9

SYSTEM BLOCK DIAGRAM



It is assumed that the motor's field current is constant so that the motor shaft position-armature voltage transfer function is the standard form used above.

The problem is to determine the allowable control signal $u(t)$ such that the error is reduced to zero in minimum time and stays at zero after this time. As before, there is a constraint on the control signal. The constraint is given by

$$|u(t)| \leq 1 \quad \text{--- IV-37}$$

The error will be given by the differential difference equation

$$\ddot{x}(t) = -a \dot{x}(t) - K x(t-\tau) + K u(t) \quad \text{--- IV-38}$$

For a unique solution to this equation, the initial state $x_{t_0}(\sigma) = \phi(\sigma)$, $\sigma \in [-\tau, 0]$, and $u(t)$ must be specified, where $\phi(\sigma)$ is a given function. Let

$$x(t) = x_1(t), \quad \dot{x}(t) = x_2(t) \quad \text{--- IV-39}$$

then the system equation IV-38 may be replaced by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -K & 0 \end{bmatrix} \begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{bmatrix} + \begin{bmatrix} 0 \\ K \end{bmatrix} u(t)$$

and for $a = 5$, $K = 5$, $\tau = 1$

--- IV-40

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} u(t)$$

--- IV-41

From Theorem 2 the system IV-41 is completely controllable to the origin of the state space since

$$a) \begin{bmatrix} 0 & 0 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} u$$

--- IV-42

with $u = -v_1$ and

$$b) [B|A_0B] = \begin{bmatrix} 0 & 5 \\ 5 & -25 \end{bmatrix}$$

--- IV-43

has rank $n = 2$.

The stability of the system is easily determined using the Nyquist criterion given in Section II, Chapter II. From II-33 and IV-41

$$\begin{aligned} \Phi(s) &= \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -5 & 0 \end{bmatrix} e^{-s} \right| = \begin{vmatrix} s & -1 \\ 5e^{-s} & s+5 \end{vmatrix} \\ &= s^2 + 5s + 5e^{-s} = 0 \end{aligned}$$

--- IV-44

This gives the Nyquist plot shown in Fig. 10.

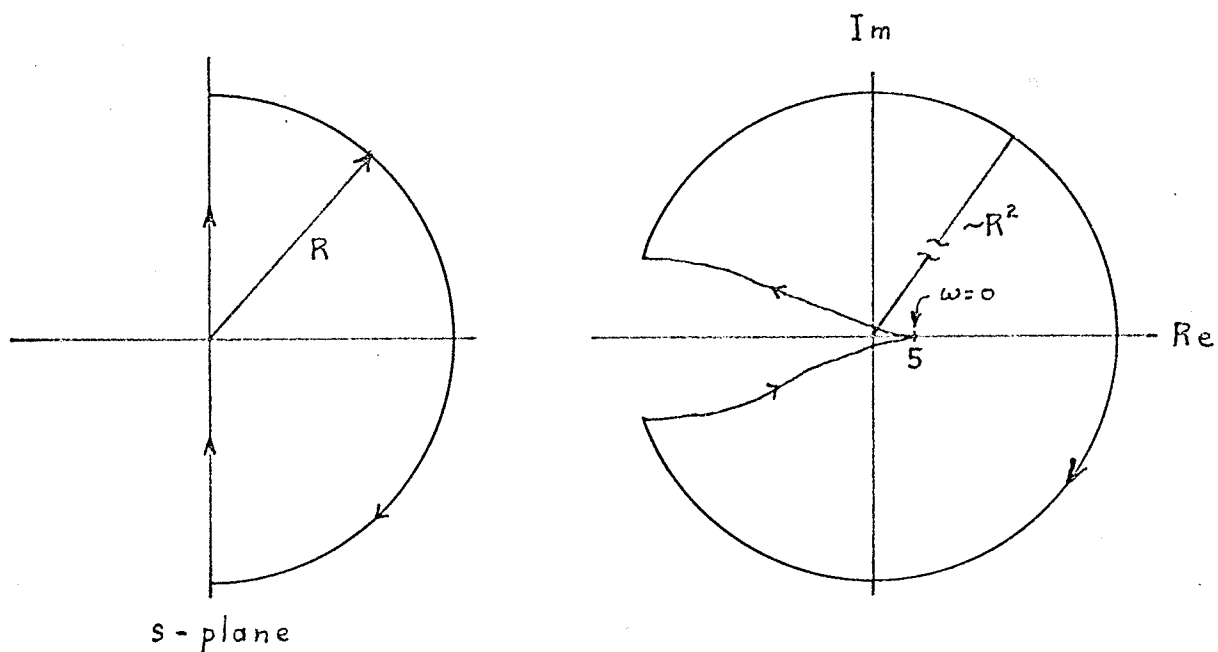


Figure 10

NYQUIST PLOT OF $s^2 + 5s + 5e^{-s}$

Since there is no encirclement of the origin there are no positive roots and the system is stable. Therefore, by Theorem 3 any continuous initial function can be reduced to the zero state by the control $u(t)$ with $|u(t)| \leq 1$.

The optimal control for this problem may be found by using Theorem 4 and the transversality condition given in Chapter III. From Theorem 4 and equation IV-41, the optimal control must satisfy

$$u(t) = - \operatorname{sgn} \left\{ \begin{bmatrix} 0 \\ 5 \end{bmatrix}^T \underline{p}(t) \right\} = - \operatorname{sgn} \{ p_2(t) \}, \quad t \in [t_0, t_f] \quad \text{--- IV-45}$$

where the adjoint $\underline{p}(t)$ must satisfy

$$\dot{\underline{p}}(t) = - \sum_{i=0}^1 A_i^T \underline{p}(t+\tau_i), \quad t \in [t_0, t_f] \quad \text{--- IV-46}$$

If the matrix

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is applied as suggested by III-42, to A_1

$$HA_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -5 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{--- IV-47}$$

then the last row of $HA_1 \underline{x}$ is identically zero, and from III-62 $\underline{p}(t)$ must satisfy

$$\dot{\underline{p}}(t) = - A_0^T \underline{p}(t), \quad t \in [t_f-1, t_f] \quad \text{--- IV-48}$$

i.e. $\underline{p}(t)$ must satisfy

$$\underline{p}(t) = 0, \quad t \in [t_f, t_f+1] \quad \text{--- IV-49}$$

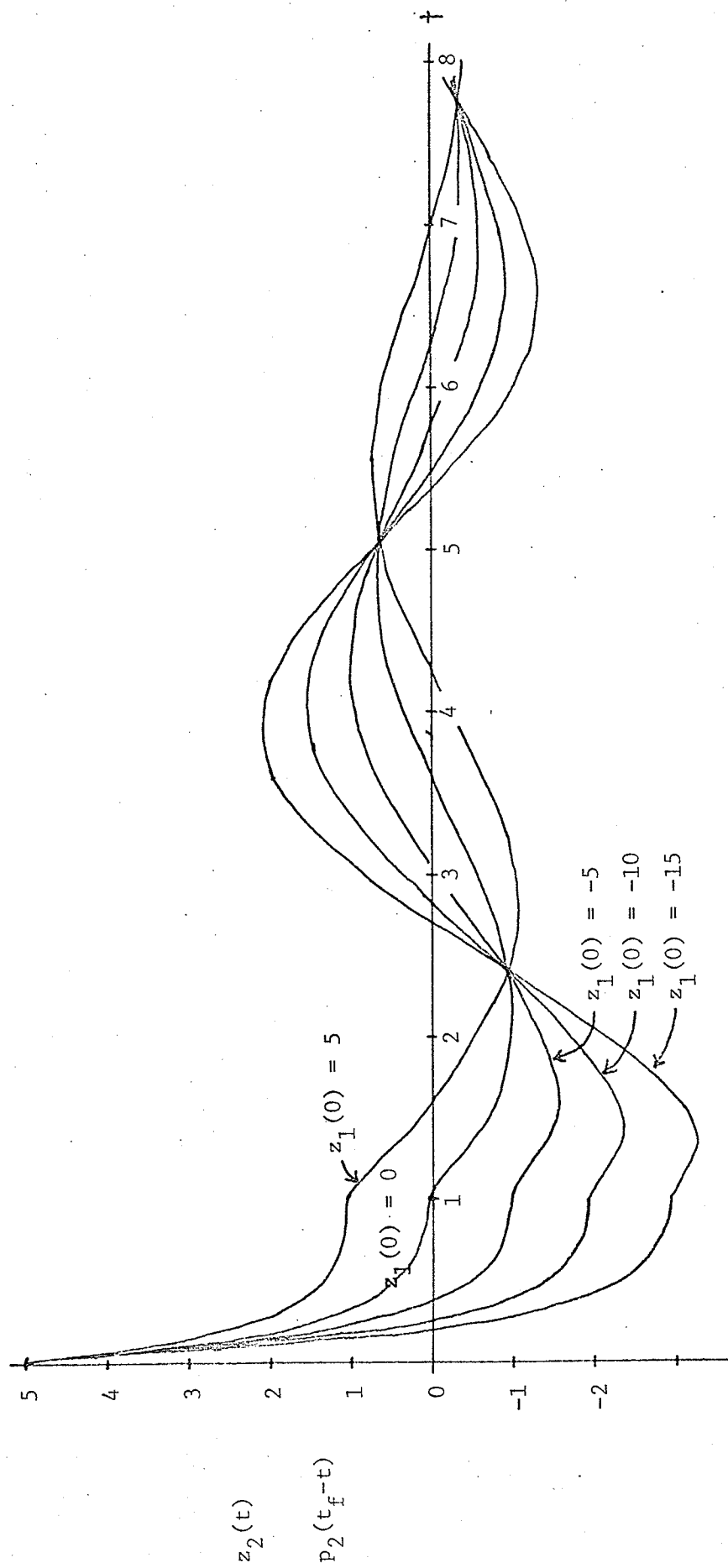


Fig. 11

$p_2(t_f - t)$

```

INIT
INCON XD=3.5,TF=3.3
CONST HZ=100.0,SZ=-100.0
      ZO=0.5*(HZ+SZ)
ZDT=5.0*INSW(XD-2.54,-1.0,1.0)
FACT=COMPAR(XD,2.54)
DYNAM
ZTD=DELAY(100,1.,ZT)
ZIN=-5.*ZTD
Z=INTGRL(ZD,ZIN)
DZT=Z-5.*ZT
ZT=INTGRL(ZOT,DZT)
TBF=(AND(ZT,2.4-TIME))*COMPAR(TF,TIME)
TAF=(AND(-ZT,TIME-2.4))*COMPAR(TF,TIME)
GA=INTGRL(2.4,TAF)
TA=(TF-GA)*FACT+GO*(1.-FACT)
GB=INTGRL(0.0,TBF)
TE=TF-GB*FACT+2.-2.*FACT
UA=COMPAR(TIME-TF,TA)
UB=COMPAR(TIME-TF,TB)
ST=COMPAR(TIME,TF-1.0-5)
DELX=DELAY(100,1.,X)
XIC=XO*ST*COMPAR(TF+.0000,TIME)
XD=XIC+DELX
VERR=U-XD
IN=5.*VERR
DXT=-5.*XT+IN
XT=INTGRL(0.0,DXT)
XINT=INTGRL(0.0,XT)
X=ST*XD+XINT
TIMA=TIME-TF
UPARA=(XT+1.0)*(AND(TIME-0.5*TF,2.54-XD))
UPAR=1.0+(XT-1.0)*(AND(TIME-TF-2.4,XD-2.54))-UPARA
TMF=COMPAR(UPAR,0.0)
U=TF*ST*(-1.0+2.*UA-2.*UB)+(1.0-TMF)*XD
TENCO=1.0-TF
TEND=INTGRL(TENCO,TMF)
TIMER DELT=0.001,INTIM=10.0
FINISH TEND=TIMA
TERMIN
      WRITE(6,12)X,XT,TF,TEND,TA,TB
12  FORMAT(5X,20H X,XT,TF,TEND,TA,TB=,6F10.4)
15  IF(HZ-SZ-0.050)24,24,16
16  IF(ABS(X)-0.01)24,24,17
17  IF(X)18,18,22
18  HZ=ZO
19  GO TO 26
22  SZ=ZO
23  GO TO 26
24  ERRTF=TEND-1.0-TF
    IF(ABS(ERRTF)-0.03)27,27,25
25  TF=TF+0.5*ERRTF
    HZ=HZ+10.0
    SZ=SZ-10.0
26  CALL RERUN
27  CONTINUE
END
      TITLE INPUT=1.1
      PLOT X(XT,TIMA,0)
END
STOP

```

Fig. 12

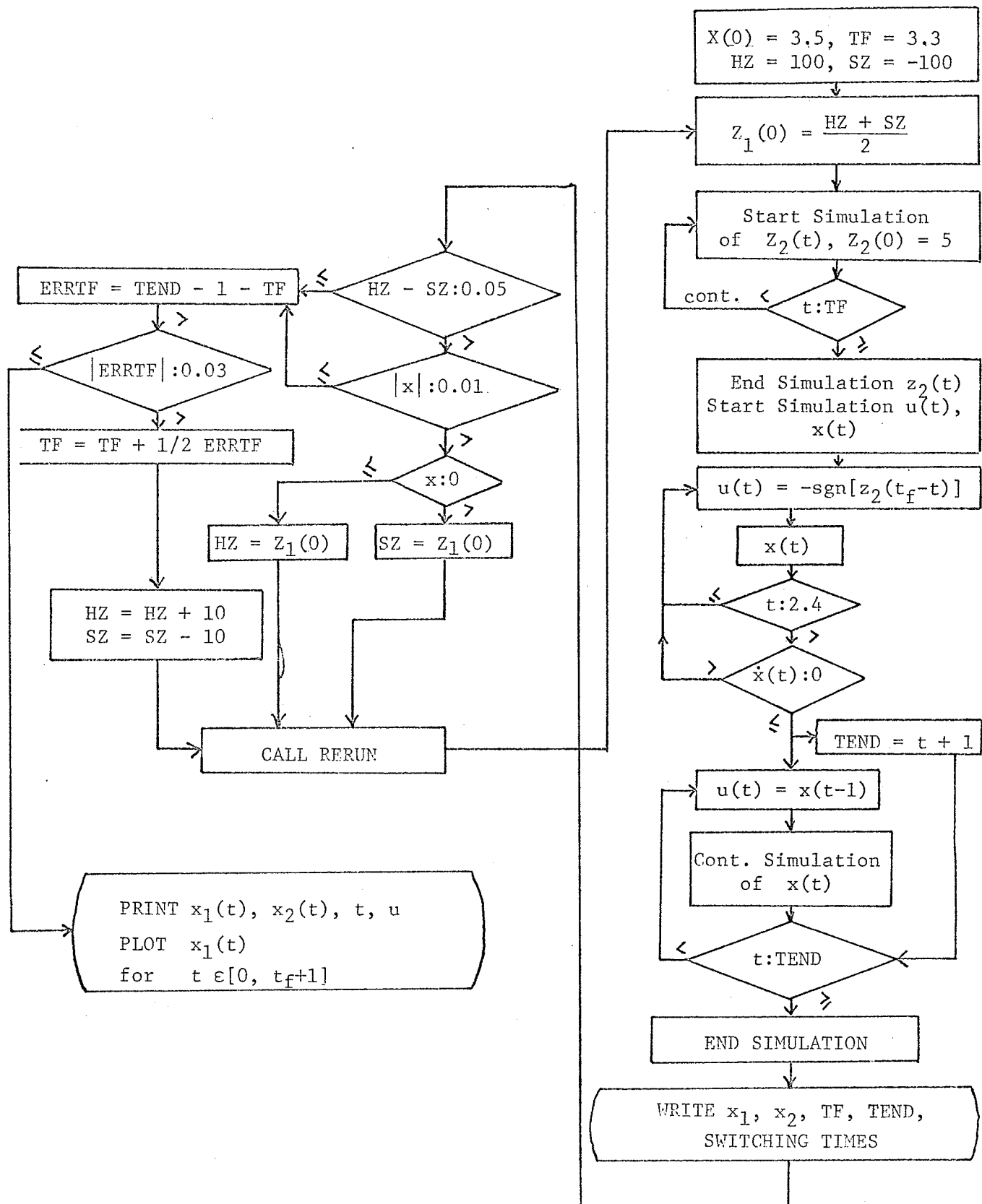


Fig. 13

Flow Chart, $x(t) = 3.5, t \in [-1, 0]$

and

$$\dot{\underline{p}}(t) = \begin{bmatrix} \dot{p}_1(t) \\ \dot{p}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_1(t+1) \\ p_2(t+1) \end{bmatrix} \quad \text{--- IV-50}$$

If $\underline{z}(t) = \underline{p}(t_f - t)$, then $\underline{z}(t)$ is the solution of

$$\dot{\underline{z}}(t) = \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & -5 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1(t-1) \\ z_2(t-1) \end{bmatrix} \quad \text{--- IV-51}$$

$$, t \in [0, t_f - t_0]$$

$$\underline{z}(t) = 0, \quad t \in [-1, 0] \quad \text{--- IV-52}$$

$\underline{z}_2(t)$ is shown in Fig. 11 for $z_2(0) = 5$ and various values of $z_1(0)$. The zero crossings of $z_2(t)$ are approximately 2.6 seconds apart, and therefore from IV-45 the optimal control will have switchings approximately 2.6 seconds apart. The system equation IV-41 can be simulated trying various initial switching times for $u(t)$ until the optimal control is found.

The optimal solution was found by using the IBM 360/65 digital computer with the Continuous System Modeling Program. The input statements are shown in Fig. 12 and the flow chart in Fig. 13 for a constant initial condition $x(t) = 3.5$ on $t \in [-1, 0]$. An initial guess of $t_f = 3.3$ is given. $z_2(t)[ZT]$ is then simulated for 3.3 seconds with $z_2(0) = 5$, $z_1(0) = 0$. The switching times [TA and TB] for $u(t)$ are then used in the simulation of $x(t)[X]$. When $\dot{x}(t) = 0$ near $t = t_f$, $u(t)$ is set equal $x(t-1)$ and one second later the simulation stops. If $x(t)$ at the end of the simulation is larger than 0.01, $z_1(0)$ is made larger and the program is run again. If $x(t)$ is smaller than -0.01, $z_1(0)$ is made

smaller and the program is rerun. If $|x(t)| \leq 0.01$ then the time when $\dot{x}(t) = 0$ [TEND-1.0] is compared to t_f . If the absolute value of the difference is greater than 0.03 a new value of t_f is tried that is equal to the old value of t_f plus half the difference. The system is rerun until all the conditions are satisfied. When all the conditions are satisfied, the optimal $x(t)$ is plotted and the optimal control is given.

For other constant initial conditions it is only necessary to replace the card with $X0 = 3.5$, $TF = 3.3$ by a new card with the new $x(0)$ and a guess for t_f . The optimum will be reached more quickly for a good guess of t_f slightly smaller than the actual t_f .

Fig. 14 shows the time optimal trajectory and control for an initial condition $x(0) = 5$. The minimum time for $x(t) = 0$, $\dot{x}(t) = 0$ is $t_f = 4.1$ seconds. Fig. 15 shows the time optimal control policy for constant initial conditions from 0.0 to 5.0. These data were obtained by running the computer program for various initial conditions in the range 0.0 to 5.0.

To determine the optimal control policy for a given constant initial condition: (1) draw a vertical line on the graph through the number on the horizontal axis corresponding to the initial condition. (2) The region directly above the horizontal axis on this vertical line gives the initial condition for $u(t)$. (3) The switching times are given by the intersection of the vertical line with lines on the graph.

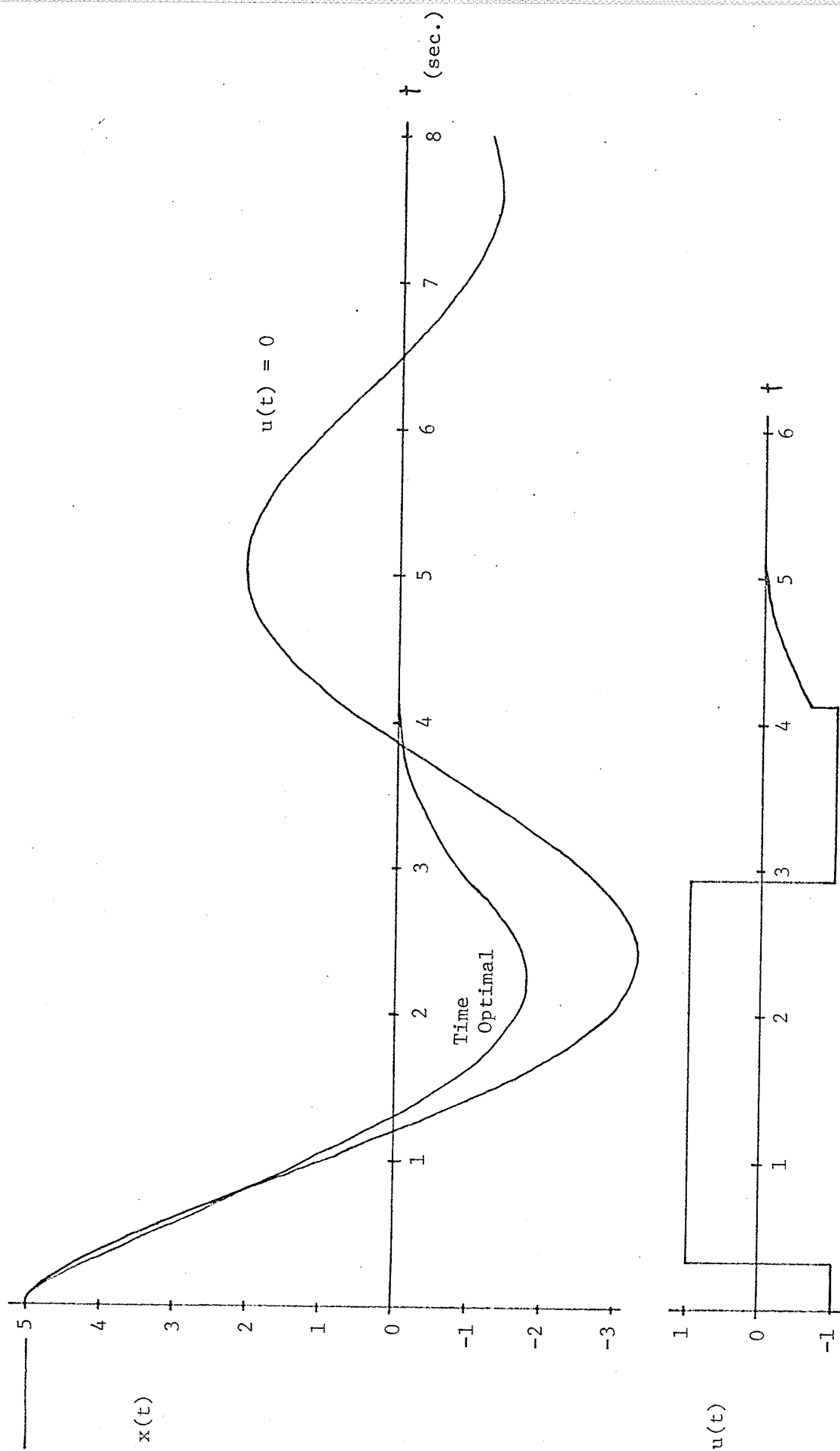


Fig. 14

Time Optimal Control and Trajectory

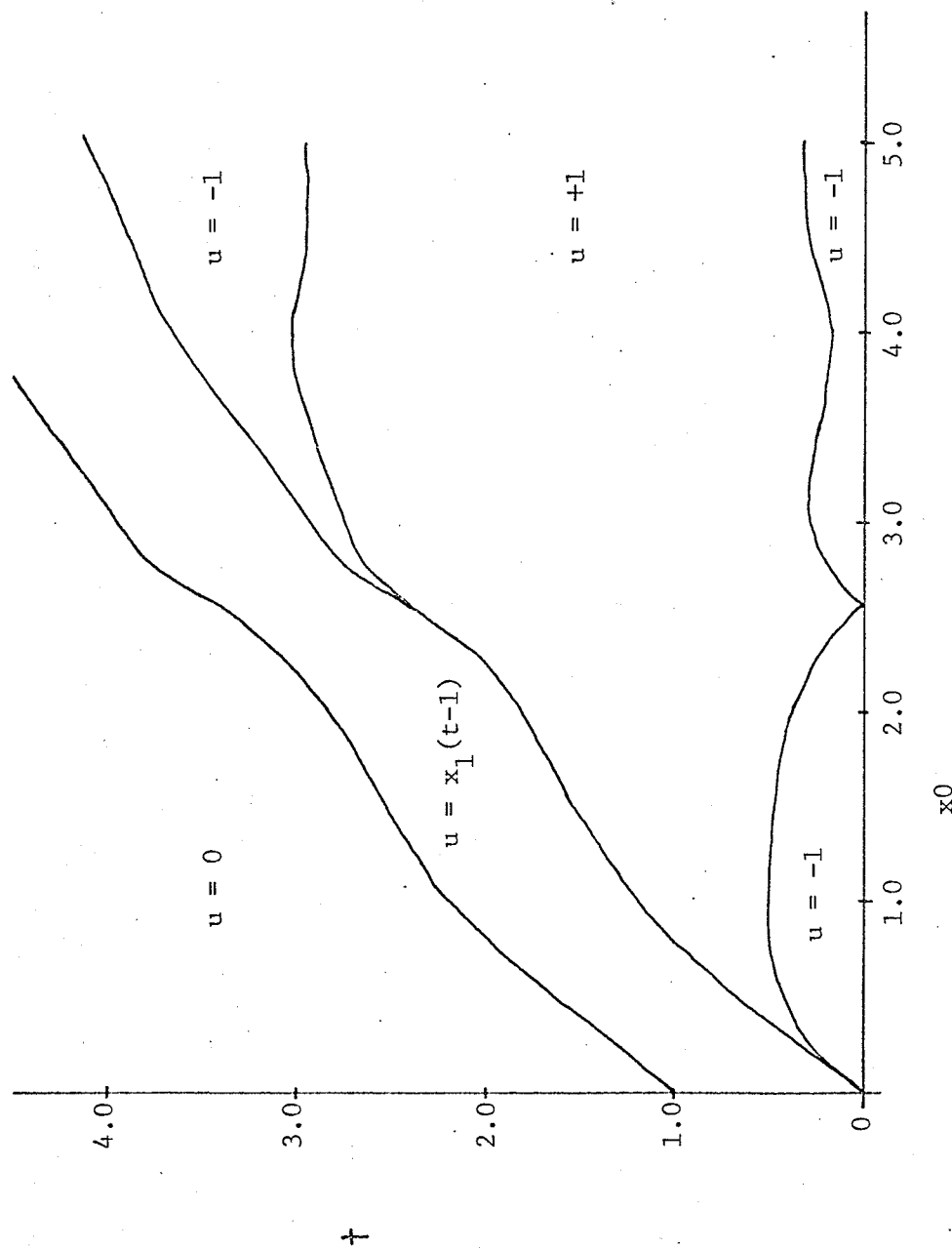


Fig. 15

Time Optimal Controls for Various Initial Conditions

CHAPTER V

CONCLUSIONS AND PROBLEMS

FOR FURTHER STUDY

Necessary conditions and sufficiency conditions have been given for system controllability. However, more general, necessary and sufficient conditions are needed that can be more easily applied.

In the first theorem on controllability it was assumed that if the original time delay system is controllable then the approximating system is controllable. This seems obvious from a practical point of view since the solutions of the two systems come arbitrarily close together as the order of the approximating system is increased. If this order is made large enough, the solutions would seem identical due to the limits of measurability. However, it would be desirable to prove this assumption mathematically. The second theorem on controllability is not new but a new proof of the theorem is presented.

Necessary conditions for optimality have been given. These could probably be extended to the variable time delay case. The transversality conditions given apply only to a small number of systems. It would be desirable to extend these to more general systems. Extra necessary conditions could possibly be obtained by a study of the set of attainability.

To satisfy the necessary conditions usually requires a lot of trial and error. It may be possible to write a computer program that would solve a general first or second order time optimal problem.

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