NONLINEAR CIRCUIT ANALYSIS

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Master of Science
in
Electrical Engineering

by
Richard Allan Johnson

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DEDICATION

This thesis is dedicated to the
memory of my father

PROFESSOR SKULI JOHNSON

without whose help, advice and
kind assistance, it never would have
been possible.

ABSTRACT

NONLINEAR CIRCUIT ANALYSIS

PREFACE

Although the following discussions are written in terms of electrical networks, the results are applicable to mechanical systems through a simple redefinition of parameters. The basic equations and concepts remain the same.

This thesis is divided into two major sections. Part A deals with the basic equations and phenomena of nonlinear circuits and Part B with a collection of methods of solution of nonlinear problems. The methods selected for inclusion in this latter part were those which seemed, to the author, to best illustrate various peculiarities of the subject. The list is by no means exhaustive.

In the text, the operational D and D^{-1} have been used for $\frac{d}{dt}$ and $\int_0 (\ )dt$, wherever possible. ω has been called simply, frequency, although it is more accurately, angular frequency.

The list of bibliography does not include all references on the study of nonlinear systems, but, by use of lists in references (6),(11), (12), (15), and (37), the number may be increased to close to five hundred.

The author wishes to thank the National Research Council of Canada for its sponsorship of this undertaking, without whose very substantial help, the completion of this thesis would have been exceedingly difficult.

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NONLINEAR CIRCUIT ANALYSIS

by

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ABSTRACT

This thesis is the result of an investigation of the fundamental concepts and phenomena and a study of the various methods of solution of nonlinear systems. As a result, it is divided into two parts.

Part A contains the generalization of the concepts of linear systems to the nonlinear case, as well as a description of two new, intrinsically nonlinear phenomena, nonlinear resonance and subharmonic oscillation. An outline of further possibilities of generalization is included. Part B is a collection of classical (as opposed to operational), graphical, and operational methods of solution which apply to different types of nonlinear systems.

The substance of Chapters I, III, and V, the arrangement of Part A, and the selection of topics in Part B are original with the author.

An extensive Bibliography is appended.

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PART A

FUNDAMENTAL CONCEPTS OF NONLINEAR SYSTEMS

CHAPTER I

DEFINITIONS

In general, the first step to the solution of any problem involves the definitions of the terms to be used. In nonlinear electrical systems this manifests itself as a problem in the definition of circuit parameters. Since nonlinear systems include linear systems as special cases, these definitions must reduce to the well-known linear parameters when the appropriate simplifying assumptions are made. What is more, the parameters so defined must be capable of manipulations which are the generalizations of those in the linear theory.

First of all, the term "nonlinear" itself must be defined. The characteristic equation of an element is the relation between the mutually dependent variables defining the operation of that element and a "nonlinear" element is, therefore, simply an element with a nonlinear characteristic equation. A nonlinear network is, by direct extension of this, a network which contains at least one nonlinear element. Characteristic equations have the general form:

$$f(v, Dv, D^2v, \ldots\ i, Di, D^2i, \ldots\ t) = 0 \qquad \text{I.1}$$

The explicit time dependence is present only when the parameters of the element have some fixed variation with time. If their values vary with one of the defining variables, voltage v, or current, i, which varies with time, the direct dependence on t will be absent. The following exposition will be limited to this type of parameter.

For elements consisting of a single parameter only such as inductance, the characteristic equation assumes the form of a differential equation of no higher than first order, linear or nonlinear as the case may be, and, if further variables are defined such as total flux linkage, ϕ, where $v = D\phi$, then all single parameter characteristics become algebraic. For example, the characteristic of a linear inductor may be written as:

$$v - LDi = 0 \qquad \text{I.2}$$

since the inductor operates on the current differentially to produce a voltage v across the element, ~~v~~, with a proportionality constant, L. If, however, this characteristic equation is integrated once and the above definition of ϕ used, then the equation becomes:

$$\phi - Li = 0 \qquad \text{I.3}$$

which is algebraic. Each characteristic, therefore, can have more than one form.

The distinction between bilateral and unilateral terminology must also be clearly understood. The bilateral element conducts in both directions but not necessarily equally well. If the element were linear, then the so-called "forward" and "backward" characteristics would be the same; if nonlinear, they would in general be different, although it is possible to have a characteristic with odd symmetry about the origin which is nonlinear. The unilateral element is a special case of the dissymmetrical nonlinear bilateral element in that it conducts in the forward direction but has a zero characteristic in the backward direction. Its characteristic in the forward direction may be either linear or nonlinear. The former is exemplified by the ac Class A operated triode or pentode which conducts in both directions about a quiescent point in a more or less linear manner once it is properly biased; the latter by a diode which theoretically conducts in one direction only in a nonlinear manner.

The characteristic equation can be further classified. If it is single-valued in both defining variables, the element is said to be "simple", if not, it is said to be "non-simple". Most elements are in the latter class to a greater or lesser degree, the simple case arising from approximations. For example, an iron-cored inductor exhibiting hysteresis may often be taken as simple for analysis purposes. Other non-simple elements include the tetrode vacuum tube which is multi-valued in current, and, in the non-electrical field, gears with backlash which are non-simple in both angular variables. <u>A connection of simple elements does not necessarily have a simple overall characteristic.</u> A case of this sort is discussed in Chapter XI, section 4.

A further classification may be made. If y and x are the forcing term and resultant displacement respectively and

$$\frac{d^2y}{dx^2} > 0,$$

when t=0+, the element is termed "hard", but if

$$\frac{d^2y}{dx^2} < 0,$$

when t=0+, it is termed "soft". The terminology originates in the example of a nonlinear spring. If the restoring force increases with displacement, the spring is of the former type; if it decreases, it is of the latter type.

Consider now the problem of defining nonlinear resistance, $R(i)$, inductance, $L(Di)$, and elastance $S(i)$ (reciprocal capacitance) and their reciprocals conductance, $G(v)$ inertiance $\Gamma(v)$ (reciprocal inductance), and capacitance $C(v)$. It is possible to define these parameters in two ways both of which reduce to the linear definitions when linearization is carried out. The parameters are defined by the following characteristic equations.

$$\begin{aligned} v(i) - iR(i) &= 0 \\ v(i) - L(Di)Di &= 0 \qquad \text{I.4} \\ v(q) - qS(q) &= 0 \end{aligned}$$

and

$$\begin{aligned} \frac{dv(i)}{di} - r(i) &= 0 \\ \frac{dv(Di)}{d(Di)} - l(Di) &= 0 \qquad \text{I.5} \\ \frac{dv(q)}{dq} - s(q) &= 0. \end{aligned}$$

The former set denoted by capital letters shall be called "instantaneous" and the latter set "incremental" parameters. If the characteristic equations are given graphically, the two sets of parameters are seen to be the slope of the line from the origin to any point on the characteristic and the slope of the characteristic at that point, respectively. The following diagrams are obtained.

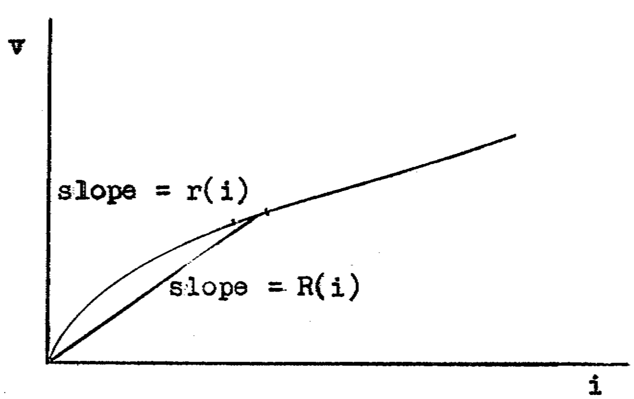
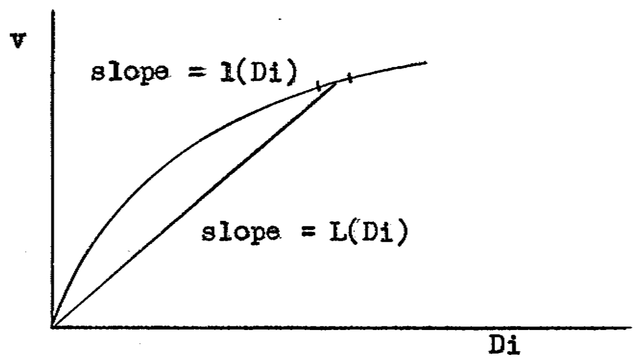
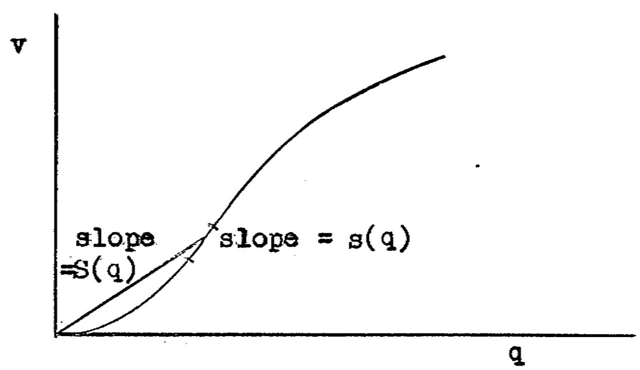


FIGURE I.1a RESISTANCE



b INDUCTANCE



c CAPACITANCE

FIGURE I.1

GRAPHICAL INTERPRETATION OF NONLINEAR PARAMETERS

If the first two equations I.4 are differentiated with respect to current i, and Di, respectively and the third by charge q, the result is:

$$\frac{dv}{di} - i\frac{dR}{di} - R = 0$$

$$\frac{dv}{dDi} - Di\frac{dL}{dDi} - L = 0 \qquad \text{I.6}$$

and

$$\frac{dv}{dq} - q\frac{dS}{dq} - S = 0,$$

which on comparison with equations I.5 yield the following relations between the two sets of parameters:

$$r(i) = R(i) + i\,\frac{dR(i)}{di}$$

$$l(Di) = L(Di) + Di\,\frac{dL(Di)}{dDi} \qquad \text{I.7}$$

and

$$s(q) = S(q) + q\,\frac{dS(q)}{dq}\ .$$

These equations may be solved for the instantaneous parameters to yield:

$$R(i) = \frac{1}{i}\int_0 r(i)\ di$$

$$L(Di) = \frac{1}{Di}\int_0 l(Di)\ dDi \qquad \text{I.7a}$$

and

$$S(q) = \frac{1}{q}\int_0 s(q)\ dq\ .$$

The reason for the incremental terminology is obvious. The instantaneous parameters were so-called because at any instant of time the value of the parameter is the ratio of the two defining variables ~~while~~ where-as the incremental parameters are the ratios of the increments of the defining variables, taken to the differential limit. These definitions when applied to the vacuum tube yield the so-called d.c. and a.c. parameters.

The reciprocal parameters, conductance G,g, inertiance Γ,γ, and capacitance C,c, are defined by the following equations:

$$i(v) - vG(v) = 0$$

$$Di(v) - v\Gamma(v) = 0 \qquad \text{I.8}$$

and

$$q(v) - vC(v) = 0,$$

and

$$\frac{di(v)}{dv} - g(v) = 0$$

$$\frac{dDi(v)}{dv} - \gamma(v) = 0 \qquad \text{I.9}$$

$$\frac{dq(v)}{dv} - c(v) = 0\ .$$

In a similar manner to the above, the following equalities may be demonstrated:

$$g(v) = G(v) + v\frac{dG(v)}{dv}$$

$$\gamma(v) = \Gamma(v) + v\frac{d\Gamma(v)}{dv} \qquad \text{I.10}$$

$$c(v) = C(v) + v\frac{dC(v)}{dv}\ ,$$

and

$$G(v) = \frac{1}{v}\int g(v)\ dv$$

$$\Gamma(v) = \frac{1}{v}\int \gamma(v)\ dv \qquad \text{I.10a}$$

$$C(v) = \frac{1}{v}\int c(v)\ dv.$$

That these parameters are the reciprocals of those previously defined is easily seen from the defining equations. For example, from the second of equations I.4 and I.8,

$$L(Di) = \frac{v}{Di} = \frac{1}{\Gamma(v)} \qquad \text{I.11}$$

and from the third of equations I.5 and I.9,

$$s(q) = \frac{dv}{dq} = \frac{1}{c(v)} \qquad \text{I.11a}$$

and similarly for the other four pairs.

CHAPTER II

LAGRANGE'S EQUATIONS

The Lagrangian form of the electric network equations will first be derived from the Kirchhoff Laws equations for the linear system and then the general nonlinear case will be demonstrated.

II.1 LINEAR SYSTEMS (3)#

Consider first the problem of the linear bilateral electric network of n meshes each containing self and mutual parameters, R_{ik}, L_{ik}, and S_{ik}. These parameters are, therefore, independent of time, voltage, current and orientation in the network. The double subscript ik signifies that the parameter is mutual to two meshes, i and k. If the subscript is of the form ii, the parameter is included in the i'th mesh only. Since the system is bilateral, the parameter is independent of the order of the subscripts; thus $L_{ik} = L_{ki}$. Both i and k may take on, independently all integral values from 1 through n.

The parameters R_{ik} and S_{ik}, $(i \neq k)$, can exist only as common elements in the two meshes i and k, but L_{ik}, $(i \neq k)$, can exist either as a common element in the two meshes, in which case it must be a positive quantity if the two currents pass through it in the same direction, or as a mutual coupling between two coils in meshes i and k, in which case it may have either algebraic sign.

By Kirchhoff's Voltage Law for the n meshes, using the element characteristics stated in Chapter I,

$$\sum_{k=1}^{n} (L_{ik}D + R_{ik} + S_{ik}D^{-1}) i_k - e_i = 0 \qquad \text{II.1.1}$$

where $i = 1,2,3,\ldots n$, and e_i is the total voltage of the sources in the i'th mesh.

If equation II.1.1 is multiplied by i_i and the summation taken over the subscript i, from 1 through n, there results:

$$\sum_{i,k} (L_{ik}i_iD + R_{ik}i_i + S_{ik}i_iD^{-1}) i_k = \sum_i e_i i_i \qquad \text{II.1.2}$$

the right hand side of which represents the total power, P, supplied by

#The numbers in parentheses refer to the Bibliography.

the voltage sources.

If the three quantities, F, T, and V are defined as:

$$2F = \sum_{i,k} R_{ik} i_i i_k \qquad \text{II.1.3}$$

$$2T = \sum_{i,k} L_{ik} i_i i_k \qquad \text{II.1.4}$$

and

$$2V = \sum_{i,k} S_{ik} q_i q_k \qquad \text{II.1.5}$$

where $q_i = D^{-1} i_i$, and are substituted into equation II.1.2, there results:

$$P = 2F + D(T + V) \qquad \text{II.1.6}$$

T is, by definition, the inductive or kinetic energy of the system, V, the capacitive or potential energy of the system, and 2F, the total power dissipation in the system. (F is the dissipation function first introduced by Rayleigh.)

Equation II.1.6 is then simply the statement of the Conservation of Energy for the general n mesh network. Thus, P, the total power supplied to the system by the voltage sources equals 2F, the total power dissipation plus D(T + V), the rate of increase of stored energy in the system. The factor 2 disappears since the double summation includes each term twice. This follows from the bilateral property of the parameters involved. For example:

$$\begin{aligned}
\sum_{i,k} L_{ik} i_i D i_k &= \frac{1}{2}\left[\sum_{i,k} L_{ik} i_i D i_k + \sum_{k,i} L_{ki} i_k D i_i\right] \\
&= \frac{1}{2} \sum_{i,k} L_{ik}(i_i D i_k + i_k D i_i) \\
&= \frac{1}{2} \sum_{i,k} L_{ik} D(i_i i_k) \\
&= D\left[\frac{1}{2} \sum_{i,k} L_{ik} i_i i_k\right] \\
&= DT .
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i,k} S_{ik} i_i q_k &= \frac{1}{2}\left[\sum_{i,k} S_{ik} i_i q_k + \sum_{k,i} S_{ki} i_k q_i\right] \\
&= \frac{1}{2} \sum_{i,k} S_{ik}(q_k D q_i + q_i D q_k) \\
&= DV .
\end{aligned}$$

If now equations II.1.3 and II.1.4 are differentiated partially with respect to i_i and equation II.1.5 with respect to q_i,

$$\frac{\partial F}{\partial i_i} = \sum_k R_{ik} i_k \qquad \text{II.1.7}$$

$$\frac{\partial T}{\partial i_i} = \sum_k L_{ik} i_k \qquad \text{II.1.8}$$

and

$$\frac{\partial V}{\partial q_i} = \sum_k S_{ik} q_k = \sum_k S_{ik} D^{-1} i_k \qquad \text{II.1.9}$$

and differentiating equation II.1.8 totally with respect to time yields:

$$D\left\{\frac{\partial T}{\partial i_i}\right\} = \sum_k L_{ik} D i_k \qquad \text{II.1.10}$$

Substituting equations II.1.7,8 and 10 above into equation II.1.1 then yields:

$$D\left\{\frac{\partial T}{\partial i_i}\right\} + \frac{\partial F}{\partial i_i} + \frac{\partial V}{\partial q_i} - e_i = 0,\ i=1,2,\ldots n. \qquad \text{II.1.11}$$

If now differentiationswith respect to i_i and q_i are considered as independent and the function $L = T - V + \sum_i e_i q_i$ II.1.12, called the Lagrangian or Kinetic Potential is introduced, then equation II.1.11 becomes:

$$D\left\{\frac{\partial L}{\partial i_i}\right\} - \frac{\partial L}{\partial q_i} = -\frac{\partial F}{\partial i_i},\ i=1,2,\ldots n, \qquad \text{II.1.13}$$

since,

$$\frac{\partial L}{\partial i_i} = \frac{\partial T}{\partial i_i},$$

and

$$\frac{\partial L}{\partial q_i} = -\frac{\partial V}{\partial q_i} + e_i.$$

The n equations II.1.13 are called the Lagrangian Equations for the dissipative system of n independent variables i_i and completely define the system.

OMIT The dissipation function F can be combined with the Lagrangian of the system by re-defining L as:

$$L' = T - V + \sum_i e_i q_i + D^{-1}F. \qquad \text{II.1.14}$$

Then,

$$D\left\{\frac{\partial L'}{\partial i_i}\right\} = D\left\{\frac{\partial T}{\partial i_i}\right\} + \frac{\partial F}{\partial i_i}$$

and equation II.1.11 becomes:

$$D\left(\frac{\partial L'}{\partial i_i}\right) - \frac{\partial L}{\partial q_i} = 0,\ i=1,2,\ldots n. \qquad \text{II.1.15}$$

This equation although it appears simpler than equation II.1.13, is actually not, and is of little interest.

The n equations II.1.15 can be expressed as a single equation in the form of Hamilton's Principle ~~of Least Action~~ which because of its conciseness can be regarded as the most fundamental formulation of the system. It takes the form:

$$\delta \int_{t_1}^{t_2} L'dt = 0^{\#} \qquad \text{II.1.16}$$

which simply states that the integral of the Lagrangian taken between two arbitrary time limits is stationary(that is, either a maximum, a minimum, or an inflection), for the natural waveforms of the currents and voltages of the network. ~~In most practical cases the value is a minimum and indicates a stable equilibrium condition, but this is not always true.~~

All the above arguments may be repeated for the general m-1 node-pair formulation by replacing i_k, q_k, e_i, L_{ik}, R_{ik}, S_{ik}, F,T, and V, by v_k, ϕ_k, j_i, C_{ik}, G_{ik}, Γ_{ik}, F_1, V_1, and T_1, respectively and by using Kirchhoff's Current Law at the m nodes. The result is in indefinite form because an arbitrary reference has been used for the unknown node-pair voltages. If one node were selected as reference, the result would be the definite m-1 independent equations. In this derivation, equations II.1.3 through II.1.6 become:

$$F_1 = \frac{1}{2}\sum_{i,k} G_{ik}v_iv_k \qquad \text{II.1.3a}$$

$$T_1 = \frac{1}{2}\sum_{i,k}\ \ {}_{ik}\phi_i\phi_k \qquad \text{II.1.4a}$$

$$V_1 = \frac{1}{2}\sum_{i,k} C_{ik}v_iv_k \qquad \text{II.1.5a}$$

and

$$P_1 = \sum_i j_iv_i = 2F_1 + D(T_1+V_1) \qquad \text{II.1.6a}$$

where ϕ_i and ϕ_k are the flux linkages in the parameter $\Gamma_{ik} = L^{-1}_{ik}$ and j_i

\# If the variation of the q_i vanish at the end points of integration.

are the current sources in the network.

Lagrange's Equations II.1.11 become:

$$D\left\{\frac{\partial V_1}{\partial v_i}\right\} + \frac{\partial F_1}{\partial v_i} + \frac{\partial T_1}{\partial \phi_i} - j_i = 0, \quad i=1,2,\ldots(m-1), \qquad \text{II.1.11a}$$

which on defining the Lagrangian Function as:

$$L = V_1 - T_1 + \sum_i j_i \phi_i \qquad \text{II.1.12a}$$

become:

$$D\left\{\frac{\partial L}{\partial v_i}\right\} - \frac{\partial L}{\partial \phi_i} = -\frac{\partial F_1}{\partial v_i}, \quad i=1,2,\ldots(m-1), \qquad \text{II.1.13a}$$

which is the same form as equation II.1.13.

Hamilton's Principle of Least Action takes the same form as for the n mesh case.

II.2 NONLINEAR SYSTEMS

The generalization of the Lagrange Equations will now be derived for the nonlinear case. First consider the following generalizations of the linear F, T, and V functions.

Content and Co-content. Consider a nonlinear resistor with a characteristic given by $f(i,v) = 0$ as shown in figure II.2.1.

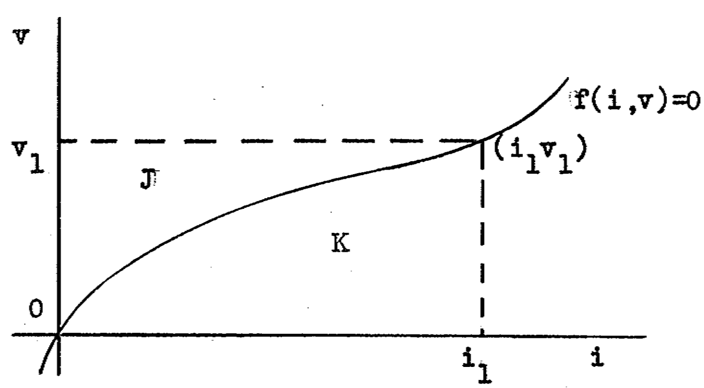


FIGURE II.2.1

CONTENT AND CO-CONTENT OF A NONLINEAR RESISTOR

The "content", K, of the resistor is defined as:

$$K = \int_0^{i_1} v \, di \qquad \text{II.2.1}$$

and the "co-content", J, as:

$$J = \int_0^{v_1} i\, dv. \qquad \text{II.2.2}$$

In the linear case, $f(i,v) = Ri - v = 0$, and $K = J = i_1 v_1/2$, for any i_1, v_1 satisfying the characteristic equation. Thus K and J in the linear case, both equal half the total energy dissipation in the resistor.

Inductive Energy and Co-energy. Consider a nonlinear inductor with a characteristic equation, $f(i,\phi) = 0$, as shown in figure II.2.2.

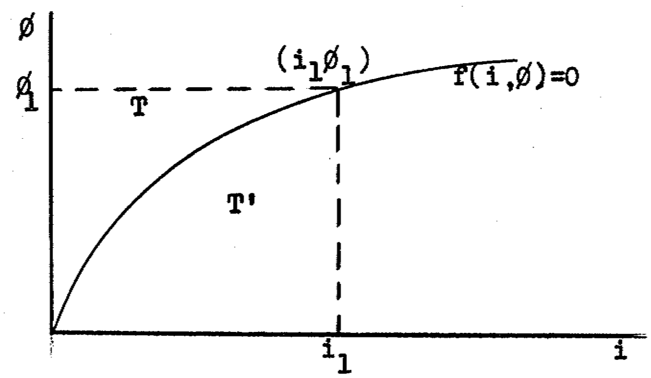


FIGURE II.2.2

INDUCTIVE ENERGY AND CO-ENERGY OF A NONLINEAR INDUCTANCE

The inductive energy, T, and the inductive co-energy, T', are defined as:

$$T = \int_0^{\phi_1} i\, d\phi \qquad \text{II.2.3}$$

and

$$T' = \int_0^{i_1} \phi\, di \qquad \text{II.2.4}$$

respectively.

Capacitive Energy and Co-energy Consider now a nonlinear capacitor with a characteristic equation $f(q,v) = 0$, as shown in figure II.2.3. The capacitive energy, V, and co-energy, V', are defined as:

$$V = \int_0^{q_1} v\, dq \qquad \text{II.2.5}$$

and

$$V' = \int_0^{v_1} q\, dv. \qquad \text{II.2.6}$$

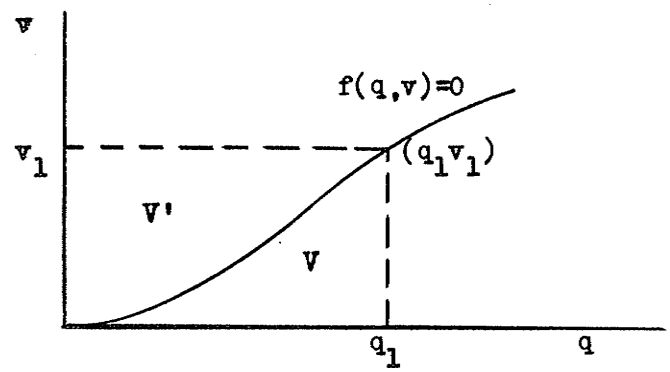


FIGURE II.2.3

CAPACITIVE ENERGY AND CO-ENERGY OF A NONLINEAR CAPACITANCE

The inductive and capacitive energies are derivable from the work required to produce a current through an inductor or impress a charge on a capacitor respectively. Thus:

$$T = \int_0^{t_1} iv\,dt = \int_0^{t_1} iD\phi\,dt = \int_0^{\phi_1} i\,d\phi \qquad \text{II.2.7}$$

and

$$V = \int_0^{t_1} vi\,dt = \int_0^{t_1} vDq\,dt = \int_0^{q_1} v\,dq. \qquad \text{II.2.7a}$$

In the linear case, the distinction between content and co-content and between energy and co-energy disappears due to their equality.

Consider now the general n mesh nonlinear network with voltage sources e_i and unknown currents i_i. By the definitions of equations I.4, the following Kirchhoff's Law Equations for the n meshes may be written:

$$\sum_k \left[L_{ik}(Di)D + R_{ik}(i) + S_{ik}(D^{-1}i)D^{-1} \right] i_k = e_i, \; i=1,2,\ldots n, \qquad \text{II.2.8}$$

where $i = i_i - i_k$, when all the mesh currents are taken in the same direction in the network and the parameters contain their appropriate algebraic sign. The parameters so involved are instantaneous and by the above assumption of currents, $L_{ik} = \pm L_{ki}$, $R_{ik} = \pm R_{ki}$, and $S_{ik} = \pm S_{ki}$ for $i \neq k$.

The co-energy of the ik'th inductive parameter is therefore:

$$T'_{ik} = \int_0^{i_i - i_k} \phi_{ik}\,di \qquad \text{II.2.9}$$

where ϕ_{ik}, the total flux linkage through it, is a function of $(i_i - i_k)$

only. Summing over all the inductances in the network, the total inductive co-energy, T', becomes:

$$T' = \frac{1}{2} \sum_{i,k} \int_0^{i_i - i_k} \phi_{ik}\, di \qquad \text{II.2.10}^{\#}$$

where the 1/2 again appears due to the doubling effect of the summation. Thence:

$$\frac{\partial T'}{\partial i_i} = \sum_k \phi_{ik} \qquad \text{II.2.11}$$

and the voltage across the inductances in the i'th mesh is then given by:

$$D \sum_k \phi_{ik} = D\left(\frac{\partial T'}{\partial i_i}\right). \qquad \text{II.2.12}$$

In a similar manner, the total charge on any elastance, S_{ik}, is a function of $(q_i - q_k)$ only and the capacitive energy becomes:

$$V_{ik} = \int_0^{q_i - q_k} v_{ik}\, dq. \qquad \text{II.2.13}$$

The total potential energy, V, of the elastances in the network is, therefore,

$$V = \frac{1}{2} \sum_{i,k} \int_0^{q_i - q_k} v_{ik}\, dq, \qquad \text{II.2.14}^{\#}$$

and hence:

$$\frac{\partial V}{\partial q_i} = \sum_k v_{ik} \qquad \text{II.2.15}$$

gives the voltage across the elastances in the i'th mesh.

Again, for the ik'th resistor in the network, the content, K_{ik}, is given by:

$$K_{ik} = \int_0^{i_i - i_k} v_{ik}\, di \qquad \text{II.2.16}$$

from which the total content of the network is:

$$K = \frac{1}{2} \sum_{i,k} \int_0^{i_i - i_k} v_{ik}\, di \qquad \text{II.2.17}^{\#}$$

and further:

$$\frac{\partial K}{\partial i_i} = \sum_k v_{ik} \qquad \text{II.2.18}$$

which gives the voltage across all the resistors in the i'th mesh.

\# (Note that in equations II.2.10, 14, and 17, differentiation by i_k in the first and last, and by q_k in the second, would not affect the sign of the right-hand side due to the fact that, for example:

$$-\sum_i \phi_{ik} = \sum_k \phi_{ik}.)$$

If Kirchhoff's Voltage Law is now applied to the n meshes, there results:

$$D\left\{\frac{\partial T'}{\partial i_i}\right\} + \frac{\partial V}{\partial q_i} + \frac{\partial K}{\partial i_i} - e_i = 0, i=1,2,\ldots n. \qquad \text{II.2.19}$$

If, as in the linear case, i_i and q_i are assumed independent with regard to partial differentiation by the other, and a Lagrangian Function is defined as:

$$L = T' - V + \sum_i e_i q_i \qquad \text{II.2.20}$$

then equation II.2.9 becomes:

$$D\left\{\frac{\partial L}{\partial i_i}\right\} - \frac{\partial L}{\partial q_i} = -\frac{\partial K}{\partial i_i},\ i=1,2,\ldots n, \qquad \text{II.2.21}$$

which are Lagrange's Equations for the nonlinear system of n meshes.

In a similar manner, the Lagrangian equations for the general m node, nonlinear network may be deduced. The form of the Kirchhoff and Lagrange equations will be:

$$\sum_k [\, C_{ik}D + G_{ik} + \Gamma_{ik}D^{-1} \,]\, v_i = j_i \qquad \text{II.2.22}$$

and

$$D\left\{\frac{\partial V_1'}{\partial v_i}\right\} + \frac{\partial T_1}{\partial \phi_i} + \frac{\partial J_1}{\partial v_i} - j_i = 0 \qquad \text{II.2.23}$$

where the v_i are the unknown node-pair voltages and j_i, the current sources, as before. V_1', T_1, and J_1 are defined in a similar manner to the corresponding variables in the linear case.

If the Lagrangian function is now defined as:

$$L = V_1' - T_1 + \sum_i j_i \phi_i,$$

then equation II.2.14 becomes:

$$D\left\{\frac{\partial L}{\partial v_i}\right\} - \frac{\partial L}{\partial \phi_i} = -\frac{\partial J_1}{\partial v_i},\ i = 1,2,\ldots (m-1). \qquad \text{II.2.24}$$

The Lagrangian equations II.2.11 and II.2.13 reduce on linearization to the linear form of equations II.1.13 and II.1.13a previously demonstrated. This fulfils the condition stated in Chapter I. The nonlinear form is the more general since the distinction between the variables and the the co-variables is lost in the linear case.

Consider the following examples.

1. The series circuit shown in figure II.2.4 is comprised of a linear capacitor of elastance S and an iron-cored, nonlinear inductor with a characteristic of the form:

$$\phi(i) = A \tan^{-1}(i/M) + Bi/M$$

as illustrated in figure II.2.5, where A, B, and M are constants.

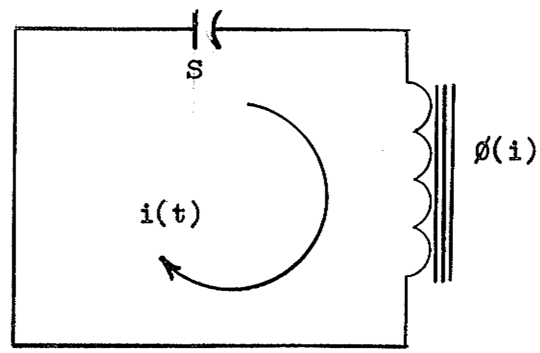


FIGURE II.2.4
SERIES CONNECTION OF A LINEAR CAPACITANCE
AND A NONLINEAR INDUCTANCE

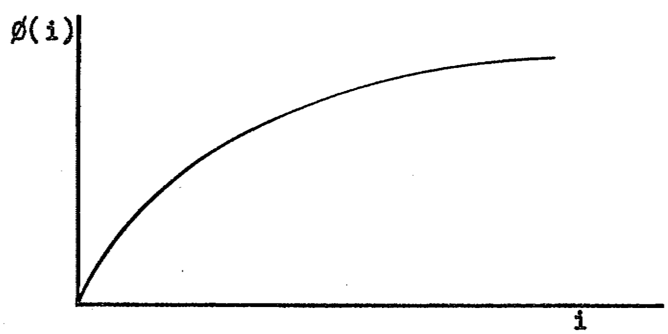


FIGURE II.2.5
GRAPHICAL CHARACTERISTIC OF A NONLINEAR INDUCTOR

By Kirchhoff' Voltage Law, neglecting any resistance present,

$$SD^{-1}i + D\phi = 0$$

or

$$Sq + D\phi = 0.$$

To reduce this to the Lagrangian form:

$$D\left\{\frac{\partial L}{\partial i}\right\} - \frac{\partial L}{\partial q} = 0,$$

the Lagrangian Function must be defined as:

$$L = \int \phi(i)di - Sq^2/2,$$

so that:

$$\frac{\partial L}{\partial i} = \phi(i),$$

$$D\left(\frac{\partial L}{\partial i}\right) = D\phi,$$

and

$$-\frac{\partial L}{\partial q} = Sq.$$

However, by equations II.2.4 and II.2.5,

$$T' = \int \phi di$$

and

$$V = \int v dq = \int Sq dq = Sq^2/2,$$

and so

$$L = T' - V,$$

which is the result expected for the mesh type of network in the force-free, conservative case. That the system is actually conservative may easily be demonstrated as follows. From the above,

$$D\phi + Sq = 0$$

where $\phi(i) = A \tan^{-1}(i/M) + Bi/M$, and $D\phi = \frac{d\phi}{di} Di$. Upon multiplying by i and integrating once with respect to t, the result is:

$$\int \frac{d\phi}{di} i\, di + \int Sq\, i\, dt = \text{a constant}, h,$$

whereupon substituting the value of $\phi(i)$ and carrying out the operations indicated, there results:

$$\frac{AM}{2} \ln(M^2+i^2) + \frac{Bi^2}{2M} + \frac{Sq^2}{2} = h.$$

This is the equation of a family of concentric ovals in the i-q plane, the so-called "phase-portrait" of the system.(1,6,7,12) The parameter h which is easily recognizable as the total energy of the system is given by T + V. The fact that these phase paths are closed proves that the system is conservative.

It is also possible to have nonlinear mesh systems in which L = T - V. For example, consider the following problem.

2. The series LC circuit shown in figure II.2.6 consists of a linear inductor of value L_0 and a nonlinear capacitor with elastance:

$$S(q) = S_0(1 + Aq + Bq^2)$$

which is illustrated in figure II.2.7.

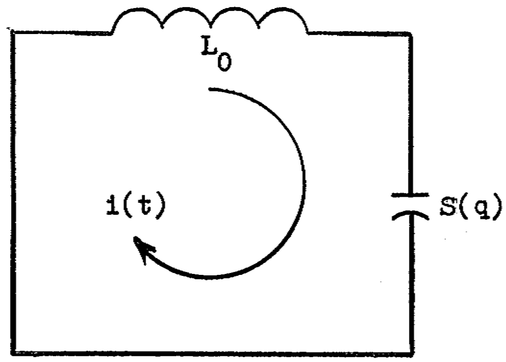


FIGURE II.2.6

SERIES CONNECTION OF A LINEAR INDUCTANCE
AND A NONLINEAR CAPACITANCE

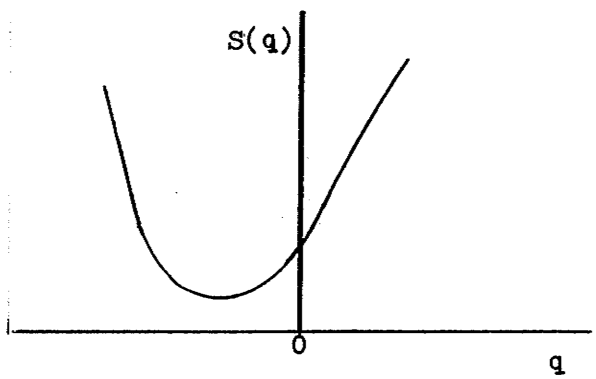


FIGURE II.2.7

GRAPHICAL CHARACTERISTIC OF A NONLINEAR CAPACITANCE

Kirchhoff's Voltage Law for the circuit yields the following equation:

$$L_0 Di + q\,S(q) = 0.$$

To reduce this to the Lagrangian form, the Lagrangian must be defined as

$$L = L_0 i^2/2 - \int qS(q)\,dq.$$

Whence,
$$\frac{\partial L}{\partial i} = L_0 i,$$

and
$$-\frac{\partial L}{\partial q} = qS(q).$$

But, by equations II.2.3 and II.2.6,

$$T = \int i\,d\phi = \int i\,\frac{d\phi}{di}\,di = L_0 i^2/2,$$

since $$L_0 = d\phi/di,$$

and $$V = \int v\, dq = \int q\, S(q)\, dq.$$

Hence, in this case, the Lagrangian takes on the form $L = T - V$. This does not contradict the theory previously presented because the inductive element is linear and so $T' = T$. Thus $L = T'-V$, is also applicable in this case.

$$(T' = \int \phi\, di = \int [\int L_0\, di]\, di = L_0 i^2/2 = T.)$$

The system above can also be shown to be conservative by deriving the phase paths in the form:

$$L_0 i^2/2 + S_0(q^2/2 + Aq^3/3 + Bq^4/4) = h = T + V,$$

these being a family of concentric non-symmetric ovals centred at the origin.

Parallel cases could be demonstrated for the node-pair system analysis.

It is now evident that the following is true. If, in a mesh type of analysis, the nonlinearity does not occur in the kinetic energy storage element, then the Lagrangian can assume its linear form $T - V$. Similarly, in a node-pair type of analysis, the absence of nonlinearity in the potential energy storage element allows the Lagrangian to take its linear form $V_1 - T_1$. Otherwise, in the other two cases, the Lagrangian will be of the forms $T' - V$, and $V_1' - T_1$.

CHAPTER III

DUALITY

In linear systems, the Principle of Duality is well known. Essentially, it is based on the fact that analysis of electrical systems by the mesh method and by the node-pair method bear a marked resemblance to each other and indeed, are identical, if certain quantities and concepts in the one are replaced by certain other quantities and concepts in the other. Such related quantities and concepts are called duals.

There are a multitude of ways in which the dual pairs may be found. The following method has been chosen because it applies itself well to nonlinear systems.

In both the mesh and the node-pair methods of the linear case discussed above, the conservation of energy equation had exactly the same form, namely,

$$P = 2F + D(T + V) \qquad \text{II.1.6}$$

and

$$P_1 = 2F_1 + D(T_1 + V_1) \qquad \text{II.1.6a}$$

but the quantities P,F,T and V had different definitions than P_1,F_1, T_1, and V_1. Since these quantities play the same part in the two analyses, they can be taken as duals. The duality of power supplied to the system from the sources in the two cases yields:

$$\sum_i e_i i_i \overset{D}{=} \sum_i j_i v_i \qquad \text{III.1}^{\#}$$

Since now any combination of elements and their derivatives and integrals has a dual which is the same combination of the duals of the constituent elements and their derivatives and integrals; and conversely, the constituent elements of any two similar combinations which are duals, are themselves duals, it follows from equation III.1 that the two sources and the two dependent variables form two dual pairs. Thus:

$$e_i \overset{D}{=} j_i \qquad \text{III.2}$$

and

$$i_i \overset{D}{=} v_i \,. \qquad \text{III.3}$$

\# The symbol $\overset{D}{=}$ is read " is the dual of".

Before the process is continued, a few remarks are necessary concerning the two statements above. They are not necessarily sufficient to define dual pairs. If the combination in question contains two elements in a symmetrical manner, for example: $x^2 + mxy + y^2$, or more to the point $e_i i_i$ and $j_i v_i$, it is possible to define dual pairs in more than one way. In the latter illustration, it is possible to define duals in the following two forms:

$$e_i \overset{D}{=} j_i \qquad \text{III.2}$$

and

$$i_i \overset{D}{=} v_i \qquad \text{III.3}$$

or

$$e_i \overset{D}{=} v_i$$

and

$$i_i \overset{D}{=} j_i \,.$$

It is necessary to know something further about one of the pairs. In the above, since e_i and j_i were known to play dual roles as sources, the equations III.2 and III.3 were then accepted as correct and the other pairing discarded.

Now since F and F_1 in the two cases are dual,

$$R_{ik} i_i i_k \overset{D}{=} G_{ik} v_i v_k \qquad \text{III.4}$$

and so

$$R_{ik} \overset{D}{=} G_{ik} \,. \qquad \text{III.5}$$

However, T and V are symmetrical in the power equations II.1.6 and II.1.6a, and so two possibilities present themselves. If, however, it is remembered that integrals or derivatives of dual pairs form dual pairs, the difficulty is easily resolved, and the following equations result from equations II.1.4,4a,5 and 5a:

$$L_{ik} i_i i_k \overset{D}{=} C_{ik} v_i v_k \qquad \text{III.6}$$

and

$$S_{ik} q_i q_k \overset{D}{=} \Gamma_{ik} \phi_i \phi_k \qquad \text{III.6a}$$

from which, since

$$q_i \overset{D}{=} \phi_i \,, \qquad \text{III.7}$$

the following dual pairs result:

$$L_{ik} \overset{D}{=} C_{ik} \qquad \text{III.8}$$

and

$$S_{ik} \overset{D}{=} \Gamma_{ik} . \qquad \text{III.9}$$

Equation III.9 could have been derived from equation III.8 by recalling that $\Gamma = L^{-1}$ and $S = C^{-1}$.

Now, since similar combinations of dual-pairs are dual-pairs, themselves, the equation:

$$\sum_k [\, L_{ik}Di_i + R_{ik}i_i + S_{ik}D^{-1}i_i \,] = e_i, \qquad \text{III.10}$$

is the dual of the equation:

$$\sum_k [\, C_{ik}Dv_i + G_{ik}v_i + \Gamma_{ik}D^{-1}v_i \,] = j_i \qquad \text{III.11}$$

But these equations are simply the mathematical statements of Kirchhoff's Voltage and Current Laws in the linear case. Hence, the two Kirchhoff Laws are dual concepts. In a similar manner the two Lagrangian Forms of Kirchhoff's Laws are dual also.

The above process will now be applied to nonlinear systems.

K and J_1 play dual roles in the two types of nonlinear analysis and so since:

$$K = \int v\, di \overset{D}{=} J_1 = \int i\, dv, \qquad \text{III.12}$$

$$i \overset{D}{=} v \qquad \text{III.13}$$

as in the linear case. Also then,

$$K = \int iR(i)\, di \overset{D}{=} J_1 = \int vG(v)\, dv,$$

or

$$R(i) \overset{D}{=} G(v). \qquad \text{III.14}$$

Now, since $\qquad \phi \overset{D}{=} q,$

$$T = \int i\, d\phi \overset{D}{=} V_1 = \int v\, dq, \qquad \text{III.15}$$

which is equivalent to:

$$T = \int il(i)\, di \overset{D}{=} V_1 = \int vc(v)\, dv,$$

or

$$l(i) \overset{D}{=} c(v), \qquad \text{III.16}$$

since

$$d\phi = \frac{d\phi}{di}di = l(i)\, di,$$

by definition.

Also, since $\gamma(v) = l(i)^{-1}$, and $s(q) = c(v)^{-1}$,

$$s(q) \overset{D}{=} \gamma(v), \qquad \text{III.17}$$

and because of equations II.1.7 and II.1.10,

$$L(i) \overset{D}{=} C(v), \qquad \text{III.18}$$

$$S(q) \overset{D}{=} \Gamma(v), \qquad \text{III.19}$$

and

$$r(i) \overset{D}{=} g(v). \qquad \text{III.20}$$

As in the linear case, the statement concerning combinations of elements is true and so it follows that the Kirchhoff equations II.2.8 and II.2.22 are duals as are their respective Lagrange forms II.2.21 and II.2.24.

The following dual-pairs have been deduced for the nonlinear network:

TABLE III.1

NONLINEAR DUAL-PAIRS

Element	Dual
i	v
$R(i)$	$G(v)$
$r(i)$	$g(v)$
$L(i)$	$C(v)$
$l(i)$	$c(v)$
$S(q)$	$\Gamma(v)$
$s(q)$	$\gamma(v)$
K	J_1
J	K_1
T	V_1
V	T_1
T'	V_1'
V'	T_1'

along with the two forms of the general network equations, and the usual dual-pairs of topology such as mesh and node-pair, branch and node, open and short circuit and series and parallel connection.

CHAPTER IV

STATIONARY FUNCTIONS (10)

A further property of the content and co-content of a nonlinear bilateral network may be demonstrated which is a generalization of Maxwell's Minimum Heat Theorem for linear networks.

The theorem for linear networks has two forms, the one the dual of the other, and can be stated as follows, the terms in parentheses indicating the dual case.

If any linear resistive, non-reactive, network is driven by voltage or current generators or both, then, of all the possible distributions of current (voltage) consistent with Kirchhoff's Current (Voltage) Law, that which gives a minimum to the quantity $W - 2P_v$, $(W - 2P_i)$ is the only one consistent with Kirchhoff's Voltage (Current) Law and is therefore, the actual distribution. W is the total power dissipated in the resistors and $P_v(P_i)$, the total power delivered by the voltage (current) generators. Hence, for such a network, the total heat, that is, $W - 2P_v$, $(W - 2P_i)$, is a stationary function.

If, in an n mesh (m node) network of generalized independent currents, i_i, (voltages, v_i), the set of equations:

$$\frac{\partial(W - 2P_v)}{\partial i_i} = 0, \quad i=1,2,\ldots n, \qquad \text{IV.1}$$

$$\left[\frac{\partial(W - 2P_i)}{\partial v_i} = 0, \quad i=1,2,\ldots m-1 \right] \qquad \text{IV.2}$$

is formed, they completely specify the system because they are sufficient to determine the n mesh currents (m-1 node-pair voltages).

In the nonlinear network, however, the function $W - 2P_v$, $(W - 2P_i)$ is not stationary. If, however, the total content of the voltage generators is denoted by $-P_v$, (total co-content of the current generators by $-P_i$) then it may be shown that, if in an active non-reactive, and in general nonlinear, network, the total content(co-content) of all the elements is expressed in terms of a complete set of generalized current (voltage) co-ordinates of the network subject to Kirchhoff's Current (Voltage) Law, then the content(co-content) is stationary for the actual distribution of currents(voltages). This reduces to the Maxwell Minimum

Heat Theorem in the linear case.

The content $-P_v$ and co-content $-P_i$ of each of the generators are illustrated in figures IV.1 and IV.2.

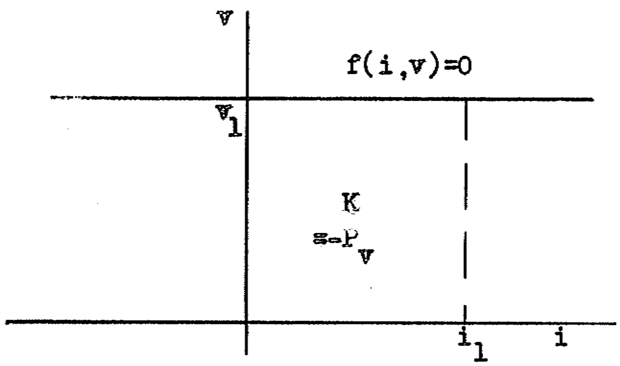


FIGURE IV.1

CONTENT OF A CONSTANT VOLTAGE SOURCE

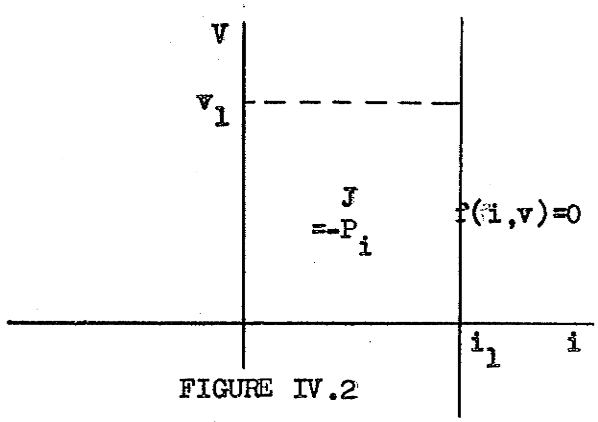


FIGURE IV.2

CO-CONTENT OF A CONSTANT CURRENT SOURCE

The co-content J of a voltage generator and the content K of a current generator have no meaning. The negative sign given to the content and co-content above denotes that the element is active since the two variables denote power dissipation.

To prove the above statement concerning stationary functions in the nonlinear case, assume that only one element of subscript jk and content K_{jk} joins any two terminals or nodes j and k. Let i_{jk} be the actual current in such a branch, and v_j, the actual voltage of node j. Suppose now that i_{jk} takes on an increment di_{jk}. Application of

Kirchhoff's Current Law at node j yields:

$$\sum_k i_{jk} = \sum_k [i_{jk} + di_{jk}] = 0 \qquad \text{IV.3}$$

which gives the condition:

$$\sum_k di_{jk} = 0 \qquad \text{IV.4}$$

Let the total content variation be dK and the constituent content variations be dK_{jk}. Then:

$$dK = \frac{1}{2} \sum_{j,k} dK_{jk} \qquad \text{IV.5}$$

$$= \frac{1}{2} \sum_{j,k} (v_j - v_k)\, di_{jk} \qquad \text{IV.6}$$

to a first order approximation of the Taylor's series expansion, by definition of content. Thence,

$$2dK = \sum_{j,k} v_j di_{jk} + \sum_{j,k} v_k di_{kj}$$

$$= 2 \sum_{j,k} v_j di_{jk} \qquad \text{IV.7}$$

since the two sums are identical.

Therefore,

$$dK = \sum_{j,k} v_j di_{jk}$$

$$= \sum_j v_j \sum_k di_{jk}. \qquad \text{IV.8}$$

But by equation IV.4, the second summation is zero, and so,

$$dK = 0. \qquad \text{IV.9}$$

Hence, K, the content of the system, is stationary to the first order, to all variations of the currents. The dual theorem is proved in an identical manner. From the stationary value of K (J), all the currents (voltages) in the n mesh (m node) network can be determined from the equations:

$$\frac{\partial G}{\partial i_i} = 0, \; i=1,2,\ldots n \qquad \text{IV.10}$$

$$\left[\frac{\partial J}{\partial v_i} = 0. i=1,2,\ldots m-1, \right] \qquad \text{IV.11}$$

which are generalizations of equations IV.1 and IV.2 of the linear case.

In the nonlinear case, if $K_R(J_R)$ is the total content (co-content) of the resistors in the network and $P_v(P_i)$ is the total power delivered by the voltage (current) sources, then, the total content (co-content) of the system is:

$$K = K_R - P_v \qquad \text{IV.13}$$

$$[\ J = J_R - P_i\ ] \qquad \text{IV.14}$$

In the linear case, since $W = 2K_R = 2J_R$, theselast two equations both reduce to the linear expression for the total heat of the system, namely, $W - 2P_v(W - 2P_i)$.

The restrictions made above that only one direct path can exist between any two nodes does not affect the generality of the result. If there were more than one direct path, say r such, between any two nodes j and k carrying currents $i_{jk}(1)$, $i_{jk}(2)$,...$i_{jk}(r)$, then only one of these must appear in the equations IV.13 or IV.14, since they are not independent variables. The same result is obtained by first combining any parallel paths between two nodes. Once the total current between these nodes is found, each constituent may be determined by Ohm's law.

The above theorem can be extended to systems containing reactance by the following argument. Equations IV.10 and IV.11 still hold for the network since the added reactance does not affect the content or co-content of the system. That is:

$$\frac{\partial K}{\partial i_i} = 0\ ,\ i=1,2,\ldots n \qquad \text{IV.10}$$

and

$$\frac{\partial J}{\partial v_i} = 0\ ,\ i=1,2,\ldots m-1. \qquad \text{IV.11}$$

The total time derivatives of K and J for the general network are:

$$\frac{dK}{dt} = \frac{\partial K}{\partial t} + \frac{\partial K}{\partial i_1}\frac{di_1}{dt} + \ldots + \frac{\partial K}{\partial i_n}\frac{di_n}{dt} \qquad \text{IV.12}$$

and

$$\frac{dJ}{dt} = \frac{\partial J}{\partial t} + \frac{\partial J}{\partial v_1}\frac{dv_1}{dt} + \ldots + \frac{\partial J}{\partial v_{m-1}}\frac{dv_{m-1}}{dt} \qquad \text{IV.13}$$

Application of equations IV.10 and IV.11, however, yields:

$$\frac{dK}{dt} = \frac{\partial K}{\partial t} \qquad \text{IV.14}$$

and

$$\frac{dJ}{dt} = \frac{\partial J}{\partial t} \qquad \text{IV.15}$$

but if the network is time-invariant in its element values, and the generators are considered as constant current or voltage, the time does not appear explicitly in the K or J and so:

$$\frac{dK}{dt} = \frac{dJ}{dt} = 0 \qquad \text{IV.16}$$

which is the required result. This, of course, can also be applied to constant current or voltage source linear networks as well.

CHAPTER V

THE IMPEDANCE CONCEPT

The concept of impedance, used extensively in linear network analyses, can also be defined in a multitude of ways. Consider, first, its definition in the linear theory. The differential equations governing such a system may be written in the form:

$$\sum_{k=1}^{n} A_{ik}(D)\ i_k = B_i e_i, \qquad \text{V.1}$$

where $i = 1,2,\ldots n$. $A_{ik}(D)$ and $B_i(D)$ are ~~constants~~ linear operators with constant coefficients which may be identified with the parameters of the system as shown in Section 1 of Chapter II. The variables i_i and e_i are either sources or unknowns depending on which system of analysis, equation V.1 represents.

If, now, the voltages and currents are assumed to be damped harmonic in form, then:

$$e_i = E_i\ e^{(\alpha+j\omega)t} = E_i e^{st} \qquad \text{V.2}$$

and

$$i_i = I_i\ e^{(\alpha+j\omega)t} = I_i e^{st} \qquad \text{V.3}$$

where α represents any damping present in the waveform, ω, the real angular frequency of oscillation, E_i and I_i, the complex amplitudes of voltage and current, respectively, and $s = \alpha+j\omega$, the complex frequency. In this form D becomes simply multiplication by s, and so, equation V.1 may be written:

$$\Sigma\ A_{ik}(s)\ I_k = B_i\ E_i. \qquad \text{V.4}$$

Thence, for each mesh or node-pair denoted by i,

$$E_i = \frac{\Sigma\ A_{ik}(s)}{B_i} I_i = Z(s)\ I_i \qquad \text{V.5}$$

where Z(s) is the impedance of the i'th mesh expressed as a function of complex frequency, when the mesh type of analysis is used, or:

$$I_i = \frac{B_i}{\Sigma\ A_{ik}(s)} E_i = Y(s)\ E_i \qquad \text{V.6}$$

where Y(s) is the admittance of the i'th node-pair when that type of analysis is used.

Z(s) and Y(s), the so-called complex impedance and admittance functions of the network, are mutually reciprocal and have a magnitude and phase which are functions of the complex frequency, s.

Since, now, e and i contain only one frequency, the impedance (or admittance) function, although defined above as the ratio of complex amplitudes, could equally well have been defined as the ratio of the amplitudes of the fundamental components of the two variables. The two results in the linear case would be identical. However, in the nonlinear case, the two values so defined would be different. The difference is due to the following fundamental property of nonlinear elements.

Consider, for example, the simple nonlinear resistor whose characteristic is given by:

$$i = i(v) \qquad \text{V.7}$$

when explicitly solved for i as a function of v. Suppose now,

$$v = V_1 \cos \omega t.$$

Then,

$$i = i(V_1 \cos \omega t) \qquad \text{V.8}$$

which is a periodic function of t and can, therefore, at least formally, be expanded as a Fourier Series:

$$i = \frac{I_0}{2} + \sum_{k=1}^{\infty} [\, I_k \cos(k\omega t + \theta_k) \,] \qquad \text{V.9}$$

where the I_k are the well known Fourier coefficients. This result can be extended to include all other types of elements.

Hence, even though the driving voltage or current as the case may be, contains a single frequency only, the resulting current or voltage can contain harmonics of all orders, the magnitudes and phases of which depend on the nature of the nonlinearity.

The magnitudes of voltage and current may be written as:

$$|v|^2 = |V_1|^2 \qquad \text{V.10}$$

and

$$|i|^2 = [\, \frac{|I_0|^2}{4} + \Sigma |I_k|^2 \,] \qquad \text{V.11}$$

and so the magnitude of the impedance defined in the former manner mentioned above is:

$$|Z| = \left|\frac{v}{i}\right| = \frac{|V_1|}{[\frac{|I_0|^2}{4} + \Sigma |I_k|^2]^{1/2}} \,. \qquad \text{V.12}$$

Introducing the harmonic coefficient, ρ, by the definition:

$$1 + \rho^2 = \frac{\frac{|I_0|^2}{4} + \Sigma |I_k|^2}{|I_1|^2}, \qquad \text{V.13}$$

the impedance magnitude may be written in the form:

$$|Z| = \frac{|V_1|}{|I_1|(1+\rho^2)^{1/2}}. \qquad \text{V.14}$$

If, however, the impedance magnitude were defined as the ratio of the magnitudes of the fundamental frequency components of the voltage and current, then:

$$|Z|_1 = \left|\frac{V_1}{I_1}\right| = |Z|(1+\rho^2)^{1/2} \qquad \text{V.15}$$

and the two definitions yield different results.

In the linear case, $I_k = 0$, $k \neq 1$, and so $\rho = 0$. The two definitions give identical results for the impedance magnitude.

The definition of phase of the impedance in the nonlinear case presents a different sort of problem, namely: what is the phase difference between two non-sinusoidal waves? In the case of the two definitions of the magnitude, both results are theoretically measurable by means of standard meters, the former value by the ratio of the rms voltage and current, measured with instruments with flat frequency response, the latter, by means of meters responding to the fundamental frequency only. In the case of phase of the impedance, only the latter type of definition yields a measurable result, namely: the phase of the impedance is given by the difference in the phases of the fundamental components of the voltage and current. That is,

$$\theta_z = \theta_{v1} - \theta_{i1} \qquad \text{V.16}$$

where the symbols are self-explanatory. The former type of definition has no meaning when applied to phase considerations.

Other definitions of impedance will be discussed in Part B along with the methods in which their definitions occur.

CHAPTER VI

NONLINEAR RESONANCE

Closely connected with the concept of impedance are the explanations of the resonance phenomena. The linear case will first be discussed from a different point of view than the usual and the process will then be extended to the nonlinear case.

VI.1 THE LINEAR DISSIPATIONLESS CASE

Consider, first, the constant voltage-(rms) driven dissipationless, series circuit, consisting of a linear inductance, L, and a linear capacitance, C. In the impedance form, the equation connecting voltage V and current I (both rms values) may be written (in magnitude):

$$I = V/|Z| \qquad \text{VI.1.1}$$

where,

$$Z = X = \omega L - 1/\omega C. \qquad \text{VI.1.2}$$

The current can have either a leading or lagging phase angle of $\pi/2$ radians with respect to the voltage but this may be neglected for the discussion of this section.

Usually I is plotted against ω directly, to give the frequency response curve, but another approach lends itself more easily to the analysis of nonlinear case.

If V is a constant, then, equation VI.1.1 describes a hyperbola in the I - Z plane as shown in figure VI.1.1. If ω is a constant, say ω_1 then Z is a constant also, and is described by a horizontal line in the figure. The intersection between the two curves gives the current flowing in the circuit when a voltage V of frequency ω_1 is applied.

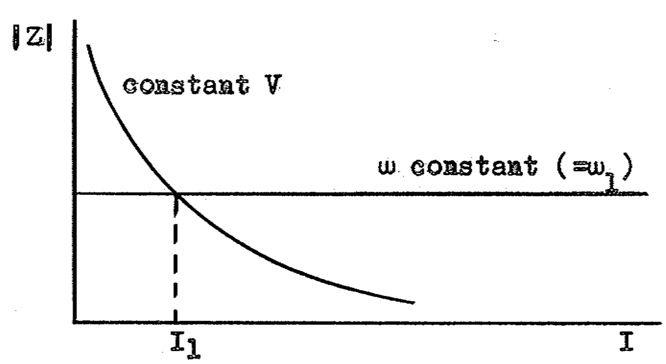


FIGURE VI.1.1

I - Z CHARACTERISTICS OF A LINEAR DISSIPATIONLESS CIRCUIT

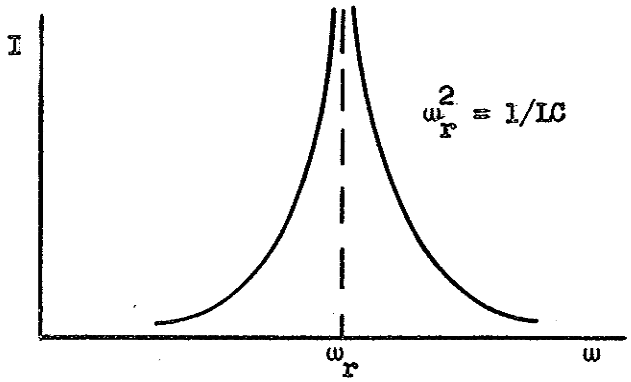


FIGURE VI.1.2

I - ω CHARACTERISTIC OF A LINEAR DISSIPATIONLESS CIRCUIT

By applying different frequencies in the above method, a sequence of points $(\omega_i I_i)$ may be found and plotted to give the more usual form of the resonance curve shown in figure VI.1.2. At $\omega = \omega_r$, the resonant frequency, $Z = 0$, and I is infinite. ω is a double-valued function of I due to the fact that as ω increases and passes resonance, the constant V curve on the I - Z characteristic is first traversed in the increasing I direction and then in the decreasing I direction.

VI.2 THE GENERAL LINEAR CASE

In the general linear case, where the circuit consists of a series connection of R, L and C, figures VI.1.1 and VI.1.2 take on the form of figures VI.2.1 and VI.2.2, respectively, where,

$$|Z| = [R^2 + (\omega L - 1/\omega C)^2]^{1/2} \qquad \text{VI.2.1}$$

and as before,

$$I = V/|Z|. \qquad \text{VI.2.2}$$

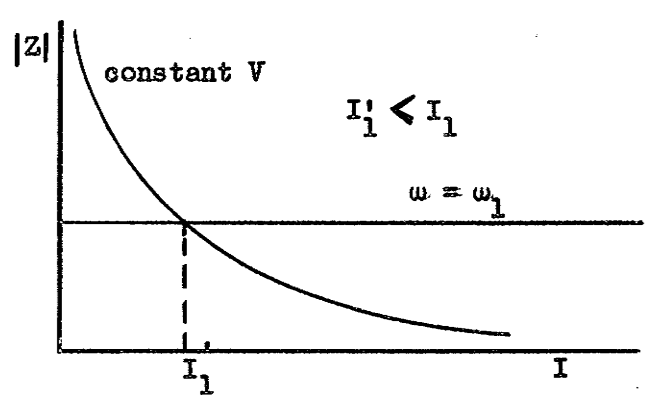


FIGURE VI.2.1

I - Z CHARACTERISTIC OF A GENERAL LINEAR CIRCUIT

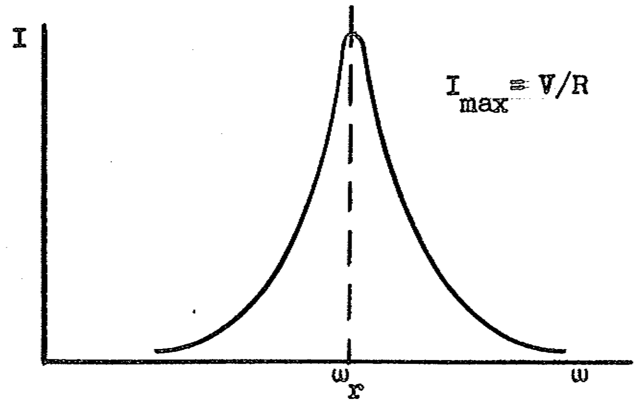


FIGURE VI.2.2

I - ω CHARACTERISTIC OF A GENERAL LINEAR CIRCUIT

VI.3 THE NONLINEAR DISSIPATIONLESS CASE

Consider now the case of the series connection of a nonlinear inductor and a nonlinear capacitor. For the sake of definiteness, let the two be such that the inductive and capacitive reactances can be written as:

$$X_L = \omega L\left(1 - \frac{3aI^2}{2}\right) \qquad \text{VI.3.1}$$

and

$$X_C = \frac{1}{\omega C}\left(1 + \frac{3bI^2}{2\omega^2} \right). \qquad \text{VI.3.2}$$

These results are easily obtained by assuming characteristics of the forms:

$$v = \frac{q}{C} + \frac{bq^3}{C} \qquad \text{VI.3.3}$$

and

$$\phi = Li - aLi^3 \qquad \text{VI.3.4}$$

for the capacitor and inductor respectively, the definition of reactance:

$$X = E_{max}/ I_{max} , \qquad \text{VI.3.5}$$

and by neglecting the third harmonic present. Also assumed in the reduction is the fact that $2I^2 = I^2_{max}$, so that the argument below is intrinsicically accurate only for small nonlinearities. The purpose of the discussion, however, is to yield qualitative results applicable in all cases.

The total impedance of such a connection is a pure reactance:

$$Z = X = [\omega L - 1/\omega C] - 3/2[a\omega L + b/\omega^3 C] I^2 \qquad \text{VI.3.6}$$

or $$Z = A - B I^2, \qquad \text{VI.3.7}$$

This equation for constant ω, has the form of a downward opening parabola in the I-Z plane since in most practical cases, a and b and hence, B, are positive. $|Z|$ is obtained by reflecting those portions of the parabola which lie below the I axis to yield a set of curves, with parameter ω, as shown in figure VI.3.1

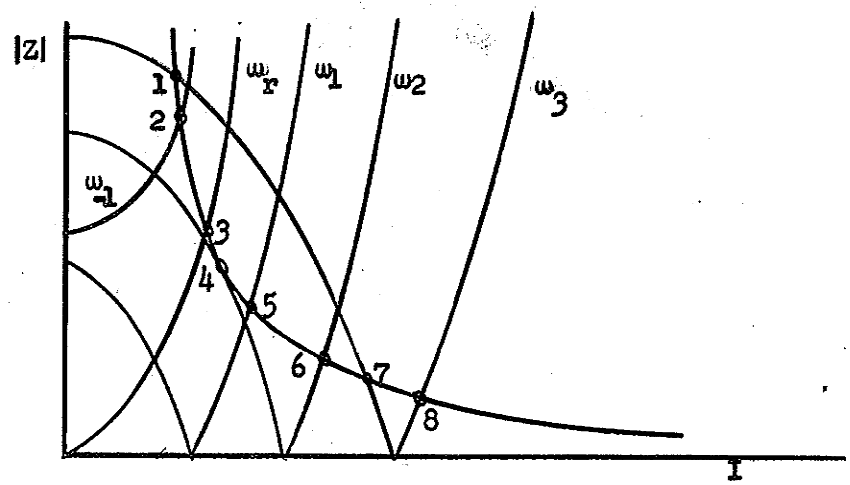


FIGURE VI.3.1

I - Z CHARACTERISTICS OF A DISSIPATIONLESS NONLINEAR CIRCUIT

The parametric dependance on ω appears through A and B being functions of ω. A is the linear reactance component of the system.

Any one of these curves plotted in figure VI.3.1 now takes the place of the constant ω, horizontal line of figure VI.1.1. Superimposing the constant V hyperbola on the figure yields a set of intersections, 1,2,3,4,5,6,7,and 8. There are three different cases present.

For $\omega = \omega_{-1}$, ω_r or ω_1,(and all lower frequencies than ω_2), there is only one intersection given by points 2,3 and 5, respectively. For $\omega = \omega_2$, there are two intersections at points 4 and 6. For $\omega = \omega_3$, (and any frequencies higher than ω_2), there are three intersections altogether, at points 1,7 and 8. Plotting I against ω yields the curve of figure VI.3.2.

The curve is obviously multiple-valued in both variables for $\omega \geqslant \omega_2$. It has the appearance of the linear frequency response curve of figure VI.1.2 if it were bent to the right.

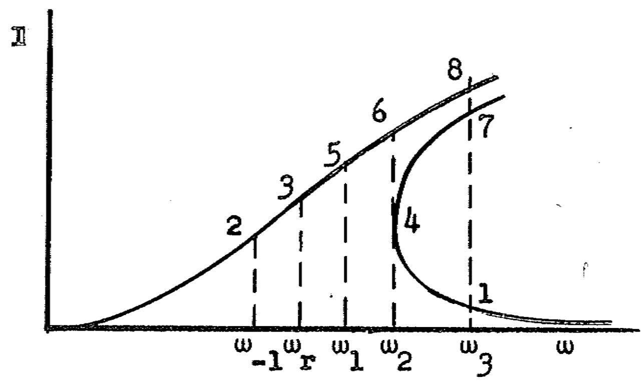


FIGURE VI.3.2

I - ω CHARACTERISTIC OF A NONLINEAR DISSIPATIONLESS CIRCUIT

VI.4 THE NONLINEAR CASE WITH LINEAR DISSIPATION

Consider, now, the more general case in which a linear resistance, R, is connected in series with the system of section VI.3. Equation VI.3.6 becomes:

$$|Z| = [\,R^2 + X^2\,]^{1/2} \qquad \text{VI.4.1}$$

where,

$$X = A - B\,I^2 \qquad \text{VI.4.2}$$

as before.

The plot of $|Z|$ versus I with ω as a parameter takes the form of figure VI.4.1.

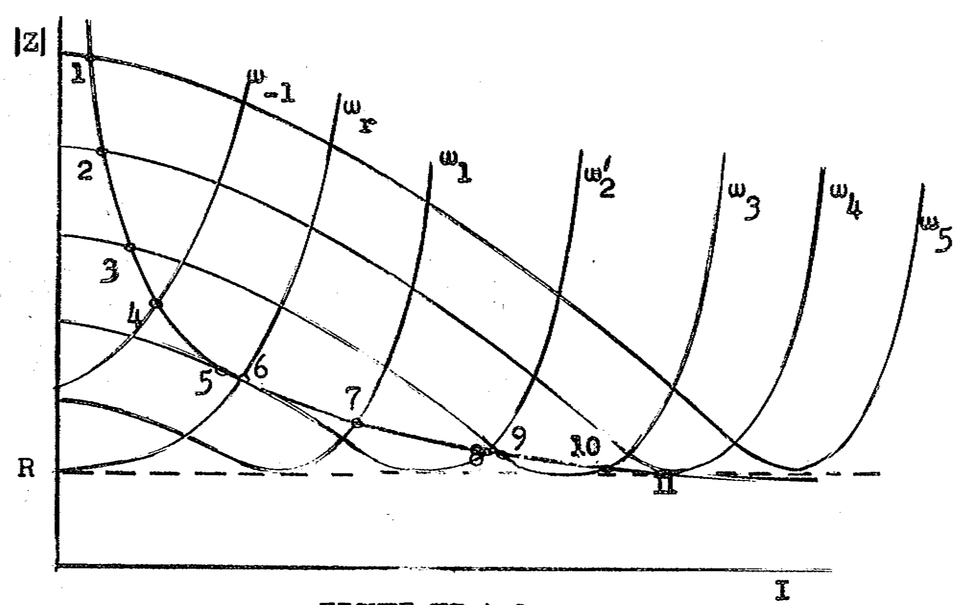


FIGURE VI.4.1

I - Z CHARACTERISTICS OF A GENERAL NONLINEAR CIRCUIT

The ω'_2 here is not the same as the ω_2 used previously, but has been changed to make it a critical value for the new system. The plot of the intersections of the hyperbola and the ω-parameter curves yield the frequency response of the system in the I - ω plane as shown in figure VI.4.2.

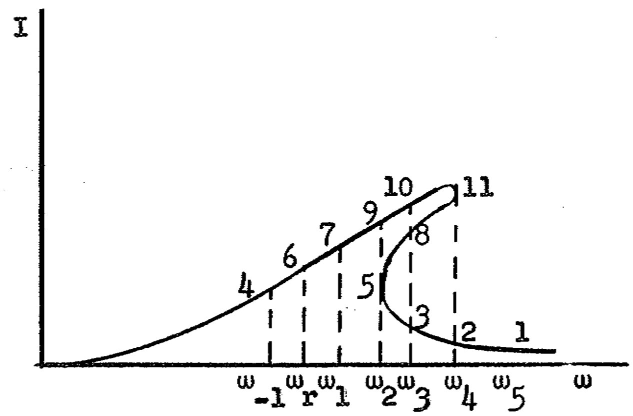


FIGURE VI.4.2

I - ω CHARACTERISTIC OF A GENERAL NONLINEAR CIRCUIT

The resistance has the effect of rounding the curve off at its upper end, so that the region of triple valued current is limited between ω_2 and ω_4 rather than infinite in extent. The curve resembles the frequency response of the linear case with dissipation, bent again to the right.

Any system with a frequency response as in figure VI.4.2 will exhibit the so-called "jump" phenomenon as the frequency is varied. If the frequency is raised from zero, the current flowing in response to the voltage, V, will follow the upper curve through points 4, 6, 7, 9, and 10 to point 11. As the frequency is increased further, the current will fall to point 2 and slowly decrease through point 1. If the frequency is now lowered, the current will follow through points 1, 2, and 3, to point 5, from which with a further decrease in frequency, it will jump to point 9 and continue on down the upper curve to zero again. The path traced out exhibits the hysteresis effect.

The section of the curve between points 5 and 11 is easily demonstrated to be unstable. Consider the current at point 9 at frequency ω_3 in figure VI.4.1. With a slight increase in current, the impedance curve falls below the constant voltage curve, and so, the circuit will take even more current from the source. This continues until the current rises to point 10. A decrease in current from its value at point 9 would result in its dropping to the value at point 3. At points 3 and 10, however, the impedance curve rises above the constant voltage curve with increasing current and vice-versa, and hence would tend to reduce any slight changes in current from either point, and the current would return to the point in question. They would, therefore, be stable.

The jump phenomenon, however, will not take place for all applied voltages. Since the voltage hyperbola will tend to approach the origin of the I - Z plane as the voltage magnitude is decreased, there will be a voltage, V_1, below which, no impedance curve intersects the hyperbola more than once and thence, a voltage, V_1, below which no jump phenomenon will take place. This value is termed the critical voltage for the system. In terms of the resonant curve in the I - ω plane, it is the lowest value of voltage for which the curve has a vertical tangent. This is indicated in figure VI.4.3, in which $V_3 > V_2 > V_1$.

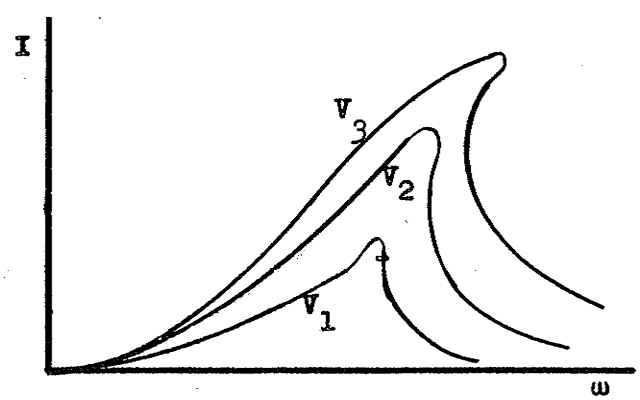


FIGURE VI.4.3

I - ω CHARACTERISTIC WITH VOLTAGE PARAMETER

The above system has a characteristic expressible as:

$$V = f(I) = AI - BI^3 \qquad \text{VI.4.3}$$

in the dissipationless case. Hence,

$$\frac{d^2V}{dI^2} = -6BI \qquad \text{VI.4.4}$$

which, because $B > 0$, is negative for $I > 0$. The system is therefore of the soft type. If B were negative, the system would be hard and the response would be bent to the left. The presence of resistance does not appreciably affect the classification.

Although the above analysis has been carried out in terms of a specified characteristic, the results are of a general nature, qualitatively speaking. Critical values computed from analytical approximations to the characteristics involved are seldom of sufficient accuracy for design. One of the basic causes for error is the complete neglect of the phase considerations.

Further work has been done by Poincare(12), Liapounoff(8), Stoker(15), Hayashi(5) and others, involving essentially complicated phase plane considerations of stability, and Mathieu functions, but the results, although mathematically elegant are of little use in design.#

It should also be noted here that the above system has a non-simple overall characteristic, even though the individual characteristics are of a simple nature. This is an example of the statement made in Chapter I, page 2.

See, for example, the deviations involved in Hayashi's predictions.

CHAPTER VII

SUBHARMONIC OSCILLATION (36)

One important feature of nonlinear circuits which has no counterpart in linear circuits is the production of subharmonic oscillations, that is, oscillations with periods longer than that of the driving function. The physical mechanism for production of such oscillations and the conditions under which they are maintained vary with cases.

VII.1 THE EXISTENCE CONDITIONS

Consider, however, the following example, the method and reasoning of which could be applied in other instances.

A nonlinear inductance, the characteristic of which is given by:

$$v = (L_0 + L_1 i^{-2/3}) Di , \qquad \text{VII.1.1}$$

is connected in series with a linear capacitance of elastance S, across a voltage source, E. The system can be considered to be normalized in time so that the frequency of the source is 1 radian per second. This system is described by the differential equation:

$$[L_0 + L_1 i^{-2/3}] Di + SD^{-1} i = E \sin(t+\theta) \qquad \text{VII.1.2}$$

where θ is the initial phase of the voltage. This may also be written as:

$$D[L_0 i + 3L_1 i^{1/3}] + SD^{-1} i = E \sin(t+\theta). \qquad \text{VII.1.3}$$

Normally, the solution sought would be of the form:

$$i_1 = I_1^3 \cos^3(t+\theta) = \frac{I_1^3}{4} [3 \cos(t+\theta) + \cos 3(t+\theta)] \qquad \text{VII.1.4}$$

which includes a fundamental and a third harmonic component, the constant I_1 to be determined.

Substituting from equation VII.1.4 into equation VII.1.3, carrying out the indicated operations and equating the coefficients of $\sin(t+\theta)$ and $\sin 3(t+\theta)$ on both sides of the resulting equation yields:

$$- \frac{3L_0 I_1^3}{4} - 3L_1 I_1 + \frac{3SI_1^3}{4} = E \qquad \text{VII.1.5}$$

and $$-\frac{3L_0 I_1^3}{4} - \frac{S I_1^3}{12} = 0 \qquad \text{VII.1.6}$$

which reduce to:

$$E + 3L_1 I_1 = \frac{3}{4} I_1^3 (S - L_0) \qquad \text{VII.1.7}$$

and $$9L_0 = S. \qquad \text{VII.1.8}$$

These are the conditions for which equation VII.1.3 has the solution of equation VII.1.4.

However, suppose a solution of the form:

$$i_2 = I_2^3 \cos^3\left(\frac{t+\theta}{3}\right) = \frac{I_2^3}{4} \left[3 \cos\left(\frac{t+\theta}{3}\right) + \cos(t+\theta)\right] \qquad \text{VII.1.9}$$

is sought for equation VII.1.3. This solution includes a fundamental and a 1/3 subharmonic term.# Carrying out the same procedure as above and equating the coefficients of the sine of like multiples of t+θ yields:

$$-\frac{L_0 I_2^3}{4} - L_1 I_2 + \frac{9 S I_2^3}{4} = 0 \qquad \text{VII.1.10}$$

and $$\frac{S I_2^3}{4} - \frac{L_0 I_2^3}{4} = E \qquad \text{VII.1.11}$$

which in turn reduce to:

$$L_1 = \frac{I_2^2}{4} [9S - L_0] \qquad \text{VII.1.12}$$

and $$4E = I_2^3 [S - L_0] . \qquad \text{VII.1.13}$$

If these conditions are satisfied then a one-third subharmonic oscillation can exist in the solution.

If all four equations VII.1.7, 8, 12 and 13 are satisfied, then equations VII.1.4 and VII.1.9 are both solutions of the original equation. One such set of parameters are:

$E = 1$, $L_0 = 0.01$, (assumed),
$S = 0.09$, (from equation VII.1.8),
$I_2 = 3.684$, (from equation VII.1.13),
$L_1 = 2.71$, (from equation VII.1.12),
and $I_1 = 11.69$, -0.123, or -11.57, (from equation VII.1.7).

No nomenclature has as yet been standardized as to whether this should be called a third subharmonic or a 1/3 harmonic. The author, therefore uses a nomenclature which is obvious in its meaning.

Using the first value of I_1 given, the two solutions become:

$$i_1 = 1176 \cos(t+\theta) + 396.2 \cos 3(t+\theta) \qquad \text{VII.1.14}$$

and

$$i_2 = 37.5 \cos\frac{(t+\theta)}{3} + 12.5 \cos(t+\theta). \qquad \text{VII.1.15}$$

Either of these solutions could be excited by use of different initial conditions. For example, if t = 0, when the current is zero, then for the equation VII.1.14:

$$\theta = \pi/2, \ (3\pi/2), \qquad i_1(0) = 0, \qquad \text{and} \quad q_1(0) = 1044(-1044),$$

so that, if the voltage sin $(t+\pi/2)$ were applied when the capacitor was charged to 1044 units with either polarity, the current would obey equation VII.1.14. For equation VII.1.15, with the same reference,

$$\theta = 3\pi/2, (\pi/2), \qquad i_2(0) = 0, \qquad \text{and} \quad q_2(0) = 100, (-100),$$

so that the current obeys that equation if the initial charge on the capacitor is 100 units. There would be a similar solution for the other two values of I_1 given in the equation above.

VII.2 THE PHYSICAL MECHANISM

Again consider the above case where the flux-linkage - current characteristic is as shown in figure VII.2.1.

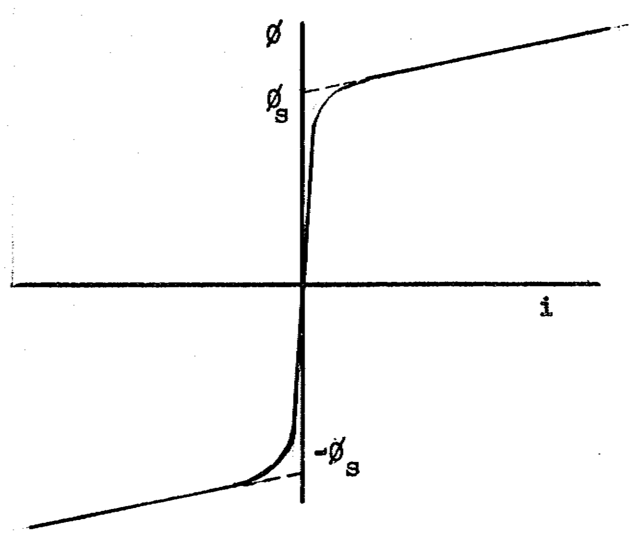


FIGURE VII.2.1

TYPICAL NONLINEAR INDUCTANCE CHARACTERISTIC WITH LINEAR APPROXIMATIONS

For the sake of argument, since i is negligible until the flux linkages reach the saturation value ϕ_s, the characteristic may be approximated by the straight lines shown dotted in the figure.

Taking $\theta = 0$, and specifying quiescent initial conditions yields the solution to the equation:

$$D\phi + SD^{-1}i = E \sin t, \qquad \text{VII.2.1}$$

in the form:

$$\phi = E [1 - \cos t] - SD^{-2}i. \qquad \text{VII.2.2}$$

For the first part of the solution, i can be taken as zero until $\phi = \phi_s$, following the above approximation. Thus, until $t = t_1$ where:

$$t_1 = \cos^{-1} \left\{\frac{E-\phi_s}{E}\right\}, \qquad \text{VII.2.3}$$

the term $SD^{-1}i = 0$, and,

$$\phi = E [1 - \cos t]. \qquad \text{VII.2.4}$$

When ϕ passes ϕ_s, current starts to flow in, say, the positive direction, charging the capacitor until ϕ decreases again to ϕ_s. The equation for i using the straight line approximations can be found but is not necessary. Since, however, the current is unidirectional, when ϕ returns to ϕ_s and current ceases, the capacitor will have acquired a charge, say q, and so for $\phi > \phi_s$ the term $SD^{-2}i$ will take the form Sqt and so the flux linkages will be given by:

$$\phi = E [1 - \cos t] - Sqt - \phi_0 \qquad \text{VII.2.5}$$

where ϕ_0 is the flux linkage value when the current ceased to flow. The flux can therefore be seen to be oscillatory with a steadily increasing negative component. If q is small enough, it will take several cycles of time before ϕ reaches the value $- \phi_s$ and allows current to flow in the negative direction thus discharging the capacitor. The charge is therefore trapped for several cycles and since the process is repetitive after the negative half cycle is traversed, the capacitor voltage will oscillate at a subharmonic frequency. The period of this frequency does not have to be a sub-multiple of the driving function frequency since it depends on q.

Although the above analysis has been applied to a single example it has a far wider application in method and can, at least, be used to detect the possibility of subharmonic oscillation in a simple manner. The entire process depends on the gating action of one non-linear element to trap stored energy in another element. This principle applies to electrical and mechanical systems alike.

A great deal of work has been done on the existence conditions for subharmonics by Cartwright(20), Levinson(23), Kryloff and Bogoliuboff(6), and others but again, the processes are highly theoretical and of little use in a practical sense because they rely mostly on analytical equations for the characteristic and thence depend critically on the form of approximation used as does the method outlined above in section 1 of this chapter.

CHAPTER VIII

CONCLUSION

The possibility of a generalization of further existing linear network properties immediately presents itself.

In the case of Foster's Reactance Theorem, which is a direct result of the invariance property discussed above, and which describes the linear system in terms of its natural modes of oscillation, if a generalization exists, it must describe the nonlinear system in terms of its characteristic oscillations. The result would, in general, involve higher transcendental functions.

For example, in the problem of the simple pendulum, if the analysis is limited to small oscillations, the solution is a linear simple harmonic oscillation. If the limitation is not made, the solution is elliptic in form. In other nonlinear cases, the result might be a hyperelliptic or a hypergeometric function depending on the type of nonlinearity involved. In most cases, the solution will be unattainable in closed form.

Since the characteristic oscillations of a linear system give rise to the so-called poles of the network in the complex frequency plane, it is possible that the characteristic oscillations of a nonlinear system could be described by a pole-zero pattern in a general complex plane, the positions of which would be dependent on initial conditions or on some variable in the system.

A generalization of many other linear network theorems is also a possibility but these will not be discussed at this time.

PART B

METHODS OF SOLUTION

CHAPTER IX

INTRODUCTION

The equations discussed in Part A were of a very general nature, the nonlinearities occurring in all three types of basic elements. The general n variable equations have not, as yet, been solved in their general form. Actually, only very few special problems have admitted a complete solution.

Nonlinear problems appear in almost every field of study but there has been little correlation in the methods of solution, due principally to the fact that few problems reduce to the same basic equations. There are, of course, a few exceptions to the rule.

As a result of this, the engineer, when faced with a nonlinear problem, can only resort to the few methods with which he is familiar, and if these fail him, he must proceed at his own risk. There are no general rules to follow and he can only rely on his intuition to lead him to any sort of an approximate solution, which, because of his lack of experience in the problem, will not satisfy him in most cases.

There exists, however, a great number of techniques each applicable to only a few problems which, although not mathematically elegant, are of much use in yielding at least an approximate solution. These techniques are wide spread throughout the literature and as a result either are unattainable or pass unnoticed. It is the purpose, therefore of this section to present a certain selection of these techniques which apply to circuit analyses of the nonlinear variety and to illustrate these methods with as many problems as possible.

The methods presented have been divided into three groups:

1. Analytical (Classical) Methods
2. Graphical Methods

and 3. Transform Methods.

The "classical" denomination has been used in the sense opposed to the more modern transform methods. All three methods are, to a greater or lesser degree, of an approximate nature.

CHAPTER X

ANALYTICAL METHODS

X.1 THE SMALL SIGNAL THEORY (16)

Perhaps the best known method of analysis of nonlinear circuits is the so-called small signal method which has its primary use in the analysis of the operation of vacuum tubes.

Fundamentally, the tube of most interest, namely the triode, is characterized by three variables: plate voltage e_b, plate current i_b, and grid voltage e_c, among which there exists a relationship expressable as the surface:

$$i_b = f(e_b, e_c) \qquad \text{X.1.1}$$

in the three dimensional space (e_b, e_c, i_b). The tube is usually biased to a certain operating or quiescent point given by (E_{bo}, E_{co}, I_{bo}), which values satisfy equation X.1.1, and then by superposition, alternating voltages and currents are applied to cause the actual point of operation to fluctuate about the initial point above. When any two of the variables are mutually constrained by external circuits, equation X.1.1 then gives one of the constrained variables as a function of the third variable. Usually, i_b and e_b are constrained by the plate load. Defining fluctuations in these variables by i_b, e_b, and e_c, or more commonly by i_p, e_p, and e_g, and expanding equation X.1.1 in a Taylor's series about the quiescent point, yields:

$$i_p = \frac{\partial i_b}{\partial e_c} e_g + \frac{\partial i_b}{\partial e_b} e_p + \frac{1}{2!}\left[\frac{\partial^2 i_b}{\partial e_c^2} e_g^2 + \frac{\partial^2 i_b}{\partial e_b^2} e_p^2 + 2\frac{\partial^2 i_b}{\partial e_c \partial e_b} \times e_g e_p\right] + \ldots \qquad \text{X.1.2}$$

where all the partial derivatives are evaluated at the operating point.

The small signal theory then stipulates that all derivatives higher than the first are negligible and so, to a first order linear approximation:

$$i_p = \frac{\partial i_b}{\partial e_c} e_g + \frac{\partial i_b}{\partial e_b} e_p. \qquad \text{X.1.3}$$

Using the definitions:

$$\frac{\partial i_b}{\partial e_c} = g_m, \qquad \text{X.1.4}$$

and

$$\frac{\partial i_b}{\partial e_b} = \frac{1}{r_p}, \qquad \text{X.1.5}$$

equation X.1.3 can be written:

$$i_p = g_m e_g + \frac{1}{r_p} e_p. \qquad \text{X.1.6}$$

The parameters g_m and r_p so defined, correspond to the incremental parameters of Part A.

If, now, the values of i_p and e_p are mutually constrained by the external load, Z, through the equation:

$$e_p = - i_p Z \qquad \text{X.1.7}$$

following the standard polarity notation, then:

$$i_p = g_m e_g + \frac{1}{r_p}(- i_p Z), \qquad \text{X.1.8}$$

which reduces to the relation:

$$i_p = \frac{g_m r_p}{r_p + Z} e_g = \frac{\mu e_g}{r_p + Z}, \qquad \text{X.1.9}$$

where $\mu = g_m r_p = \frac{e_b}{e_c}$ is the amplification factor of the tube.

From equation X.1.9, the linear constant voltage equivalent plate circuit is as shown in figure X.1.1.

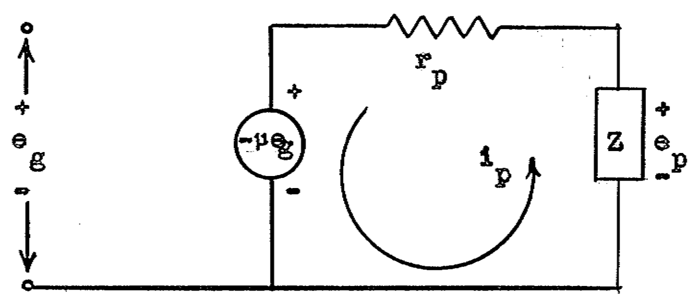


FIGURE X.1.1

LINEAR EQUIVALENT VACUUM TUBE PLATE CIRCUIT

The forms of the corresponding higher order equivalent circuits may be derived as follows. From equation X.1.2, using the definitions of equations X.1.4 and 5, and considering μ as a constant, there results:

$$i_p = g_m(e_g + e_p/\mu) + \frac{1}{2!}\left[\frac{\partial g_m}{\partial e_c} e_g^2 + 2\frac{\partial g_m}{\partial e_b} e_g e_p + \frac{\partial(\frac{1}{r_p})}{\partial e_b} e_p^2\right] + \dots \qquad \text{X.1.10}^{\#}$$

If $\frac{\partial g_m}{\partial e_c}$ is factored out of the quadratic term, the equation becomes:

$$i_p = g_m(e_g + e_p/\mu) + \frac{1}{2!}\,[e_g + e_p/\mu]^2 \frac{g_m}{e_c} + \frac{1}{3!}[\,e_g + e_p/\mu]^3 \frac{\partial^2 g_m}{\partial e_c^2} + \dots \qquad \text{X.1.11}$$

after some allowable manipulation. If now the constraint of equation X.1.7 is imposed on equation X.1.11, there results an equation giving i_p as an implicit function of e_g which can be reduced to the form:

$$i_p = a_1 e_g + a_2 e_g^2 + a_3 e_g^3 + \dots \qquad \text{X.1.12}$$

where the coefficients have the form:

$$a_1 = \frac{\mu}{r_p + Z_1} = \frac{b_1}{r_p + Z_1}, \text{ say,} \qquad \text{X.1.13a}$$

$$a_2 = \frac{1}{2\,\mu g_m}\,\frac{g_m}{e_c}\,\frac{(\mu r_p)^2}{(r_p+Z_1)^2(r_p+Z_2)}$$

$$= \frac{b_2}{r_p + Z_1}\,, \qquad \text{X.1.13b}$$

and so on. $Z_1, Z_2, \dots$ are the impedances of the load offered to the different order components of plate current. These values will be different, since, in general, the external elements will be frequency sensitive, and, therefore, the appropriate value of Z should be used for each frequency.

The equations X.1.13 then give rise to the equivalent plate circuit of figure X.1.2.

\# The order of differentiation in the second term of the quadratic is interchangeable because the surface is well behaved.

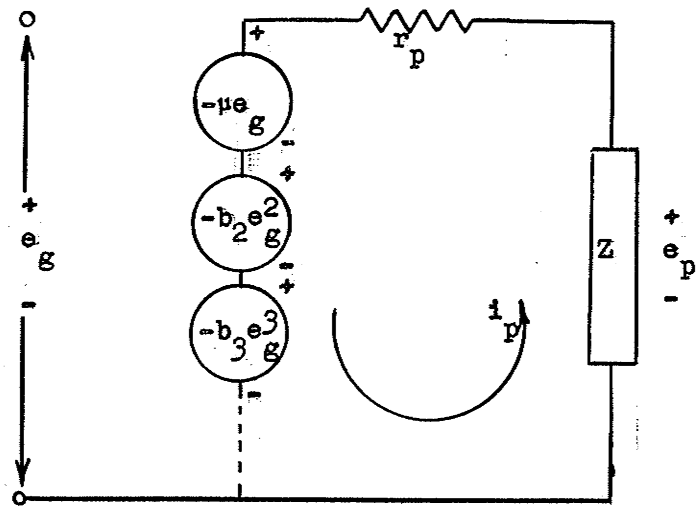


FIGURE X.1.2

NONLINEAR EQUIVALENT VACUUM TUBE CIRCUIT

X.2 THE EQUIVALENT GENERATORS METHOD (22)

The above extension of the small signal theory of the vacuum tube leads directly to the following generalization which has been called the equivalent generators method.

Consider the element whose characteristic may be written as:

$$e = r[\, i + \alpha(\beta i^2 + \gamma i^3 + \delta D i + \epsilon D^{-1} i)]. \qquad \text{X.2.1}$$

There are two possible equivalent circuits for the element, the one consistent with sinusoidal current, the other with sinusoidal voltage. For sinusoidal current, the equivalent circuit can be obtained as follows. Let:

$$i = I \cos \omega t. \qquad \text{X.2.2}$$

Then, by equation X.2.1, after some manipulation,

$$e = \frac{r\alpha\beta I^2}{2} + rI\left(1 + \frac{3\alpha\gamma I^2}{4}\right)\cos \omega t - r\alpha I(\delta\omega - \epsilon/\omega)\sin \omega t$$
$$+ \frac{r\alpha\beta I^2}{2}\cos 2\omega t + \frac{r\alpha\gamma I^3}{4}\cos 3\omega t \qquad \text{X.2.3}$$

or
$$\frac{e}{rI} = A_0 + \cos \omega t + A_1 \cos \omega t + A_1' \sin \omega t + A_2 \cos 2\omega t$$
$$+ A_3 \cos 3\omega t. \qquad \text{X.2.4}$$

This equation leads to the equivalent circuit shown in figure X.2.1, in which it has been assumed that the current $I\cos \omega t$ flowing through the linear resistor r causes a voltage drop $E_a \cos \omega t$. The equivalent current generator is given in figure X.2.1 also.

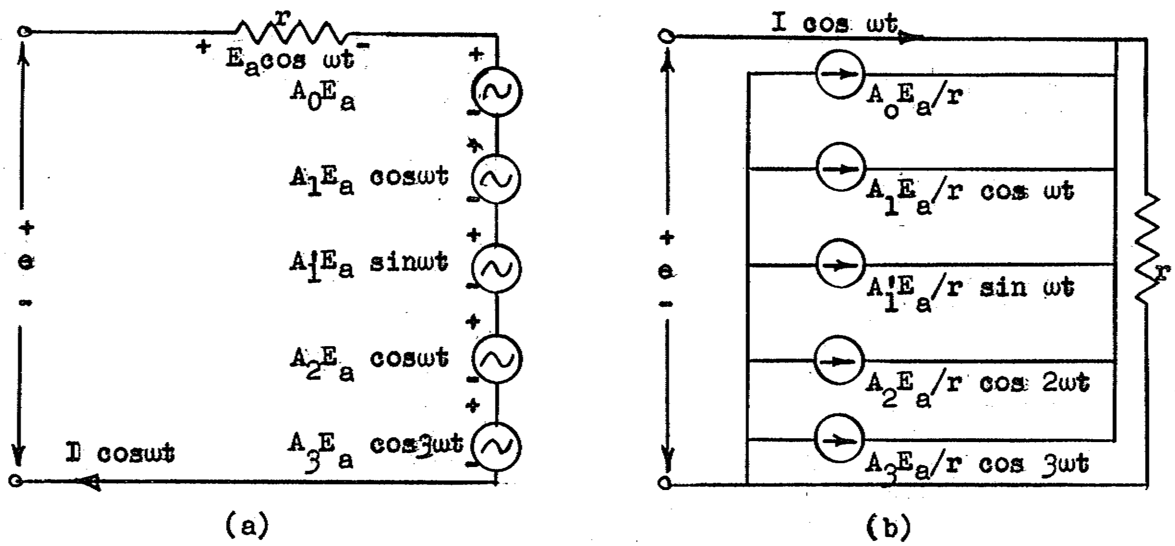


FIGURE X.2.1

EQUIVALENT VOLTAGE AND CURRENT GENERATOR CIRCUITS (COSINUSOIDAL CURRENT)

The sinusoidal voltage equivalent circuit can be found in an analgous manner. Let

$$e = E \cos \omega t \qquad \text{X.2.5}$$

The current can then be found by successive approximations from the characteristic X.2.1. As a first approximation let:

$$i = i_o + \alpha\, i_1 \qquad \text{X.2.6}$$

where it is assumed that $\alpha \ll 1$. Substitution in equation X.2.1 yields:

$$\frac{E \cos \omega t}{r} = i_o + \alpha(i_1 + \beta i_o^2 + \gamma i_o^3 + \delta D i_o + \epsilon D^{-1} i_o) \qquad \text{X.2.7}$$

where quadratic and higher order terms in α have been neglected.

This equation must hold true in the linear case, and so, with $\alpha = 0$,

$$\frac{E \cos \omega t}{r} = i_o = I_a \cos \omega t \qquad \text{X.2.8}$$

where $I_a = E/r$. For small α then, equation X.2.7 yields:

$$i_1 = -\beta\, i_o^2 - \gamma i_o^3 - \delta D i_o - \epsilon D^{-1} i_o \qquad \text{X.2.9}$$

$$= -\frac{1}{2}\beta\, I_a^2 - \frac{3}{4}\gamma I_a^3 \cos \omega t + (\delta\omega - \epsilon/\omega)\, I_a \sin \omega t$$

$$-\frac{1}{2}\beta\, I_a^2 \cos 2\omega t - \frac{1}{4}\gamma I_a^3 \cos 3\omega t. \qquad \text{X.2.10}$$

Thence, to the second order of approximation, equation X.2.6 yields:

$$i = I_a \cos \omega t - \alpha\,[\,\frac{1}{2}\beta\, I_a^2 + \frac{3}{4}\gamma I_a^3 \cos \omega t -$$

$$I_a(\delta\omega - \epsilon/\omega) \sin \omega t + \frac{1}{2}\beta\, I_a^2 \cos 2\omega t +$$

$$\frac{1}{2}\gamma\, I_a^3 \cos 3\omega t] \qquad \text{X.2.11}$$

or

$$\frac{i}{I_a} = -\frac{1}{2}\alpha\beta I_a + \cos \omega t - \frac{3}{4}\alpha\,\gamma I_a^2 \cos \omega t + \alpha(\delta\omega - \epsilon/\omega) \sin \omega t$$

$$-\frac{1}{2}\alpha\beta I_a \cos 2\omega t - \frac{1}{4}\alpha\,\gamma I_a^2 \cos 3\omega t \qquad \text{X.2.12}$$

$$= B_0 + \cos \omega t + B_1 \cos \omega t + B_1' \sin \omega t + B_2 \cos 2\omega t$$

$$+ B_3 \cos 3\omega t \qquad \text{X.2.13}$$

which equation yields the two equivalent circuits shown in figure X.2.2.

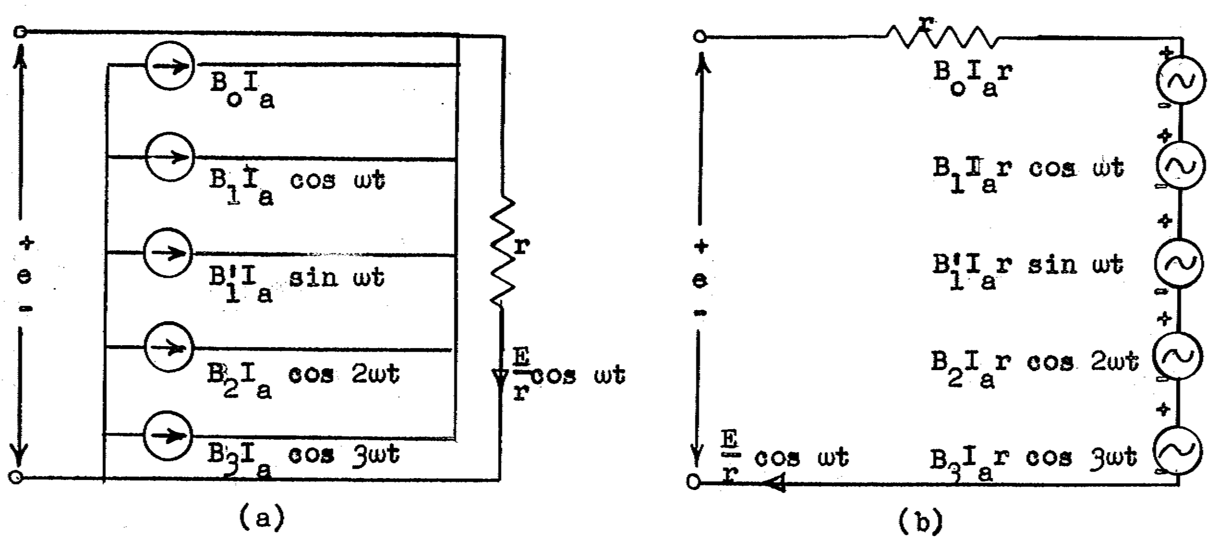


FIGURE X.2.2

EQUIVALENT CURRENT AND VOLTAGE GENERATOR CIRCUIT(COSINUSOIDAL VOLTAGE)

In figures X. .1a and b, the external impedance is infinite (current generator) and therefore, the voltage across the terminals is distorted ~~while as~~ whereas no distorted current flows in the external system. In figures X.2.2a and b, however, the external impedance is zero (voltage generator) and therefore, although the distorted currents flow through the external circuit, no distortion in voltage appears across the terminals.

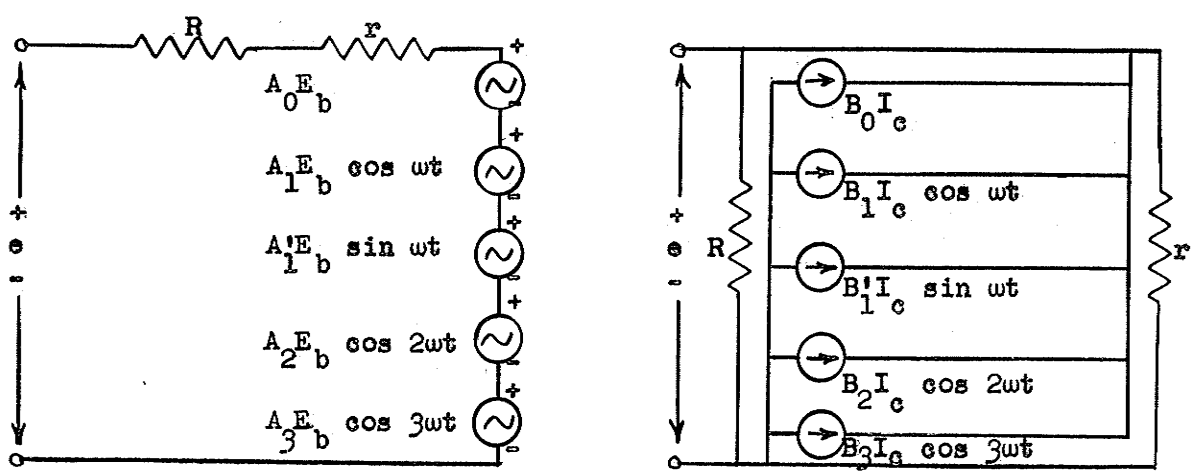
The above analysis was, of necessity, only approximate, and therefore holds for small nonlinearities only. If the nonlinearity were larger, higher order terms of current would have to be included in equation X.2.1, and the magnitudes of the various components of voltage would change. In the second case, the approximation to the current of equation X.2.6 would not be sufficient, and higher order terms in α would have to be included.

The process is easily extended to the case where an additional linear resistance R is added in series or parallel with the nonlinear element. The equivalent circuits of these combinations are given in figure X.2.3 a and b, where:

$$I_b = E/(r+R),\ E_b = I_b r, \qquad X.2.14$$

and

$$I_c = IR/(r+R),\ E_c = I_c r. \qquad X.2.15$$



$e = E \cos \omega t$ (a)

$e = E_c \cos \omega t$ (b)

FIGURE X.2.3

EQUIVALENT VOLTAGE AND CURRENT GENERATOR CIRCUITS WITH LINEAR RESISTANCE

In these cases, the voltage across and the current through each individual element must both be nonlinear. The extension of the method to series and parallel combinations of reactances and resistances involves new difficulties which have not as yet been resolved. The concepts of distorted voltage without distorted current and vice-versa, however, are directly applicable to reactive cases, for example to the different properties of suppressing harmonic voltages or currents by different three phase transformer connections.

X.3 HARMONIC ANALYSES (24)

The above discussion leads logically to the consideration of harmonic analysis of the waveforms in the nonlinear elements. Consider the case where one of the mutually dependent variables defining the element characteristic is sinusoidal, thus causing the other to be distorted. Suppose the characteristic is expressible as:

$$y = f(x) \qquad \text{X.3.1}$$

where f(x) may be expressed by the infinite power series:

$$f(x) = \sum_n a_n x^n \qquad \text{X.3.2}$$

or by a polynomial whose coefficients are adjusted to give the best fit over a given range. This is usually accomplished by making the agreement exact between the polynomial and the actual curve at the required number of points, in which case, the polynomial could be called an expansion about a finite number of points ~~while as~~ whereas the power expansion would be termed an expansion about a single point.

However, whatever form f(x) has, so long as it obeys the Dirichlet conditions, when x has a sinusoidal form in time, f(x) can be expanded by Fourier series to obtain its constituent time harmonics. If $\theta = \omega t$, and $x = \sin\theta$, then:

$$f(\sin\theta) = \frac{A_0}{2} + \Sigma\,[\,A_n \cos n\theta + B_n \sin n\theta\,] \qquad \text{X.3.3}$$

where:

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\sin\theta)\cos n\theta\, d\theta \qquad \text{X.3.4}$$

and

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\sin\theta)\sin n\theta\, d\theta, \qquad \text{X.3.5}$$

which coefficients depend on the nature of f(x) in a manner that is not readily visualizable.

There is, however, a serious disadvantage to this system of computation of harmonics. The amplitude of each harmonic is dependent on the degree of the approximating polynomial. An example of this is present in the three-point and five-point Fischer- Hinnen method of harmonic analysis used in distortion computations for vacuum tubes. The outcome of this problem is that to obtain accurate values for the average and lower harmonic terms of the wave in question, an approximation must be used that includes all powers of the independent variable with magnitudes of the order of the allowable error. This involves the solution of a number of simultaneous equations one greater than the highest power involved. Such a solution is obviously unwieldy.

The following method, however, eliminates this difficulty by establishing a value for each harmonic dependent only on the function f(x) itself, and not on the other harmonics. It involves at the worst, graphical integration along the x-axis. The procedure is based on equations X.3.4 and X.3.5. By the following substitutions,

$$x = \sin\theta, \qquad \text{X.3.6}$$

$$d\theta = (1-x^2)^{-1/2}\,dx\ , \qquad \text{X.3.7}$$

$$\cos n\theta = \cos(n \sin^{-1} x) = \phi_n(x), \qquad \text{X.3.8}$$

and

$$\sin n\theta = \sin(n \sin^{-1} x) = \psi_n(x). \qquad \text{X.3.9}$$

equations X.3.4 and X.3.5 become:

$$A_n = \frac{2}{\pi}\int_{-1}^{1} f(x)\,\phi_n(x)\,(1-x^2)^{-1/2}dx, \qquad \text{X.3.10}$$

and

$$B_n = \frac{2}{\pi}\int_{-1}^{1} f(x)\psi_n(x)\,(1-x^2)^{-1/2}dx. \qquad \text{X.3.11}$$

These equations give the coefficients of the expansion of f(x) in terms of the orthogonal functions $\phi_n(x)$ and $\psi_n(x)$ which is

$$f(x) = \frac{A_o}{2}\phi_o + \Sigma[A_n\phi_n(x) + B_n\psi_n(x)]. \qquad \text{X.3.12}$$

The functions ϕ_n and ψ_n are related to the well-known Tschebyscheff polynomials T_n and U_n by the equations:

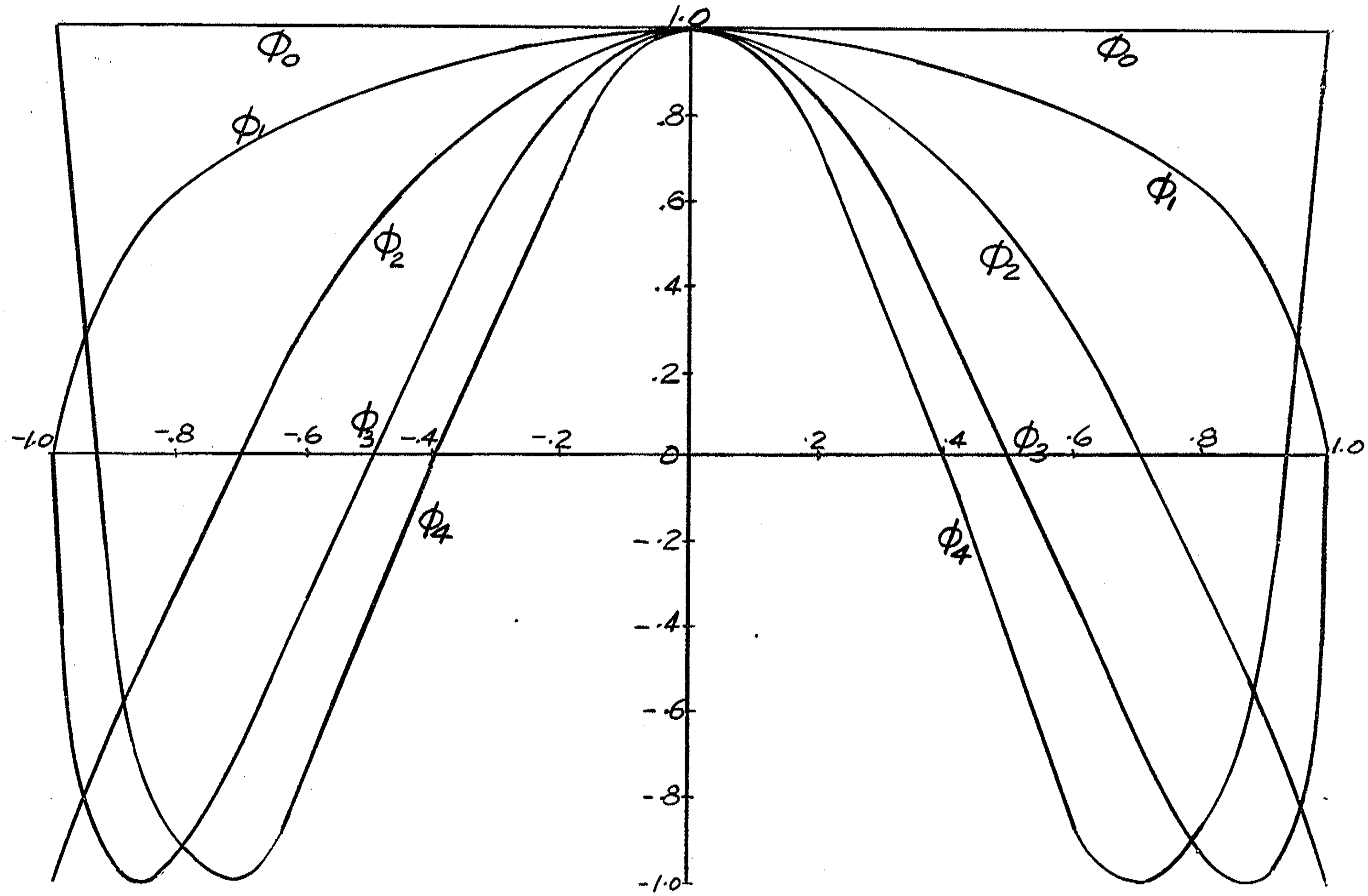

1.0
Φ_0
Φ_0
Φ_1
Φ_1
Φ_2
Φ_2
Φ_3
Φ_3
Φ_4
Φ_4
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
-1.0
-.8
-.6
-.4
-.2
.2
.4
.6
.8
1.0

FIGURE X.3.1

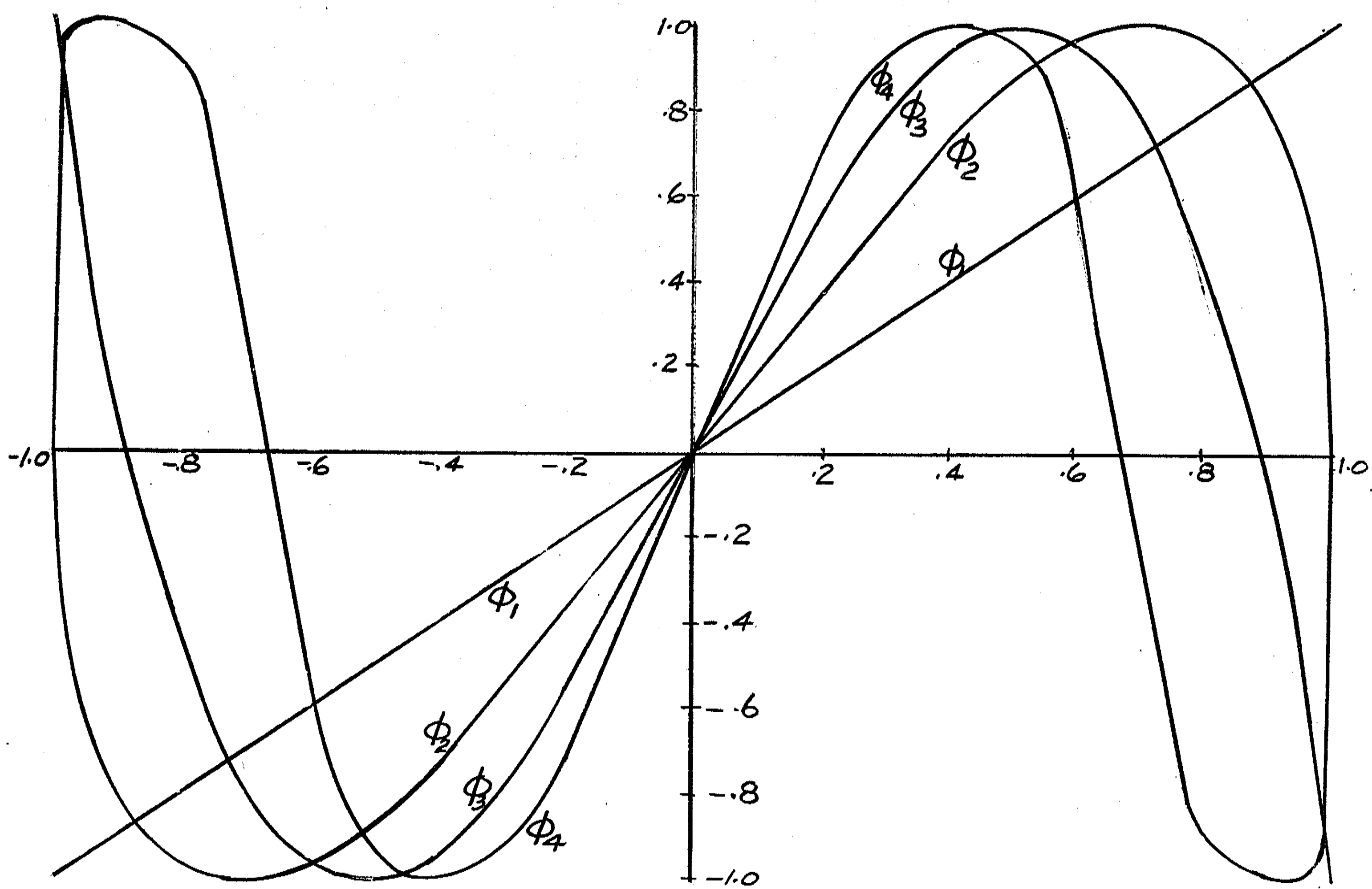


FIGURE X.3.2

$$\phi_n(x) = (-1)^{n/2} T_n(x), \quad n=0,2,4,\ldots \qquad X.3.13$$

$$\Psi_n(x) = (-1)^{(n+2)/2} U_n(x), n=0,2,4\ldots \qquad X.3.14$$

and

$$\phi_n(x) = (-1)^{(n-1)/2} U_n(x), n=1,3,5,\ldots \qquad X.3.15$$

$$\Psi_n(x) = (-1)^{(n-1)/2} T_n(x), n=1,3,5,\ldots \qquad X.3.16$$

and so can be plotted by use of tables. The functions for $n = 1,2,3$ and 4 are shown in figures X.3.1 and X.3.2. They are also expressible as:

$$\phi_o = 1, \qquad \Psi_1 = x, \qquad X.3.17$$

$$\phi_1 = \cos\theta = (1-\sin^2\theta)^{1/2} = (1-x^2)^{1/2}, \qquad X.3.18$$

and

$$\Psi_2 = \sin 2\theta = 2\sin\theta\cos\theta = 2x(1-x^2)^{1/2} \qquad X.3.19$$

and the recursion formulae:

$$\phi_{n+1} = \phi_{n-1} - 2x\Psi_n \qquad X.3.20$$

and

$$\Psi_{n+1} = 2x\phi_n + \Psi_{n-1} \qquad X.3.21$$

are valid for $n > 1$.

To find any amplitude coefficient in the expansion X.3.12, once f(x) is known graphically or otherwise, the new function:

$$f(x)\,\phi_n(x)\,(1-x^2)^{1/2},$$

or

$$f(x)\,\Psi_n(x)\,(1-x^2)^{1/2}$$

is plotted and the area beneath it computed. Since the surd increases indefinitely as x approaches ± 1, the integration can be carried out in sections, using the mean-value theorem for the end sections. The accuracy to which a particular amplitude is computed depends only on that individual computation and not on any other approximations.

The method is applicable to double valued functions if the following definitions are used:

$$f(x) = f_1(x), \quad x \text{ increasing}, \qquad X.3.22$$

$$f(x) = f_2(x), \quad x \text{ decreasing}, \qquad X.3.23$$

and

$$f_A(x) = 1/2[f_1(x) + f_2(x)] \qquad X.3.24$$

$$f_B(x) = 1/2[f_1(x) \mp f_2(x)]. \qquad X.3.25$$

The above process may be applied to $f_A(x)$ and $f_B(x)$ which are single-valued functions.

The method has a still further extension. In either the single or double valued case, since ϕ_n and Ψ_n can be plotted for reference, the nonlinear function f(x) can be synthesised by inspection to yield its harmonic content, and what is more, since the expansion X.3.12 is in terms of x, the coefficients may be determined to reasonable accuracy by comparison with the approximation X.3.2. The value of x can be considered in all the above discussions to be normalized so that the treatment is of more general nature than actually appears.

For example, consider the case where f(x) may be approximated as:

$$y = f(x) \doteq ax - bx^3 \qquad \text{X.3.26}$$

where the graphical form is that of the B-H curve for an iron-cored inductor shown in figure X.3.3.

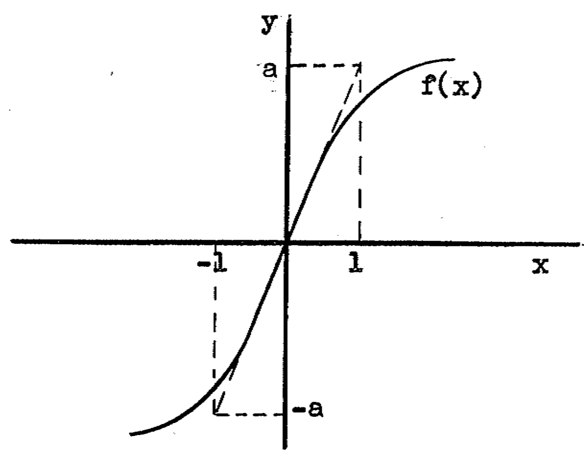


FIGURE X.3.3

TYPICAL NONLINEAR CHARACTERISTIC

By consideration of the forms of f(x), Ψ_1, and Ψ_3, the following equation may be written from inspection:

$$f(x) \doteq B_1\Psi_1 + B_3\Psi_3. \qquad \text{X.3.27}$$

But, $\Psi_1(x) = x$ X.3.17

and $\Psi_3(x) = 3x - 4x^3.$ X.3.28

Hence, $f(x) = B_1x + 3B_3x - 4B_3x^3.$ X.3.29

By comparison of equations X.3.26 and X.3.29:

$$B_1 + 3B_3 = a \qquad X.3.30$$

and

$$-4B_3 = -b \qquad X.3.31$$

which yield:

$$B_3 = b/4 \qquad X.3.32$$

and

$$B_1 = a - 3b/4. \qquad X.3.33$$

Hence, equation X.3.27 becomes:

$$f(x) = (a - 3b/4)\Psi_1 + b/4\Psi_3 \qquad X.3.34$$

or

$$f(\sin\theta) = (a - 3b/4)\sin\theta + b/4 \sin 3\theta, \qquad X.3.35$$

since the functions are simply the sine functions of the variable θ expressed in terms of $x = \sin\theta$.

This same result could have been deduced by substituting $x = \sin\theta$ into equation X.3.26 and eliminating the cubic term in $\sin\theta$ by trigonometric identities. It can easily be seen for higher degree polynomials that this latter process becomes unwieldy.

It must be remembered, however, that this extension introduces the same type of error sources that were present in the Fischer-Hinnen method for the obvious reason that it too is based on an approximate expansion of $f(x)$. However, much less actual manipulation is required to yield the same results, and so, more accurate results may be obtained with no increase in the work involved.

X.4 EQUIVALENT LINEARIZATION (6)

The considerations of the small signal theory outlined in section 1 above may be generalized in another way by what is known as the method of equivalent linearization. One of the important aspects of this theory is the extension to the linear impedance concept outlined in Chapter V, known as the principles of harmonic balance and equality of energy.

The method of equivalent linearization will be considered for systems of small nonlinearity only. A method has been devised (26) applying to large nonlinearity but it may be used only in special cases. Consider the nonlinear system governed by the differential equation: $\ddot{y} + \mu\, g(y,\dot{y}) + \omega^2 y = 0$ X.4.1

in which it is assumed that the nonlinear term, $\mu\, g(y,\dot{y})$ is small. Neglecting this term the "zero'th" order equivalent linear equation of equation X.4.1 becomes:

$$\ddot{y} + \omega^2 y = 0 \qquad \text{X.4.2}$$

which has the solution:

$$y = A \sin(\omega t + \phi) \qquad \text{X.4.3}$$

where A and ϕ are arbitrary constants. Defining $q = \omega t + \phi$, equation X.4.3 becomes:

$$y = A \sin q \qquad \text{X.4.4}$$

and so:

$$\dot{y} = A\omega \cos q. \qquad \text{X.4.5}$$

Suppose now, A and ϕ be allowed to vary slowly in time so that the solution of equation X.4.1 may be written as:

$$y = A(t) \sin[\,\omega t + \phi(t)] \qquad \text{X.4.6}$$

or:

$$y = A(t) \sin q(t). \qquad \text{X.4.7}$$

Thence,

$$\dot{y} = \dot{A} \sin q + A(\omega + \dot{\phi}) \cos q. \qquad \text{X.4.8}$$

Upon equating equations X.4.5 and X.4.8, then,

$$\dot{A} \sin q + A\, \dot{\phi} \cos q = 0. \qquad \text{X.4.9}$$

Differentiating equation X.4.5 yields:

$$\ddot{y} = \dot{A}\, \omega \cos q + A\omega\, (\omega + \dot{\phi}) \sin q, \qquad \text{X.4.10}$$

and substituting equations X.4.4, X.4.5 and X.4.10 into equation X.4.1 yields:

$$\dot{A} \cos q - A\, \dot{\phi} \sin q = -\frac{\mu}{\omega} g[\,A \sin q,\ A\omega \cos q\,]. \qquad \text{X.4.11}$$

Adding equation X.4.9 multiplied by cos q, to equation X.4.11 multiplied by (- sin q) yields:

$$\dot{A} = -\frac{\mu}{\omega} g\,[\,A \sin q,\ A\omega \cos q\,] \cos q \qquad \text{X.4.12}$$

and adding equation X.4.9 multiplied by cos q to equation X.4.11 multiplied by (- sin q) yields:

$$\dot{\phi} = \frac{\mu}{\omega A} g\,[\,A \sin q,\ A\omega \cos q\,] \sin q. \qquad \text{X.4.13}$$

The results for the time derivatives of A and ϕ are consistent with the assumptions that they vary slowly with time since μ is small by hypothesis. Since $\dot{A}$ and $\dot{\phi}$ will vary very little in one period, they may be replaced to a first order approximation by their mean values for that interval, which are given by:

$$\bar{\dot{A}} = -\frac{\mu}{2\pi\omega}\int_0^{2\pi} g[\ A \sin \phi_1,\ A\omega \cos \phi_1] \cos \phi_1\, d\phi_1, \qquad X.4.14$$

and

$$\bar{\dot{\phi}} = +\frac{\mu}{2\pi\omega A}\int_0^{2\pi} g[\ A \sin \phi_1,\ A\omega \cos \phi_1] \sin \phi_1\, d\phi_1, \qquad X.4.15$$

wherethe integration variable is ωt.

Equation X.4.15 then yields:

$$\bar{\dot{q}} = \omega + \bar{\dot{\phi}}$$

$$= \omega + \frac{\mu}{2\pi\omega A}\int_0^{2\pi} g[\ A \sin \phi_1,\ A\omega \cos \phi_1] \sin \phi_1\, d\phi_1.$$

X.4.16

These equations form the basis for the method of equivalent linearization which follows. Equation X.4.8 may be written as:

$$\dot{y} = \dot{A} \sin q + A\dot{q} \cos q \qquad X.4.8$$

and so,

$$-\frac{2\dot{A}}{A}\dot{y} = -\frac{2\dot{A}^2}{A}\sin q - 2\dot{A}\dot{q} \cos q. \qquad X.4.17$$

Differentiating equation X.4.8 :

$$\ddot{y} = \ddot{A} \sin q + 2\dot{A}\dot{q} \cos q + A\ddot{q} \cos q - A\dot{q}^2 \sin q \quad X.4.18$$

and from equation X.4.7, by multiplying by $\dot{q}^2$, there results:

$$\dot{q}^2 y = A\,\dot{q}^2 \sin q. \qquad X.4.19$$

Adding equations X.4.17, 18, and 19 yields:

$$\ddot{y} - \frac{2\dot{A}}{A}\dot{y} + \dot{q}^2 y = [\ \ddot{A} - \frac{2\dot{A}^2}{A}] \sin q + A\,\ddot{q} \cos q. \quad X.4.20$$

But by equations X.4.14 and X.4.15, $\dot{A}^2/A$, $\ddot{A}$, and $\ddot{q}$ are all of the order of μ^2, which is very small.

Thus, the linear equation:

$$\ddot{y} + 2\,\bar{\alpha}\,\dot{y} + \bar{\omega}^2 y = 0 \qquad X.4.22$$

where $\bar{\alpha} = -\dot{A}/A$, and $\bar{\omega} = \dot{q}$, is equivalent to the nonlinear equation X.4.1 to within the order of μ^2. The coefficients are functions of time and do not introduce nonlinearity in y.

Effectively, the above process has been to replace the nonlinear force $\mu\, g(y,\dot{y})$ by the linear force:

$$2\,\bar{\alpha}\,\dot{y} + (\,\bar{\omega}^2 - \omega^2)\,y. \qquad X.4.23$$

The reason for this particular choice lies in the so-called Principle of Equivalent Balance of Energy.

Since the two equations X.4.1 and X.4.22 are equivalent, the work per cycle must be equal to within the order of μ^2. Equation X.4.1 may be written as:

$$\dot{y}\,\frac{d\dot{y}}{dy} + \mu\, g(y,\dot{y}) + \omega^2 y = 0 \qquad X.4.24$$

and integrating over one cycle $T = 2\pi/\omega$ yields:

$$\int_0^T \dot{y}\, d\dot{y} + \int_0^T \omega^2 y\, dy + \mu \int_0^T g(y,\dot{y})\dot{y}\, dt = C, \text{a constant.} \qquad X.4.25$$

That is:

$$\frac{1}{2}\left[\,\dot{y}^2 + \omega^2 y^2\right]_0^T + \mu \int_0^T g(y,\dot{y})\dot{y}\, dt = C. \qquad X.4.26$$

But the first term vanishes due to the periodicity of the motion and thence:

$$\mu \int_0^T g(y,\dot{y})\dot{y}\, dt = C\ \# \qquad X.4.27$$

which is the energy dissipated in heat during one cycle. Equation X.4.22 integrated over one cycle $T_0 = 2\pi/\bar{\omega}$, which may be taken as T in this approximation, yields:

$$\frac{1}{2}\left[\,\dot{y}^2 + \bar{\omega}^2 y\right]_0^T + 2\,\bar{\alpha} \int_0^T \dot{y}^2\, dt = C_1, \qquad X.4.28$$

another constant. As before, the first term is zero and the last term then becomes:

$$2\,\bar{\alpha} \int_0^T \dot{y}^2 dt = C_1. \qquad X.4.29$$

Now replacing y and $\dot{y}$ in equation X.4.27 and X.4.29 by $A \sin q$ and $A\omega \cos q$ respectively, there results:

$$C = \mu A \int_0^{2\pi} g[\,A \sin q,\ A\omega \cos q]\, \cos q\, dq, \qquad X.4.30$$

\# Although the damping present invalidates this statement, it is correct to within the required order of approximation.

and $$C_1 = 2\,\bar{\bar{\alpha}}\,A^2\omega^2 \int_0^{2\pi/\omega} \cos^2 q \; dt$$

$$= 2\pi\,\bar{\bar{\alpha}}\,A^2\omega^2. \qquad X.4.31$$

If these two energy dissipations are to be equal, $C = C_1$, and so:

$$\bar{\bar{\alpha}} = \frac{\mu}{2\pi A\omega} \int_0^{2\pi} g[\,A \sin q,\; A\omega \cos q]\, \cos q \; dq \qquad X.4.32$$

which is equivalent to the definition given on page 62.

If the system examined were electrical in nature, the power dissipated would be given as watts and for any period T, voltage e, and current i, would be given by:

$$P_a = \frac{1}{T}\int_0^T e(t)\; i(t)\; dt. \qquad X.4.33$$

In a similar manner, the reactive or wattless power would be given by:

$$P_r = \frac{1}{T}\int_0^T e(t)\; i(t-T/4)\; dt. \qquad X.4.34$$

For a mechanical system with force $F(t)$ and velocity $\dot{y}(t)$, these become:

$$P_a = \frac{1}{T}\int_0^T F(t)\; \dot{y}(t)\; dt \qquad X.4.35$$

and $$P_r = \frac{1}{T}\int_0^T F(t)\; \dot{y}(t-T/4)\; dt, \qquad X.4.36$$

the first of which equations was used above to give a physical interpretation to $\bar{\bar{\alpha}}$. The second may be used to give a physical interpretation to $\bar{\bar{\omega}}$.

If the wattless powers for the two equivalent forces are equated there results:

$$\frac{\mu\omega}{2\pi}\int_0^T g[y(t),\dot{y}(t)]\; \dot{y}[t-T/4]\; dt =$$

$$\frac{\omega}{2\pi}\int_0^T [(\bar{\bar{\omega}}^2-\omega^2)\; y(t) + 2\bar{\bar{\alpha}}\; \dot{y}(t)]\; \dot{y}[t-T/4]\; dt, \quad X.4.37$$

which upon substitution of the values of y and $\dot{y}$ yields:

$$\bar{\bar{\omega}}^2 = \omega^2 + \frac{\mu}{\pi A}\int_0^{2\pi} g(\,A \sin q,\; A\omega \cos q)\, \sin q \; dq. \quad X.438$$

By taking the square root of this equation and expanding the surd on the right side by the binomial theorem, neglecting all terms containing higher powers of μ than the first, this equation becomes:

$$\bar{\omega} = \omega + \frac{\mu}{2\pi\omega A}\int_0^{2\pi} g(A \sin q, A\omega \cos q) \sin q \, dq \qquad \text{X.4.39}$$

which is the definition used above.

Thus, the definitions used in the method of equivalent linearization are consistent with the Principle of Equivalent Balance of Energy applied to the original nonlinear system and its linear equivalent.

Also, since the solution for y is periodic in nature, the linear force may be written as:

$$F = F_L \sin (\omega t + \theta_L) \qquad \text{X.4.40}$$

and the nonlinear force may be expanded as a Fourier series with the fundamental term:

$$F = F_1 \sin (\omega t + \theta_1). \qquad \text{X.4.41}$$

By equating the two forces in magnitude and phase angle, two simultaneous equations in $\bar{\alpha}$ and $\bar{\omega}$ resulting from this yield the same results as were used above. This equivalence is known as the Principle of Harmonic Balance. The result is a natural outcome of the fact that with a solution of fundamental frequency all higher harmonics contribute nothing to the power dissipated per cycle of fundamental frequency.

The above principles may be used with great effectiveness in many electrical problems giving accurate results for systems of small nonlinearity and qualitative results for systems of large nonlinearity. Consider for example, the case of a nonlinear resistor, the characteristic of which may be written as:

$$v = f_1(i) \qquad \text{X.4.42}$$

connected across a constant current source denoted by:

$$i = I_0 \sin \omega t$$
$$= I_0 \sin \theta \qquad \text{X.4.43}$$

as shown in figure X.4.1.

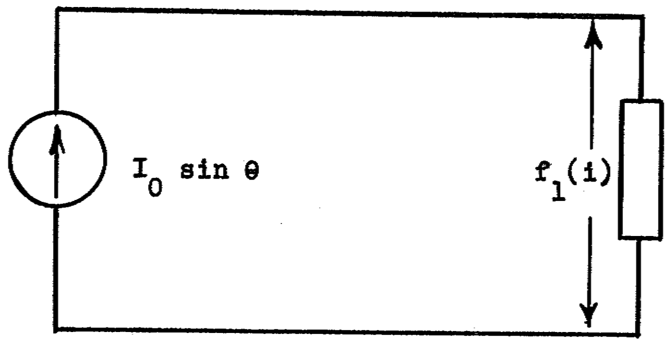


FIGURE X.4.1

NONLINEAR RESISTOR CONNECTED ACROSS A CONSTANT CURRENT SOURCE

The equivalent resistance for such an element may be defined by the equation:

$$\frac{1}{2} I_0^2 R_e = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} f(i)\, i\, dt$$

$$= \frac{I_0}{2\pi} \int_0^{2\pi} f(I_0 \sin \theta) \sin \theta \, d\theta \qquad \text{X.4.44}$$

which becomes:

$$R_e = \frac{1}{\pi I_0} \int_0^{2\pi} f_1(I_0 \sin \theta) \sin \theta \, d\theta. \qquad \text{X.4.45}$$

The value of R_e may be positive or negative depending on whether the element dissipates or generates power. This depends directly on the form of the characteristic.

In a similar manner, the equivalent inductance of a nonlinear inductor, the characteristic of which is given by:

$$\phi = f_2(i), \qquad \text{X.4.46}$$

may be shown to be:

$$L_e = \frac{1}{\pi I_0} \int_0^{2\pi} f_2(I_0 \sin \theta) \sin \theta \, d\theta. \qquad \text{X.4.47}$$

Also, the equivalent elastance of a nonlinear capacitor, characterized by:

$$v = f_3(q), \qquad \text{X.4.47}$$

may be written as:

$$S_e = \frac{1}{\pi Q_0} \int_0^{2\pi} f_3(Q_0 \sin \theta) \sin \theta \, d\theta \qquad \text{X.4.48}$$

where $q = Q_0 \sin \theta$. X.4.49

All the above results require a sinusoidal current approximation. The reciprocal equivalent parameters, namely, conductance, inertiance and capacitance, G_e, Γ_e, and C_e, may be defined by assuming sinusoidal voltage waveform. There results:

$$G_e = \frac{1}{\pi E_0} \int_0^{2\pi} f_4(E_0 \sin\theta) \sin\theta \, d\theta, \qquad X.4.50$$

$$\Gamma_e = \frac{1}{\pi E_0} \int_0^{2\pi} f_5(E_0 \sin\theta) \sin\theta \, d\theta, \qquad X.4.51$$

and

$$C_e = \frac{1}{\pi E_0} \int_0^{2\pi} f_6(E_0 \sin\theta) \sin\theta \, d\theta, \qquad X.4.52$$

where the nonlinear characteristics are defined as:

$$i = f_4(v), \qquad X.4.53$$

$$i = f_5(v), \qquad X.4.54$$

and

$$q = f_6(v), \qquad X.4.55$$

for the three different types of elements.

The process may be extended to mutual parameters such as the transconductance of a vacuum tube or iron-cored coupled coils. The definitions assume the same form as those above.

X.5 THE MINIMUM MEAN SQUARED ERROR METHOD (28)

Closely allied with the principle of equivalent linearization and the associated principles of Equivalent Balance of Energy and Harmonic Balance is the minimum mean squared error method of solution of nonlinear systems. The method will be demonstrated for a more or less general series circuit, including a nonlinear capacitor.

Consider the equation:

$$LD^2q + G(Dq) + V(q) = E_0 + E_1 \sin \omega t + E_2 \cos \omega t, \qquad X.5.1$$

describing a series connection of a linear inductance L, a nonlinear resistance $G(Dq)$, and a nonlinear capacitance $V(q)$. Let the approximate solution take the form:

$$q_1 = A \sin \omega t + B \qquad X.5.2$$

and be substituted into equation X.5.1 to yield:

$$Q(\omega t) \equiv LD^2 q_1 + G(Dq_1) + V(q_1) = E_0 + E_1 \sin \omega t + E_2 \cos \omega t + P(\omega t) \qquad X.5.3$$

where $P(\omega t)$ is the error introduced by the approximation q_1. Thus:

$$P(\omega t) = - LA\omega^2 \sin \omega t + G(A\omega \cos \omega t) + V(A \sin \omega t + B) = - E_0 - E_1 \sin \omega t - E_2 \cos \omega t, \qquad X.5.3$$

and the mean squared error is defined as:

$$I = \frac{1}{2\pi} \int_0^{2\pi} P^2(\omega t)\, d\omega t. \qquad X.5.4$$

The constants of the approximate solution X.5.2, that is, A and B are now to be determined to give I a minimum value.

Considering I as a function of E_0, E_1 and E_2, the necessary conditions:

$$\frac{\partial I}{\partial E_0} = \frac{\partial I}{\partial E_1} = \frac{\partial I}{\partial E_2} = 0 \qquad X.5.5$$

yield the equations:

$$\int_0^{2\pi} P(\omega t)\, d\omega t = 0, \qquad X.5.6$$

$$\int_0^{2\pi} P(\omega t) \sin \omega t\, d\omega t = 0, \qquad X.5.7$$

and

$$\int_0^{2\pi} P(\omega t) \cos \omega t\, d\omega t = 0, \qquad X.5.8$$

which become, upon utilizing equations X.5.2 and X.5.3:

$$\int_0^{2\pi} Q(\omega t)\, d\omega t = 2\pi E_0, \qquad X.5.9$$

$$\int_0^{2\pi} Q(\omega t) \sin \omega t\, d\omega t = \pi E_1, \qquad X.5.10$$

and

$$\int_0^{2\pi} Q(\omega t) \cos \omega t\, d\omega t = \pi E_2. \qquad X.5.11$$

These three equations give the constants A and B in terms of E_0 and $(E_1^2 + E_2^2)^{1/2}$, the amplitude of the alternating driving function.

Consider again the problem of a linear capacitor and inductor in series with a nonlinear resistor, the characteristic of which is given by:

$$v = R_0 i^2, \qquad X.5.12$$

connected across a voltage source, $E_0 + E_1 \sin \omega t$. The differential equation is:

$$LD^2q + Sq + R_0(Dq)^2 = E_0 + E_1 \sin \omega t \qquad \text{X.5.13}$$

where, for this case, the functions $G(Dq)$ and $V(q)$ become:

$$G(Dq) = R_0(Dq)^2 \qquad \text{X.5.14}$$

and

$$V(q) = Sq. \qquad \text{X.5.15}$$

Also,

$$Q(\omega t) = -\omega^2 LA \sin \omega t + R_0 A^2\omega^2 \sin^2\omega t + SA \sin \omega t + SB. \qquad \text{X.5.16}$$

The two equations X.5.9 and X.5.10 yield the relations:

$$R_0\omega^2A^2 + 2SB = 2E_0 \qquad \text{X.5.17}$$

and

$$(S - \omega^2L)A = E_1. \qquad \text{X.5.18}$$

That is:

$$A = (S - \omega^2L)^{-1}E_1 \qquad \text{X.5.19}$$

and

$$B = \frac{E_0}{S} - \frac{R_0\omega^2E_1^2}{2S(S - \omega^2L)^2}. \qquad \text{X.5.20}$$

Hence, the solution giving a minimum mean squared error is:

$$q_1 = (S - \omega^2L)^{-1}E_1 \sin \omega t + \frac{E_0}{S} - \frac{R_0\omega^2E_1^2}{2S(S-\omega^2L)^2}. \qquad \text{X.5.21}$$

The magnitude of the mean squared error I can be found by carrying out the integration of equation X.5.4. This yields in this case:

$$I = \frac{1}{2\pi}\int_0^{2\pi} [Q(\omega t) - (E_0 + E_1 \sin \omega t)]^2 d\omega t$$

$$= \frac{1}{2\pi}\int_0^{2\pi} Q^2(\omega t)\, d\omega t - [E_0^2 + E_1^2/2] \qquad \text{X.5.22}$$

by the use of equations X.5.9 and X.5.10. Substituting for $Q(\omega t)$ yields upon integration:

$$I = E_1^2/2 - \frac{R_0^2}{8}\left\{\frac{E_1}{X}\right\}^4 + R_0E_0\left\{\frac{E_1}{X}\right\}^2. \qquad \text{X.5.23}$$

after some substitution from equations X.5.18 and X.5.19, where X is the reactance of the LC series circuit combination, that is:

$$X = S/\omega - \omega L \qquad \text{X.5.24}$$

The magnitude of this error can be determined by substitution of the known values of E_0, E_1, R_0, and X.

CHAPTER XI

GRAPHICAL METHODS

Graphical methods are not of a general nature, each type of problem requiring a specialized graphical procedure for its solution. There are, however, several graphical procedures which apply well to different groups of problems.

XI.1 THE VACUUM TUBE GRAPHICAL PROCEDURES (16)

Perhaps the best known of all graphical procedures in electrical engineering is the graphical analysis of vacuum tubes. The method may be applied to any circuit involving a nonlinear resistance and is based on the separation of the nonlinear characteristic in the differential equation describing the system.

Consider first the case of the nonlinear resistor element in series with a linear element across a voltage supply $e(t)$. If the characteristic of the nonlinear element is given by the equation:

$$v = f(i) \qquad \text{XI.1.1}$$

then the equation relating the voltage and current in the system of figure XI.1.1 is:

$$e(t) = f(i) + Ri, \qquad \text{XI.1.2}$$

or

$$f(i) = e(t) - Ri. \qquad \text{XI.1.3}$$

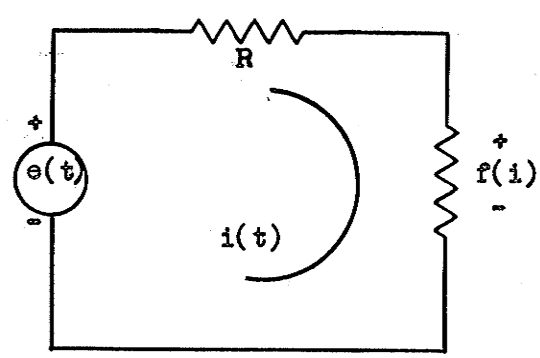


FIGURE XI.1.1

SERIES CONNECTION OF A LINEAR AND A NONLINEAR RESISTANCE

It is on this last equation that the graphical method is based. Consider for example, the case of the diode rectifier in series with a resistive load and driven by a voltage, $E \sin \theta$, where $\theta = \omega t$. The tube characteristic, $v = f(i)$, is shown in figure XI.1.2, and is known as the static characteristic. The system is governed by the equation:

$$E \sin \theta - Ri = f(i). \qquad \text{XI.1.4}$$

Selecting various values of $E \sin \theta$ for selected t values and plotting the load lines given by:

$$v = E \sin \theta - Ri \qquad \text{XI.1.5}$$

yields a set of intersection points with the static characteristic as shown. The lines are all parallel with a slope equal to $-R^{-1}$, and different intercepts on the voltage axis. These intersection points yield the current values flowing in the circuit due to the different applied voltages. When plotted against the applied voltages, the points form a curve known as the dynamic characteristic which is actually the characteristic $i - v$ curve for the compound element composed of the tube and the linear resistor R. Each value of R obviously results in a different dynamic characteristic.

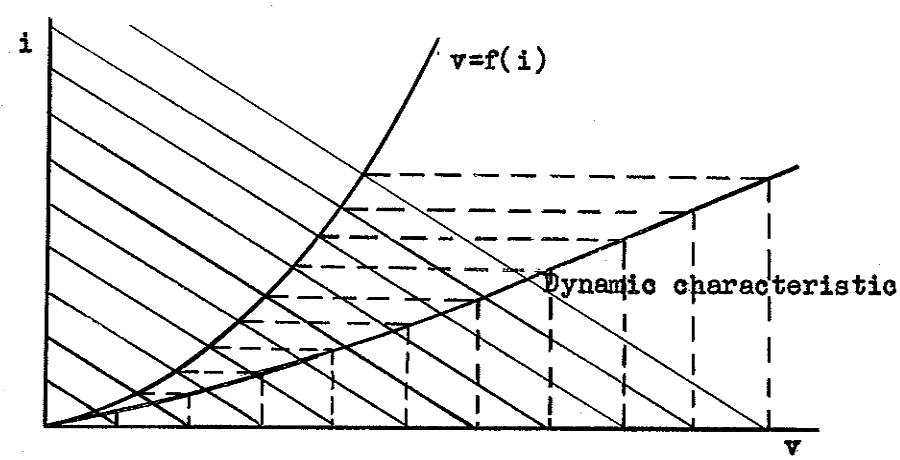


FIGURE XI.1.2

STATIC AND DYNAMIC CHARACTERISTICS

The results may be applied to any circuit consisting of a linear resistor in series with a nonlinear resistor to obtain the overall characteristic of the system. It may also be applied to such cases as the triode and pentode tubes to obtain the dynamic transfer curves between say, the control grid voltage and the plate current. The results are well known in electronics.

It is possible, once the dynamic characteristic of the system is known, to plot the resulting current waveshape for any applied

It is possible, once the dynamic characteristic is known, to plot the resulting waveshape of current for any applied voltage waveshape or vice-versa. The procedure is elementary and obvious.

XI.2 METHOD OF ISOCLINES AND LIÉNARD'S METHOD

The method of isoclines# may be used for the solution of any system reducible to the equation:

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} = N(x,y). \qquad \text{XI.2.1}$$

The chief advantage of the method lies in the fact that the equation does not have to be solved to yield the qualitative types of solution possible. Its chief disadvantage lies in its inherent inaccuracy, but since it is a graphical method, this is to be expected.

The method is as follows. The function,

$$N(x,y) = A_1, \text{ a constant}, \qquad \text{XI.2.1a}$$

is plotted in the x,y plane. The values of dy/dx along the resulting curve are by definition of equation XI.2.1, equal to A_1. The process is repeated for different values of A_1 and the final result is a field of vectors or tangents in the x,y plane giving the direction of dy/dx.

Then, starting from any initial conditions, x_0, y_0, a continuous curve may be drawn following the directions of dy/dx in the field, the result being the relation between y and x, which is the solution of equation XI.2.1 for the initial conditions specified.

The method is, in general, applicable to the second order system:

$$D^2x + \omega^2 x = \mu\, \phi(x)\, Dx, \qquad \text{XI.2.2}$$

by use of the substitution:

$$Dx = v\ , \qquad \text{XI.2.3}$$

so that equation XI.2.2 becomes:

$$Dv + \omega^2 x = \mu\, v\, \phi(x), \qquad \text{XI.2.4}$$

or

$$\frac{dv}{dx} = -\omega^2 \frac{x}{v} + \mu\, \phi(x) \qquad \text{XI.2.5}$$

which is of the form of equation XI.2.1. This actually gives the phase

---

Lines with constant slope.

portrait representation of the system.#

Consider first of all the linear case, where $\phi(x) = 0$. Equation XI.2.5 becomes:

$$\frac{dv}{dx} = -\omega^2 \frac{x}{v} \qquad \text{XI.2.6}$$

which gives the so-called trajectories the form of ellipses in the v,x plane as would be expected in the dissipationless case. The existence of closed paths indicates the presence of a periodic motion.

Suppose now, $\phi(x) = b$, a constant; the system now represents a damped linear circuit and the equation XI.2.5 becomes:

$$\frac{dv}{dx} = -\omega^2 \frac{x}{v} + \mu b . \qquad \text{XI.2.7}$$

This equation yields a set of spiral curves in the phase plane. If $\mu b < 0$, the damping is positive and the result is a decaying oscillation and if $\mu b > 0$, the damping is negative and the resulting oscillation builds up. The spirals are traced out in opposite directions for the two cases, the stable motion characterized by the inwardly traced curve.

When $\phi(x) = 1 - x^2$, equation XI.2.2 becomes the celebrated Van der Pol equation (37) which may be used to represent the triode oscillator. The condition on $\phi(x)$ to yield a closed path and therefore a periodic motion of finite amplitude may easily be ascertained. Multiplying equation XI.2.5 by v dx and integrating over a specified closed path yields:

$$\mu \oint \phi(x)\, v\, dx = \mu \oint \phi(x)\, v^2 dt = 0 \qquad \text{XI.2.8}$$

the other integrals vanishing.

This result demands that $\phi(x)$ change sign over the path traversed.## This is seen to be satisfied by the Van der Pol case, and so periodic motions can exist and actually do.

The solution for equations of this type may also be carried out in the Liénard plane (12) defined by the variables x, $y = Dx - \frac{\mu}{\phi} \times \int_0^x \phi(x)\, dx$. The method is as follows.

From equation XI.2.5, by substituting:

$$y = v - \mu \int_0^x \phi(x)\, dx \qquad \text{XI.2.9}$$

See Chapter II

The condition is necessary but not sufficient.

which is: $$y = v + F(x),\qquad \text{XI.2.10}$$

there results:

$$\frac{dy}{dx} + \frac{\omega^2 x}{y - F(x)} = 0. \qquad \text{XI.2.11}$$

This may be verified by substitution from equation XI.2.10 into XI.2.11. Equation XI.2.11 defines a set of isoclines:

$$\frac{dy}{dx} = -\frac{\omega^2 x}{y - F(x)} = A, \text{ a constant,} \qquad \text{XI.2.12}$$

and the method then takes on a form similar to the isocline method. Since, however, the equation of the normal at point x,y to the trajectory is:

$$\cancel{\frac{dy}{dx}} \neq -\frac{\omega^2(x-x_0)}{y-y_0} = A \qquad \text{XI.2.13}^{\#}$$

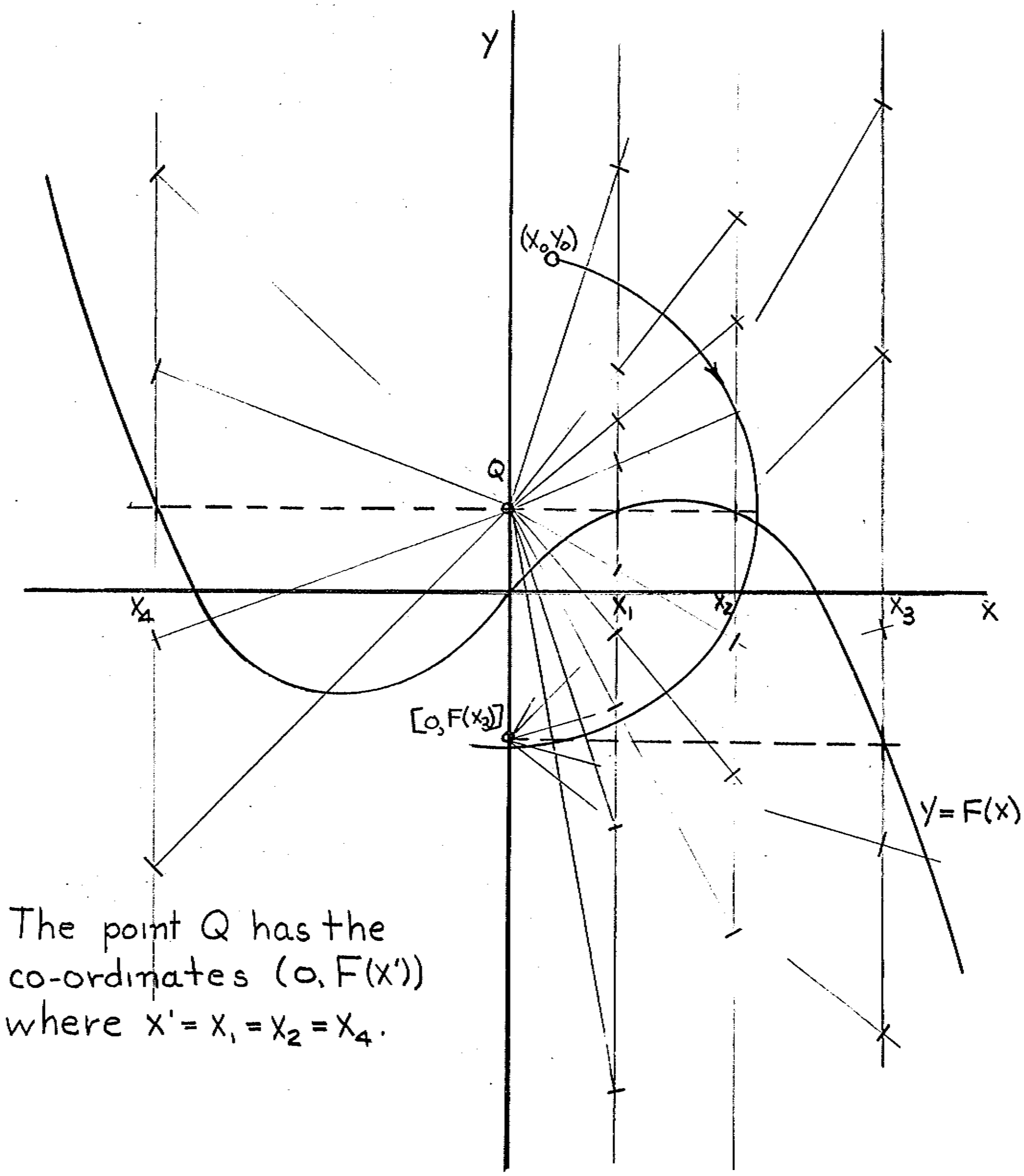
comparison with equation XI.2.12 yields the result that the normal passes through the point $x_0=0$, $y_0= F(x)$. Thus for any abscissa x_1, the tangents to the trajectories are all tangents to circles centred at [0, $F(x_1)$].

If then, $y = F(x)$ is drawn, x_1 is selected and its projection through $F(x)$ is located on the y-axis, the dy/dx directions may all be drawn in along $x = x_1$ as arcs of circles centred at [0, $F(x_1)$]. This at times yields a simpler method of solution than does the usual isocline method. ~~It should also be noted that y = F(x) is the locus of all vertical tangents to the trajectories.~~ The tangents of the trajectories are vertical at the points where the trajectories intersect $y = F(x)$. Any trajectory initially at x_0, y_0 may therefore be drawn in a manner similar to that above. Many examples of phase plane trajectories may be found in references (1), (5), (6), (11), (12) and (37). The construction is illustrated in figure XI.2.1.

XI.3 LINEAR APPROXIMATIONS (19)

In the graphical approximate linearization technique, the non-linear characteristic (plotted graphically) is replaced by a piecewise linear function and the solutions in the various regions of linearity are made continuous at the boundaries by adjusting the boundary conditions. The process will be demonstrated by use of the example of Chapter VII, section 2.

The factor ω^2 has been included so that the scale of y and the scale of $\omega^2 x$ are identical.



The point Q has the co-ordinates $(0, F(X'))$ where $X' = X_1 = X_2 = X_4$.

FIGURE XI.2.1
LIÉNARD DIAGRAM

Consider the saturation curve of an iron-cored inductor as given in figure XI.3.1, and approximated by two straight lines also shown in the figure.

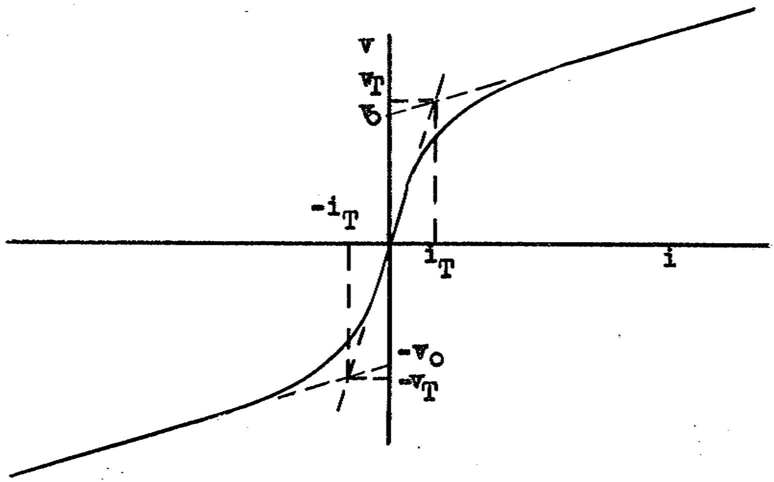


FIGURE XI.3.1

LINEAR APPROXIMATIONS

The analysis must, therefore be carried out for two regions. In region 1, $|i| \leqslant i_T$ and:

$$v = X_1 i \qquad \text{XI.3.1}$$

and in region 2, $|i| \geqslant i_T$, and:

$$v = v_0 + X_2 i \qquad \text{XI.3.2}$$

where X_1 and X_2 are the slopes of the two straight line approximations.

Consider the case where this reactor is connected across a voltage source, $E \sin \theta$, where $E > v_T$ so that both regions must be used. Phase considerations are, of course, neglected.

In region 1, the current will follow the equation:

$$i = \frac{E}{X_1} \sin \theta \qquad \text{XI.3.3}$$

and reach i_T when:

$$\theta = \sin^{-1}\left\{\frac{X_1 i_T}{E}\right\} = \theta_1. \qquad \text{XI.3.4}$$

It will then enter region 2, and will there be given by:

$$i = \frac{E \sin \theta - v_0}{X_2}, \qquad \text{XI.3.5}$$

until:

$$\theta = \sin^{-1}\left\{\frac{i_T X_2 + v_0}{E}\right\} = \theta_2, \qquad \text{XI.3.6}$$

when it again returns to region 1. Since i_T, v_T must satisfy equation XI.3.5, v_o may be shown to be given by:

$$v_o = v_T - i_T X_2, \qquad \text{XI.3.7}$$

and so, the waveshape, for a half cycle would take the form shown in figure XI.3.2.

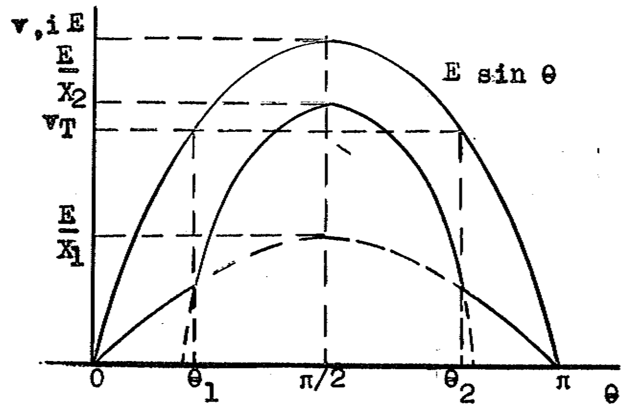


FIGURE XI.3.2

CURRENT AND VOLTAGE WAVEFORM APPROXIMATIONS

The form of the current wave indicates at least qualitatively the shape of the exciting current wave of a transformer, neglecting hysteresis. Hysteresis introduces losses and makes the two halves of the wave un-symmetrical.

This type of analysis may be applied to a great number of non-linear elements such as nonlinear arcs, figure XI.3.3, and transfer functions of vacuum tubes, figure XI.3.4

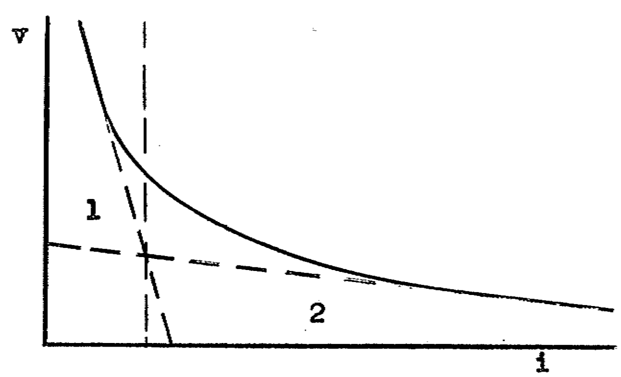


FIGURE XI.3.3

TYPICAL ARC CHARACTERISTIC

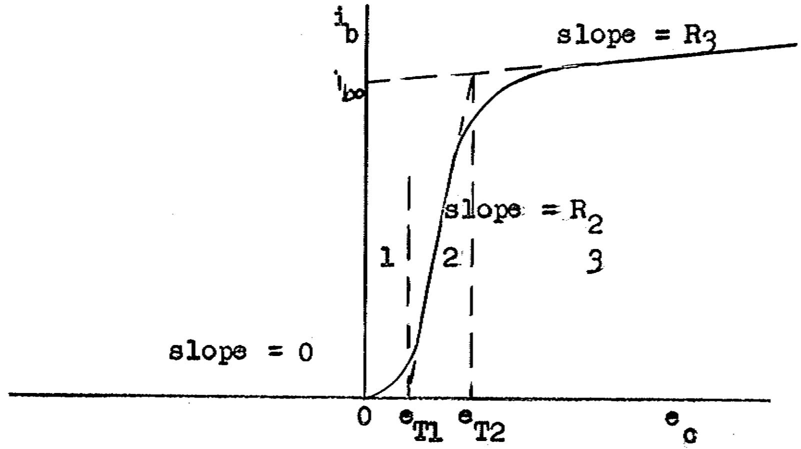


FIGURE XI.3.4

TYPICAL VACUUM TUBE CHARACTERISTIC

Take, as a second example, the case of a vacuum tube with the grid voltage given by:

$$e_c = E_c + E \sin \theta. \qquad \text{XI.3.8}$$

Suppose E_c lies in region 2 of figure XI.3.4 and E is sufficient to drive the system into regions 1 and 3.

The analysis is carried out in three parts and is as follows:

Region 1:
$$e_c \leqslant e_{T1},$$
$$i_b = 0$$
and
$$\sin \theta \leqslant \frac{e_{T1} - E_c}{E} .$$

Region 2:
$$e_{T1} \leqslant e_c \leqslant e_{T2} ,$$
$$i_b = G_2 E_c + G_2 E \sin \theta, - G_2 e_{T1}$$
and
$$(e_{T1} - E_c)/E \leqslant \sin \theta \leqslant (e_{T2} - E_c)/E.$$

Region 3:
$$e_c \geqslant e_{T2},$$
$$i_b = G_3 E_c + G_3 E \sin \theta, + i_{bo}$$
and
$$\sin \theta \geqslant (e_{T2} - E_c)/E.$$

The waveform construction is shown in figure XI.3.4.

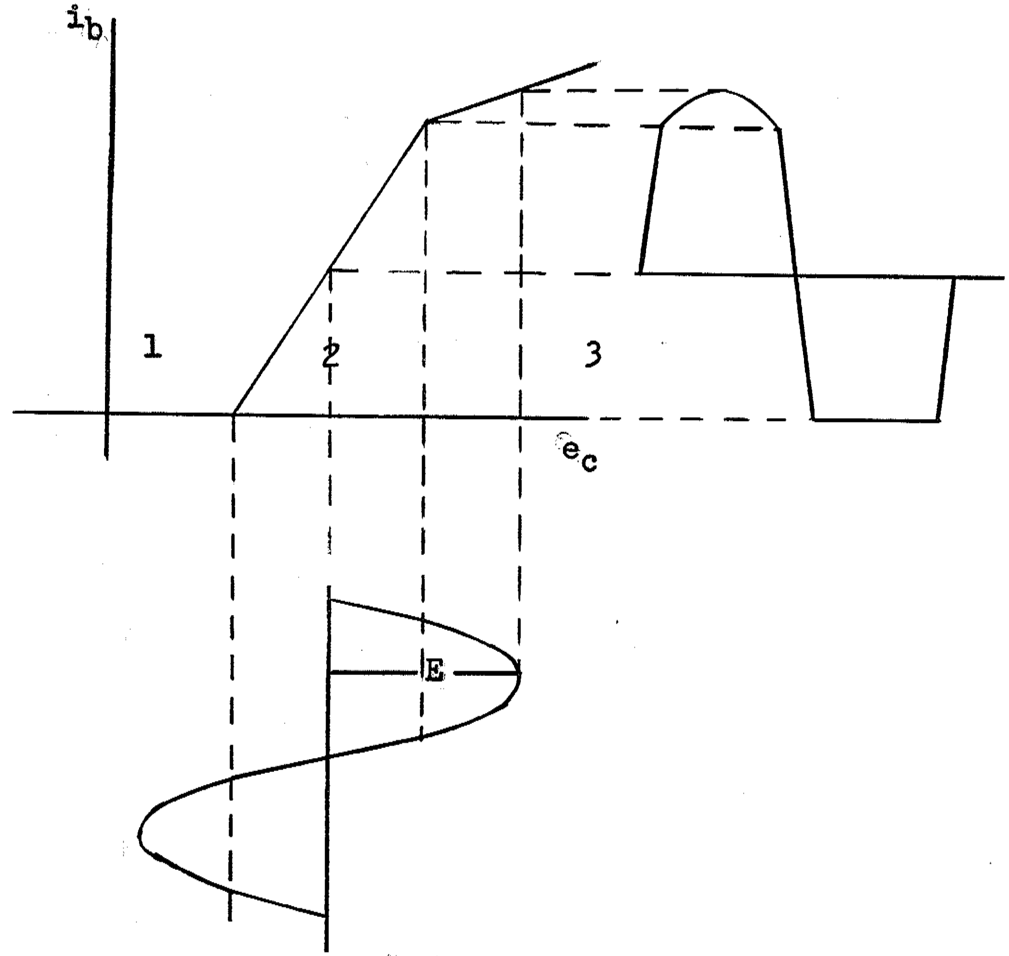


FIGURE XI.3.5

VACUUM TUBE CURRENT AND VOLTAGE WAVESHAPES

Obviously, for very complex characteristics, the number of linear approximations would increase and in so doing would complicate the procedure. The method, however, can be used quite satisfactorily in many cases, including those with a mixture of linear and nonlinear elements in which cases, the combined dynamic characteristic must be used.

XI.4 GRAPHICAL IMPEDANCE METHOD (27)

The above methods in sections 1 and 3 may be used most effectively in circuits where there is either resistance or reactance but when both are present, the analysis becomes very complex very quickly. The isocline methods are of course, applicable to any circuit with reactance, whether it is dissipationless or not. These methods, however, dealt directly with the differential equation of the system. The following method, developed extensively by Odessey and Weber (27) is also applicable to circuits of a mixed nature, but depends on a basis of impedance characteristics and impedance equations. The procedure is outlined in the following example.

Consider the series connection of a linear resistor, R, a linear capacitor, C, and a nonlinear reactor, the characteristic of which is given by:

$$E_L = f(I) \qquad \text{XI.4.1}$$

where the E_L and I are rms values. The hysteresis is neglected so that equation XI.4.1 is simple. A voltage of rms value E and frequency ω is applied to the circuit.

In the linear case, the equation:

$$E^2 = I^2R^2 + I^2[\,\omega L - \frac{1}{\omega C}\,]^2 \qquad \text{XI.4.2}$$

describes the system. This is carried over into the nonlinear case by simply calling $I\omega L$, E_L so that:

$$E^2 = (IR)^2 + [\,E_L - I/\omega C]^2. \qquad \text{XI.4.3}$$

The graphical method is then based on the two representations of E_L, given by:

$$E_L = f(I), \qquad \text{XI.4.3}$$

and

$$E_L = [\,E^2 - (IR)^2]^{1/2} + I/\omega C. \qquad \text{XI.4.4}$$

The radical term in equation XI.4.4 is an ellipse in the E - I plane and the second term is a straight line through the origin of slope $1/\omega C$.

If, at first, the resistance $R = 0$, then, when the characteristics $f(I)$ and $I/\omega C$ are plotted as in figure XI.4.1, the voltage E required to support any current I, in the system is given by the difference between $f(I)$ and $I/\omega C$. The dynamic or combined characteristic is as shown in the figure. Phase considerations have been neglected.

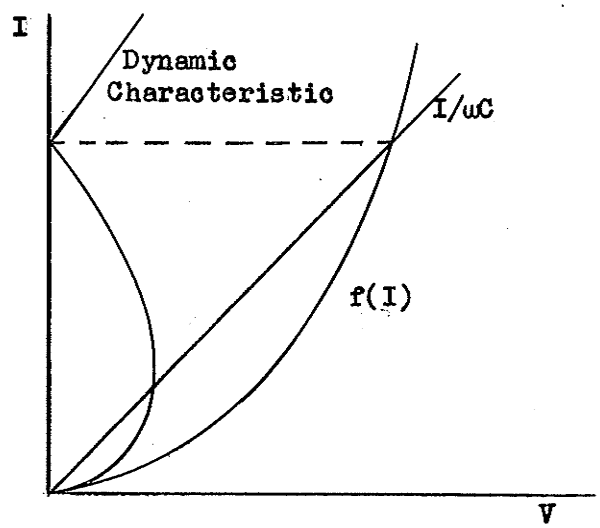


FIGURE XI.4.1

STATIC AND DYNAMIC CHARACTERISTICS OF A REACTIVE NONLINEAR CIRCUIT

The point at which $f(I) = I/\omega C$ is a resonant point where the current satisfying the equality flows with no voltage applied. This is not the same type of resonance as occurs in a linear circuit at a given frequency. The resonant point in the nonlinear case will occur for any frequency for which the two characteristics intersect. If L_o is the initial value of the incremental inductance, that is,

$$\omega L_o = \left.\frac{dV}{dI}\right)_{I=0},$$

the condition for intersection for a characteristic of the type above is that:

$$\omega L_o > \frac{1}{\omega C}. \qquad \text{XI.4.5}$$

In the case where resistance is present, the above procedure is repeated, except the equation :

$$E_L = [\,E^2 - (IR)^2]^{1/2} + I/\omega C \qquad XI.4.4$$

is used instead of just the last term. As noted before, the right side represents an ellipse with a sloped axis $I/\omega C$. For a specific value of E, the typical graphical plot is shown in figure XI.4.2.

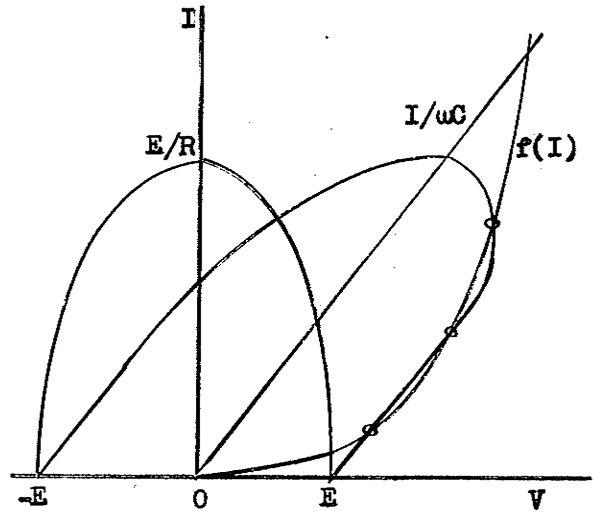


FIGURE XI.4.2

STATIC AND DYNAMIC CHARACTERISTICS OF A NONLINEAR CIRCUIT WITH RESISTANCE

The intersections of the oblique ellipse and f(I) are the points corresponding to an applied voltage E on the dynamic curve. The points may be three, two or one in number. If sufficient values of E are used the points may be used to give the dynamic characteristic.

Several points should be noted regarding the above characteristic. First of all, as E increases from the value shown in figure XI.4.2, the ellipse will expand until the two lower intersections will merge at tangency, and then for higher E will disappear. This explains the obtained result of jumps or discontinuities in the dynamic characteristic.

Secondly, as E decreases from the value shown, the two upper intersections will merge to the point of tangency and then will disappear. There is, thus, a bounded region along the E axis for which there are multiple current values. For low and high values of E, the

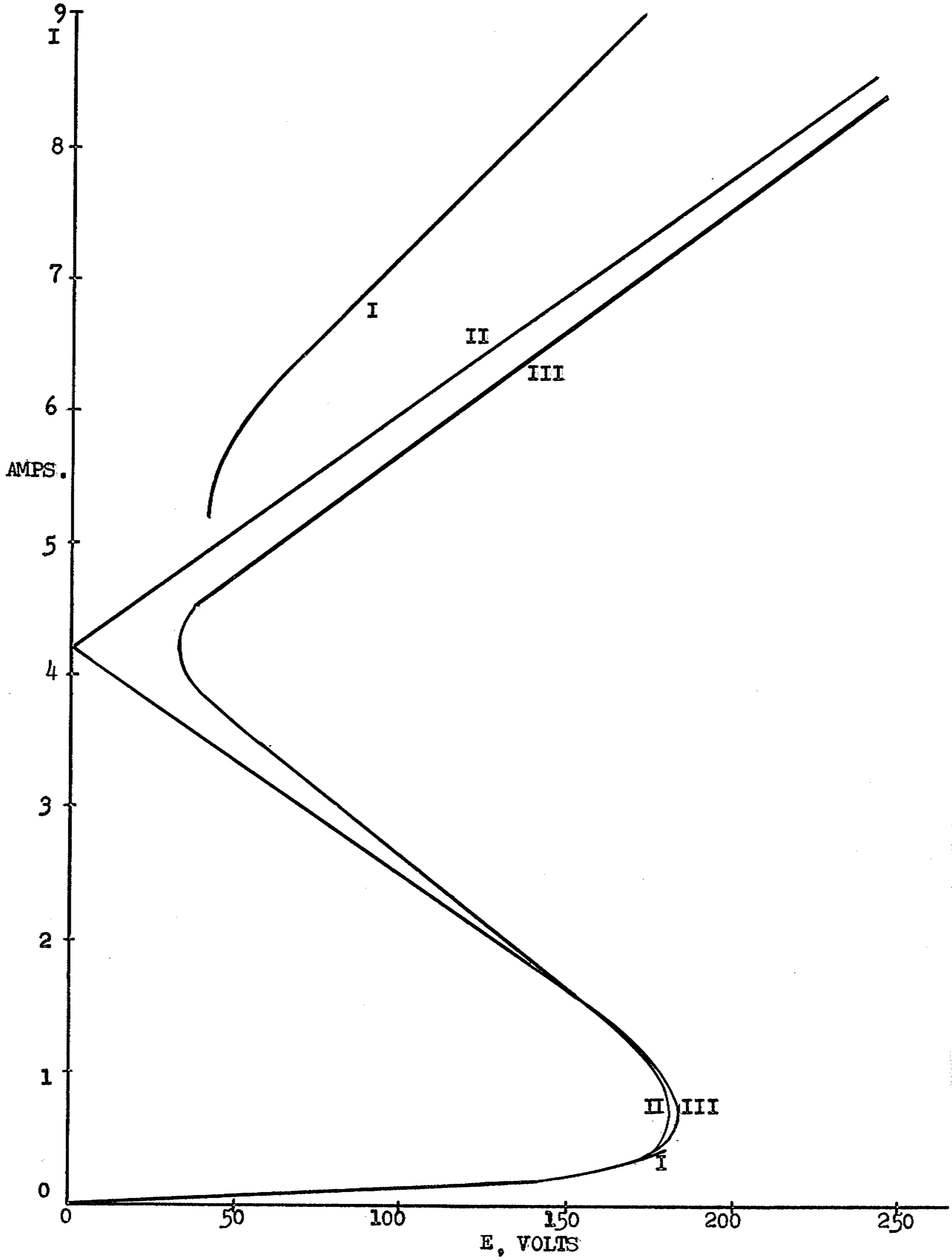


FIGURE XI.4.3

EXPERIMENTAL AND PREDICTED CURVES COMPARISON.

dynamic characteristic is single-valued. This exemplifies the statement made in Chapter I that the combination of two simple characteristics is not always simple.

In a case studied by the author, using a 3 KVA 220/110, power transformer and a capacitor of 70.5 ohms impedance at 60 cps., the experimental results are compared with the predicted values in figure XI.4.3. The curve 1 in the figure is the experimental curve, curve 2 and curve 3 are predicted, the former, neglecting the transformer resistance and the latter including it.

The experimental curve has no centre section. This is due to its being unstable, a fact which may be ascertained in a manner similar to that used in Chapter VI for multi-valued frequency response.

The curves, although accurate for low current values, are very inaccurate for the higher values. This is due to several reasons.

1. The waveform of the current at high saturation contains harmonics which will resonate at different points in the I - V plane. The rms current was measured using a moving iron instrument which reads only the square root of the sum of the squares of the harmonics with debatable accuracy. Corrections would depend on the peak values of current and voltage, and so would vary over the range of variables.

2. Hysteresis was not taken into account since the characteristic was taken as simple. The presence of hysteresis would change the phase angle between voltage and current and would, therefore, introduce an equivalent resistance which varied with frequency.

3. The nonlinear element subjected to a single-frequency oscillation will act as a generator of all harmonics of that frequency, (see Chapter X, section 2). As a result any currents flowing due to the nonlinearity will introduce losses in any resistor in the circuit. This will actually change the characteristics of the system through the variation in equivalent resistance.

The results of figure XI.4.3 indicate that this graphical method may be used to analyze a system in the qualitative sense but not in the quantitative. Further refinements on all these methods will, in general, yield no better results while, at the same time, introducing severe complications.

This jump phenomenon has been successfully applied to practical design, by Suits (32), Summers (33), and others, but has not as yet been fully exploited. A great deal has been done in computing critical conditions by Odessey and Weber (27) and Thomson (35).

CHAPTER XII

OPERATIONAL AND TRANSFORM METHODS

XII.1 THE OPERATIONAL METHOD (14)

The operational approach to the solution of nonlinear circuit equations lies between the method of successive approximations (iteration) and transform methods. Instead of applying iteration to the solution of equations in the domain of the independent variable, which is usually time, it is applied to the Laplace transform or Heaviside operational form of the independent variable in what is usually the complex frequency plane. The extensive tabulation of transforms reduces the labour of calculation to a minimum. The method is as follows.

Consider a series circuit as shown in figure XII.1.1, consisting of a linear network with an impedance operator $Z(D)$ where $D = \frac{d}{dt}$, and a nonlinear element with a characteristic $v = f(i)$, driven by an arbitrary voltage source $e(t)$.

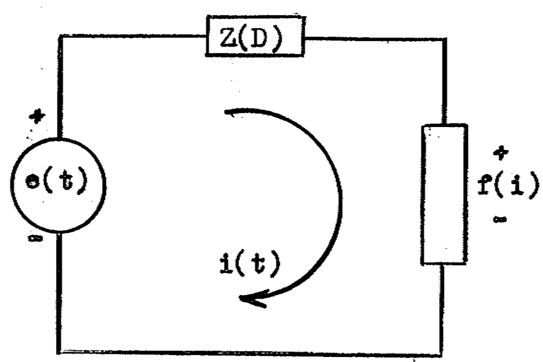


FIGURE XII.1.1

SERIES CONNECTION OF A LINEAR AND A NONLINEAR ELEMENT ACROSS AN ARBITRARY VOLTAGE SOURCE

Kirchhoff' Voltage Law yields:

$$Z(D)\, i(t) + f[\, i(t)] = e(t). \qquad \text{XII.1.1}$$

Applying the Laplace Transform to this equation then results in:

$$Z(s)\, I(s) + F(s) = E(s), \qquad \text{XII.1.2}$$

where:

$$L[\, i(t)\,] = I(s), \qquad \text{XII.1.3}$$

$$L[\, e(t)\,] = E(s), \qquad \text{XII.1.4}$$

and $$L[\, f(i(t))\,] = F(s) \qquad \text{XII.1.5}$$

assuming quiescent initial conditions. If the initial conditions are non-zero, they enter the term E(s) and so the treatment is actually quite general.

From equation XII.1.2, there results:

$$I(s) = \frac{E(s)}{Z(s)} - \frac{F(s)}{Z(s)} \qquad \text{XII.1.6}$$

or $$I(s) = Y(s)\,E(s) - Y(s)\,F(s). \qquad \text{XII.1.7}$$

Applying the inverse transform to this equation yields:

$$i(t) = L^{-1}[\,Y(s)E(s)] - L^{-1}[\,Y(s)F(s)], \qquad \text{XII.1.8}$$

which upon application of the Faltung theorem of Laplace Transform theory, becomes:

$$i(t) = \int_0^t y(t-u)\,e(u)\,du - \int_0^t y(t-u)f[i(u)]\,du \qquad \text{XII.1.9}$$

where: $$y(t) = L^{-1}[Y(s)]. \qquad \text{XII.1.10}$$

For the zero'th approximation, the nonlinear element is removed yielding:

$$i_o(t) = \int_0^t y(t-u)\,e(u)\,du \qquad \text{XII.1.11}$$

and so, equation XII.1.9 becomes:

$$i(t) = i_o(t) - \int_0^t y(t-u)\,f[i(u)]\,du, \qquad \text{XII.1.12}$$

which is known as Lalesco's nonlinear integral equation.

The limit of the sequence:

$$i_o(t) = \int_0^t y(t-u)\,e(u)\,du, \qquad \text{XII.1.11}$$

$$i_1(t) = i_o(t) - \int_0^t y(t-u)\,f[i_o(u)]du \qquad \text{XII.1.12a}$$

$$\vdots$$

$$i_n(t) = i_o(t) - \int_0^t y(t-u)f[i_{n-1}(u)]\,du, \qquad \text{XII.1.13}$$

as n approaches infinity, is the solution of equation XII.1.12.

The process converges most rapidly when the nonlinearity is small. The first order approximation is another form of the principle of equivalent linearization.

The above sequence, however is in the time domain. It can be carried out in a usually simpler manner in operational form in the frequency domain. Equations XII.1.11 and XII.1.12 may be written as:

$$i_o(t) = L^{-1}[\ Y(s)\ E(s)\ ] \qquad \text{XII.1.14}$$

and

$$i(t) = i_o(t) - L^{-1}[\ Y(s) Lf[i(t)\ ]. \qquad \text{XII.1.15}$$

The sequence then becomes:

$$i_1(t) = i_o(t) - L^{-1}[Y(s)\ Lf[i_o(t)\ ], \qquad \text{XII.1.16}$$

$$i_2(t) = i_o(t) - L^{-1}[Y(s)\ Lf[i_1(t)\ ], \qquad \text{XII.1.17}$$

$$\vdots$$

$$i_n(t) = i_o(t) - L^{-1}[Y(s)\ Lf[i_{n-1}(t)\ ]. \qquad \text{XII.1.18}$$

which may be more easily computed using a suitable table of Laplace Transforms than in the time domain. The function y(t) is the well-known Green's function for the circuit.

The above method theoretically may be applied to a system containing any forcing function e(t) and so will yield the transient or the steady-state response or both. However, when a steady-state voltage function is applied, the process again becomes cumbersome due to the complications in the terms:

$$L^{-1}[\ Y(s)\ L[\ f(i_n)\ ]\ ].$$

The results may be obtained in a much simpler manner which uses a short-hand notation for the Laplace Transforms of trigonometric functions based on the definition:

$$T(\omega) = 1/(s^2 + \omega^2), \qquad \text{XII.1.19}$$

so that:

$$L[\ \sin \omega t] = \omega\ T(\omega) \qquad \text{XII.1.20}$$

and so on.# The process is, in reality, the operational form of the series expansion method of Liapounoff (8).

A short table of transforms and identities in $T(\omega)$ may be found in reference (13) page 600.

Consider the problem of the linear inductor, L, connected in series with a nonlinear resistor, the characteristic of which is given by:

$$v = f(i) = R_o i^2 \qquad \text{XII.1.21}$$

across a constant voltage, E. The differential equation describing the system is then:

$$L\,Di + R_o i^2 = E. \qquad \text{XII.1.22}$$

By changing variables to:

$$\rho = (ER_o)^{1/2}\, t\,/L, \qquad \text{XII.1.23}$$

and

$$x = (R_o/E)^{1/2} i, \qquad \text{XII.1.24}$$

equation XII.1.22 becomes:

$$\frac{dx}{d\rho} + x^2 = 1. \qquad \text{XII.1.25}$$

For the solution:

$$Y(s) = 1/\ L[\frac{dx}{d\rho}] = 1/s, \qquad \text{XII.1.26}$$

$$f(x) = x^2,$$

and

$$E(s) = 1/s. \qquad \text{XII.1.28}$$

Thence, the zero'th order approximation is:

$$i_o(\rho) = L^{-1}[\ \frac{1}{s}\cdot\frac{1}{s}]$$

$$= \rho. \qquad \text{XII.1.29}$$

The higher order approximations are:

$$i_1(\rho) = \rho - L^{-1}[\ \frac{1}{s}\, L[\ \rho^2]\ ]$$

$$= \rho - \rho^3/3, \qquad \text{XII.1.30}$$

$$i_2(\rho) = \rho - L^{-1}[\ \frac{1}{s}\, L[\ \rho^2 - \frac{2\rho^4}{3} + \frac{\rho^6}{9}]\ ]$$

$$= \rho - \rho^3/3 + 2\rho^5/15 - \rho^7/63, \qquad \text{XII.1.31}$$

$$i_3(\rho) = \rho - L^{-1}[\ \frac{1}{s}\, L[\ \rho^2 - 2\rho^4/3 + 17\rho^6/45 - 38\rho^8/315 \ldots$$

$$\ldots + \rho^{14}/3969]\ ]$$

$$= \rho - \rho^3/3 + 2\rho^5/15 - 17\rho^7/315 + \ldots$$

and so on.

As a comparison, the exact solution to equation XII.1.25 is:

$$x = \tanh \rho, \qquad \text{XII.1.33}$$

which when expanded by power series in ρ yields:

$$x = \rho - \rho^3/3 + 2\rho^5/15 - 17\rho^7/315 + \dots \ . \qquad \text{XII.1.34}$$

The two solutions are seen to agree exactly in the first four terms. Higher ordered approximations are possible to yield more accurate results but they become unwieldy, a fact which was mentioned above.

XII.2 THE Z TRANSFORM METHOD

The z transform method for solving linear and nonlinear systems had its origin in the so-called time-series of Tustin and the method of Madwed (9). This older method was exceedingly complicated to use due to the cumbersome algebra involved.

The process was to reduce the driving functions of time to a number series which actually represented the magnitude of the function at certain specific times. The network was then described by an operator which was applied to the input time series to give the response in the form of another number series. The process could be made as accurate as required by reducing the sampling period and, of course, thereby increasing the amount of computation involved.

The next development in the method was made by Ragazzini and Zadeh (31) and Ragazzini and Bergen (30). It eliminated the operational algebra by use of a function generator which approximated the original sampled function by straight line segments. This reduced the system to a Laplace Transform problem involving transforms containing e^{-sT} where T is the sampling time interval. However, the function generator complicated the system function considerably.

The z transform was then defined using:

$$z = e^{-sT}. \qquad \text{XII.2.1}$$

The function generator was eliminated in the method presented by Boxer and Thaler (18), the result being a much simpler procedure. This method will be discussed below.

Consider a time function f(t) as shown in figure XII.2.1.

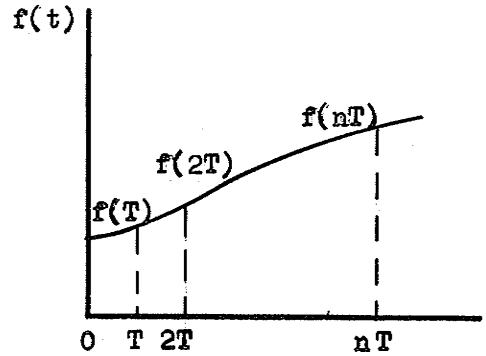


FIGURE XII.2.1

SAMPLING INTERVALS OF A CONTINUOUS FUNCTION

The sampling of such a function can be carried out as a sequence of shifted impulse or delta functions at intervals of T, the areas or strengths of which are equal to the magnitude of the function at the sampling points. The result is that the sampled function denoted by $f^{\#}(t)$ may be written as:

$$f^{\#}(t) = \sum_{n=0}^{\infty} f(nT)\,\delta(t-nT) \qquad \text{XII.2.2}$$

where $\delta(t-nT)$ is the Dirac delta function at the point $t = nT$. The Laplace Transform of such a function is then:

$$F^{\#}(e^{sT}) = \sum_{n} f(nT)\, e^{-snT} \qquad \text{XII.2.3}$$

or using the definition of equation XII.2.1, the so-called z-form of f(t) is given by:

$$F^{\#}(z) = \sum_{n} f(nT)\, z^{-n}. \qquad \text{XII.2.4}$$

The inverse Laplace Transform of a function F(s) is written as:

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)\, e^{st}\, ds \qquad \text{XII.2.5}$$

being the contour integral along the line s = c, c real. The inverse z-form may be written as:

$$f(nT) = \frac{1}{2\pi j} \oint F^{\#}(z)\, z^{n-1} dz \qquad \text{XII.2.6}$$

where the contour is now taken as a circle centred at the origin and including all the singularities of $F^{\#}(z)$.

The infinite form of equation XII.2.4 can usually be written in closed form so that $F^{\#}(z)$ becomes the ratio of two polynomials in z^{-1}. If the long division is carried out for the ratio, the result is a series in z^{-1}, the coefficients of which are simply the values of $f(t)$ at the sampling intervals, a result derived without the use of the contour integration of equation XII.2.6.

This simple result may be applied to nonsampled system analysis by use of the Inverse Laplace Transform of equation XII.2.5, so that:

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)\, e^{st} ds \qquad \text{XII.2.5}$$

$$= \frac{1}{2\pi j} \int_{-j\frac{\pi}{T}}^{j\pi/T} F(s)\, e^{st} ds \; +$$

$$\frac{1}{2\pi j} \int_{j\pi/T}^{j\infty} [F(s)\, e^{st} + F(-s)\, e^{-st}]\, ds, \qquad \text{XII.2.7}$$

which, if the time T is small enough, may be approximated by :

$$f_A(t) = \frac{1}{2\pi j} \int_{-j\pi/T}^{j\pi/T} F(s)\, e^{st}\, ds. \qquad \text{XII.2.8}$$

Writing $t = nT$ and $s = \ln z/T$, equation XII.2.8 becomes:

$$f_A(nT) = \frac{1}{2\pi j} \int_{-j\pi/T}^{j\pi/T} F\left(\frac{1}{T} \ln z \right) e^{nT(\frac{1}{T} \ln z)} d\left(\frac{1}{T} \ln z\right) \qquad \text{XII.2.9}$$

$$= \frac{1}{2\pi j} \oint \frac{1}{T} F\left(\frac{1}{T} \ln z\right) z^{n-1}\, dz \qquad \text{XII.2.10}$$

where the truncated contour of equation XII.2.9 becomes the unit circle contour in equation XII.2.10. This, however, has the form of the z-transform of equation XII.2.4 divided by T. $F(\ln z/T)$, however, is transcendental and not of the form of a ratio of two polynomials.

Using the infinite series expansion:

$$\ln z = 2[\, u + u^3/3 + u^5/5 + \ldots] \,, \qquad \text{XII.2.11}$$

where:

$$u = (1 - z^{-1})/(1 + z^{-1}), \qquad \text{XII.2.12}$$

yields:

$$s^{-1} = \frac{T}{\ln z} = \frac{T}{2} [\, u^{-1} - \frac{u}{3} - \frac{4}{45} u^3 - \ldots], \qquad \text{XII.2.13}$$

which converges relatively rapidly. Other higher negative powers of s can be obtained by raising equation XII.2.13 to the required power. Retaining only the principal part and the constant term of the Laurent series yields:

$$s^{-k} = \frac{P_k(z^{-1})}{(1-z^{-1})^k} \qquad \text{XII.2.14}$$

where $P_k(z^{-1})$ is a polynomial in z^{-1}. This is normally called the z-form of s^{-k}. Additional terms retained in the expansion actually increase the error of this second approximation.(18)

The process of solution of a problem is then as follows.

1. Obtain the response of a system in terms of a rational function F(s) in terms of inverse powers of s.
2. Substitute for s^{-k} from Table XII.2.1 of z-forms based on equation XII.2.14, to give a rational fraction in z^{-1}.
3. Divide by T, the required sampling interval and reduce(by division) the rational fraction to a power series in z^{-1}. The coefficient of z^{-n} is then the approximate value of the response at $t = nT$.

TABLE XII.2.1

Z-FORMS

s^{-k}	z-form
s^{-1}	$\frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}$
s^{-2}	$\frac{T^2}{12}\frac{1+10z^{-1}+z^{-2}}{(1-z^{-1})^2}$
s^{-3}	$\frac{T^3}{2}\frac{z^{-1}+z^{-2}}{(1-z^{-1})^3}$
s^{-4}	$\frac{T^4}{6}\frac{z^{-1}+4z^{-2}+z^{-3}}{(1-z^{-1})^4} - \frac{T^4}{720}$
s^{-5}	$\frac{T^5}{24}\frac{z^{-1}+11z^{-2}+11z^{-3}+z^{-4}}{(1-z^{-1})^5}$

The method is applicable to the nonlinear case with the following modification. Any nonlinear differential equation can be expressed as:

$$D^n[C_n x] + D^{n-1}[C_{n-1}x] + \dots + D[C_1 x] + C_0 x = f(t), \qquad \text{XII.2.15}$$

where:

$$C_i = f_i[x, Dx, \dots]. \qquad \text{XII.2.16}$$

The Laplace Transform of equation XII.2.15 is taken, regarding the C_i as constants. The process outlined above is carried out in an identical manner except, in the long division, the values of the C_i are to be changed at each step to the most recent values available. The method is demonstrated in the following example.

Consider the case of a series connection consisting of a linear inductor L and a nonlinear resistor, the characteristic of which may be written as:

$$v = f(i) = R_0 i^2, \qquad \text{XII.2.17}$$

connected across a constant dc voltage E. The differential equation describing the system is, as in section 1 of this chapter:

$$L\,Di + R_0 i^2 = E. \qquad \text{XII.2.18}$$

By defining the new variables as in the above mentioned section, there results the equation:

$$\frac{dx}{d\rho} + x^2 = 1. \qquad \text{XII.2.18}$$

Written in the form of equation XII.2.15, this becomes:

$$\frac{dx}{d\rho} + Cx = 1, \qquad \text{XII.2.19}$$

where:

$$C = x. \qquad \text{XII.2.20}$$

Taking the Laplace Transform, assuming the initial condition, $x(0) = 0$, yields:

$$(s + C)\,X(s) = 1/s, \qquad \text{XII.2.21}$$

where:

$$X(s) = L[x(\rho)], \qquad \text{XII.2.22}$$

Thence,

$$X(s) = 1/s(s + C)$$

$$= \frac{s^{-2}}{1 + Cs^{-1}}. \qquad \text{XII.2.23}$$

Substituting from the table and dividing by T, yields the z-form of X(s):

$$X_A^{\#}(z) = \frac{\frac{1}{T}\left[\frac{\frac{T^2}{12}(1+10z^{-1}+z^{-2})}{(1-z^{-1})^2}\right]}{1 + C\frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}}$$

$$= \frac{T(1 + 10z^{-1} + z^{-2})}{(12+6CT)-24z^{-1}+(12-6CT)z^{-2}}. \quad \text{XII.2.24}$$

Carrying outthe long division with T = 0.1, assigning the latest values of C = x in each step, starting with the initial condition that C(0) = x(0) = 0, yields:

$$X_A^{\#}(z) = 0.00833 + 0.0996z^{-1} + 0.1989z^{-2} + 0.2950z^{-3} + 0.3864z^{-4} + 0.4716z^{-5} + 0.5495z^{-6} + 0.6195z^{-7} + 0.6814z^{-8} + 0.7352z^{-9} + 0.7815z^{-10} + \ldots \quad \text{XII.2.25}$$

This yields the values of the solution to the equation XII.2.20 tabulated below. The exact solution to the equation is x = tanh ρ, and is also tabulated along with the percentage error between the two.

TABLE XII.2.2

ρ	tanh ρ	From $X_A^{\#}(z)$	% error
0.0	0.00000	0.00833	-------
0.1	0.09967	0.09996	0.29
0.2	0.19738	0.1989	0.76
0.3	0.29131	0.2950	1.27
0.4	0.37995	0.3864	1.68
0.5	0.46212	0.4716	2.05
0.6	0.53705	0.5495	2.31
0.7	0.60437	0.6195	2.50
0.8	0.66404	0.6814	2.62
0.9	0.71630	0.7352	2.64
1.0	0.76159	0.7815	2.61

The error is very small indeed,especially when the components in any typical network are of 5% or more tolerance.

The increasing percentage error is due to the cumulative effect on the error of the division process.

It may be seen, from the above example, that the z-transform method is very powerful. The example does, however have a slowly varying solution and is, therefore, readily obtainable by such a sampling method. For a solution which varies rapidly in time, the oscillations are, of course, smoothed out by the procedure. A good example of this is the output of a delay line with unity feedback, driven by a unit step of voltage. The output is a set of pulses of length equal to the delay in the line. For analysis, the sampling time must be taken as a submultiple of the delay time. The actual output and the sampling points are shown in figure XII.2.2, where T = 1/3 × the delay time.

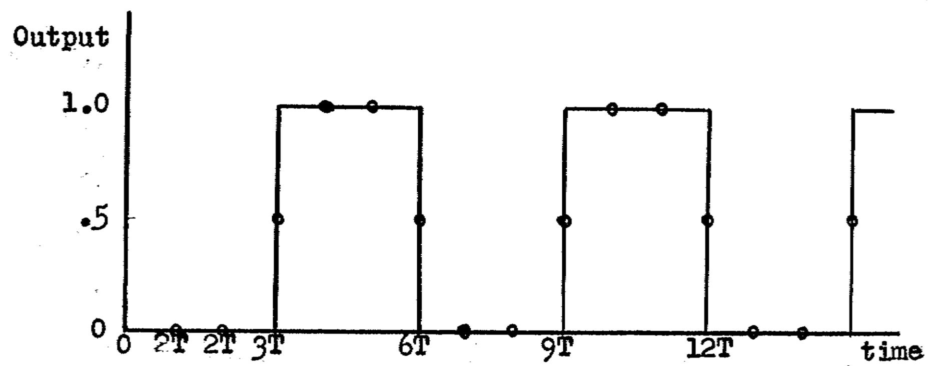


FIGURE XII.2.2

ACTUAL AND SAMPLED OUTPUT OF A DELAY LINE

To obtain an accurate solution, the sampling time T must be taken as a very small fraction of the period of oscillation which in this case is twice the delay time.

The method, however, is capable of tremendous application.

CHAPTER XIII

CONCLUSION

The number of methods outlined above are by no means complete and exhausti~~ng~~ve. Notably absent is any mention of analogue methods of solution. These were omitted because (1) they require specialized computing apparatus not normally available, and, (2) they contribute nothing to the understanding of nonlinear phenomena. All the methods in Part B may be used with no special apparatus. They are all theoretical in nature as opposed to experimental.

Many other methods similar to those included, and, in some cases, variations of them have been omitted also, with the fear that their inclusion would have rendered this work unwieldy. Furthermore, no matter how lengthy the list of methods become, new methods are constantly being brought forth.

Another major field in nonlinear investigation omitted is the consideration of stability of solutions. This topic has received a great deal of attention from the pure mathematicians such as in references (7) and (8), and is such a wide field that it could not be included.

The field of nonlinear analysis is at one time extensive and narrow. It is extensive in the number of researchers involved in its studies and narrow in each individual's immediate scope of investigation. It is hoped that this thesis succeeds in consolidating both the theoretical aspects of nonlinear analysis and the various methods of solution.

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