THE UNIVERSITY OF MANITOBA

HYPERIDENTITIES OF LATTICES, SEMILATTICES AND DIAGONAL SEMIGROUPS

by

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ABSTRACT

Following W. Taylor, we define a hyperidentity ε to be formally the same as an identity. However, a variety V is said to satisfy a hyperidentity ε, iff whenever the operation symbols of ε are replaced by arbitrary polynomials (of appropriate arity) in the operations of V, then the resulting identity holds in V in the usual sense. In order to avoid confusing hyperidentities with identities, we underline the operation symbols of ε whenever ε is being viewed as a hyperidentity.

In Chapter One, we introduce the concepts of hyperidentities and hypervarieties. A hypervariety is a class of varieties defined (up to equivalence) by a set of hyperidentities. We also give a summary of some of the known results in this area of research.

Let λ and μ be binary operation symbols and define Σ^2 to be the following set of three hyperidentities:

$$x \ \underline{\lambda} \ x = x,$$

$$x \ \underline{\lambda} \ (y \ \underline{\lambda} \ z) = (x \ \underline{\lambda} \ y) \ \underline{\lambda} \ z,$$

$$(x \ \underline{\lambda} \ y) \ \underline{\mu} \ (z \ \underline{\lambda} \ t) = (x \ \underline{\mu} \ z) \ \underline{\lambda} \ (y \ \underline{\mu} \ t).$$

We prove, in Chapter Two (the content of which is also due to R. Padmanabhan), that Σ^2 is a basis for all binary semilattice hyperidentities (that is, all semilattice hyperidentities of type <2,2,...>). We also prove structure

theorems for algebras of type <2> or <2,2> satisfying all semilattice hyperidentities.

Let Σ^n be the following set of four hyperidentities:

(i) $\underline{F}(x,\ldots,x) = x$,

(ii) $\underline{F}(\underline{F}(x_1,\ldots,x_n),\underline{F}(y_1,\ldots,y_n),z_1,\ldots,z_{n-2}) =$
$\underline{F}(\underline{F}(x_1,y_1,x_3,x_4,\ldots,x_n),\underline{F}(x_2,y_2,y_3,\ldots,y_n),z_1,\ldots,z_{n-2})$

(iii) $\underline{F}(\underline{F}(x_1,\ldots,x_n),\underline{F}(u,u,y_1,\ldots,y_{n-2}),z_1,\ldots,z_{n-2}) =$
$\underline{F}(\underline{F}(x_1,\ldots,x_n),\underline{F}(x_2,u,y_1,\ldots,y_{n-2}),z_1,\ldots,z_{n-2})$

(iv) $\underline{F}(\underline{G}(x_{11},\ldots,x_{1n}),\ldots,\underline{G}(x_{n1},\ldots,x_{nn})) =$
$\underline{G}(\underline{F}(x_{11},\ldots,x_{n1}),\ldots,\underline{F}(x_{1n},\ldots,x_{nn}))$

In Chapter Three, we show that Σ^n is a basis for all n-ary semilattice hyperidentities, and consequently, that $\Sigma = \bigcup(\Sigma^n \mid n < \omega)$ is a basis for the class of all semilattice hyperidentities. We also prove a structure theorem for algebras of type <n> which satisfy all semilattice hyperidentities.

Let V be any nontrivial lattice variety, or the variety of all semilattice varieties. In Chapter Four, we show that the hyperidentities of V are not finitely based. This was proved in Chapter Two for the variety of all semilattices and also for the variety of diagonal semigroups.

The class of all hypervarieties (up to equivalence) is a complete lattice. Let $V_h(A)$, $V_h(D)$, $V_h(SL)$ and $V_h(DS)$ be the smallest hypervarieties containing the varieties of abelian groups, distributive lattices, semilattices, and diagonal semigroups, respectively. We show, in Chapter Five, that $V_h(A) \vee V_h(D) = V_h(SL)$ and that $V_h(DS) < V_h(SL)$ is a covering relationship.

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CHAPTER I

INTRODUCTION TO HYPERIDENTITIES AND HYPERVARIETIES

A hyperidentity, as defined by W. Taylor [25], is formally the same as an identity. However, a variety V is said to satisfy a hyperidentity ϵ in the operation symbols $F_1, F_2, \ldots, F_k$ of arity $n_1, n_2, \ldots, n_k$ respectively, iff whenever each $\underline{F}_i$ is replaced by an arbitrary n_i-ary polynomial in the operations of V, the resulting identity holds in V in the usual sense.

By way of example, a proof that the hyperidentity

$$\epsilon\colon\ F(G(x,y),x) = G(x,F(y,x))$$

holds in L, the variety of all lattices, consists of a case by case examination of $(F,G) \in \{x, y, x \vee y, x \wedge y\}^2$, the set of all ordered pairs of binary lattice polynomials.

We shall follow the convention that hyperidentities have no nullary operation symbols. Later in this chapter, when we discuss hyperidentities in terms of clones, the reason for this will become apparent. The set of all hyperidentities satisfied by a variety V, will be denoted by H(V). We shall let $H^m(V)$ be the set of all hyperidentities holding in V with operation symbols all of arity m, and

$H_n(V)$ will denote the set of all hyperidentities of V with at most n distinct variables. The hyperidentities of $H^m(V)$ will be said to be m-ary. The set of hyperidentities of V which are of type $\langle m,m,\ldots,m\rangle$ will be denoted by $H(V)\langle m,m,\ldots,m\rangle$. We thus have $H(V)\langle m,m,\ldots,m\rangle \subseteq H^m(V)$. To avoid confusing hyperidentites with identities we shall underline the function symbols of ϵ whenever ϵ is being viewed as a hyperidentity. For example, if ϵ is the lattice hyperidentity discussed above we shall write

$$\epsilon\colon\ \underline{F}(\underline{G}(x,y),x) = \underline{G}(x,\underline{F}(y,x)).$$

An expression such as $\underline{P} = \underline{F}(\underline{G}(x,y),x)$, where $\underline{F}$ and $\underline{G}$ are again being treated as variables to be replaced by polynomials of corresponding arity, will be referred to as a hyperpolynomial. When the polynomials p and q are substituted for the symbols $\underline{F}$ and $\underline{G}$ of $\underline{P}$, the resulting polynomial will be denoted by $\underline{P}(p,q)$.

The study of hyperidentities has risen naturally out of the study of classes of varieties, in particular, the study of varieties which are definable by Mal'cev conditions (for example, W. D. Neumann [15], W. Taylor [22]).

A. I. Mal'cev proved (See G. Grätzer [12] for a proof.) that all algebras in a variety V have permutable congruence relations iff there exists a ternary polynomial p (in the operation symbols of V), such that

$$p(x,x,y) = y$$
$$p(x,y,y) = x$$

holds identically in V. That is, the class of all varieties satisfying the permutable congruence condition is the class of all varieties satisfying

$$\exists p(p(x,x,y) = y \wedge p(x,y,y) = x).$$

More generally, a class K of varieties is said to be defined by a strong Mal'cev condition (W. Taylor [22]) iff there exists a conjunction α of identities in function symbols f_i and variables x_j such that a variety V is in K iff

$$\exists f_1 f_2 \ldots f_n(\alpha).$$

That is, there exist polynomials f_i such that α holds identically in V.

In the same spirit, one can ask whether there are interesting classes of varieties which can be defined by a universally quantified conjunction of identities. For example,

$$\alpha\colon \forall f(f(x,x,x) = x)$$

defines W. D. Neumann's [15] class of idempotent varieties, and is exactly the class of varieties satisfying the hyperidentity $\underline{F}(x,x,x) = x$, since, as W. Taylor points out in [25], it is easy to show that if $\underline{F}(x,x,x) = x$ holds in a variety V, then all polynomials of V will be idempotent.

In [3], J. T. Baldwin and J. Berman point out this correspondence between strong Mal'cev conditions and positive existential Horn sentences and between hyperidentities and universal positive Horn sentences, and explore to what an extent first order model theory can be exploited in this situation.

The term hyperidentity appears as early as V. D. Belousov [4] and J. Aczél [2]. However, their notion differs somewhat from that of W. Taylor in that they require a hyperidentity ϵ to hold for some but not all polynomials. They say that a hyperidentity ϵ is satisfied by an algebra **A** iff whenever the operation symbols of ϵ are replaced by any choice of operations of A of corresponding arity, then the resulting identity holds in **A**. When this is the case, we shall say that the hyperidentity ϵ is <u>satisfied (possibly vacuously) by the operations of A</u>. Thus, a set H of hyperidentities is satisfied by the operations of an algebra **A**, iff those hyperidentities of H which are of the same type as A, are satisfied by the operations of A.

The median hyperidentities of type <m,n>

$$\alpha\langle m,n\rangle: \quad \underline{F}(\underline{G}(x_{1,1},\ldots,x_{1,n}),\ldots,\underline{G}(x_{m,1},\ldots,x_{m,n})) = \underline{G}(\underline{F}(x_{1,1},\ldots,x_{m,1}),\ldots,\underline{F}(x_{1,n},\ldots,x_{m,n})),\ 1 \le m,n < \omega$$

play an important role when investigating semilattice hyperidentities. They are a generalization of the semigroup identity xyzt = xzyt, sometimes referred to as the entropic or medial law, which has been investigated by J. Aczel [1], T. Evans [9] and others. The median hyperidentities have the interesting property that if the set $M = \{\alpha\langle m,n\rangle \mid 1 \le m,n < \omega\}$ is satisfied by the operations of an algebra **A** then M is satisfied by **A**, that is, if M is satisfied by the operations of A then M is satisfied by the polynomials of A. This was pointed out to the author by B. Wolk and follows easily from the results of T. Evans [8].

A <u>hypervariety</u>, again following W. Taylor [25], is what W. D. Neumann [15] defines as a variety of varieties, that is, a class of varieties closed under the formation of equivalent varieties, product varieties, subvarieties, and reduct varieties. Rather than define these concepts (whose definitions can be found in W. Taylor [22]), we shall, following Taylor [25], develop the concept of the clone of a variety and then define hypervarieties in terms of clones.

If V is a variety, the <u>clone</u> of V is the heterogeneous (many-sorted) algebra

$$C(V) = \langle F_1(V),\ F_2(V),\ldots;\ C_m^n,\ e_i^n\rangle,$$

where each $F_n(V)$ is the universe of the V-free algebra on the n generators $x_1,\ldots,x_n$, each e_i^n is a nullary operation denoting $x_i \in F_n(V)$, and each C_m^n is a composition operation with domain $F_n(V) \times (F_m(V))^n$ and range $F_m(V)$, defined

$$C_m^n(f,g_1,\ldots,g_n) = f(g_1,\ldots,g_n).$$

In W. Taylor [22], it is shown that the family of all clones of varieties can be defined (up to isomorphism) by the following three identities (cf. Cohn [7]):

$$\text{(i)}\ C_m^p(z,C_m^n(y_1,x_1,\ldots,x_n),\ldots,C_m^n(y_p,x_1,\ldots,x_n)) = C_m^n(C_n^p(z,y_1,\ldots,y_p),x_1,\ldots,x_n),\quad 1 \le m,n,p < \omega,$$

$$\text{(ii)}\ C_m^n(e_i^n,x_1,\ldots,x_n) = x_i,\quad m = 1,2,\ldots,\quad 1 \le i \le n < \omega,$$

$$\text{(iii)}\ C_n^n(y,e_1^n,\ldots,e_n^n) = y,\quad 1 \le n < \omega.$$

The concepts of homomorphism, subalgebra, and product are the obvious extensions of the homogeneous case. A homomorphism ρ from a clone $A = \langle A_1,A_2,\ldots;\ C_n^m,\ e_i^n\rangle$ to a clone $B = \langle B_1,B_2,\ldots;\ C_n^m,\ e_i^n\rangle$ is a set of mappings $\rho_i\colon A_i \to B_i$, $i < \omega$, such that

$$\rho_m[C_m^n(a_1,a_2,\ldots,a_{n+1})] = C_m^n[\rho_n(a_1),\rho_m(a_2),\ldots,\rho_m(a_{n+1})].$$

A product $A = \prod A_i$ of clones $A_i = \langle A_1^i,A_2^i,\ldots;\ C_m^n,\ e_i^n\rangle$, $i \in I$, is defined to be the clone $A = \langle \prod A_1^i,\ \prod A_2^i,\ldots;\ C_m^n,\ e_i^n\rangle$, where the operations are defined componentwise, that is,

$$C_m^n(f_1,f_2,\ldots,f_{n+1})(i) = C_m^n(f_1(i),\ldots,f_{n+1}(i)),$$

and $B = \langle B_1, B_2, \ldots;\ C_n^m,\ e_i^n\rangle$ is a subclone of $A = \langle A_1, A_2, \ldots;\ C_n^m,\ e_i^n\rangle$ iff B is a heterogeneous algebra, $B_i \subseteq A_i$ for each i, $i < \omega$, and the operations of B are restrictions of the operations of A. The usual theorems for products, subalgebras, congruences, etc., can easily be extended to the many sorted case. (See G. Birkhoff and J. D. Lipson [6].)

If we define two varieties to be <u>equivalent</u> iff they have isomorphic clones, and we define the <u>product</u> of two varieties V_1 and V_2 via the product of the clones $C(V_1)$ and $C(V_2)$, then the concepts of product varieties, subvarieties and reduct varieties correspond exactly to products of clones, homomorphic image clones and subclones respectively. (See W. Taylor [22], [23] for equivalent definitions of equivalent varieties and products of varieties.) Thus a class K of varieties is a hypervariety iff the corresponding class of clones is closed under the formation of products, homomorphic images and subclones. If V is a variety, $V_h(V)$ will denote the smallest hypervariety containing V.

Birkhoff's Theorem for varieties of homogeneous algebras states that a variety of algebras is definable by identities. This has an easy extension to classes of heterogeneous algebras such as clones.

<u>Theorem 1.1</u>. (Taylor [25]). For each hypervariety V_h, there exists a set I_c of clone identities such that V_h is precisely the class of varieties whose clones obey I_c.

Note that not every clone identity can be directly converted into a hyperidentity. For example, the clone identity

$$\epsilon_c:\ C_3^3(x, C_1^1(e_1^1, C_3^2(y, e_1^3, e_1^3)), e_1^3, e_1^3) = e_1^3$$

cannot be directly converted into a hyperidentity. However, by the above three clone identities, ϵ_c is obviously equivalent to the clone identity

$$\epsilon_c':\ C_3^3(x, C_3^2(y, e_1^3, e_1^3), e_1^3, e_1^3) = e_1^3$$

and the identity ϵ_c' will hold in a clone C(V) iff the hyperidentity

$$\epsilon_h:\ \underline{F}(\underline{G}(x_1, x_1), x_1, x_1) = x_1$$

holds in V. It can thus easily be seen that for every clone identity ϵ_c there exists a hyperidentity ϵ_h, such that ϵ_c holds in the clone C(V), of a variety V, iff ϵ_h holds in V, so we have the following theorem:

Theorem 1.2 (Taylor [25]). Every hypervariety is definable by hyperidentities.

Since hyperpolynomials often become somewhat cumbersome to work with, we shall often refer to their formation trees. We inductively define the formation tree of a hyperpolynomial $\underline{P}$ to be a finite set of nodes and directed edges satisfying the following conditions:

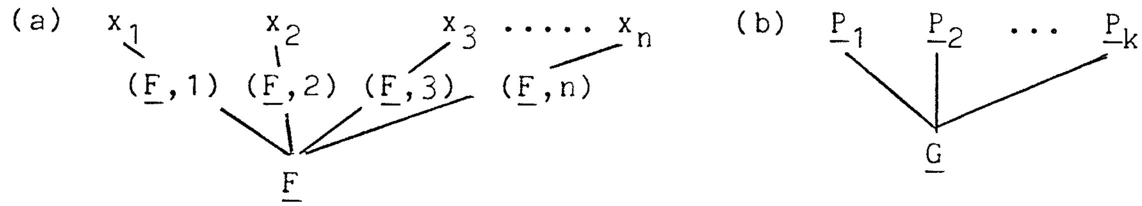


Figure 1.1

(i) The formation tree of $\underline{F}(x_1,\dots,x_n)$ consists of n+1 nodes and n edges. One node (the root) is labelled with the symbol $\underline{F}$ and the remaining nodes (the leaves) are labelled with the symbols $x_1,\dots,x_n$. Each leaf x_k, $k \in \{1,\dots,n\}$, is joined to the root by an edge labelled $(\underline{F},k)$. The edge $(\underline{F},k)$ is said to leave the node $\underline{F}$ and enter the node x_k. When depicting a formation tree in a diagram, the n edges leaving a node $\underline{F}$ will be labelled $(\underline{F},1),\dots,(\underline{F},n)$ starting from the left (see Figure 1.1(a)).

(ii) If $\underline{P} = \underline{G}(\underline{P}_1,\underline{P}_2,\dots,\underline{P}_m)$, then the formation tree of $\underline{P}$ is obtained by replacing the leaves of the tree of $\underline{G}(x_1,\dots,x_m)$ by the trees of $\underline{P}_1,\dots,\underline{P}_m$, so that the roots of $\underline{P}_1,\dots,\underline{P}_m$ are at the nodes previously labelled by $x_1,\dots,x_m$ respectively (see Figure 1.1(b)). Usually we shall label only the nodes of the diagram of a formation tree, and the edges will be understood to be labelled by the above convention.

For a diagram of the hyperidentity ϵ_1: $\underline{G}(\underline{F}(x,y),y) = \underline{F}(\underline{G}(x,y),y)$ see Figure 1.2 on page 12. When considering a hyperpolynomial $\underline{P}$ of type $\langle n \rangle$, we shall label only the leaves, and the remaining nodes will be understood to be labelled by the operation symbol of $\underline{P}$ (see Figure 1.4, page 15).

If α is a hyperidentity, and H is a set of hyperidentities, $H \vdash \alpha$ will denote that α is a consequence of H, that is, whenever an algebra **A** satisfies H then the algebra **A** will also satisfy α. A set H of hyperidentities (identities) is said to be <u>closed</u> if $H \vdash \alpha$ implies that $\alpha \in H$ and if H is any set of hyperidentities, [H] will denote the smallest closed set of hyperidentities containing H. Birkhoff's rules of proof for identities (see G. Grätzer [12], page 170) have the following obvious extension to clones.

<u>Lemma 1.3</u>. Let α_c be a clone identity, H_c a set of clone identites and let the variables of sort i be x^i_j, $1 \leq i,j < \omega$. Then $H_c \vdash \alpha_c$ iff α_c is a consequence of H_c by the following five rules of proof (where P, Q, R, P_i, and Q_i are clone polynomials):

(i) $x^i_j = x^i_j$ for all $i,j < \omega$,

(ii) $(P = Q) \vdash (Q = P)$,

(iii) $(P = Q \text{ and } Q = R) \vdash (P = R)$,

(iv) If $P_1 = Q_1, \ldots, P_{n+1} = Q_{n+1}$, where the codomain of P_1 and Q_1 is $F_n(V)$ and the codomain of $P_2, Q_2, \ldots, P_{n+1}, Q_{n+1}$, is $F_m(V)$ then $C_m^n(P_1, P_2, \ldots, P_{n+1}) = C_m^n(Q_1, Q_2, \ldots, Q_{n+1})$,

(v) If the variable x_i^n occurs in $P = Q$ and if the clone identity $P' = Q'$ is obtained by replacing every occurrence of x_i^n in $P = Q$ by a clone polynomial R with codomain $F_n(V)$, then $(P = Q) \vdash (P' = Q')$.

If for hyperidentities ϵ_h and α_h we now (following W. Taylor [25]) say that $\epsilon_h \vdash \alpha_h$ whenever for the corresponding clone identities ϵ_c and α_c we have $\epsilon_c \vdash \alpha_c$, then we obviously have a complete set of rules of proof for identities. The following example illustrates Lemma 1.3(iv):

<u>Example 1.1</u>. If we are given the following hyperidentites:

ϵ_1: $\underline{G}(\underline{F}(x,y),y) = \underline{F}(\underline{G}(x,y),y)$,

ϵ_2: $\underline{H}(x,x) = x$,

ϵ_3: $\underline{F}(\underline{F}(x,y),z) = \underline{F}(x,\underline{F}(y,z))$,

ϵ_4: $\underline{G}\{\underline{F}[\underline{H}(x,x),\underline{F}(\underline{F}(x,y),z)],\underline{F}(\underline{F}(x,y),z)\} =$
$\underline{F}\{\underline{G}[x,\underline{F}(x,\underline{F}(y,z))],\underline{F}(x,\underline{F}(y,z))\}$,

and if we let $\underline{P}_1 = \underline{G}(\underline{F}(x,y),y)$, $\underline{Q}_1 = \underline{F}(\underline{G}(x,y),y)$, $\underline{P}_2 = \underline{H}(x,x)$, $\underline{Q}_2 = x$, $\underline{P}_3 = \underline{F}(\underline{F}(x,y),z)$, and $\underline{Q}_3 = \underline{F}(x,\underline{F}(y,z))$ with $P_1, Q_1, P_2, Q_2, P_3, Q_3$ the corresponding clone polynomials, then by Lemma 1.3(iv), we have $C_3^2(P_1, P_2, P_3) = C_3^2(Q_1, Q_2, Q_3)$, which implies that $\{\epsilon_1, \epsilon_2, \epsilon_3\} \vdash \epsilon_4$ (see Figure 1.2).

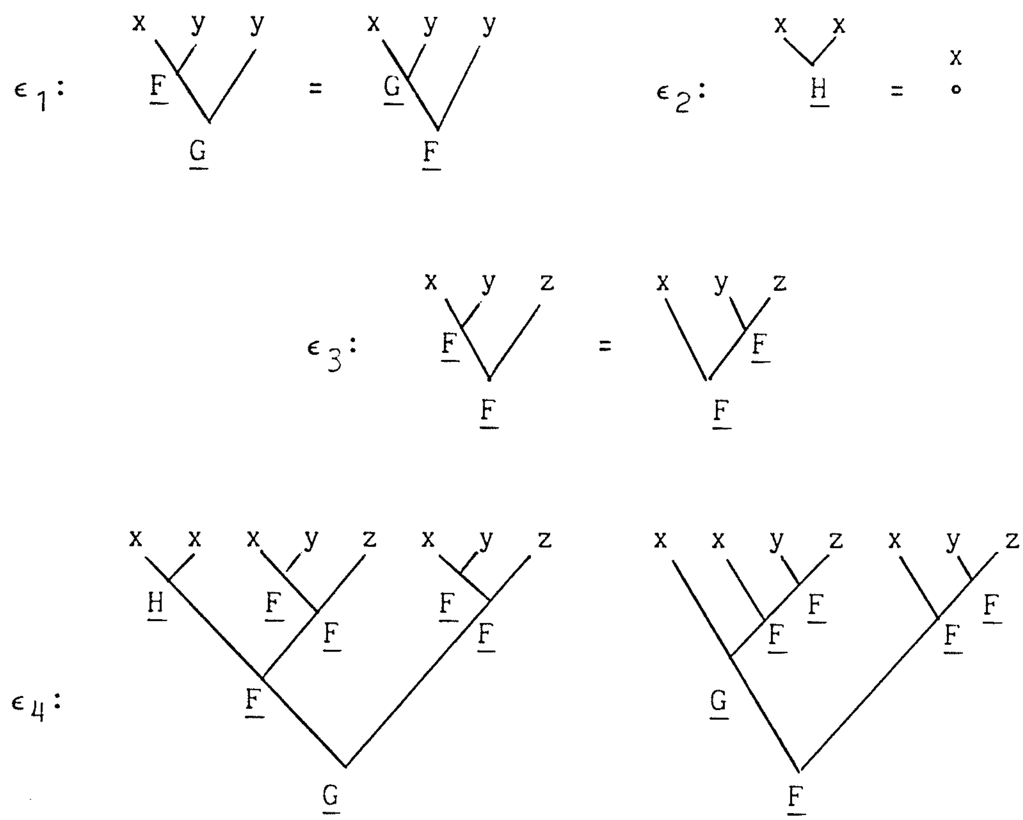


Figure 1.2

It can easily be seen that if H is a set of hyperidentities and ϵ is a hyperidentity then the following two statements are equivalent:

(i) H $\vdash \epsilon$ follows from (i) to (iv) of the above rules of proof;

(ii) If we view H as a set of identities and ϵ as an identity then H $\vdash \epsilon$.

As can be seen in the example below, Lemma 1.3(v) will in general yield hyperidentities which, when viewed as identities, do not follow from H viewed as a set of idenities.

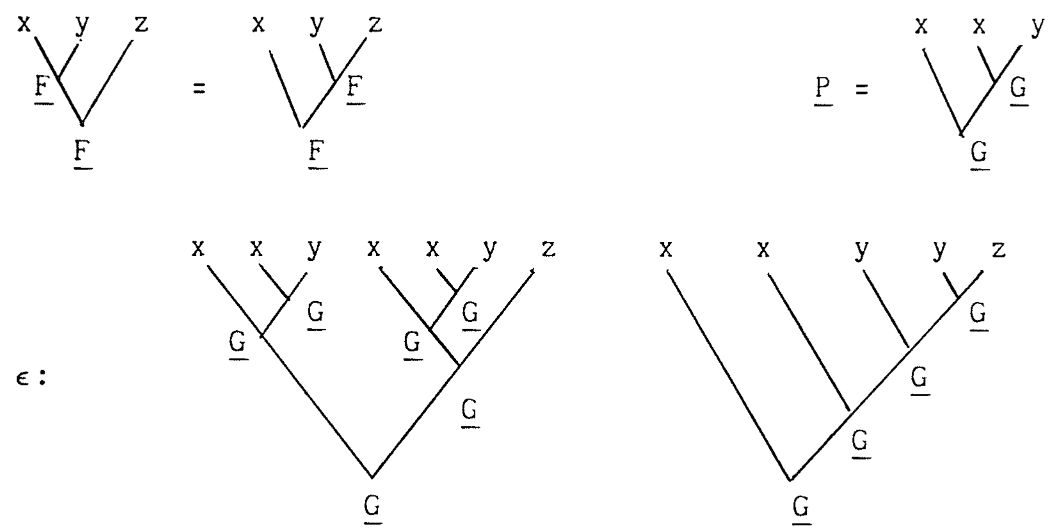


Figure 1.3

Example 1.2. If $H = \{\underline{F}(\underline{F}(x,y),z) = \underline{F}(x,\underline{F}(y,z))\}$, and if we substitute the hyperpolynomial $\underline{P} = \underline{G}(x,\underline{G}(x,y))$ for the symbol $\underline{F}$ we obtain (see Figure 1.3)

$$\epsilon:\ \underline{G}[\underline{G}(x,\underline{G}(x,y),\underline{G}\{\underline{G}(x,\underline{G}(x,y)),z\}] = \underline{G}[x,\underline{G}\{x,\underline{G}(y,\underline{G}(y,z))\}].$$

This example is much more instructive if we use infix notation, that is, if we replace $\underline{F}$ by $\underline{\circ}$ to obtain $H = \{(x\underline{\circ}y)\underline{\circ}z = x\underline{\circ}(y\underline{\circ}z)\}$ and then substitute the polynomial $x^2\underline{\circ}y$ for the symbol $\underline{\circ}$. Letting x^2 denote $x\underline{\circ}x$ and using associativity, we obtain the hyperidentity

$$\epsilon:\ (x^2\underline{\circ}y)^2\underline{\circ}z = x^2\underline{\circ}y^2\underline{\circ}z$$

which, as an identity, would clearly not be a consequence of H when viewing H as set of identities.

The following two lemmas are immediate consequences of Birkhofff's Rules of Proof.

Lemma 1.4. Let H be a closed set of hyperidentities and let the operations of an algebra A satisfy H. Then A satisfies H.

Proof. Let ($\underline{P} = \underline{Q}$) ϵ H, with $\underline{P}$ and $\underline{Q}$ hyperpolynomials in $\underline{F}_1,\ldots,\underline{F}_n$ and let $p_1,\ldots,p_n$ be polynomials of corresponding arity in the operations $f_1,\ldots,f_m$ of A. The identity

$$\alpha\colon \underline{P}(p_1,\ldots,p_n) = \underline{Q}(p_1,\ldots,p_n)$$

if viewed as a hyperidentity α_h in the symbols $\underline{f}_1,\ldots,\underline{f}_m$, is also in H by (v) of Birkhoff's Rules of Proof. The operations of A thus satisfy α_h and α is thus an identity of A, completing the proof of the lemma.

Lemma 1.5. Let V be any variety and let $\alpha \in H(V)$ be a hyperidentity with operation symbols of arity at most n. Then $H^n(V) \vdash \alpha$.

Proof. Let $\alpha \in H(V)$ be a hyperidentity in the symbols $\underline{F}_1,\ldots,\underline{F}_m$, such that the arity of $\underline{F}_i$, $1 \le i \le m$, is at most n. By Lemma 1.3(v), if $\underline{F}_i$ is of arity k, $k < n$, then $\underline{F}_i$ can be replaced by $\underline{G}_i(x_1,\ldots,x_k,\ldots,x_k)$, where $\underline{G}_i$ is an n-ary operation symbol. We thus obtain an n-ary hyperidentity α' which holds in V iff α holds in V, completing the proof.

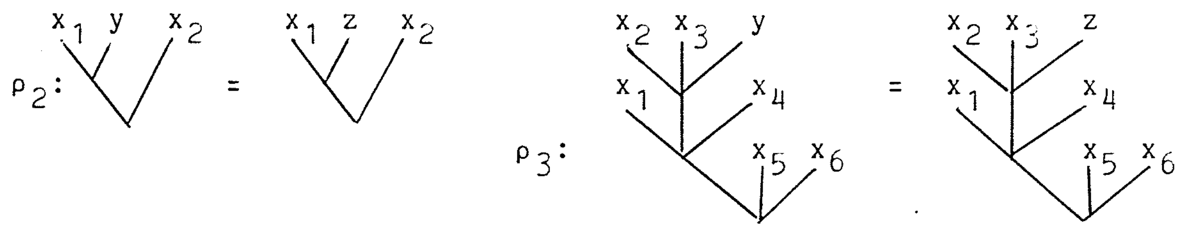


Figure 1.4

In [15], W. D. Neumann gave two examples of hypervarieties, the class of idempotent varieties (that is, those varieties satisfying $\underline{F}(x,x) = x$), and the commutative varieties (those satisfying the median hyperidentities, $\alpha\langle m,n\rangle$, $1 \le m,n < \omega$), and asked whether the lattice of hyperidentities is possibly quite sparse. W. Taylor [24] showed that there are $2^{\aleph_0}$ hypervarieties. In [25], Taylor gives two more proofs of this result. The first proof (in [25]) consists of showing that the $2^{\aleph_0}$ self-dual lattice varieties generate distinct hypervarieties. The idea behind the hyperidentities of the second proof occurs frequently in this thesis, so we give a brief outline of this proof.

First we define ρ_i, $1 < i < \omega$, as follows:

ρ_2: $\underline{F}(\underline{F}(x_1,y),x_2) = \underline{F}(\underline{F}(x_1,z),x_2)$

ρ_3: $\underline{F}(\underline{F}(x_1,\underline{F}(x_2,x_3,y),x_4),x_5,x_6) =$
$\underline{F}(\underline{F}(x_1,\underline{F}(x_2,x_3,z),x_4),x_5,x_6)$

etc. (See Figure 1.4.)

For $k \ge 2$ we let $p_k(V)$ denote the number of polynomials of V which depend on exactly k variables. (For a general theory of the sequence of numbers $p_0, p_1, \ldots$, see G. Grätzer [10].) We say $g(x_1,..,x_i,..,x_k)$ depends on x_i if $g(x_1,..,x_i,..,x_k) = g(x_1,..,x_i',..,x_k)$ does not hold in V. It is easily seen that if $p_k(V) = 0$ then V satisfies the hyperidentity ρ_k. The converse is false since a product of k non-trivial varieties will never satisfy $p_k(V) = 0$. (For example, if A is an algebra with $|A| > 1$, then $\langle A^k;\ (e_1^k,..,e_k^k)\rangle$ has an essentially k-ary operation.) If we now let $H_p = \{\rho_{k+1} \mid k \geq 2,\ k \text{ prime}\}$ then H_p is irredundant, that is, ρ_{k+1} is not a consequence of $H_p - \{\rho_{k+1}\}$ for any prime k, $k \geq 2$. The irredundancy of H_p is proved by exhibiting an algebra A which satisfies $H_p - \{\rho_{k+1}\}$ but not ρ_{k+1}.

Another interesting result of W. Taylor [25] is that R, the variety of rings, is generic. That is, if ϵ is a nontrivial hyperidentity, then ϵ is not satisfied by R.

The remaining part of this chapter will consist of some known results on hyperidentities of G, the variety of all groups, and A, the variety of all abelian groups. The median hyperidentities $\alpha\langle m,n\rangle$ are examples of abelian group hyperidentities. Another example of a member of H(A) is the hyperidentity ρ:

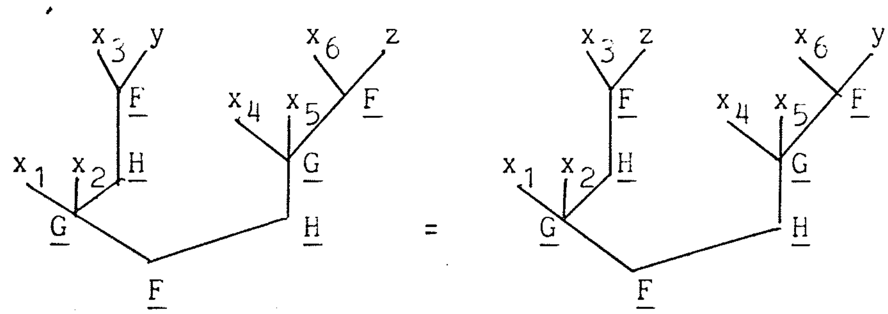


Figure 1.5

$$\rho: \underline{F}(\underline{G}[x_1,x_2,\underline{H}(\underline{F}(x_3,y))],\underline{H}[\underline{G}(x_4,x_5,\underline{F}(x_6,z))]) =$$
$$\underline{F}(\underline{G}[x_1,x_2,\underline{H}(\underline{F}(x_3,z))],\underline{H}[\underline{G}(x_4,x_5,\underline{F}(x_6,y))]),$$

depicted in Figure 1.5. To verify, for example, that $\rho \in H(A)$, note that in A we have $F(x,y) = x^m y^n$, $G(x,y,z) = x^p y^q z^r$ and $H(x) = x^s$, m,n,p,q,r,s integers. It is now an easy exercise to prove that the resulting identity will hold in A.

In order to give a syntactic characterization of H(A) we need a few definitions. Let $\underline{P}$ be a hyperpolynomial containing the operation symbols $\underline{F}_1,\ldots,\underline{F}_m$. Recall that if $\underline{F}_i$ is an n-ary symbol, then the edges leaving any node labelled by $\underline{F}_i$ are labelled $(\underline{F}_i,1),\ldots,(\underline{F}_i,n)$ starting from the left. If x is a variable of $\underline{P}$, then <u>the string terminating with x</u> will be defined to be the sequence whose terms are in order, the labels of the edges leading from the root of $\underline{P}$ to the variable x, and the variable x. For example, if $\underline{P}$ is the left hand side of the above hyperidentity ρ, then $\langle(\underline{F},1),(\underline{G},3),(\underline{H},1),(\underline{F},2),y\rangle$ is the string terminating with y. There will be exactly one string terminating with x for each occurrence of x as a variable of $\underline{P}$. We shall let $st_P(x)$ denote the set of all strings of $\underline{P}$ terminating with the variable x and $st(\underline{P})$ will denote the set of all strings of $\underline{P}$. If two strings α_1 and α_2 are the same except for the order of the terms, that is, the sequence α_1 is a rearrangement of the sequence α_2, we shall say that α_1 and α_2 are <u>balanced equal</u> and shall write $\alpha_1 =_b \alpha_2$. For example, $\langle(\underline{F},1),(\underline{G},3),(\underline{F},2),x\rangle =_b \langle(\underline{F},2),(\underline{F},1),(\underline{G},3),x\rangle$, but $\langle(\underline{F},1),(\underline{F},1),y\rangle \neq_b \langle(\underline{F},1),y\rangle$ and $\langle(\underline{F},1),y\rangle \neq_b \langle(\underline{F},1),z\rangle$.

The following characterization of H(A) is due to W. Taylor [25]:

<u>Lemma 1.6</u>. Let $\underline{P} = \underline{Q}$ be a hyperidentity. Then $(\underline{P} = \underline{Q}) \in H(A)$ iff there exists a bijection $f: st(\underline{P}) \to st(\underline{Q})$ such that $\alpha =_b f(\alpha)$ for all $\alpha \in st(\underline{P})$.

Since in G, the variety of all groups, any group generated by a single element is abelian, we have $H_1(A) \subseteq H(G)$. In G. M. Bergman [5], it is proved that $H_1(A)$ is in fact a basis for H(G) and that H(G) = H(M) = H(MA), where M is the variety of all monoids and MA (metabelian groups) is the subvariety of G satisfying ((x,y),(u,v)) = 1. It is also shown that H(G) is not finitely based but that H(G) follows from (but is not equivalent to) the existence of a nullary operation, two hyperidentities and a "hyper-Horn-sentence". The reader who is interested in hyperidentities of other interesting varieties such as module varieties, k'th power varieties and varieties generated by some quasiprimal algebras is referred to W. Taylor [25].

CHAPTER II

BINARY HYPERIDENTITIES OF SEMILATTICES[1]

In this chapter, we shall introduce binary semilattice hyperidentities, exhibit a finite basis Σ^2 for $H^2(SL)$, the set of all binary semilattice hyperidentities, and prove structure theorems for algebras of type <2> or <2,2> which satisfy H(SL).

However, before discussing $H^2(SL)$, it should be instructive to have a brief look at H(DS), where DS is the variety of diagonal semigroups (rectangular bands). That is, DS is defined by the following three identities:

$$x + x = x,$$
$$x + (y + z) = (x + y) + z,$$
$$x + y + z = x + z.$$

It is known (see R. A. Knoebel [13]) that DS is generated by the two classes P_l and P_r of right and left binary projection algebras (P_l is the class of all algebras $A = \langle A; x_l \rangle$ of type <2>, where x_l is the left projection, and P_r is the class of all algebras $A = \langle A; x_r \rangle$ of type <2>, where x_r is the right projection). We shall say that the hyperidentity

[1]The content of this chapter is due jointly to R. Padmanabhan and the author [17].

$\underline{P} = \underline{Q}$ in the operation symbols $\underline{F}_1,\ldots,\underline{F}_n$, models projections if whenever $\underline{F}_1,\ldots,\underline{F}_n$ are replaced by any choice of projections we obtain the trivial identity $x_i = x_i$, for some $i < \omega$. Obviously, this is a necessary condition for $\underline{P} = \underline{Q}$ to be satisfied by any variety V. The set of all hyperidentities modelling projections will be denoted by H_p.

Proposition 2.1. $H(DS) = H_p$.

Proof. By the above discussion $H(DS) \subseteq H_p$. Since the only polynomials of a binary projection algebra are projections, H_p is satisfied by P_l and P_r, and thus by DS. We thus have $H_p \subseteq H(DS)$, completing the proof of the proposition.

In this chapter we shall use infix notation for binary hyperpolynomials, that is, the hyperpolynomial $\underline{F}(\underline{G}(x,y),y)$ will be written $(x\ \underline{\lambda}\ y)\ \underline{\mu}\ y$. Let $\underline{\lambda}_i$, $i \in I$, be a countable set of binary operation symbols. Let us call this type $\underline{2}$, that is, $\underline{2}$ is just an abreviation for the infinite vector $\langle 2,2,2,\ldots\rangle$. We define inductively a class of syntactical transforms for hyperpolynomials of type $\underline{2}$:

$$L_i(x) = R_i(x) = S_{ij}(x) = x \quad \text{for all variables } x;$$
$$L_i(\underline{P}\underline{\lambda}_j\underline{G}) = L_i(\underline{P}) \quad \text{if } j = i,$$
$$= (L_i(\underline{P}))\underline{\lambda}_j(L_i(\underline{G})) \quad \text{otherwise;}$$
$$R_i(\underline{P}\underline{\lambda}_j\underline{G}) = R_i(\underline{G}) \quad \text{if } j = i,$$
$$= (R_i(\underline{P}))\underline{\lambda}_j(R_i(\underline{G})) \quad \text{otherwise;}$$

$$S_{ij}(\underline{P}\underline{\lambda}_k\underline{G}) = (S_{ij}(\underline{P}))\underline{\lambda}_i(S_{ij}(\underline{G})) \quad \text{if } k=j,$$
$$= (S_{ij}(\underline{P}))\underline{\lambda}_k(S_{ij}(\underline{G})) \quad \text{otherwise.}$$

Thus the transform S_{ij} substitutes $\underline{\lambda}_i$ for $\underline{\lambda}_j$ for every occurrence of $\underline{\lambda}_j$ and the transform L_i (R_i) replaces each occurrence of the operation $\underline{\lambda}_i$ by the left (right) projection. If the hyperpolynomial $\underline{P}$ contains the operation symbols $\underline{\lambda}$ and $\underline{\mu}$, we shall write $S_{\lambda\mu}(\underline{P})$, $L_\lambda(\underline{P})$, etc., for the above transforms. If $\underline{P} = \underline{G}$ is a hyperidentity of type $\underline{\varepsilon}$, then for $T \in \{S_{ij}, L_i, R_i : i,j \in I\}$, we define $T(\underline{P} = \underline{G})$ to be the hyperidentity $T(\underline{P}) = T(\underline{G})$. A class H of hyperidentities is closed for a transform T iff whenever $(\underline{P} = \underline{G}) \in H$ then $T(\underline{P} = \underline{G}) \in H$.

The class $H^2(DS)$ is thus the largest class of binary hyperidentities closed under $\{L_i, R_i \mid i \in I\}$. Note however, that many of the members of $H^2(DS)$ are not members of H(SL). For example,

$$\rho: x\underline{\lambda}(y\underline{\lambda}z) = x\underline{\lambda}z$$

is satisfied by DS since it models projections, but when $\underline{\lambda}$ is replaced by the binary semilattice polynomial $x \vee y$, we obtain the identity $x \vee y \vee z = x \vee z$, which is not regular, and thus not satisfied by SL. To insure regularity we need the transforms S_{ij} and the reader can easily verify that $H^2(SL)$ is exactly the largest subset of $H^2(DS)$ which is closed under $\{S_{ij}, R_i, L_i \mid i,j \in I\}$.

Examples of semilattice hyperidentities are the median hyperidentities

$$\alpha\langle m,n\rangle: \quad \underline{F}(\underline{G}(x_{1,1},..,x_{1,n}),...,\underline{G}(x_{m,1},..,x_{m,n})) = \underline{G}(\underline{F}(x_{1,1},..,x_{m,1}),...,\underline{F}(x_{1,n},..,x_{m,n})).$$

Recall that verifying that $\alpha\langle 3,3\rangle$ is a semilattice hyperidentity consists of a case by case examination of

$$(\underline{F},\underline{G}) \in \{x,\ y,\ z,\ x \vee y,\ x \vee z,\ y \vee z,\ x \vee y \vee z\}^2.$$

The median hyperidentities lead to an easy proof (similar to the one used by G. M. Bergman [5] to prove that the hyperidentities of groups are not finitely based) that H(SL) and H(DS) are not finitely based.

Lemma 2.2. H(SL) and H(DS) are not finitely based.

Proof. For a given integer n, we construct an algebra **A** which satisfies $H^m(SL)$ and $H^m(DS)$, $m < n$, but neither $H^n(SL)$ nor $H^n(DS)$. Let A be the set consisting of all subwords of the expression

$$P = f(f(x_{1,1},x_{1,2},...,x_{1,n}),...,f(x_{n,1},x_{n,2},...,x_{n,n})).$$

We define an n-ary function f on A in the natural way where possible (that is, whenever $f(x_1,...,x_n)$ is a subword of P), and we set $f(x_1,..,x_n) = x_1$ otherwise. It is easily seen that the algebra **A** = <A; f> satisfies all diagonal semigroup (semilattice) hyperidentities of arity less than n, since all polynomials of arity less than n will be equal to projections. However, if the symbols $\underline{F}$ and $\underline{G}$ of $\alpha\langle n,n\rangle$ are replaced by the polynomial $f(x_1,\ldots,x_n)$, we obtain the expression $P = f(x_{11},x_{12},\ldots,x_{1n})$, so A does not satisfy the n-ary median hyperidentity $\alpha\langle n,n\rangle$. Since any finite basis of H(SL) or H(DS) would be of bounded arity, this completes the proof of the lemma.

In Chapter Four, we shall show that H(V) is not finitely based where V is any nontrivial variety of lattices or the variety of semilattices. For semilattices, however, we shall also prove that for any fixed integer n, $H^n(SL)$ is finitely based. In order to help the reader visualize the behaviour of semilattice hyperidenties, we shall in the remainder of this chapter, using the familiar infix notation, exhibit a basis Σ^2 for $H^2(SL)$ and also prove some structure theorems for algebras of type <2> and <2,2> which satisfy H(SL). In Chapter Three, we shall prove analogous results for $H^n(SL)$. With the exception of Theorem 2.14, the remaining part of this chapter is thus contained in the more general setting of Chapter Four.

By $\Sigma\langle 2\rangle$ we denote the set of three hyperidentities

$$x\underline{\lambda}x = x,$$

$$x\underline{\lambda}(y\underline{\lambda}z) = (x\underline{\lambda}y)\underline{\lambda}z,$$

$$(x\underline{\lambda}y)\underline{\lambda}(z\underline{\lambda}t) = (x\underline{\lambda}z)\underline{\lambda}(y\underline{\lambda}t).$$

we shall be talking about two classes of semilattice hyperidentities, the class H(SL)<2> of all semilattice hyperidentities of type <2>, and the class of all (semilattice) hyperidentities of type <2> which are a consequence of Σ<2>. For the sake of clarity, if $\underline{P} = \underline{Q}$ is a semilattice hyperidentity which is a consequence of Σ<2>, we shall write $\underline{P} \Leftrightarrow \underline{Q}$. The following lemma thus states that if $\underline{P} = \underline{Q}$ is a semilattice hyperidentity of type <2> then $\underline{P} \Leftrightarrow \underline{Q}$.

Lemma 2.3. The set Σ<2> of three hyperidentities is a basis for the set H(SL)<2>.

Proof. Certainly Σ<2> is a subset of H(SL)<2>. Let $\underline{P}(x_1, x_2, \ldots x_n)$ be any hyperpolynomial with variables $\{x_1, x_2, \ldots, x_n\}$ and the operation symbol $\underline{\lambda}$. Let x_1 and x_n, respectively, be the first and last variables of $\underline{P}$. Then it is easy to see the validity of the hyperidentity

$$\underline{P}(x_1, x_2, \ldots, x_n) \Leftrightarrow x_1 \underline{\lambda} x_2 \underline{\lambda} \ldots \underline{\lambda} x_{n-1} \underline{\lambda} x_n .$$

For, by the associative law we can rearrange all the parenthesis and by the median law we can rearrange all the inside variables so that we can pool all the x_1's together, then all the x_2's together and so on. Finally, by the idempotent law, just one x_i will remain for each $i=1,2,\ldots,n$ yielding precisely the hyperpolynomial on the right hand side. If $\underline{P} = \underline{Q}$ is an arbitrary semilattice hyperidentity of type <2>, then it is regular, $L_\lambda \underline{P} = L_\lambda \underline{Q}$ and $R_\lambda \underline{P} = R_\lambda \underline{Q}$, so we have $\underline{P} \Leftrightarrow \underline{Q}$.

Corollary 2.4. The set $\Sigma\langle 2\rangle$ when viewed as a set of identities will make the binary polynomial $p(x,y) = x\lambda y\lambda x$ a partition function in the sense of J. Płonka [20]. This means that $\Sigma\langle 2\rangle$, viewed as a set of identities, has the following identities as consequences (see Lemma 1 of [21]):

(i) $p(x,x) = x$,

(ii) $p(x,p(y,z)) = p(x,p(z,y))$,

(iii) $p(x,p(y,z)) = p(p(x,y),z)$,

(iv) $p(x\lambda y,z) = x\lambda p(y,z)$,

(v) $p(x\lambda y,z) = p(x,z)\lambda y$,

(vi) $p(x,p(y,x\lambda y)) = p(x,y)$.

Proof. All the identities (i) to (vi) are regular and have the same first and last variable on either side. Hence, since the proof of Lemma 2.3 is also valid when $\Sigma\langle 2\rangle$ and $H(SL)\langle 2\rangle$ are viewed as sets of identities, they all are consequences of $\Sigma\langle 2\rangle$.

Theorem 2.5. For an algebra $A = \langle A;\lambda\rangle$ of type $\langle 2\rangle$ the following are equivalent:

(i) The operation of A satisfies $\Sigma\langle 2\rangle$, that is, A satisfies $\Sigma\langle 2\rangle$ as a set of identities;

(ii) A is a Płonka sum of diagonal semigroups (see [14] for the definition of this concept);

(iii) $A \in SL \vee DS$;

(iv) A satisfies all the semilattice hyperidentities.

Proof. (i) implies (ii). By Corollary 2.4, $p(x,y) = x\lambda y\lambda x$ is a partition function for **A** and hence **A** is a Płonka sum of algebras satisfying $\sum\langle 2\rangle$ and further, the identity $p(x,y) = x$. Now

$$x\lambda y\lambda z = x\lambda z\lambda y\lambda z \quad \text{by } \textstyle\sum\langle 2\rangle \text{ viewed as a set of identities}$$
$$= x\lambda z \quad \text{since } z\lambda y\lambda z = z$$

and hence **A** is a Plonka sum of diagonal semigroups.

(ii) implies (iii). Since DS satisfies the identity $x\lambda y\lambda x = x$, by the result of J. Płonka [20], the variety SL $\vee$ DS defined by the regular identities of diagonal semigroups, is precisely the Płonka sum of diagonal semigroups and hence $A \in$ SL $\vee$ DS if A satisfies (ii).

(iii) implies (iv). Id(SL $\vee$ DS) = Id(SL) $\cap$ Id(DS). Now "being regular", "having the same L_λ" and "having the same R_λ" are syntactical properties of identities which remain invariant under equational consequences (see Birkhoff's rules of proof [12]). Hence Id(SL) $\cap$ Id(DS) has all the three syntactical properties and is obviously the largest such class of identities. Hence, (iii) implies (iv) by Lemma 1.4.

Since all the members of $\sum\langle 2\rangle$ are semilattice hyperidentities of type $\langle 2\rangle$, clearly (iv) implies (i). The proof of the theorem is complete.

Recall that a semilattice hyperidentity $\underline{P} = \underline{Q}$ of type $\langle 2,2\rangle$ is an identity $\underline{P} = \underline{Q}$ in the language of two binary operation symbols, say $\underline{\lambda}$ and $\underline{\mu}$, such that $\underline{P} = \underline{Q}$ is valid in every semilattice $\mathbf{A} = \langle A;\vee\rangle$ where $\underline{\lambda}$ and $\underline{\mu}$ are to be interpreted as arbitrary binary polynomnials of the semilattice $\mathbf{A}$. Of course, x, y, $x \vee y$ are the only binary polynomials of any semilattice $\langle A; \vee\rangle$. Hence, $H(SL)\langle 2,2\rangle$, the class of all semilattice hyperidentities of type $\langle 2,2\rangle$, is simply the largest class of regular identities $\underline{P} = \underline{Q}$ of type $\langle 2,2\rangle$ such that $L_\lambda\underline{P} = L_\lambda\underline{Q}$, $R_\lambda\underline{P} = R_\lambda\underline{Q}$, $L_\mu\underline{P} = L_\mu\underline{Q}$ and $R_\mu\underline{P} = R_\mu\underline{Q}$ are in $H(SL)\langle 2\rangle$. By Σ^2 we mean the set of four hyperidentities

$$\Sigma\langle 2\rangle \cup \{(x\underline{\lambda}y)\underline{\mu}(z\underline{\lambda}t) = (x\underline{\mu}z)\underline{\lambda}(y\underline{\mu}t)\}.$$

In the remainder of this chapter, if $\underline{P} = \underline{Q}$ is a consequence of Σ^2 we shall write $\underline{P} \Leftrightarrow \underline{Q}$. We can write Σ^2 even more compactly as

$$x\underline{\lambda}x \Leftrightarrow x,$$
$$x\underline{\lambda}(y\underline{\lambda}z) \Leftrightarrow (x\underline{\lambda}y)\underline{\lambda}z,$$
$$(x\underline{\lambda}y)\underline{\mu}(z\underline{\lambda}t) \Leftrightarrow (x\underline{\mu}z)\underline{\lambda}(y\underline{\mu}t).$$

To get the median law of type $\langle 2\rangle$ we simply "put $\underline{\lambda} = \underline{\mu}$" and it is this concept we abstracted as $S_{\lambda\mu}$.

<u>Theorem 2.6</u>. The set Σ^2 is an equational basis for the set $H(SL)\langle 2,2\rangle$.

<u>Remark</u>. This theorem can be construed as providing an

algorithmic solution to the word problem for the variety of type <2,2> defined by Σ^2. For the theorem says that a hyperidentity $\underline{P} = \underline{Q}$ in the operation symbols $\underline{\lambda}$ and $\underline{\mu}$ is a consequence of Σ^2 iff it is regular, $L_{\underline{\alpha}}\underline{P} \Leftrightarrow L_{\underline{\alpha}}\underline{Q}$ and $R_{\underline{\alpha}}\underline{P} \Leftrightarrow R_{\underline{\alpha}}\underline{Q}$ for all $\underline{\alpha} \in \{\underline{\lambda},\underline{\mu}\}$. We first establish a few lemmas before proving the theorem. For the sake of convenience we shall use the operation symbols "$\underline{\lambda}$" and "$\underline{\cdot}$", instead of "$\underline{\lambda}$" and "$\underline{\mu}$", with the convention that $xy \ \underline{\lambda}\ zt$ means $(x\underline{\cdot}y)\underline{\lambda}(z\underline{\cdot}t)$, and if $X = \{x_1,x_2,\ldots,x_n\}$ we shall write πX for the product $x_1x_2\ldots x_n$.

Lemma 2.7. Σ^2 implies

$$x \ \underline{\lambda}\ yt \ \underline{\lambda}\ zu \Leftrightarrow x \ \underline{\lambda}\ yzt \ \underline{\lambda}\ zu,$$

$$xz \ \underline{\lambda}\ yt \ \underline{\lambda}\ u \Leftrightarrow xz \ \underline{\lambda}\ yzt \ \underline{\lambda}\ u.$$

Proof. First we note that the distributive law

$$x(y\underline{\lambda}z) \Leftrightarrow xy \ \underline{\lambda}\ xz$$

is a consequence of Σ^2 since:

$x(y\underline{\lambda}z)$	$\Leftrightarrow (x\underline{\lambda}x)(y\underline{\lambda}z)$	idempotence
	$\Leftrightarrow xy \ \underline{\lambda}\ xz$	median law.

Now, $x \ \underline{\lambda}\ yt \ \underline{\lambda}\ zu$	$\Leftrightarrow x \ \underline{\lambda}\ (y\underline{\lambda}z)(t\underline{\lambda}u)$	median law
	$\Leftrightarrow (x\underline{\lambda}y\underline{\lambda}z)(x\underline{\lambda}t\underline{\lambda}u)$	distributivity
	$\Leftrightarrow (x\underline{\lambda}y)x \ \underline{\lambda}\ z(t\underline{\lambda}u)$	median law
	$\Leftrightarrow ((x\underline{\lambda}y)x(xx) \ \underline{\lambda}\ (zz)z(t\underline{\lambda}u)$	idempotence

$$\Leftrightarrow ((x\underline{\lambda}y)x \ \underline{\lambda} \ zz)(xx \ \underline{\lambda} \ z(t\underline{\lambda}u)) \qquad \text{median law}$$

$$\Leftrightarrow (x\underline{\lambda}y\underline{\lambda}z)(x\underline{\lambda}z)(x\underline{\lambda}z)(x\underline{\lambda}t\underline{\lambda}u) \qquad \text{median law}$$

$$\Leftrightarrow (x\underline{\lambda}y\underline{\lambda}z)(x\underline{\lambda}z)(x\underline{\lambda}t\underline{\lambda}u) \qquad \text{idempotence}$$

$$\Leftrightarrow x \ \underline{\lambda} \ ((y\underline{\lambda}z)z(t\underline{\lambda}u)) \qquad \text{distributivity}$$

$$\Leftrightarrow x \ \underline{\lambda} \ (yz\underline{\lambda}z)(t\underline{\lambda}u) \qquad \text{distributivity}$$

$$\Leftrightarrow x \ \underline{\lambda} \ yzt \ \underline{\lambda} \ zu \qquad \text{median law.}$$

The second hyperidentity follows in a similar manner, and this completes the proof of the lemma.

Note that the hyperidentity

$$x \ \underline{\lambda} \ yzt \ \underline{\lambda} \ z \Leftrightarrow x \ \underline{\lambda} \ yt \ \underline{\lambda} \ z$$

is an immediate consequence of Lemma 2.7 if we let $u = z$.

<u>Lemma 2.8</u>. If $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$ then Σ^2 implies

$$\Pi X \ \underline{\lambda} \ zt \ \underline{\lambda} \ \Pi Y \Leftrightarrow \Pi X \ \underline{\lambda} \ zx_ht \ \underline{\lambda} \ \Pi Y, \ h \in \{1,2,\ldots,n\},$$

$$\Pi X \ \underline{\lambda} \ zt \ \underline{\lambda} \ \Pi Y \Leftrightarrow \Pi X \ \underline{\lambda} \ zy_ht \ \underline{\lambda} \ \Pi Y, \ h \in \{1,2,\ldots,m\}.$$

<u>Proof</u>. If $h \neq 1$ or m then by Lemma 2.7

$$\Pi X \ \underline{\lambda} \ zy_ht \ \underline{\lambda} \ \Pi Y \Leftrightarrow \Pi X \ \underline{\lambda} \ zy_ht \ \underline{\lambda} \ (y_1y_2\ldots y_{h-1})y_h(y_{h+1}\ldots y_m)$$

$$\Leftrightarrow \Pi X \ \underline{\lambda} \ z(y_1\ldots y_{h-1})y_ht \ \underline{\lambda} \ (y_1\ldots y_h)(y_{h+1}\ldots y_m)$$

$$\Leftrightarrow \Pi X \ \underline{\lambda} \ zt \ \underline{\lambda} \ \Pi Y.$$

If $h = 1$ the lemma again easily follows from Lemma 2.7, and if $h = m$,

$$\Pi X + zy_h t + \Pi Y \Leftrightarrow \Pi X + zy_h t + (y_1 \ldots y_{m-1})y_m$$
$$\Leftrightarrow \Pi X + z\Pi Y t + \Pi Y$$
$$\Leftrightarrow \Pi X + zt + \Pi Y.$$

The proof for the first hyperidentity is similar, completing the proof of the lemma.

Lemma 2.9. If $\underline{P} = \underline{P}_1 \underline{\lambda} \underline{P}_2 \underline{\lambda} \ldots \underline{\lambda} \underline{P}_n$ and if $\underline{P}_h = x_r z x_s$ and $\underline{P}_k = x_p z x_q$, $1 < h,k < n$, then Σ^2 implies that

$$\underline{P} \Leftrightarrow \underline{P}_1 \underline{\lambda} \underline{P}_2 \underline{\lambda} \ldots \underline{\lambda} \underline{P}_k' \underline{\lambda} \ldots \underline{\lambda} \underline{P}_h' \underline{\lambda} \ldots \underline{\lambda} \underline{P}_n$$

where $\underline{P}_h' = x_p z x_s$ and $\underline{P}_k' = x_r z x_q$.

Proof. Due to the median law, we can without loss of generality, assume that $k = h + 1$. Thus we have

$$\underline{P} \Leftrightarrow \underline{P}_1 \underline{\lambda} \ldots \underline{\lambda} x_r z x_s \underline{\lambda} x_p z x_q \underline{\lambda} \ldots \underline{\lambda} \underline{P}_n$$

$$\Leftrightarrow \underline{P}_1 \underline{\lambda} \ldots \underline{\lambda} (x_r z \underline{\lambda} x_p z)(x_s \underline{\lambda} x_q) \underline{\lambda} \ldots \underline{\lambda} \underline{P}_n \qquad \text{median law}$$

$$\Leftrightarrow \underline{P}_1 \underline{\lambda} \ldots \underline{\lambda} x_r z x_s \underline{\lambda} x_r z x_q \underline{\lambda} x_p z x_s \underline{\lambda} x_p z x_q \underline{\lambda} \ldots \underline{\lambda} \underline{P}_n \qquad \text{distributivity}$$

$$\Leftrightarrow \underline{P}_1 \underline{\lambda} \ldots \underline{\lambda} x_r z x_q \underline{\lambda} x_r z x_s \underline{\lambda} x_p z x_q \underline{\lambda} x_p z x_s \underline{\lambda} \ldots \underline{\lambda} \underline{P}_n \qquad \text{median law}$$

$$\Leftrightarrow \underline{P}_1 \underline{\lambda} \ldots \underline{\lambda} (x_r z \underline{\lambda} x_p z)(x_q \underline{\lambda} x_s) \underline{\lambda} \ldots \underline{\lambda} \underline{P}_n \qquad \text{distributivity}$$

$$\Leftrightarrow \underline{P}_1 \underline{\lambda} \ldots \underline{\lambda} x_r z x_q \underline{\lambda} x_p z x_s \underline{\lambda} \ldots \underline{\lambda} \underline{P}_n \qquad \text{median law}$$

completing the proof of the lemma.

Lemma 2.10. If $\underline{P}$ is a hyperpolynomial consisting of two binary operation symbols $\underline{\lambda}$ and $\underline{\cdot}$, and if

$$L_\lambda(\underline{P}) = \underline{P}_1,$$
$$R_\lambda(\underline{P}) = \underline{P}_k,$$
$$X = \mathrm{Var}(\underline{P}),$$

$\mathrm{Var}(L.(\underline{P})) \times \mathrm{Var}(R.(\underline{P})) = \{(a_1,b_1),(a_2,b_2),\ldots,(a_h,b_h)\}$,

then

$$\underline{P} \Leftrightarrow \underline{P}_1 \ \underline{\lambda}\ a_1 \Pi X b_1 \ \underline{\lambda}\ a_2 \Pi X b_2 \ \underline{\lambda} \cdots \underline{\lambda}\ a_h \Pi X b_h \ \underline{\lambda}\ \underline{P}_k.$$

Proof. By distributivity

$$\underline{P} \Leftrightarrow \underline{P}_1 \ \underline{\lambda}\ \underline{P}_2 \ \underline{\lambda} \cdots \underline{\lambda}\ \underline{P}_k$$

where each $\underline{P}_i$, $i \in \{1,2,\ldots,k\}$, is a product of variables, $L_\lambda(\underline{P}) = \underline{P}_1$ and $R_\lambda(\underline{P}) = \underline{P}_k$. By idempotence we can write $\underline{P} \Leftrightarrow \underline{P}_1 \ \underline{\lambda}\ \underline{P}_1 \ \underline{\lambda}\ \underline{P}_2 \ \underline{\lambda}\ \underline{P}_3 \ \underline{\lambda} \cdots \underline{\lambda}\ \underline{P}_{k-1} \ \underline{\lambda}\ \underline{P}_k \ \underline{\lambda}\ \underline{P}_k$, so we will assume without loss of generality that $\underline{P}_1 = \underline{P}_2$ and $\underline{P}_{k-1} = \underline{P}_k$.

If $x_m \in X$ and $x_m \notin \mathrm{Var}(\underline{P}_j)$ for some $j \in \{2,3,\ldots,k-1\}$, then $x_m \in \mathrm{Var}(\underline{P}_i)$ for some i, $1 < i < k$, and by the median law we can write

$$\underline{P} \Leftrightarrow \underline{P}_1 \ \underline{\lambda} \cdots \underline{\lambda}\ \underline{P}_{j-1} \ \underline{\lambda}\ \underline{P}_j \ \underline{\lambda}\ \underline{P}_i \ \underline{\lambda} \cdots \underline{\lambda}\underline{P}_k$$

and by Lemma 2.8 $\underline{P}_{j-1} \ \underline{\lambda}\ \underline{P}_j \ \underline{\lambda}\ \underline{P}_i \Leftrightarrow \underline{P}_{j-1} \ \underline{\lambda}\ \underline{P}_j' \ \underline{\lambda}\ \underline{P}_k$ where $\underline{P}_j'$ is obtained by "inserting" x_m into $\underline{P}_j$. This process can be continued until we obtain

$$\underline{P} <=> \underline{P}_1 \ \underline{\lambda} \ \underline{P}_2' \ \underline{\lambda} \ \underline{P}_3' \ \underline{\lambda} \ \cdots \ \underline{\lambda} \ \underline{P}_{k-1}' \ \underline{\lambda} \ \underline{P}_k$$

where each $\underline{P}_j'$, $j \in \{2,3,\ldots,k-1\}$, is of the form $a\Pi Xb$, $a = L.(\underline{P}_j)$ and $b = R.(\underline{P}_j)$.

If for some $i \in \{1,2,\ldots,h\}$, $a_i\Pi Xb_i$ is not a term of $\underline{P}'$, then there are terms $a_i\Pi Xb_k$ and $a_r\Pi Xb_i$, so by Lemma 2.9 we can "switch" a_i and a_r to obtain a term $a_i\Pi Xb_i$. Then, by idempotence, we can insert a second term $a_i\Pi Xb_i$ into $\underline{P}'$and then again apply Lemma 2.9 to $a_r\Pi Xb_k$ and one of the terms $a_i\Pi Xb_i$, to get back the terms $a_i\Pi Xb_k$ and $a_r\Pi Xb_i$. This process can be continued until we have

$$\underline{P} <=> \underline{P}_1 \ \underline{\lambda} \ a_1\Pi Xb_1 \ \underline{\lambda} \ a_2\Pi Xb_2 \ \underline{\lambda} \ \cdots \ \underline{\lambda} \ \underline{P}_k,$$

completing the proof of the lemma.

Proof of Theorem 2.6. If $\underline{P} = \underline{Q}$ is a semilattice hyper-identity in the two operation symbols $\underline{\lambda}$ and $\underline{\cdot}$, then the following conditions will hold:

(i) $Var(\underline{P}) = Var(\underline{Q})$,

(ii) $(L_\lambda(\underline{P}) = L_\lambda(\underline{Q})) \in H(SL)<2>$,

(iii) $(R_\lambda(\underline{P}) = R_\lambda(\underline{Q})) \in H(SL)<2>$,

(iv) $(L.(\underline{P}) = L.(\underline{Q})) \in H(SL)<2>$ and thus $Var(L.(\underline{P})) = Var(L.(\underline{Q}))$,

(v) $(R.(\underline{P}) = R.(\underline{Q})) \in H(SL)<2>$ and thus $Var(R.(\underline{P})) = Var(R.(\underline{Q}))$.

Conditions (ii) to (v) follow from Lemma 2.3, and thus by Lemma 2.10, $P \Leftrightarrow Q$.

We shall now prove that all binary semilattice hyperidentities are a consequence of Σ^2, and prove a structure theorem for algebras of type <2,2> satisfying these hyperidentities. First, we need to establish a lemma. Let T and P be hyperpolynomials in the symbols λ, $\cdot$, $\mu_1,\ldots,\mu_{n-2}$, such that the variables of T are a subset of the variables of P. If the variables of T' are a subset of the variables of P' whenever T' and P' are obtained from T and P by replacing any subset of $\{\mu_1,\ldots,\mu_{n-2}\}$ by any combination of left and right projections, we shall write $\Pi_\mu(T) < \Pi_\mu(P)$.

Lemma 2.11. Let P be a hyperpolynomial in the operation symbols $\lambda, \cdot, \mu_1,\ldots,\mu_{n-2}$, $n>2$, and T a hyperpolynomial in the symbols $\cdot$, $\mu_1,\ldots,\mu_{n-2}$, such that $\Pi_\mu(T) < \Pi_\mu(P)$, and assume that all binary semilattice hyperidentities of n-1 operation symbols are a consequences of Σ^2. If

$$P = Q_1 \lambda Q_2 \lambda \cdots \lambda Q_k \lambda SR \lambda Q_{k+1} \lambda \cdots \lambda Q_j$$

where $Q_1,\ldots,Q_j$, S, R, are hyperpolynomials in the operation symbols $\cdot$, $\mu_1,\ldots,\mu_{n-2}$, then

$$P \Leftrightarrow Q_1 \lambda \cdots \lambda Q_k \lambda STR \lambda Q_{k+1} \lambda \cdots \lambda Q_j.$$

Proof. Let $Q = Q_1 Q_2 \cdots Q_j$. If we obtain P' by replacing the symbols $Q_1,\ldots,Q_j$, S, and R by the variables $x_1,\ldots,x_j,y$, and z respectively and let $x = x_1 x_2 \ldots x_j$, then

$$\underline{P}' = x_1 \,\underline{\lambda}\, \cdots \,\underline{\lambda}\, x_k \,\underline{\lambda}\, yxz \,\underline{\lambda}\, x_{k+1} \,\underline{\lambda}\, \cdots \,\underline{\lambda}\, x_j$$

is a semilattice hyperidentity in two operations, and thus a consequence of Σ^2. By Lemma 1.3(iv) we thus have

$$\underline{P} \iff \underline{Q}_1 \,\underline{\lambda}\, \cdots \,\underline{\lambda}\, \underline{Q}_k \,\underline{\lambda}\, \underline{SQR} \,\underline{\lambda}\, \underline{Q}_{k+1} \,\underline{\lambda}\, \cdots \,\underline{\lambda}\, \underline{Q}_j.$$

Since $\Pi_\mu(\underline{T}) < \Pi_\mu(\underline{P})$, we thus also have $\Pi_\mu(\underline{T}) < \Pi_\mu(\underline{SQR})$, and

$$\underline{SQR} = \underline{STQR}$$

is thus a semilattice hyperidentity in n-1 operation symbols, and thus a consequence of Σ^2. It therefore follows that

$$\underline{P} \iff \underline{Q}_1 \,\underline{\lambda}\, \cdots \,\underline{\lambda}\underline{Q}_k \,\underline{\lambda}\, \underline{STQR} \,\underline{\lambda}\, \underline{Q}_{k+1} \,\underline{\lambda}\, \cdots \,\underline{\lambda}\, \underline{Q}_j$$

and, as in the beginning of the proof, that

$$\underline{P} \iff \underline{Q}_1 \,\underline{\lambda}\, \cdots \,\underline{\lambda}\, \underline{Q}_k \,\underline{\lambda}\, \underline{STR} \,\underline{\lambda}\, \underline{Q}_{k+1} \,\underline{\lambda}\, \cdots \,\underline{\lambda}\, \underline{Q}_j$$

completing the proof of the lemma.

Theorem 2.12. The class of all binary semilattice hyperidentities is a consequence of Σ^2.

Proof. We will prove the theorem by inducting on n, the number of binary operation symbols in the hyperidentity. The cases n = 1 and n = 2 have already been proved so we will now assume that all binary semilattice hyperidentities with at most n-1 symbols, n > 2, are a consequence of Σ^2. Let $\underline{P} = \underline{Q}$ be a binary semilattice hyperidentity in the n operation symbols $\underline{\lambda}$, $\underline{\cdot}$, and $\underline{\mu}_1, \ldots, \underline{\mu}_{n-2}$.

By distributivity $\underline{P} \Leftrightarrow \underline{P}_1 \,\underline{\lambda}\, \ldots \,\underline{\lambda}\, \underline{P}_k$ and $\underline{Q} \Leftrightarrow \underline{Q}_1 \,\underline{\lambda}\, \ldots \,\underline{\lambda}\, \underline{Q}_s$ where $\underline{P}_1, \ldots, \underline{P}_k$ and $\underline{Q}_1, \ldots, \underline{Q}_s$ are hyperpolynomials in the n-1 symbols $\underline{\cdot}, \underline{\mu}_1, \ldots, \underline{\mu}_{n-2}$. Since $(L_\lambda(\underline{P}) = L_\lambda(\underline{Q})) \in H(SL)$, we have $(\underline{P}_1 = \underline{Q}_1) \in H(SL)$, and this is thus a consequence of Σ^2 by the induction hypothesis. Similarly, $\underline{P}_k \Leftrightarrow \underline{Q}_s$. Thus, by distributivity and idempotence, we can write

$$\underline{P} \Leftrightarrow \underline{H}_1\underline{H}_2\underline{H}_3 \,\underline{\lambda}\, \underline{K}_1\underline{L}_1\underline{R}_1 \,\underline{\lambda}\, \underline{K}_2\underline{L}_2\underline{R}_2 \,\underline{\lambda}\, \cdots \,\underline{\lambda}\, \underline{K}_m\underline{L}_m\underline{R}_m \,\underline{\lambda}\, \underline{T}_1\underline{T}_2\underline{T}_3$$

and $$\underline{Q} \Leftrightarrow \underline{H}_1\underline{H}_2\underline{H}_3 \,\underline{\lambda}\, \underline{U}_1\underline{V}_1\underline{W}_1 \,\underline{\lambda}\, \underline{U}_2\underline{V}_2\underline{W}_2 \,\underline{\lambda}\, \cdots \,\underline{\lambda}\, \underline{U}_m\underline{V}_m\underline{W}_m \,\underline{\lambda}\, \underline{T}_1\underline{T}_2\underline{T}_3$$

where $\underline{H}_1$, $\underline{H}_3$, $\underline{T}_1$, $\underline{T}_3$, and $\underline{K}_i$, $\underline{R}_i$, $\underline{U}_i$, $\underline{W}_i$, $i \in \{1, \ldots, m\}$, are hyperpolynomials in only the symbols $\underline{\mu}_1, \ldots, \underline{\mu}_{n-2}$.

Clearly, for $i \in \{1, \ldots, m\}$, $\Pi_\mu(\underline{L}_i) < \Pi_\mu(\underline{P})$, and thus also $\Pi_\mu(\underline{L}_i) < \Pi_\mu(\underline{Q})$. Similarly, $\Pi_\mu(\underline{V}_i) < \Pi_\mu(\underline{P})$ and $\Pi_\mu(\underline{V}_i) < \Pi_\mu(\underline{Q})$. Thus, by Lemma 2.11, if we let $\underline{S} = \underline{L}_1\underline{L}_2 \cdot\cdot \underline{L}_m\underline{V}_1\underline{V}_2 \cdot\cdot \underline{V}_m$,

$$\underline{P} \Leftrightarrow \underline{H}_1\underline{H}_2\underline{H}_3 \,\underline{\lambda}\, \underline{K}_1\underline{S}\underline{R}_1 \,\underline{\lambda}\, \cdots \,\underline{\lambda}\, \underline{K}_m\underline{S}\underline{R}_m \,\underline{\lambda}\, \underline{T}_1\underline{T}_2\underline{T}_3$$

and $$\underline{Q} \Leftrightarrow \underline{H}_1\underline{H}_2\underline{H}_3 \,\underline{\lambda}\, \underline{U}_1\underline{S}\underline{W}_1 \,\underline{\lambda}\, \cdots \,\underline{\lambda}\, \underline{U}_m\underline{S}\underline{W}_m \,\underline{\lambda}\, \underline{T}_1\underline{T}_2\underline{T}_3$$

are consequences of Σ^2.

As in the proof of Lemma 2.11, we can replace hyper-polynomials $\underline{H}_i$, $\underline{K}_i$, $\underline{S}$, $\underline{R}_i$, $\underline{U}_i$, $\underline{W}_i$, and $\underline{T}_i$ by the variables

h_i, k_i, s, r_i, u_i, w_i, and t_i, and use Lemma 1.3(iv) to obtain

$$\underline{P} \Leftrightarrow (\underline{H}_1 \underline{\lambda} \underline{K}_1 \underline{\lambda} \cdots \underline{\lambda} \underline{K}_m \underline{\lambda} \underline{T}_1)(\underline{H}_2 \underline{\lambda} \underline{S} \underline{\lambda} \cdots \underline{\lambda} \underline{S} \underline{\lambda} \underline{T}_2)(\underline{H}_3 \underline{\lambda} \underline{R}_1 \underline{\lambda} \cdots \underline{\lambda} \underline{R}_m \underline{\lambda} \underline{T}_3)$$

$$\underline{Q} \Leftrightarrow (\underline{H}_1 \underline{\lambda} \underline{U}_1 \underline{\lambda} \cdots \underline{\lambda} \underline{U}_m \underline{\lambda} \underline{T}_1)(\underline{H}_2 \underline{\lambda} \underline{S} \underline{\lambda} \cdots \underline{\lambda} \underline{S} \underline{\lambda} \underline{T}_2)(\underline{H3} \underline{\lambda} \underline{W}_1 \underline{\lambda} \cdots \underline{\lambda} \underline{W}_m \underline{\lambda} \underline{T}_3)$$

since the reader can easily check that

$$h_1h_2h_3 \underline{\lambda} k_1sr_1 \underline{\lambda} \cdots \underline{\lambda} k_msr_m \underline{\lambda} t_1t_2t_3 = (h_1 \underline{\lambda} k_1 \underline{\lambda} \cdots \underline{\lambda} k_m \underline{\lambda} t_1)(h_2 \underline{\lambda} s \underline{\lambda} \cdots \underline{\lambda} s \underline{\lambda} t_2)(h_3 \underline{\lambda} r_1 \underline{\lambda} \cdots \underline{\lambda} r_m \underline{\lambda} t_3)$$

is a binary semilattice hyperidentity.

Since $\underline{H}_1$, $\underline{T}_1$, $\underline{H}_3$, $\underline{T}_3$, and $\underline{U}_i$, $\underline{K}_i$, $\underline{R}_i$, $\underline{W}_i$, $i \in \{1,\ldots,m\}$ are polynomials in only the symbols $\underline{\mu}_1,\ldots,\underline{\mu}_{n-2}$,

$$L.(\underline{P}) = \underline{H}_1 \underline{\lambda} \underline{K}_1 \underline{\lambda} \cdots \underline{\lambda} \underline{K}_m \underline{\lambda} \underline{T}_1$$

and

$$L.(\underline{Q}) = \underline{H}_1 \underline{\lambda} \underline{U}_1 \underline{\lambda} \cdots \underline{\lambda} \underline{U}_m \underline{\lambda} \underline{T}_1.$$

These are polynomials in n-1 symbols and thus their equality is a consequence of Σ^2. Similarly

$$\underline{H}_3 \underline{\lambda} \underline{R}_1 \underline{\lambda} \cdots \underline{\lambda} \underline{R}_m \underline{\lambda} \underline{T}_3 \Leftrightarrow \underline{H}_3 \underline{\lambda} \underline{W}_1 \underline{\lambda} \cdots \underline{\lambda} \underline{W}_m \underline{\lambda} \underline{T}_3$$

and this completes the proof.

We shall now prove a structure theorem similar to Theorem 2.5. In order to do so we introduce some more notation and prove a lemma. Let $D_{\langle 2,2\rangle}$ be the smallest equational class containing algebras $A = \langle A; \lambda, \cdot \rangle$ of type $\langle 2,2 \rangle$, where $\langle A; \vee \rangle$ is a semilattice, and $\{\lambda, \cdot\} \subseteq \{x, y, x \vee y\}$, the set of binary semilattice polynomials, such that not both λ and $\cdot$ are equal to $x \vee y$. Similarly, we define $S_{\langle 2,2 \rangle}$ to be the smallest equational class containing algebras of type $\langle 2,2 \rangle$ in the operations λ and $\cdot$, with both λ and $\cdot$ equal to the semilattice polynomial $x \vee y$. If H is any set of hyperidentities in the symbols $\underline{\lambda}_i$, we define H* to be the corresponding set of identities, that is, the set of identities obtained from H by replacing all the operation symbols $\underline{\lambda}_i$ by λ_i.

<u>Lemma 2.13</u>. Let $A = \langle A; \lambda, \cdot \rangle$ be an algebra of type $\langle 2,2 \rangle$ satisfying H(SL). Then the polynomial

$$f(x,y) = x \ \lambda \ xyx \ \lambda \ x$$

is a partition function of A.

<u>Proof</u>. By Lemma 1 of [21], f(x,y) will be a partition function if it satisfies the identities (i) - (vi) of Corollary 2.4 plus the identities obtained from (iv), (v) and (vi) when the operation symbol λ is replaced by the symbol $\cdot$.

If the operations λ and $\cdot$ in f(x,y) are replaced by any semilattice polynomials p(x,y) and q(x,y), we obtain f(x,y) = x, except when $p = q = x \vee y$. Consequently, the above identities (when viewed as hyperidentities) are semilattice hyperidentities, completing the proof of the lemma.

Theorem 2.14. For an algebra **A** of type <2,2> the following are equivalent:

(i) A satisfies all semilattice hyperidentities;

(ii) A is a Płonka sum of algebras of $D_{<2,2>}$;

(iii) A ε $D_{<2,2>} \vee S_{<2,2>}$;

(iv) The operations of A satisfy Σ^2.

Proof. (i) implies (ii). By Lemma 2.13, and a result of Płonka [20], A is a Płonka sum of algebras of type <2,2> which satisfy f(x,y) = x (where f(x,y) is the partition function of Lemma 2.13) and (H(SL)<2,2>)*. Now the identities of $D_{<2,2>}$ are exactly those identities in the symbols λ and $\cdot$, which yield semilattice identities when λ and $\cdot$ are replaced by any pair of binary semilattice polynomials, not both equal to $x \vee y$. Thus, for example, if $r = m \lambda xyz \lambda n$, and if y is replaced by any polynomial in λ and $\cdot$ to obtain a polynomial r', then r = r' will be an identity of $D_{<2,2>}$. By distributivity and associativity, the identities of $D_{<2,2>}$ are exactly those which can be obtained from (H(SL)<2,2>)* in this way. To prove that (i) implies (ii), it is thus enough to prove that

$$m \lambda xyz \lambda n = m \lambda xwz \lambda n$$

is a consequence of $\Sigma = (H(SL)\langle 2,2\rangle)^* \cup \{f(x,y) = x\}$. (In this proof only, $p \Leftrightarrow q$ will denote $p = q$ is a consequence of Σ.)

$$\begin{aligned} m \wedge xyz \wedge n &\Leftrightarrow m \wedge x(x \wedge xyx \wedge x)z \wedge n, && \text{since it is in } (H(SL)\langle 2,2\rangle)^* \\ &\Leftrightarrow m \wedge x(x \wedge xwx \wedge x)z \wedge n, && \text{since } f(x,y) = x, \\ &\Leftrightarrow m \wedge xwz \wedge n, && \text{since this is again in } (H(SL)\langle 2,2\rangle)^*. \end{aligned}$$

(ii) implies (iii) follows immediately from Płonka's theorem [20] that if A is a Płonka sum of algebras of $D_{\langle 2,2\rangle}$ then A will satisfy all the regular identities of $D_{\langle 2,2\rangle}$, that is, all the identities of $Id(S_{\langle 2,2\rangle}) \cap Id(D_{\langle 2,2\rangle})$.

(iii) implies (iv). Let Σ' be the set of six identities obtained by substituting all possible choices of the operations of A for the operation symbols of Σ^2. Σ' is obviously a subset of $Id(S_{\langle 2,2\rangle}) \cap Id(D_{\langle 2,2\rangle})$, and thus (iii) implies (iv).

(iv) implies (i). If $(p = q) \in (H(SL)\langle 2,2\rangle)^*$ then we can show, exactly as in the proof of Theorem 2.6, that $p = q$ is a consequence of Σ', proving that A satisfies $p = q$ and completing the proof by Lemma 1.4.

The partition function of Lemma 2.13 can also be shown to be a partition function for algebras of type $\langle 2,2\rangle$ which satisfy H(L), the set of all lattice hyperidentities, and we believe that an approach similar to that used in this chapter should be applicable when investigating binary lattice hyperidentities.

CHAPTER III

N-ARY SEMILATTICE HYPERIDENTITIES

In this chapter we shall show that for any integer n, the set of all semilattice hyperidentities of type $\langle n\rangle$ has a finite basis $\Sigma\langle n\rangle$ similar to $\Sigma\langle 2\rangle$ of the binary case, and we shall also prove a structure theorem similar to Theorem 2.5. For example, in the case n = 3, $\Sigma\langle 3\rangle$ is the following set of hyperidentities:

$$\text{(i)}\quad \underline{F}(x,x,x) = x,$$

$$\text{(ii)}\quad \underline{F}(\underline{F}(a,x,b),\underline{F}(y,c,d),u) = \underline{F}(\underline{F}(a,y,b),\underline{F}(x,c,d),u),$$

$$\text{(iii)}\quad \underline{F}(\underline{F}(x,y,z),\underline{F}(u,u,v),p) = \underline{F}(\underline{F}(x,y,z),\underline{F}(y,u,v),p).$$

We shall also show that for any positive integer n, the set of all n-ary semilattice hyperidenties has a finite basis Σ^n similar to Σ^2, where in the case n = 3,

$$\Sigma^3 = \Sigma\langle 3\rangle \cup \{(\underline{F}(\underline{G}(x_1,x_2,x_3),\underline{G}(y_1,y_2,y_3),\underline{G}(z_1,z_2,z_3)) = G(\underline{F}(x_1,y_1,z_1),\underline{F}(x_2,y_2,z_2),\underline{F}(x_3,y_3,z_3))\}.$$

Rather than proving the above assertions for an arbitrary positive integer n, we shall first prove them for n = 3 and then discuss the easy generalization to an arbitrary positive integer n.

We shall frequently be referring to the <u>formation trees</u> of hyperpolynomials and will often use the terms tree and

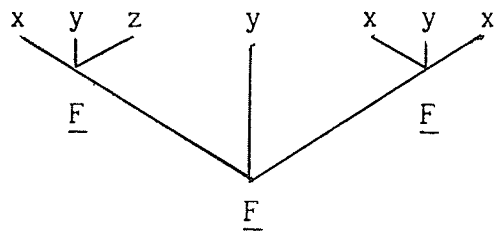


Figure 3.1

hyperpolynomial interchangeably (See Figure 3.1 for the formation tree of $\underline{P} = \underline{F}(\underline{F}(x,y,z),y,\underline{F}(x,y,x)))$. We shall follow the convention that whenever the nodes of a formation tree are left unlabeled they will be understood to be labeled by the operation symbol $\underline{F}$. Recall that if x is a variable of $\underline{P}$, then the string terminating with x is the sequence of operation symbols and turnings (that is, the labels of the edges) leading from the root of $\underline{P}$ to the variable x, and the variable x. For example, in Figure 3.1, the string terminating with z is $\langle(\underline{F},1),(\underline{F},3),z\rangle$. Similarly, if $\underline{P}_1$ is a subtree of $\underline{P}$, we shall speak of the string terminating with $\underline{P}_1$. For example, in Figure 3.1, the string terminating with the subtree $\underline{P}_1 = \underline{F}(x,y,z)$ is $\langle(\underline{F},1),\underline{P}_1\rangle$. A string α with n+1 terms, counting multiple occurrences, shall be said to be of length $l(\alpha) = n$. For example, the string $\langle(\underline{F},3),(\underline{F},3),x\rangle$ has the terms $(\underline{F},3)$ and x, and is of length 2. The length $l(\underline{P})$, of a hyperpolynomial $\underline{P}$, will be defined to be the length of its longest string, and if all the strings of $\underline{P}$ have the same length we shall say that $\underline{P}$ is of uniform length. Let $\underline{P}$ and $\underline{Q}$ be hyperpolynomials consisting of operation symbols $\underline{F}_1,\ldots,\underline{F}_n$. If α and β are strings of $\underline{P}$ and/or $\underline{Q}$ (that is, not necessarily from the same tree) such that $(\underline{F}_i,k)$, $k \in \{1,2,3\}$, is a member of α implies that $(\underline{F}_i,k)$ is a member of β, we shall write $\alpha < \beta$. If $\alpha < \beta$ and $\beta < \alpha$ we shall write $\alpha =_r \beta$ and say that α and β are <u>regular equal</u>. If $p(x_1,x_2,x_3) = x_i \vee x_j$, $i,j \in \{1,2,3\}$, we shall denote $V(\underline{P}(p))$, the set of variables of $\underline{P}(p)$, by $V_{i,j}(\underline{P})$. Similarly, if $p = x_i$, $i \in \{1,2,3\}$, $V_i(\underline{P})$ shall denote the set $V(\underline{P}(p))$. Clearly, the set $V(\underline{P})$, of all variables of $\underline{P}$, is equal to $V_{1,2,3}(\underline{P})$, the set $V(\underline{P}(p))$ for $p = x_1 \vee x_2 \vee x_3$. It can easily be seen that $x \in V_i(\underline{P})$, $i \in \{1, 2, 3\}$, iff $\underline{P}$ has a string $\alpha = \langle(\underline{F},i), (\underline{F},i), \ldots, (\underline{F},i), x\rangle$. Similarly, it can be shown by inducting on the length $l(\underline{P})$ of $\underline{P}$ that $x \in V_{i,j}(\underline{P})$, $i,j \in \{1, 2, 3\}$, iff $\underline{P}$ has a string α terminating with x, such that the set of terms of α is $\{(\underline{F},i), (\underline{F},j),x\}$, $\{(\underline{F},i),x\}$ or $\{(\underline{F},j),x\}$.

If $\underline{T}$ is a hyperpolynomial of type $\langle 3\rangle$, $\underline{F}$ the operation symbol and s a permutation with domain $\{1,2,3\}$, we define $s(\underline{T})$ inductively as follows:

(i) $s(x_i) = x_i$,

(ii) $s(\underline{F}(\underline{P}_1,\underline{P}_2,\underline{P}_3)) = \underline{F}(s(\underline{P}_{s(1)}),s(\underline{P}_{s(2)}),s(\underline{P}_{s(3)}))$.

That is, $s(\underline{P})$ is defined to be the hyperpolynomial obtained by replacing each occurrence of the symbol $\underline{F}$ in $\underline{P}$ by $\underline{F}(x_{s(1)},x_{s(2)},x_{s(2)})$. If p is a ternary polynomial, $s(p)$ is defined to be $p(x_{s(1)}, x_{s(2)}, x_{s(3)})$. We now prove the following lemma:

<u>Lemma 3.1</u>. If s is a permutation with domain {1, 2, 3}, then the hyperidentity $\underline{P} = \underline{Q}$ holds in a variety V iff $s(\underline{P}) = s(\underline{Q})$ holds in V.

<u>Proof</u>. We shall induct on the length of $\underline{P}$. If $l(\underline{P}) = 1$ and p is a ternary polynomial then

$$s(\underline{P})(p) = \underline{P}(s(p)).$$

Now assume that $s(\underline{P})(p) = \underline{P}(s(p))$ for all hyperpolynomials of length less than n and let $l(p) = n$. Then if $\underline{P} = \underline{F}(\underline{Q}, \underline{R}, \underline{T})$,

$$\begin{aligned} s(\underline{P})(p) &= s(p)(s(\underline{Q})(p), s(\underline{R})(p), s(\underline{T})(p)) \\ &= s(p)(\underline{Q}(s(p)), \underline{R}(s(p)), \underline{T}(s(p))) \quad \text{by the induction hypothesis} \\ &= \underline{P}(s(p)). \end{aligned}$$

Therefore, $s(\underline{P})(p) = s(\underline{Q})(p)$ for all appropriate polynomials p iff $\underline{P}(s(p)) = \underline{Q}(s(p))$ iff $\underline{P} = \underline{Q}$ holds in V, completing the proof.

We now define $\Sigma\langle 3\rangle$ to be the following set of hyperidentities:

(i) $\underline{F}(x,x,x) = x$,

(ii) $\underline{F}(\underline{F}(a,x,b,),\underline{F}(y,c,d),u) = \underline{F}(\underline{F}(a,y,b,),\underline{F}(x,c,d),u)$,

(iii) $\underline{F}(\underline{F}(x,y,z),\underline{F}(u,u,v),p) = \underline{F}(\underline{F}(x,y,z),\underline{F}(y,u,v),p)$.

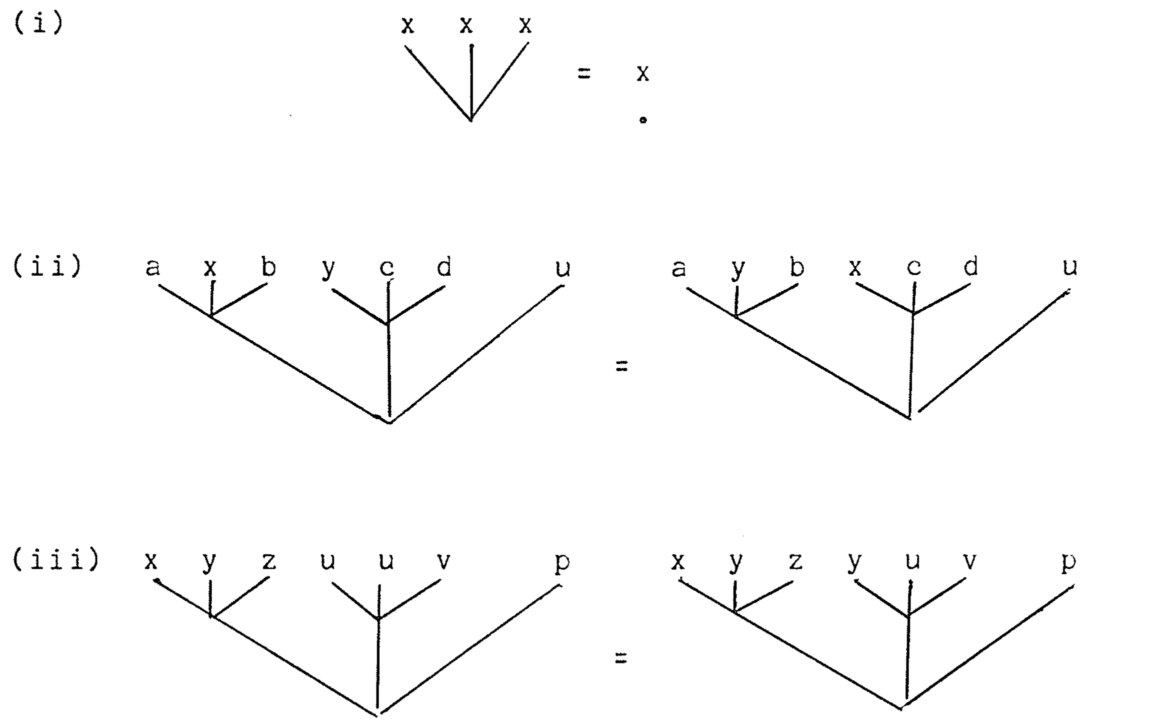


Figure 3.2

See Figure 3.2 for the formation trees depicting these hyperidentities. Whenever a hyperidentity $\underline{P} = \underline{Q}$ is a consequence of $\Sigma\langle 3\rangle$ we shall write $\underline{P} \Leftrightarrow \underline{Q}$.

<u>Theorem 3.2</u>. The set $\Sigma\langle 3\rangle$ is an equational basis for the set of all semilattice hyperidentities of type $\langle 3\rangle$.

In order to prove this theorem we shall first establish some lemmas.

<u>Lemma 3.3</u>. The following hyperidentities are consequences of $\Sigma\langle 3\rangle$:

(i) $\underline{F}(\underline{F}(x,y,z),\underline{F}(u,u,v),p) \Leftrightarrow \underline{F}(\underline{F}(x,y,z),\underline{F}(y,u,v),p)$,

(ii) $\underline{F}(\underline{F}(x,y,z),\underline{F}(u,u,v),p) \Leftrightarrow \underline{F}(\underline{F}(x,y,z),\underline{F}(x,u,v),p)$,

(iii) $\underline{F}(\underline{F}(x,u,z),\underline{F}(y,u,v),p) \Leftrightarrow \underline{F}(\underline{F}(x,y,z),\underline{F}(y,u,v),p)$,

(iv) $\underline{F}(\underline{F}(x,u,z),\underline{F}(y,u,v),p) \Leftrightarrow \underline{F}(\underline{F}(x,x,z),\underline{F}(y,u,v),p)$,

(v) $\underline{F}(\underline{F}(x,u,z),\underline{F}(u,y,v),p) \Leftrightarrow \underline{F}(\underline{F}(x,u,z),\underline{F}(y,y,v),p)$,

(vi) $\underline{F}(\underline{F}(x,u,z),\underline{F}(u,y,v),p) \Leftrightarrow \underline{F}(\underline{F}(x,y,z),\underline{F}(u,y,v),p)$,

(vii) $\underline{F}(\underline{F}(x,u,z),\underline{F}(u,y,v),p) \Leftrightarrow \underline{F}(\underline{F}(x,u,z),\underline{F}(x,y,v),p)$,

(viii) $\underline{F}(\underline{F}(x,u,z),\underline{F}(u,y,v),p) \Leftrightarrow \underline{F}(\underline{F}(x,x,z),\underline{F}(u,y,v),p)$.

<u>Proof</u>. (i) This is exactly (iii) of $\Sigma\langle 3\rangle$.

(ii) $\underline{F}(\underline{F}(u,y,v),\underline{F}(x,x,z),p) \Leftrightarrow \underline{F}(\underline{F}(u,y,v),\underline{F}(y,x,z)$ follows from $\Sigma\langle 3\rangle$(iii), and if we let $s = \{(1,2),(2,1),(3,3)\}$, then Lemma 3.1 yields the hyperidentity

$$\varepsilon\colon\ \underline{F}(\underline{F}(x,x,z),\underline{F}(y,u,v),p) \Leftrightarrow \underline{F}(\underline{F}(x,y,z),\underline{F}(y,u,v),p).$$

Thus $\underline{F}(\underline{F}(x,y,z),\underline{F}(u,u,v),p) \Leftrightarrow \underline{F}(\underline{F}(x,y,z),\underline{F}(y,u,v),p)$ by $\Sigma\langle 3\rangle$(iii)

$\Leftrightarrow \underline{F}(\underline{F}(x,x,z),\underline{F}(y,u,v),p)$ by ε

$\Leftrightarrow \underline{F}(\underline{F}(x,y,z),\underline{F}(x,u,v),p)$ by $\Sigma\langle 3\rangle$(ii).

(iii) Apply $\Sigma\langle 3\rangle$(ii) to $\Sigma\langle 3\rangle$(iii).

(iv) Apply $\Sigma\langle 3\rangle$(ii) to (ii) of this lemma.

(v) Follows from $\Sigma\langle 3\rangle$(iii).

(vi) Apply $\Sigma\langle 3\rangle$(ii) to (v).

(vii) $\underline{F}(\underline{F}(x,u,z),\underline{F}(u,y,v),p)$

$\Leftrightarrow \underline{F}(\underline{F}(x,u,z),\underline{F}(y,y,v),p)$ by §<3>(iii)

$\Leftrightarrow \underline{F}(\underline{F}(x,u,z),\underline{F}(x,y,v),p)$ by (ii) of this lemma.

(viii) Apply §<3>(ii) to (vii).

Lemma 3.4. Let $\underline{P}$ be a hyperpolynomial of type <3>. If $\underline{P}'$ is obtained from $\underline{P}$ in any of the following ways then $\underline{P} \Leftrightarrow \underline{P}'$.

(i) $\underline{P}$ is a hyperpolynomial with strings $\alpha_1 = \langle(\underline{F},i),(\underline{F},j),x\rangle$ and $\alpha_2 = \langle(\underline{F},j),(\underline{F},i),y\rangle$, $i,j \in \{1,2,3\}$, and $\underline{P}'$ is obtained from $\underline{P}$ by switching the variables of α_1 and α_2.

(ii) $\underline{P}$ is a hyperpolynomial with strings $\alpha_1 = \langle(\underline{F},i),(\underline{F},i),x\rangle$, $\alpha_2 = \langle(\underline{F},i),(\underline{F},j),y\rangle$, $\alpha_3 = \langle(\underline{F},j),(\underline{F},i),u\rangle$, and $\alpha_4 = \langle(\underline{F},j),(\underline{F},j),u\rangle$ and $\underline{P}'$ is obtained by replacing the variable u of α_3 by either x or y.

(iii) $\underline{P}$ has strings $\alpha_1 = \langle(\underline{F},i),(\underline{F},i),x\rangle$, $\alpha_2 = \langle(\underline{F},i),(\underline{F},j),u\rangle$, $\alpha_3 = \langle(\underline{F},j),(\underline{F},i),y\rangle$, and $\alpha_4 = \langle(\underline{F},j),(\underline{F},j),u\rangle$ and $\underline{P}'$ is obtained by replacing the variable of α_2 by either x or y.

(iv) $\underline{P}$ has strings $\alpha_1 = \langle(\underline{F},i),(\underline{F},i),x\rangle$, $\alpha_2 = \langle(\underline{F},i),(\underline{F},j),u\rangle$, $\alpha_3 = \langle(\underline{F},j),(\underline{F},i),u\rangle$, and $\alpha_4 = \langle(\underline{F},j),(\underline{F},j),y\rangle$ and $\underline{P}'$ is obtained by replacing the variable of α_2

by x or y, or $\underline{P}'$ is obtained by replacing the variable of α_3 by x or y.

Proof. Apply Lemma 3.1 to $\Sigma\langle 3\rangle$ and Lemma 3.3.

At this point it should be instructive to compare $\Sigma\langle 3\rangle$ with $\Sigma\langle 2\rangle$. The hyperidentity of $\Sigma\langle 3\rangle$(i) is the idempotent hyperidentity and obviously the ternary counterpart of $\underline{F}(x,x) = x$. The hyperidentity

$$\underline{F}(\underline{F}(x_1,x_2,x_3),\underline{F}(y_1,y_2,y_3),\underline{F}(z_1,z_2,z_3)) \Leftrightarrow \\ \underline{F}(\underline{F}(x_1,y_1,z_1),\underline{F}(x_2,y_2,z_2),\underline{F}(x_3,,y_3,z_3)),$$

which is a consequence of $\Sigma\langle 3\rangle$(ii) by Lemma 3.4(i), is the median hyperidentity of type $\langle 3\rangle$, and the counterpart of the third hyperidentity of $\Sigma\langle 2\rangle$. Substituting any binary (and thus also ternary) polynomial $f(x,y)$ for $\underline{F}$ in $\Sigma\langle 3\rangle$(iii) yields $f(f(x,y),u) = f(f(x,y),f(y,u))$, and substituting $f(x,y)$ into ε of the proof of Lemma 3.3(ii) yields $f(x,f(y,u)) = f(f(x,y),f(y,u))$. Since ε was a consequence of $\Sigma\langle 3\rangle$(iii) we obtain the binary associative hyperidentity $\underline{f}(\underline{f}(x,y),u) \Leftrightarrow \underline{f}(x,\underline{f}(y,u))$ as a consequence of $\Sigma\langle 3\rangle$(iii) and $\Sigma\langle 3\rangle$(i). Note that if ε is a hyperidentity holding in a variety V, and if the operation symbols of ε are replaced by polynomials of appropriate arity, the resulting expression is again a hyperidentity holding in V by Lemma 1.4.

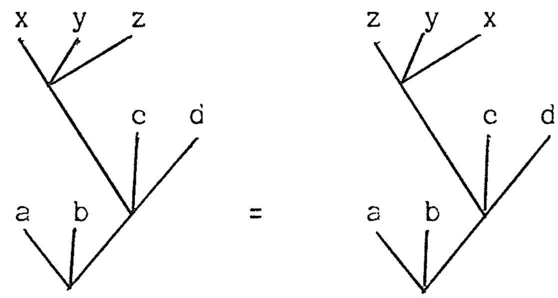


Figure 3.3

Five more lemmas need to be established in order to prove Theorem 2.2.

<u>Lemma 3.5</u>. The hyperidentity

$$\underline{F}(a,b,\underline{F}(\underline{F}(x,y,z),c,d)) \Leftrightarrow \underline{F}(a,b,\underline{F}(\underline{F}(z,y,x),c,d))$$

is a consequence of $\Sigma\langle 3\rangle$. (See Figure 3.3.)

<u>Proof</u>. By idempotence and the results of Lemma 3.4 we have

$$\begin{aligned}
&\underline{F}(a,b,\underline{F}(\underline{F}(x,y,z),c,d))\\
&\quad\Leftrightarrow \underline{F}(\underline{F}[\underline{F}(a,a,a),a,\underline{F}(a,a,a)],b,\underline{F}[\underline{F}(x,y,z),c,\underline{F}(d,d,d)])\\
&\quad\Leftrightarrow \underline{F}(\underline{F}[\underline{F}(a,a,a),a,\underline{F}(a,a,a)],b,\underline{F}[\underline{F}(x,y,z),c,\underline{F}(z,d,d)])\\
&\quad\Leftrightarrow \underline{F}(\underline{F}[\underline{F}(a,a,a),a,\underline{F}(a,a,a)],b,\underline{F}[\underline{F}(x,y,x),c,\underline{F}(z,d,d)])\\
&\quad\Leftrightarrow \underline{F}(\underline{F}[\underline{F}(a,a,a),a,\underline{F}(x,y,x)],b,\underline{F}[\underline{F}(a,a,a),c,\underline{F}(z,d,d)])\\
&\quad\Leftrightarrow \underline{F}(\underline{F}[\underline{F}(a,a,a),a,\underline{F}(a,y,x)],b,\underline{F}[\underline{F}(a,a,a),c,\underline{F}(z,d,d)])\\
&\quad\Leftrightarrow \underline{F}(\underline{F}[\underline{F}(a,a,a),a,\underline{F}(a,a,a)],b,\underline{F}[\underline{F}(a,y,x),c,\underline{F}(z,d,d)]).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\underline{F}(a,b,\underline{F}(\underline{F}(z,y,x),c,d))\\
&\quad\Leftrightarrow (\underline{F}[\underline{F}(a,a,a),a,\underline{F}(a,a,a)],b,\underline{F}[\underline{F}(a,y,z),c,\underline{F}(x,d,d)]),
\end{aligned}$$

and since

$$\underline{F}[\underline{F}(a,y,z),c,\underline{F}(x,d,d)] \Leftrightarrow \underline{F}[\underline{F}(a,y,x),c,\underline{F}(z,d,d)]$$

by Lemma 3.4(i), we are done.

Lemma 3.6. Let $\underline{P}$ be a hyperpolynomial of type <3> with strings $\alpha = \langle(\underline{F},i),(\underline{F},j),(\underline{F},i),x\rangle$ and $\beta = \langle(\underline{F},i),(\underline{F},j),(\underline{F},j),y\rangle$, $i,j \in \{1,2,3\}$. If $\underline{P}'$ is obtained from $\underline{P}$ by switching the variables of α and β, then $\underline{P} \Leftrightarrow \underline{P}'$.

Proof. Lemmas 3.5 and 3.1.

Lemma 3.7. Let $\underline{P}$ be a hyperpolynomial of type <3> and of uniform length, with strings

$$\alpha = \langle(\underline{F},j),(\underline{F},i),\ldots,(\underline{F},i),(\underline{F},i),x\rangle,$$
$$\beta = \langle(\underline{F},j),(\underline{F},i),\ldots,(\underline{F},i),(\underline{F},j),y\rangle,$$

with the term $(\underline{F},j)$ occurring once in α and twice in β. If P' is obtained from $\underline{P}$ by switching the variables of α and β, then $\underline{P} \Leftrightarrow \underline{P}'$.

Proof. If $l(\underline{P}) = 3$, the statement of the lemma follows from Lemma 3.6.

We now assume that the statement of the lemma holds for hyperpolynomials of length less than n, n > 3, and let

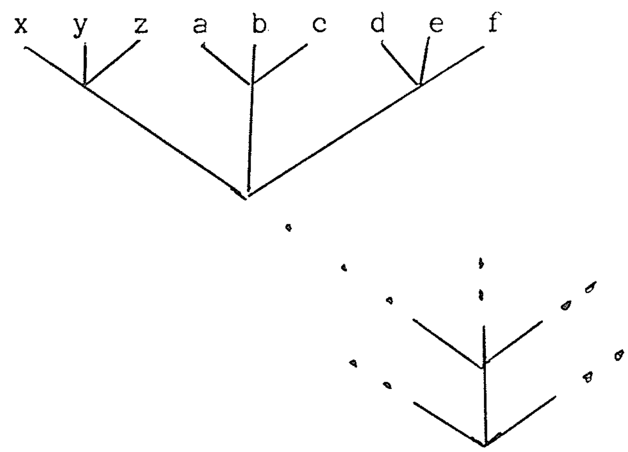


Figure 3.4

$l(\underline{P}) = n$. Since the variables x and y may occur more than once in a hyperpolynomial $\underline{P}$, we shall assume that whenever we refer to the variable x or y, that we are referring to the particular variables in question. By Lemma 3.1, we can assume that $i = 1$ and $j = 2$. Let $\underline{P}_1$ be the subtree $\underline{F}(x,y,z)$ of $\underline{P}$ which is of length one and contains x and y as variables. Let $\underline{P}_2 = \underline{F}(a,b,c)$ be the subtree of length one which is to the immediate right of $\underline{P}_1$ (See Figure 3.4). By the induction hypothesis the subtrees $\underline{P}_1$ and $\underline{P}_2$ can be switched to obtain a tree $\underline{P}^*$. The strings corresponding to α and β (that is, the strings terminating with x and y) are

$$\alpha^* = \langle(\underline{F},2),(\underline{F},1),\ldots,(\underline{F},1),(\underline{F},2),(\underline{F},1),x\rangle,$$

$$\beta^* = \langle(\underline{F},2),(\underline{F},1),\ldots,(\underline{F},1),(\underline{F},2),(\underline{F},2),y\rangle.$$

Now consider the subtree

$$\underline{F}(\underline{F}\{\underline{F}(a,b,c),\underline{F}(x,y,z),\underline{F}(d,e,f)\},\ldots,\ldots)$$

of $\underline{P}^*$, which is of length three and contains x and y as variables. By Lemma 3.6, x and y can be switched to obtain the subtree $\underline{P}_3 = \underline{F}(y,x,z)$ of length one, and by the induction hypothesis we can now switch $\underline{P}_3$ and $\underline{P}_2$ to obtain the desired $\underline{P}'$, completing the proof of the lemma.

Lemma 3.8. Let $\underline{P}$ be a hyperpolynomial of type <3> whose formation tree is of uniform length. Let α and β be strings of $\underline{P}$ such that both α and β contain the terms $(\underline{F},i)$ and $(\underline{F},j)$, $i,j \in \{1,2,3\}$, and neither contains a term $(\underline{F},k)$ with $k \neq i$ or j. If $\underline{P}'$ is obtained from $\underline{P}$ by switching the variables of α and β, then $\underline{P} \Leftrightarrow \underline{P}'$.

Proof. We shall induct on the length $l(\underline{P})$ of $\underline{P}$. If $l(\underline{P})$ = 0 or 1, the statement of the lemma is true vacuously, and for $l(\underline{P}) = 2$ it follows from Σ<3>(ii). Let the variables of α and β be x and y respectively, and assume, without loss of generality, that $i = 1$ and $j = 2$. Assume the lemma is true for $l(\underline{P}) < n$, $n > 2$, and let $\underline{P}$ be a tree with $l(\underline{P}) = n$.

Let $\underline{P} = \underline{F}(\underline{P}_1,\underline{P}_2,\underline{P}_3)$. If α and β both terminate in the same subtree $\underline{P}_i$, assume, without loss of generality, that they both terminate in $\underline{P}_1$, and let the strings α_1 and β_1 be the respective strings α and β restricted to $\underline{P}_1$, that is, the strings obtained by removing the first terms from α and β. If both α_1 and β_1 contain the term $(\underline{F},1)$ we can switch x and y by the induction hypothesis (since $l(\underline{P}_i) = n-1$).

If not, assume, without loss of generality, that $\beta_1 = \langle(\underline{F},2), \ldots,(\underline{F},2),y\rangle$. If x is to the immediate left of y, they can now be switched by Lemma 3.7. If the variable to the immediate left of y is z, $z \neq x$, we can switch y and z by Lemma 3.7, then y and x by the induction hypothesis, and then x and z by Lemma 3.7, to obtain $\underline{P}'$.

Now assume that α terminates in $\underline{P}_i$ and β in $\underline{P}_j$, $i \neq j$. By Lemma 3.7 and the induction hypothesis, we can assume that the strings α and β restricted to $\underline{P}_i$ and $\underline{P}_j$ are

$$\alpha_i = \langle(\underline{F},i),\ldots,(\underline{F},i),(\underline{F},j),x\rangle,$$
$$\beta_j = \langle(\underline{F},i),\ldots,(\underline{F},i),y\rangle.$$

By $\Sigma\langle 3\rangle$(ii) we can now switch the j'th branch of $\underline{P}_i$ and the i'th branch of $\underline{P}_j$. Then, by the induction hypothesis, we can switch x and y, and by $\Sigma\langle 3\rangle$(ii) we can again switch the i'th and j'th branches of the new $\underline{P}_i$ and $\underline{P}_j$ to obtain the desired $\underline{P}'$, completing the proof of the lemma.

Lemma 3.9. Let $\underline{P}$ be a hyperpolynomial of type $\langle 3\rangle$ and of uniform length. If α and β are strings both containing the terms $(\underline{F},1)$, $(\underline{F},2)$ and $(\underline{F},3)$, and if the variables of α and β are switched to obtain a hyperpolynomial $\underline{P}'$, then $\underline{P} \Leftrightarrow \underline{P}'$.

Proof. The lemma is true vacuously for $l(\underline{P}) = 0, 1$ or 2. Assume the lemma is true for hyperpolynomials of length less

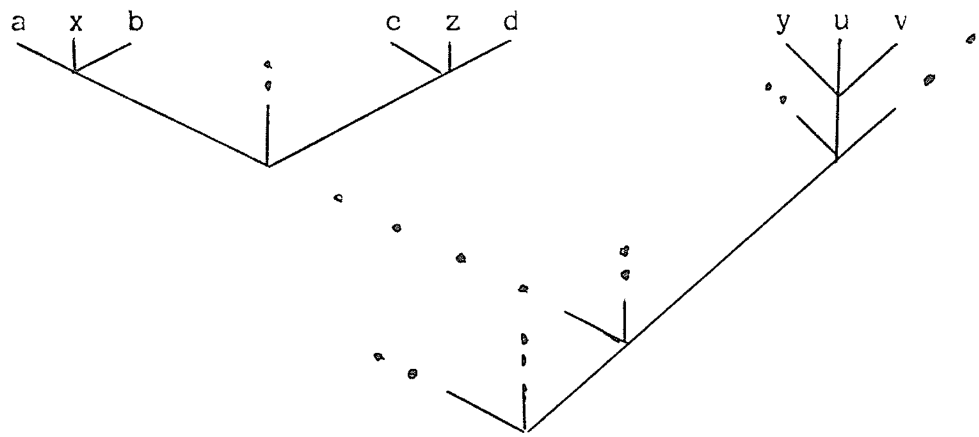


Figure 3.5

than n, n > 2, let $l(\underline{P}) = n$, and let $\underline{P} = \underline{F}(\underline{P}_1,\underline{P}_2,\underline{P}_3)$. We now consider two cases:

(i) If α and β terminate in the same subtree $\underline{P}_i$, we can assume, with out loss of generality, that they terminate in $\underline{P}_3$. If $\alpha_3 =_r \beta_3$, that is, the strings α and β restricted to $\underline{P}_3$, have exactly the same terms (except for the variables, and the order and frequency of occurrence of the terms), we are done by either Lemma 3.8 or the induction hypothesis.

If, without loss of generality, we assume that α_3 has the turns $(\underline{F},1)$ and $(\underline{F},2)$ and β_3 has the turns $(\underline{F},1)$, $(\underline{F},2)$, and $(\underline{F},3)$, then by Lemma 3.8 and the induction hypothesis, we can assume that $\alpha_3 = \langle(\underline{F},1), ..,(\underline{F},1),(\underline{F},2),x\rangle$, with possibly $(\underline{F},1)$ occurring only twice, and $\beta_3 = \langle(\underline{F},3),...,(\underline{F},3),(\underline{F},2),(\underline{F},1),y\rangle$, with possibly $(\underline{F},3)$ occurring only once (see Figure 3.5).

Let z be the variable of the string $\gamma_3 = <(\underline{F},1),\ldots$ $\ldots,(\underline{F},1),(\underline{F},3),(\underline{F},2),z>$, of $\underline{P}_3$. By Lemma 3.8, we can switch the subtrees $\underline{P}_x = \underline{F}(a,x,b)$ and $\underline{P}_z = \underline{F}(c,z,d)$ of $\underline{P}$. By the induction hypothesis, we now can switch the variables x and y, and then again by Lemma 3.8, we can switch the new subtree $\underline{P}_x' = \underline{F}(a,y,b)$ with $\underline{P}_z$, to obtain the desired P'.

(ii) If α and β terminate in different subtrees $\underline{P}_i$ of $\underline{P}$, we can without loss of generality, assume that α terminates in $\underline{P}_1$ and β terminates in $\underline{P}_2$. If α_1 and β_2 are the strings α and β restricted to $\underline{P}_1$ and $\underline{P}_2$, we can assume by Lemma 3.8 or the induction hypothesis, that the first term of α_1 is $(\underline{F},3)$, and the first term of β_2 is $(\underline{F},1)$.

If we now switch the middle branch of $\underline{P}_1$ with the left branch of $\underline{P}_2$, we can switch the variables x and y by (i) of this proof, and then switch back the above branches to obtain the desired $\underline{P}'$, completing the proof.

We are now ready to prove Theorem 3.2.

<u>Proof of Theorem 3.2</u>. Let $\underline{P} = \underline{Q}$ be a semilattice hyperidentity of type <3>. Then $\underline{P} \Leftrightarrow \underline{Q}$.

<u>Proof</u>. Obviously $V(\underline{P}) = V(\underline{Q})$. Let $V(\underline{P}) = \{x_1,x_2,\ldots,x_n\}$, where for $i \in \{1,2,3\}$, $x_i \in V_i(\underline{P})$, and let $m = \sup\{l(\underline{P}), l(\underline{Q})\}$. Whenever a string of $\underline{P}$ or $\underline{Q}$ terminates in some variable x_i, $x_i \in V(\underline{P})$, and the length of the string is less

than m+2, we can, by $\Sigma\langle 3\rangle(i)$, replace x_i by $\underline{F}(x_i,x_i,x_i)$, obtaining trees $\underline{P}^*$ and $\underline{Q}^*$ of uniform length m+2.

Let $S_{i,j}(\underline{P})$ be the set of strings of $\underline{P}$ which contain both $(\underline{F},i)$ and $(\underline{F},j)$ as terms, but no terms $(\underline{F},k)$ with $k \neq i$ or j. We similarly define $S_i(\underline{P})$ and $S_{1,2,3}(\underline{P})$. Due to $l(\underline{P}^*) = m+2$, we have $V_{i,j}(\underline{P}) = V_{i,j}(\underline{Q})$ = the set of variables of the strings in $S_{i,j}(\underline{P}^*)$, $i,j \in \{1,2,3\}$, and $V(\underline{P}) = V(\underline{Q})$ = the set of variables of the strings in $S_{1,2,3}(\underline{P}^*)$.

By Lemma 3.9, the variables of $S_{1,2,3}(\underline{P}^*)$ can be rearranged in any desirable order. If $x_j \in V(\underline{P})$, $j \neq 1$, occurs more than once as a variable terminating a string of $S_{1,2,3}(\underline{P}^*)$, we can replace all but one of the x_j's by x_1's as follows. Let

$$\underline{P}_1^* = \underline{F}(\underline{F}(x_1,x_1,x_1),\underline{F}(x_1,x_1,x_1),\underline{F}(x_1,x_1,x_1))$$

be the subtree of $\underline{P}^*$ of length two, terminating the string $\langle(\underline{F},1),\ldots,(\underline{F},1),\underline{P}_1^*\rangle$. By Lemma 3.9, we can switch two of the x_j's with two x_1's, to obtain the subtree

$$\underline{P}_1^{*\prime} = \underline{F}(\underline{F}(x_1,x_1,x_1),\underline{F}(x_1,x_1,x_j),\underline{F}(x_1,x_j,x_1)),$$

and then we can replace one of the x_j's of $\underline{P}_1^{*\prime}$ by x_1. We can continue in this manner until x_1 is the only variable terminating more than one string of $S_{1,2,3}(\underline{P}^*)$. The variables of the strings of $S_{1,2,3}(\underline{P}^*)$ can now be arranged in any desirable order, for example, in the order $x_1, x_1, \ldots, x_1, x_2, x_3, \ldots, x_n$.

We shall now consider the strings of $S_{i,j}(\underline{P}^*)$, $i,j \in \{1,2,3\}$. We shall assume, without loss of generality, that $i = 2$ and $j = 3$. If x_k, $k \neq 2$, occurs more than once as a variable of $S_{2,3}(\underline{P}^*)$, then as above, all but one of the x_k's can be replaced by x_2's, by considering the subtree

$$\underline{P}_2^* = \underline{F}(\underline{F}(x_2,x_2,x_2),\underline{F}(x_2,x_2,x_2),\underline{F}(x_2,x_2,x_2))$$

of $\underline{P}^*$, where $\underline{P}_2^*$ terminates the string $\langle(\underline{F},2),(\underline{F},2),\ldots,(\underline{F},2),\underline{P}_2^*\rangle$. As above, the variables of $S_{1,2}(\underline{P}^*)$ can again be arranged in any desired order. We can use this procedure to obtain hyperpolynomials $\underline{P}'$ and $\underline{Q}'$ such that $\underline{P} \Leftrightarrow \underline{P}'$, $\underline{Q} \Leftrightarrow \underline{Q}'$, and $\underline{P}'$ and $\underline{Q}'$ are formally equal, completing the proof of the theorem.

In Chapter Two it was proved that an algebra A satisfies all semilattice hyperidentities of type <2> iff it is a Płonka sum of diagonal algebras. In the remainder of this section we shall prove a similar structure theorem for algebras of type <3>.

Lemma 3.10. Let $A = \langle A; F\rangle$ be an algebra of type <3> satisfying all semilattice hyperidentities. Then the polynomial

$$f(x,y) = F(F(x,x,F(x,y,x)),x,x)$$

is a partition function for A in the sense of J. Płonka [20]. This means that the following are identities of A:

(i) $f(f(x,y),z) = f(x,f(y,z))$,

(ii) $f(x,x) = x$,

(iii) $f(x,f(y,z)) = f(x,f(z,y))$,

(iv) $f(F(x_1,x_2,x_3),y) = F(f(x_1,y),f(x_2,y),f(x_3,y))$,

(v) $f(y,F(x_1,x_2,x_3)) = f(y,F(f(y,x_1),f(y,x_2),f(y,x_3)))$,

(vi) $f(F(x_1,x_2,x_3),x_k) = F(x_1,x_2,x_3)$, $k \in \{1,2,3\}$,

(vii) $f(y,F(y,y,y)) = y$.

Proof. Note that the string terminating with y contains all three turns (F,1), (F,2) and (F,3) of F, and consequently, if F is replaced by any ternary semilattice polynomial other than $F(x,y,z) = x \vee y \vee z$, we have $f(x,y) = x$. From this it is obvious that (i) to (vii) (viewed as hyperidentities) are semilattice hyperidentities, and consequently, identities of A, completing the proof.

Consider the class of all algebras $A = \langle A; F\rangle$ of type $\langle 3\rangle$, where A has a semilattice operation $\vee$ and F is the polynomial $x_i \vee x_j$, for two fixed integers (possibly equal) $i,j \in \{1,2,3\}$. The smallest variety generated by these algebras will be denoted by $D_{i,j}$. Let S be the variety generated by all algebras $A = \langle A; F\rangle$ of type $\langle 3\rangle$, where A is a semilattice, and F is the semilattice polynomial $F(x,y,z) = x \vee y \vee z$. Let $s(\Sigma\langle 3\rangle)$ denote the set of all hyperidentities obtained by applying Lemma 3.1 to $\Sigma\langle 3\rangle$, and recall that $(s(\Sigma\langle 3\rangle))^*$ is the set of identities obtained from $s(\Sigma\langle 3\rangle)$ by replacing each occurrence of $\underline{F}$ by F.

Theorem 3.11. If A is an algebra of type $\langle 3\rangle$ the following are equivalent:

(i) A satisfies all the semilattice hyperidentities;

(ii) A is a Płonka sum (See [14] for the definition of this concept) of algebras of the variety

$$D = \bigvee(D_{i,j} \mid i,j \in \{1,2,3\});$$

(iii) $A \in S \vee D$;

(iv) The operation F of A satisfies $s(\Sigma\langle 3\rangle)$, that is, A satisfies $(s(\Sigma\langle 3\rangle))^*$.

Proof. (i) implies (ii): By Lemma 3.10 and a result of Płonka [20], A is a Plonka sum of algebras $A = \langle A; F\rangle$ of type $\langle 3\rangle$, satisfying $(H(SL))^*$ and $f(x,y) = x$ (where $f(x,y)$ is the partition function of Lemma 3.10). Now the identities of D are those which will be regular whenever F is replaced by any semilattice polynomial $x_i \vee x_j$, $i,j \in \{1,2,3\}$. That means that if P is a polynomial, P_1 is a subtree of P such that the string terminating with P_1 contains the terms (F,i), (F,j), (F,k), $\{i,j,k\} = \{1,2,3\}$, and if P' is obtained by replacing P_1 by any tree P_2 (in the operation F) whatsoever, then P' = P will be an identity holding in D. The smallest class of identities generated by $(H(SL)\langle 3\rangle)^*$ in this way will be exactly the class of all identities of D. We claim that these identities are a consequence of

$$\Sigma' = (H(SL)\langle 3\rangle)^* \cup \{f(x,y) = x\}.$$

In order to prove the claim, we first prove that the identity

$$\varepsilon: F(a,b,F(F(m,y,m),c,d)) = F(a,b,F(F(m,z,m),c,d))$$

is a consequence of Σ'. (In this proof only, $P \Leftrightarrow Q$ will denote $P = Q$ is a consequence of Σ'.)

$F(a,b,F(F(m,y,n),c,d))$

$\Leftrightarrow$ $F(a,b,F(F\{F[F(m,m,F(m,y,m)),m,m],n,n\},c,d))$ since it is a semilattice hyperidentity,

$\Leftrightarrow$ $F(a,b,F(F\{F[F(m,m,F(m,z,m)),m,m],n,n\},c,d))$ since $f(x,y) = x$,

$\Leftrightarrow$ $F(a,b,F(F(m,z,n),c,d))$ since it is again a semilattice hyperidentity.

If P1 is a subtree of P, such that the string terminating with P1 contains the terms (F,1), (F,2) and (F,3), then we can assume (By the identities of $(H(SL)\langle 3\rangle)^*$) that the last three terms preceding P1 are (F,1), (F,2) and (F,3), and our claim now follows from the identity ε, thus proving that (i) implies (ii).

(ii) implies (iii) follows immediately from Płonka's theorem [20], that if A is a Plonka sum of algebras of D then A will satisfy the regular identities of D, that is, all the identities of Id(S) $\cap$ Id(D).

(iii) implies (iv). $(s(\Sigma\langle 3\rangle))^*$ is a set of semilattice identities, and it is clear from the proof of (i) implies (ii), that $(s(\Sigma\langle 3\rangle))^*$ is also satisfied by D, so $(s(\Sigma 3))^* \subseteq$ Id(S) $\cap$ Id(D).

(iv) implies (i). Let the algebra A = <A; F> satisfy $(s(\Sigma\langle 3\rangle))^*$. Using exactly the same methods as were used in proving Theorem 3.2, one can show that $(H(SL)\langle 3\rangle)^*$ is a consequence of $(s(\Sigma\langle 3\rangle))^*$. The operation F of A thus satisfies H(SL)<3>, and consequently H(SL), completing the proof by Lemma 1.4.

We now define Σ^3 to be the set of hyperidentities consisting of $\Sigma\langle 3\rangle$ and the median hyperidentity of type <3,3>, that is,

$$\Sigma^3 = \Sigma\langle 3\rangle \cup \{\underline{F}(\underline{G}(x_1,x_2,x_3),\underline{G}(y_1,y_2,y_3),\underline{G}(z_1,z_2,z_3)) = \underline{G}(\underline{F}(x_1,y_1,z_1),\underline{F}(x_2,y_2,z_2),\underline{F}(x_3,y_3,z_3))\}$$

In the remaining part of this chapter $\underline{P} \Leftrightarrow \underline{Q}$ will denote $\underline{P} = \underline{Q}$ is a consequence of Σ^3.

In order to show that Σ^3 is a basis for all semilattice hyperidentities of type $\underline{3} = \langle 3,3,\ldots\rangle$, we need a few additional definitions. The <u>height</u> of a symbol $\underline{F}$ (or of the corresponding node of the formation tree), in a hyperpolynomial $\underline{P}$, will be said to be the length of the string terminating with the subtree $\underline{P}'$, whose root is at the node in question. If $\underline{P}$ is a hyperpolynomial of type $\underline{n}$, with symbols $\underline{F}_1$, $\underline{F}_2$, ..., $\underline{F}_n$, we shall say that $\underline{P}$ is <u>uniform</u> iff the following conditions hold:

(i) $\underline{P}$ is of uniform length,

(ii) If $\underline{F}_i$, $i \in \{1,2,\ldots,n\}$, occurs in $\underline{P}$ then $\underline{F}_i$ occurs the same number of times in each string of $\underline{P}$,

(iii) If each symbol $\underline{F}_i$ occurs n_i times in every string of $\underline{P}$, $i \in \{1,\ldots,n\}$, and if α is any string of $\underline{P}$, then the first n_1 terms of α will have $\underline{F}_1$ as a function symbol, the next n_2 terms will have $\underline{F}_2$ as a function symbol, and so on.

That is, if P is uniform, the formation tree of $\underline{P}$ will have a layer of $\underline{F}_1$'s at the bottom, a layer of $\underline{F}_2$'s next, and so on, with all the $\underline{F}_n$'s appearing in the top layer.

<u>Lemma 3.12</u>. Let $\underline{P}$ be a uniform hyperpolynomial in the ternary operation symbols $\underline{F}_1$, $\underline{F}_2,\ldots,\underline{F}_n$. If α and β are strings of $\underline{P}$ such that $\alpha =_r b$, and if the variables of α and β are switched to obtain a hyperpolynomial P', then $\underline{P} \Leftrightarrow \underline{P}'$.

Proof. We shall induct on $l(\underline{P})$. If $l(\underline{P}) = 0$, the lemma is true vacuously (if $l(\underline{P}) = 1$ or 2 the lemma is also obviously true). We shall now assume that the lemma is true for any uniform hyperpolynomial of length less than m and let $l(\underline{P}) = m$. Let x and y be the variables of α and β respectively.

If α and β terminate in the same proper subtree $\underline{P}_i$ of $\underline{P} = \underline{F}_1(\underline{P}_1,\underline{P}_2,\underline{P}_3)$, let α_i and β_i be the substrings α and β restricted to $\underline{P}_i$. If $\alpha_i =_r \beta_i$, we are done by the induction hypothesis. If $\alpha_i \neq_b \beta_i$, either α_i or β_i has a term $(\underline{F}_1,k)$ occurring at least twice. By Theorem 3.2, we can thus switch subtrees (in the symbols $\underline{F}_2,\ldots,\underline{F}_n$) of $\underline{P}$, to obtain a tree $\underline{P}^*$ with strings α^* and β^* corresponding to α and β (that is, terminating in the variables x and y in question), such that $\alpha_i^* =_r \beta_i^*$, where α_i^* and β_i^* are the restrictions of α^* and β^* to $\underline{P}_i^*$, and we're done by the induction hypothesis.

If α and β do not terminate in the same subtree $\underline{P}_i$, we can assume that $n > 1$, since otherwise we are done by Theorem 3.2. By Theorem 3.2, we can also assume that the first occurrence of the symbol $\underline{F}_n$ in both α and β is the term $(\underline{F}_n,k)$, for some $k \in \{1,2,3\}$, and that the term $(\underline{F}_n,k)$ occurs the same number of times in α and β. By the ternary median hyperidentity, the bottom row of $\underline{F}_n$'s, of the tree depicting $\underline{P}$, can be moved down to obtain a tree $\underline{P}^*$, with $\underline{F}_n$

at the root, then a layer consisting of all the $\underline{F}_1$'s, and so on, with the remaining $\underline{F}_n$'s at the top. If α^* and β^* are the strings corresponding to α and β, that is, the strings terminating in x and y, then α^* and β^* will now both have $(\underline{F}_n, k)$ as their first term. Thus they both terminate in the subtree $\underline{P}_k^*$ of $\underline{P}^*$, where $\underline{P}^* = \underline{F}_n(\underline{P}_1^*, \underline{P}_2^*, \underline{P}_3^*)$. Since $\alpha_k^* =_r \beta_k^*$, we can now switch x and y, and then again use the median hyperidentity to obtain the desired hyperpolynomial $\underline{P}'$, completing the proof of this lemma.

Lemma 3.13. Let $\underline{P}$ and $\underline{Q}$ be ternary hyperpolynomials containing the operation symbols $\underline{F}_1, \ldots, \underline{F}_n$. Then there is a sequence of integers $m_1, m_2, \ldots, m_n$, and there are uniform hyperpolynomials $\underline{P}'$ and $\underline{Q}'$, such that $\underline{P} \Leftrightarrow \underline{P}'$, $\underline{Q} \Leftrightarrow \underline{Q}'$ and for every $k \in \{1,2,\ldots,n\}$, $\underline{F}_k$ occurs m_k times in each string of $\underline{P}'$ and $\underline{Q}'$.

Proof. If for some i, $0 < i < n+1$, the number of occurrences of $\underline{F}_i$ in each string of $\underline{P}$ and $\underline{Q}$ is not the same, then let m_i be the maximum number of times that F_i occurs in any string of $\underline{P}$ or $\underline{Q}$. If a string α has not got enough $\underline{F}_i$'s, then by idempotence, the variable x of α can be replaced by $\underline{F}_i(x,x,x)$. This procedure can be continued until condition (ii) of the definition of uniformity is met.

We now use the median property to "layer" $\underline{P}$. Of all the $\underline{F}_1$'s which have an $\underline{F}_j$, $j \neq 1$, immediately below it, pick one of maximal height. The subtree $\underline{P}_s$, whose root is immediately below this $\underline{F}_1$, will be of the form

$$\underline{P}_s = \underline{F}_j(\underline{F}_1(\underline{P}_1,\underline{P}_2,\underline{P}_3),\underline{F}_1(\underline{P}_4,\underline{P}_5,\underline{P}_6),\underline{F}_1(\underline{P}_7,\underline{P}_8,\underline{P}_9),$$

since otherwise, the maximality of the height of our choice of $\underline{F}_1$ would be contradicted. We can now move $\underline{F}_1$ down to the root of $\underline{P}_s$ by the median hyperidentity, and continue this procedure until we have the desired $\underline{P}'$.

Lemma 3.14. Let $\underline{P}$ be a uniform hyperpolynomial with operation symbols $\underline{F}_1,\dots,\underline{F}_n$, such that for each $i < n$, the symbol $\underline{F}_i$ occurs at least three times in each string. Let α, β and γ be strings of $\underline{P}$ such that $\alpha =_r \beta =_r \gamma$, with α and β terminating with the same variable x, and γ terminating with z. Then if $\underline{P}'$ is obtained from $\underline{P}$ by replacing the x of either α or β by z, $\underline{P} \Leftrightarrow \underline{P}'$.

Proof. We shall induct on the length of $\underline{P}$. If $l(\underline{P}) = 0$ or 1, the lemma is obviously true, so assume that it is true for $l(\underline{P}) < m$ and let $l(\underline{P}) = m$. If $n = 1$, the lemma follows from Theorem 3.2, so assume that $n > 1$.

Let the number of occurrences of the symbol $\underline{F}_1$ in any string of $\underline{P}$ be k. Since $k > 2$, we can assume by Lemma 3.12, that the $(k + 1)$'th nodes of α, β and γ are distinct (if this is not possible, then α, β and γ terminate in a common subtree in the symbols $\underline{F}_2,\dots,\underline{F}_n$, and we're done by the induction hypothesis). We can thus also assume that the first occurrence of $\underline{F}_n$ in α, β or γ is the term $(\underline{F}_n,k)$, for some $k \in \{1,2,3\}$, and that $(\underline{F}_n,k)$ occurs the same number of times in each of these three strings. If we now apply the median hyperidentity of type $\langle 3,3\rangle$ to move the bottom row of $\underline{F}_n$'s to the root of $\underline{P}$, we obtain the hyperpolynomial $\underline{P}^* = \underline{F}_n(\underline{P}_1^*,\underline{P}_2^*,\underline{P}_3^*)$, with α^*, β^* and γ^*, the strings corresponding to α, β and γ, all terminating in $\underline{P}_k^*$. If α_k^*, β_k^* and γ_k^* are the restrictions of α^*, β^* and γ^* to $\underline{P}_k^*$, then $\alpha_k^* =_r \beta_k^* =_r \gamma_k^*$, and we are done by the induction hypothesis.

<u>Theorem 3.15</u>. The set Σ^3 is a basis for all ternary semilattice hyperidentities.

<u>Proof</u>. let $\underline{P} = \underline{Q}$ be a semilattice hyperidentity in the symbols $\underline{F}_1,\ldots,\underline{F}_n$. We can assume, by Lemma 3.13, that $\underline{P}$ and $\underline{Q}$ are uniform, and that for every $k \in \{1,\ldots,n\}$, $\underline{F}_k$ occurs the same number of times in $\underline{P}$ and $\underline{Q}$. By idempotence, we can add two more rows of $\underline{F}_k$'s to $\underline{P}$ and $\underline{Q}$ for every $k \in \{1,\ldots,n\}$, to obtain trees $\underline{P}^*$ and $\underline{Q}^*$. This insures that if $\alpha < \beta$ are strings of $\underline{P}^*$, then there is a string γ of $\underline{P}^*$, such that $\gamma =_r \beta$, and α and γ terminate with the same variable. Thus, if each $\underline{F}_i$, $i \in \{1,\ldots,n\}$, is replaced by a semilattice polynomial $p_i = \bigvee(x_k \mid k \in I_i)$, $I_i \subseteq \{1,2,3\}$, then x is a variable of $\underline{P}(p_1,\ldots,p_n)$ iff $\underline{P}^*$ has a string γ, such that the set of terms of γ is $\{(\underline{F}_i,k) \mid i \in \{1,\ldots,n\}, k \in I_i\} \cup \{x\}$.

Let $S_\alpha(\underline{P}^*) = \{\beta |\ \beta$ is a string of $\underline{P}^*$ and $\beta =_r \alpha\}$ and let $V(S_\alpha(\underline{P}^*))$ be the set of variables of the strings in $S_\alpha(\underline{P}^*)$. It is clear from the above paragraph that if α and β are strings of $\underline{P}^*$ and $\underline{Q}^*$ such that $\alpha =_r \beta$, then $V(S_\alpha(\underline{P}^*)) = V(S_\beta(Q^*))$.

For any set $S_\alpha(\underline{P}^*)$ of strings of $\underline{P}^*$, we can pick a variable $z \in V(S_\alpha(\underline{P}^*))$, and by Lemma 3.14, we can obtain a tree $\underline{P}''$, such that z is the only variable occurring more than once in a string of $S_\alpha(P'')$. By Lemma 3.12, these can be rearranged in any desirable order. This procedure can thus be used to obtain hyperpolynomials $\underline{P}'$ and $\underline{Q}'$, such that $\underline{P} \Leftrightarrow \underline{P}'$, $\underline{Q} \Leftrightarrow \underline{Q}'$ and $\underline{P}'$ and $\underline{Q}'$ are formally equal, completing the proof of the theorem.

Rather than proving the general result, we shall show how the results of this chapter easily generalize to the case n = 4, or any higher arity.

The basis $\Sigma\langle 4\rangle$ for all semilattice hyperidentities of type $\langle 4\rangle$ is the following set of hyperidentities:

(i) $\underline{F}(x,x,x,x) = x$,

(ii) $\underline{F}(\underline{F}(a,b,c,d),\underline{F}(x,y,z,u),p,r) =$
$\underline{F}(\underline{F}(a,x,c,d),\underline{F}(b,y,z,u),p,r)$,

(iii) $\underline{F}(\underline{F}(a,b,c,d),\underline{F}(x,x,y,z),p,r) =$
$\underline{F}(\underline{F}(a,b,c,d),\underline{F}(b,x,y,z),p,r)$,

and $\Sigma^4 = \Sigma\langle 4\rangle \cup$

$$\{\underline{F}(\underline{G}(x_1,..,x_4),\underline{G}(y_1,..,y_4),\underline{G}(z_1,..,z_4),\underline{G}(u_1,..,u_4)) = \underline{G}(\underline{F}(x_1,y_1,z_1,u_1),\underline{F}(x_2,...,u_2),\underline{F}(x_3,..,u_3),\underline{F}(x_4,..,u_4))\}.$$

The proof that $\Sigma\langle 4\rangle$ is a basis for the semilattice hyperidentities of type $\langle 4\rangle$ is almost identical to the case $n = 3$, except that Lemma 3.9 would have to be replaced by a lemma which examines $\alpha =_r \beta$, where α and β contain three or four different turns. This can be generalized to the n-ary case, where we consider strings α and β containing up to n distinct turns, by an easy inductive proof. In general, we thus have that $\Sigma\langle n\rangle$ is a basis for $H(SL)\langle n\rangle$ and Σ^n is a basis for $H^n(SL)$, where $\Sigma\langle n\rangle$ is the following set of n-ary hyperidentities:

(i) $\underline{F}(x,...,x) = x$,

(ii) $\underline{F}(\underline{F}(x_1,..,x_n),\underline{F}(y_1,..,y_n),z_1,..,z_{n-2}) = \underline{F}(\underline{F}(x_1,y_1,x_3,x_4,..,x_n),\underline{F}(x_2,y_2,y_3,..,y_n),z_1,..,z_{n-2})$,

(iii) $\underline{F}(\underline{F}(x_1,..,x_n),\underline{F}(u,u,y_1,..,y_{n-2}),z_1,..,z_{n-2}) = \underline{F}(\underline{F}(x_1,..,x_n),\underline{F}(x_2,u,y_1,..,y_{n-2}),z_1,..,z_{n-2})$,

and $\Sigma^n = \Sigma\langle n\rangle \cup \{\underline{F}(\underline{G}(x_{11},..,x_{1n}),...,\underline{G}(x_{n1},..,x_{nn})) = \underline{G}(\underline{F}(x_{11},..,x_{n1}),...,\underline{F}(x_{1n},..,x_{nn}))\}$.

By Lemma 1.5, $\Sigma = \bigcup(\Sigma^n \mid 0 < n < \omega)$ is a basis for $H(SL)$.

The partition function for $n = 4$ is

$$f(x,y) = F(x,x,x,F(x,x,F(F(x,y,x,x),x,x,x),x)),$$

and again $f(x,y) = x$ whenever F is replaced by any lattice polynomial other than $F = x_1 \vee x_2 \vee x_3 \vee x_4$. In general, the formation tree of the partition function $f(x,y)$, has one leaf labelled with y, all the other leaves labelled with the variable x, and the string terminating with y contains all of the n distinct terms $(\underline{F},1),(\underline{F},2),\ldots,(\underline{F},n)$.

The structure theorem in the case $n = 4$, is like the case $n = 3$, with

$$D = \bigvee(D_{i,j,k} \mid i,j,k \in \{1,2,3,4\}),$$

where the $D_{i,j,k}$'s are defined analogously to the ternary case. The generalization to an arbitrary n is again obvious.

CHAPTER IV

LATTICE HYPERIDENTITIES

Recall that a hyperidentity α is called binary if $\alpha \in H^2(V)$. Since L, the variety of all lattices, has only four binary polynomials, it is easy to check whether or not a binary hyperidentity holds in L. As was pointed out in Chapter One, a proof that

$$\epsilon: \quad \underline{F}(\underline{G}(x,y),x) = \underline{G}(x,\underline{F}(y,x))$$

holds in L, consists of a case by case examination of $(\underline{F},\underline{G}) \in \{x, y, x \vee y, x \wedge y\}^2$. For example, if $(\underline{F},\underline{G}) = (x, x \vee y)$, we obtain the lattice identity $x \vee y = x \vee y$. Lattice hyperidentities in two or more binary operation symbols, such as the above hyperidentity ϵ, rely strongly on the absorption property of lattices for their validity. Note however, that the absorption hyperidentity $\underline{F}(\underline{G}(x,y),x) = x$ is not satisfied by L, the variety of all lattices, since it is not regular and thus does not yield a lattice identity when $(\underline{F},\underline{G}) = (x \vee y, x \vee y)$.

The associative hyperidentity,

$$\alpha: \quad \underline{F}(x,\underline{F}(y,z)) = \underline{F}(\underline{F}(x,y),z),$$

is another example of a binary hyperidentity which holds in L. It is easily seen that there are hyperidentities which hold in some but not all lattice varieties. For example, the hyperidentity

$$\epsilon_1: \underline{F}(\underline{G}(x,y),z) = \underline{G}(\underline{F}(x,z),\underline{F}(y,z))$$

is satisfied by D, the variety of all distributive lattices, but none of the other nontrivial lattice varieties, and

$$\epsilon_2: \underline{F}(\underline{G}[x,\underline{F}(y,z)],\underline{G}(y,z)) = \underline{G}(\underline{F}[x,\underline{G}(y,z)],\underline{F}(y,z))$$

is satisfied by all modular lattices, but not L. In [25], W. Taylor proves that there exist $2^{\aleph_0}$ hypervarieties, all containing different lattice varieties, by proving that the map $V \to V_h(V)$, from varieties to the hypervarieties generated by the varieties, is one-one on self-dual lattice varieties.

In the above examples we have been discussing only binary lattice hyperidentities. Lattice hyperidentities involving ternary symbols are, in general, much more difficult to check, since there are infinitely many polynomials to substitute. It is thus natural to ask whether all lattice hyperidentities are determined by those involving only binary functions [25, Problem 3]. The first result of this chapter answers this question in the negative.

<u>Theorem 4.1</u>. Let V be a nontrivial variety of lattices or the variety of all semilattices. Then for any integer m, there exists a hyperidentity ϵ, such that ϵ holds in V but ϵ is not a consequence of $H^m(V)$.

Hyperidentities with at most two variables are also relatively easy to verify in L. Consider, for example, the lattice hyperidentity

$$\alpha: \underline{F}(x,x,y) = \underline{F}(x,\underline{F}(x,x,y),\underline{F}(y,x,y)).$$

Since α has only two variables, it is obvious that α will be a lattice hyperidentity iff α is a distributive lattice hyperidentity. Thus to verify that α holds in L, it is enough to check α for each of the 18 ternary distributive lattice polynomials. It is thus natural to ask, analogously to [25, Problem 3], whether $H_2(L)$ contains a basis for $H(L)$. This is not the case either.

<u>Theorem 4.2</u>. Let V be a nontrivial variety of lattices or the variety of all semilattices. Then for any integer n, there exists a hyperidentity ϵ, such that ϵ holds in V but ϵ is not a consequence of $H_n(V)$.

We shall prove Theorems 4.1 and 4.2 by constructing a lattice hyperidentity ϵ which is not a consequence of $H^m(V) \cup H_n(V)$, thus proving the following theorem.

<u>Theorem 4.3</u>. Let V be the variety of all semilattices or a nontrivial variety of lattices. For any two integers m and n, there exists a hyperidentity ϵ, such that ϵ holds in V but ϵ is not a consequence of $H^m(V) \cup H_n(V)$.

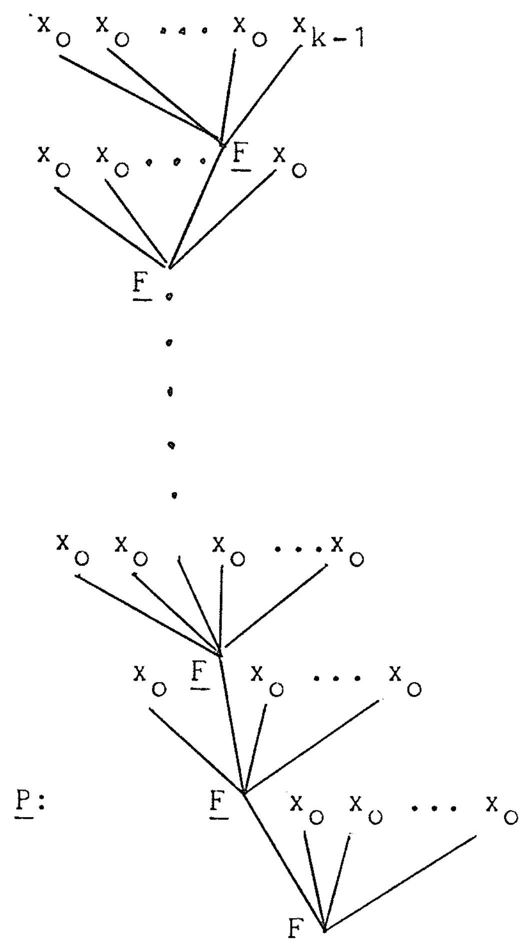


Figure 4.1

Corollary 4.4. The hyperidentities of V are not finitely based.

Proof of Theorem 4.3.

Given V, m, and n, we shall construct a hyperidentity ϵ which is satisfied by L, the variety of all lattices. From this we shall deduce that ϵ is satisfied by any nontrivial variety of lattices, and also by SL, the variety of all semilattices. Finally, we shall exhibit an algebra A which satisfies $H^m(V) \cup H_n(V)$ but not ϵ.

We begin by defining a hyperpolynomial:

$$\underline{P} = \underline{F}(\underline{F}(x_o,\underline{F}(x_o,x_o,\underline{F}(\ldots,\underline{F}(x_o,\ldots,x_o,\underline{F}(x_o,\ldots,x_o,x_{k-1}),x_o),\ldots \ldots)\ldots),\ldots),x_o,\ldots,x_o,x_o),$$

where $k = 2^{\sup\{m,n\}}$, $\underline{F}$ is k-ary, and x_o is the only variable except for the single occurrence of x_{k-1}. See Figure 4.1 for the tree depicting $\underline{P}$. We also set $t = x_o \vee x_1 \vee x_2 \vee \ldots \vee x_{k-1}$, $b = x_o \wedge x_1 \wedge x_2 \wedge \ldots \wedge x_{k-1}$, and if p is any k-ary lattice polynomial, for $0 \leq i < k$, we set

$$p_i = p(x_o,x_o,\ldots x_o,x_{k-1},x_o,\ldots x_o),$$

where x_{k-1} is substituted for the i'th variable of p, and x_o is substituted for all the other variables. Obviously $p_i \in \{x_o, x_{k-1}, x_o \vee x_{k-1}, x_o \wedge x_{k-1}\}$, for $0 \leq i < k$.

We now claim that if p is a lattice polynomial, then $\underline{P}(p) = x_o \vee x_{k-1}$ if $p = t$, $\underline{P}(p) = x_o \wedge x_{k-1}$ if $p = b$, and $\underline{P}(p) = x_o$ otherwise. We discuss several cases:

(i) If for some i $(0 \leq i < k)$, $p_i = x_o$, then $\underline{P}(p) = x_o$.

(ii) If for some i $(0 \leq i < k)$, $p_i = x_{k-1}$, then since p is isotone,

$$x_i = p(b,\dots,b,x_i,b,\dots b) \le p(x_o,x_1,\dots,x_{k-1})$$
$$\le p(t,\dots,t,x_i,t,\dots,t) = x_i.$$

Thus, if $p_i = x_{k-1}$, then $p = x_i$, and again $\underline{P}(p) = x_o$.

(iii) If for all i $(0 \le i < k)$, $p_i = x_o \vee x_{k-1}$ or $p_i = x_o \wedge x_{k-1}$, and for some i,j $(0 \le i,j < k)$, $p_i \ne p_j$, then due to the absorption identity we again obtain $\underline{P}(p) = x_o$.

(iv) Finally, if for all i $(0 \le i < k)$, $p_i = x_o \vee x_{k-1}$, then

$$p = p(x_o,x_1,\dots,x_{k-1}) \vee p(x_o,b,b,\dots,b) \vee$$
$$p(b,x_1,b,\dots,b) \vee p(b,\dots,b,x_{k-1})$$
$$= x_o \vee x_1 \vee \dots \vee x_{k-1},$$

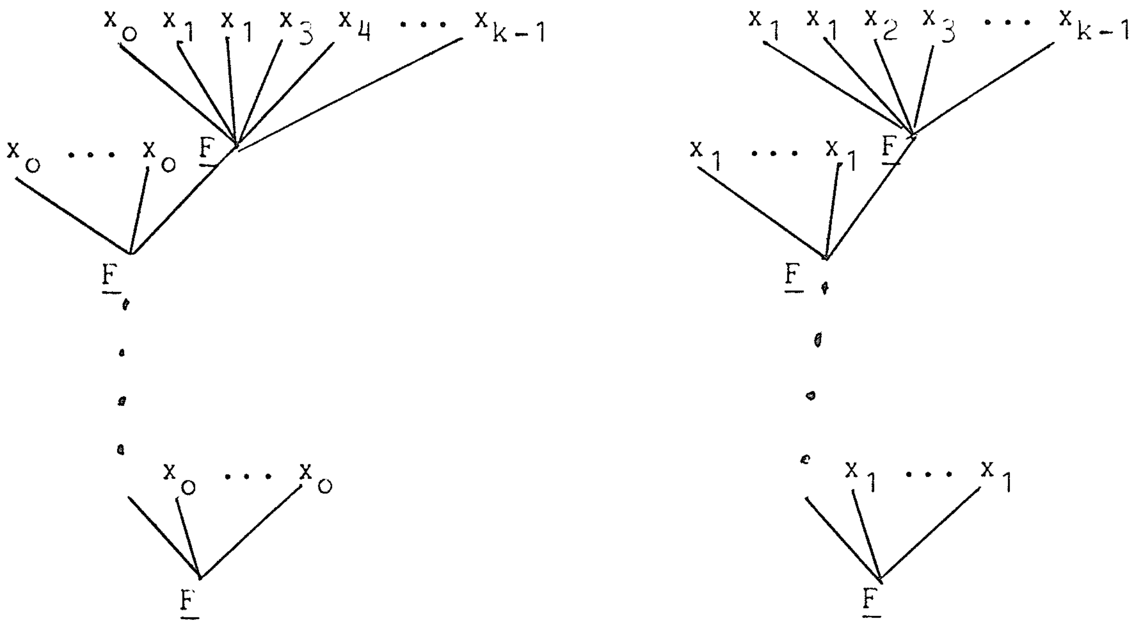
and $\underline{P}(p) = x_o \vee x_{k-1}$ (and dually if $p_i = x_o \wedge x_{k-1}$ for all i, $0 \le i < k$).

We now define $\underline{Q}_i$, for $0 \le i < k-1$, to be the hyperpolynomial obtained from $\underline{P}$ by replacing x_{k-1} by

$$\underline{G}_i = \underline{F}(x_o,x_1,\dots,x_{i-1},x_i,x_i,x_{i+2},\dots,x_{k-1}),$$

where $\underline{G}_i$ is obtained from $\underline{F}$ by replacing x_{i+1} by x_i.

$\underline{Q}_{k-1}$ is obtained from $\underline{P}$ by replacing all the x_o's by



$\underline{Q}_1$: $\underline{Q}_{k-1}$:

Figure 4.2

x_1, and replacing x_{k-1} by $\underline{F}(x_1, x_1, x_2, x_3, \ldots, x_{k-1})$. (See Figure 4.2.)

For $0 \le i < k-1$, and $p = t$, we have

$$\underline{Q}_i(p) = x_o \vee x_1 \vee \ldots \vee x_i \vee x_{i+2} \vee \ldots \vee x_{k-1},$$

which is one of the dual atoms of $F_L(k)$ (and dually if $p = b$), and if $p \ne t$ or b, then

$$\underline{Q}_i(p) = x_o.$$

Finally, if $p = t$, then

$$\underline{Q}_{k-1}(p) = x_1 \vee x_1 \vee x_2 \vee x_3 \vee \ldots \vee x_{k-1}$$

Figure 4.3

(and dually if $p = b$), and if $p \neq t$ or b, then

$$\underline{Q}_{k-1} = x_1.$$

We are now ready to define our hyperidentity ϵ. Let

$$\underline{Q} = \underline{F}(\underline{Q}_0, \underline{Q}_1, \underline{Q}_2, \ldots, \underline{Q}_{k-1}),$$

$$R = \underline{F}(\underline{Q}_0, \underline{Q}_0, \ldots \underline{Q}_0, \underline{Q}_{k-1}),$$

as shown in Figure 4.3, and let ϵ be the hyperidentity $\underline{Q} = \underline{R}$.

It is easily seen that $\underline{Q}(t) = \underline{R}(t) = t$, $\underline{Q}(b) = \underline{R}(b) = b$, and if $p \neq t$ or b,

$$\underline{Q}(p) = \underline{R}(p) = p(x_0, x_0, \ldots, x_0, x_1).$$

Thus ϵ is satisfied by L. Obviously, it follows that ϵ will be satisfied by any nontrivial variety of lattices, and also by SL.

We now define **A** = $\langle A; f \rangle$ by letting $A = F_{SL}(k)$, the free semilattice on k generators, $a_0, a_1, \ldots, a_{k-1}$, and defining f as follows:

(i) $f(x_0, x_1, \ldots, x_{k-1}) = x_0$, if $\{x_0, x_1, \ldots, x_{k-1}\}$ is the set of dual atoms of A,

(ii) $f(x_0, x_1, \ldots, x_{k-1}) = x_0 \vee x_1 \vee \ldots \vee x_{k-1}$ otherwise.

If p is any A-polynomial in the variables x_i, $i \in I$, we shall say that $q = \bigvee(x_i \mid i \in I)$, is the semilattice polynomial corresponding to p.

Clearly

$$\underline{Q}(f)(a_0, a_1, \ldots, a_{k-1}) = a_0 \vee a_2 \vee \ldots \vee a_{k-1},$$

$$\underline{R}(f)(a_0, a_1, \ldots, a_{k-1}) = a_0 \vee a_1 \vee \ldots \vee a_{k-1},$$

and thus ϵ is not satisfied by A.

Since the free semilattice on $\sup\{m,n\}$ generators has k-1 elements, any subalgebra of A, generated by at most $\sup\{m,n\}$ elements, cannot contain the set of k dual atoms. Thus every A-polynomial in the variables x_i, $i \in I$, $|I| \leq \sup\{m,n\}$, is equal to the corresponding semilattice polynomial $\bigvee(x_i \mid i \in I)$.

If $\underline{H}$ is a hyperpolynomial with h distinct symbols of arity at most m, $p_0, \ldots, p_{h-1}$ are A-polynomials of appropriate arity, and $q_o, \ldots, q_{h-1}$, are the corresponding semilattice polynomials, then it can easily be shown by induction on the rank of $\underline{H}$, that

$$\underline{H}(p_o, \ldots, p_{h-1}) = \underline{H}(q_o, \ldots, q_{h-1}),$$

the corresponding semilattice polynomial.

Thus A satisfies $H^m(SL)$ and since $H^m(SL)$ clearly contains $H^m(V)$, **A** also satisfies $H^m(V)$.

If $\underline{H}$ is a semilattice hyperidentity with at most n variables, and if $\underline{H}$ has h distinct operation symbols, then for all **A**-polynomials $p_o, \ldots, p_{h-1}$, of appropriate arity, the polynomial $\underline{H}(p_o, \ldots, p_{h-1})$ will have at most n variables and will thus be equal to the corresponding semilattice polynomial. The algebra A thus satisfies all semilattice hyperidentities with at most n variables, and, as in the previous paragraph, this means that A satisfies $H_n(V)$. The algebra **A** thus satisfies $H^m(V) \cup H_n(V)$, but not ϵ, which completes the proof of theorem.

In the beginning of this chapter we showed that in the variety of all lattices, hyperidentities are easier to check if the number of variables, or the arity of their symbols, is restricted to two. It is also easy to verify hyperidentities whose formation trees have the property that the number of distinct branches leaving any node is at most two.

If $\underline{P}$ is a hyperpolynomial of this type, with k operation symbols, and if $p_o,\ldots,p_{k-1}$ and $q_o,\ldots,q_{k-1}$, are two sets of lattice polynomials of appropriate arity such that $p_i = q_i$ $(0 \leq i < k)$ in the variety of distributive lattices, then

$$\underline{P}(p_o,\ldots,p_{k-1}) = \underline{P}(q_o,\ldots q_{k-1})$$

in the variety of lattices. This can easily be proved inductively by using the fact that if p_i and q_i are equal as distributive lattice polynomials, and if all the variables of p_i and q_i are replaced by x_o and x_1 (so that the corresponding variables of p_i and q_i are replaced by the same variable x_i, $i \in \{0,1\}$), then p_i and q_i will yield the same binary lattice polynomial. Thus hyperidentities of this type can again be verified by making a finite number of substitutions. An example of such a lattice hyperidentity is

$$\alpha:\ \underline{F}(\underline{F}(x,y,x),\underline{F}(x,y,x),z) = \underline{F}(\underline{F}(x,x,z),\underline{F}(y,y,z),\underline{F}(x,x,z)).$$

It is clear from the proof of Theorem 4.3 that the class of all lattice hyperidentities with the property that the number of distinct branches leaving any node is bounded, will not contain a basis for H(V).

In Chapter Three we showed that $H^n(SL)$ is finitely based. Whether or not this is the case for $H^n(L)$ remains an open question.

CHAPTER V

THE LATTICE OF HYPERVARIETIES

In this chapter we shall examine the order relationships between $V_h(SL)$, the hypervariety generated by the variety of all semilattices, and some other members of the lattice of hypervartieties such as $V_h(A)$, $V_h(D)$, $V_h(DS)$ and $V_h(M)$ where A is the variety of abelian groups, D the variety of distributive lattices, DS the variety of diagonal semigroups and M is the variety generated by the ternary median algebra. We shall prove that in the lattice of hypervarieties, $V_h(SL) = V_h(A) \wedge V_h(D)$ and $V_h(SL)$ covers $V_h(DS)$, obtaining a basis for H(DS) as an easy corollary.

In order to do so we first give a definition and prove an easy lemma. If $\underline{P}$ is a hyperpolynomial in the operation symbols $\underline{F}_1,\ldots,\underline{F}_n$, $\alpha \in st(\underline{P})$ is a string of $\underline{P}$, and

$$S = \{p_i \mid 0 < i \leq n,\ p_i = \bigvee(x_j \mid j \in J_i)\}$$

is a set of semilattice polynomials such that the arity of p_i is equal to the arity of $\underline{F}_i$, $0 < i \leq n$, then we shall say that <u>α is contained in S</u> iff $(\underline{F}_i,k)$ is a term of α implies that $k \in J_i$. For example, if $\underline{P} = \underline{F}_1(\underline{F}_2(x,y),z)$, $\alpha = \langle(\underline{F}_1,1),(\underline{F}_2,2),y\rangle$ and $S = \{p_1,p_2\}$ where $p_1 = x \vee y$ and $p_2 = y$, then α is contained in S. Note that $y \in V(\underline{P}(p_1,p_2))$.

Lemma 5.1. Let $\underline{P}$ be a hyperpolynomial in the symbols $\underline{F}_1,\ldots,\underline{F}_n$, and $S = \{p_i \mid 0 < i \leq n,\ p_i = v(x_j \mid j \in J_i)\}$ a set of semilattice polynomials of corresponding arities. Then $x \in V(\underline{P}(p_1,\ldots,p_n))$, the set of variables of $\underline{P}(p_1,\ldots,p_n)$, iff there exists a string $\alpha \in st_P(x)$, the set of strings of $\underline{P}$ terminating with x, such that α is contained in S.

Proof. We induct on the length $l(\underline{P})$ of $\underline{P}$. If $l(\underline{P}) = 0$, the statement is obviously true. We now assume that the lemma holds for hyperpolynomials of length less than n and that $l(\underline{P}) = n$. We let $\underline{P} = \underline{F}_k(\underline{P}_1,\ldots,\underline{P}_m)$. Then $x \in V(P(p_1,\ldots,p_n))$ iff $x \in V(P_j(p_1,\ldots,p_n))$ for some $j \in J_k$ iff there exists a string $\alpha \in st_P(x)$ such that α terminates in $\underline{P}_j$, $j \in J_k$ (that is, the first term of α is (F_k,j), $j \in J_k$), and α', the string α restricted to P_j, is contained in S (by the induction hypothesis) iff α is contained in S, completing the proof of the lemma.

Lemma 5.2. $V_h(SL) \leq V_h(A) \wedge V_h(D)$ in the lattice of hypervarieties.

Proof. A hyperidentity $\underline{P} = \underline{Q}$ in the symbols $\underline{F}_1,\ldots,\underline{F}_n$ holds in SL iff $V(\underline{P}(p_1,\ldots,p_n)) = V(\underline{Q}(p_1,\ldots,p_n))$ for all sets $S = \{p_1,\ldots,p_n\}$ of semilattice polynomials of appropriate arity. By Lemma 1.6 and Lemma 5.1, this is the case for any hyperidentity $(\underline{P} = \underline{Q}) \in H(A)$, so we have $H(A) \subseteq H(SL)$ (that is,

$V_h(SL) \subseteq V_h(A))$. Since the clone C(SL) is isomorphic to a subclone of C(D), we have $V_h(SL) \subseteq V_h(D)$, completing the proof of the lemma.

Since the associative and idempotent binary hyperidentities are satisfied by D but not A, and the binary median hyperidentity is satisfied by A but not D, we have $H(SL) < H(A)$, $H(SL) < H(D)$ and $[H^2(SL)] = [H^2(A)] \vee [H^2(D)]$ (in fact, $[H^2(SL)] = [H^2(A)] \vee [H^2(L)]$) in the lattice of closed sets of hyperidentities. We shall prove that $[H^n(SL)] = [H^n(A)] \vee [H^n(D)]$ for arbitrary n, from which it follows that $V_h(SL) = V_h(A) \wedge V_h(D))$ (whether it is also true that $V_h(SL) = V_h(A) \wedge V_h(L)$ remains an open question). To prove this, it is enough to show that Σ^n(iii) is a consequence of $H^n(D) \cup H^n(A)$, since Σ^n(i), the idempotent hyperidentity, holds in D, and the other two hyperidentities of Σ^n hold in A.

<u>Lemma 5.3</u>. The hyperidentity

$$\alpha: \underline{F}(x_1,\ldots,x_n) = \underline{F}(x_1,\underline{F}(x_1,x_2,\ldots,x_n),x_3,\ldots,x_n)$$

is satisfied by the variety of all distributive lattices.

<u>Proof</u>. Let p be an n-ary distributive lattice polynomial. We write p in the form $p = \bigvee(\wedge K_i \mid i \in I,\ K_i \subseteq \{x_1,\ldots,x_n\})$ and define K_i' to be the following set:

$$K_i' = K_i, \text{ if } x_2 \notin K_i;$$
$$K_i' = (K_i - \{x_2\}) \cup \{p\}, \text{ if } x_2 \in K_i.$$

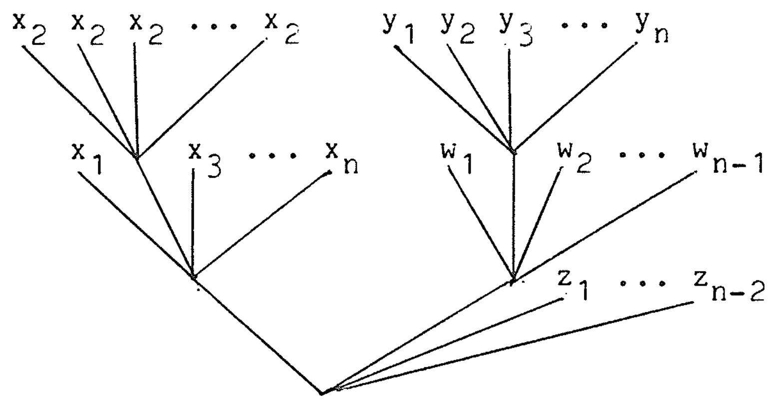


Figure 5.1

That is, whenever x_2 is a member of K_i we obtain K_i' by replacing x_2 by the polynomial p. Clearly, $\bigwedge K_i' \leq p$, for each $i \in I$, and consequently $\bigvee(\bigwedge K_i') \leq p$, that is, $p(x_1,p(x_1,\ldots,x_n),x_3,\ldots,x_n) \leq p$.

We can also write p in the form $p = \bigwedge(\bigvee L_j \mid j \in J, L_j \subseteq \{x_1,..,x_n\})$, and if we now proceed as above, we obtain $p(x_1,p(x_1,..,x_n)x_3,..,x_n) \geq p$, completing the proof.

Lemma 5.4. Let

$$\underline{P} = \underline{F}(\underline{F}[x_1,\underline{F}(x_2,x_2,..,x_2),x_3,\ldots,x_n],\underline{F}[w_1,\underline{F}(y_1,\ldots,y_n),w_2,..,w_{n-1}],z_1,..,z_{n-2}).$$

(See Figure 5.1 for the corresponding formation tree.) If the variable x_2 of the string $\alpha = \langle(\underline{F},1),(\underline{F},2),(\underline{F},2),x_2\rangle$ is replaced by y_2 to obtain the hyperpolynomial

$$\underline{P}' = \underline{F}(\underline{F}[x_1,\underline{F}(x_2,y_2,x_2,..,x_2),x_3,...,x_n],\underline{F}[w_1,\underline{F}(y_1,...,y_n),w_2,..,w_{n-1}],z_1,..,z_{n-2})$$

then $(\underline{P} = \underline{P}') \in H(D)$.

Proof. Let p be an n-ary distributive lattice polynomial, and let $\underline{P}_1 = \underline{F}(x_2,..,x_2)$ and $\underline{P}_1' = \underline{F}(x_2,y_2,x_2,..,x_2)$. We now examine four cases:

(i) If $\underline{P}_1'(p) = x_2$, then obviously, $\underline{P}'(p) = \underline{P}(p)$.

(ii) $\underline{P}_1'(p) = x_2 \vee y_2$. If we write p in the form $p = \bigwedge(\bigvee L_j \mid j \in J,\ L_j \subseteq \{x_1,..,x_n\})$, then each L_j, $j \in J$, contains both x_2 and some variable x_k, $k \neq 2$. That is, we can write $p = x_2 \vee q(x_1,x_3,..,x_n)$ for some (n-1)-ary distributive lattice polynomial q. Consequently,

$$\underline{P}(p) = y_2 \vee q(y_1,y_3,..,y_n) \vee q(w_1,..,w_{n-1}) \vee q(x_2 \vee q(x_1,x_3,..,x_n),z_1,..,z_{n-2}),$$

$$\underline{P}'(p) = y_2 \vee q(y_1,y_3,..,y_n) \vee q(w_1,..,w_{n-1}) \vee q(x_2 \vee y_2 \vee q(x_1,x_3,..,x_n),z_1,..,z_{n-2}).$$

Now $\underline{P}(p) = \underline{P}'(p)$ is a distributive lattice identity iff $\underline{P}(p) = \underline{P}'(p)$ holds for the two element lattice $\langle\{0,1\}; \vee,\wedge\rangle$. If $y_2 = 0$, then $x_2 \vee y_2 = x_2$, so the identity $\underline{P}(p) = \underline{P}'(p)$ holds for $y_2 = 0$. If $y_2 = 1$, then $\underline{P}(p) = \underline{P}'(p) = 1$, and again the identity holds, so $\underline{P}(p) = \underline{P}'(p)$ holds in D.

(iii) The case $\underline{P}'(p) = x_2 \wedge y_2$ is the dual of (ii).

(iv) $\underline{P}'(p) = y_2$ is only possible when p is the second projection x_2, so the proof of the lemma is complete.

Lemma 5.5. The hyperidentity

$$\epsilon:\ \underline{F}(\underline{F}(x_1,..,x_n),\underline{F}(u,u,y_1,...,y_{n-2}),z_1,..,z_{n-2}) = \underline{F}(\underline{F}(x_1,..,x_n),\underline{F}(x_2,u,y_1,..,y_{n-2}),z_1,..,z_{n-2})$$

is a consequence of H(D) ∪ H(A).

Proof. By Lemma 5.3 and idempotence, if

$$\underline{P} = \underline{F}(\underline{F}(x_1,..,x_n),\underline{F}(u,u,y_1,...,y_{n-2}),z_1,..,z_{n-2}),$$
$$\underline{Q} = \underline{F}(\underline{F}(x_1,\underline{F}(x_2,..,x_2),x_3,..,x_n),\underline{F}[\underline{F}(x_1,..,x_n),\underline{F}(u,u,y_1, ...,y_{n-2}),z_1,..,z_{n-2}],z_1,..,z_{n-2}),$$

then $\underline{P}$ <=> $\underline{Q}$, where $\underline{P}$ <=> $\underline{Q}$ denotes (in this proof only) that $\underline{P} = \underline{Q}$ is a consequence of H(D) ∪ H(A). By Lemma 1.6, we can now switch the variables x_2 and u of the strings $\langle(\underline{F},1),(\underline{F},2),(\underline{F},2),x_2\rangle$ and $\langle(\underline{F},2),(\underline{F},2),(\underline{F},1),u\rangle$ of Q to obtain a tree $\underline{Q}'$ with $\underline{Q}$ <=> $\underline{Q}'$, and then by Lemma 5.4, we can replace the variable u of the string $\langle(\underline{F},1),(\underline{F},2),(\underline{F},2),u\rangle$ of Q' by x_2, completing the proof.

Theorem 5.6. $V_h(SL) = V_h(A) \wedge V_h(D)$.

Proof. The theorem follows immediately from Lemma 5.5 and the results of Chapter Three.

As was pointed out in the discussion preceding Lemma 5.3, $[H^n(SL)] = [H^n(A)] \vee [H^n(L)]$ for $n = 2$, but we do not know whether this holds in general.

<u>Problem 5.1</u>. Is $V_h(SL) = V_h(A) \wedge V_h(L)$?

It is fairly easy to see that neither $V_h(A)$ or $V_h(D)$ are covers of $V_h(SL)$. In the case of $V_h(D)$, consider the variety M generated by the median algebra $M = \langle F_D(3), m\rangle$, where $F_D(3)$ is the free distributive lattice on 3 generators and m is the ternary median polynomial $m(x,y,z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$. Since the only ternary polynomials of M are m, x, y, and z, and since $x = m(x,x,y) = m(x,y,x) = m(y,x,x)$, $m(x,y,z) = m(x,z,y) = \ldots\ldots = m(z,y,x)$ and $m(m,x,y) = m$, it is easily seen that

$$\alpha_m\colon\ \underline{F}(x,x,\underline{F}(z,z,y)) = \underline{F}(x,x,y)$$

is satisfied by M but not SL (since α_m is not regular), and the ternary median hyperidentity

$$\alpha\langle 3\rangle\colon\ \underline{F}(\underline{F}(x_1,x_2,x_3),\underline{F}(y_1,y_2,y_3),\underline{F}(z_1,z_2,z_3)) = \underline{F}(\underline{F}(x_1,y_1,z_1),\underline{F}(x_2,y_2,z_2),\underline{F}(x_3,y_3,z_3))$$

is satisfied by SL but not M (set $\underline{F} = m$, $x_1 = y_2 = z_1 = z_2 = z_3 = 0$ and all the other variables equal to 1). This proves that $V_h(SL)$ and $V_h(M)$ are incomparable. The semilattice hyperidentity

$$\epsilon: \underline{F}(\underline{F}(x,x,a),\underline{F}(y,y,b),\underline{F}(z,z,c)) = \underline{F}(\underline{F}(x,y,z),\underline{F}(x,y,z),\underline{F}(a,b,c))$$

is satisfied by M but fails in D (set $\underline{F} = z \vee (x \wedge y)$ and substitute 1 for x and b, and 0 for the remaining variables of the resulting identity), so we have

$$V_h(SL) < V_h(SL) \vee V_h(M) < V_h(D).$$

To show that $V_h(SL)$ is not covered by $V_h(A)$, consider Z_2, the additive group of integers modulo 2. We claim that the semilattice hyperidentity

$$\rho: \underline{F}(\underline{F}(\ldots(\underline{F}(x))\ldots)) = \underline{F}(x),$$

where $\underline{F}$ occurs n times on the left (for some arbitrary integer n), is satisfied by Z_2. For if p is any unary polynomial of Z_2, p is of the form $p = mx$, and substituting p for $\underline{F}$ yields the identity $m^n x = mx$. This holds in Z_2 if $m^n - m$ is even, which is always the case. Consequently, $V_h(SL) \vee V_h(Z_2) < V_h(A)$. It is also easily seen that the hyperidentity

$$\alpha: \underline{F}(\underline{F}(x,y),\underline{F}(y,x)) = \underline{F}(x,x)$$

is satisfied by Z_2 but not SL. Thus, since Z_2 is not idempotent, $V_h(Z_2)$ and $V_h(SL)$ are incomparable, and

$$V_h(SL) < V_h(SL) \vee V_h(Z_2) < V_h(A).$$

We saw in Chapter 2 that $V_h(DS) < V_h(SL)$. We shall now prove that this is a covering relationship.

Theorem 5.7. Let $\Sigma = \bigcup(\Sigma^n \mid n<\omega)$ where Σ^n is defined as in Chapter Three and let $(\underline{P} = \underline{Q}) \in H(DS)$ such that $(\underline{P} = \underline{Q}) \notin H(SL)$. Then $\Sigma \cup \{\underline{P} = \underline{Q}\}$ is a basis for H(DS).

Proof. Let $(\underline{P} = \underline{Q}) \in H(DS)$ be a hyperidentity in the operation symbols $\underline{F}_1,\ldots,\underline{F}_n$ such that $(\underline{P} = \underline{Q}) \notin H(SL)$. We shall (in this proof only) write $\underline{R} \Leftrightarrow \underline{Q}$ whenever $\underline{R} = \underline{Q}$ is a consequence of $\Sigma \cup \{\underline{P} = \underline{Q}\}$. To prove the theorem, we need to show that if $\underline{R} = \underline{S}$ is an m-ary hyperidentity (for arbitrary m) which models projections, then $\underline{R} \Leftrightarrow \underline{S}$.

Since $(\underline{P} = \underline{Q}) \notin H(SL)$ there exists a set $S = \{p_1, \ldots, p_n\}$ of semilattice polynomials of appropriate arity such that the identity

$$\alpha_I\colon\ \underline{P}(p_1,\ldots,p_n) = \underline{Q}(p_1,\ldots,p_n)$$

is not regular. By Lemma 1.3(v) α_I can be viewed as a hyperidentity α in the operation symbol $\underline{v}$. Let y be a variable which appears only in one side of α. Since α models projections, replacing all variables except y by the variable x yields

$$\alpha'\colon\ (x \,\underline{v}\, y) \,\underline{v}\, x \Leftrightarrow x.$$

If the symbol $\underline{v}$ of α' is replaced by the $\underline{K}(x,y,\ldots,y)$, where

$\underline{K}$ is m-ary, we obtain

$$\underline{K}(\underline{K}(x,y,\ldots,y),x,\ldots,x) \Leftrightarrow x.$$

We now let $\underline{T} = \underline{K}(\underline{K}(x,y,\ldots,y),x,\ldots,x)$. We thus have

$\underline{K}(\underline{K}(x,y,\ldots,y),x,\ldots,x) \Leftrightarrow \underline{K}(\underline{K}(\underline{T},\underline{T},\ldots,\underline{T},y),x,\ldots,x)$ since this is a semilattice hyperidentity

$\Leftrightarrow \underline{K}(\underline{K}(x,x,\ldots,x,y),x,\ldots,x)$ since $\underline{T} \Leftrightarrow x$,

and we thus obtain

$$\alpha_m: \underline{K}(\underline{K}(x,x,\ldots,x,y),x,\ldots,x) \Leftrightarrow x.$$

We thus also have, letting $\underline{N}_y = \underline{K}(\underline{K}(x_1,\ldots,x_1,y),x_1,\ldots,x_1)$,

$\underline{K}(\underline{K}(x_1,\ldots,x_{m-1},y),x_m,\ldots,x_{2m-2})$

$\Leftrightarrow \underline{K}(\underline{K}(x_1,\ldots,x_{m-1},\underline{N}_y),x_m,\ldots,x_{2m-2})$ since this is a semilattice hyperidentity

$\Leftrightarrow \underline{K}(\underline{K}(x_1,\ldots,x_{m-1},\underline{N}_z),x_m,\ldots,x_{2m-2})$ by α_m

$\Leftrightarrow \underline{K}(\underline{K}(x_1,\ldots,x_{m-1},z),x_m,\ldots,x_{2m-2})$ since this is again a semilattice hyperidentity,

thus obtaining the hyperidentity

$$\epsilon_m: \underline{K}(\underline{K}(x_1,\ldots,x_{m-1},y),x_m,\ldots,x_{2m-2}) \Leftrightarrow \underline{K}(\underline{K}(x_1,\ldots,x_{m-1},z),x_m,\ldots,x_{2m-2}).$$

Let $\underline{R} = \underline{S}$ be a hyperidentity in the m-ary symbols

$\underline{F}_1,\ldots,\underline{F}_r$ which models projections. By Lemma 3.13, we can obtain hyperpolynomials $\underline{R}'$ and $\underline{S}'$ such that $\underline{R}'$ and $\underline{S}'$ are uniform, and for each k, $k \leq r$, the symbol $\underline{F}_k$ occurs the same number of times in $\underline{R}'$ and $\underline{S}'$. If α is any string containing more than two distinct turns of some operation symbol $\underline{F}_k$ (that is, α has terms $(\underline{F}_k,i)$ and $(\underline{F}_k,j)$, $i \neq j$), then by ϵ_n and the results of Chapter Three, the variable of α can be replaced by x_1. We can thus obtain hyperpolynomials $\underline{R}''$ and $\underline{S}''$ such that $\underline{R}''$ and $\underline{S}''$ are formally equal, $\underline{R} \Leftrightarrow \underline{R}''$ and $\underline{S} \Leftrightarrow \underline{S}''$, completing the proof of the theorem.

Corollary 5.7. $V_h(SL)$ covers $V_h(DS)$ in the lattice of hypervarieties.

Corollary 5.8. $V_h(SL) \wedge V_h(M) = V_h(DS)$ and $V_h(SL) \wedge V_h(Z_2) = V_h(DS)$.

Corollary 5.9. $\Sigma \cup \{\underline{F}(\underline{F}(x,y),x) = x\}$ is a basis for H(DS).

For a summary of the results of this chapter see Figure 5.2. Meets and joins in the lattice of hypervarieties are as shown, with the solid line indicating that $V_h(SL)$ covers $V_h(DS)$, and the dotted lines indicating that we do not know whether or not they are covering relationships.

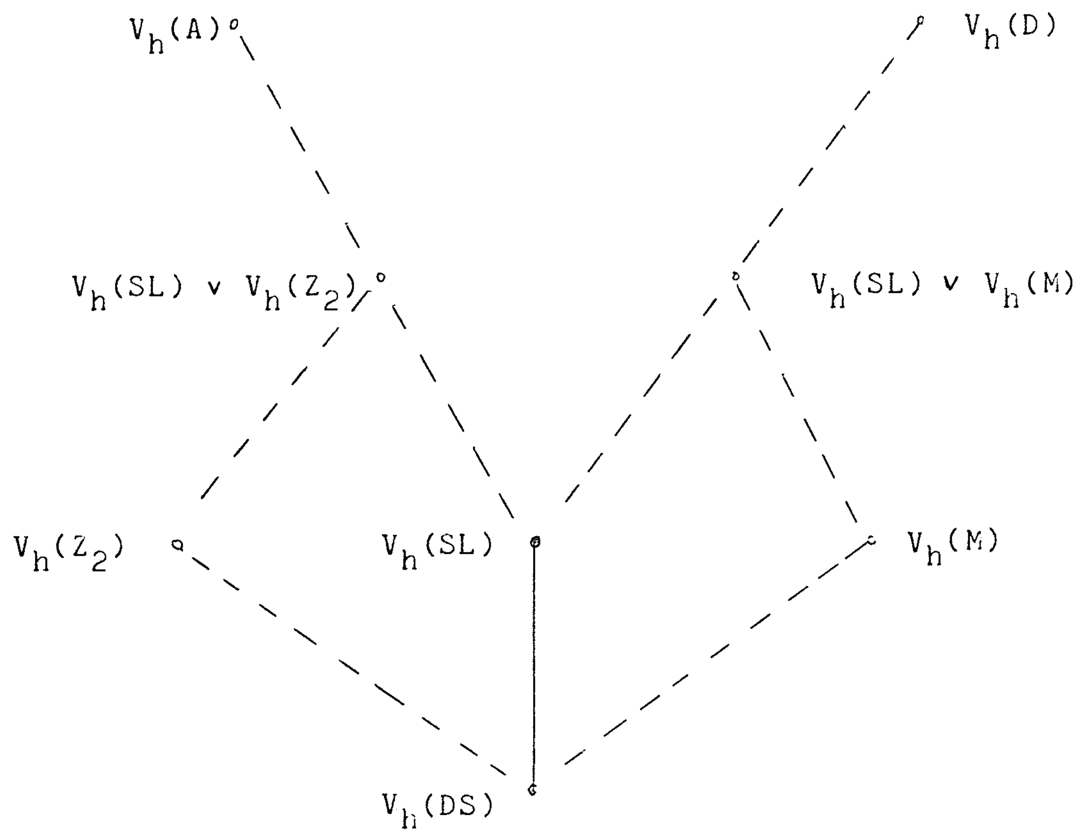


Figure 5.2

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