THE UNIVERSITY OF MANITOBA

# QUALITY OF RESPONSE FOR SYSTEMS INVOLVING TIME DELAY

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF ELECTRICAL ENGINEERING

WINNIPEG, MANITOBA

MAY, 1971

## ABSTRACT

This thesis deals with the quality of response of linear control systems involving time delay.

An extensive study, carried out to find the correlation of quality of step response with frequency response and with open loop root location is described. This study reveals that the phase margin fails to be a measure of relative stability and quality of response for some systems involving time delay; instead, gain margin may be used to serve the purpose.

The estimates of the quality of response for a large class of systems, based upon the principal pair of closed loop complex roots, have been found to be good. These estimates are especially good when an open loop pole is located at the origin.

The effects of delay approximands on the quality of response and the stability limit are investigated.

A design technique based upon gain margin is developed to design systems involving time delay. It is particularly useful for the design of linear control systems involving time delay to which the long-established design technique is simply not applicable.

## ACKNOWLEDGEMENT

The author is deeply indebted to Professor Richard Allan Johnson for his encouragement and many helpful suggestions at every stage of this work.

Thanks are also extended to the University of Manitoba whose financial support in the form of a Graduate Fellowship, allowed this work to be done.

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# CHAPTER I

## GENERAL INTRODUCTION

### 1.1 INTRODUCTION

In feedback control systems, especially in the field of process control, one often encounters transportation delay; such delay may arise as a result of the time consumed while a system component travels for a finite time from one point in space to another.

In recent years, systems involving time delay have received considerable attention, but many of the studies have been devoted to the mathematical discussion of the differential-difference equation (3)#. The characteristic equations which describe systems with delay are transcendental in nature; the solutions are usually difficult to obtain and time consuming in computation. In fact, a whole book devoted to these equations has been written by Bellman and Cooke(2).

However, a designer is often more interested in obtaining a quick estimate of system response characteristics by an analysis that does not require the actual solution of the differential-difference equations.

Wright (26) has dealt with the effect of time delay simulation methods upon the stability. Elgerd and

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# Numbers in parentheses refer to items in the Bibliography.

Stephens(12) have dealt with the effect of closed loop transfer function pole and zero locations on the transient response of linear control systems without delay.

In this thesis, the commonly used methods of root locus analysis and frequency analysis have been applied to various types of linear control systems involving time delay. The analysis and experimental results presented in Chapters II and III permit an estimate of the quality of response of some delay systems to be made once the parameter values are established, and in some cases, it enables one to realize practical system components when some of the desired performance characteristics and system parameter values are known.

In Chapter IV, the quality of response and the stability limit of various systems involving time delay using different types of delay approximands are compared with those of the actual delay.

The results presented in Chapters II and III are used in Chapter V to develop a design technique for realizing linear control systems using practical components which will satisfy the desired performance criteria, and to indicate the manner or method by which the systems must be adjusted or compensated to produce the desired performance characteristics.

## 1.2 MATHEMATICAL TIME DELAY APPROXIMATION

In the study of time delay control systems many different approximations have been developed. One of the most commonly used methods involves the polynomial approximation technique in which the Laplace Transform of the transportation delay is approximated by the rational polynomial fraction

$$e^{-sT} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots\ldots + b_n}{s^n + a_1 s^{n-1} + \ldots\ldots\ldots + a_n} \qquad 1.1$$

### (A) PADE APPROXIMAND

Expansion of the transfer function in a Maclaurin series produces

$$e^{-sT} = 1 - Ts + \frac{(Ts)^2}{2!} - \frac{(Ts)^3}{3!} + \frac{(Ts)^4}{4!} \ldots \qquad 1.2$$

Pade (19) approximated this well-known power series by defining two polynomials $N_{u,v}(Ts)$ and $D_{u,v}(Ts)$ such that

$$e^{-sT} = \lim_{\substack{u \to \infty \\ v \to \infty}} \frac{N_{u,v}(Ts)}{D_{u,v}(Ts)} \qquad 1.3$$

where $N_{u,v}(Ts) = 1 - \frac{u(Ts)}{(u+v)1!} + \frac{u(u-1)(Ts)^2}{(u+v)(u+v-1)2!}$

$$-\ldots + \frac{(-1)^u u(u-1)\cdots 2\cdot 1(Ts)^u}{(u+v)(u+v-1)\cdots(v+1)u!} \qquad 1.4$$

and $D_{u,v}(Ts) = 1 - \frac{v(Ts)}{(u+v)1!} + \frac{v(v-1)(Ts)^2}{(u+v)(u+v-1)2!}$

$$-\ldots + \frac{v(v-1)\cdots 2\cdot 1(Ts)^v}{(u+v)(u+v-1)\cdots(u+1)v!} \qquad 1.5$$

Normal types of Pade approximations are obtained by setting u=v=n, where n is the order of approximation. The first three of these are

$$P_1(s) = \frac{2 - Ts}{2 + Ts} \qquad 1.6$$

$$P_2(s) = \frac{12 - 6Ts + T^2s^2}{12 + 6Ts + T^2s^2} \qquad 1.7$$

and $$P_3(s) = \frac{120 - 60Ts + 12T^2s^2 - T^3s^3}{120 + 60Ts + 12T^2s^2 + T^3s^3} \qquad 1.8$$

The power series obtained from the division of the above fraction does not deviate from Eq.1.2 for a number of terms which depends on the value of n. In general, for higher values of n and smaller values of ωT, a better approximation is obtained.

(B) CASCADED LAG APPROXIMATION

The transfer function of transportation delay may also be expressed as

$$e^{-sT} = \lim_{n\to\infty} \frac{1}{(1+\frac{T}{n}s)^n} \qquad 1.9$$

and approximated by limiting n to some finite value. In particular, if n is taken to be one and two, then the first order cascaded lag is

$$C_1(Ts) = \frac{1}{(1 + Ts)} \qquad 1.10$$

and the second order cascaded lag is

$$C_2(Ts) = \frac{1}{(1 + \frac{Ts}{2})^2} \qquad 1.11$$

The degree of accuracy of this approximation is again dependent upon the values of n and Ts.

For a given frequency ω, Pade one and second order cascaded lag approximands contribute the same amount of phase lag, viz:

$$\theta(\omega) = -2\tan^{-1} \frac{\omega T}{2} \qquad 1.12$$

At this frequency, the n-th order cascaded lag causes a gain attenuation of $1/(1+(\omega T/n)^2)^{n/2}$ but with any order of Pade approximand there is no gain attenuation at any frequency. As a result, the Pade approximands are advantageous over cascaded lags.

## 1.3 METHOD OF DYNAMIC SIMULATION

### (A) ANALOG SIMULATION

Pade delay circuits

Analog circuits for Pade delay are obtained in Reference (7,14) and also suggested in (15). In general, the approximands to $e^{-sT}$ may be written as

$$e^{-sT} = \frac{ABC....N - ABC....(N-1)s + ABC....(N-2)s^2 - ...As^{n-1} + s^n}{ABC....N + ABC....(N-1)s + ABC....(N-2)s^2 + ...As^{n-1} + s^n} \qquad 1.13$$

where A,B,C,...N are gain coefficients.

The simulation diagrams for the first ,second and n-th order Pade approximands are shown in Figures 1.1, 1.2 and 1.3, respectively.

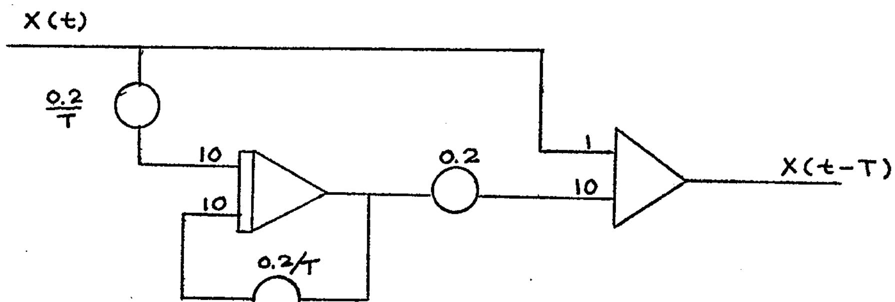


FIGURE 1.1 Diagram of first order Pade Approximand.

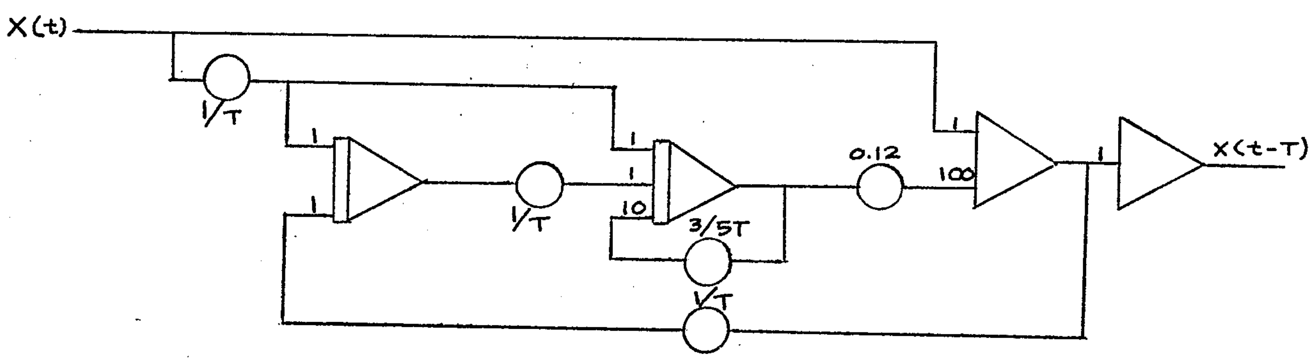


FIGURE 1.2 Diagram of second order Pade Approximand.

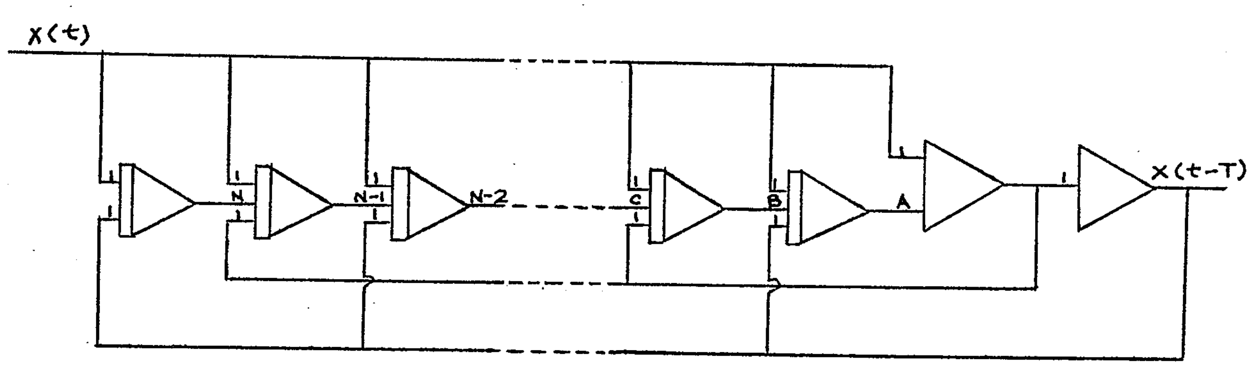


FIGURE 1.3 Diagram of n-th. order Pade Approximand.
(See reference 15)

Cascaded lag circuits

In general, the transfer function of the approximand is

$$C_n(s) = \frac{1}{(1 + \frac{sT}{n})^n} \qquad 1.14$$

for which the computer circuit is shown in Figure 1.4.

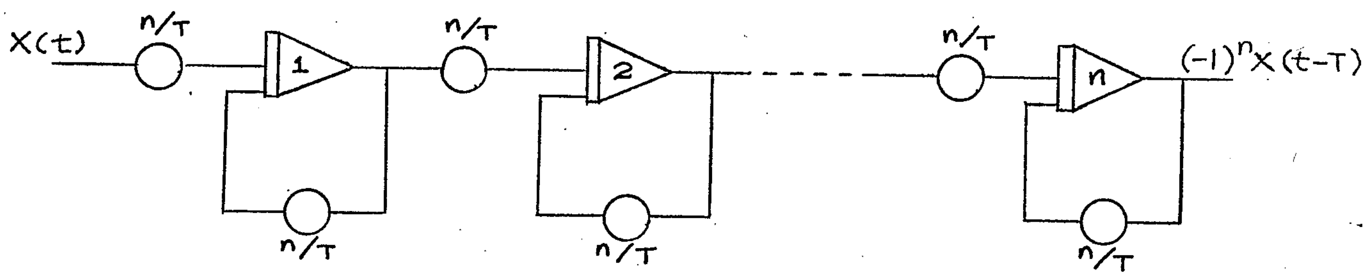


FIGURE 1.4 Diagram of nth. order cascaded lag Approximand.

(B) DIGITAL SIMULATION

The advancement of digital computing techniques has made various types of transfer functions including that of delay directly available in the continuous system modelling programme, i.e. C.S.M.P.(22).

Though the cost of digital progamming is probably higher than analog or hybrid simulation, simulation of time delay control systems using C.S.M.P. is rather simple, and the dynamic responses can be more accurately obtained and recorded. Therefore, the digital simulation method is used throughout most of the study in this thesis, especially when a high degree of accuracy is required.

Consider the rational transfer function

$$W(s) = \frac{N(s)}{D(s)} = \frac{b_o s^n + b_1 s^{n-1} + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots + a_n} = \frac{Y(s)}{R(s)} \qquad 1.15$$

In order to eliminate the derivatives in the control force r(t), the transformation technique outlined in Athans and Falb (1) may be used. First define a state vector $\underline{x(t)}$ with components $x_1(t), x_2(t), \ldots x_n(t)$, such that

$$\begin{aligned} x_1 &= y - b_o r \\ x_2 &= \dot{x}_1 - h_1 r \\ x_3 &= \dot{x}_2 - h_2 r \\ &\cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \\ x_n &= \dot{x}_{n-1} - h_{n-1} r \end{aligned} \qquad 1.16$$

where

$$\begin{aligned} h_1 &= (b_1 - a_1 b_o) \\ h_2 &= (b_2 - a_2 b_o) - a_1 h_1 \\ h_3 &= (b_3 - a_3 b_o) - a_2 h_1 - a_1 h_2 \\ &\cdot \\ &\cdot \\ &\cdot \\ h_n &= (b_n - a_n b_o) - a_{n-1} h_1 - a_{n-2} h_2 \\ &\quad - \ldots - a_2 h_{n-2} - a_1 h_{n-1} \end{aligned} \qquad 1.17$$

The state equations become

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_n & -a_{n-1} & -a_{n-2} & \cdot & \cdot & \cdot & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ \cdot \\ \cdot \\ \cdot \\ h_n \end{bmatrix} r \qquad 1.18$$

and the system output is given by

$$y = x_1 + b_0 r$$

With these state equations, any type of delay approximand can be simulated without differentiation. A continuous system modelling programme for simulating a second or third order system is given in Appendix I for various types of delay approximands.

## 1.4 NORMALIZATION

In this thesis, many types of linear control systems involving time delay were investigated. To make the analysis applicable to any system independent of the speed of response and dependent on a minimum number of parameters, the well-known Laplace Transformation relation

$$L[f(\tfrac{t}{a})] = aL[f(t)] \qquad 1.19$$

was used to normalize time and thereby reduce the number of parameters by one ,by suitable choice of a. If the radial distance from the origin to the root of the system characteristic equation in the s plane is reduced by a factor 1/a, the corresponding speed of response is increased by a factor a.

Consider a general second order system involving time delay; the transfer function is represented by

$$\frac{C(S)}{R(S)} = \frac{K}{S(S+A)} e^{-S\tau} \qquad 1.20$$

Eq. 1.20 may be written as

$$\frac{C(S)}{R(S)} = \frac{K/A^2}{\frac{S}{A}(1+\frac{S}{A})} e^{-S\tau} \qquad 1.21$$

which with the following substitutions

$$\frac{S}{A} = s$$

$$\frac{K}{A^2} = k \qquad 1.22$$

and $A\tau = T$

becomes

$$\frac{c(s)}{r(s)} = \frac{k\ e^{-sT}}{s(s+1)} \qquad 1.23$$

Thus the equation with three running parameters A, $\tau$ and K has been reduced to a new equation with T and k as the running parameters.

Therefore a whole set of second order systems satisfying the relation $a\tau = T$ may be assessed by consulting the performance characteristics of Eq. 1.23 having T as its delay time and $K/A^2$ as its forward gain, the response time being increased by a factor of A, and the root locations reduced by a factor of 1/A.

Similarly, systems such as

$$\frac{Ke^{-S}}{(S+A)}, \quad \frac{Ke^{-S\tau}}{(S^2+2A\omega_n S+\omega_n^2)} \quad \text{and} \quad \frac{Ke^{-S\tau}}{S(S^2+2A\omega_n S+\omega_n^2)}$$

may be normalized to give

$$\frac{K/A}{(s+1)}\, e^{-(\tau A)s}, \quad \frac{(K/\omega_n^2)\, e^{-(\tau\omega_n)s}}{(s^2+2As+1)} \quad \text{and} \quad \frac{(K/\omega_n^3)\, e^{-(\tau\omega_n)s}}{s(s^2+2As+1)}$$

respectively, with response time again being increased in the ratio of 1 to A.

CHAPTER II

# FREQUENCY ANALYSIS

## II.1 INTRODUCTION

Frequency analysis consists of a number of analytical and graphical procedures based on the sinusoidal steady state response of a system. Amongst other things it provides information about the relative stability of the system.

In 1932, Nyquist (18) developed his frequency domain stability criterion; it and its subsequent generalization (9) continue to be among the most useful techniques in the investigation of the system stability.

In particular, it has been found that the frequency analysis method is particularly useful in the determination of stability characteristics of systems involving time delay. The delay operator $e^{-sT}$ has no pole or zero in the finite s-plane, and so only contributes a phase shift proportional to the frequency to the response function of the system.

Consider the single loop feedback control system shown below, where G(s) and H(s) are the transfer functions of the forward and feedback paths, respectively.

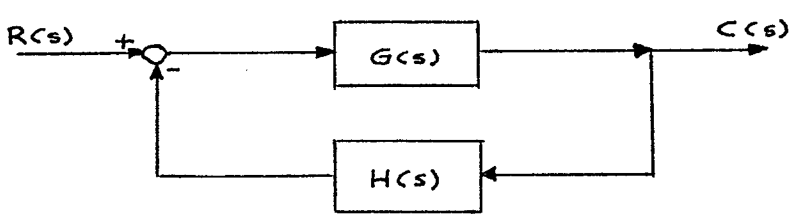


Figure 2.1 Block diagram of a single loop feedback control system.

The Nyquist stability Criterion states that a feedback control system is stable if, and only if, the contour $G(j\omega)H(j\omega)$ does not encircle the (-1,0) point in the $G(j\omega)H(j\omega)$ plane, where $G(s)H(s)$ has no pole in the right-hand s plane.

The phase crossover frequency, $\omega_m$, is defined as the frequency for which the phase angle of $G(j\omega)H(j\omega)$ is $180^0$.

The gain margin, $G_m$, is mathematically defined as

$$G_m = \left| \frac{1}{G(j\omega_m)\ H(j\omega_m)} \right| \qquad 2.1$$

The gain crossover frequency, $\omega_c$, is the frequency at which the gain magnitude is unity.

The phase margin, $\Phi_m$, is defined as

$$\Phi_m = 180^0 - \arg\left[G(j\omega_c)H(j\omega_c)\right]_{degree} \qquad 2.2$$

These quantities are illustrated in Figure 2.2.

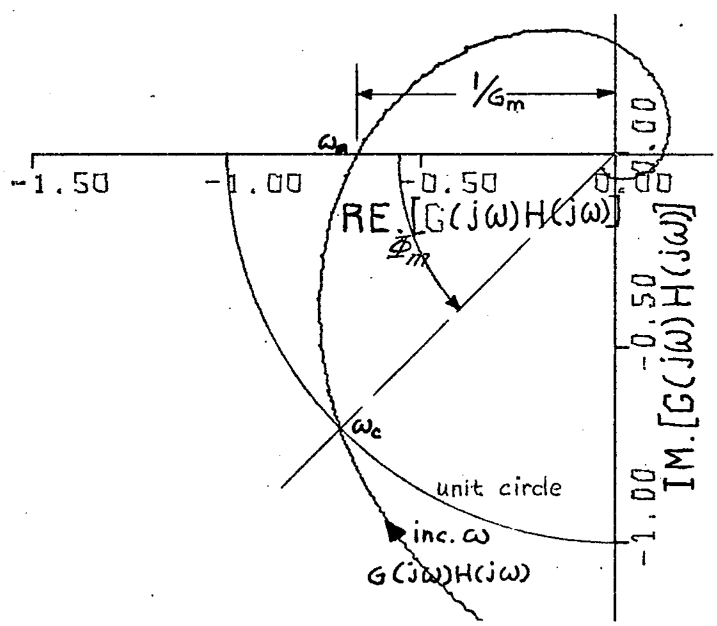


FIGURE 2.2 Nyquist diagram for $G(j\omega)H(j\omega)$.

The basic part of the study in this chapter is the examination of the adequacy of the phase margin and the gain margin in predicting the quality of response of linear control systems involving time delay.

## II.2 EVALUATION OF PHASE MARGIN AND GAIN MARGIN

Gain Margin

To obtain the gain margin, $G_m$, it is necessary to locate $\omega_m$, where

$$\arg\left[G(j\omega_m)\ H(j\omega_m)\right] = 180^\circ \qquad 2.3$$

For each given form of transfer function $G(j\omega)H(j\omega)$, $\omega_m$ is independent of the gain constant K.

Phase Margin

To obtain the phase margin, $\Phi_m$, it is necessary to locate $\omega_c$.

The negative of the open loop transfer function of any second order system without numerator dynamics involving time delay may be written as:

$$G_1(s)H(s) = \frac{K_1}{(As^2 + Bs + C)} \begin{bmatrix}\text{delay or delay} \\ \text{approximand trans-} \\ \text{fer function}\end{bmatrix} \qquad 2.4$$

Now the Laplace transformation representing the actual delay $e^{-sT}$ and the Pade delay approximands only introduce phase lag without altering the magnitude of the gain.

Since $|G_1(j\omega_{c_1})H(j\omega_{c_1})| = 1$ , 2.5

$$K_1^2 = (-A\omega_{c_1}^2 + C)^2 + B^2\omega_{c_1}^2 \quad , \qquad 2.6$$

$$\text{or} \quad A^2\omega_{c_1}^4 + (B^2 - 2AC)\omega_{c_1}^2 + (C^2 - K_1^2) = 0 \qquad 2.7$$

$$\text{and so} \quad \omega_{c_1} = \left[ \frac{-(B^2 - 2AC) \pm [(B^2 - 2AC)^2 - 4A^2(C^2 - K_1^2)]^{\frac{1}{2}}}{2A^2} \right]^{\frac{1}{2}} \qquad 2.8$$

For $B^2-2AC>0$, $|K_1|$ must be greater than $|C|$ for $\omega_{c_1}$ to be real, and the positive sign preceding the interior radical must be used. When $(B^2-2AC)<0$ and $|K_1|<C$ both positive and negative signs preceding the interior radical may be used, in which case two different values of $\omega_{c_1}$ may be obtained; consequently, two values for the phase margin result. Moreover, when $B^2-2AC>0$ and $|K_1|<|C|$, no phase margin can be defined.

The phase margins of any second order system with either an actual delay, or Pade 1, or Pade 2 approximands, in the feedback loop are, respectively

$$\Phi_{m_{\text{actual delay}}} = 180^0 - [\tan^{-1}(\frac{B\omega_c}{C - A\omega_c^2}) + \omega_c T]^0 \qquad 2.9$$

$$\Phi_{m_{\text{Pade 2}}} = 180^0 - [\tan^{-1}(\frac{B\omega_c}{C - A\omega_c^2}) + 2\tan^{-1}(\frac{6T\omega_c}{12 - T^2\omega_c^2})]^0 \qquad 2.10$$

and

$$\Phi_{m_{\text{Pade 1}}} = 180^0 - [\tan^{-1}(\frac{B\omega_C}{C-A\omega_C^2}) + 2\tan^{-1}(\frac{\omega_C T}{2})]^0 \qquad 2.11$$

The cascaded lag delay approximands, on the other hand, not only introduce phase lag but also change the magnitude of gain.

If the loop gain of a second order system with double cascaded lag delay approximand is increased by a factor of $(1+\frac{\omega_C{}^2T^2}{4})$ from that of the Pade 1 delay approximand, the phase margin equal to that of the Pade 1 delay control system is obtained.

Consider now the third order system with a pole at the origin,

$$G_2(s)H(s) = \frac{K_2}{s(As^2+Bs+C)} \left(\begin{array}{l}\text{delay or delay}\\ \text{approximand transfer}\\ \text{function}\end{array}\right) \qquad 2.12$$

The crossover frequency, $\omega_{C_2}$, is a solution of the equation

$$K_2{}^2 = \omega_{C_2}{}^2\ [(C - A\omega_{C_2}{}^2)^2 + B^2\omega_{C_2}{}^2] \qquad 2.13$$

which with the substitution $X = \omega_{C_2}{}^2$ is reduced to

$$K_2{}^2 = X[(C - AX)^2 + B^2 X)] \qquad 2.14$$

This is a cubic in X which, in practice, is time consuming to solve.

Let $\omega_{c_1}$ and $\omega_{c_2}$ be, respectively, the gain crossover frequencies for $G_1(s)H(s)$ and $G_2(s)H(s)$ as defined earlier.

Now,

$$\arg\left[G_2(j\omega)H(j\omega)\right] = \arg\left[G_1(j\omega)H(j\omega)\right] - 90^0 \qquad 2.15$$

and, furthermore,

$$\omega_{c_2} = \omega_{c_1} \qquad 2.16$$

if, and only if $\quad K_2 = K_1\omega_{c_1} \qquad 2.17$

In order to determine the phase margin, $\Phi_{m_2}$, for a given $K_2$, instead of solving for $\omega_{c_2}$ in Eq.2 14, the values of $\omega_{c_1}$ and $K_1$ satisfying Eq. 2.8 may be found, and then $\omega_{c_2}$ is found by searching for the particular values of $\omega_{c_1}$ and $K_1$ satisfying the relation $\omega_{c_1} K_1 = K_2$. The value of $\Phi_{m_2}$ is $90^0$ less than the phase margin of $G_1(s)H(s)$ with $\omega_{c_1}$ as crossover frequency.

Similarly, in order to find $K_2$ for a given $\Phi_{m_2}$ one may search for $\omega_{c_1}$ and $K_1$ for which $\Phi_{m_1} = \Phi_{m_2} - 90^0$, and $K_2 = \omega_{c_1} K_1$.

Since the phase crossover frequency $\omega_{\bar{m}}$ is independent of K for each particular feedback control system, whereas $\omega_c$ generally varies with K, the special technique described above for evaluating gain margin is comparatively easier to apply.

## II.3 PHASE MARGIN AND TRANSIENT RESPONSE

For a second order closed loop linear control system,