# Free Spectrahedra and Completely Bounded Maps

by

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**Abstract**

In this study, we present proofs of results of Helton, Kelp, and McCullough from [8] on inclusions of free spectrahedra and their relationship to completely positive maps. Our original contributions extended some of their results to the setting of completely bounded maps.

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# 1

# Introduction

## 1.1 Linear Matrix Inequalities

Recall that a linear matrix inequality for the variable $x = (x_1, x_2, \dots, x_g) \in \mathbb{C}^g$ has the form

$$L_A(x) = A_0 + \sum_{j=1}^{g} x_j A_j \geq 0,$$

where $A_0, A_1, \dots, A_g \in S\mathbb{C}^{n\times n}$ and $S\mathbb{C}^{n\times n}$ denotes all complex self-adjoint $n \times n$ matrices. Moreover, the expression $L_A(x) = A_0 + \sum_{j=1}^{g} x_j A_j$ is called a linear pencil.

J. C. Willems [4, p.27] was the first person that introduced using of the term "Linear Matrix Inequalities" which is abbreviated as LMIs. The current interest in this subject is due to the pivotal role that LMIs have played in solving numerous engineering problems in terms of system and control theory and its connection to convex optimisation problems [4],[9]. For instance, solving some system and control theory problems can lead to solving a semi-definite program which can be written as a LMI. Consider the problem of minimizing a linear function of the variable $x \in \mathbb{R}^g$:

$$\begin{cases} \text{minimize } C^T x \\ \text{subject to } L_A(x) = A_0 + \sum_{j=1}^{g} x_j A_j \geq 0, \end{cases}$$

where $C \in \mathbb{C}^g$ and $A_0, A_1, \ldots, A_g \in S\mathbb{C}^{d\times d}$ are fixed [4]. In recent years, various investigations have been conducted on semi-definite programming based on LMIs and spectrahedra where the spectrahedra is the set of $x \in \mathbb{C}^g$ for which $L_A(x) \geq 0$ [14].

Free spectrahedra is the generalization of the spectrahedron in the sense that we use g-tuples of matrices $X \in (S\mathbb{C}^{n\times n})^g$ in our pencils instead of vector $x$. Recall that the positivity domain of the linear pencil $L_A(X) = A_0 \otimes I_n + \sum_{j=1}^{g} A_j \otimes X_j$ is defined as

$$\mathcal{D}_{L_A} = \left\{ X \in (S\mathbb{C}^{n\times n})^g |\ L_A(X) \geq 0, n \in \mathbb{N} \right\},$$

this is also called free spectrahedra.

An easy calculation shows that spectrahedra is a convex set. A free spectrahedra fulfills a stronger convexity property namely matrix convexity. A set $E$ of matrices is matrix convex if and only if every matrix convex combinations of elements of $E$ are in $E$, i.e. for every matrices $A_1, \ldots, A_g \in E$ and $A_j \in \mathbb{M}_{k_j}$, the sum of the form $\sum_{j=1}^{g} V_j^* A_j V_j \in E$ with matrices $V_j \in \mathbb{M}_{k_j,p}$ such that $\sum_{j=1}^{g} V_j^* V_j = I_p$. In [6], the geometry of matrix convex sets and their relationship to completely positive maps and dilation theory were investigated. In particular, for two given matrix convex sets $\mathcal{S} = \bigcup_{n\geq 1} \mathcal{S}_n$ and $\mathcal{T} = \bigcup_{n\geq 1} \mathcal{T}_n$, they identified the geometric conditions on $\mathcal{S}$ or $\mathcal{T}$ such that $\mathcal{S}_1 \subseteq \mathcal{T}_1$ implies $\mathcal{S} \subseteq C\mathcal{T}$ for some constant $C$.

When it comes to free spectrahedra and LMIs, we can ask the following questions for two given pencils

$$L_A(X) = I_{d_1} \otimes I_n + \sum_{j=1}^{g} A_j \otimes X_j, L_B(X) = I_{d_2} \otimes I_n + \sum_{j=1}^{g} B_j \otimes X_j, X \in (S\mathbb{C}^{n\times n})^g :$$

(a) When does the inequality $L_A(X) \geq 0$ imply $L_B(X) \geq 0$?

(b) When do inequalities $L_A(X) \geq 0$ and $L_B(X) \geq 0$ have the same solutions?

Ben-Tal and Nemiroski's paper [3] is an example of answering the first question.

More precisely, they determined for which linear pencils $L_A(x) = I + \sum_{j=1}^{g} x_j A_j$ the inclusion $\big((-1,1)\big)^g \subseteq \mathcal{D}_{L_A}(1) = \{x \in \mathbb{C}^g | \ L_A(x) \geq 0\}$ holds. Their interest in this problem originated from a semi-definite programming problem.

Later in 2013, Helton, Klep and McCullough [8] verified that $L_A, L_B$ give rise to the same positivity domain(free spectrahedra) if and only if there exist matrices $E_1, \ldots, E_\mu \in \mathbb{C}^{nd_1 \times nd_2}$ $(\mu \in \mathbb{N})$ such that

$$L_B(X) = E_1^* L_A(X) E_1 + \ldots + E_\mu^* L_A(X) E_\mu \text{ and} \sum_{j=1}^{\mu} E_j^* E_j = I.$$

Besides that, they identified that $L_A(X) \geq 0$ implies $L_B(X) \geq 0$ if and only if there exists a natural completely positive map between the sets $\mathcal{S}_A := \text{span}\{I, A_1, \ldots, A_g\}$ and $\mathcal{S}_B := \text{span}\{I, B_1, \ldots, B_g\}$ where $A_1, \ldots, A_g \in S\mathbb{C}^{d_1 \times d_1}$ and $B_1, \ldots, B_g \in S\mathbb{C}^{d_2 \times d_2}$. In this thesis, the relationship between inclusions of the free spectrahedra in terms of the existence of the completely positive and completely bounded maps between $S_A$ and $S_B$ will be investigated. This thesis is organized as follows:

Chapter 2 is dealing with some basic definitions and technical preliminaries regarding completely positive and completely bounded maps.

In Chapter 3, which follows the Helton, Klep, and McCullough paper [8] closely, the conditions under which question (a) might be answered will be discussed. In addition, we will use one important tool in operator theory, namely complete positivity, to deal with question (b). For instance, under some conditions, $L_A(X) \geq 0$ implies $L_B(X) \geq 0$ if and only if there exists a natural completely positive map between the subspace $\mathcal{S}_A$ and $\mathcal{S}_B$ spanned by matrix coefficients of $L_A$ and $L_B$ respectively.

Chapter 4 contains the original contributions of this thesis. Therein, we will find a relationship between

$$\Lambda_A := \left\{X \in \left(\mathbb{C}^{n \times n}\right)^g \mid \|L_A^{(1)}(X)\| \leq 1\right\}$$

and

$$\Lambda_B := \left\{ X \in \left(\mathbb{C}^{n\times n}\right)^g \mid \|L_B^{(1)}(X)\| \leq 1 \right\}$$

that characterizes when there exists a natural completely bounded map between the sets $\mathcal{S}_A$ and $\mathcal{S}_B$.

Moreover, we will exhibit that there is a relationship between $\mathcal{D}_{L_A}$ and $\mathcal{D}_{L_B}$ that characterizes when there exists a natural completely bounded map between $\mathcal{S}_A$ and $\mathcal{S}_B$. Basic tools from [8] will be used to accomplish this goal.

# 2

# Preliminaries

In this chapter, we will review and recall some basic preliminaries along with some technical theorems that we need in the following chapters.

## 2.1 Basic operator theory

Recall that if $T$ is a bounded linear operator on some Hilbert space $H$, then $T$ is positive if $\langle Tf, f\rangle \geq 0$ and $T$ is positive definite if $\langle Tf, f\rangle > 0$ for $f \in H (f \neq 0)$.

For instance, the direct sum of two matrices $A = (a_{ij})_{i,j=1}^{n}$ and $B = (b_{ij})_{i,j=1}^{m}$, i.e.,

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

and their tensor product

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \vdots & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{bmatrix}$$

are positive definite whenever $A, B \geq 0$.

Now, we state the Schur complement condition for positive definite matrices.

**Theorem 2.1.** *Assume $Y$ is a self-adjoint block matrix such that* $Y = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$. *Then the following hold:*

(1) *If $A > 0$, then $Y$ is positive semi-definite if and only if $C - B^*A^{-1}B \geq 0$,*

(2) *If $C > 0$, then $Y$ is positive semi-definite if and only if $A - BC^{-1}B^* \geq 0$.*

*Proof.* (1) Consider the following decomposition:

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ B^*A^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & C - B^*A^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}.$$

Let $A > 0$ and let $Y$ be positive semi-definite, then by the virtue of the above decomposition we have:

$$\begin{bmatrix} A & 0 \\ 0 & C - B^*A^{-1}B \end{bmatrix} \geq 0,$$

which is equivalent to saying $C - B^*A^{-1}B \geq 0$. On the other hand, if we have $C - B^*A^{-1}B \geq 0$, then by using the above decomposition we obtain $Y$ is positive definite.

(2) follows the same argument as in the previous part. □

**Example 2.2.** *Consider the following linear pencil* $L(x) = \begin{bmatrix} x_1 + x_3 & x_2 - 1 \\ x_2 - 1 & x_1 \end{bmatrix} \geq 0.$ *Here we have*

$$A_0 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

*If $x_1 + x_3 > 0$, then by theorem 2.1 the LMI $L(x) \geq 0$ is equivalent to*

$$(x_1 + x_3)x_1 - (x_2 - 1)^2 \geq 0.$$

$\triangle$

A unital Banach algebra $\mathcal{B}$ is an algebra over $\mathbb{C}$ endowed with a norm $\|\cdot\|$ such that $\mathcal{B}$ is a Banach space under the norm $\|\cdot\|$ which satisfies the following additional conditions

1. $\|I\| = 1$,

2. $\|ab\| \leq \|a\|\|b\|$ for $a, b \in \mathcal{B}$.

Let $\mathcal{B}$ be a Banach algebra over $\mathbb{C}$ endowed with a conjugate linear involution map $a \to a^*$ such that $\|a^*a\| = \|a\|^2,\ a \in \mathcal{B}$. Then, we say that $\mathcal{B}$ is a $C^*$-algebra. Recall that $B(H)$ is a $C^*$-algebra.

**Lemma 2.3.** [5, Corollary 1.10.6, Theorem 1.10.7] *Every $*$-representation of $\mathbb{M}_n$ is unitarily equivalent to a multiple of the identity representation. Moreover, every finite dimensional $C^*$-algebra is $*$-isomorphic to $\bigoplus_{i=1}^{n} \mathbb{M}_{k_i}$. In particular, if the $C^*$-algebra is simple and finite dimensional, then it is $*$-isomorphic to $\mathbb{M}_n$.*

**Proposition 2.4.** [7, Proposition 4.33, Corollary 4.34] *Let $A$ be an operator on some Hilbert space $H$. Then the following hold:*

(1) *If $A$ is positive, then there exists a unique positive operator $B$ such that $B^2 = A$.*

(2) *$A$ is positive if and only if there exists an operator $E$ such that $A = E^*E$.*

Recall that if $A$ is an operator on some Hilbert space $H$ and $M$ is a closed subspace of $H$, then:

(1) $M$ is an invariant subspace for $A$ if $A(M) \subseteq M$.

(2) $M$ is a reducing subspace if $A(M) \subseteq M$ and $A(M^\perp) \subseteq M^\perp$.

**Lemma 2.5.** *Consider a positive and self-adjoint linear operator $A : H \to H$ such that there exists $x \in H$ with the property that $\langle Ax, x\rangle = 0$. Then, $Ax = 0$.*

*Proof.* By using the proposition 2.4 we obtain $A = A^{\frac{1}{2}}A^{\frac{1}{2}}$. Consequently, we have

$$\langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}x\rangle = \langle Ax, x\rangle = 0,$$

which is equivalent to saying $\|A^{\frac{1}{2}}x\| = 0$. Therefore, $Ax = 0$. $\square$

For a bounded linear operator $T$ on some Hilbert space $H$, the numerical range $W(T)$ and numerical radius $w(T)$ are defined as follow:

(1) $W(T) := \{\langle Tx, x\rangle : \quad x \in H, \|x\| = 1\}$,

(2) $w(T) := \sup\{|x| : \ x \in W(T)\}$.

Recall that for every bounded operator $T$ on Hilbert space $H$, we have:

$$w(T) \leq \|T\| \leq 2w(T),$$

which implies that the numerical range of a non-zero operator $T$ is non-zero [11, p.428].

**Proposition 2.6.** [7, Proposition 4.42] *Let $A$ be an operator on some Hilbert space $H$, $M$ be a closed subspace of $H$, and $P_M$ be the orthogonal projection onto $M$. Then:*

(1) *$M$ is an invariant subspace for $A$ if and only if $P_MAP_M = AP_M$, which in turn is equivalent to $A^*(M^\perp) \subseteq M^\perp$,*

(2) *$M$ is a reducing subspace for $A$ if and only if $P_MA = AP_M$, which in turn is equivalent to saying $A^*(M^\perp) \subseteq M^\perp$ and $A^*(M) \subseteq M$.*

Note that if $A$ is an operator on Hilbert space $H$ and $M$ is a closed subspace of $H$, then according to the decomposition $H = M \oplus M^\perp$, we have

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where $A_{11} = P_MAP_M$, $A_{12} = P_MAP_{M^\perp}$, $A_{21} = P_{M^\perp}AP_M$, and $A_{22} = P_{M^\perp}AP_{M^\perp}$.

Furthermore, if we suppose $M$ is an invariant subspace for $A$, then:

$$A = \begin{bmatrix} A|_M & P_M A|_{M^\perp} \\ 0 & P_{M^\perp} A|_{M^\perp} \end{bmatrix}.$$

## 2.2 Convexity

Recall that the closed line segment joining two points $a$ and $b$ in $\mathbb{R}^n$ is the set $[a, b] = \{(1-\lambda)a + \lambda b : 0 \le \lambda \le 1\}$ and the open line segment joining two points $a$ and $b$ is $(a, b) = \{(1-\lambda)a + \lambda b : 0 < \lambda < 1\}$. Moreover, the convex hull of the points $a_1, \ldots, a_j$ in $\mathbb{R}^n$ is the intersection of all convex sets containing $a_1, \ldots, a_j$, which is denoted by $\text{con}\{a_1, \ldots, a_j\}$. Furthermore, the topological interior, and topological boundary of the set $T_1 \in \mathbb{R}^n$ will be denoted by int $T_1$ and $\partial T_1$ respectively.

**Lemma 2.7.** [8, Lemma 3.10] *Assume $T_1, T_2 \in \mathbb{R}^n$ are closed convex sets such that $T_1 \subseteq T_2$, $0 \in \text{int } T_1 \cap \text{int } T_2$ and $\partial T_1 \subseteq \partial T_2$. Then $T_1 = T_2$.*

*Proof.* Assume $T_1 \subset T_2$ but $T_1 \neq T_2$. Then, there is $z \in T_2, z \notin T_1$. We know that $[0, z] \cap T_1 = [0, \mu z]$ for some $0 < \mu < 1$( because $[0, z] \cap T_1$ is a convex subset of a line segment $[0, z]$ and $z \notin T_1$). That is to say, $\mu z \in \partial T_1$ and $\partial T_1 \subseteq \partial T_2$. Therefore, $\mu z \in \partial T_2$. Moreover, since $0 \in \text{int } T_1$, there is $\epsilon > 0$ such that the disk $D(0, \epsilon) \subset T_1$. Now, we define $V = \text{con}\left(\{z\}, D(0, \epsilon)\right)$, then by convexity of $T_2$, we conclude that $V \subset T_2$. On the other hand, by convexity of $V$, $(0, z) \in V$ and $V \subsetneq T_2$. Thus, $\mu z + (1-\mu)0 \in V$. We claim that $\mu z \in \text{int } V$ which is equivalent to saying that there exists $\delta > 0$ such that $D(\mu z, \delta) \subset V$. To see this, consider $x = (1-\mu)\left(\frac{x-\mu z}{1-\mu}\right) + \mu z$. Therefore, $x \in V$ if $\left(\frac{x-\mu z}{1-\mu}\right) \in D(0, \epsilon)$ which is equivalent to saying $\left\|\frac{x-\mu z}{1-\mu}\right\| < \epsilon$ which in turn is equivalent to $\|x - \mu z\| < \epsilon(1-\mu)$. Also, we have int $V \subseteq$ int $T_2$, and therefore $\mu z \in \text{int } T_2$, but this is in contradiction with $\mu z \in \partial T_2$. □

## 2.3 Completely bounded maps

In this section, first we will define completely positive and completely bounded maps. Then, we state some important theorems regarding these maps which are our tools in the next chapters.

If $\mathcal{A}$ is a $C^*$-algebra and $S$ is a self-adjoint subspace of $\mathcal{A}$ such that $I \in S$, then $S$ is an operator system. A linear map $\tau : S \to \mathcal{B}$, where $\mathcal{B}$ is a $C^*$-algebra, is called positive if for every positive element $a \in S$ we have $\tau(a) \geq 0$.

**Theorem 2.8.** [10, Proposition 2.1] *Let $\tau$ be a positive map as in the preceding. Then $\tau$ is bounded and $\|\tau\| \leq 2\|\tau(I)\|$.*

Assume $\tau : S \to \mathcal{B}$ is a linear map between an operator system $S$ and a $C^*$-algebra $\mathcal{B}$. We say that $\tau$ is $n-$positive if

$$\begin{aligned} \tau^{(n)} : \mathbb{M}_n(S) &\to \mathbb{M}_n(\mathcal{B}) \\ (a_{ij}) &\mapsto (\tau(a_{ij})) \end{aligned}$$

is positive and call $\tau$ completely positive if $\tau^{(n)}$ is $n$-positive for every $n$, where $\mathbb{M}_n$ denotes the set of all $n \times n$ complex matrices and $\mathbb{M}_n(S)$ denotes the set of all $n \times n$ matrices with entries from $S$. Moreover, we call $\tau$ completely bounded if $\|\tau\|_{cb} = \sup_n \|\tau^{(n)}\| < \infty$ and completely contractive if $\|\tau\|_{cb} \leq 1$. Recall that an element $a$ in $C^*$-algebra $\mathcal{A}$ is positive if we have $a = b^*b$ for some $b$ in $\mathcal{A}$.

**Lemma 2.9.** [10, Lemma 3.1] *Let $\mathcal{A}$ be $C^*$-algebra and let $a \in \mathcal{A}$. Then, $\|a\| \leq 1$ if and only if $\begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix}$ is positive in $\mathbb{M}_2(\mathcal{A})$.*

**Proposition 2.10.** [10, Theorem 3.9] *Let $S$ be an operator system and let $\tau : S \to C(X)$ be a bounded linear map, where $C(X)$ denotes the $C^*$-algebra of all continuous complex-valued functions on some compact Hausdorff space $X$ equipped with* *the supremum norm. Then* $\|\tau\|_{cb} = \|\tau\|$. *Also, if* $\tau$ *is positive, then* $\tau$ *is completely positive.*

Recall that $\{E_{ij}\}_{i,j=1}^n$ denote the system of matrix units for $\mathbb{M}_n$ where the $(i, j)$th entry of $E_{ij}$ is 1 and all other entries are zero.

**Theorem 2.11.** [10, Theorem 3.14]*(Choi). Suppose that* $\mathcal{A}$ *is a* $C^*$*-algebra and the map* $\tau : \mathbb{M}_n \to \mathcal{A}$ *is linear. Then the following are equivalent:*

(1) $\tau$ *is completely positive,*

(2) $(\tau(E_{i,j}))_{i,j=1}^n$ *is positive in* $\mathbb{M}_n(\mathcal{A})$.

The following theorem shows that we can represent a completely positive map on a $C^*$-algebra as a compression of a $*$-homomorphism.

**Theorem 2.12.** [10, Theorem 4.1] *Let* $\mathcal{A}$ *be a unital* $C^*$*-algebra and let* $\tau : \mathcal{A} \to \mathcal{B}(H)$ *be a completely positive map. Then, there exists a Hilbert space* $K$*, a unital* $*$*-homomorphism* $\Pi : \mathcal{A} \to \mathcal{B}(K)$ *and a bounded operator* $V : H \to K$ *with* $\|\tau(I)\| = \|V\|^2$ *such that*

$$\tau(a) = V^*\Pi(a)V, \quad a \in \mathcal{A}.$$

*If* $\tau$ *is unital, then* $V$ *is an isometry. In addition, if we assume* $\mathcal{A}, H$ *are finite dimensional, then* $K$ *is also finite dimensional. Last two sentences can be extracted from the proof.*

The following theorem will exhibit that there is an extension(completely positive) for the completely positive map from an operator system into $\mathcal{B}(H)$.

**Theorem 2.13.** [10, Theorem 7.5] *Suppose that* $\mathcal{A}$ *is a* $C^*$*-algebra,* $S \subset \mathcal{A}$ *is an operator system and* $\psi : S \to \mathcal{B}(H)$ *is a completely positive map. Then, there exists a completely positive map* $\phi : \mathcal{A} \to \mathcal{B}(H)$ *extending* $\psi$.

# 3

# Inclusion of free Spectrahedra

This section follows Helton, Klep, and McCullough's paper [8] closely.

## 3.1 Preliminaries and Introduction on LMIs

We define a linear pencil as follows:

$$L_A(x) = A_0 + \sum_{j=1}^{g} A_j x_j,$$

where $A_0, A_1, \ldots, A_g \in S\mathbb{C}^{d\times d}$ and $x \in \mathbb{C}^g$. We say $L_A$ is monic when $A_0 = I$ and its truly linear part $\sum_{j=1}^{g} A_j x_j$ is denoted by $L_A^{(1)}$.

For g-tuples of matrices $X \in (S\mathbb{C}^{n\times n})^g$, we define

$$L_A(X) = A_0 \otimes I_n + \sum_{j=1}^{g} A_j \otimes X_j.$$

We define the free spectrahedron of $L_A$ as:

$$\mathcal{D}_{L_A} = \bigcup_{n\in\mathbb{N}} \left\{ X \in \left(S\mathbb{C}^{n\times n}\right)^g \mid L(X) \geq 0 \right\}.$$

Moreover, the set $\mathcal{D}_{L_A}(n)$ which we call spectrahedron is defined as

$$\mathcal{D}_{L_A}(n) = \left\{X \in \left(S\mathbb{C}^{n\times n}\right)^g \mid L(X) \geq 0\right\} \subset (\mathbb{M}_n)^g.$$

Also, we define boundaries for spectrahedron and free spectrahedra as follows:

$$\partial\mathcal{D}_{L_A}(n) = \left\{X \in \left(S\mathbb{C}^{n\times n}\right)^g \mid L(X) \geq 0, L(X) \not> 0\right\},$$

$$\partial\mathcal{D}_{L_A} = \bigcup_{n\in\mathbb{N}} \partial\mathcal{D}_{L_A}(n).$$

We say a free spectrahedron is bounded if there is $K \in \mathbb{N}$ such that $\|X\| \leq K,$ for $X \in \mathcal{D}_{L_A}$. Note that in the previous definition we have $\|X\| = \max_i \|X_i\|$.

**Example 3.1.** *Consider $L(x) = E_{11} + xE_{11}$. Then, $L(0) = E_{11} \geq 0$, so that $0 \in \mathcal{D}_L$, and $E_{11}$ is not strictly positive which means that $0 \in \partial\mathcal{D}_L$. However, we see that $L(t) \geq 0$ for $t \geq -1$, so in particular the point $0$ is in the topological interior of $\mathcal{D}_L(1)$.* △

By the previous example, if the linear pencil $L$ is not monic then the definition of $\partial\mathcal{D}_L$ does not always coincide with the topological boundary of $\mathcal{D}_L$. However, the following lemma will establish that if we consider a monic linear pencil $L_A$, then the definition of $\partial\mathcal{D}_{L_A}$ coincides with the topological boundary of $\mathcal{D}_{L_A}$ which is denoted by $\mathrm{b}(\mathcal{D}_{L_A})$.

**Lemma 3.2.** *Suppose $L_A(x) = I + \sum_{j=1}^{g} A_j x_j$, where $A_1, \ldots, A_g \in S\mathbb{C}^{d_1\times d_1}$ is a linear pencil. Then,*

$$b(\mathcal{D}_{L_A}(n)) = \partial\mathcal{D}_{L_A}(n) = \left\{X \in \left(S\mathbb{C}^{n\times n}\right)^g \mid L_A(X) \geq 0, L_A(X) \not> 0\right\}.$$

*Proof.* Let $X \in \mathcal{D}_{L_A}(n) \setminus \partial\mathcal{D}_{L_A}(n)$, i.e. $L_A(X) = I + L_A^{(1)}(X) > 0$. At this part, our goal is to show that there exist $r > 0$ such that if $\|Y - X\| < r$, then $I + L_A^{(1)}(Y) > 0$

which is equivalent to saying $X \in \text{int}\,(D_{L_A}(n)) = \mathcal{D}_{L_A}(n) \backslash \text{b}(\mathcal{D}_{L_A}(n))$. Once we know the set of all positive definite $n \times n$ matrices $S^n_{++}$ is an open set, then since $X \in S^n_{++}$, there exists $\epsilon > 0$ such that $B_\epsilon(L_A(X)) \subset S^n_{++}$. Moreover, we know $L_A$ is continuous and therefore there exists $\delta$ such that if $Y \in B_\delta(X)$, then $L_A(Y) \in B_\epsilon(L_A(X)) \subset S^n_{++}$ which says $L_A(Y) > 0$. Now, we establish the proof of the openness of $S^n_{++}$. Since all eigenvalues of $L_A(X)$ are greater than zero, there exist $\epsilon > 0$ such that $L_A(X) - \epsilon I > 0$.

On the other hand, let $X \in \partial \mathcal{D}_{L_A}(n)$, then we have $L_A(X) \geq 0,\ L_A(X) \not> 0$. Our next goal is to show that for every $r > 0$ there exist $X_r,\ Y_r$ such that

$$L_A(X_r) \geq 0,\ L_A(Y_r) \not\geq 0,\ \|X - X_r\| < r,\ \text{and}\ \|X - Y_r\| < r.$$

If we assume $Y_r = (1 + \frac{r}{2\|X\|})X,\ X_r = X$, then we have

$$L_A(Y_r) = L_A((1 + \frac{r}{2\|X\|})X) = (1 + \frac{r}{2\|X\|})L_A(X) - (\frac{r}{2\|X\|})I,$$

so we have a positive semi-definite but not positive definite matrix $(1+\frac{r}{2\|X\|})L_A(X)$ minus $(\frac{r}{2\|X\|})I$, and thus at least one of the eigenvalues of $L_A(Y)$ is negative which implies $L_A(Y_r) \not\geq 0$. Hence, $X \in \text{b}(\mathcal{D}_{L_A}(n))$.

□

**Lemma 3.3.** [8, Proposition 2.1] *Assume $L$ is a linear pencil such that $0$ belongs the interior of its free spectrahedron ($0 \in \mathcal{D}_L \setminus \partial \mathcal{D}_L$). Then there is a monic pencil $L'$ such that $\mathcal{D}_L = \mathcal{D}_{L'}$.*

*Proof.* We know that $0 \in \mathcal{D}_L$, so $L(0) = A_0 \geq 0$. Moreover, we have $0 \in \text{int}(\mathcal{D}_L(1))$, therefore there is $\epsilon > 0$ such that if $\|X\| \leq \epsilon$ then $X \in \mathcal{D}_L(1)$. Fix $1 \leq j \leq g$ and let $Y_j = (0, 0, \ldots, -\epsilon, 0, \ldots, 0) \in \mathbb{C}^g$. Then we have $Y_j \in \mathcal{D}_L(1)$ which means $L(Y_j) \geq 0$. As a result, $A_0 - \epsilon A_j \geq 0$. Now, let $E = \text{ran}\ A_0$ and $A'_j = A_j|_E$, $j = 0, 1, \ldots, g$. We claim $A_0' : E \to E$ is invertible. That is because: $E = \text{ran } A_0 \subset \mathbb{C}^d$ is closed and $\text{ran } A_0 = (\ker A_0^*)^\perp = (\ker A_0)^\perp$, therefore we have

$$\text{ran } A_0' = A_0 \left(\text{ran } A_0\right) = A_0 \left(\ker A_0\right)^\perp = A_0(\mathbb{C}^d) = \text{ran } A_0$$

which states that $A_0'$ is surjective. On the other hand, we have $\text{ran } A_0 = (\ker A_0)^\perp$ which says $A_0'$ in injective. Hence, $A_0'$ is positive definite. In the next step, we want to show that $\text{ran } A_j \subseteq \text{ran } A_0$ for $j = 0, 1, \ldots, g$. If $x \in (\text{ran } A_0)^\perp$ then $x \in \ker A_0, A_0 x = 0$. Also, we know $\pm\epsilon\langle A_j x, x\rangle \le \langle A_0 x, x\rangle = 0$ (because $A_0 - \epsilon A_j \ge 0$). As a result, $\langle A_j x, x\rangle = 0$ and we want to show that $A_j x = 0$. To this end, we have $A_0 + \epsilon A_j \ge 0$ and $\underbrace{\langle A_0 x, x\rangle}_{0} + \epsilon \underbrace{\langle A_j x, x\rangle}_{0} = \langle (A_0 + \epsilon A_j)x, x\rangle = 0$. Therefore, by using the lemma 2.5 we conclude that $(A_0 + \epsilon A_j)x = 0$ and thus $A_j x = 0$ which is equivalent to saying $x \in (\text{ran } A_j)^\perp$. Hence, by knowing the fact that $\text{ran } A_j \subseteq \text{ran } A_0$ along with the argument after proposition 2.6 we conclude that

$$A_j = \begin{bmatrix} A_j|_E & 0 \\ 0 & 0 \end{bmatrix}.$$

On the other hand, for every $a, b \in E$ and $j = 0, 1, \ldots, g$ we have

$$\langle A_j' a, b\rangle = \langle A_j a, b\rangle = \langle a, A_j b\rangle = \langle a, A_j' b\rangle,$$

which says $A_j' : E \to E$ is self-adjoint for $j = 0, 1, \ldots, g$.

Now, we define another pencil $L'$ as $L'(x) = A_0' + \sum_{j=1}^{g} A_j' x_j$. Since $L = L' \oplus 0$, we obtain $\mathcal{D}_L = \mathcal{D}_{L'}$. At this step, we need to construct a monic pencil $\widetilde{L}$ such that $\mathcal{D}_{L'} = \mathcal{D}_{\widetilde{L}}$. To this end, we know $A_0'$ is positive definite and therefore we can factor it with invertible matrix $C$ as $A_0' = C^* C$. Let $\widetilde{A}_j = (C^{-1})^* A_j' C^{-1}$ for $j = 0, 1, \ldots, g$. In addition, if we define monic linear pencil $\widetilde{L} = I + \sum_{j=1}^{g} \widetilde{A}_j x_j$, then if $x \in \mathcal{D}_{\widetilde{L}}$ we have:

$$I + \sum_{j=1}^{g} \left(C^{-1}\right)^* A_j' C^{-1} x_j \geq 0,$$

which is equivalent to

$$C^* \left( I + \sum_{j=1}^{g} \left(C^{-1}\right)^* A_j' C^{-1} x_j \right) C \geq 0,$$

which in turn is equivalent to saying

$$A_0' + \sum_{j=1}^{g} A_j' x_j \geq 0,$$

which is equivalent to

$$x \in \mathcal{D}_{L'}.$$

Therefore, $\mathcal{D}_{\widetilde{L}} = \mathcal{D}_{L'}$. $\square$

**Lemma 3.4.** [8, Lemma 2.2] *Suppose* $L_A(x) = A_0 + \sum_{j=1}^{g} A_j x_j$*, where* $A_0, \ldots, A_g \in S\mathbb{C}^{d_1 \times d_1}$ *is a linear pencil. Also, assume* $L_B(x) = B_0 + \sum_{j=1}^{g} B_j x_j$*, where* $B_0, \ldots, B_g \in S\mathbb{C}^{d_2 \times d_2}$ *is a linear pencil. Then for* $w = d_2(1+g)$ *the following hold:*

(1) $L_B|_{\mathcal{D}_{L_A}} > 0$ *if and only if* $L_B|_{\mathcal{D}_{L_A}(w)} > 0$*,*

(2) $L_B|_{\mathcal{D}_{L_A}} \geq 0$ *if and only if* $L_B|_{\mathcal{D}_{L_A}(w)} \geq 0$*.*

*Proof.* (1)($\Rightarrow$) is trivial.

($\Leftarrow$): We will prove that if $L_B|_{\mathcal{D}_{L_A}} \not> 0$, then $L_B|_{\mathcal{D}_{L_A}(w)} \not> 0$. Assume $L_B|_{\mathcal{D}_{L_A}} \not> 0$, then there is $t \in \mathbb{N}$ such that $X \in \mathcal{D}_{L_A}(t)$ and $z = (z_1, \ldots, z_{d_2}) \in (\mathbb{C}^t)^{d_2}$ with the property that: $\langle L_B(X)z, z \rangle \leq 0$. Let

$$K = \operatorname{span}\left(\{X_i z_j |\ i = 1, \ldots, g, j = 1, \ldots, d_2\} \cup \{z_j |\ j = 1, \ldots, d_2\}\right).$$

Let $p$ be the orthogonal projection onto $K$. Note that $\dim K \leq d_2(1+g)$. We have:

$$\begin{aligned}(B_j \otimes pX_jp)\, z &= \begin{bmatrix} (b_{11})_j pX_jp & \dots & (b_{1d_2})_j pX_jp \\ \vdots & \ddots & \vdots \\ (b_{d_21})_j pX_jp & \dots & (b_{d_2d_2})_j pX_jp \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_{d_2} \end{bmatrix} \\ &= \begin{bmatrix} (b_{11})_j pX_jpz_1 + (b_{12})_j pX_jpz_2 + \ldots + (b_{1d_2})_j pX_jpz_{d_2} \\ \vdots \\ (b_{d_21})_j pX_jpz_1 + (b_{d_22})_j pX_jpz_2 + \ldots + (b_{d_2d_2})_j pX_jpz_{d_2} \end{bmatrix}.\end{aligned}$$

Since $p$ is a projection, we have $pz_i = z_i$ and $pX_jz_i = X_jz_i$ for $i = 1, 2, \ldots, d_2$. Hence,

$$\begin{aligned}(B_j \otimes pX_jp)\, z &= \begin{bmatrix} (b_{11})_j pX_jz_1 + \ldots + (b_{1d_2})_j pX_jz_{d_2} \\ \vdots \\ (b_{d_21})_j pX_jz_1 + \ldots + (b_{d_2d_2})_j pX_jz_{d_2} \end{bmatrix} \\ &= \begin{bmatrix} (b_{11})_j X_jz_1 + \ldots + (b_{1d_2})_j X_jz_{d_2} \\ \vdots \\ (b_{d_21})_j X_jz_1 + \ldots + (b_{d_2d_2})_j X_jz_{d_2} \end{bmatrix} = (B_j \otimes X_j)z,\end{aligned}$$

which says:

$$\langle L_B(pXp)z, z\rangle = \langle L_B(X)z, z\rangle \leq 0.$$

Consequently, $pXp \notin \mathcal{D}_{L_B}(t)$.

Let $\eta \in I_{d_1} \otimes \operatorname{ran} p$, then we have

$$\begin{aligned}\langle L_A(pXp)\eta, \eta\rangle &= \left\langle \left( A_0 \otimes I_{\text{ran p}} + \sum_{j=1}^{g} A_j \otimes pX_jp \right) \eta, \eta \right\rangle \\ &= \left\langle \left( A_0 \otimes I + \sum_{j=1}^{g} A_j \otimes X_j \right) (\eta \oplus 0), (\eta \oplus 0) \right\rangle \\ &= \langle L_A(X)(\eta \oplus 0), (\eta \oplus 0)\rangle \geq 0,\end{aligned}$$

thus $pXp \in \mathcal{D}_{L_A}(w)$.

For part (2) also we can use the same method as in part (1). $\square$

It is interesting to know that at what level the boundedness of the spectrahedron is equivalent to the boundedness of free spectrahedra. The following lemma provides a detailed answer for this conjecture.

**Lemma 3.5.** [8, Lemma 2.3] *For a linear pencil $L$ the following are equivalent:*

(1) $\mathcal{D}_L$ *is bounded,*

(2) $\mathcal{D}_L((1+g)^2)$ *is bounded.*

*Proof.* (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) : For a given $n \in \mathbb{N}$ we define a monic linear pencil as

$$L_n(x) = \frac{1}{n}\begin{bmatrix} n & x_1 & x_2 & x_3 & \dots & x_g \\ x_1 & n & 0 & 0 & \dots & 0 \\ x_2 & 0 & n & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ x_g & 0 & \dots & & 0 & n \end{bmatrix} = \frac{1}{n}\begin{bmatrix} n & x^* \\ x & nI_g \end{bmatrix}.$$

We want to demonstrate that $\mathcal{D}_L$ is bounded if and only for some $n, L_n\Big|_{\mathcal{D}_L} \geq 0$. Assume $X \in \mathcal{D}_L$ and $L_n(X) \geq 0$, the latter being equivalent to saying $\begin{bmatrix} nI_d & X^* \\ X & nI_g \end{bmatrix} \geq 0$.

We know by Theorem 2.1,

$$\begin{bmatrix} n & X^* \\ X & nI_g \end{bmatrix} \geq 0 \text{ if and only if } nI_g - X\frac{1}{n}X^* \geq 0,$$

which in turn is equivalent to $n^2 I_g - XX^* \geq 0$, so $XX^* \leq n^2 I_g$. Moreover, we have

$$\|X^*(v)\|^2 = \langle X^*(v), X^*(v)\rangle = \langle v, XX^*(v)\rangle \leq n^2\|v\|^2.$$

Thus,

$$\|X^*(v)\|^2 \leq n^2\|v\|^2.$$

So, we have $\|X^*\| \leq n$, which says $\mathcal{D}_L$ is bounded. By using previous lemma, our assumption $\mathcal{D}_L((1+g)^2)$ is bounded is equivalent to saying $L_n\Big|_{\mathcal{D}_L} \geq 0$ which in turn is equivalent to saying $\mathcal{D}_L$ is bounded. □

For a linear pencil $L$ we define matricial ball as follows:

$$\mathcal{B}_L := \bigcup_{n\in\mathbb{N}} \left\{X \in \left(S\mathbb{C}^{n\times n}\right)^g \mid \|L(X)\| \leq 1\right\}.$$

In lemma 3.5, we obtained an equivalent condition for the boundedness of the free spectrahedra of a linear pencil. Now, in the following proposition, we will obtain the relationship between the boundedness of the free spectrahedra and the boundedness of the spectrahedron at first level.

**Proposition 3.6.** [8, Proposition 2.4] *Let $L$ be a monic linear pencil then:*

(1) *$\mathcal{D}_L$ is bounded if and only if $\mathcal{D}_L(1)$ is bounded.*

(2) *$\mathcal{B}_L$ is bounded if and only if $\mathcal{B}_L(1)$ is bounded.*

*Proof.* (1)($\Rightarrow$) is trivial.

(1)($\Leftarrow$) We want to prove that if $\mathcal{D}_L$ is not bounded, then $\mathcal{D}_L(1)$ is not bounded.

Assume $\mathcal{D}_L$ is not bounded. By using the lemma 3.5, there exists $N \in \mathbb{N}$ such that $\mathcal{D}_L(N)$ is not bounded. Consider a sequence $\{X^k\} \subset \left(S\mathbb{C}^{N\times N}\right)^g$ with the property that $\|X^k\| = 1$ and a sequence $\{t_k\} \subset \mathbb{R}^+$ with $\lim_{k\to\infty} t_k = \infty$, such that $L(t_k X^k) \geq 0$. The sequence $\{X^k\}$ has a convergent subsequence which converges to

$X = (X_1, \ldots, X_g) \in (S\mathbb{C}^{N\times N})^g$, with $\|X\| = 1$.

For every $t > 0$, $tX^k \to tX$. We claim that for big enough $k$, $tX^k \in \mathcal{D}_L(N)$. To see that, since $L$ is monic, then $0 \in \mathcal{D}_L$. Moreover, we know $t_k X^k \in \mathcal{D}_L$. For big enough $k$, we have $0 \leq t \leq t_k$, then $tX^k = \dfrac{t}{t_k} t_k X^k + \left(1 - \dfrac{t}{t_k}\right) 0$. By using the convexity of $\mathcal{D}_L$ which could be obtained by an easy calculation, we have $tX^k \in \mathcal{D}_L$.

Thus, $L(tX) \geq 0$ (for $t \in \mathbb{R}^+$). If we write $X = (X_1, \ldots, X_g)$, then because $\|X\| = 1$ at least one of $X_i$ is non-zero and by using the fact that $W(X_i) = \{\langle X_i v, v\rangle : \ \|v\| = 1\}$ is non-zero, we can infer that there is a non-zero vector $v$ such that at least for one $i$ we have $\langle X_i v, v\rangle \neq 0$.

We define $z := (\langle X_1 v, v\rangle, \ldots, \langle X_g v, v\rangle) \in \mathbb{C}^g \setminus \{0\}$ and the map

$$\begin{aligned} u : \mathbb{C} &\to \mathbb{C}^N \\ r &\mapsto rv, \end{aligned}$$

then for $m \in \mathbb{C}$ we have

$$\langle u^* X_i v, m\rangle = \langle X_i v, um\rangle = \langle X_i v, mv\rangle = \overline{m}\langle X_i v, v\rangle = \langle \langle X_i v, v\rangle, m\rangle.$$

Therefore, we have $u^* X_i v = \langle X_i v, v\rangle$ and

$$u^* X_i u(t) = u^* X_i t v = t\langle X_i v, v\rangle = t z_i.$$

Consequently,

$$L(tz) = (I \otimes u^*)\, L(tX)\, (I \otimes u) \tag{3.1}$$

is non-negative for all $t > 0$, so $tz \in \mathcal{D}_L(1)$ $(tz \to \infty)$ and therefore $\mathcal{D}_L(1)$ is not bounded.

(2)($\Rightarrow$) is trivial.

(2)($\Leftarrow$) consider the pencil $L_1$ as follows:

$$L_1 = \begin{bmatrix} I & L \\ L & I \end{bmatrix}.$$

Furthermore, we have

$$\mathcal{B}_L := \bigcup_{n\in\mathbb{N}} \left\{X \in \left(S\mathbb{C}^{n\times n}\right)^g \mid \|L(X)\| \leq 1\right\} = \left\{X \mid I - L(X)^2 \geq 0\right\}.$$

By applying lemma 2.9, we conclude that $\mathcal{B}_L = \mathcal{D}_{L_1}$, now we can use the previous part to get the desired result.

$\square$

A linear pencil $L$ is said to be non-degenerate, if $L(X) = L(Y)$ implies $X = Y$ for all $n \in \mathbb{N}$ and $X, Y \in (S\mathbb{C}^{n\times n})^g$. In particular, a truly linear pencil $L$ is nondegenerate if and only if $L(X) \neq 0$ for $X \neq 0$.

For matrices $A \in \mathbb{M}_n$ and $B \in \mathbb{M}_m$, we have

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{bmatrix}, \quad B \otimes A = \begin{bmatrix} b_{11}A & \dots & b_{1m}A \\ \vdots & \ddots & \vdots \\ b_{1m}A & \dots & b_{mm}A \end{bmatrix}.$$

Block matrices $A \otimes B$ and $B \otimes A$ are unitarily equivalent. We call this rearrangement the canonical shuffle [10, p.30].

**Lemma 3.7.** [8, Lemma 2.5] *For a linear pencil* $L(X) = A_0 + \sum_{j=1}^{g} A_j x_j$ *the following are equivalent:*

(1) $L$ *is non-degenerate,*

(2) $L(z) = L(w)$ *implies* $z = w$ *for all* $z, w \in \mathbb{C}^g$*,*

(3) *the set* $\{A_j \mid \ j = 1, 2, \dots, g\}$ *is linearly independent,*

(4) $L^{(1)}$ *is non-degenerate.*

*Proof.* The proof of (1) $\Leftrightarrow$ (4), (1) $\Rightarrow$ (2), and (2) $\Rightarrow$ (3) are straightforward. To complete the proof, we prove that the implication (3) $\Rightarrow$ (1). Let $L(X) = L(Y)$ for $X, Y \in (S\mathbb{C}^{n\times n})^g$, then $L^{(1)}(X - Y) = 0$. By using the canonical shuffle, we get $\sum_{j=1}^{g} (X_j - Y_j) \otimes A_j = 0$. Using the linear independence of the set $\{A_j |\ j = 1, 2, \ldots, g\}$, we infer that $X_j - Y_j = 0$ for every $j \in \{1, 2, \ldots, g\}$, and $X = Y$. $\square$

The following propositions will exhibit a relationship between the boundedness of the free spectrahedra of $L(x)$ and matrices $A_1 \ldots, A_g \in S\mathbb{C}^{n\times n}$ that we used to construct the linear pencil $L(x)$.

**Proposition 3.8.** [8, Proposition 2.6] *Assume* $L = I + \sum_{j=1}^{g} A_j x_j \in S\mathbb{C}^{d\times d}$ *is a monic linear pencil and* $L^{(1)}$ *denotes its truly linear part. Then:*

(1) $\mathcal{B}_{L^{(1)}}$ *is bounded if and only if* $L^{(1)}$ *is non-degenerate,*

(2) *If* $\mathcal{D}_L$ *is bounded, then* $\{I, A_j |\ \ j = 1, 2, \ldots, j\}$ *is linearly independent.*

*Proof.* (1) Suppose $L^{(1)}$ is degenerate, $\sum_{j=1}^{g} z_j A_j = 0$ for some $z_j \in \mathbb{C}$ such that at least one of $z_j$ is non-zero. Let $z = (z_1, \ldots, z_g) \in \mathbb{C}^g \setminus \{0\}$. We know $tz \in \mathcal{B}_{L^{(1)}}$ for every $t$. Thus, $\mathcal{B}_{L^{(1)}}$ is not bounded.

Conversely, if $\mathcal{B}_{L^{(1)}}$ is unbounded, then by Proposition 3.6, $\mathcal{B}_{L^{(1)}}(1)$ is not bounded. So, there is a sequence $\{z^k\} \subset \mathbb{C}^g$ such that $\|z^k\| = 1$ and a sequence $t_k \in \mathbb{R}^+$ such that $t_k \to \infty$, $\left\|L^{(1)}(t_k z^k)\right\| \leq 1$. There is a subsequence of $\{z^k\}$ which converges to $z \in \mathbb{C}^g$ which has norm 1. So, we have $\left\|L^{(1)}(z^k)\right\| \leq \frac{1}{|t_k|}$. Let $k \to \infty$ we have $\left\|L^{(1)}(z)\right\| \leq 0$ which says $\left\|L^{(1)}(z)\right\| = 0$. Consequently, $L^{(1)}$ is not non-degenerate.

(2) Suppose $\lambda I + \sum_j x_j A_j = 0$ with $\lambda, x_j \in \mathbb{C}$ not all zero. We may assume for at least one index $j$ we have $x_j \neq 0$. Let $z = (x_1, \ldots, x_g) \neq 0$. If $\lambda = 0$, then $L^{(1)}(tz) = 0$ is positive semi-definite for every $t \in \mathbb{R}$. So, $tz \in \mathcal{D}_L$ and if $t \to \infty$, then $tz \to \infty$ which shows that $\mathcal{D}_L$ is not bounded. If $0 \neq \lambda \in \mathbb{C}$, then by assumption we have,

$$L(z/\lambda) = I + \frac{1}{\lambda} \sum_j x_j A_j = 0.$$

Therefore, we have $L(tz/\lambda) \geq 0$ for all $t \in \mathbb{R}^-$ which shows that $\mathcal{D}_L$ is unbounded.

$\square$

For two linear pencils:

$$L_A(x) = I + \sum_{j=1}^{g} A_j x_j, \qquad A_1, \dots, A_g \in S\mathbb{C}^{d_1 \times d_1},$$
$$L_B(x) = I + \sum_{j=1}^{g} B_j x_j, \qquad B_1, \dots, B_g \in S\mathbb{C}^{d_2 \times d_2},$$

we will study the following two inclusions for free spectrahedra,

$$\mathcal{D}_{L_A} \subseteq \mathcal{D}_{L_B} \text{ (i.e., for all } n \in \mathbb{N} \text{ and } X \in (S\mathbb{C}^{n\times n})^g,\ L_A(X) \geq 0 \text{ implies } L_B(X) \geq 0), \tag{3.2}$$

$$\partial\mathcal{D}_{L_A} \subseteq \partial\mathcal{D}_{L_B} \text{ (i.e., for all } n \in \mathbb{N} \text{ and } X \in (S\mathbb{C}^{n\times n})^g,\ L_A(X) \geq 0 \text{ and } L_A(X) \not> 0 \text{ implies } L_B(X) \geq 0 \text{ and } L_B(X) \not> 0) \tag{3.3}$$

In this part, we will characterize the relationship between $L_A$ and $L_B$ which satisfies (3.2) and (3.3). By using lemma 3.4, to check (3.2), it is enough to use matrices of large enough size.

**Example 3.9.** [8, Example 3.1] *In this example, we will show that to verify condition* (3.2)*, it is not sufficient to use a scalar vector $X \in \mathbb{C}^g$. To this end, let*

$$\begin{aligned} \triangle(X_1, X_2) &= I + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} X_1 + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} X_2 \\ &= \begin{bmatrix} 1 & X_1 & X_2 \\ X_1 & 1 & 0 \\ X_2 & 0 & 1 \end{bmatrix}, \end{aligned}$$

*and*

$$\begin{aligned}\Gamma(X_1,X_2) &= I+\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}X_1+\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}X_2\\ &= \begin{bmatrix}1+X_1 & X_2\\ X_2 & 1-X_1\end{bmatrix}.\end{aligned}$$

*Then, by the theorem 2.1, we have*

$$\mathcal{D}_\triangle=\bigcup_n\left\{(X_1,X_2)\in(S\mathbb{C}^{n\times n})^2|\ \ I-X_1^2-X_2^2\geq 0\right\},$$
$$\mathcal{D}_\triangle(1)=\left\{(X_1,X_2)\in\mathbb{R}^2|\ \ X_1^2+X_2^2\leq 1\right\},$$
$$\mathcal{D}_\Gamma(1)=\left\{(X_1,X_2)\in\mathbb{R}^2|\ \ X_1^2+X_2^2\leq 1\right\},$$

*and we have* $\mathcal{D}_\triangle(1)=\mathcal{D}_\Gamma(1)$.

*Consider* $X=(X_1,X_2)=\left(\begin{bmatrix}\frac{1}{2} & 0\\ 0 & 0\end{bmatrix},\begin{bmatrix}0 & \frac{3}{4}\\ \frac{3}{4} & 0\end{bmatrix}\right)$. *We have:*

$$I-\begin{bmatrix}\frac{1}{2} & 0\\ 0 & 0\end{bmatrix}^2-\begin{bmatrix}0 & \frac{3}{4}\\ \frac{3}{4} & 0\end{bmatrix}^2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}\frac{13}{16} & 0\\ 0 & \frac{9}{16}\end{bmatrix}\geq 0$$

*which means* $X\in\mathcal{D}_\triangle$.

*Furthermore,*

$$\Gamma(X)=\begin{bmatrix}I+X_1 & X_2\\ X_2 & I-X_1\end{bmatrix}=\begin{bmatrix}\overbrace{\begin{pmatrix}\frac{3}{2} & 0\\ 0 & 1\end{pmatrix}}^{A} & \overbrace{\begin{pmatrix}0 & \frac{3}{4}\\ \frac{3}{4} & 0\end{pmatrix}}^{B}\\ \underbrace{\begin{pmatrix}0 & \frac{3}{4}\\ \frac{3}{4} & 0\end{pmatrix}}_{B^*} & \underbrace{\begin{pmatrix}\frac{1}{2} & 0\\ 0 & 1\end{pmatrix}}_{C}\end{bmatrix}.$$

*By using lemma 2.1, we infer that* $\Gamma(X) \geq 0$ *if and only if* $C - B^*A^{-1}B \geq 0$. *However, a simple calculation shows that* $C - B^*A^{-1}B \ngeq 0$, *and therefore* $X \notin \mathcal{D}_\Gamma$. *Thus,* $\triangle(X) \geq 0$, *does not imply* $\Gamma(X) \geq 0$. $\triangle$

The previous example exhibits that although two given linear pencils have the same spectrahedra at first level, the free spectrahedra of $\triangle$ is not the subset of the free spectrahedra of $\Gamma$. One may consider [8, Proposition 5.3] and the algorithm following it to construct the pencil $L_{\mathcal{C}_1}$ from the given pencil $L_\eta$ such that $\mathcal{D}_{L_{\mathcal{C}_1}}(1) \subseteq \mathcal{D}_{L_\eta}(1)$ but for $n \geq 2$, $\mathcal{D}_{L_{\mathcal{C}_1}}(n) \nsubseteq \mathcal{D}_{L_\eta}(n)$.

## 3.2 Main Results

In this part, we introduce the subspaces $\mathcal{S}_A, \mathcal{S}_B$ as follow:

$$\mathcal{S}_A = \operatorname{span}\{I, A_j \mid j = 1, \ldots, g\},$$

$$\mathcal{S}_B = \operatorname{span}\{I, B_j \mid j = 1, \ldots, g\},$$

where $A_j \in S\mathbb{C}^{d_1 \times d_1}$ and $B_j \in S\mathbb{C}^{d_2 \times d_2}$ for $j = 1, \ldots, g$. Further, we define a unital map $\tau$ between $\mathcal{S}_A, \mathcal{S}_B$ as:

$$\begin{aligned} \tau : \mathcal{S}_A &\to \mathcal{S}_B \\ A_j &\mapsto B_j \end{aligned}.$$

Note that in order to have $\tau$ well-defined, the set $\mathcal{S}_A$ needs to be linearly independent. By using proposition 3.8, the boundedness of $\mathcal{D}_{L_A}$ implies that the set $\mathcal{S}_A$ is linearly independent.

Recall that the tensor product of two Hilbert spaces $H, K$ is a Hilbert space $H \otimes K$ such that for vectors $(x_1 \otimes y_1) \in H \otimes K$ and $(x_2 \otimes y_2) \in H \otimes K$ we have

$$\langle (x_1 \otimes y_1), (x_2 \otimes y_2) \rangle = \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle. \tag{3.4}$$

**Lemma 3.10.** [8, Lemma 3.6] *Let* $L_A(x) = I + \sum_{j=1}^{g} A_j x_j$ *be a monic linear pencil and* $\mathcal{D}_{L_A}$ *be a bounded set, where* $A_1, \dots, A_g \in S\mathbb{C}^{d_1 \times d_1}$. *Let* $L = I \otimes \Lambda + L_A^{(1)}(X)$, *where* $X \in (\mathbb{C}^{n\times n})^g$ *and* $\Lambda \in \mathbb{C}^{n\times n}$. *Then the followings hold:*

(1) *Assume* $L$ *is self-adjoint and* $X \in (S\mathbb{C}^{n\times n})^g$. *Then* $\Lambda = \Lambda^*$.

(2) *If* $L$ *is positive semi-definite and* $X \in (S\mathbb{C}^{n\times n})^g$, *then* $\Lambda$ *is also positive semi-definite.*

(3) *Suppose* $L' = \Lambda \otimes I + \sum_{j=1}^{g} X_j \otimes A_j \geq 0$ *for* $\Lambda \in \mathbb{C}^{n\times n}$, *and* $X \in (S\mathbb{C}^{n\times n})^g$. *Then,* $\Lambda$ *is positive semi-definite.*

*Proof.* (1) If we assume $L = I\otimes\Lambda+\sum_{j=1}^{g} A_j\otimes X_j$ is self-adjoint then we have: $0 = L^*-L$ which implies $0 = I \otimes (\Lambda^* - \Lambda)$, so $\Lambda^* = \Lambda$.

(2) Let $\Lambda \ngeq 0$, then there is a vector $z \in \mathbb{C}^n$ such that $\langle \Lambda z, z\rangle < 0$ with $\|z\| = 1$. In the next part, we intend to exhibit that $\mathcal{D}_{L_A}$ is not bounded. To this end, we define the projection

$$Q : \mathbb{C}^n \to \mathbb{C}z.$$

Let $F = (\langle X_j z, z\rangle)_{j=1}^g \in \mathbb{C}^g$ and $M = (QX_jQ)_{j=1}^g$. We know $L \geq 0$, so $(I \otimes Q)L(I \otimes Q) \geq 0$, and thus

$$I \otimes Q\Lambda Q + \sum_{j=1}^{g} A_j \otimes QX_jQ \geq 0.$$

Equivalently, we have:

$$\left\langle \Big(I \otimes Q\Lambda Q + L_A^{(1)}(M)\Big)\, t \otimes z, t \otimes z \right\rangle \geq 0, \quad t \otimes z \in \mathbb{C}^{d_1} \otimes \mathbb{C}^n.$$

Thus, by using 3.4 we obtain

$$\left\langle L_A^{(1)}(M) t \otimes z, t \otimes z \right\rangle \geq -\|t\|^2 \langle \Lambda z, z\rangle > 0. \tag{3.5}$$

On the other hand, for $t \in \mathbb{C}^{d_1}$ we have

$$\begin{aligned}
\left\langle L_A^{(1)}(F)t, t \right\rangle &= \left\langle \left( \sum_{j=1}^{g} (A_j \langle X_j z, z\rangle) \right) t, t \right\rangle \\
&= \left\langle \left( \sum_{j=1}^{g} A_j \otimes QX_jQ \right) t \otimes z, t \otimes z \right\rangle \\
&= \left\langle L_A^{(1)}(M) t \otimes z, t \otimes z \right\rangle \\
&\geq -\|t\|^2 \langle \Lambda z, z\rangle \qquad \text{(by using 3.5).}
\end{aligned}$$

We proved that $L_A^{(1)}(F) > 0$, which implies $vF \in \mathcal{D}_{L_A}$ for $v > 0$ because $L_A(vF) = I + L_A^{(1)}(vF) \geq 0$, so $\mathcal{D}_{L_A}$ is not bounded.

(3) If we apply the canonical shuffle on $L'$, then it has the form of $L$, so we can apply part (2) to get the desired result. □

**Proposition 3.11.** [8, Proposition 3.9] *Suppose $L_A, L_B$ are linear monic pencils such that $\mathcal{D}_{L_A}$ is bounded and $\partial\mathcal{D}_{L_A} \subseteq \partial\mathcal{D}_{L_B}$. Then $\mathcal{D}_{L_A} = \mathcal{D}_{L_B}$.*

*Proof.* If we assume $T_1 = \mathcal{D}_{L_A}$ and $T_2 = \mathcal{D}_{L_B}$, then we have

$$\partial T_1 = \left\{ X \in \mathcal{D}_{L_A} \middle|\ L_A(X) \geq 0, L_A(X) \not> 0 \right\},$$

$$\partial T_2 = \left\{ X \in \mathcal{D}_{L_B} \middle|\ L_B(X) \geq 0, L_B(X) \not> 0 \right\}.$$

In addition, we have $T_1$ is closed, bounded and convex. So, by using lemma [13, 1.4.1], we infer that $T_1 = \text{con}\ \partial T_1$. Moreover, we know that $\partial T_1 \subseteq \partial T_2$. Therefore,

$$\text{con}\ \partial T_1 \subseteq \text{con}\ \partial T_2 \subseteq T_2,$$

which implies that $T_1 \subseteq T_2$. On the other hand, we know $0 \in \text{int}\ T_1 \cap \text{int}\ T_2$ because both $L_A$ and $L_B$ are monic, so we can apply lemma 2.7 and attain $T_1 = T_2$. □

As we stated in chapter 1, it is of the interest to clarify when $L_A(X) \geq 0$ implies $L_B(X) \geq 0$.

The next part provides an answer to our question by relating complete positivity to LMIs. In particular, the following theorem will show that $L_A(X) \geq 0$ implies $L_B(X) \geq 0$ if and only if there exists a natural completely positive map between the sets $\text{span}\{I, A_1, \ldots, A_g\}$ and $\text{span}\{I, B_1, \ldots, B_g\}$.

**Theorem 3.12.** [8, Theorem 3.5] *Assume* $L_A(x) = I + \sum_{j=1}^{g} A_j x_j$*, where* $A_1, \ldots, A_g \in S\mathbb{C}^{d_1 \times d_1}$ *with* $\mathcal{D}_{L_A}$ *bounded and* $L_B(x) = I + \sum_{j=1}^{g} B_j x_j$*, where* $B_1, \ldots, B_g \in S\mathbb{C}^{d_2 \times d_2}$*. Consider the unital linear map* $\tau : S_A \to S_B$ *defined as* $A_j \mapsto B_j$*.*

*Then, we have:*

(1) $\tau$ *is* $n$*-positive if and only if* $\mathcal{D}_{L_A}(n) \subseteq \mathcal{D}_{L_B}(n)$*,*

(2) $\tau$ *is completely positive if and only if* $\mathcal{D}_{L_A} \subseteq \mathcal{D}_{L_B}$*,*

(3) $\tau$ *is completely isometric if and only if* $\partial\mathcal{D}_{L_A} \subseteq \partial\mathcal{D}_{L_B}$*.*

*Proof.* (1): Assume $\tau$ is $n$-positive. Then, for every $X \in \mathcal{D}_{L_A}(n)$, $L_A(X) \geq 0$, thus $L_B(X) = \tau_n(L_A(X)) \geq 0$.

($\Leftarrow$): For fixed $n \in \mathbb{N}$, suppose $L' \in \mathbb{M}_n(\mathcal{S}_A)$ is a positive element of the form $L' = \Lambda \otimes I + \sum_{j=1}^{g} X_j \otimes A_j \geq 0$ for $\Lambda \in \mathbb{C}^{n \times n}$ and $X \in (S\mathbb{C}^{n \times n})^g$. In the previous part, the reason why we have $X \in (S\mathbb{C}^{n \times n})^g$ is since $L'$ is a positive operator, $L'$ is self-adjoint which implies $X \in (S\mathbb{C}^{n \times n})^g$ .

Then, by Lemma 3.10 we have that $\Lambda \geq 0$. By applying the canonical shuffle, we get $L = I \otimes \Lambda + \sum_{j=1}^{g} A_j \otimes X_j \geq 0$. In order to make $\Lambda$ invertible, we change it to $\Lambda_\epsilon = \Lambda + \epsilon I$ and let $L_\epsilon = I \otimes \Lambda_\epsilon + \sum_{j=1}^{g} A_j \otimes X_j$. Consequently,

$$\left(I \otimes \Lambda_\epsilon^{-\frac{1}{2}}\right) L_\epsilon \left(I \otimes \Lambda_\epsilon^{-\frac{1}{2}}\right) = I \otimes I + \sum_{j=1}^{g} A_j \otimes \left(\Lambda_\epsilon^{-\frac{1}{2}} X_j \Lambda_\epsilon^{-\frac{1}{2}}\right) \geq 0.$$

Moreover, we have $\mathcal{D}_{L_A}(n) \subseteq \mathcal{D}_{L_B}(n)$. Hence,

$$I \otimes I + \sum_{j=1}^{g} B_j \otimes \left(\Lambda_\epsilon^{-\frac{1}{2}} X_j \Lambda_\epsilon^{-\frac{1}{2}}\right) \geq 0,$$

which implies

$$I \otimes \Lambda_\epsilon^{\frac{1}{2}} \left(I \otimes I + \sum_{j=1}^{g} B_j \otimes \left(\Lambda_\epsilon^{-\frac{1}{2}} X_j \Lambda_\epsilon^{-\frac{1}{2}}\right)\right) I \otimes \Lambda_\epsilon^{\frac{1}{2}} \geq 0,$$
$$I \otimes \Lambda_\epsilon + \sum_{j=1}^{g} B_j \otimes X_j \geq 0.$$

At this step, if we apply the canonical shuffle, we have:

$$\tau(L_\epsilon) = \Lambda_\epsilon \otimes I + \sum_{j=1}^{g} X_j \otimes B_j \geq 0,$$
$$\tau\left(\Lambda_\epsilon \otimes I + \sum_{j=1}^{g} X_j \otimes A_j\right) \geq 0.$$

Therefore, we have proved that for $L'_\epsilon = \Lambda_\epsilon \otimes I + \sum_{j=1}^{g} X_j \otimes A_j \in \mathbb{M}_n(\mathcal{S}_A)$ with $\Lambda_\epsilon > 0$ if $L'_\epsilon > 0$ then $\tau(L'_\epsilon) \geq 0$. On the other hand, we have $L' + \epsilon I > 0$, so $\tau(L' + \epsilon I) \geq 0$. Moreover, we have $\lim_{\epsilon \to 0} \tau(L' + \epsilon I) \geq 0$ which is equivalent to saying $\lim_{\epsilon \to 0} (\tau(L') + \epsilon I) \geq 0$. Therefore, $\tau(L') \geq 0$ which in turn states that $\tau$ is $n$-positive.

(2) This is a consequence of (1).

(3) Assume we have $\partial\mathcal{D}_{L_A} \subseteq \partial\mathcal{D}_{L_B}$, then by using proposition 3.10 we conclude that $\mathcal{D}_{L_A} = \mathcal{D}_{L_B}$. Therefore, by using part (2) we infer that $\tau$ and $\tau^{-1}$ are UCP maps which is equivalent to saying $\tau$ is a unital completely isometric map [10, Proposition 3.6].

Assume $\tau$ is a unital completely isometric map which is equivalent to saying $\tau$ and $\tau^{-1}$ are UCP maps. Let $X \in \partial\mathcal{D}_{L_A}$, so $L_A(X) \geq 0$ and $L_A(X) \not> 0$. By using that $\tau$

is a UCP map, we obtain that

$$L_B(X) = \tau\left(L_A(X)\right) \geq 0.$$

On the other hand, we know that $\tau^{-1}\left(L_B(X)\right) = L_A(X) \not\geq 0$ and $\tau^{-1}$ is a UCP map, and therefore $L_B(X) \not\geq 0$. That is to say, $X \in \partial\mathcal{D}_{L_B}$.

□

In the following part, we will establish another equivalent condition for two monic linear pencils under which we will have $\mathcal{D}_{L_A} \subseteq \mathcal{D}_{L_B}$.

**Corollary 3.13.** [8, Corollary 3.7] *Assume $L_A, L_B$ are the same as in the previous theorem and $\mathcal{D}_{L_A}$ is bounded. If $\mathcal{D}_{L_A} \subseteq \mathcal{D}_{L_B}$, then there is $k \in \mathbb{N}$ and an isometry $z \in \mathbb{C}^{kd_1 \times d_2}$ such that $L_B(x) = z^*\left(I_k \otimes L_A(x)\right)z$.*

*Proof.* Assume $\mathcal{D}_{L_A} \subseteq \mathcal{D}_{L_B}$. Then, by theorem 3.12, there exists a UCP map $\tau : \mathcal{S}_1 \to \mathcal{S}_2, \tau(A_j) = B_j$, and by using theorem 2.13 we obtain a UCP map $\psi : \mathbb{C}^{d_1 \times d_1} \to \mathbb{C}^{d_2 \times d_2}$ extending $\tau$. By considering $\psi$, we realize that all conditions of Stinespring representation theorem 2.12 are fulfilled. Therefore, by applying 2.12 we obtain that there is unital $*$-homomorphism $\pi : \mathbb{C}^{d_1 \times d_1} \to \mathbb{C}^{d_3 \times d_3}$ and isometry $V : \mathbb{C}^{d_1} \to \mathbb{C}^{d_3}$ such that

$$\psi(t) = V^*\pi(t)V, \quad t \in \mathbb{C}^{d_1 \times d_1}.$$

Since all representations of $\mathbb{C}^{d_1 \times d_1}$ are unitarily equivalent to a multiple of the identity representation by using lemma 2.3, then

$$\psi(t) = \mathcal{V}^*(I_\mu \otimes t)\mathcal{V}, \quad \mathcal{V} \in \mathbb{C}^{\mu d_1 \times \mu d_1},\ t \in \mathbb{C}^{d_1 \times d_1}.$$

□

We call the projection $p$ in $C^*(\mathcal{S}_A)$ central if for every $B \in C^*(\mathcal{S}_A)$, $Bp = pB$.

Also, we say the projection $\widetilde{p} \in \mathcal{B}(H)$ is a reducing projection for $C^*(\mathcal{S}_A)$ if for every $B \in C^*(\mathcal{S}_A)$, $B\widetilde{p} = \widetilde{p}B$.

**Lemma 3.14.** *Let $\widetilde{p}$ be a minimal reducing projection for $C^*(\mathcal{S}_A)$ such that $\widetilde{p} \notin C^*(\mathcal{S}_A)$. Then, there is a minimal central projection $p$ for $C^*(\mathcal{S}_A)$ such that $\widetilde{p} \leq p$.*

*Proof.* Let $Q_{\widetilde{p}} = \{e \in C^*(\mathcal{S}_A) |\ e$ is a central projection such that $\widetilde{p} \leq e\}$ and we know $Q_{\widetilde{p}}$ is not an empty set because $I \in Q_{\widetilde{p}}$. Therefore, if $e \in Q_{\widetilde{p}}$, then $\widetilde{p} \leq e$ which is equivalent to say ran $\widetilde{p} \subset$ ran $e \subset \mathbb{C}^{d_1}$. Hence, we know the set $\left\{\dim(\text{ran } e) | e \in Q_{\widetilde{p}}\right\} \subset \mathbb{N} \cap [\dim \text{ran } \widetilde{p}, d_1]$ is a bounded set of natural numbers, so it has a minimum $\delta$. Thus, there is $p \in Q_{\widetilde{p}}$ such that $\dim(\text{ran } p) = \delta$. □

**Lemma 3.15.** *Let $p$, $\widetilde{p}$ be same as in the previous lemma and the $C^*$-algebra $C = C^*(\mathcal{S}_A)p$ be a $*$-algebra of operators on the range $H$ of $p$. Then, the map $c \in C \mapsto c\widetilde{p}$ is injective.*

*Proof.* Assume $c \mapsto c\widetilde{p}$ is not one-one, so it has a non-trivial kernel $J$. For $T \in \mathcal{B}(H)$ and every $j \in J$, $h \in H$ we have $Tjh \in JH$, and $T^*jh \in JH$, so the subspace $K = JH$ reduces $C$. Moreover, assume $R$ is the projection onto $K$. In the following parts we will show:

(1) $R$ is the unit of $J$,

(2) $R \leq I - p$,

(3) $R = I - p$.

(1) Since $J$ is a finite dimensional $C^*$-algebra, by using lemma 2.3 it has a unit $R' \in J$. Note that $JH \supset R'H$. Consequently, ran $R \supset$ ran $R'$, so $R \geq R'$. If $X \in J$, then $X = R'X$ because $R'$ is the unit of $J$ and therefore ran $X \subset$ ran $R'$ which implies $XH \subset$ ran $R'$. Consequently, ran $R = JH \subset$ ran $R'$ which is equivalent to saying $R \leq R'$. As a result, we infer that $R = R'$.

(2) $R$ is the unit of $J$, so $R\widetilde{p} = 0$ by definition of $J$ which says ran $R \subset (\text{ran } \widetilde{p})^{\perp}$. Moreover, we know that ran $\widetilde{p} \subseteq (\text{ran } R)^{\perp} = \text{ran}(I - R)$, thus $\widetilde{p} \leq I - R$ and

minimality of $p$ yields $\widetilde{p} < p \leq I - R$. Consequently, we have $R \leq I - p$.

(3) We have $(I-p) \cdot \widetilde{p} = \widetilde{p} - p\widetilde{p} = \widetilde{p} - \widetilde{p} = 0$, so $(I-p) \in J$. Consequently, we have $(I-p)R = I - p$ and by using part (2) we get $(I-p)R = R$. Therefore, we conclude that $R = I - p$.

On the other hand, $R \in C$ which shows $R = Rp = (I-p)p = 0$. All in all, we showed that $R = 0$ so that $J = 0$ which says the map $c \mapsto c\widetilde{p}$ is injective. □

**Lemma 3.16.** *The $C^*$-algebra $C^*(\mathcal{S}_A)$ contains all reducing projections if and only if $C^*(\mathcal{S}_A)$ contains all minimal reducing projections.*

*Proof.* If $q \in B(\mathbb{C}^{d_1})$ is a reducing projection for $C^*(\mathcal{S}_A)$ which is not minimal, then there exists a minimal central projection $q'$ for $C^*(\mathcal{S}_A)$ such that $q' \lneq q$. Therefore, $q - q'$ is also reducing and $q = q' \oplus (q - q')$. By using induction and the finite dimensionality of $\mathbb{C}^{d_1}$ we find that $q = \oplus q_i$ where $q_i$ is a minimal reducing projection. □

**Definition 3.17.** *Assume $K$ is the biggest two sided ideal of $C^*(S)$ such that the natural map $C^*(S) \to C^*(S)/K$, $r \mapsto r + K$ is completely isometric on $S$, then $K$ is called the Šilov ideal for $S$ in $C^*(S)$.*

For a linear pencil $L_A(x) = I + \sum_{j=1}^{g} A_j x_j$ where $A_1, A_2, \ldots, A_g \in S\mathbb{C}^{d_1 \times d_1}$, the pencil $\widetilde{L}_A(x) = I + \sum_{j=1}^{g} \widetilde{A}_j x_j$ is a subpencil if there exists a non-trivial reducing subspace $M$ for $\mathcal{S}_A$ such that for every $j$, $\widetilde{A}_j = T^* A_j T$ where $T : M \hookrightarrow \mathbb{C}^{d_1}$ is an inclusion map. We call the pencil $L_A$ minimal if there is not a subpencil $\widetilde{L}_A$ such that $\mathcal{D}_{L_A} = \mathcal{D}_{\widetilde{L}_A}$.

**Theorem 3.18.** [8, Proposition 3.17] *Assume $L_A(x) = I + \sum_{j=1}^{g} A_j x_j$, $A_1, A_2, \ldots, A_g \in S\mathbb{C}^{d_1 \times d_1}$ is a linear pencil and $\mathcal{D}_{L_A}$ is bounded. Then, $L_A$ is minimal if and only if every minimal reducing projection $\widetilde{p}$ is in $C^*(\mathcal{S}_A)$ and the Šilov ideal of $C^*(\mathcal{S}_A)$ is zero.*

*Proof.* (⇒) Assume $\widetilde{p}$ is a minimal reducing projection for $C^*(\mathcal{S}_A)$ such that $\widetilde{p} \notin C^*(\mathcal{S}_A)$. By using lemma 3.14, there is a minimal central projection $p$ for $C^*(\mathcal{S}_A)$ such that $\widetilde{p} \leq p$.

Now, we define $C^*$-algebra $C = C^*(\mathcal{S}_A)p$ as a $*$-algebra of operators on the range $H$ of $p$ and by using lemma 3.15 the map $c \in C \mapsto c\widetilde{p}$ is injective. At this step, we want to show that the map $D : c \mapsto c(I - p) + c\widetilde{p}$ where $c \in C^*(\mathcal{S}_A)$ is one-one. We know that for all $x \in \ker D$ we have:

$$0 = x(I - p) + x\widetilde{p} \tag{3.6}$$

and since $\widetilde{p} \leq p$, then Ran $\widetilde{p} \in (\text{Ran } (I - p))^{\perp}$. For $h \in \text{ran } \widetilde{p}$ we have $\widetilde{p}h = h$ and $(I - p)h = 0$, then if we apply (3.6) to $h$, we infer that $x\widetilde{p}h = 0$. Thus, $x\widetilde{p} = 0$ and $x = xp$. Because of $x = xp$ we have $x \in C$ and by the lemma 3.15 $c \mapsto c\widetilde{p}$ is injective, so we conclude that $x \in J = 0$ ($J$ is the kernel of the map $c \mapsto c\widetilde{p}$) which is equivalent to saying that $D$ is one-one. If we consider [12, Theorem 2.4] along with the fact that $D$ is an injective $*$-homomorphism, it can be inferred that $D$ is a completely isometric map. We know that restriction of the self-adjoint operators to a reducing subspace is also self-adjoint operator, so if we restrict $L_A$ to $\text{ran}(I - p) \oplus \text{ran } \widetilde{p}$, then by using the theorem 3.12, we obtain a new pencil $L'$ such that $\mathcal{D}_{L'} = \mathcal{D}_{L_A}$ which contradicts the minimality of $L_A$.

We assume the Šilov ideal of $C^*(\mathcal{S}_A)$ is non-zero. Let $J \subseteq C^*(\mathcal{S}_A)$ be the Šilov ideal, so $\psi : C^*(\mathcal{S}_A) \to C^*(\mathcal{S}_A)/J,\ c \mapsto c + J$ is completely isometric on $\mathcal{S}_A$. Let $K = J\mathbb{C}^{d_1}$and let p be the projection onto $K$. As shown above, we know that $p$ is reducing for $C^*(S_A)$. By what we showed above, we conclude that $p \in C^*(S_A)$.

Also, $p \in J = \ker \psi$ (by the same argument as in the proof of the lemma 3.15 used to show that $R \in J$) which says $\psi(p) = 0$, so

$$\psi(s(I - p)) = \psi(s)\psi(I - p) = \psi(s), \text{ for every } s \in S_A. \tag{3.7}$$

We claim that $\tau : \mathcal{S}_A \to C^*(\mathcal{S}_A),\ X \mapsto X(I-p)$ is completely isometric. To see this, first we show that $\tau$ is completely contractive:

$$\begin{aligned} \left\|\tau^{(n)}\left([X_{ij}]\right)\right\| &= \|[X_{ij}(I-p)]\| \\ &= \left\|[X_{ij}]\begin{bmatrix} I-p & & 0 \\ & \ddots & \\ 0 & & I-p \end{bmatrix}\right\| \qquad (3.8) \\ &\leq \|[X_{ij}]\|\,\|I-p\| \\ &\leq \|[X_{ij}]\|\,. \qquad (3.9) \end{aligned}$$

Now, let $[X_{ij}] \in \mathbb{M}_n(\mathcal{S}_A)$, then we have

$$\begin{aligned} \|[X_{ij}]\|_{\mathbb{M}_n(\mathcal{S}_A)} &= \|[\psi(X_{ij})]\|_{\mathbb{M}_n(C^*(\mathcal{S}_A)/J)} \\ &= \|[\psi(X_{ij}(I-p))]\|_{\mathbb{M}_n(C^*(\mathcal{S}_A)/J)}\,, \qquad (By\ 3.7) \\ &= \|[\psi(\tau(X_{ij}))]\| \\ &\leq \|[\tau(X_{ij})]\| \\ &\leq \|[X_{ij}]\| \qquad \text{(By 3.9)}. \end{aligned}$$

Thus, $\tau$ is completely isometric, so by considering theorem 3.12and the minimality of $L_A$ we obtain $p = 0$ which is contradiction.

($\Leftarrow$) Suppose the Šilov ideal of $C^*(\mathcal{S}_A)$ is zero and every minimal reducing projection is in $C^*(\mathcal{S}_A)$. Moreover, assume $L_A$ is not minimal, so that there is subpencil $\widetilde{L}$ corresponding to a reducing subspace $K \subsetneq \mathbb{C}^{d_1}$ such that $\mathcal{D}_{\widetilde{L}} = \mathcal{D}_{L_A}$.

Let $\widetilde{p}$ be the projection onto $K$. We know that $\mathcal{D}_{\widetilde{L}} = \mathcal{D}_{L_A}$, so by using the theorem 3.12 the mapping $\mathcal{S}_A \to \{s\widetilde{p} | s \in \mathcal{S}_A\}$ such that $s \mapsto s\widetilde{p}$ is completely isometric. Let $p'$ be the projection onto some reducing subspace containing $K$. We claim that the map $s \mapsto sp'$ is completely isometric. To see this, we know ran $\widetilde{p} \subset$ ran $p'$ which says

$p'\widetilde{p} = \widetilde{p}$ and for $s \in \mathcal{S}_A$ we have:

$$\|s\| = \|s\widetilde{p}\| = \|sp'\widetilde{p}\| \leq \|sp'\|\|\widetilde{p}\| = \|sp'\| \leq \|s\|,$$

which proves that the map $s \mapsto sp'$ is completely isometric. Note that the previous calculation only works at scalar level and for the other matrix levels we can use the same method.

If we assume $W$ is a minimal orthogonal projection onto $K'$ which is a reducing subspace of $K^{\perp}$, then by assumption $W \in C^*(\mathcal{S}_A)$. In the following part, we show that $C^*(\mathcal{S}_A)W$ has no non-trivial ideal. That is because, if $J \subset C^*(\mathcal{S}_A)W$, then due to finite dimensionality we have $J = (C^*(\mathcal{S}_A)W)\,R$ where $R$ is the unit of $J$. Furthermore, for every $X \in C^*(\mathcal{S}_A)W$ we have $XW = X$ and $WX = X$ for every $X \in J$.

Thus, $WR = RW = R$ which is equivalent to saying ran $R \subseteq$ ran $W$, so $R \leq W$. Therefore, $R = 0$ or $R = W$, so $J = 0$ or $J = C^*(\mathcal{S}_A)W$. As a result, $C^*(\mathcal{S}_A)W$ is a minimal two-sided ideal of $C^*(\mathcal{S}_A)$. Also, $(I - W)$ is a projection onto a reducing subspace which contains $K$, and thus $s \mapsto s(I - W)$ is completely isometric as shown above. For $A = (a_{ij}) \in \mathbb{M}_n(\mathcal{S}_A)$ and

$$Q = (q_{ij})(I_n \otimes W) \in \mathbb{M}_n\left(C^*(\mathcal{S}_A)\right)(I \otimes W),$$

we have:

$$\|A + Q\| = \|(A + Q)(I_n \otimes (I - W)) \oplus (A + Q)(I_n \otimes W)\|$$

and

$$\begin{aligned} Q\left(I_n \otimes (I - W)\right) &= (q_{ij})(I_n \otimes W)(I_n \otimes (I - W)) \\ &= (q_{ij})\left(I_n \otimes W(I - W)\right) = 0. \end{aligned}$$

Thus,

$$\begin{aligned} \|A + Q\| &= \|A\left(I_n \otimes (I - W)\right) \oplus (A + Q)(I_n \otimes W)\| \\ &= \max\left\{\|A\left(I_n \otimes (I - W)\right)\|, \|(A + Q)(I_n \otimes W)\|\right\}. \end{aligned}$$

Since $s \mapsto s(I - W)$ is completely isometric, we have

$$\|A + Q\| = \max\left\{\|A\|, \|(A + Q)(I_n \otimes W)\|\right\},$$

hence, $\|A\| \leq \|A + Q\|$.

Our next goal is to show that

$$\psi : C^*(\mathcal{S}_A) \to C^*(\mathcal{S}_A)/C^*(\mathcal{S}_A)W$$

is completely isometric on $\mathcal{S}_A$. Let $X \in C^*(\mathcal{S}_A)$, then

$$\|\psi(X)\| = \inf\left\{\|X + q\| : \quad q \in C^*(\mathcal{S}_A)W\right\}.$$

If $X \in \mathcal{S}_A$, then previous part yields that $\|X\| \leq \|X + q\|$ for $q \in C^*(\mathcal{S}_A)W$. That is to say,

$$\|x\| \geq \|\psi(x)\| \geq \|x\|,$$

and if we do the similar argument for $[X_{ij}] \in \mathbb{M}_n(\mathcal{S}_A)$ we obtain that $\psi$ is completely isometric which says $0 \neq C^*(\mathcal{S}_A)W$ is a subset of Šilov ideal. But, this is in contradiction with our assumption that the Šilov ideal is zero. □

It is of interest to investigate under which conditions two monic linear pencils that share the same free spectrahedra, have the matrix coefficients of the same matrix level. The following theorem will answer this question.

**Theorem 3.19.** [8, Theorem 3.12] *Let* $L_A(x) = I + \sum_{j=1}^{g} A_j x_j,\ A_1, \dots, A_g \in S\mathbb{C}^{d_1 \times d_1}$ *and* $L_B(x) = I + \sum_{j=1}^{g} B_j x_j,\ B_1, \dots, B_g \in S\mathbb{C}^{d_2 \times d_2}$ *be linear pencils with* $\mathcal{D}_{L_A} = \mathcal{D}_{L_B}$ *bounded. If we assume* $L_A$ *and* $L_B$ *are minimal, then* $d_1 = d_2$ *and there exists a unitary matrix* $U$ *such that* $U^* L_A U = L_B$.

*Proof.* Let $S_1 = \{A_1, \dots A_g\}$, $S_2 = \{B_1, \dots, B_g\}$. Moreover, let $\beta_1$ and $\beta_2$ be maximal families of minimal non-zero reducing projections for $C^*(S_1), C^*(S_2)$ respectively. The reason why $\beta_1$ and $\beta_2$ exist will be explained later in this proof. For $p_1, p_2 \in \beta_1$ by using the theorem 3.18, we infer that $p_1$ and $p_2$ are central, therefore we have

$$p_1 p_2 \le p_1, \quad p_1 p_2 \le p_2.$$

On the other hand, minimality yields

$$\begin{cases} p_1 p_2 = p_1 \\ \text{or} \\ p_1 p_2 = 0 \end{cases}, \qquad \begin{cases} p_1 p_2 = p_2 \\ \text{or} \\ p_2 p_1 = 0 \end{cases}$$

which is equivalent to saying that $p_1 = p_2$ or $p_1 \perp p_2$. The reason why $\beta_1$ exists is all minimal non-zero reducing projections for $C^*(S_1)$ are non-zero pairwise orthogonal reducing projections, so $\bigoplus_{p \in \beta_1} \operatorname{ran} p = \mathbb{C}^{d_1}$ which says we have at most $d_1$ non-zero pairwise orthogonal reducing projections. The same argument holds for $\beta_2$. We claim that $\bigoplus_{p \in \beta_1} p = I$, $\bigoplus_{\widetilde{p} \in \beta_2} \widetilde{p} = I$. To see this, if we assume $\bigoplus_{p \in \beta_1} p = R \ne I$, then $I - R$ is another reducing projection which says there is minimal reducing projection

$S \leq (I - R)$. Thus, $S \cup \beta_1$ is a family of minimal reducing projections, but this contradicts maximality of $\beta_1$. The other equality follows in a similar way. For every $a \in C^*(S_1)$ we have:

$$a = IaI = \bigoplus_{p \in \beta_1} pa \bigoplus_{q \in \beta_1} q = \bigoplus_{p,q \in \beta_1} paq = \bigoplus_{pq \in \beta_1} apq = \bigoplus_{p' \in \beta_1} ap',$$

so $C^*(S_1) = \bigoplus_{p' \in \beta_1} C^*(S_1)p'$ and likewise $C^*(S_2) = \bigoplus_{q \in \beta_2} C^*(S_2)q$.

At the next step, we will prove that a minimal ideal in $C^*(S_1)$ is of the form $C^*(S_1)p,\ p \in \beta_1$. Assume $J \subset C^*(S_1)$ is a minimal ideal, so there is a projection $p \in C^*(S_1)$ such that $J = C^*(S_1)p$ (by the same argument as in the proof of the theorem 3.18). If $R$ is a minimal reducing projection for $C^*(S_1)$ such that $R \leq p$, then $Rp = R$. On the other hand, by minimality of $L_A$ and theorem 3.18, we have $R \in C^*(S_1)$, thus $R = Rp \in C^*(S_1)p = J$. Therefore, $RC^*(S_1) \subset J$. Since $J$ is minimal this shows that $J = RC^*(S_1)$. Accordingly, we have $R = p$ which is equivalent to saying $p$ is minimal. Since $\mathcal{D}_{L_A} = \mathcal{D}_{L_B}$, we have the unital map $\tau : S_1 \to S_2, \tau(A_j) = B_j$ is completely isometric isomorphism by theorem 3.12. Recall that by [1, Theorem 7.1] we know that the generating sets $S_1, S_2$ for $C^*(S_1)$ and $C^*(S_2)$ are admissible, so by using [2, Theorem 2.2.5], we can extend $\tau$ to a *-isomorphism $\gamma : C^*(S_1) \to C^*(S_2)$. If we assume $C^*(S_1)p,\ p \in \beta_1$ is a minimal ideal for $C^*(S_1)$, then $\gamma(C^*(S_1)p) = C^*(S_2)q$ for some $q \in \beta_2$. Conversely, for every $q \in \beta_2$ there is $p \in \beta_1$ such that $C^*(S_2)q = \gamma(C^*(S_1)p)$ because $*$-isomorphism preserves minimality of projections.

Consider $\psi_p : C^*(S_1)p \to C^*(S_2)q$ such that $\gamma|_{C^*(S_1)p} = \psi_p$. Since $C^*(S_1)p$ is a simple algebra and finite dimensional $C^*$-algebra, by applying lemma 2.3 we infer that there exists a unitary map $u_p$ such that for every $A \in C^*(S_1)p$, $\psi_p(A) = u_p^* A u_p$.

Moreover, for $A_j = \bigoplus_{p\in\beta_1} A_j p \in C^*(S_1)$ we have

$$\gamma(A_j) = \gamma\left(\bigoplus_{p\in\beta_1} pA_jp\right) = \bigoplus_{p\in\beta_1} u_p^* pA_jpu_p.$$

Accordingly, If we assume $V = \bigoplus_{p\in\beta_1} pu_p$, then we infer that $\gamma(A_j) = V^*A_jV$. On the other hand, we have

$$d_2 = tr(I_{d_2}) = tr(\sum_{q\in\beta_2} q) = tr(\sum_{p\in\beta_1} \gamma(p)) = tr(\sum_{p\in\beta_1} u_p^* pu_p) = tr(\sum_{p\in\beta_1} p) = d_1.$$

□

# 4

# Completely Bounded Maps and Inclusion of Free Spectrahedra

We introduced subspaces $\mathcal{S}_A, \mathcal{S}_B \subseteq S\mathbb{C}^{d_p \times d_p}$, $p = 1, 2$ in chapter 3 as:

$\mathcal{S}_A = \text{span}\{I, A_j | \ j = 1, \ldots, g\}, \mathcal{S}_B = \text{span}\{I, B_j | \ j = 1, \ldots, g\}$ where $A_j \in S\mathbb{C}^{d_1 \times d_1}$ and $B_j \in S\mathbb{C}^{d_2 \times d_2}$ for $j = 1, \ldots, g$. In chapter 3, we proved that there exists a unital completely positive map $\mathcal{S}_A \to \mathcal{S}_B$, $A_j \mapsto B_j$ if and only if $\mathcal{D}_{L_A} \subset \mathcal{D}_{L_B}$. Since UCB(unital completely bounded) maps are more general than UCP(unital completely positive) maps, it would be interesting if in analogy with theorem 3.12 we could explain the relationship between $\mathcal{D}_{L_A}$, $\mathcal{D}_{L_B}$ in terms of the existence of a completely bounded map between $\mathcal{S}_A$ and $\mathcal{S}_B$. The reason why this chapter is developed is to investigate the possible relationship between the free spectrahedra of $L_A$ and $L_B$ when the map $\mathcal{S}_A \to \mathcal{S}_B, A_j \mapsto B_j$ is merely completely bounded, but not necessarily completely positive.

Recall that for a linear pencil $L_A(x) = I + \sum_{j=1}^{g} A_j x_j$ we defined $L_A^{(1)}(x) = \sum_{j=1}^{g} A_j x_j$. Moreover, we define

$$\Lambda_A := \left\{ X \in \left(\mathbb{C}^{n \times n}\right)^g \mid \|L_A^{(1)}(X)\| \leq 1 \right\},$$

and

$$\mathcal{B}_{A,r} := \left\{ X \in \left(\mathbb{C}^{n\times n}\right)^g \mid \|L_A(X)\| \leq r \right\}.$$

Note that in order to have a well defined map $\tau$ between $\mathcal{S}_A$ and $\mathcal{S}_B$ such that $A_j \mapsto B_j$, it is equivalent to have the set $\{I, A_j | \, j = 1, \ldots, g\}$ be linearly independent. Therefore, by using the proposition 3.8 the map $\tau$ is well defined if the set $\mathcal{D}_{L_A}$ is bounded.

**Theorem 4.1.** *Let $L_A(x) = I + \sum_{j=1}^{g} A_j x_j$, and $L_B(x) = I + \sum_{j=1}^{g} B_j x_j$ be monic linear pencils such that $\mathcal{D}_{L_A}$ and $\mathcal{D}_{L_B}$ are bounded. Let $\tau$ be the unital linear map between $\mathcal{S}_A, \mathcal{S}_B$ such that:*

$$\tau(A_j) = B_j. \tag{4.1}$$

*Then, the following are equivalent:*

(1) *$\tau$ is a completely bounded map,*

(2) *there is $K > 0$ such that $\Lambda_A \subset K\Lambda_B$,*

(3) *there is $\lambda > 0$ such that $\mathcal{B}_{A,2} \subset \mathcal{B}_{B,2\lambda}$,*

(4) *there is $\lambda' > 0$ such that $\mathcal{B}_{A,r} \subset \mathcal{B}_{B,\lambda' r}$ for every $r \geq 0$.*

*Proof.* (2) $\Rightarrow$ (1) We define a projection $P$ as follows:

$$P : \mathcal{S}_A \to \mathbb{C}I$$
$$\left(\lambda I + \sum_{j=1}^{g} A_j x_j\right) \mapsto \lambda I.$$

It is well-defined because the set $\{I, A_j | \quad j = 1, \ldots, g\}$ is linearly independent

(Lemma 3.8). For $\Lambda \in \mathbb{M}_n, X \in (\mathbb{M}_n)^g$ we have:

$$\begin{aligned} (I-P)^{(n)} \left( I \otimes \Lambda + \sum_{j=1}^{g} A_j \otimes X_j \right) &= \sum_{j=1}^{g} A_j \otimes X_j = L_A^{(1)}(X), \\ P^{(n)} \left( I \otimes \Lambda + \sum_{j=1}^{g} A_j \otimes X_j \right) &= I \otimes \Lambda. \end{aligned}$$

Taking an arbitrary element $S = I \otimes \Lambda + \sum_{j=1}^{g} A_j \otimes X_j$ from $\mathbb{M}_n(\mathcal{S}_A)$ such that $\|S\| = 1$, we have:

$$\tau^{(n)}(S) = \tau^{(n)} \left( P^{(n)}(S) \right) + \tau^{(n)} \left( (I-P)^{(n)}(S) \right).$$

Thus,

$$\|\tau^{(n)}(S)\| \leq \left\| \tau^{(n)}(I \otimes \Lambda) \right\| + \left\| \tau^{(n)} \left( (I-P)^{(n)}(S) \right) \right\|.$$

In addition, we have:

$$\|I \otimes \Lambda\| \leq \|P^{(n)}\| \cdot \left\| I \otimes \Lambda + \sum_{j=1}^{g} A_j \otimes X_j \right\|, \tag{4.2}$$

$$\left\| \sum_{j=1}^{g} A_j \otimes X_j \right\| \leq \|(I-P)^{(n)}\| \left\| I \otimes \Lambda + \sum_{j=1}^{g} A_j \otimes X_j \right\|. \tag{4.3}$$

Therefore, using that $\tau$ is a unital map, and using (4.2) we have:

$$\begin{aligned} \|\tau^{(n)}(S)\| &\leq \|I \otimes \Lambda\| + \left\| \tau^{(n)}(I-P)^{(n)}(S) \right\| \\ &\leq \|P^{(n)}\| + \left\| \tau^{(n)}(I-P)^{(n)}(S) \right\|. \end{aligned} \tag{4.4}$$

Furthermore, if we assume $\|(I-P)\|_{cb} = t$, then by (4.3) we have $\left\| L_A^{(1)} \left( \frac{X}{t} \right) \right\| \leq 1$, thus $\frac{X}{t} \in \Lambda_A$ and by using our assumption there is $K$ s.t $\frac{X}{t} \in K\Lambda_B$ which is equivalent to saying that: there is $X' \in \Lambda_B$ such that $X' = \frac{X}{Kt}$. Consequently

$$\left\| \tau^{(n)} \left( L_A^{(1)} \left( X/Kt \right) \right) \right\| = \left\| L_B^{(1)} \left( X/Kt \right) \right\| \leq 1,$$

hence

$$\left\|\tau^{(n)}\left(L_A^{(1)}(X)\right)\right\| \leq Kt. \tag{4.5}$$

By using (4.4),(4.5) we obtain that

$$\left\|\tau^{(n)}(S)\right\| \leq \|P^{(n)}\| + K\|(I-P)\|_{cb}. \tag{4.6}$$

Also we know $P, (I-P)$ are linear operators between finite dimensional spaces, and so are bounded. By using [10, Proposition 3.8], we get:

$$\begin{aligned}
\|P^{(n)}\| &= \|P\|, \\
\|(I-P)\|_{cb} &\leq \|I\|_{cb} + \|P\|_{cb} \\
&= 1 + \|P\|.
\end{aligned}$$

Accordingly, we conclude that $\left\|\tau^{(n)}(S)\right\| \leq \|P\| + K(1+\|P\|)$.

(1) $\Rightarrow$ (4): Let $X \in \mathcal{B}_{A,r}$. Then

$$\begin{aligned}
\|L_B(X)\| &= \|\tau^{(n)}(L_A(X))\| \\
&\leq \|\tau\|_{cb} \cdot \|L_A(X)\| \\
&\leq \|\tau\|_{cb} \cdot r.
\end{aligned}$$

Hence, $\mathcal{B}_{A,r} \subset \mathcal{B}_{B,\|\tau\|_{cb}\cdot r}$ for $r \geq 0$.

(4) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (2): Let $X \in \Lambda_A$ and let the projection $P_B$ be as follows:

$$P_B : S_B \to \mathbb{C}I$$
$$\left(\lambda + \sum_{j=1}^{g} B_j x_j\right) \mapsto \lambda.$$

As above, we know that $I - P_B$ is completely bounded, so that

$$\|L_B^{(1)}(X)\| \leq \|I - P_B\|_{cb}\|L_B(X)\|.$$

But

$$\|L_A(X)\| \leq 1 + \|L_A^{(1)}(X)\| \leq 2,$$

so $X \in \mathcal{B}_{A,2}$ which says $X \in \mathcal{B}_{B,2\lambda}$. As a result,

$$\|L_B^{(1)}(X)\| \leq \|I - P_B\|_{cb} 2\lambda.$$

□

In the preceding theorem, we established a relationship between $\Lambda_A$ and $\Lambda_B$ when there exists a natural completely bounded map between $\mathcal{S}_A$ and $\mathcal{S}_B$. In the following theorem, we will find a relationship between $\mathcal{D}_A$ and $\mathcal{D}_B$, which are more commonly used than $\Lambda_A$ and $\Lambda_B$, when there exists a natural completely bounded map between $\mathcal{S}_A$ and $\mathcal{S}_B$.

We make some elementary preliminary observations. Note that for every $\Lambda \in \mathbb{M}_n$ with $\|\Lambda\| < 1$ by using the Lemma 2.9 we have:
$\begin{bmatrix} I_n & \Lambda \\ \Lambda^* & I_n \end{bmatrix} > 0$. Furthermore, we claim that $\begin{bmatrix} cI_n & \Lambda \\ \Lambda^* & cI_n \end{bmatrix} > 0$ if $c > 1$. That is because:

$$\begin{aligned} \begin{bmatrix} cI_n & \Lambda \\ \Lambda^* & cI_n \end{bmatrix} &= \begin{bmatrix} (c-1)I_n & 0 \\ 0 & (c-1)I_n \end{bmatrix} + \begin{bmatrix} I_n & \Lambda \\ \Lambda^* & I_n \end{bmatrix} \\ &\geq (c-1) \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} > 0. \end{aligned}$$

Likewise, we have $\begin{bmatrix} cI & I\otimes\Lambda \\ I\otimes\Lambda^* & cI \end{bmatrix} > 0$ for $c > 1$ whenever $\|\Lambda\| < 1$.

**Lemma 4.2.** *Let $L_A$ and $L_B$ be the same as in the previous theorem. Moreover, assume $n \in \mathbb{N}$, $\Lambda \in \mathbb{M}_n$ and $\|\Lambda\| < 1$. We define*

$$E(A,\Lambda,n,c) := \begin{bmatrix} cI & \Lambda \\ \Lambda^* & cI \end{bmatrix}^{\frac{1}{2}} \mathcal{D}_A(2n) \begin{bmatrix} cI & \Lambda \\ \Lambda^* & cI \end{bmatrix}^{\frac{1}{2}}, c > 1.$$

*Then,* $X = \left\{ \begin{bmatrix} 0 & X_j \\ X_j & 0 \end{bmatrix} \right\}_{j=1}^{g} \in E(A,\Lambda,n,C)$ *if and only if* $\|I_{d_1}\otimes\Lambda + \sum_{j=1}^{g} A_j\otimes X_j\| \leq c.$

*Proof.* Assume $S = I_{d_1}\otimes\Lambda + \sum_{j=1}^{g} A_j\otimes X_j \in \mathbb{M}_n(S_A), \|S\| \leq 1$. By using Lemma 2.9, we obtain that
$\begin{bmatrix} I_{nd_1} & S \\ S^* & I_{nd_1} \end{bmatrix} \geq 0$ in $\mathbb{M}_2\left(\mathbb{M}_n(\mathcal{S}_A)\right)$. On the other hand,

$$\begin{aligned} \begin{bmatrix} I_{nd_1} & S \\ S^* & I_{nd_1} \end{bmatrix} &= \begin{bmatrix} I_{nd_1} & I_{d_1}\otimes\Lambda + \sum_{j=1}^{g} A_j\otimes X_j \\ I_{d_1}\otimes\Lambda^* + \sum_{j=1}^{g} A_j\otimes X_j & I_{nd_1} \end{bmatrix} \\ &= \begin{bmatrix} I_{nd_1} & I_{d_1}\otimes\Lambda \\ I_{d_1}\otimes\Lambda^* & I_{nd_1} \end{bmatrix} + \sum_{j=1}^{g} A_j\otimes \begin{bmatrix} 0 & X_j \\ X_j & 0 \end{bmatrix} \geq 0. \end{aligned}$$

By using lemma 3.10 we obtain that $\begin{bmatrix} I_{nd_1} & I_{d_1}\otimes\Lambda \\ I_{d_1}\otimes\Lambda^* & I_{nd_1} \end{bmatrix} \geq 0$, and by using lemma 2.9 we infer that $\|\Lambda\| < 1$.

Suppose $c > 1$ and consequently $\Omega_c := \begin{bmatrix} cI_n & \Lambda \\ \Lambda^* & cI_n \end{bmatrix}$ is invertible as discussed before the lemma. We find

$$I_{d_1} \otimes \Omega_c + \sum_{j=1}^{g} A_j \otimes \begin{bmatrix} 0 & X_j \\ X_j & 0 \end{bmatrix} = (c-1) \begin{bmatrix} I_{nd_1} & 0 \\ 0 & I_{nd_1} \end{bmatrix} + \begin{bmatrix} I_{nd_1} & S \\ S^* & I_{nd_1} \end{bmatrix} \geq 0,$$

thus

$$I_{d_1} \otimes \Omega_c^{\frac{1}{2}} \left( I_{d_1} \otimes I_{2n} + \sum_{j=1}^{g} A_j \otimes \Omega_c^{-\frac{1}{2}} \begin{bmatrix} 0 & X_j \\ X_j & 0 \end{bmatrix} \Omega_c^{-\frac{1}{2}} \right) I_{d_1} \otimes \Omega_c^{\frac{1}{2}} \geq 0,$$

which in turn implies that

$$I_{d_1} \otimes I_{2n} + \sum_{j=1}^{g} A_j \otimes \Omega_c^{-\frac{1}{2}} \begin{bmatrix} 0 & X_j \\ X_j & 0 \end{bmatrix} \Omega_c^{-\frac{1}{2}} \geq 0.$$

As a result, for every $c > 1$,

$$\left\{ \Omega_c^{-\frac{1}{2}} \begin{bmatrix} 0 & X_j \\ X_j & 0 \end{bmatrix} \Omega_c^{-\frac{1}{2}} \right\}_{j=1}^{g} \in \mathcal{D}_A(2n).$$

□

An immediate consequence of the previous lemma is that

$$X = \left\{ \begin{bmatrix} 0 & X_j \\ X_j & 0 \end{bmatrix} \right\}_{j=1}^{g} \in \bigcap_{c>1} E(A, \Lambda, n, c) \text{ if and only if } X \in E(A, \Lambda, n, 1).$$

**Theorem 4.3.** *Let $L_A$ and $L_B$ be the same as in the theorem . Moreover, let the unital linear map $\tau$ be defined as (4.1). Let $k > 0$, then the following statements are equivalent:*

*1. If*

*$E(A,\Lambda,n,1) \subset E(B,\Lambda,n,k)$ for every $n \in \mathbb{N}, \Lambda \in \mathbb{M}_n$ with $\|\Lambda\| < 1$.*

*2. The map $\tau : \mathcal{S}_A \to \mathcal{S}_B$ is completely bounded, $\|\tau\|_{cb} \leq k$.*

*Proof.* ($\Rightarrow$) Assume $X = \left\{\begin{bmatrix} 0 & X_j \\ X_j & 0 \end{bmatrix}\right\}_{j=1}^{g} \in E(A,\Lambda,n,1)$ and by using the assumption we have:

$$\left\{\begin{bmatrix} 0 & X_j \\ X_j & 0 \end{bmatrix}\right\}_{j=1}^{g} \in E(B,\Lambda,n,k).$$

Therefore,

$$I_{d_2} \otimes \Omega_k^{\frac{1}{2}} \left( I_{d_2} \otimes I_{2n} + \sum_{j=1}^{g} B_j \otimes \Omega_k^{-\frac{1}{2}} \begin{bmatrix} 0 & X_j \\ X_j & 0 \end{bmatrix} \Omega_k^{-\frac{1}{2}} \right) I_{d_2} \otimes \Omega_k^{\frac{1}{2}} \geq 0,$$

so

$$I_{d_2} \otimes \Omega_k + \sum_{j=1}^{g} B_j \otimes \begin{bmatrix} 0 & X_j \\ X_j & 0 \end{bmatrix} = (z-k) \begin{bmatrix} I_{nd_2} & 0 \\ 0 & I_{nd_2} \end{bmatrix} + \begin{bmatrix} kI_{nd_2} & \tau(S) \\ \tau(S)^* & kI_{nd_2} \end{bmatrix} \geq 0, \quad (4.7)$$

for every $z > k$. Let $z \downarrow k$, we infer $\begin{bmatrix} kI_{nd_2} & \tau^{(n)}(S) \\ \tau^{(n)}(S)^* & kI_{nd_2} \end{bmatrix} \geq 0$, which asserts that $\|\tau^{(n)}(S)\| \leq k$ (Lemma 2.9).

($\Leftarrow$) Assume $\|\tau\|_{cb} \leq K$. Moreover, suppose $\Omega_c = \begin{bmatrix} cI_n & \Lambda \\ \Lambda^* & cI_n \end{bmatrix}$, $c > 1$ and $\Lambda \in \mathbb{M}_n$ with $\|\Lambda\| < 1$. Let $X = \left\{\begin{bmatrix} 0 & X_j \\ X_j & 0 \end{bmatrix}\right\}_{j=1}^{g} \in \Omega_c^{\frac{1}{2}} \mathcal{D}_A(2n) \Omega_c^{\frac{1}{2}}$, for every $c > 1$ and let $S = I_{d_1} \otimes \Lambda + \sum_{j=1}^{g} A_j \otimes X_j$. Thus, we have

$$I_{d_1} \otimes \Omega_c^{\frac{1}{2}} \left( I_{d_1} \otimes I_{2n} + \sum_{j=1}^{g} A_j \otimes \Omega_c^{-\frac{1}{2}} \begin{bmatrix} 0 & X_j \\ X_j & 0 \end{bmatrix} \Omega_c^{-\frac{1}{2}} \right) I_{d_1} \otimes \Omega_c^{\frac{1}{2}} \geq 0,$$

which in turn implies that

$$(c-1)\begin{bmatrix} I_{nd_1} & 0 \\ 0 & I_{nd_1} \end{bmatrix} + \begin{bmatrix} I_{nd_1} & S \\ S^* & I_{nd_1} \end{bmatrix} = I_{d_1} \otimes \Omega_c + \sum_{j=1}^{g} A_j \otimes \begin{bmatrix} 0 & X_j \\ X_j & 0 \end{bmatrix} \geq 0,$$

thus $\begin{bmatrix} cI_{nd_1} & S \\ S^* & cI_{nd_1} \end{bmatrix} \geq 0$ for every $c > 1$. That is to say, $\|S\| \leq 1$. Accordingly, $\|\tau^{(n)}(S)\| \leq K$ and by using lemma 2.9 we infer that:

$$\begin{bmatrix} qI_{nd_2} & \tau^{(n)}(S) \\ \tau^{(n)}(S)^* & qI_{nd_2} \end{bmatrix} \geq 0, \qquad q > K.$$

Therefore,

$$(I_{d_2} \otimes \Omega_q^{\frac{1}{2}}) L_B \left( \Omega_q^{-\frac{1}{2}} \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \Omega_q^{-\frac{1}{2}} \right) (I_{d_2} \otimes \Omega_q^{\frac{1}{2}}) \geq 0,$$

so we have:

$$X = \left\{ \begin{bmatrix} 0 & X_j \\ X_j & 0 \end{bmatrix} \right\}_{j=1}^{g} \in \Omega_q^{\frac{1}{2}} \mathcal{D}_B(2n) \Omega_q^{\frac{1}{2}}$$

for $q > K$. □

The main limitation that we encountered in chapter 3 was that most of the theorems related to this study are for completely positive maps. To overcome this issue, we took the advantage of Lemma 2.9 along with some other techniques to translate boundedness to positivity and positivity to boundedness.

The future work regarding this study can concentrate on the following questions:

1. In theorem 4.3, is there any other relationship between the free spectrahedra of the linear pencils $L_A$ and $L_B$ if there exists a natural completely bounded map between $\mathcal{S}_A$ and $\mathcal{S}_B$?

2. In theorem 4.3, is it possible to relate $\|P\|$ to properties of matrices $A_0, A_1, \ldots, A_g \in$

$S\mathbb{C}^{n\times n}$?

3. What is the relationship between $\Lambda_A$ and $\mathcal{B}_A := \{X \in (S\mathbb{C}^{n\times n})^g \mid \|L_A(X)\| \leq 1\}$?

# Bibliography

[1] Arveson, W. The noncommutative Choquet boundary. *J. Amer. Math. Soc. 21*, 4 (2008), 1065–1084.

[2] Arveson, W. B. Subalgebras of $C^*$-algebras. *Acta Math. 123* (1969), 141–224.

[3] Ben-Tal, A., and Nemirovski, A. On tractable approximations of uncertain linear matrix inequalities affected by interval uncertainty. *SIAM J. Optim. 12*, 3 (2002), 811–833.

[4] Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V. *Linear matrix inequalities in system and control theory*, vol. 15 of *SIAM Studies in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994.

[5] Davidson, K. R. *$C^*$-algebras by example*, vol. 6 of *Fields Institute Monographs.* American Mathematical Society, Providence, RI, 1996.

[6] Davidson, K. R., Dor-On, A., Shalit, O. M., and Solel, B. Dilations, inclusions of matrix convex sets, and completely positive maps. *Int. Math. Res. Not. IMRN*, 13 (2017), 4069–4130.

[7] Douglas, R. G. *Banach algebra techniques in operator theory*, second ed., vol. 179 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1998.

[8] Helton, J. W., Klep, I., and McCullough, S. The matricial relaxation of a linear matrix inequality. *Math. Program. 138*, 1-2, Ser. A (2013), 401–445.

[9] Nemirovski, A. Advances in convex optimization: conic programming. In *International Congress of Mathematicians. Vol. I*. Eur. Math. Soc., Zürich, 2007, pp. 413–444.

[10] Paulsen, V. *Completely bounded maps and operator algebras*, vol. 78 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002.

[11] Rassias, T. M., and Brzdęk, J., Eds. *Functional equations in mathematical analysis*, vol. 52 of *Springer Optimization and Its Applications*. Springer, New York, 2012.

[12] Schaefer, H. H., and Wolff, M. P. *Topological vector spaces*, second ed., vol. 3 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.

[13] Schneider, R. *Convex bodies: the Brunn-Minkowski theory*, vol. 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

[14] Wolkowicz, H., Saigal, R., and Vandenberghe, L., Eds. *Handbook of semidefinite programming*, vol. 27 of *International Series in Operations Research & Management Science*. Kluwer Academic Publishers, Boston, MA, 2000. Theory, algorithms, and applications.