

ON THE NUMBER OF

ESSENTIALLY n -ARY POLYNOMIALS

OF IDEMPOTENT ALGEBRAS

A Thesis

Presented to

the Faculty of Graduate Studies and Research

University of Manitoba

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

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May 1971

ACKNOWLEDGMENT

It is my pleasure to acknowledge my debt of gratitude to my supervisor, Professor G. Grätzer, for his invaluable advice and constant encouragement during the course of this research. My sincere thanks go to Professor J.A. Gerhard for his help in checking this dissertation.

Financial assistance from the Department of Mathematics and the Faculty of Graduate Studies of this University is appreciated.

To Chin-Boey, my wife, I express my heartfelt thanks for her patience, encouragement, support and her help in typing this thesis despite unfamiliar terms.

INTRODUCTION

A universal algebra, or briefly, algebra \mathcal{A} is an ordered pair $\langle A; F \rangle$ where A is a non-empty set and F is a family of finitary operations on A . For each natural number n , we can consider the set $P^{(n)}(\mathcal{A})$ of n -ary polynomials of \mathcal{A} which are certain functions from A^n to A built up from the variables x_i , $i = 1, 2, \dots, n$ by substituting them in the operations f , $f \in F$, successively in a finite number of steps.

An n -ary polynomial p over \mathcal{A} is said to depend on x_i if there exist $a_1, \dots, a_i, a_i', \dots, a_n$ in A such that

$$p(a_1, \dots, a_i, \dots, a_n) \neq p(a_1, \dots, a_i', \dots, a_n).$$

By an essentially n -ary polynomial over \mathcal{A} is meant a n -ary polynomial over \mathcal{A} which depends on each variable x_i , $i = 1, \dots, n$. For $n > 1$, let $p_n(\mathcal{A})$ designate the number of essentially n -ary polynomials over \mathcal{A} . We denote by $p_1(\mathcal{A})$ and $p_0(\mathcal{A})$ the number of non-constant unary polynomials excluding x_1 and the number of constant unary polynomials respectively. Thus, with any algebra \mathcal{A} , there is associated an ω -sequence of cardinals $\langle p_0(\mathcal{A}), p_1(\mathcal{A}), \dots, p_n(\mathcal{A}), \dots \rangle$.

Let \mathcal{C} be a class of algebra. A sequence $\langle p_0, p_1, \dots, p_n, \dots \rangle$ of cardinals is said to representable in \mathcal{C} if there exists an algebra \mathcal{A} in \mathcal{C} with $p_n = p_n(\mathcal{A})$ for each $n \geq 0$. If \mathcal{C} is the class of all algebras, then we say that the sequence

$\langle p_0, p_1, \dots, p_n, \dots \rangle$ is representable. An algebra $\mathcal{A} = \langle A; F \rangle$

is said to be idempotent if $f(x, \dots, x) = x$, for any f in F . Thus, it is easy to see that an algebra \mathcal{A} is idempotent if and only if $p_0(\mathcal{A}) = p_1(\mathcal{A}) = 0$. We shall say that an algebra $\mathcal{A} = \langle A; F \rangle$ can be represented as an algebra $\mathcal{A}^* = \langle A; f_1, f_2, \dots \rangle$ of type τ if it is possible to choose a sequence (f_1, f_2, \dots) of polynomials from F in such a way that the sequence of the arities of the f_i equals τ . Note that the set of polynomials $\{f_i\}$ can be taken as a set of operations in \mathcal{A} .

Our basic problem is to study and characterize representable sequences. An easy combinatorial argument shows that this problem is equivalent to Problem 42 in [6] which can be stated as follows: Let \underline{K} be an equational class, and let F_n denote the cardinal of the free algebra over \underline{K} on n generators. Characterize the sequence $\langle F_n \rangle$.

The development of the study of $\langle p_n \rangle$ sequence may briefly be divided into three stages.

The period that started in 1910 may be considered as the initial stage. In this period, even though there were no significant contributions to the theory, the idea was foreshadowed by the work of S. Sierpinski. He published a series of articles between 1918 and 1945 for the purpose of investigating the composition of functions. One of his typical results (see [41]) says that given any set A and any function $f : A^n \longrightarrow A$, f can

be obtained by an appropriate composition of binary functions. Recently, R.W. Quackenbush studied the corresponding problem for idempotent functions. He proved [37] that every idempotent function on a given set A can be obtained by composition of binary idempotent functions provided $|A| > 2$. For $|A| = 2$, the role of binary functions is replaced by ternary functions.

The explicit formulation of the basic problem, given by E. Marczewski in 1963 - 1964, may be considered as the beginning of the second stage. Since it was considered too difficult to deal with explicitly; E. Marczewski and his colleagues in Wroclaw studied only problems associated with it. In particular, he himself defined for each \mathcal{A} , the zero set

$$Z(\mathcal{A}) = \{ n / p_n(\mathcal{A}) = 0 \}$$

and showed, for instance, in [22] that for algebras without constants and with one essentially n -ary symmetry (or even quasi-symmetric) polynomial the complement of the zero set $Z(\mathcal{A})$ contains the arithmetical progression $n + k(n - 1)$, ($k = 0, 1, \dots$). This generalizes a result of J. Płonka [28] for $n = 2$. One of the deepest results was obtained by K. Urbanik [42] who gave a complete description of all possible sets $Z(\mathcal{A})$.

It was in 1968 that the present stage began with a systematic and intensive investigation of the basic problem in G. Grätzer's seminar at the University of Manitoba. Influenced by the first paper due to G. Grätzer, J. Płonka and A. Sekanina

[11] , a steady flow of contributions to it, by the members of the seminar, has appeared in 1968 - 1969. G. Grätzer, J. Płonka and R. Padmanabhan have especially enriched and clarified the subject. The results were summarized by G. Grätzer [8] who gave a survey lecture at the Conference on Universal Algebras held at Queen's University in October, 1969.

In this period, the investigation of the basic problem was naturally split into two categories : (1). Study the basic problem for non-idempotent algebras; (2). Study the same for idempotent algebras. The first case was attacked by G. Grätzer, J. Płonka, A. Sekanina in [11] , [12] and [33] . Some of their results were of the type that sequences satisfying some mild condition (e.g. $p_0 > 0$) are all representable, and so the p_i are independent. However, the situation completely changes when we deal with the idempotent case. As a matter of fact, the cardinals $p_n(\mathcal{A})$, for idempotent algebra \mathcal{A} , turn out to be quite strongly interrelated (see, for instance, [13] and [14]). Because of this extremely interesting fact, recently, most papers were devoted to the study of idempotent algebra ; this study can be separated roughly into two parts : (A). Investigate the behaviour and the maximum asymptotic rate of growth of the general sequence $\langle p_n \rangle$; (B). Description of all algebras representing a given sequence with application to the Minimal Extension Property (for definitin, see Chapter two of Part IV).

The purpose of this thesis is to provide some results for the second part of category 2 in a systematic way with emphasis on applications to the Minimal Extension Property. The equivalent problem of $\langle p_n \rangle$ sequences in Group Theory, the so called growth function of free groups, has been extensively studied for equational classes of groups by British mathematicians ; for instance, G. Higman (see [16] and [23]), P. Neumann and his students. The application of $\langle p_n \rangle$ sequences to semilattices was considered in G. Grätzer and J. Płonka [15] while the case of idempotent semigroups was settled by J.A. Gerhard [5] . In this thesis, we make a first attack in applying the $\langle p_n \rangle$ sequences to Lattice Theory.

This thesis falls into four parts with nine chapters altogether. Since a short description of the content is given at the beginning of each chapter, we shall include here only a brief outline. Part I, which consists of three chapters, is devoted to study idempotent algebras with one essentially binary polynomial. The sequence $\langle 0,0,1,2 \rangle$ has, in particular, very interesting properties. Thus, we restrict our attention to this sequence in the first two chapters. Some results of Part I are generalized to Part II in which we consider idempotent algebras with one essentially m -ary polynomial for $m \geq 2$. All of these are applied to derive the function $F(n,k)$ with the property that $F(n,k)$ is the least value such that the sequence $\langle 0,0,1,\dots,1,n,F(n,k) \rangle$ is representable by symmetric

algebra. Part III consists of two chapters. Płonka's basic lemmas are used in Chapter 1 to derive some results about the sequences $\langle 0,0,2,m \rangle$. The considerations of Chapter 2 center around the algebras representing $\langle 0,0,3,m \rangle$. Part IV consists of two chapters. Previous results are applied to Lattice Theory in Chapter 1 and the Minimal Extension Property in Chapter 2.

Cross references are given in the form (III,1,2) where III stands for Part III, 1 for Chapter 1 and 2 for section 2. The part and chapter numerals will be omitted in case the reference is made in the same chapter.

For those basic concepts and notations, we refer to G. Grätzer's books [6] and [9] .

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----- PART I -----

IDEMPOTENT ALGEBRAS WITH ONE ESSENTIALLY BINARY POLYNOMIAL

CHAPTER 1

ALGEBRAS REPRESENTING $\langle 0,0,1,2 \rangle$

The sequence $\langle 0,0,1,1 \rangle$ is, evidently, representable. This can be seen simply by taking a non-trivial semilattice. To go one step further, we are interested in the case where $p_3 = 2$. Thus, the following questions naturally arises :

- (1). Is the sequence $\langle 0,0,1,2 \rangle$ representable ?
- (2). If the answer to (1) is in the affirmative, what can we say about those algebras representing $\langle 0,0,1,2 \rangle$?

It is the main object of this chapter to provide solutions to the above questions. We shall see that the sequence $\langle 0,0,1,2 \rangle$ is indeed representable. As a matter of fact, it is shown that there exist exactly two equational classes of algebras \underline{K}_1 and \underline{K}_2 such that an algebra \mathcal{A} represents $\langle 0,0,1,2 \rangle$ if and only if \mathcal{A} can be represented as an algebra belonging to either \underline{K}_1 or \underline{K}_2 .

1. Basic Lemmas.

Let \mathcal{A} be an algebra representing $\langle 0,0,1 \rangle$. Then \mathcal{A} has one and only one essentially binary polynomial which is commutative and idempotent. There are two possible cases, namely, the binary polynomial is either associative or non-

associative. Lemma 2.2 gives a sufficient condition for the former case to be happen. We need the following:

Lemma 1.1 (J. Płonka [28]).

let \mathcal{K} be an algebra without constants. If $p(x_0, x_1)$ is an essentially commutative binary polynomial over \mathcal{K} , then $p(x_0, p(x_1, (\dots, p(x_{n-2}, x_{n-1}) \dots)))$ is essential n -ary, for each $n=2,3, \dots$.

Lemma 1.2 .

Let \mathcal{K} be an algebra representing $\langle 0,0,1 \rangle$. If there exist $n \in \{3,4,5, \dots\}$ such that $p_n(\mathcal{K}) < \frac{1}{3}(2^n - (-1)^n)$, then \mathcal{K} has a semilattice operation.

Proof: Suppose that the binary Operation " \circ " is non-associative. We claim that $p_n(\mathcal{K}) \geq \frac{1}{3}(2^n - (-1)^n)$ for each $n=3,4, \dots$.

First of all, consider the following ternary polynomials:
 $(xy)z$, $(yz)x$, $(zx)y$.

It follows from Lemma 1.1 that they are all essential. By the commutativity of " \circ ", it is easy to see that the equality of any two would imply the associativity of " \circ ", which contradicts our assumption. Thus, we have $p_3(\mathcal{K}) \geq 3 = \frac{1}{3}(2^3 - (-1)^3)$.

Since $p_3(\mathcal{K}) \geq 3 > 2$, we can apply a result (Theorem 4 of [10]) and obtain $p_n(\mathcal{K}) \geq \frac{1}{3}(2^n - (-1)^n)$, for $n \geq 4$, as required.

Hence, "." must be associative and therefore \mathcal{K} has a semilattice operation.

Suppose that \mathcal{K} is an algebra representing $\langle 0,0,1,2 \rangle$. By Lemma 1.2, \mathcal{K} has a semilattice operation ".". By Lemma 1.1, we have already one essentially ternary polynomial $x \cdot y \cdot z$ over \mathcal{K} . Thus, if $p_3(\mathcal{K})=2$, there must exist one and only one essentially ternary polynomial $f(x,y,z)$ which is distinct from xyz . Our aim, here, is to investigate the general properties of the polynomial $f(x,y,z)$.

Clearly, we have

(1) $f(x,y,z)$ is idempotent.

Observe that if $p = p(x_0, \dots, x_{n-1})$ is an essentially n -ary polynomial over \mathcal{K} , then so is $p = p(x_{0\alpha}, \dots, x_{(n-1)\alpha})$ for each $\alpha \in S(n)$, where $S(n)$ is the symmetric group on n symbols.

Thus, $f(y,x,z)$ is essentially ternary. If $f(y,x,z) = xyz$, then $f(x,y,z) = yxz = xyz$, a contradiction. Hence, it follows that

(2) $f(x,y,z)$ is symmetric.

By identifying any two variables in $f(x,y,z)$, the resulting polynomial is binary. The following crucial result shows that it is essentially binary.

(3) $f(x,y,y) = xy$

Proof : As \mathcal{K} represents $\langle 0,0,1 \rangle$, we have only the following three cases:

$$f(x,y,y) = \begin{cases} x \\ y \\ xy \end{cases}$$

Case 1. $f(x,y,y) = x$

First of all, we claim that the following polynomials

$$(*) \quad f(x,y,z)x, \quad f(x,y,z)y, \quad f(x,y,z)z$$

are pairwise distinct.

Assume that $f(x,y,z)x = f(x,y,z)y$. Setting $x=z$, we get $yx=y$, a contradiction. By symmetry of $f(x,y,z)$ it follows that polynomials in (*) are pairwise distinct.

Next, we assert that each polynomial in (*) is essentially ternary. By symmetry, we need only check for $f(x,y,z)x$. Clearly, $f(x,y,z)x$ depends on x . Moreover, it depends on y if, and only if it depends on z . Thus, if $f(x,y,z)x$ is not essentially ternary, we then get

$$f(x,y,z)x = x.$$

Setting $x=y$, it follows from (2) that $zx=x$, which is impossible.

Hence $f(x,y,z)x$ is essentially ternary, as was to be shown.

Accordingly, if $f(x,y,y) = x$ holds, we would have $p_3(\mathcal{K}) \geq 3$, a contradiction.

Case 2. $f(x,y,y) = y$

In analogy to case 1, we claim that the polynomials in (*) are pairwise distinct.

For this purpose, assume that $f(x,y,z)x = f(x,y,z)y$. Setting $x=y$, we obtain $x=xy$, a contradiction. Thus, they are

pairwise distinct.

If one of them is essentially ternary, then so are the other two and hence $p_3(\mathcal{U}) \geq 3$, a contradiction. Since, for example, $f(x,y,z)x$ is not essentially ternary we have

$$f(x,y,z)x = \begin{cases} x \\ y \\ z \\ xy \\ yz \\ zx \end{cases}$$

From the fact that $f(x,y,z)x$ depends on x and is symmetric with respect to y,z , it follows that

$$f(x,y,z)x = x .$$

Set $y=z$. Then we obtain $yx = x$, a contradiction.

Thus, we conclude that the case $f(x,y,y) = y$ is impossible.

Therefore, it is necessary that $f(x,y,y) = xy$, proving (3).

(4) $f(x,y,z)$ is not diagonal.

Proof : If $f(x,y,z)$ were diagonal, we would have

$$f(x,y,z) = f(f(x,y,z), f(x,y,z), f(x,y,z)) \quad (1)$$

$$= f(f(x,y,z), f(y,x,z), f(y,z,x)) \quad (2)$$

$$= f(x,x,x) \quad (\text{diagonality})$$

$$= x , \quad (1)$$

which is a contradiction.

The following result will be of great use in deriving other identities.

(5) $f(xy,x,y) = xy$.

Proof : As $p_0(\mathcal{U})=0$, $f(xy,x,y)$ is not a constant. However, by symmetry, $f(xy,x,y)$ depends on x if, and only if

it depends on y . Hence $f(xy, x, y)$ is essentially binary and thus $f(xy, x, y) = xy$, since $p_2(\mathcal{U}) = 1$.

From (5), we obtain

$$(6) \quad f(xyz, xy, z) = xyz .$$

$$(7) \quad f(xyz, xy, xz) = xyz .$$

Consider the following ternary polynomial:

$$f(xy, y, z) .$$

It is easy to check that $f(xy, y, z)$ is essentially ternary. Thus we have the two possible cases:

$$f(xy, y, z) = \begin{cases} f(x, y, z) \\ xyz \end{cases}$$

$$\text{Suppose } f(xy, y, z) = f(x, y, z) \quad \text{-----} \quad (A)$$

We observe that

$$xyz = f(xyz, xy, xz) \quad (7)$$

$$= f(z, xy, xy, xz)$$

$$= f(z, xy, xz) \quad (A)$$

$$= f(xz, z, xy) \quad (2)$$

$$= f(x, z, xy) \quad (A)$$

$$= f(yx, x, z) \quad (2)$$

$$= f(x, y, z) \quad (A), (2)$$

which is impossible. Thus, we have

$$(8) \quad f(xy, y, z) = xyz .$$

The following are immediate consequences of the above identities:

$$(9) \quad f(xy, xz, z) = xyz \text{ .}$$

$$(10) \quad f(xy, xz, x) = xyz \text{ .}$$

$$(11) \quad f(xyz, y, z) = xyz \text{ .}$$

$$(12) \quad f(xyz, xy, y) = xyz \text{ .}$$

Though most of the identities of $f(x, y, z)$ are easy consequences of the previous ones, the following seems to be an exception.

$$(13) \quad f(xy, yz, zx) = xyz \text{ .}$$

Proof : Consider $f(xy, yz, zx)$. Set $x=y$. Then we obtain $f(x, xz, xz) = xz$ by (3). Thus, $f(xy, yz, zx)$ depends on z . By symmetry, it also depends on x and y . Hence $f(xy, yz, zx)$ is essentially ternary.

$$\text{If } f(xy, yz, zx) = f(x, y, z) \text{ ----- (B)}$$

$$\text{then } xyz = f(xyz, xyz, xyz) \text{ (1)}$$

$$= f(xy \cdot yz, yz \cdot zx, zx \cdot xy)$$

$$= f(xy, yz, zx) \text{ (B)}$$

$$= f(x, y, z) \text{ (B)}$$

which is a contradiction. Therefore, (13) follows.

A ternary operation f is associative if the following property holds:

$$f(f(x, y, z), u, v) = f(x, f(y, z, u), v) = f(x, y, f(z, u, v)).$$

Clearly, we have

$$(14) \quad f(x, y, z) \text{ is non-associative.}$$

2. Two Types of Algebras.

In this section, we continue our study of ternary polynomials built up from "." and "f". As a result, we obtain two types of algebras which are both compatible with our hypothesis.

To begin with, let us consider the polynomial $f(x,y,z) \cdot x$. It turns out that $f(x,y,z)x$ is essentially ternary. Thus, we have

$$f(x,y,z)x = \begin{cases} f(x,y,z) & \text{-----} & \text{I} \\ xyz & \text{-----} & \text{II} \end{cases}$$

From now on, we shall naturally split our investigation into two parts, each of which deals with each of the two possibilities in detail. We shall call those algebras satisfying the identity I, Type I algebras and those satisfying II, Type II algebras.

TYPE I. $f(x,y,z)x = f(x,y,z)$ ----- I

In this case, by the symmetry of $f(x,y,z)$, we get

$$(15) \quad f(x,y,z)x = f(x,y,z)xy = f(x,y,z)xyz = f(x,y,z).$$

Consider the polynomial $f(f(x,y,z),y,z)$. We have

$$f(f(x,y,z),y,z) = f(f(x,y,z)y,y,z) \quad (15)$$

$$= f(x,y,z)yz \quad (8)$$

$$= f(x,y,z) \quad (15)$$

Thus, it follows that

$$(16) \quad f(f(x,y,z),y,z) = f(x,y,z).$$

By applying the same argument, using (8) and (15) the following identities can be derived immediately.

$$(17) \quad f(f(x,y,z),xy,z) = f(x,y,z) .$$

$$(18) \quad f(f(x,y,z),xyz,z) = f(x,y,z) .$$

$$(19) \quad f(f(x,y,z),xy,x) = f(x,y,z) .$$

$$(20) \quad f(f(x,y,z),xy,xz) = f(x,y,z) .$$

$$(21) \quad f(f(x,y,z),xyz,xy) = f(x,y,z) .$$

TYPE II. $f(x,y,z)x = xyz$ _____ II

In this case, as $f(x,y,z)$ is symmetric, we have

$$(22) \quad f(x,y,z)x = f(x,y,z)y = f(x,y,z)z = xyz .$$

Now, consider the polynomial $f(f(x,y,z),y,z)$. It can be easily checked that it is essentially ternary.

$$\text{If} \quad f(f(x,y,z),y,z) = f(x,y,z) \quad \text{_____} \quad (C)$$

$$\begin{aligned} \text{then} \quad f(x,y,z) &= f(x,y,z) \cdot f(x,y,z) \\ &= f(f(x,y,z),y,z) f(x,y,z) && (C) \\ &= f(x,y,z)yz && (22) \\ &= xyz && (22) \end{aligned}$$

which is a contradiction. Thus it follows that

$$(23) \quad f(f(x,y,z),y,z) = xyz .$$

Similarly, we get

$$(24) \quad f(f(x,y,z),xy,z) = xyz .$$

$$(25) \quad f(f(x,y,z),xy,xz) = xyz .$$

Observe that

$$\begin{aligned} f(f(x,y,z),xyz,z) &= f(f(x,y,z),xy \cdot z, z) \\ &= xyz f(x,y,z) && (8) \end{aligned}$$

$$= xyz \quad (22)$$

Thus, we have

$$(26) \quad f(f(x,y,z),xyz,z) = xyz .$$

Similar arguments can be applied to yield the following:

$$(27) \quad f(f(x,y,z),xy,x) = xyz .$$

$$(28) \quad f(f(x,y,z),xyz,xy) = xyz .$$

Let \mathcal{A} be an algebra representing $\langle 0,0,1,2 \rangle$. Let $p(x,y,z)$ be an arbitrary ternary polynomial over \mathcal{A} . Then $p(x,y,z)$ is built up from the set of symbols $\{x,y,z\}$ by substituting them in two operation symbols "." and "f" . If \mathcal{A} is a Type I algebra, then by making use of those identities hold in \mathcal{A} , $p(x,y,z)$ can be reduced to one of the ternary polynomials $\{xyz, f(x,y,z)\}$. If \mathcal{A} is a Type II algebra, the same situation holds. For clarity, we now give the following list:

| | | TYPE I | TYPE II |
|---|-------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------|------------------------------------------------------------------------------|
| A | $f(xy, y, z)$ $f(xy, xz, x)$ $f(xy, yz, z)$ $f(xy, yz, zx)$ $f(xyz, y, z)$ $f(xyz, xy, z)$ $f(xyz, xy, y)$ $f(xyz, xy, yz)$ | $\left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} = xyz$ | $\left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} = xyz$ |
| B | $f(x, y, z)x$ $f(x, y, z)xy$ $f(x, y, z)xyz$ | $\left. \begin{array}{l} \\ \\ \end{array} \right\} = f(xy, z)$ | $\left. \begin{array}{l} \\ \\ \end{array} \right\} = xyz$ |
| C | $f(f(x, y, z), y, z)$ $f(f(x, y, z), xy, z)$ $f(f(x, y, z), xy, x)$ $f(f(x, y, z), xy, xz)$ $f(f(x, y, z), xyz, z)$ $f(f(x, y, z), xyz, xy)$ | $\left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array} \right\} = f(x, y, z)$ | $\left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array} \right\} = xyz$ |

3. Characterization Theorem and Applications.

We are now in a position to establish some of the main results of this chapter. Summarizing all the results in the previous sections, we arrive at the following

Theorem 3.1

Let \mathcal{A} be an algebra representing $\langle 0,0,1,2 \rangle$. Then \mathcal{A} can be represented as an algebra $\langle A; \cdot, f \rangle$ of type $\langle 2,3 \rangle$ where " \cdot " is the semilattice operation belonging to one of the equational classes $\underline{K}_1, \underline{K}_2$ of algebras where

$$\text{Id}(\underline{K}_1) = \left\{ \begin{array}{l} (1) \quad f(x,y,z) = f(y,x,z) = f(y,z,x) \\ (2) \quad f(xy,y,z) = xyz \\ (3) \quad f(xy,yz,zx) = xyz \\ (4) \quad f(x,y,z)x = f(x,y,z) \end{array} \right.$$

$$\text{Id}(\underline{K}_2) = \left\{ \begin{array}{l} (1) \quad f(x,y,z) = f(y,x,z) = f(y,z,x) \\ (2) \quad f(xy,y,z) = xyz \\ (3) \quad f(xy,yz,zx) = xyz \\ (4) \quad f(x,y,z)x = xyz \\ (5) \quad f(f(x,y,z),y,z) = xyz \\ (6) \quad f(f(x,y,z),xy,z) = xyz \\ (7) \quad f(f(x,y,z),xy,xz) = xyz \end{array} \right.$$

Moreover, if $\mathcal{A} \in \underline{K}_1$ and $p(x,y,z)$ is an essentially ternary polynomial over \mathcal{A} then

$$p(x,y,z) = \begin{cases} f(x,y,z) & \text{if the whole factor } f(x,y,z) \text{ appears} \\ & \text{in } p(x,y,z) \\ xyz & \text{otherwise} \end{cases}$$

If $\mathcal{A} \in \underline{K}_2$ and $p(x,y,z)$ is an essentially ternary polynomial over \mathcal{A} then

$$p(x,y,z) = \begin{cases} f(x,y,z) & \text{if } p(x,y,z) \text{ is of the form } f(x,y,z) \\ xyz & \text{otherwise} \end{cases}$$

We will now prove the converse of Theorem 3.1. The two types of algebras will be considered separately.

Theorem 3.2 (Type I).

Let $\mathcal{A} = \langle A; \cdot, f \rangle$ be an algebra of type $\langle 2,3 \rangle$ where " \cdot " is the semilattice operation and $f(x,y,z)$ is the ternary operation satisfying $\text{Id}(\underline{K}_1)$ of Theorem 3.1. Then \mathcal{A} represents $\langle 0,0,1,2 \rangle$.

Proof: Since " \cdot " is idempotent and $f(x,x,x) = x$ by (2) of $\text{Id}(\underline{K}_1)$, it follows that \mathcal{A} is idempotent. This is equivalent to saying that $p_0(\mathcal{A}) = p_1(\mathcal{A}) = 0$.

(2) of $\text{Id}(\underline{K}_1)$ implies $f(x,y,y) = xy$ and $f(xy,x,y) = xy$. Combine these with (1) of $\text{Id}(\underline{K}_1)$. Then it follows that 'xy' is the only essentially binary polynomial over \mathcal{A} . Thus, $p_2(\mathcal{A}) = 1$.

Finally, we have to prove that $p_3(\mathcal{A}) = 2$. Since $f(x,y,z) \neq xyz$, $p_3(\mathcal{A}) \geq 2$. On the other hand, according to the results in sections 1 and 2, we see that (1), (2) and (3) of $\text{Id}(\underline{K}_1)$ imply that all the forms of ternary polynomials in category A (see section 2) are the same and equal to xyz .

Moreover, from (2) and (4) of $\text{Id}(\underline{K}_1)$, it follows that all the forms of the ternary polynomials in categories B and C are the same and equal to $f(x,y,z)$. Hence, $p_3(\mathcal{A}) = 2$, proving our theorem.

Theorem 3.3 (Type II).

Let $\mathcal{A} = \langle A; \cdot, f \rangle$ be an algebra of type $\langle 2,3 \rangle$ where " \cdot " is the semilattice operation and $f(x,y,z)$ is the ternary operation satisfying $\text{Id}(\underline{K}_2)$ of Theorem 3.1. Then \mathcal{A} represents $\langle 0,0,1,2 \rangle$.

Proof : In analogy to the proof of Theorem 3.2, we see that

(1) and (2) of $\text{Id}(\underline{K}_2)$ imply that \mathcal{A} represents $\langle 0,0,1 \rangle$.

To prove that $p_3(\mathcal{A}) = 2$, observe that (1),(2) and (3) guarantee that all the forms of the ternary polynomials in category A are all the same and equal to xyz . Furthermore, (4) of $\text{Id}(\underline{K}_2)$ implies that all the forms of the ternary polynomials in category B are all the same and equal to xyz . Finally, (2),(4), (5),(6) and (7) imply that all the forms of the ternary polynomials in category C are all the same and again equal to xyz . Hence, $p_3(\mathcal{A}) = 2$, as was to be shown.

Combining the above three results, we have the following characterization theorem.

Theorem 3.4

There exist two equational classes of algebras \underline{K}_1 and \underline{K}_2