

ON THE NUMBER OF

ESSENTIALLY  $n$ -ARY POLYNOMIALS

OF IDEMPOTENT ALGEBRAS

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## INTRODUCTION

A universal algebra, or briefly, algebra  $\mathcal{A}$  is an ordered pair  $\langle A; F \rangle$  where  $A$  is a non-empty set and  $F$  is a family of finitary operations on  $A$ . For each natural number  $n$ , we can consider the set  $P^{(n)}(\mathcal{A})$  of  $n$ -ary polynomials of  $\mathcal{A}$  which are certain functions from  $A^n$  to  $A$  built up from the variables  $x_i$ ,  $i = 1, 2, \dots, n$  by substituting them in the operations  $f$ ,  $f \in F$ , successively in a finite number of steps.

An  $n$ -ary polynomial  $p$  over  $\mathcal{A}$  is said to depend on  $x_i$  if there exist  $a_1, \dots, a_i, a_i', \dots, a_n$  in  $A$  such that

$$p(a_1, \dots, a_i, \dots, a_n) \neq p(a_1, \dots, a_i', \dots, a_n).$$

By an essentially  $n$ -ary polynomial over  $\mathcal{A}$  is meant a  $n$ -ary polynomial over  $\mathcal{A}$  which depends on each variable  $x_i$ ,  $i = 1, \dots, n$ . For  $n > 1$ , let  $p_n(\mathcal{A})$  designate the number of essentially  $n$ -ary polynomials over  $\mathcal{A}$ . We denote by  $p_1(\mathcal{A})$  and  $p_0(\mathcal{A})$  the number of non-constant unary polynomials excluding  $x_1$  and the number of constant unary polynomials respectively. Thus, with any algebra  $\mathcal{A}$ , there is associated an  $\omega$ -sequence of cardinals  $\langle p_0(\mathcal{A}), p_1(\mathcal{A}), \dots, p_n(\mathcal{A}), \dots \rangle$ .

Let  $\mathcal{C}$  be a class of algebra. A sequence  $\langle p_0, p_1, \dots, p_n, \dots \rangle$  of cardinals is said to representable in  $\mathcal{C}$  if there exists an algebra  $\mathcal{A}$  in  $\mathcal{C}$  with  $p_n = p_n(\mathcal{A})$  for each  $n \geq 0$ . If  $\mathcal{C}$  is the class of all algebras, then we say that the sequence

$\langle p_0, p_1, \dots, p_n, \dots \rangle$  is representable. An algebra  $\mathcal{A} = \langle A; F \rangle$

is said to be idempotent if  $f(x, \dots, x) = x$ , for any  $f$  in  $F$ . Thus, it is easy to see that an algebra  $\mathcal{A}$  is idempotent if and only if  $p_0(\mathcal{A}) = p_1(\mathcal{A}) = 0$ . We shall say that an algebra  $\mathcal{A} = \langle A; F \rangle$  can be represented as an algebra  $\mathcal{A}^* = \langle A; f_1, f_2, \dots \rangle$  of type  $\tau$  if it is possible to choose a sequence  $(f_1, f_2, \dots)$  of polynomials from  $F$  in such a way that the sequence of the arities of the  $f_i$  equals  $\tau$ . Note that the set of polynomials  $\{f_i\}$  can be taken as a set of operations in  $\mathcal{A}$ .

Our basic problem is to study and characterize representable sequences. An easy combinatorial argument shows that this problem is equivalent to Problem 42 in [6] which can be stated as follows: Let  $\underline{K}$  be an equational class, and let  $F_n$  denote the cardinal of the free algebra over  $\underline{K}$  on  $n$  generators. Characterize the sequence  $\langle F_n \rangle$ .

The development of the study of  $\langle p_n \rangle$  sequence may briefly be divided into three stages.

The period that started in 1910 may be considered as the initial stage. In this period, even though there were no significant contributions to the theory, the idea was foreshadowed by the work of S. Sierpinski. He published a series of articles between 1918 and 1945 for the purpose of investigating the composition of functions. One of his typical results ( see [41] ) says that given any set  $A$  and any function  $f : A^n \longrightarrow A$ ,  $f$  can

be obtained by an appropriate composition of binary functions. Recently, R.W. Quackenbush studied the corresponding problem for idempotent functions. He proved [37] that every idempotent function on a given set  $A$  can be obtained by composition of binary idempotent functions provided  $|A| > 2$ . For  $|A| = 2$ , the role of binary functions is replaced by ternary functions.

The explicit formulation of the basic problem, given by E. Marczewski in 1963 - 1964, may be considered as the beginning of the second stage. Since it was considered too difficult to deal with explicitly; E. Marczewski and his colleagues in Wroclaw studied only problems associated with it. In particular, he himself defined for each  $\mathcal{A}$ , the zero set

$$Z(\mathcal{A}) = \{ n / p_n(\mathcal{A}) = 0 \}$$

and showed, for instance, in [22] that for algebras without constants and with one essentially  $n$ -ary symmetry ( or even quasi-symmetric ) polynomial the complement of the zero set  $Z(\mathcal{A})$  contains the arithmetical progression  $n + k(n - 1)$ , (  $k = 0, 1, \dots$  ). This generalizes a result of J. Płonka [28] for  $n = 2$ . One of the deepest results was obtained by K. Urbanik [42] who gave a complete description of all possible sets  $Z(\mathcal{A})$ .

It was in 1968 that the present stage began with a systematic and intensive investigation of the basic problem in G. Grätzer's seminar at the University of Manitoba. Influenced by the first paper due to G. Grätzer, J. Płonka and A. Sekanina

[ 11 ] , a steady flow of contributions to it, by the members of the seminar, has appeared in 1968 - 1969. G. Grätzer, J. Płonka and R. Padmanabhan have especially enriched and clarified the subject. The results were summarized by G. Grätzer [ 8 ] who gave a survey lecture at the Conference on Universal Algebras held at Queen's University in October, 1969.

In this period, the investigation of the basic problem was naturally split into two categories : (1). Study the basic problem for non-idempotent algebras; (2). Study the same for idempotent algebras. The first case was attacked by G. Grätzer, J. Płonka, A. Sekanina in [ 11 ] , [ 12 ] and [ 33 ] . Some of their results were of the type that sequences satisfying some mild condition ( e.g.  $p_0 > 0$  ) are all representable, and so the  $p_i$  are independent. However, the situation completely changes when we deal with the idempotent case. As a matter of fact, the cardinals  $p_n(\mathcal{A})$ , for idempotent algebra  $\mathcal{A}$  , turn out to be quite strongly interrelated ( see, for instance, [ 13 ] and [ 14 ] ). Because of this extremely interesting fact, recently, most papers were devoted to the study of idempotent algebra ; this study can be separated roughly into two parts : (A). Investigate the behaviour and the maximum asymptotic rate of growth of the general sequence  $\langle p_n \rangle$  ; (B). Description of all algebras representing a given sequence with application to the Minimal Extension Property ( for definitin, see Chapter two of Part IV ).

The purpose of this thesis is to provide some results for the second part of category 2 in a systematic way with emphasis on applications to the Minimal Extension Property. The equivalent problem of  $\langle p_n \rangle$  sequences in Group Theory, the so called growth function of free groups, has been extensively studied for equational classes of groups by British mathematicians ; for instance, G. Higman ( see [ 16 ] and [ 23 ] ), P. Neumann and his students. The application of  $\langle p_n \rangle$  sequences to semilattices was considered in G. Grätzer and J. Płonka [ 15 ] while the case of idempotent semigroups was settled by J.A. Gerhard [ 5 ] . In this thesis, we make a first attack in applying the  $\langle p_n \rangle$  sequences to Lattice Theory.

This thesis falls into four parts with nine chapters altogether. Since a short description of the content is given at the beginning of each chapter, we shall include here only a brief outline. Part I, which consists of three chapters, is devoted to study idempotent algebras with one essentially binary polynomial. The sequence  $\langle 0,0,1,2 \rangle$  has, in particular, very interesting properties. Thus, we restrict our attention to this sequence in the first two chapters. Some results of Part I are generalized to Part II in which we consider idempotent algebras with one essentially  $m$ -ary polynomial for  $m \geq 2$ . All of these are applied to derive the function  $F(n,k)$  with the property that  $F(n,k)$  is the least value such that the sequence  $\langle 0,0,1,\dots,1,n,F(n,k) \rangle$  is representable by symmetric

algebra. Part III consists of two chapters. Płonka's basic lemmas are used in Chapter 1 to derive some results about the sequences  $\langle 0,0,2,m \rangle$  . The considerations of Chapter 2 center around the algebras representing  $\langle 0,0,3,m \rangle$  . Part IV consists of two chapters. Previous results are applied to Lattice Theory in Chapter 1 and the Minimal Extension Property in Chapter 2.

Cross references are given in the form (III,1,2) where III stands for Part III, 1 for Chapter 1 and 2 for section 2. The part and chapter numerals will be omitted in case the reference is made in the same chapter.

For those basic concepts and notations, we refer to G. Grätzer's books [6] and [9] .



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----- PART I -----

IDEMPOTENT ALGEBRAS WITH ONE ESSENTIALLY BINARY POLYNOMIAL

## CHAPTER 1

### ALGEBRAS REPRESENTING $\langle 0,0,1,2 \rangle$

The sequence  $\langle 0,0,1,1 \rangle$  is, evidently, representable. This can be seen simply by taking a non-trivial semilattice. To go one step further, we are interested in the case where  $p_3 = 2$ . Thus, the following questions naturally arises :

- (1). Is the sequence  $\langle 0,0,1,2 \rangle$  representable ?
- (2). If the answer to (1) is in the affirmative, what can we say about those algebras representing  $\langle 0,0,1,2 \rangle$  ?

It is the main object of this chapter to provide solutions to the above questions. We shall see that the sequence  $\langle 0,0,1,2 \rangle$  is indeed representable. As a matter of fact, it is shown that there exist exactly two equational classes of algebras  $\underline{K}_1$  and  $\underline{K}_2$  such that an algebra  $\mathcal{A}$  represents  $\langle 0,0,1,2 \rangle$  if and only if  $\mathcal{A}$  can be represented as an algebra belonging to either  $\underline{K}_1$  or  $\underline{K}_2$ .

#### 1. Basic Lemmas.

Let  $\mathcal{A}$  be an algebra representing  $\langle 0,0,1 \rangle$ . Then  $\mathcal{A}$  has one and only one essentially binary polynomial which is commutative and idempotent. There are two possible cases, namely, the binary polynomial is either associative or non-

associative. Lemma 2.2 gives a sufficient condition for the former case to be happen. We need the following:

Lemma 1.1 (J. Płonka[28 ]).

let  $\mathcal{K}$  be an algebra without constants. If  $p(x_0, x_1)$  is an essentially commutative binary polynomial over  $\mathcal{K}$ , then  $p(x_0, p(x_1, (\dots, p(x_{n-2}, x_{n-1}) \dots)))$  is essential  $n$ -ary, for each  $n=2,3, \dots$ .

Lemma 1.2 .

Let  $\mathcal{K}$  be an algebra representing  $\langle 0,0,1 \rangle$ . If there exist  $n \in \{3,4,5, \dots\}$  such that  $p_n(\mathcal{K}) < \frac{1}{3}(2^n - (-1)^n)$ , then  $\mathcal{K}$  has a semilattice operation.

Proof: Suppose that the binary Operation " $\circ$ " is non-associative. We claim that  $p_n(\mathcal{K}) \geq \frac{1}{3}(2^n - (-1)^n)$  for each  $n=3,4, \dots$ .

First of all, consider the following ternary polynomials:  
 $(xy)z$ ,  $(yz)x$ ,  $(zx)y$ .

It follows from Lemma 1.1 that they are all essential. By the commutativity of " $\circ$ ", it is easy to see that the equality of any two would imply the associativity of " $\circ$ ", which contradicts our assumption. Thus, we have  $p_3(\mathcal{K}) \geq 3 = \frac{1}{3}(2^3 - (-1)^3)$ .

Since  $p_3(\mathcal{K}) \geq 3 > 2$ , we can apply a result (Theorem 4 of [10]) and obtain  $p_n(\mathcal{K}) \geq \frac{1}{3}(2^n - (-1)^n)$ , for  $n \geq 4$ , as required.

Hence, "." must be associative and therefore  $\mathcal{K}$  has a semilattice operation.

Suppose that  $\mathcal{K}$  is an algebra representing  $\langle 0,0,1,2 \rangle$ . By Lemma 1.2,  $\mathcal{K}$  has a semilattice operation ".". By Lemma 1.1, we have already one essentially ternary polynomial  $x \cdot y \cdot z$  over  $\mathcal{K}$ . Thus, if  $p_3(\mathcal{K})=2$ , there must exist one and only one essentially ternary polynomial  $f(x,y,z)$  which is distinct from  $xyz$ . Our aim, here, is to investigate the general properties of the polynomial  $f(x,y,z)$ .

Clearly, we have

(1)  $f(x,y,z)$  is idempotent.

Observe that if  $p = p(x_0, \dots, x_{n-1})$  is an essentially  $n$ -ary polynomial over  $\mathcal{K}$ , then so is  $p = p(x_{0\alpha}, \dots, x_{(n-1)\alpha})$  for each  $\alpha \in S(n)$ , where  $S(n)$  is the symmetric group on  $n$  symbols.

Thus,  $f(y,x,z)$  is essentially ternary. If  $f(y,x,z) = xyz$ , then  $f(x,y,z) = yxz = xyz$ , a contradiction. Hence, it follows that

(2)  $f(x,y,z)$  is symmetric.

By identifying any two variables in  $f(x,y,z)$ , the resulting polynomial is binary. The following crucial result shows that it is essentially binary.

(3)  $f(x,y,y) = xy$

Proof : As  $\mathcal{K}$  represents  $\langle 0,0,1 \rangle$ , we have only the following three cases:

$$f(x,y,y) = \begin{cases} x \\ y \\ xy \end{cases}$$

Case 1.  $f(x,y,y) = x$

First of all, we claim that the following polynomials

$$(*) \quad f(x,y,z)x, \quad f(x,y,z)y, \quad f(x,y,z)z$$

are pairwise distinct.

Assume that  $f(x,y,z)x = f(x,y,z)y$ . Setting  $x=z$ , we get  $yx=y$ , a contradiction. By symmetry of  $f(x,y,z)$  it follows that polynomials in (\*) are pairwise distinct.

Next, we assert that each polynomial in (\*) is essentially ternary. By symmetry, we need only check for  $f(x,y,z)x$ . Clearly,  $f(x,y,z)x$  depends on  $x$ . Moreover, it depends on  $y$  if, and only if it depends on  $z$ . Thus, if  $f(x,y,z)x$  is not essentially ternary, we then get

$$f(x,y,z)x = x.$$

Setting  $x=y$ , it follows from (2) that  $zx=x$ , which is impossible.

Hence  $f(x,y,z)x$  is essentially ternary, as was to be shown.

Accordingly, if  $f(x,y,y) = x$  holds, we would have  $p_3(\mathcal{K}) \geq 3$ , a contradiction.

Case 2.  $f(x,y,y) = y$

In analogy to case 1, we claim that the polynomials in (\*) are pairwise distinct.

For this purpose, assume that  $f(x,y,z)x = f(x,y,z)y$ . Setting  $x=y$ , we obtain  $x=xy$ , a contradiction. Thus, they are



pairwise distinct.

If one of them is essentially ternary, then so are the other two and hence  $p_3(\mathcal{U}) \geq 3$ , a contradiction. Since, for example,  $f(x,y,z)x$  is not essentially ternary we have

$$f(x,y,z)x = \begin{cases} x \\ y \\ z \\ xy \\ yz \\ zx \end{cases}$$

From the fact that  $f(x,y,z)x$  depends on  $x$  and is symmetric with respect to  $y,z$ , it follows that

$$f(x,y,z)x = x .$$

Set  $y=z$ . Then we obtain  $yx = x$ , a contradiction.

Thus, we conclude that the case  $f(x,y,y) = y$  is impossible.

Therefore, it is necessary that  $f(x,y,y) = xy$ , proving (3).

(4)  $f(x,y,z)$  is not diagonal.

Proof : If  $f(x,y,z)$  were diagonal, we would have

$$f(x,y,z) = f(f(x,y,z), f(x,y,z), f(x,y,z)) \quad (1)$$

$$= f(f(x,y,z), f(y,x,z), f(y,z,x)) \quad (2)$$

$$= f(x,x,x) \quad (\text{diagonality})$$

$$= x , \quad (1)$$

which is a contradiction.

The following result will be of great use in deriving other identities.

(5)  $f(xy,x,y) = xy$ .

Proof : As  $p_0(\mathcal{U})=0$ ,  $f(xy,x,y)$  is not a constant. However, by symmetry,  $f(xy,x,y)$  depends on  $x$  if, and only if

it depends on  $y$ . Hence  $f(xy, x, y)$  is essentially binary and thus  $f(xy, x, y) = xy$ , since  $p_2(\mathcal{U}) = 1$ .

From (5), we obtain

$$(6) \quad f(xyz, xy, z) = xyz .$$

$$(7) \quad f(xyz, xy, xz) = xyz .$$

Consider the following ternary polynomial:

$$f(xy, y, z) .$$

It is easy to check that  $f(xy, y, z)$  is essentially ternary. Thus we have the two possible cases:

$$f(xy, y, z) = \begin{cases} f(x, y, z) \\ xyz \end{cases}$$

$$\text{Suppose } f(xy, y, z) = f(x, y, z) \quad \text{-----} \quad (A)$$

We observe that

$$xyz = f(xyz, xy, xz) \quad (7)$$

$$= f(z, xy, xy, xz)$$

$$= f(z, xy, xz) \quad (A)$$

$$= f(xz, z, xy) \quad (2)$$

$$= f(x, z, xy) \quad (A)$$

$$= f(yx, x, z) \quad (2)$$

$$= f(x, y, z) \quad (A), (2)$$

which is impossible. Thus, we have

$$(8) \quad f(xy, y, z) = xyz .$$

The following are immediate consequences of the above identities:

$$(9) \quad f(xy, xz, z) = xyz \text{ .}$$

$$(10) \quad f(xy, xz, x) = xyz \text{ .}$$

$$(11) \quad f(xyz, y, z) = xyz \text{ .}$$

$$(12) \quad f(xyz, xy, y) = xyz \text{ .}$$

Though most of the identities of  $f(x, y, z)$  are easy consequences of the previous ones, the following seems to be an exception.

$$(13) \quad f(xy, yz, zx) = xyz \text{ .}$$

Proof : Consider  $f(xy, yz, zx)$ . Set  $x=y$ . Then we obtain  $f(x, xz, xz) = xz$  by (3). Thus,  $f(xy, yz, zx)$  depends on  $z$ . By symmetry, it also depends on  $x$  and  $y$ . Hence  $f(xy, yz, zx)$  is essentially ternary.

$$\text{If } f(xy, yz, zx) = f(x, y, z) \text{ ----- (B)}$$

$$\text{then } xyz = f(xyz, xyz, xyz) \text{ (1)}$$

$$= f(xy \cdot yz, yz \cdot zx, zx \cdot xy)$$

$$= f(xy, yz, zx) \text{ (B)}$$

$$= f(x, y, z) \text{ (B)}$$

which is a contradiction. Therefore, (13) follows.

A ternary operation  $f$  is associative if the following property holds:

$$f(f(x, y, z), u, v) = f(x, f(y, z, u), v) = f(x, y, f(z, u, v)).$$

Clearly, we have

$$(14) \quad f(x, y, z) \text{ is non-associative.}$$

## 2. Two Types of Algebras.

In this section, we continue our study of ternary polynomials built up from "." and "f". As a result, we obtain two types of algebras which are both compatible with our hypothesis.

To begin with, let us consider the polynomial  $f(x,y,z) \cdot x$ . It turns out that  $f(x,y,z)x$  is essentially ternary. Thus, we have

$$f(x,y,z)x = \begin{cases} f(x,y,z) & \text{-----} & \text{I} \\ xyz & \text{-----} & \text{II} \end{cases}$$

From now on, we shall naturally split our investigation into two parts, each of which deals with each of the two possibilities in detail. We shall call those algebras satisfying the identity I, Type I algebras and those satisfying II, Type II algebras.

TYPE I.  $f(x,y,z)x = f(x,y,z)$  ----- I

In this case, by the symmetry of  $f(x,y,z)$ , we get

$$(15) \quad f(x,y,z)x = f(x,y,z)xy = f(x,y,z)xyz = f(x,y,z).$$

Consider the polynomial  $f(f(x,y,z),y,z)$ . We have

$$f(f(x,y,z),y,z) = f(f(x,y,z)y,y,z) \quad (15)$$

$$= f(x,y,z)yz \quad (8)$$

$$= f(x,y,z) \quad (15)$$

Thus, it follows that

$$(16) \quad f(f(x,y,z),y,z) = f(x,y,z).$$

By applying the same argument, using (8) and (15) the following identities can be derived immediately.

$$(17) \quad f(f(x,y,z),xy,z) = f(x,y,z) .$$

$$(18) \quad f(f(x,y,z),xyz,z) = f(x,y,z) .$$

$$(19) \quad f(f(x,y,z),xy,x) = f(x,y,z) .$$

$$(20) \quad f(f(x,y,z),xy,xz) = f(x,y,z) .$$

$$(21) \quad f(f(x,y,z),xyz,xy) = f(x,y,z) .$$

TYPE II.       $f(x,y,z)x = xyz$       \_\_\_\_\_      II

In this case, as  $f(x,y,z)$  is symmetric, we have

$$(22) \quad f(x,y,z)x = f(x,y,z)y = f(x,y,z)z = xyz .$$

Now, consider the polynomial  $f(f(x,y,z),y,z)$ . It can be easily checked that it is essentially ternary.

$$\text{If} \quad f(f(x,y,z),y,z) = f(x,y,z) \quad \text{_____} \quad (C)$$

$$\begin{aligned} \text{then} \quad f(x,y,z) &= f(x,y,z) \cdot f(x,y,z) \\ &= f(f(x,y,z),y,z) f(x,y,z) && (C) \\ &= f(x,y,z)yz && (22) \\ &= xyz && (22) \end{aligned}$$

which is a contradiction. Thus it follows that

$$(23) \quad f(f(x,y,z),y,z) = xyz .$$

Similarly, we get

$$(24) \quad f(f(x,y,z),xy,z) = xyz .$$

$$(25) \quad f(f(x,y,z),xy,xz) = xyz .$$

Observe that

$$\begin{aligned} f(f(x,y,z),xyz,z) &= f(f(x,y,z),xy \cdot z, z) \\ &= xyz f(x,y,z) && (8) \end{aligned}$$

$$= xyz \quad (22)$$

Thus, we have

$$(26) \quad f(f(x,y,z),xyz,z) = xyz .$$

Similar arguments can be applied to yield the following:

$$(27) \quad f(f(x,y,z),xy,x) = xyz .$$

$$(28) \quad f(f(x,y,z),xyz,xy) = xyz .$$

Let  $\mathcal{A}$  be an algebra representing  $\langle 0,0,1,2 \rangle$  . Let  $p(x,y,z)$  be an arbitrary ternary polynomial over  $\mathcal{A}$  . Then  $p(x,y,z)$  is built up from the set of symbols  $\{x,y,z\}$  by substituting them in two operation symbols "." and "f" . If  $\mathcal{A}$  is a Type I algebra, then by making use of those identities hold in  $\mathcal{A}$  ,  $p(x,y,z)$  can be reduced to one of the ternary polynomials  $\{xyz, f(x,y,z)\}$  . If  $\mathcal{A}$  is a Type II algebra, the same situation holds. For clarity, we now give the following list:

		TYPE I	TYPE II
A	$f(xy, y, z)$ $f(xy, xz, x)$ $f(xy, yz, z)$ $f(xy, yz, zx)$ $f(xyz, y, z)$ $f(xyz, xy, z)$ $f(xyz, xy, y)$ $f(xyz, xy, yz)$	$\left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} = xyz$	$\left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} = xyz$
B	$f(x, y, z)x$ $f(x, y, z)xy$ $f(x, y, z)xyz$	$\left. \begin{array}{l} \\ \\ \end{array} \right\} = f(xy, z)$	$\left. \begin{array}{l} \\ \\ \end{array} \right\} = xyz$
C	$f(f(x, y, z), y, z)$ $f(f(x, y, z), xy, z)$ $f(f(x, y, z), xy, x)$ $f(f(x, y, z), xy, xz)$ $f(f(x, y, z), xyz, z)$ $f(f(x, y, z), xyz, xy)$	$\left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array} \right\} = f(x, y, z)$	$\left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array} \right\} = xyz$

### 3. Characterization Theorem and Applications.

We are now in a position to establish some of the main results of this chapter. Summarizing all the results in the previous sections, we arrive at the following

#### Theorem 3.1

Let  $\mathcal{A}$  be an algebra representing  $\langle 0,0,1,2 \rangle$ . Then  $\mathcal{A}$  can be represented as an algebra  $\langle A; \cdot, f \rangle$  of type  $\langle 2,3 \rangle$  where " $\cdot$ " is the semilattice operation belonging to one of the equational classes  $\underline{K}_1, \underline{K}_2$  of algebras where

$$\text{Id}(\underline{K}_1) = \left\{ \begin{array}{l} (1) \quad f(x,y,z) = f(y,x,z) = f(y,z,x) \\ (2) \quad f(xy,y,z) = xyz \\ (3) \quad f(xy,yz,zx) = xyz \\ (4) \quad f(x,y,z)x = f(x,y,z) \end{array} \right.$$

$$\text{Id}(\underline{K}_2) = \left\{ \begin{array}{l} (1) \quad f(x,y,z) = f(y,x,z) = f(y,z,x) \\ (2) \quad f(xy,y,z) = xyz \\ (3) \quad f(xy,yz,zx) = xyz \\ (4) \quad f(x,y,z)x = xyz \\ (5) \quad f(f(x,y,z),y,z) = xyz \\ (6) \quad f(f(x,y,z),xy,z) = xyz \\ (7) \quad f(f(x,y,z),xy,xz) = xyz \end{array} \right.$$

Moreover, if  $\mathcal{A} \in \underline{K}_1$  and  $p(x,y,z)$  is an essentially ternary polynomial over  $\mathcal{A}$  then



$$p(x,y,z) = \begin{cases} f(x,y,z) & \text{if the whole factor } f(x,y,z) \text{ appears} \\ & \text{in } p(x,y,z) \\ xyz & \text{otherwise} \end{cases}$$

If  $\mathcal{A} \in \underline{K}_2$  and  $p(x,y,z)$  is an essentially ternary polynomial over  $\mathcal{A}$  then

$$p(x,y,z) = \begin{cases} f(x,y,z) & \text{if } p(x,y,z) \text{ is of the form } f(x,y,z) \\ xyz & \text{otherwise} \end{cases}$$

We will now prove <sup>a</sup>the converse of Theorem 3.1 . The two types of algebras will be considered separately.

Theorem 3.2 (Type I).

Let  $\mathcal{A} = \langle A; \cdot, f \rangle$  be an algebra of type  $\langle 2,3 \rangle$  where " $\cdot$ " is the semilattice operation and  $f(x,y,z)$  is the ternary operation satisfying  $\text{Id}(\underline{K}_1)$  of Theorem 3.1 . Then  $\mathcal{A}$  represents  $\langle 0,0,1,2 \rangle$  .

Proof : Since " $\cdot$ " is idempotent and  $f(x,x,x) = x$  by (2) of  $\text{Id}(\underline{K}_1)$ , it follows that  $\mathcal{A}$  is idempotent. This is equivalent to saying that  $p_0(\mathcal{A}) = p_1(\mathcal{A}) = 0$  .

(2) of  $\text{Id}(\underline{K}_1)$  implies  $f(x,y,y) = xy$  and  $f(xy,x,y) = xy$  . Combine these with (1) Of  $\text{Id}(\underline{K}_1)$ . Then it follows that 'xy' is the only essentially binary polynomial over  $\mathcal{A}$  . Thus,  $p_2(\mathcal{A}) = 1$  .

Finally, we have to prove that  $p_3(\mathcal{A}) = 2$ . Since  $f(x,y,z) \neq xyz$ ,  $p_3(\mathcal{A}) \geq 2$ . On the other hand, according to the results in sections 1 and 2, we see that (1),(2) and (3) of  $\text{Id}(\underline{K}_1)$  imply that all the forms of ternary polynomials in category A (see section 2) are the same and equal to  $xyz$ .

Moreover, from (2) and (4) of  $\text{Id}(\underline{K}_1)$ , it follows that all the forms of the ternary polynomials in categories B and C are the same and equal to  $f(x,y,z)$ . Hence,  $p_3(\mathcal{A}) = 2$ , proving our theorem.

Theorem 3.3 (Type II).

Let  $\mathcal{A} = \langle A; \cdot, f \rangle$  be an algebra of type  $\langle 2,3 \rangle$  where " $\cdot$ " is the semilattice operation and  $f(x,y,z)$  is the ternary operation satisfying  $\text{Id}(\underline{K}_2)$  of Theorem 3.1. Then  $\mathcal{A}$  represents  $\langle 0,0,1,2 \rangle$ .

Proof : In analogy to the proof of Theorem 3.2, we see that

(1) and (2) of  $\text{Id}(\underline{K}_2)$  imply that  $\mathcal{A}$  represents  $\langle 0,0,1 \rangle$ .

To prove that  $p_3(\mathcal{A}) = 2$ , observe that (1),(2) and (3) guarantee that all the forms of the ternary polynomials in category A are all the same and equal to  $xyz$ . Furthermore, (4) of  $\text{Id}(\underline{K}_2)$  implies that all the forms of the ternary polynomials in category B are all the same and equal to  $xyz$ . Finally, (2),(4), (5),(6) and (7) imply that all the forms of the ternary polynomials in category C are all the same and again equal to  $xyz$ . Hence,  $p_3(\mathcal{A}) = 2$ , as was to be shown.

Combining the above three results, we have the following characterization theorem.

Theorem 3.4

There exist two equational classes of algebras  $\underline{K}_1$  and  $\underline{K}_2$

such that an algebra  $\mathcal{A}$  represents the sequence  $\langle 0,0,1,2 \rangle$  if, and only if  $\mathcal{A}$  can be represented as an algebra  $\langle A; \cdot, f \rangle$  of type  $\langle 2,3 \rangle$  where " $\cdot$ " is the semilattice operation belonging to either  $\underline{K}_1$  or  $\underline{K}_2$ .

Applying our main Theorem, some simple results which show the behavior of the sequence  $\langle 0,0,1,2, p_4, \dots, p_n, \dots \rangle$  can easily be derived.

In [13], G. Grätzer and J. Płonka proved the following result: Let  $\mathcal{A}$  be an idempotent algebra having a commutative and associative binary polynomial. If  $p_n(\mathcal{A}) \neq 1$  ( $n \geq 2$ ) then  $p_{n+1}(\mathcal{A}) \geq p_n(\mathcal{A}) + 1 + \max. \{ p_n(\mathcal{A}), n+1 \}$

From this, we have

Corollary 3.5

Let  $\mathcal{A}$  be an algebra representing  $\langle 0,0,1,2 \rangle$ . Then  $p_n(\mathcal{A}) \geq 2^{n-1} - 1$  for all  $n \geq 4$ .

Proof: We prove the corollary by induction on  $n$ .

If  $n=4$ , then  $p_4(\mathcal{A}) \geq p_3(\mathcal{A}) + 1 + \max. \{ p_3(\mathcal{A}), 4 \}$   
 $= 7 = 2^{4-1} - 1$ .

Assume the statement is true for  $n=k$ , that is  $p_k(\mathcal{A}) \geq 2^{k-1} - 1$ .

Consider the case when  $n=k+1$ . We have

$$\begin{aligned} p_{k+1}(\mathcal{A}) &\geq p_k(\mathcal{A}) + 1 + \max. \{ p_k(\mathcal{A}), k+1 \} \\ &= 2p_k(\mathcal{A}) + 1 \\ &\geq 2(2^{k-1} - 1) + 1 \\ &= 2^k - 1. \end{aligned}$$

Hence the Corollary follows.

Corollary 3.6 .

Let  $\mathcal{A}$  be an algebra representing  $\langle 0,0,1,2 \rangle$  . Then for each integer  $k$ ,  $p_n(\mathcal{A}) > k$  for all but finitely many  $n$ .

Corollary 3.7 .

The sequence  $\langle 0,0,1,2,p_4(\mathcal{A}),p_5(\mathcal{A}), \dots,p_n(\mathcal{A}),\dots \rangle$  is unbounded.

Following [13], we say that a sequence  $\langle p_i \rangle$  is conditionally strictly increasing (C.S.I) if  $1 \leq p_i \leq \aleph_0$  implies  $p_i < p_{i+1}$ . Thus, we have

Corollary 3.8 .

Let  $\mathcal{A}$  be an algebra representing  $\langle 0,0,1,2 \rangle$  . Then  $\langle p_n(\mathcal{A}),p_{n+1}(\mathcal{A}), \dots \rangle$  is C.S.I. for each  $n \geq 1$ .

## CHAPTER 2.

### ALGEBRAS REPRESENTING $\langle 0,0,1,2 \rangle$ ( CONTINUED )

The study of the two equational classes of algebras representing  $\langle 0,0,1,2 \rangle$  is continued in this chapter. By making use of previous results, the structures of those algebras representing  $\langle 0,0,1,2 \rangle$  will be considered here.

This chapter falls into five sections. Though Theorem 3.4 (I,1) characterizes those algebras representing  $\langle 0,0,1,2 \rangle$ , whether these exist algebras in  $\underline{K}_1$  or  $\underline{K}_2$  has not so far been discussed. In section 1, we establish two Existence Theorems, one for each equational class. Several algebras representing  $\langle 0,0,1,2 \rangle$  will be furnished and some relations between them will be indicated in section 2. The major result of this chapter states that if  $\mathcal{K}$  is an algebra representing  $\langle 0,0,1,2 \rangle$  then  $\mathcal{K}$  contains one of the eight algebras as a subalgebra. This is shown in section 3. Applying this main result, we are able to provide in section 4 a lower bound for  $\langle p_n(\mathcal{K}) \rangle$  which is much stronger than that in Corollary 3.5 (I,1). Finally, finite subdirectly irreducible algebras in  $\underline{K}_1$  and  $\underline{K}_2$  will be studied in section 5.

#### 1. Existence Theorems.

Let  $K(\tau)$  be the class of all algebras of type  $\tau$ . The  $n$ -ary polynomial algebra,

$$\mathcal{B}^{(n)}(\tau) = \langle P^{(n)}(\tau); F \rangle$$

where the underlying set  $P^{(n)}(\tau)$  is the set of all  $n$ -ary polynomial symbols and the operations on  $P^{(n)}(\tau)$  are defined in a natural way (see [ 6 ] ), is known to be an element of  $K(\tau)$ .

Let us now confine ourselves to a special case where  $\tau = \langle 2,3 \rangle$  and  $n=3$ . In this situation, we have an algebra in  $K(\langle 2,3 \rangle)$ , namely,

$$\mathcal{B}^{(3)}(\langle 2,3 \rangle)$$

Denote by "." and "f" the binary and ternary operations of  $\mathcal{B}^{(3)}(\langle 2,3 \rangle)$  respectively. Consider the following set  $\Sigma_1$  of identities:

$$\Sigma_1 : \left\{ \begin{array}{l} (1) \quad x \cdot x = x \\ (2) \quad (x \cdot y)_z = (yz)x \\ (3) \quad f(x,y,z) = f(y,x,z) = f(y,z,x) \\ (4) \quad f(xy,y,z) = xyz \\ (5) \quad f(xy,yz,zx) = xyz \\ (6) \quad f(x,y,z)x = f(x,y,z) \end{array} \right.$$

Remark : It is proved in [ 24 ] that the two identities (1) and (2) of  $\Sigma_1$  characterize the semilattice operation.

We shall now define a binary relation  $\Theta$  on  $\mathcal{B}^{(3)}(\langle 2,3 \rangle)$  as follows : For any two elements  $p, q$  in  $P^{(3)}(\langle 2,3 \rangle)$ , we put  $p \equiv q(\Theta)$  if, and only if the identity  $p=q$  is provable from the set  $\Sigma_1$ .

It turns out that  $\Theta$  is an equivalence relation on  $P^{(3)}(\langle 2,3 \rangle)$ .

Furthermore, one can check that  $\mathbb{H}$  has the Substitution Property.

Thus,  $\mathbb{H}$  is a congruence relation on the algebra  $\mathcal{B}^{(3)}(\langle 2,3 \rangle)$ .

From this, we get a quotient algebra, namely,

$$\mathcal{B}^{(3)}(\langle 2,3 \rangle) / \mathbb{H}$$

Now, by making use of the results in Chapter 1 (I) and the identities of  $\Sigma_1$ , we can describe the set of all ternary polynomials over the algebra  $\mathcal{B}^{(3)}(\langle 2,3 \rangle) / \mathbb{H}$  which turns out to be the following:

$$\left\{ \begin{array}{lll} x & y & z \\ xy & yz & zx \\ xyz & & f(x,y,z) \end{array} \right.$$

Evidently,  $xyz$  and  $f(x,y,z)$  are the only essentially ternary polynomials. Thus  $P_3(\mathcal{B}^{(3)}(\langle 2,3 \rangle) / \mathbb{H}) = 2$ . The only essentially binary polynomial is  $xy$  and there are no constants and no unary polynomials which are distinct from the projections. Consequently, we have the following :

Theorem 1.1 (Existence Theorem of Type 1 Algebras)

The algebra  $\mathcal{B}^{(3)}(\langle 2,3 \rangle) / \mathbb{H}$  represents the sequence  $\langle 0,0,1,2 \rangle$ . Moreover,  $\mathcal{B}^{(3)}(\langle 2,3 \rangle) / \mathbb{H} \in \mathcal{K}_1$ .

On the other hand, instead of the set  $\Sigma_1$ , let us take the following:

$$\Sigma_2 : \left\{ \begin{array}{l} (1) \quad x \cdot x = x \\ (2) \quad (x \cdot y)z = (yz)x \\ (3) \quad f(x,y,z) = f(y,x,z) = f(y,z,x) \\ (4) \quad f(xy,y,z) = xyz \\ (5) \quad f(xy,yz,zx) = xyz \\ (6) \quad f(x,y,z)x = xyz \\ (7) \quad f(f(x,y,z),y,z) = xyz \\ (8) \quad f(f(x,y,z),xy,z) = xyz \\ (9) \quad f(f(x,y,z),xy,xz) = xyz. \end{array} \right.$$

Moreover, instead of  $\mathbb{H}$ , we define a binary relation  $\Phi$  on  $\mathcal{S}^{(3)}(\langle 2,3 \rangle)$  as follows: For any elements  $p, q$  in  $\mathcal{P}^{(3)}(\langle 2,3 \rangle)$  we put  $p \equiv q (\Phi)$  if, and only if the identity  $p = q$  is provable from the set  $\Sigma_2$ .

In analogy to the first case, it follows that  $\Phi$  is a congruence relation on  $\mathcal{S}^{(3)}(\langle 2,3 \rangle)$  and the algebra  $\mathcal{S}^{(3)}(\langle 2,3 \rangle) / \Phi$  has the following properties.

Theorem 1.2. (Existence Theorem of Type 2 Algebras).

The algebra  $\mathcal{S}^{(3)}(\langle 2,3 \rangle) / \Phi$  represents the sequence  $\langle 0,0,1,2 \rangle$ . Moreover  $\mathcal{S}^{(3)}(\langle 2,3 \rangle) / \Phi \in \mathcal{K}_2$ .

2. Other Examples.

In this section, we shall construct eight algebras  $I(j)$ ,  $II(j)$ ,  $j = 1,2,3,4$ , where  $5 \leq |I(j)|$ ,  $|II(j)| \leq 8$ , four for each equational class  $\mathcal{K}_i$ ,  $i = 1,2$  and each of which represents the



sequence  $\langle 0,0,1,2 \rangle$  .

(A) Examples in  $\underline{K}_1$ .

1) Algebra  $I(1)$  where  $|I(1)| = 5$

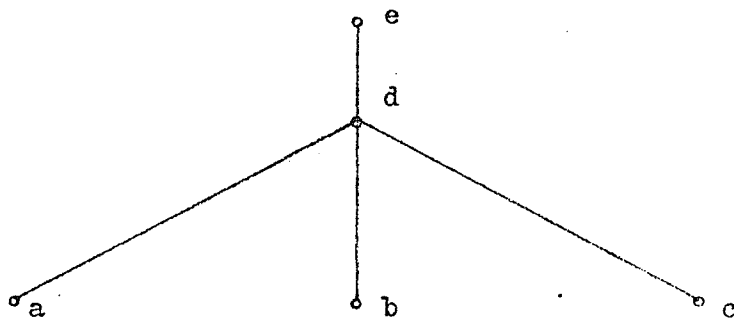


Fig. 1

2) Algebra  $I(2)$  where  $|I(2)| = 6$

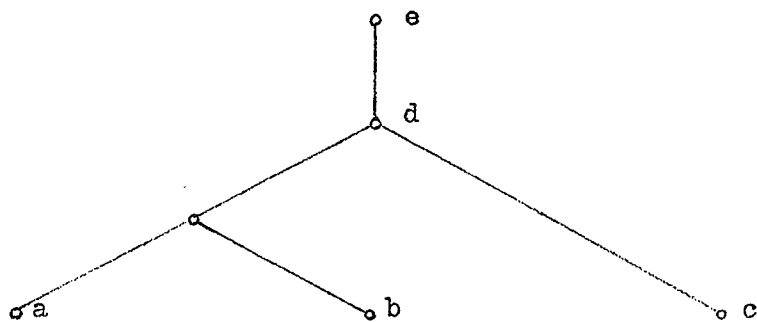


Fig. 2

3) Algebra  $I(3)$  where  $|I(3)| = 7$

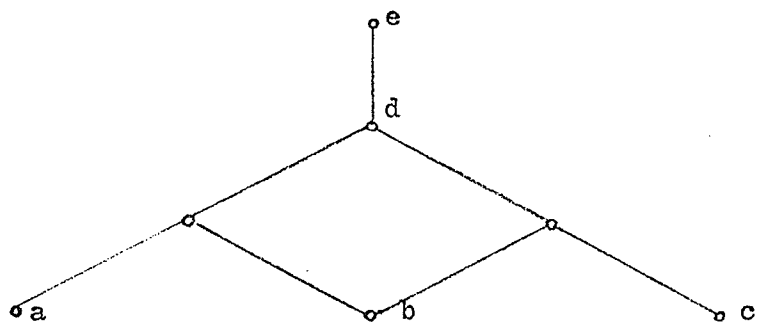


Fig. 3

4) Algebra  $I(4)$  where  $|I(4)| = 8$ .

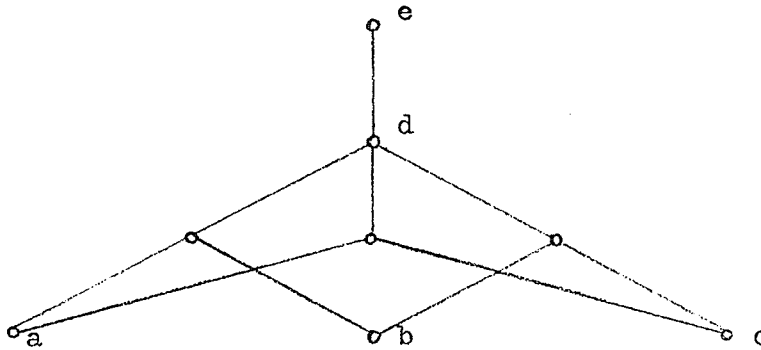


Fig. 4

For each  $j = 1, 2, 3, 4$ ,  $I(j)$  is an algebra of type  $\langle 2, 3 \rangle$  where the base set is shown in Fig.  $j$ . The binary operation "." in  $I(j)$  regarded as a join semilattice operation while the ternary operation "f" is defined as follows:

$$f(x, y, z) = \begin{cases} e & \text{if } \{x, y, z\} = \{a, b, c\} \\ xyz & \text{otherwise} \end{cases} .$$

It follows immediately from the above definition of  $f$  that  $f(x, y, z)$  is an essentially ternary polynomial over  $I(j)$  and  $f(x, y, z) \neq xyz$ . To show that each algebra  $I(j)$ ,  $j = 1, 2, 3, 4$  is an element in  $\underline{K}_1$ , we have to show by Theorem 3.2 (I, 1) that the ternary operation "f" defined above satisfies the set  $\text{Id}(\underline{K}_1)$  of Theorem 3.1 (I, 1). We shall now give a proof for the algebra  $I(4)$ . The other three can be proved in a similar way.

Clearly,  $f(x, y, z)$  is symmetric. Thus (1) of  $\text{Id}(\underline{K}_1)$  holds. To see that  $f(xy, y, z) = xyz$ , we note that  $\{xy, y, z\} (S) \neq \{a, b, c\}$  for any substitution  $S$ . For if  $\{xy, y, z\} (S) = \{a, b, c\}$  then  $(xy)(S) = a$  say, and we would get  $y(S) = a$ , a contradiction.

Thus by definition,  $f(xy,y,z) = xy.y.z = xyz$ , which was to be shown. Similarly, we have  $f(xy,yz,zx) = xyz$ . Finally, we claim that  $f(x,y,z)x = f(x,y,z)$ . To this end, observe that if  $\{x,y,z\} (S) = \{a,b,c\}$ , then  $f(a,b,c)a = ea = e = f(a,b,c)$ . If  $\{x,y,z\} (S) \neq \{a,b,c\}$ , then  $(f(x,y,z)x)(S) = [(xyz)x] (S) = (xyz)(S) = f(x,y,z)(S)$ . Thus the identity  $f(x,y,z)x = f(x,y,z)$  follows.

Hence, we have

Theorem 2.1.

For each  $j = 1,2,3,4$ ,  $I(j) \in \underline{K}_1$ .

(B) Example in  $\underline{K}_2$ .

1) Algebra II(1) where  $|II(1)| = 5$

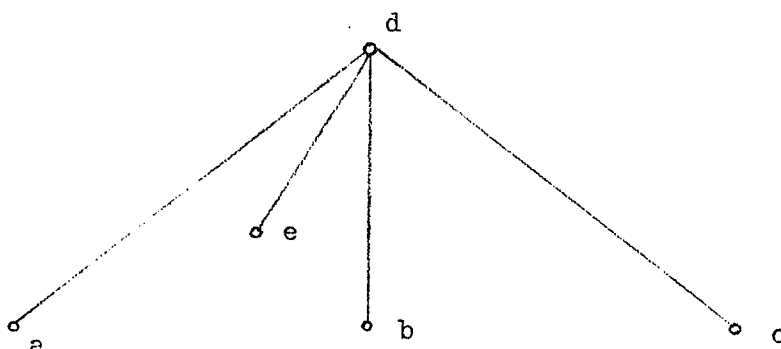


Fig. 5.

2) Algebra II(2) where  $|II(2)| = 6$

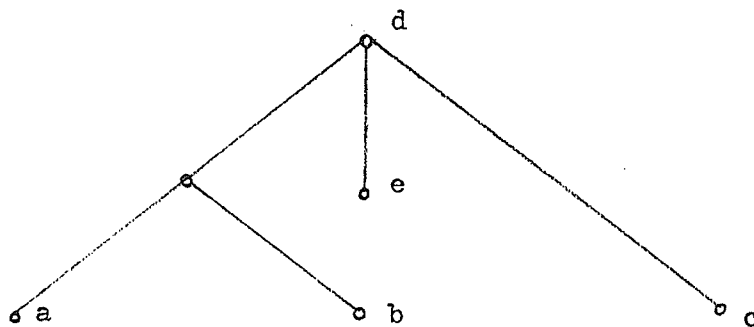


Fig. 6.

3) Algebra  $II(3)$  where  $|II(3)| = 7$

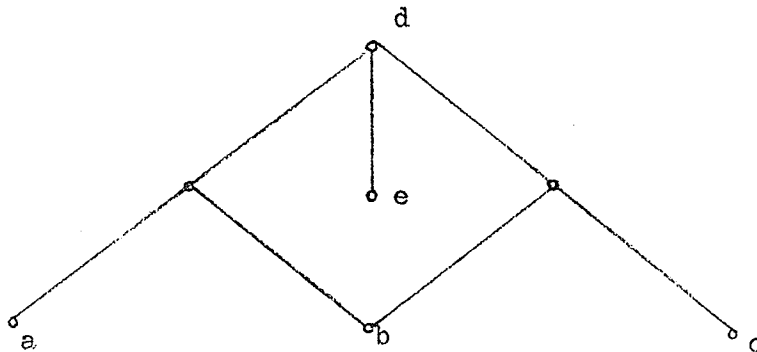


Fig. 7.

4) Algebra  $II(4)$  where  $|II(4)| = 8$

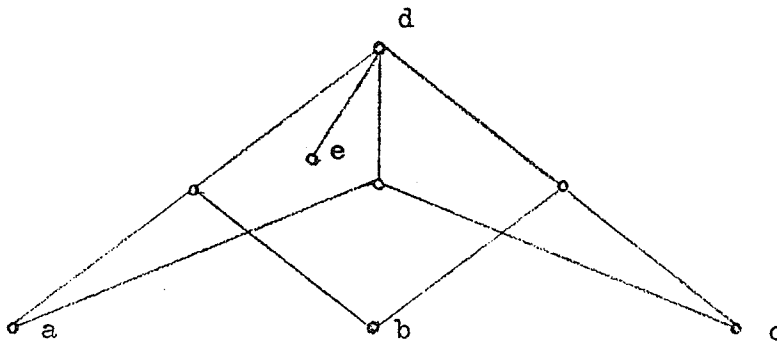


Fig. 8.

For each  $j = 1, 2, 3, 4$ ,  $II(j)$  is an algebra of type  $\langle 2, 3 \rangle$  where the base set is shown in Fig(4+j). The binary operation "." in  $II(j)$  is regarded as a join semilattice operation while the ternary operation is defined as follows:

$$f(x, y, z) = \begin{cases} e & \text{if } \{x, y, z\} = \{a, b, c\} \\ xyz & \text{otherwise} \end{cases}$$

Clearly,  $f(x, y, z)$  is an essentially ternary polynomial over  $II(j)$  and  $f(x, y, z) \neq xyz$ . By using the similar argument as before, it can be shown that the set  $Id(\underline{K}_2)$  of Theorem 3.1 (I, 1) is

satisfied by "f" and thus we get

Theorem 2.2.

For each  $j = 1, 2, 3, 4$ ,  $II(j) \in \underline{K}_2$ .

It is quite interesting to note that the structures of the four algebras, for each equational class  $\underline{K}_i$ ,  $i = 1, 2$  are closely related. Thus, it is perhaps worthwhile to point out some relationships between them.

Remark. 1

Observe that the cardinality of the algebra  $\mathcal{B}^{(3)}(\langle\langle 2,3 \rangle\rangle) / \Theta$  shown in section 1 is eight. In fact, this algebra is isomorphic to the algebra  $I(4)$  and it turns out that both of them are the free algebra over  $\underline{K}_1$  with the free generating set which consists three unordered elements. In notation,

$$\mathcal{B}^{(3)}(\langle\langle 2,3 \rangle\rangle) / \Theta \cong F_{\underline{K}_1}(3) \cong I(4).$$

Similarly, we have

$$\mathcal{B}^{(3)}(\langle\langle 2,3 \rangle\rangle) / \Phi \cong F_{\underline{K}_2}(3) \cong II(4)$$

Remark 2.

It is easily seen that for each  $j = 1, 2, 3, 4$ ,  $I(j) - \{e\}$ , considered as a semilattice, is a homomorphic image of the free semilattice  $I(4) - \{e\}$ . Moreover, they are the only homomorphic images of  $I(4) - \{e\}$ .

The same relation holds for  $II(j) - \{e\}$  and  $II(4) - \{e\}$ .

Remark 3.

If we consider only the semilattice structure, then it is clear that  $I(1)$  is isomorphic to a sub-semilattice of  $I(4)$ .

Indeed,  $I(1)$  can be embedded in  $I(4)$  and is isomorphic to the five-element sub-semilattice  $\{ab, bc, ca, d, e\}$  of  $I(4)$ .

However  $I(1)$  is no longer a subalgebra of  $I(4)$  if we consider both of them as algebras of type  $\langle 2, 3 \rangle$ , for  $f(ab, bc, ca) = abc = d$  in  $I(4)$  while in  $I(1)$ , we have  $f(a, b, c) = e$ .

The same remark is true for  $II(1)$  and  $II(4)$ .

### 3. A Theorem on Subalgebras.

The eight algebras  $I(j)$ ,  $II(j)$ ,  $j = 1, 2, 3, 4$ , play central roles in the study of the equational classes  $\underline{K}_1$  and  $\underline{K}_2$ . In Lattice Theory, it is well-known that a lattice is non-distributive if and only if it contains  $M_5$  or  $N_5$  as sublattices. In our case, we have a similar result for the "only if" part.

This can be seen from the following

#### Theorem 3.1.

Let  $\mathcal{A}$  be an algebra of type  $\langle 2, 3 \rangle$  which represents  $\langle 0, 0, 1, 2 \rangle$ . If  $\mathcal{A} \in \underline{K}_1$ , then  $\mathcal{A}$  contains one of the  $I(j)$ ,  $j = 1, 2, 3, 4$  as subalgebra. If  $\mathcal{A} \in \underline{K}_2$ , then  $\mathcal{A}$  contains one of the  $II(j)$ ,  $j = 1, 2, 3, 4$  as subalgebra.

Proof: Let  $f$  be the ternary operation in  $\mathcal{A}$ . Since  $f(x, y, z) \neq xyz$ , there exist  $a, b, c \in A$  such that

$$f(a, b, c) \neq abc \quad \text{in } A.$$

Claim 1.  $a, b, c$  are pairwise distinct.

If this is not the case, say  $a = b$ , then we would have

$$\begin{aligned}
f(a,b,c) &= f(a,a,c) \\
&= ac \\
&= abc
\end{aligned}$$

which is a contradiction. Hence  $a, b$  and  $c$  are pairwise distinct.

A partial ordering can be defined in  $A$  in a natural way, namely,  $p \leq q$  if and only if  $pq = q$ .

Claim 2. The set  $\{a, b, c\}$  is unordered.

Otherwise, say  $b \leq a$ , i.e.,  $ab = a$ , then it follows that

$$f(a,b,c) = f(ab,b,c) = abc,$$

a contradiction. The other possible cases can be proved similarly.

Thus,  $\{a, b, c\}$  is unordered.

Now, if we set  $d = abc$ ,  $e = f(a,b,c)$ , we assert that  $d$  and  $e$  are comparable.

Indeed, if  $\mathcal{K} \in \underline{K}_1$ , then  $f(x,y,z)xyz = f(x,y,z)$  holds in  $\mathcal{K}$ . From this, it follows that  $e.d = e$  and so  $e > d$ .

If  $\mathcal{K} \in \underline{K}_2$ , then  $f(x,y,z)xyz = xyz$  holds in  $\mathcal{K}$ . Thus we get  $ed = d$ , i.e.,  $e < d$ , as required.

Now,  $\mathcal{K}$ , being an algebra containing  $a, b$  and  $c$ , must contain the subalgebra generated by  $\{a, b, c\}$ .

If  $\mathcal{K} \in \underline{K}_1$ , then  $e > d$  and the subalgebra generated by  $\{a, b, c\}$  must be one of the  $I(j)$ ,  $j = 1, 2, 3, 4$ .

If  $\mathcal{K} \in \underline{K}_2$ , then  $e < d$ . Observe that  $e$  and each of the elements in  $\{a, b, c\}$  are incomparable. For :

$$f(a,b,c)a = abc = d$$

i.e.,  $ea = d$

If  $e$  and  $a$  are comparable then either  $e = d$  or  $e = a$  which are impossible. Thus, knowing this fact, it is easy to see that the subalgebra generated by  $\{a,b,c\}$  is one of the  $II(j)$ ,  $j = 1,2,3,4$ . This completes the proof of the theorem.

The converse of Theorem 3.1 is tempting, however false, in general. In what follows, we shall construct two counter examples.

Example 1.

Consider the following algebra  $\mathcal{A} = \langle A; \cdot, f \rangle$  (see Fig. 9) where " $\cdot$ " is the semilattice operation and  $f$  is the ternary operation defined by

$$f(x,y,z) = \begin{cases} e & \text{if } \{x,y,z\} = \{a,b,c\} \\ e' & \text{if } \{x,y,z\} = \{a',b,c\} \\ xyz & \text{otherwise} \end{cases}$$

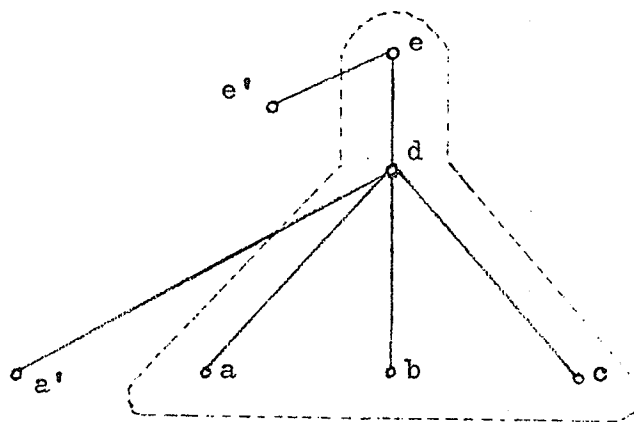


Fig. 9.

It is easy to check that  $\mathcal{A}$  contains  $I(1)$  as subalgebra. But  $\mathcal{A} \notin K_1$  since



$$f(a',b,c)a' = e'a' = e \neq e' = f(a',b,c),$$

i.e.,  $f(x,y,z)x = f(x,y,z)$  does not hold in  $\mathcal{K}$ .

Example 2.

Consider the following algebra  $\mathcal{K} = \langle A; \cdot, f \rangle$  (see Fig. 10) where " $\cdot$ " is the semilattice operation and the ternary operation  $f$  is defined as follows:

$$f(x,y,z) = \begin{cases} e & \text{if } \{x,y,z\} = \{a,b,c\} \\ e' & \text{if } \{x,y,z\} = \{a',b,c\} \\ xyz & \text{otherwise.} \end{cases}$$

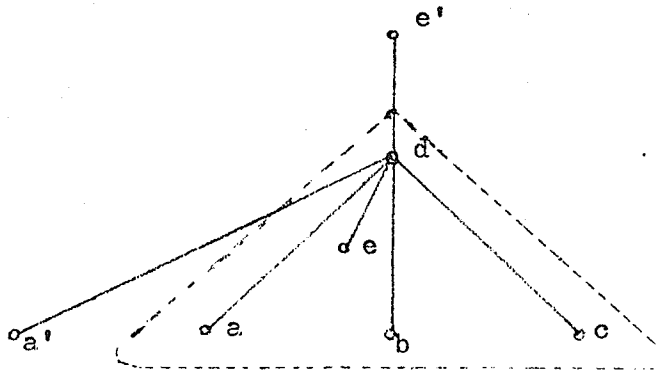


Fig. 10.

Clearly,  $\text{II}(1)$  is a subalgebra of  $\mathcal{K}$ . However,  $\mathcal{K} \notin \underline{\mathbb{K}}_2$

since  $f(a',b,c)a' = e'a' = e' \neq d = a'bc$ ,

i.e.,  $f(x,y,z)x = xyz$  does not hold in  $\mathcal{K}$ .

In view of Theorem 3.1, in order to get some information about the sequence  $\langle p_n(\mathcal{K}) \rangle$  where  $\mathcal{K}$  is an algebra representing  $\langle 0,0,1,2 \rangle$ , it is necessary to study the relationship between the sequences  $\{p_n(I(j)), p_n(\text{II}(j))\}$ ,  $j = 1,2,3,4$ .

There are redundant essential polynomials over  $I(j)$  and  $\text{II}(j)$ .

Indeed, if for  $n \geq 2$ , we let  $p(x_1, \dots, x_n)$  be a non-trivial  $n$ -ary polynomial (i.e., with exactly  $n$  variables). Then we have

Lemma 3.2.

For each  $j = 1, 2, 3, 4$ ,  $p(x_1, \dots, x_n)$  is an essentially  $n$ -ary polynomial over  $I(j)$  and  $II(j)$ .

Proof : It suffices to prove that  $p(x_1, \dots, x_n)$  depends on  $x_1$ . Observe that if we put  $x_1 = \dots = x_n = a$  (see Fig. 1 - 8), then by the idempotence of "." and "f", it follows that  $p(a, \dots, a) = a$ . On the other hand, if we set  $x_1 = d$ ,  $x_2 = \dots = x_n = a$ , then by using an inductive argument, we will get  $p(d, a, \dots, a) = d$  in  $I(j)$  and  $II(j)$ . Thus, as  $p(a, a, \dots, a) \neq p(d, a, \dots, a)$ ,  $p(x_1, \dots, x_n)$  depends on  $x_1$ , as required.

Corollary 3.3.

Let  $\mathcal{K}$  be an algebra representing  $\langle 0, 0, 1, 2 \rangle$ . Then  $p(x_1, \dots, x_n)$  is essential over  $\mathcal{K}$ .

Proof : Let us note that if  $\mathcal{B}$  is a subalgebra of  $\mathcal{K}$ , then if  $p$  is an essentially  $n$ -ary polynomial over  $\mathcal{B}$ , so is  $p$  over  $\mathcal{K}$ . Combining this fact with Theorem 3.1 and Lemma 3.2, the Corollary follows.

Let  $p$  be a polynomial over  $\mathcal{K}$ . For simplicity, we denote by  $p_{\mathcal{K}}(S)$ , an element in  $A$  which is obtained from  $p$  under the

substitution  $S$ .

Lemma 3.4.

Let  $\mathcal{B}$  be a homomorphic image of  $\mathcal{A}$ . Then  $p_n(\mathcal{B}) \leq p_n(\mathcal{A})$ , for each  $n = 0, 1, 2, \dots$ .

Proof : Let  $\varphi: \mathcal{A} \rightarrow \mathcal{B}$  be a homomorphism of  $\mathcal{A}$  onto  $\mathcal{B}$ .

Let  $p(x_1, \dots, x_n)$  be an essentially  $n$ -ary polynomial over  $\mathcal{B}$ .

In what follows, we shall prove that  $p(x_1, \dots, x_n)$  is an essentially  $n$ -ary polynomial over  $\mathcal{A}$ . It suffices to prove that  $p$

depends on  $x_1$ . By assumption, there exist  $b_1, b_1', b_2, \dots, b_n \in \mathcal{B}$

such that  $p_{\mathcal{B}}(b_1, b_2, \dots, b_n) \neq p_{\mathcal{B}}(b_1', b_2, \dots, b_n)$ . As  $\varphi$  is onto,

there exist  $a_1, a_1', a_2, \dots, a_n \in \mathcal{A}$  with  $a_i \varphi = b_i$   $i = 1, \dots, n$

and  $a_1' \varphi = b_1'$ . Thus  $p_{\mathcal{A}}(a_1, \dots, a_n) \varphi = p_{\mathcal{B}}(a_1 \varphi, \dots, a_n \varphi)$

$= p_{\mathcal{B}}(b_1, b_2, \dots, b_n) \neq p_{\mathcal{B}}(b_1', b_2, \dots, b_n) = p_{\mathcal{B}}(a_1' \varphi, a_2 \varphi, \dots, a_n \varphi)$

$= p_{\mathcal{A}}(a_1', a_2, \dots, a_n) \varphi$ . Hence  $p_{\mathcal{A}}(a_1, a_2, \dots, a_n) \neq p_{\mathcal{A}}(a_1', a_2, \dots, a_n)$ ,

as was to be shown.

Now, let  $p, q$  be two distinct essentially  $n$ -ary polynomials over  $\mathcal{B}$ . Then there is a substitution  $S$  such that  $p_{\mathcal{B}}(S) \neq q_{\mathcal{B}}(S)$ .

Let  $T \subseteq \mathcal{A}$  with  $T \varphi = S$ . Then  $p_{\mathcal{A}}(T) \varphi = p_{\mathcal{B}}(T \varphi) = p_{\mathcal{B}}(S) \neq q_{\mathcal{B}}(S)$

$= q_{\mathcal{B}}(T \varphi) = q_{\mathcal{A}}(T) \varphi$ . Hence  $p_{\mathcal{A}}(T) \neq q_{\mathcal{A}}(T)$  and so  $p \neq q$  over  $\mathcal{A}$ .

From these, it follows that  $p_n(\mathcal{B}) \leq p_n(\mathcal{A})$  for each  $n$ , proving Lemma 3.4.

Theorem 3.5.

$$(1) \quad p_n(I(1)) = p_n(I(2)) = p_n(I(3)) = p_n(I(4));$$

$$(2) \quad p_n(II(1)) = p_n(II(2)) = p_n(II(3)) = p_n(II(4)),$$

for each  $n = 0, 1, 2, \dots$

Proof : We will prove only (2). The proof of (1) is similar.

It is easy to check that for  $i \leq j$ ,  $i, j = 1, 2, 3, 4$ ,  $II(i)$  is a homomorphic image of  $II(j)$ . Thus, invoking Lemma 3.4, we have  $p_n(II(1)) \leq p_n(II(2)) \leq p_n(II(3)) \leq p_n(II(4))$ , for each  $n$ . Hence, to get (2), it suffices to prove that

$$p_n(II(4)) \leq p_n(II(1)), \text{ for each } n.$$

To this end, let  $p, q$  be any two essentially  $n$ -ary polynomials with  $p \neq q$  over  $II(4)$ . By Corollary 3.3,  $p$  and  $q$  are essentially  $n$ -ary polynomials over  $I(1)$ . Thus, what we have to prove is that  $p \neq q$  over  $I(1)$ .

As  $p \neq q$  over  $II(4)$ , there exist a non-trivial substitution  $S$  such that  $p_{II(4)}(S) \neq q_{II(4)}(S)$  (see Fig. 8). Observe that  $p_{II(4)}(S), q_{II(4)}(S) \notin \{a, b, c\}$  since  $S$  is non-trivial. Hence, by symmetry, we have only four possible cases:

$$\begin{cases} p_{II(4)}(S) \\ q_{II(4)}(S) \end{cases} = \begin{cases} (1) & \begin{cases} d \\ ab \end{cases} \\ (2) & \begin{cases} e \\ ab \end{cases} \\ (3) & \begin{cases} ab \\ ac \end{cases} \\ (4) & \begin{cases} d \\ e \end{cases} \end{cases}$$

Suppose (1) holds, i.e.,

$$\begin{aligned} p_{II(4)}(S) &= d \\ q_{II(4)}(S) &= ab. \end{aligned}$$

Then, since  $q_{II(4)}(S) = ab$ , we have  $S \subseteq \{a,b,ab\}$ . However, in this case,  $p_{II(4)}(S) < d$ , which is a contradiction. Thus, (1) is impossible.

Suppose (2) holds, i.e.,

$$p_{II(4)}(S) = e$$

$$q_{II(4)}(S) = ab.$$

Since  $p_{II(4)}(S) = e$  and  $S$  is non-trivial; it follows that

$$\{a,b,c\} \subseteq S.$$

On the other hand,  $q_{II(4)}(S) = ab$  implies

$$S \subseteq \{a,b,ab\}.$$

Combining the two inclusions, we have

$$\{a,b,c\} \subseteq \{a,b,ab\},$$

which is a contradiction. Thus, (2) is impossible.

Suppose (3) holds, i.e.,

$$p_{II(4)}(S) = ab$$

$$q_{II(4)}(S) = ac.$$

Clearly,  $p_{II(4)}(S) = ab$  implies  $S \subseteq \{a,b,ab\}$  and

$$q_{II(4)}(S) = ac \text{ implies } S \subseteq \{a,c,ac\}$$

Hence,  $S \subseteq \{a,b,ab\} \cap \{a,c,ac\} = \{a\}$  and so  $S = \{a\}$ ,

which contradicts the fact that  $S$  is non-trivial. Thus, (3) is impossible.

In conclusion, we must have (4), i.e.,

$$p_{II(4)}(S) = d$$

$$q_{II(4)}(S) = e.$$

Since  $q_{II(4)}(S) = e$  and  $S$  is non-trivial, it follows that

$$S \subseteq \{a, b, c, e\}.$$

In this situation, we can use the same substitution  $S$  for  $II(1)$

as  $\{a, b, c, e\} \subseteq II(1)$ . Clearly, we have

$$p_{II(1)}(S) = d$$

$$\text{and } q_{II(1)}(S) = e.$$

Hence  $p \neq q$  over  $II(1)$ , which was to be shown.

Theorem 3.6.

Let  $p(x_1, \dots, x_n), q(x_1, \dots, x_n)$  be two  $n$ -ary polynomials.

Then  $p \neq q$  over  $\mathcal{K}$ , for each algebra  $\mathcal{K}$  representing  $\langle 0, 0, 1, 2 \rangle$

if and only if  $p \neq q$  over  $I(1)$  and  $II(1)$ .

Proof: One implication is trivial. Thus, assume that  $p \neq q$  over  $I(1)$  and  $II(1)$ . Let  $\mathcal{K}$  be an arbitrary algebra representing

$\langle 0, 0, 1, 2 \rangle$ , we shall prove that  $p \neq q$  over  $\mathcal{K}$ .

By Theorem 3.1,  $\mathcal{K}$  contains one of the algebras  $I(j), II(j)$   $j = 1, 2, 3, 4$  as subalgebra. Thus, if we know that

$$p \neq q \text{ over } I(j), II(j), j = 1, 2, 3, 4,$$

it is clear that  $p \neq q$  over  $\mathcal{K}$ .

Observe that  $I(1), II(1)$  are homomorphic images of  $I(j), II(j), j = 1, 2, 3, 4$  respectively. Hence, if  $p \neq q$  over  $I(1)$  and  $II(1)$ , then  $p \neq q$  over  $I(j)$  and  $II(j)$  for each  $j = 1, 2, 3, 4$ , by Lemma 3.4. Therefore the proof of Theorem 3.6 is complete.

Theorem 3.7.

Let  $\mathcal{A}$  be an algebra representing  $\langle 0,0,1,2 \rangle$ . Then  
 $p_n(\mathcal{A}) \geq \min. \{ p_n(I(1)), p_n(II(1)) \}$  for each  $n = 0,1,2, \dots$ .

Proof : If  $\mathcal{A} \in \underline{K}_1$  then  $\mathcal{A}$  contains  $I(j)$  as a subalgebra, for some  $j = 1,2,3,4$ . Thus

$$p_n(\mathcal{A}) \geq p_n(I(j)) = p_n(I(1)) \geq \min \{ p_n(I(1)); p_n(II(1)) \}$$

If  $\mathcal{A} \in \underline{K}_2$ , then there exists a  $j = 1,2,3,4$  such that

$$p_n(\mathcal{A}) \geq p_n(II(j)) = p_n(II(1)) \geq \min. \{ p_n(I(1), p_n(II(1)) \}.$$

Hence the theorem follows.

4. A Lower Bound for  $p_n(\mathcal{A})$ .

Though Theorem 3.7 gives us a lower bound for  $\langle p_n(\mathcal{A}) \rangle$  in terms of  $\langle p_n(I(1)) \rangle$  and  $\langle p_n(II(1)) \rangle$ , we still do not know the exact rate of increase of the sequence. In this section, we shall fill this gap by providing a lower bound for  $\langle p_n(\mathcal{A}) \rangle$ . It turns out that this lower bound is much stronger than that in Corollary 3.5(II,1).

Our main result is the following:

Theorem 4.1.

Let  $\mathcal{A}$  be an algebra representing the sequence  $\langle 0,0,1,2 \rangle$ . Then  $p_n(\mathcal{A}) \geq 11 \cdot \frac{n!}{4!}$  for each  $n \geq 4$ .

The proof of Theorem 4.1 is based on the following

construction of polynomials and Lemmas 4.2, 4.3.

Construction of Polynomials.

It is a simple matter to check that the following eleven polynomials are essential and distinct over  $I(1)$  and  $II(1)$ .

Thus, by Theorem 3.6, it follows that for each algebra representing  $\langle 0,0,1,2 \rangle$ ,  $\mathcal{U}$  has at least eleven essentially 4 - ary polynomials.

$$\begin{aligned}
 & x_1 x_2 x_3 x_4, \\
 & f(x_1, x_2, x_3) x_4, f(x_2, x_3, x_4) x_1, f(x_3, x_4, x_1) x_2, f(x_4, x_1, x_2) x_3 \\
 & f(x_1 x_2, x_3, x_4), f(x_1 x_3, x_2, x_4), f(x_1 x_4, x_2, x_3) \\
 & f(x_2 x_3, x_1, x_4), f(x_2 x_4, x_1, x_3), f(x_3 x_4, x_1, x_2)
 \end{aligned}$$

Now, for each polynomial listed above, we claim that we can construct at least five 5-ary polynomials. For instance,

1) From  $x_1 x_2 x_3 x_4$ , we construct

$$\left[ \begin{array}{l}
 x_1 x_2 x_3 x_4 x_5 \\
 f(x_1, x_2 x_3 x_4, x_5) \quad f(x_1 x_2, x_3 x_4, x_5) \\
 f(x_2, x_1 x_3 x_4, x_5) \quad f(x_1 x_3, x_2 x_4, x_5) \\
 f(x_3, x_1 x_2 x_4, x_5) \quad f(x_1 x_4, x_2 x_3, x_5) \\
 f(x_4, x_1 x_2 x_3, x_5)
 \end{array} \right.$$

2) From  $f(x_1, x_2, x_3) x_4$ , we construct



$$\left[ \begin{array}{l} f(x_1, x_2, x_3) x_4 x_5 \\ f(x_1 x_5, x_2, x_3) x_4 \\ f(x_1, x_2 x_5, x_3) x_4 \\ f(x_1, x_2, x_3 x_5) x_4 \\ f(f(x_1, x_2, x_3), x_4, x_5) \end{array} \right.$$

3) From  $f(x_1 x_2, x_3, x_4)$ , we construct

$$\left[ \begin{array}{l} f(x_1 x_2, x_3, x_4) x_5 \\ f(x_1 x_2 x_5, x_3, x_4) \\ f(x_1 x_2, x_3 x_5, x_4) \\ f(x_1 x_2, x_3, x_4 x_5) \\ f(f(x_1, x_2, x_5), x_3, x_4) \end{array} \right.$$

Similar constructions can be given for the other polynomials.

Suppose that we are given a 5-ary polynomial  $p(x_1, \dots, x_5)$  which is obtained from one of the eleven 4-ary polynomials by using the above Construction. Applying the same argument, it is not difficult to construct six 6-ary polynomials out of  $p$ . For example,

1) If  $p = x_1 x_2 x_3 x_4 x_5$ , we construct

$$\left[ \begin{array}{l} \prod_{i=1}^5 x_i x_6 \\ f(x_1, \prod_{i=2}^5 x_i, x_6), f(x_1 x_2, x_3 x_4 x_5, x_6) \\ f(x_2, x_1 x_3 x_4 x_5, x_6), f(x_1 x_3, x_2 x_4 x_5, x_6) \\ \vdots \\ \vdots \end{array} \right.$$

2) If  $p = f(x_1, x_2, x_3)x_4x_5$ , we construct

$$\left[ \begin{array}{l} f(x_1, x_2, x_3)x_4x_5x_6 \\ f(x_1x_6, x_2, x_3)x_4x_5 \\ f(x_1, x_2x_6, x_3)x_4x_5 \\ f(x_1, x_2, x_3x_6)x_4x_5 \\ f(f(x_1, x_2, x_3), x_4x_5, x_6) \\ f(x_4, f(x_1x_2x_3)x_5, x_6) \\ f(x_5, f(x_1, x_2, x_3)x_4, x_6) \end{array} \right].$$

We are now in a position to construct, by using an inductive argument, at least  $n + 1$   $(n+1)$ -ary polynomials out of a given  $n$ -ary polynomial.

Let  $p(x_1, \dots, x_n)$  be such a  $n$ -ary polynomial. Consider the following three types of constructions :

(A)  $p(x_1, \dots, x_n) \cdot x_{n+1}$  ;

(B) If there is a factor  $f(A, B, C)$  in  $p$ , we set

1)  $f(A \cdot x_{n+1}, B, C)$ ,

2)  $f(A, B \cdot x_{n+1}, C)$ ,

3)  $f(A, B, C \cdot x_{n+1})$  in  $p(x_1, \dots, x_n)$ .

(C) If there is a product  $\prod_{\alpha \in \Lambda} A_\alpha$  in  $p(x_1, \dots, x_n)$  where  $|\Lambda| > 1$  and  $\Lambda$  is maximal (in the sense that if there is a product  $\prod_{\beta \in \Delta} A_\beta$  in  $p(x_1, \dots, x_n)$  with  $\Lambda \subseteq \Delta$  then  $\Lambda = \Delta$ ), we set

$f\left(\prod_{\alpha \in I} A_\alpha, \prod_{\alpha \in J} A_\alpha, x_{n+1}\right)$  in  $p(x_1, \dots, x_n)$   
 where  $\{I, J\}$  is a partition of the index set  $\Lambda$ .

Let us note that under these constructions, those  $(n+1)$ -ary polynomials have the following properties:

- 1) Let  $p(x_1, \dots, x_{n+1})$  be such a  $(n+1)$ -ary polynomial, the number of occurrences of each variable  $x_i$  in  $p$  is exactly one.
- 2) By Corollary 3.3, all such  $(n+1)$ -ary polynomials are essential.
- 3) All such  $(n+1)$ -ary polynomials are of different forms.

Lemma 4.2.

Let  $p(x_1, \dots, x_n)$  be an  $n$ -ary polynomial. Then the number of  $(n+1)$ -ary polynomials obtained by using the constructions (A), (B) and (C) is greater than or equal to  $n+1$ .

Proof : Let  $r$  be the number of occurrences of the symbol "f" in  $p(x_1, \dots, x_n)$ . Then, by applying the constructions (A) and (B) we have at least  $1 + 3r$   $(n+1)$ -ary polynomials.

If  $r \geq \frac{n}{3}$ , then  $1+3r \geq 1+n$ , and the proof is complete.

Thus, we may assume  $r < \frac{n}{3}$ , in other words,  $3r < n$ .

In order to have  $n+1$   $(n+1)$ -ary polynomials, we need  $(n+1) - (3r+1) = n-3r$  more  $(n+1)$ -ary polynomials. Observe that we have at most  $3r$  positions in all the forms  $f(\quad, \quad, \quad)$ , therefore

at least  $n-3r$  variables appear in the product form  $\prod A_i$  in  $p$ . Hence, by construction (C) we get at least  $n-3r$   $(n+1)$ -ary polynomials. This completes the proof of Lemma 4.2.

For a given  $n$ -ary polynomial which is constructed by induction, we obtain, by Lemma 4.2, at least  $n+1$   $(n+1)$ -ary polynomials different in forms. The question arises: are these  $(n+1)$ -ary polynomials distinct over algebra representing  $\langle 0,0,1,2 \rangle$ ? The answer to this question is in the affirmative. Indeed, this can be seen from the following somewhat stronger version:

Lemma 4.3.

Let  $\mathcal{A}$  be an algebra representing  $\langle 0,0,1,2 \rangle$ . Let  $p, q$  be two essentially  $n$ -ary polynomials over  $\mathcal{A}$  such that

- 1) The number of occurrences of each variable  $x_i$  ( $i=1, \dots, n$ ) in both  $p$  and  $q$  is exactly one;
- 2)  $p$  and  $q$  are in different forms.

Then  $p \neq q$  over  $\mathcal{A}$ .

Proof: In view of Theorem 3.6, it suffices to show that  $p \neq q$  over  $I(1)$  and  $II(1)$ .

If  $p = \prod_{i=1}^n x_i$ , then by condition 2), there is a factor  $f(A,B,C)$  in  $q$ . Let us construct a substitution  $S$  for the variables  $x_i$ 's with respect to the algebra  $II(1)$  in such a

way that

$$f(A,B,C)(S) = f(a,b,c)$$

and  $x_i(S) = e$  for every  $x_i$  not in  $f(A,B,C)$ .

Observe that according to the condition 1) such a substitution exists.

In this situation, we get

$$p_{II(1)}(S) = \left\{ \begin{array}{l} abc \\ \text{or} \\ abce \end{array} \right\} = d;$$

while  $q_{II(1)}(S) = e$ .

Thus,  $p \neq q$  over  $II(1)$ .

To show that  $p \neq q$  over  $I(1)$ , we construct a substitution  $T$  for the variables  $x_i$ 's with respect to the algebra  $I(1)$  in such a way that

$$f(A,B,C)(T) = f(a,b,c)$$

and  $x_i(T) = d$  for every  $x_i$  not in  $f(A,B,C)$ .

Again, such a  $T$  exists by 1). In this case, we get

$$p_{I(1)}(T) = \left\{ \begin{array}{l} abc \\ \text{or} \\ abcd \end{array} \right\} = d;$$

while  $q_{I(1)}(T) = e \cdot d = e$ .

Hence,  $p \neq q$  over  $I(1)$ , which was to be shown.

Now, we may assume that both  $p$  and  $q$  are different from  $\prod_{i=1}^n x_i$ . In other words, both  $p$  and  $q$  include  $f( , , )$  as a factor. By assumption 2) there is a factor  $f_*(A_1, A_2, A_3)$  in  $p$ ,

say, such that for each factor  $f(B_1, B_2, B_3)$  in  $q$ ,  $A_j \neq B_i$  for some  $i, j = 1, 2, 3$ . Let us choose such a  $f_*(A_1, A_2, A_3)$  in which the number of occurrences of variables is minimum. If we construct a substitution  $S$  for the variables  $x_i$ 's w.r.t  $II(1)$  in such a way that

$$f_*(A_1, A_2, A_3)(S) = f(a, b, c)$$

and  $x_i(S) = e$  if  $x_i$  is not in  $f_*(A_1, A_2, A_3)$ ,

then we get  $p_{II(1)}(S) = e$

while

$$q_{II(1)}(S) = \left\{ \begin{array}{l} abc \\ \text{or} \\ abce \end{array} \right\} = d.$$

(Note that there is no factor  $f(a, b, c)$  in  $q_{II(1)}(S)$ ; for otherwise, if  $f_A(B_1, B_2, B_3)$  exists in  $q$  such that  $f_A(B_1, B_2, B_3)(S) = f(a, b, c)$ , we would change the roles of  $f_A$  and  $f_*$ ).

Hence,  $p \neq q$  over  $II(1)$ .

To show that  $p \neq q$  over  $I(1)$ , we use the same argument as about but instead of the substitution  $S$ , we construct  $T$  in such a way that

$$f_*(A_1, A_2, A_3)(T) = f(a, b, c)$$

and  $x_i(T) = d$  if  $x_i$  is not in  $f_*(A_1, A_2, A_3)$ .

It then follows that

$$p_{I(1)}(T) = e \cdot d = e$$

while  $q_{I(1)}(T) = d$ .

Thus,  $p \neq q$  over  $I(1)$ .

The proof of Lemma 4.2 is therefore complete.

Proof of Theorem 4.1 : For  $n \geq 4$ , let  $\overline{p_n(\mathcal{K})}$  be the set of all essentially  $n$ -ary polynomials over  $\mathcal{K}$ . If  $N$  is the set of essentially  $n$ -ary polynomials constructed by induction, let  $R(N)$  denote the set of all essentially  $(n+1)$ -ary polynomials induced by some element in  $N$  by using the constructions (A), (B) or (C). Then by Lemmas 4.2 and 4.3, we have

$$|R(N)| \geq (n+1) |N|.$$

Evidently,

$$\overline{p_{n+1}(\mathcal{K})} \supseteq \underbrace{R(R(\dots(R(E))\dots))}_{n-3},$$

where  $E$  is the set of those eleven 4-ary polynomials.

From this, it follows that

$$\begin{aligned} p_{n+1}(\mathcal{K}) &= |\overline{p_{n+1}(\mathcal{K})}| \\ &\geq (n+1)n(n-1) \dots 5|E| \\ &= 11 \cdot \frac{(n+1)!}{4!} \quad \text{for each } n \geq 3. \end{aligned}$$

This completes the proof of our main Theorem.

## 5. Finite Subdirectly Irreducible Algebras.

An algebra  $\mathcal{A}$  is called subdirectly irreducible if the relation  $\bigwedge (\mathbb{H}_i \mid i \in I) = \omega$  implies that  $\mathbb{H}_i = \omega$  for some  $i \in I$ , where for each  $i \in I$ ,  $\mathbb{H}_i \in \mathcal{C}(\mathcal{A})$ . Equivalently,  $\mathcal{A}$  is subdirectly irreducible if there exist  $u, v \in A$  such that  $u \neq v$  and  $u \equiv v(\mathbb{H})$  for each  $\mathbb{H} > \omega$ .

From Birkhoff's classical result (see [1]) on subdirect decompositions which asserts that every algebra having more than

one element is a subdirect product of subdirectly irreducible algebras, it thus follows that subdirectly irreducible algebras play an important role in the study of structure of algebras.

The purpose of this section is to give a characterization theorem for finite subdirectly irreducible algebras in the equational class  $\underline{K}_1$ . To this end, we shall first define some notation and basic concepts.

Let  $\mathcal{A} \in \underline{K}_1$  such that  $|A|$  is finite. Consider the following set of triples:

$$T = \{(a_i, b_i, c_i) / a_i, b_i, c_i \in A, f(a_i, b_i, c_i) \neq a_i b_i c_i\}.$$

Let us make the following elementary observations:

- 1) Since  $f(x, y, z) \neq xyz$  in  $\mathcal{A}$ , thus,  $T \neq \emptyset$  and  $|T|$  is finite.
- 2) As "f" and "." are symmetric, hence if  $(a, b, c) \in T$ , so does  $(a\alpha, b\alpha, c\alpha)$ , where  $\alpha$  is any permutation on the set  $\{a, b, c\}$ .
- 3) If  $(a, b, c) \in T$ , then it follows from the proof of Theorem 3.1 that  $\{a, b, c\}$  are pairwise distinct and incomparable. Thus  $abc > a, b, c$ .

For each  $i$  and for each  $(a_i, b_i, c_i) \in T$ , denote

$$e_i = f(a_i, b_i, c_i) \quad ; \quad d_i = a_i b_i c_i.$$

Since  $f(x, y, z)xyz = f(x, y, z)$  holds in  $\underline{K}_1$ , we have  $e_i > d_i$ , for each  $i$ . Let



$$E = \{ e_i / (a_i, b_i, c_i) \in T \}.$$

Then  $E$  is a finite subposet of  $A$ . Thus, the set of maximal elements in  $E$  is not empty. By the Axiom of Choice, let  $e^*$  be a maximal element of  $E$ .

Let  $d_j^* = a_j b_j c_j$  where  $\{a_j, b_j, c_j\}$  is a triple of elements of  $A$  such that  $f(a_j, b_j, c_j) = e^*$ . Let  $d^*$  be one of the maximal elements in  $\{d_j^*\}$ . We have  $e^* > d^*$ .

Let  $a \in A$ . Then  $a$  is said to be a component of an element of  $T$  if there exist  $b, c \in A$  such that  $(a, b, c) \in T$ . Thus, it follows that if  $a$  is not a component of any element of  $T$ , then

$$f(a, u, v) = auv \quad \text{for all } u, v \in A.$$

We are, now, ready to establish the following

Theorem 5.1.

Let  $\mathcal{A}$  be a finite algebra in  $\mathcal{K}_1$ . Then  $\mathcal{A}$  is subdirectly irreducible if and only if  $\mathcal{A}$  satisfies all the following conditions:

- 1)  $e^* \succ d^*$
- 2) Let  $u, v \in A$  such that  $u \prec v$  and  $\{u, v\} \neq \{d^*, e^*\}$ .

If both  $u$  and  $v$  are not components of elements of  $T$ , then there exists  $p \in A$  such that  $u < p$  but  $v \not\leq p$ .

- 3) For each  $c, c'$  in  $A$  such that i)  $c \succ c'$  and ii)  $(a, b, c) \in T$  if, and only if  $(a, b, c') \in T$ , if  $f(a, b, c) = f(a, b, c')$  for  $(a, b, c) \in T$ , then there

exists  $p \in A$  such that  $c' < p$  but  $c \not\leq p$ .

Proof: Let  $\mathcal{A}$  be a finite subdirectly irreducible algebra in  $\mathcal{K}_1$ . We shall show that conditions 1), 2) and 3) hold in  $\mathcal{A}$ .

Firstly, we have Claim:  $e^*$  is the maximum element of  $\mathcal{A}$ .

Suppose that this is not the case, let  $m$  be the largest element of  $\mathcal{A}$ . Then  $m > e^*$ . If

- (1) There exist  $m_1, m_2 \in A$ ,  $m_1 \neq m_2$  such that  $m_1 \prec m$ ,  $m_2 \prec m$  (see Fig. 11), then consider the following two equivalence relations

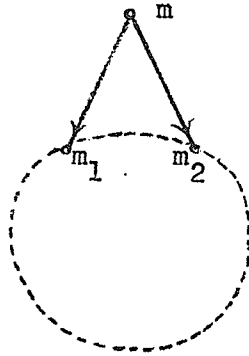


Fig. 11

$$\Phi_1 = \{ (x, x) / x \in A \} \cup \{ (m_1, m), (m, m_1) \} ,$$

$$\Phi_2 = \{ (x, x) / x \in A \} \cup \{ (m_2, m), (m, m_2) \} .$$

It is a simple matter to check that both  $m_1$  and  $m_2$  cannot be components of any elements of  $T$ . Thus, from the previous observation 3), it follows that  $\Phi_1$  and  $\Phi_2$  have the Substitution Property with respect to "." and "f". Hence,  $\Phi_1$  and  $\Phi_2$  are congruence relations of  $\mathcal{A}$ . Clearly,  $\Phi_1, \Phi_2 > \omega$ , but

$\Phi_1 \wedge \Phi_2 = \omega$ . This contradicts the fact that  $\mathcal{A}$  is subdirectly irreducible. Therefore (1) is not the case, but then we have

(2) there exist  $m_1, m_2 \in A$  such that  $m_2 \prec m_1 \prec m$  ( see

Fig: 12 )

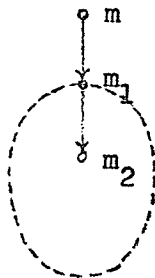


Fig. 12

Consider the following two equivalence relations :

$$\bar{\Phi}_1 = \{ (x,x) / x \in A \} \cup \{ (m, m_1), (m_1, m) \} ,$$

$$\bar{\Phi}_2 = \{ (x,x) / x \in A \} \cup \{ (m_1, m_2), (m_2, m_1) \} .$$

Clearly,  $m_1$  cannot be a component of elements of  $T$ . The same is true of  $m_2$ . For, if it were, then there exist  $b, c \in A$  such that  $(m_2, b, c) \in T$ . So,  $e_2 = f(m_2, b, c) > m_2 bc > m_2$ . Since  $m > e^*$ , thus  $m > e_2$  and so  $m_1 \succ e_2$  ( note that we assume (1) is not the case ! ). We have

$$m > m_1 \succ e_2 > m_2 bc > m_2 ,$$

which contradicts the fact that  $m_1 \succ m_2$ .

As  $m_1$  and  $m_2$  are not components of elements of  $T$ , it follows that  $\bar{\Phi}_1$  and  $\bar{\Phi}_2$  have the Substitution Property with respect to "f" and ".". Thus,  $\bar{\Phi}_1, \bar{\Phi}_2 \in C(\mathcal{A})$ . However, the fact that  $\bar{\Phi}_1, \bar{\Phi}_2 > \omega$  and  $\bar{\Phi}_1 \wedge \bar{\Phi}_2 = \omega$  contradicts our assumption that  $\mathcal{A}$  is subdirectly irreducible.

Hence, we conclude that  $e^*$  is the largest element in  $\mathcal{A}$ , as was to be shown.

By applying the same argument as in the first part, it follows

that if  $e^* \succ p$  for some  $p$ , then  $e^* \succ x$  implies  $p \geq x$ . In other words, the algebra  $\mathcal{A}$  is illustrated in Fig. 13.

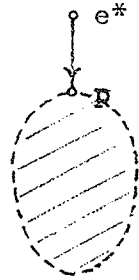


Fig. 13

We are now in a position to prove the statements 1), 2) and 3). Assume that 1) is not the case, i.e.,  $e^* \not\succeq d^*$ , then there exist  $p_1, p_2 \in [d^*, e^*)$  such that  $e^* \succ p_1 \succ p_2$ . Clearly,  $p_1$  is not a component of elements of  $T$ . The same is true of  $p_2$ . For, if it were, then there exist  $b, c \in A$  such that  $(p_2, b, c) \in T$ . Thus  $e_2 = f(p_2, b, c) \succ p_2 bc \succ p_2$ . If  $e_2 = e^*$ , then  $p_2 bc = d_2^* \succ p_2 \geq d^*$ , contradicting the fact that  $d^*$  is maximal in  $\{d_j^*\}$ . If  $e_2 < e^*$ , then by the above observation,  $e_2 \leq p_1$  and so  $e^* \succ p_1 \geq e_2 \succ p_2 bc \succ p_2$ , which contradicts  $p_1 \succ p_2$ . Consequently,  $p_2$  is not a component of elements of  $T$ . From this, it follows that equivalence relations

$$\begin{aligned} \bar{\Phi}_1 &= \{(x, x) \mid x \in A\} \cup \{(e^*, p_1), (p_1, e^*)\}, \\ \bar{\Phi}_2 &= \{(x, x) \mid x \in A\} \cup \{(p_1, p_2), (p_2, p_1)\} \end{aligned}$$

have the Substitution Property with respect to "f" and ".". Thus  $\bar{\Phi}_1, \bar{\Phi}_2 \in C(\mathcal{A})$ . As  $\bar{\Phi}_1 \wedge \bar{\Phi}_2 = \omega$ , while  $\bar{\Phi}_1, \bar{\Phi}_2 \succ \omega$ , we get a contradiction. Hence,  $e^* \succ d^*$ , proving 1).

To prove 2), suppose to the contrary that there exist  $u, v \in A$  such that  $u \prec v$ ,  $\{u, v\} \neq \{d^*, e^*\}$ , both  $u$  and  $v$  are not

components of elements of  $T$  and that  $u < x$  implies  $v \leq x$  for each  $x \in A$ .

Let us consider the following equivalence relation

$$\bar{\Phi} = \{(x,x) / x \in A\} \cup \{(u,v), (v,u)\}.$$

Claim :  $\bar{\Phi} \in C(\mathcal{K})$ .

We first prove that  $\bar{\Phi}$  has the Substitution Property with respect to " $\cdot$ ". It suffices to show that  $u \equiv v (\bar{\Phi})$  and  $x \equiv x (\bar{\Phi})$  imply  $ux \equiv vx (\bar{\Phi})$ . (1). If  $x \leq u$ , then  $x < v$ . Thus  $ux = u \equiv v = vx (\bar{\Phi})$ . (2). If  $x \parallel u$ , then  $ux > u$ . By assumption,  $ux \geq v$ . Thus,  $ux \geq vx$ . On the other hand,  $u < v$  implies  $ux \leq vx$ . Hence  $ux = vx$  and so  $(ux, vx) \in \bar{\Phi}$ . (3). If  $x > u$ , then by assumption  $x \geq v$ . Thus,  $ux = x = vx$  and so  $(ux, vx) \in \bar{\Phi}$ .

It remains to prove that  $\bar{\Phi}$  has the Substitution Property with respect to " $f$ ". Observe that  $u \equiv v (\bar{\Phi})$ ,  $v \equiv u (\bar{\Phi})$  and  $x \equiv x (\bar{\Phi})$  imply that  $f(u,v,x) \equiv f(v,u,x) (\bar{\Phi})$ , since  $f(u,v,x) = f(v,u,x)$ . Thus, it suffices to show that  $u \equiv v (\bar{\Phi})$ ,  $x \equiv x (\bar{\Phi})$  and  $y \equiv y (\bar{\Phi})$  imply  $f(u,x,y) \equiv f(v,x,y) (\bar{\Phi})$ . To see this, we note that as  $u$  and  $v$  are not components of elements of  $T$ , by a previous observation, we obtain

$$f(u,x,y) = uxy \text{ and } f(v,x,y) = vxy.$$

Hence,  $f(u,x,y) = uxy \equiv vxy = f(v,x,y) (\bar{\Phi})$ , as required.

Therefore,  $\bar{\Phi} \in C(\mathcal{K})$ .

Consider the following congruence relation of  $\mathcal{K}$ ,

$$\bar{\mathbb{H}}(e^*, d^*) = \{(x,x) / x \in A\} \cup \{(e^*, d^*), (d^*, e^*)\}.$$

Clearly,  $\bar{\mathbb{H}}(e^*, d^*) \cdot \bar{\Phi} > \omega$ , while  $\bar{\mathbb{H}}(e^*, d^*) \wedge \bar{\Phi} = \omega$ ,

since  $\{u,v\} \neq \{d^*,e^*\}$  and  $u \prec v, d^* \prec e^*$ . Thus,  $\mathcal{A}$  is subdirectly reducible, a contradiction. Hence, 2) follows.

Again, assume that 3) is not the case. Then there exist  $c, c'$  in  $A$  such that  $c \succ c', (a,b,c) \in T$  iff  $(a,b,c') \in T$  and  $f(a,b,c) = f(a,b,c')$  for  $(a,b,c) \in T$ , but  $c' \prec x$  implies  $c \leq x$  for each  $x \in A$ .

In what follows, we shall prove that the following equivalence relation  $\bar{\Phi} = \{(x,x) \mid x \in A\} \cup \{(c,c'),(c',c)\}$  is in  $C(\mathcal{A})$ . The method used above can be applied again to show that  $\bar{\Phi}$  has Substitution Property with respect to " $\cdot$ ". To show the same for " $f$ ", we need only check that

$$\left. \begin{array}{l} c \equiv c' \ (\bar{\Phi}) \\ x \equiv x \ (\bar{\Phi}) \\ y \equiv y \ (\bar{\Phi}) \end{array} \right\} \text{ imply } f(x,y,c) \equiv f(x,y,c') \ (\bar{\Phi}).$$

This is, indeed, the case for (1) If  $(x,y,c) \in T$ , then  $(x,y,c') \in T$  and  $f(x,y,c) = f(x,y,c')$  by assumption. Thus,  $f(x,y,c) \equiv f(x,y,c') \ (\bar{\Phi})$ . (2) If  $(x,y,c) \notin T$ , then  $(x,y,c') \notin T$  by assumption. Hence,  $f(x,y,c) = xyc$  and  $f(x,y,c') = xyc'$ . From the fact that  $xyc \equiv xyc' \ (\bar{\Phi})$ , it follows that  $f(x,y,c) \equiv f(x,y,c') \ (\bar{\Phi})$ . Hence  $\bar{\Phi} \in C(\mathcal{A})$ .

Since  $e^*$  and  $d^*$  cannot be components of elements of  $T$  and  $\{e^*,d^*\} \neq \{c,c'\}$ ; thus we have

$$\textcircled{H} (e^*,d^*) > \omega, \ \bar{\Phi} > \omega \text{ but } \textcircled{H} (e^*,d^*) \wedge \bar{\Phi} = \omega,$$

which is a contradiction. Therefore 3) follows.

Conversely, let  $\mathcal{A}$  be a finite algebra in  $\underline{K}_1$  satisfying the

conditions 1), 2), and 3). What we are going to prove is that is subdirectly irreducible. To achieve this, we prove, for each  $\mathbb{H} \in C(\mathcal{A}), \mathbb{H} > \omega$ , that  $d^* \equiv e^* (\mathbb{H})$  always holds.

For preparation, we make the following observation:

Claim 1.  $e^*$  is the greatest element of  $\mathcal{A}$ .

Suppose that this is false. Let  $m$  be the greatest. Then  $m > e^*$ . Let  $s \in A$  be such that  $m \succ s \geq e^*$ . Evidently,  $m, s$  cannot be components of elements of  $T$  and  $\{m, s\} \neq \{e^*, d^*\}$ . Thus, by virtue of 2), there exists  $p$  in  $A$  such that  $s < p$  but  $m \not\leq p$ . The fact that  $m$  is the greatest element implies  $p < m$ . Thus, we have  $s < p < m$ , contradicting the fact that  $s \prec m$ . Hence,  $e^*$  is greatest.

Claim 2. If  $x < e^*$ , then  $x \leq d^*$  for each  $x$  in  $A$ .

Suppose to the contrary that there is a  $y$  in  $A$  such that  $y < e^*$  but  $y \not\leq d^*$ . Let  $q$  be in  $A$  such that  $y \leq q \prec e^*$ . Clearly,  $q \not\leq d^*$ . Furthermore,  $\{q, e^*\} \neq \{d^*, e^*\}$  and both  $q$  and  $e^*$  are not components of elements of  $T$ . Thus, according to the condition 2), there is a  $p$  in  $A$  with  $q < p$  but  $e^* \not\leq p$ . Since  $e^*$  is the greatest element of  $A$ , it follows that  $e^* > p > q$ . This, however, contradicts the fact that  $e^* \succ q$ . Thus, Claim 2 follows.

From the above two observations, it is now clear that  $\mathcal{A}$  is illustrated in Fig. 13.

Let  $\mathbb{H} \in C(\mathcal{A})$  such that  $\mathbb{H} > \omega$ . Then there exist  $s, t$  in  $A$  such that  $s \neq t$  and  $s \equiv t (\mathbb{H})$ . Since, (1)  $s \equiv t (\mathbb{H})$  implies  $st \equiv t (\mathbb{H})$  and (2)  $s \equiv t (\mathbb{H})$  implies  $x \equiv y (\mathbb{H})$  for each  $x,$

y in  $[s, t]$  if  $s < t$ ; hence, without loss of generality, we may assume  $s \prec t$ .

Suppose that  $e^* \neq d^* (\textcircled{H})$ . Then  $\{s, t\} \neq \{d^*, e^*\}$ . In fact,  $s \prec t \leq d^*$ . Let us choose a pair  $\{s, t\}$  such that  $s$  is maximal. ( Note that this is possible as  $\mathcal{A}$  is finite. )

We have the following four cases :

Case I. Both  $s$  and  $t$  are not components of elements of  $T$ .

Case II.  $s$  is a component of an element of  $T$  but  $t$  not.

Case III.  $t$  is a component of an element of  $T$  but  $s$  not.

Case IV. Both  $s$  and  $t$  are components of elements of  $T$ .

Case I. In this case, by 2), there is a  $p$  in  $A$  such that  $s < p$  but  $t \not\leq p$ , ( see Fig. 14 ). Now,  $s \equiv t (\textcircled{H})$  implies

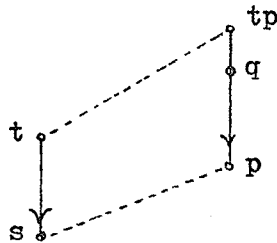


Fig. 14

that  $sp \equiv tp (\textcircled{H})$  and so  $p \equiv tp (\textcircled{H})$ . Clearly,  $tp > p > s$ . Let  $q \in (p, tp]$  such that  $q \succ p$ . As  $p \equiv q (\textcircled{H})$ , it follows that  $\{p, q\} \neq \{d^*, e^*\}$  where  $p > s$ .

Case II. Since  $s$  is a component of elements of  $T$ ; thus, there exist  $b, c$  in  $A$  such that  $(s, b, c) \in T$ . We have

$$e_s = f(s, b, c) > sbc.$$

As  $t$  is not a component of any element of  $T$ ; thus,  $(t, b, c) \notin T$  and so  $f(t, b, c) = tbc$ .



Observe that  $s \equiv t \ (\mathbb{H})$  implies  $f(s,b,c) \equiv f(t,b,c) \ (\mathbb{H})$ , i.e.,  $e_s \equiv tbc \ (\mathbb{H})$ . If (1)  $e_s \neq tbc$ , then as  $e_s > s$  and  $tbc \geq t > s$ , we can choose  $p, q$  in  $A$  such that  $p \prec q$ ,  $p \equiv q$  where  $p, q \in [\text{minimal}(e_s, tbc), e_s \cdot (tbc)]$ . Obviously,  $p > s$  and  $\{p, q\} \neq \{d^*, e^*\}$ . If (2)  $e_s = tbc$ , then as  $tbc = e_s > sbc$  and  $t \equiv s \ (\mathbb{H})$  implies  $tbc \equiv sbc \ (\mathbb{H})$ , there exist  $p, q$  in  $[sbc, tbc]$  such that  $p \prec q$  and  $p \equiv q \ (\mathbb{H})$ . Observe that  $(s,b,c) \in T$  implies  $sbc > s$  and thus  $p > s$ .

Case III. Since  $t$  is a component of element of  $T$ ; there exist  $b, c$  in  $A$  such that  $(t,b,c) \in T$ . Thus,  $e_t = f(t,b,c) > tbc$ . Because  $s$  is not a component of any element of  $T$ , we obtain  $f(s,b,c) = sbc$ . As  $s \equiv t \ (\mathbb{H})$  implies  $f(s,b,c) \equiv f(t,b,c) \ (\mathbb{H})$ , we get  $sbc \equiv e_t \ (\mathbb{H})$ . Observe that  $sbc > s$ . For, if  $sbc = s$  then  $bc \leq s$  and so  $tbc \leq ts = t$ . On the other hand, as  $(t,b,c) \in T$ , it follows that  $tbc > t$ , a contradiction. Hence  $sbc > s$ , as required. From this, we have ( see Fig. 15 )

$$e_t > tbc \geq sbc > s .$$

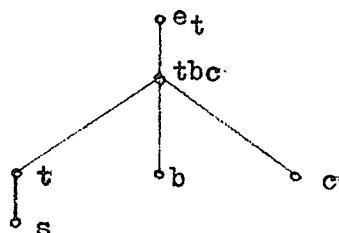


Fig. 15

Let  $p, q \in [sbc, e_t]$  such that  $p \prec q$ . Clearly,  $p \equiv q \ (\mathbb{H})$  and  $\{p, q\} \neq \{d^*, e^*\}$  where  $p > s$ .

Case IV. In this case, we assume that both  $s$  and  $t$  are components of elements of  $T$ .

If (1) there exist  $a, b$  in  $A$  such that  $(a, b, s) \in T$  but  $(a, b, t) \notin T$ , then the same argument as in Case II can be applied. If (2) there exist  $a, b$  in  $A$  such that  $(a, b, t) \in T$  but  $(a, b, s) \notin T$ , then the situation is same as Case III. If (1) and (2) are not the case, we have (3),  $(a, b, s) \in T$  iff  $(a, b, t) \in T$  for  $a, b \in A$ .

If  $f(a, b, s) \neq f(a, b, t)$  for some  $a, b \in A$ , then  $s \equiv t$  ( $\mathbb{H}$ ) implies  $f(a, b, s) \equiv f(a, b, t)$  ( $\mathbb{H}$ ) where

$$f(a, b, s) > abs > s$$

$$\text{and } f(a, b, t) > abt > t > s.$$

Let  $p, q$  be in  $[\text{minimal } \{f(a, b, s), f(a, b, t)\}, f(a, b, s)f(a, b, t)]$  such that  $p \prec q$ . Clearly,  $p \equiv q$  ( $\mathbb{H}$ ),  $\{p, q\} \neq \{d^*, e^*\}$  and  $p > s$ .

Now, if  $f(a, b, s) = f(a, b, t)$  for all such  $a, b$  in  $A$ , then by 3) there is a  $p$  in  $A$  such that  $s < p$  but  $t \not\equiv p$  ( see Fig 16 ).

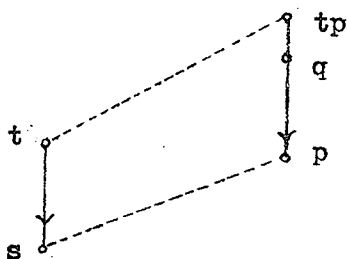


Fig. 16

Clearly,  $s \equiv t$  ( $\mathbb{H}$ ) implies  $sp \equiv tp$  ( $\mathbb{H}$ ), i.e.,  $p \equiv tp$  ( $\mathbb{H}$ ).

As  $tp > p$ , let  $q \in (p, tp]$  such that  $p \prec q$ . We have  $p \equiv q$  ( $\mathbb{H}$ ),  $\{p, q\} \neq \{d^*, e^*\}$  and  $p > s$ .

Hence, in any case, we obtain a pair  $\{p, q\}$  with  $p \prec q$ ,  $\{p, q\} \neq \{d^*, e^*\}$ ,  $p \equiv q$  ( $\mathbb{H}$ ) and  $p > s$ . This, however, contradicts the maximality of  $s$ . Therefore the assumption that

$e^* \neq d^* (\mathbb{H})$  is false. Consequently, for each  $\mathbb{H} \in C(\mathcal{A}), \mathbb{H} > \omega$  we have  $e^* \equiv d^* (\mathbb{H})$ . Hence,  $\mathcal{A}$  is subdirectly irreducible, which was to be shown.

Thus, the proof of Theorem 5.1 is complete.

Following from the proof of Theorem 5.1, we have

Corollary 5.2.

Let  $\mathcal{A}$  be a finite algebra in  $\underline{K}_1$ . If  $\mathcal{A}$  is subdirectly irreducible, then  $e^*$  is the greatest element of  $\mathcal{A}$ . Moreover,  $\mathcal{A}$  has one and only one dual atom  $d^*$  which contains every element other than  $e^*$ .

Let  $\mathcal{A} \in \underline{K}_1$ . By Theorem 3.1,  $\mathcal{A}$  contains one of the four algebras  $I(j)$ ,  $j = 1, 2, 3, 4$  as subalgebra. In particular, however, if  $\mathcal{A}$  is assumed to be a finite subdirectly irreducible algebra, then by virtue of the Characterization Theorem 5.1, we are able to prove that  $\mathcal{A}$  must contain  $I(1)$  as subalgebra. Indeed, we have the following :

Corollary 5.3.

The algebra  $I(1)$  is subdirectly irreducible. The algebra  $I(j)$  is subdirectly reducible, for each  $j = 2, 3, 4$ .

Corollary 5.4.

Let  $\mathcal{A}$  be a finite algebra in  $\underline{K}_1$ . If  $\mathcal{A}$  is subdirectly irreducible, then  $\mathcal{A}$  contains  $I(1)$  as subalgebra.

Proof : Let  $\mathcal{A}$  be a finite subdirectly irreducible algebra in  $\underline{K}_1$ .

In view of Corollary 5.2,  $\mathcal{A}$  can be represented by Fig. 17.

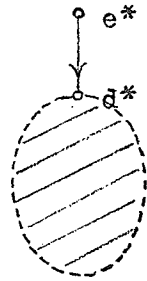


Fig. 17

Let us choose a triple  $(a^*, b^*, c^*)$  in  $T$  such that

$$f(a^*, b^*, c^*) = e^* \text{ and } a^*b^*c^* = d^*.$$

If  $a^*b^* = a^*c^* = b^*c^* = d^*$ , then, clearly,  $\mathcal{A}$  contains the algebra of Fig. 18 as subalgebra. In this case, the proof is complete.

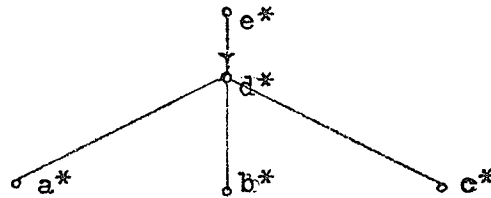


Fig. 18

Thus, we may assume, say,  $a^*b^* < d^*$ . Let  $s$  be in  $[a^*b^*, d^*)$  such that  $s \prec d^*$  ( see Fig. 19 ).

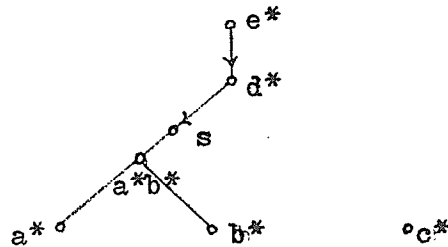


Fig. 19

If  $s$  is not a component of any element of  $T$ , then by condition 2) of Theorem 5.1, there exists  $p$  in  $A$  such that  $s < p$  but  $d^* \not\leq p$ . This implies that  $p \neq e^*$ ; i.e.,  $p < e^*$ . As  $p \neq d^*$ , by Corollary 5.2,  $p < d^*$ . Thus, it follows that  $s < p < d^*$ , which contradicts the fact that  $s \prec d^*$ .

Therefore,  $s$  must be a component of some element of  $T$ , and so there exist  $b_s, c_s$  in  $A$  such that  $(s, b_s, c_s) \in T$  ( see Fig. 20 ).

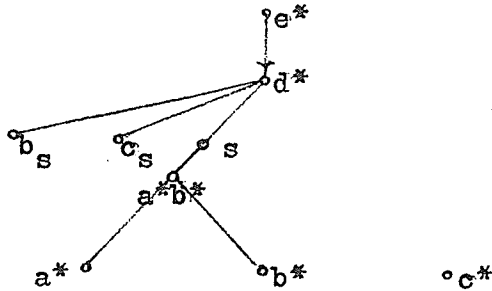


Fig. 20

Evidently,  $f(s, b_s, c_s) > sb_s c_s > s$ . Moreover, as  $b_s, c_s \neq e^*$ , we have  $b_s, c_s \leq d^*$ . Thus,  $b_s, c_s < d^*$  since  $d^*$  cannot be a component of any element of  $T$ . Clearly,  $sb_s c_s \leq d^*$ . If  $sb_s c_s < d^*$ , we would obtain  $s < sb_s c_s < d^*$ , contradicting the fact that  $s \prec d^*$ . Hence, it follows that  $sb_s c_s = d^*$ . But then  $f(s, b_s, c_s) > sb_s c_s = d^*$  and therefore  $f(s, b_s, c_s) = e^*$ . Now, if the elements  $s, b_s, c_s$  are such that  $sb_s = sc_s = b_s c_s = d^*$ , then contains the algebra of Fig. 21 as subalgebra. In this case, the proof is complete.

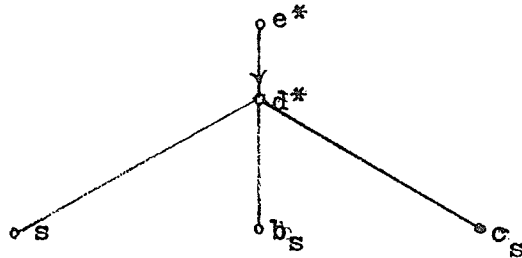


Fig. 21

Thus, we may assume  $b_s c_s < d^*$  ( note that as  $s \prec d^*$ ,  $sb_s = sc_s = d^*$  ! ). Following the same argument, we will obtain elements  $t_1, t_2, t_3 \in A$  such that  $(t_1, t_2, t_3) \in T$ ,  $f(t_1, t_2, t_3) = e^* > t_1 t_2 t_3 = d^* > t_i$ ,  $i = 1, 2, 3$  and  $d^* \succ t_1 \geq b_s c_s$ . If

$t_1 t_2 = t_2 t_3 = t_3 t_1 = d^*$ , the proof is complete. Otherwise, we would continue this process. Since  $\mathcal{A}$  is finite, the process will stop after finitely many steps. Hence, the Corollary follows.

The following examples 1 and 2 are subdirectly irreducible algebras in  $\underline{K}_1$  while example 3 is an algebra in  $\underline{K}_1$  which is subdirectly reducible.

Example 1.

The algebra is the semilattice of Fig. 22 with the ternary operation "f" defined as follows :

$$f(x,y,z) = \begin{cases} e_i & \text{if } \{x,y,z\} = \{a_i,b_i,c_i\} \\ xyz & \text{otherwise} \end{cases}$$

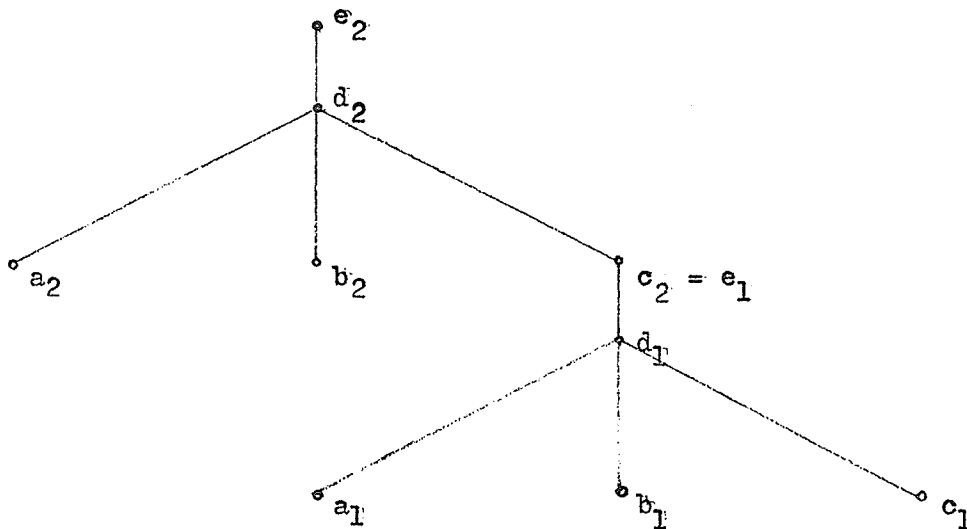


Fig. 22

Example 2.

The algebra is the semilattice of Fig. 23 with the ternary operation "f" defined as above.

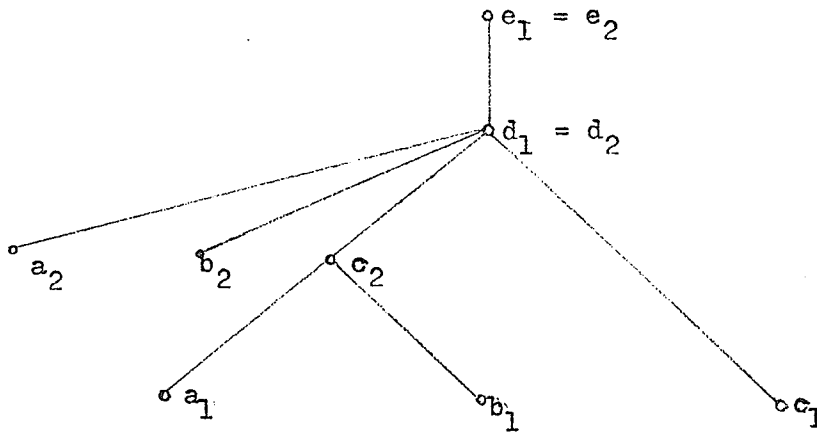


Fig. 23

Example 3.

The algebra is the semilattice of Fig. 24 with the ternary operation "f" defined as above.

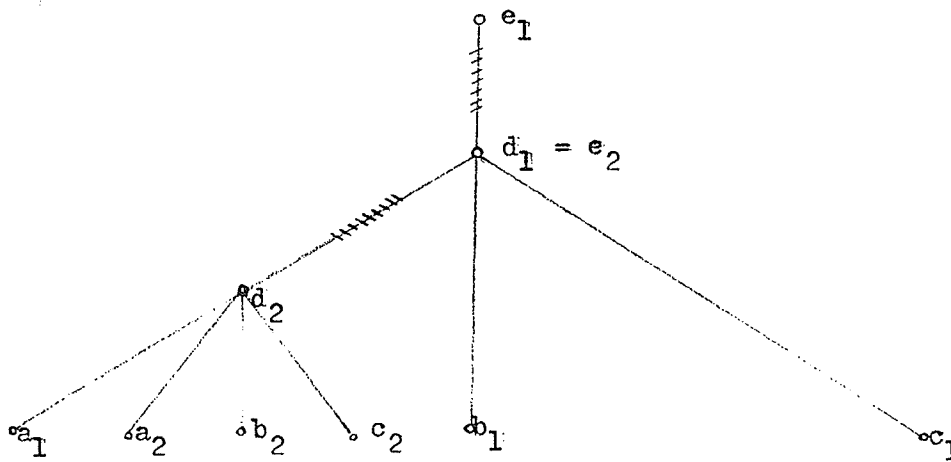


Fig. 24

Remark.

It is conjectured that the corresponding results on finite subdirectly irreducible algebras can similarly be obtained in the equational class  $\underline{K}_2$ . In this case, of course, the role that  $I(1)$  plays in  $\underline{K}_1$  will be replaced by  $II(1)$  in  $\underline{K}_2$ .

### CHAPTER 3

#### ALGEBRAS REPRESENTING $\langle 0,0,1,m \rangle$

In this chapter, we shall deal with the representability of the sequence  $\langle 0,0,1,m \rangle$ , for an arbitrary natural number  $m$ . The case  $m = 1$  and  $m = 2$  have been considered before. By a result of G. Grätzer and R. Padmanabhan ( see [10] ), it is known that if  $\mathcal{K}$  is the idempotent reduct of an abelian group of exponent 3, then  $p(\mathcal{K}) = \langle 0,0,p_2,p_3, \dots, p_n, \dots \rangle$  where  $p_n = \frac{1}{3} ( 2^n - (-1)^n )$ . Thus, in particular, the sequence  $\langle 0,0,1,3 \rangle$  is representable. It is, therefore, natural to ask : given an arbitrary natural  $m$ , is the sequence  $\langle 0,0,1,m \rangle$  always representable ?

Let  $\mathcal{K}$  be an algebra representing  $\langle 0,0,1 \rangle$ . Then  $\mathcal{K}$  has one and only one idempotent, commutative essentially binary polynomial. If  $p_3(\mathcal{K}) \geq 3$ , then the binary polynomial is not necessary associative. However, if we assume that it is associative, then for each natural number  $m$ , it is possible to find such algebra  $\mathcal{K}$  such that  $p_3(\mathcal{K}) = m$ . More precisely, we are able to construct a semilattice with certain ternary operations defined on it so that the resulting algebra has exactly  $m$  essentially ternary polynomials. Thus, the sequence  $\langle 0,0,1,m \rangle$  is, a fortiori, representable, for each natural number  $m$ , which solves the above problem.



1. Construction of Algebras.

In this section, we start to construct algebras which will be shown to meet the requirement in next section.

For each natural number  $m$ , let us consider the four types of semilattices  $\langle A; \cdot \rangle$  described by Fig. 25—28

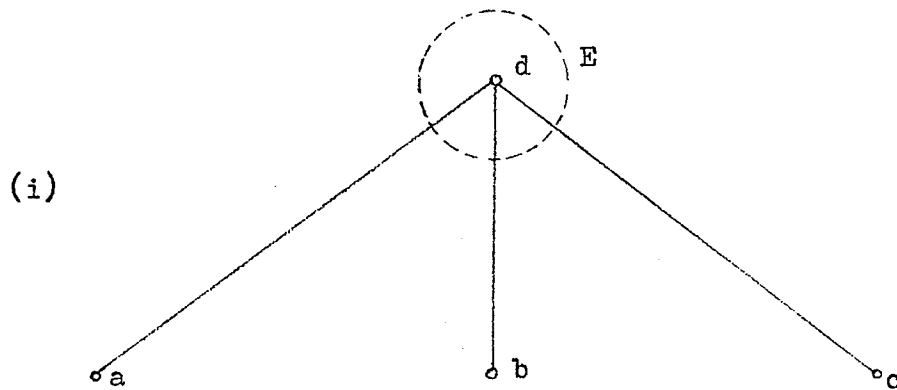


Fig. 25

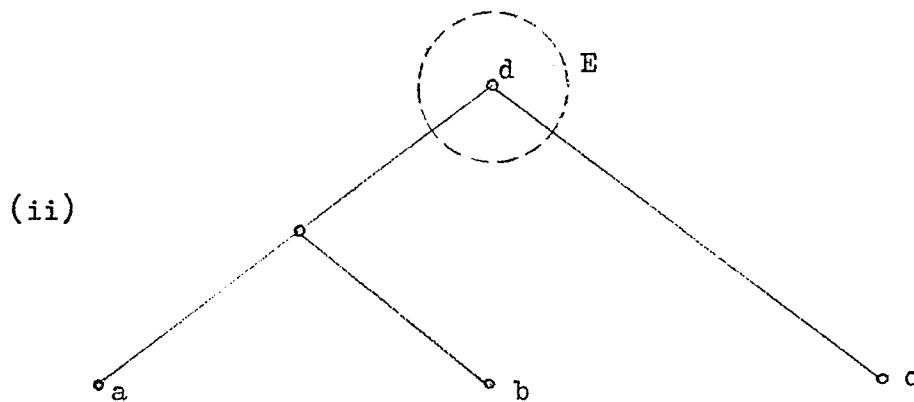


Fig. 26

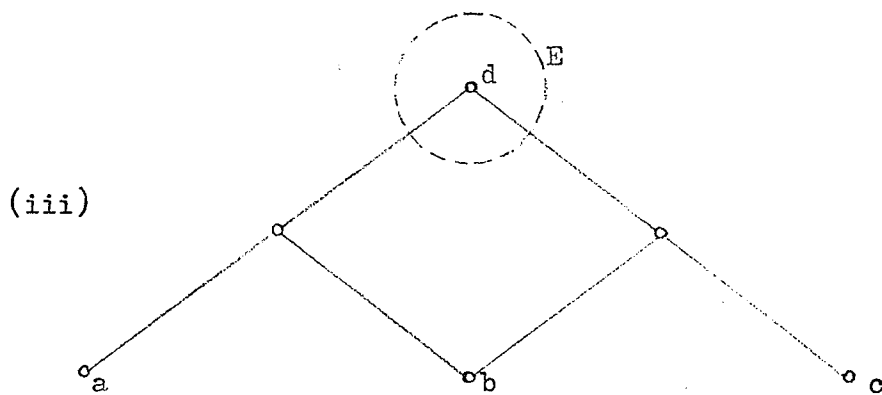


Fig. 27

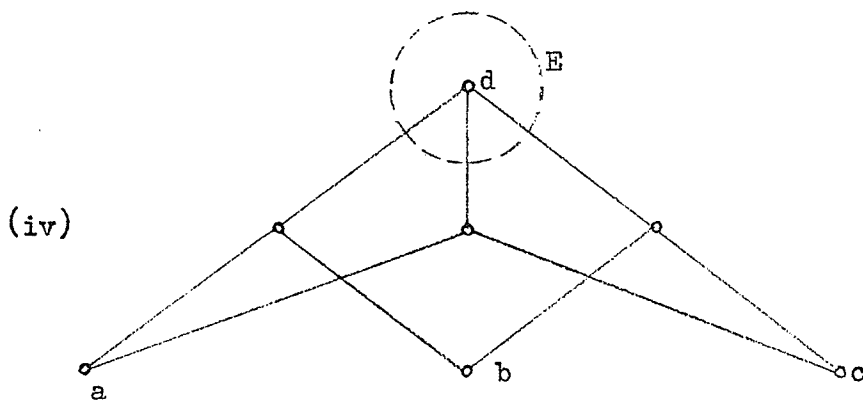


Fig. 28

For each type of semilattice, the subset

$$E = \{ d, e_1, e_2, \dots, e_{m-1} \}$$

is a  $m$ -element subsemilattice such that for each  $i = 1, 2, \dots, m-1$ ,

either (a)  $e_i > d$ ;

or (b)  $e_i < d$  but  $e_i$  is incomparable with every element of  $A - E$ .

Or (c)  $e_i$  is incomparable with every element of  $(A - E) \cup \{d\}$ .

Given any of the semilattices  $\langle A; \cdot \rangle$ , we shall define,

for each  $i = 1, 2, \dots, m-1$ , a ternary operation  $f_i$  on  $A$  as follows:

$$f_i(x, y, z) = \begin{cases} e_i & \text{if } \{x, y, z\} = \{a, b, c\} \\ xyz & \text{otherwise} \end{cases}$$

Evidently, we have the following elementary observations:

- (1) The polynomial  $f_i(x, y, z)$  is essentially ternary, for each  $i = 1, 2, \dots, m-1$ .
- (2)  $f_i(x, x, x) = x$  for each  $i = 1, 2, \dots, m-1$ .
- (3)  $f_i(x, y, z)$  is symmetry, for each  $i = 1, 2, \dots, m-1$ .
- (4) If  $i \neq j$  then  $f_i(x, y, z) \neq f_j(x, y, z)$  over  $A$ .

Now, to any of the semilattices  $\langle A; \cdot \rangle$ , we can associate an idempotent algebra

$$\mathcal{A} = \langle A; F \rangle$$

where  $F = \{\cdot, f_1, f_2, \dots, f_{m-1}\}$  consists of one semilattice operation and  $m-1$  ternary operations which are defined as above.

Remark.

If  $m = 2$ , there are exactly eight different algebras (up to isomorphism) obtained from the construction. They are isomorphic to  $I(j)$ ,  $II(j)$   $j = 1, 2, 3, 4$  respectively.

If  $m = 3$ , we obtain exactly twenty different algebras from the construction. For instance, the following are constructed from the first type of semilattice,

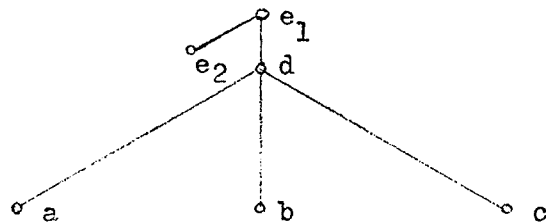
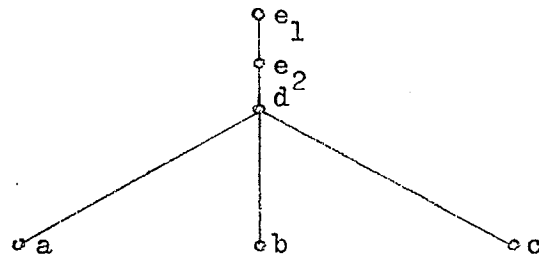
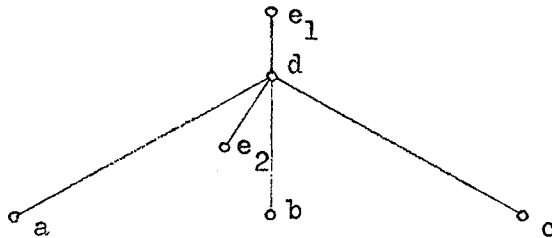
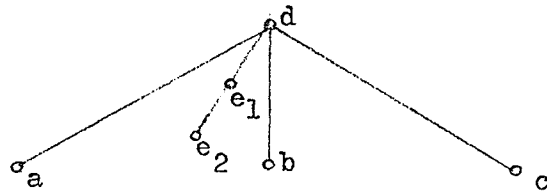
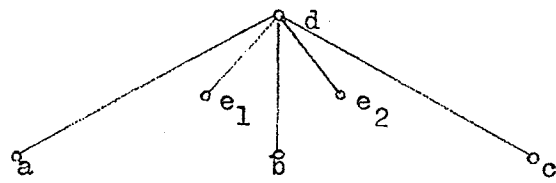


Fig. 29

Where for each  $i = 1, 2$ ,  $f_i$  is defined as above.

2. Main Theorem.

We are now in a position to prove the following

Theorem 2.1.

For each natural number  $m$ , any one of the algebras where  $|E| = m$  as constructed in section 1 has exactly  $m$  essentially ternary polynomials.

Corollary 2.2.

Let  $\underline{K}$  be the class of all idempotent algebras with a semilattice operation. Then, for each natural number  $m$ , the sequence  $\langle 0,0,1,m \rangle$  is representable in  $\underline{K}$ .

Proof of Theorem 2.1 : Let  $\mathcal{A}$  be such a given algebra with  $|E| = m$ . Then  $\mathcal{A}$  has at least the following  $m$  essentially ternary polynomials:

$$xyz, f_1(x,y,z), \dots, f_{m-1}(x,y,z).$$

Thus, to show that  $p_3(\mathcal{A}) = m$ , we have to show that the collection of all these  $m$  essentially ternary polynomials is closed under substitution. In other words, let  $p(x,y,z)$  be an arbitrary essentially ternary polynomial over  $\mathcal{A}$ . Our purpose is to show that  $p(x,y,z)$  is the same as one of the above  $m$  essentially ternary polynomials.

To this end, let  $p(x,y,z)$  be given. First of all, we claim that:

$$p(a,b,c) \in E.$$

Suppose that this is not the case. Then  $p(a,b,c) \in A - E$   
 $\subseteq \{a,b,c,ab,ac,bc\}$  (note that the later set depends on  $\mathcal{A}$  ).  
 However, by definition of  $f_i(x,y,z)$  and the fact that  $abc = d$ ,  
 the above inclusion is impossible. Hence,  $p(a,b,c) \in E$ , as  
 required.

Similarly,  $p(\alpha a, \alpha b, \alpha c)$  is an element of  $E$  for any permutation  
 $\alpha$  on the set  $\{a,b,c\}$  .

Next, we prove that  $p(x,y,z)$  is symmetric over  $\mathcal{A}$  . It  
 suffices to show that  $p(x,y,z) = p(y,x,z)$ . The other cases can  
 be shown similarly.

Let  $S$  be a substitution for  $\{x,y,z\}$  .

If  $S \neq \{a,b,c\}$  , then evidently by definition of  $f_i$ , we have

$$p(x,y,z)(S) = (xyz)(S) = (yxz)(S) = p(y,x,z)(S).$$

If  $S = \{a,b,c\}$  , say  $x(S) = a$ ,  $y(S) = b$ ,  $z(S) = c$ , we have  
 to prove  $p(a,b,c) = p(b,a,c)$ .

For simplicity, let us make the following conventions:

- (1) If there is a factor  $f_i(a,b,c)$  in  $p(a,b,c)$ , we denote  
 $f_i(a,b,c)$  by  $e_i$ ;
- (2) If there is a factor  $f_i(u,v,w)$  in  $p(a,b,c)$  where  
 $\{u,v,w\} \neq \{a,b,c\}$  , we denote  $f_i(u,v,w)$  by  $uvw$ .
- (3) If there is a factor  $abc$  in  $p(a,b,c)$ , we denote  $abc$   
 by  $d$ .

Thus, from these and the fact that "." is a semilattice  
 operation, it follows immediately that

$$p(a,b,c) = \begin{cases} \prod_{i \in I} e_i \\ \text{or} & (\prod_{i \in I} e_i)a \quad \left[ \text{or} \left( \prod_{i \in I} e_i \right)b \quad \text{or} \left( \prod_{i \in I} e_i \right)c \right] \\ \text{or} & (\prod_{i \in I} e_i)ab \quad \left[ \text{or} \left( \prod_{i \in I} e_i \right)bc \quad \text{or} \left( \prod_{i \in I} e_i \right)ac \right] \\ \text{or} & (\prod_{i \in I} e_i)d \end{cases}$$

where  $I$  is a finite subset of  $1, 2, \dots, m-1$ , possibly empty.

If  $p(a,b,c) = \prod_{i \in I} e_i$  or  $(\prod_{i \in I} e_i)d$ , then from the fact that  $f_i(x,y,z)$  and  $xyz$  are symmetric, the result follows.

If  $p(a,b,c) = (\prod_{i \in I} e_i)a$ , then  $p(b,a,c) = (\prod_{i \in I} e_i)b$ . We shall prove that  $(\prod_{i \in I} e_i)a = (\prod_{i \in I} e_i)b$ .

For simplicity, write  $e = \prod_{i \in I} e_i$ . Since  $E$  is a subsemilattice of  $A$ . Thus,  $e_i \in E$  imply  $e \in E$

Case 1.  $e < d$ .

In this case, as  $e \in E$  and  $e < d$  we have

$$ea = d = eb.$$

Case 2.  $e \not< d$

In this case, clearly, we have

$$ea = ed = eb.$$

Thus,  $p(a,b,c) = p(b,a,c)$ , as was to be shown.

For the other possible values of  $p(a,b,c)$ , the proof is similar. Hence, we conclude that  $p(x,y,z)$  is, indeed, symmetric.

Since  $p(a,b,c) \in E$ , it follows that  $p(a,b,c) = d$  or  $p(a,b,c) = e_i$ , for some  $i = 1, \dots, m-1$ .

(1) If  $p(a,b,c) = d$ , we claim that  $p(x,y,z) = xyz$ .

Let  $S$  be a substitution for the variables  $x, y, z$ .

If  $S \neq \{a, b, c\}$  then clearly

$$p(x, y, z)(S) = (xyz)(S).$$

If  $S = \{a, b, c\}$ , then by symmetry of  $p(x, y, z)$ , we have

$$p(x, y, z)(S) = p(a, b, c) = d = abc = (xyz)(S).$$

Hence  $p(x, y, z) = xyz$ , as required.

(2) If  $p(a, b, c) = e_i$ , for some  $i = 1, \dots, m-1$ . We claim that  $p(x, y, z) = f_i(x, y, z)$ .

Let  $S$  be a substitution for the variables  $x, y, z$ . If  $S \neq \{a, b, c\}$ , then clearly,

$$p(x, y, z)(S) = (xyz)(S) = (f_i(x, y, z))(S).$$

If  $S = \{a, b, c\}$ , then by symmetry of  $p(x, y, z)$ , we have

$$\begin{aligned} p(x, y, z)(S) &= p(a, b, c) \\ &= e_i \\ &= f_i(a, b, c) \\ &= f_i(x, y, z)(S). \end{aligned}$$

Hence  $p(x, y, z) = f_i(x, y, z)$ .

Therefore, we conclude that

$$p(x, y, z) = \begin{cases} xyz & \text{if } p(a, b, c) = d \\ f_i(x, y, z) & \text{if } p(a, b, c) = e_i \end{cases}$$

which completes the proof of Theorem 2.1.



----- PART II -----

IDEMPOTENT ALGEBRAS WITH ONE ESSENTIALLY  $m$ -ARY POLYNOMIAL,  $m \geq 2$

## CHAPTER 1

### ALGEBRAS REPRESENTING $\langle 0, 0, a_1, \dots, a_k, m \rangle$ WITH $a_1 = \dots = a_k = 1$

In this chapter, we shall, naturally, deal with the same problems as in Part I for the more general sequence  $\langle 0, 0, a_1, \dots, a_k, m \rangle$  with  $a_1 = \dots = a_k = 1$  where  $k, m$  are arbitrary positive integers. It turns out that almost all the results in Part I can be extended to this general case.

The case that  $m = 2$  will be examined in section 1. We obtain a generalization of Theorem 3.4(I,1). Moreover, a lower bound for sequence  $\langle p_n \rangle$  is provided in this case. In section 2, the representability Theorem for the sequence  $\langle 0, 0, a_1, \dots, a_k, m \rangle$  with  $a_1 = \dots = a_k = 1$ ,  $k, m$  are arbitrary positive integers, is established. It is applied, in section 3, to prove a characterization Theorem about the sequence  $\langle 0, 0, 1, \dots, 1, m, n \rangle$

#### 1. The Case $m = 2$ .

Let  $\mathcal{A}$  be an algebra representing the sequence  $\langle 0, 0, \overbrace{1, \dots, 1}^k, 2 \rangle$  where  $k$  is a positive integer (we may assume  $k > 1$ ). By Lemma 1.2(I,1),  $\mathcal{A}$  has a unique semilattice operation. Since  $p_{k+2}(\mathcal{A}) = 2$ , it follows that there is one and only one essentially  $(k+2)$ -ary polynomial, denoted by  $f(x_1, \dots, x_{k+2})$ , which is distinct from  $\prod_{i=1}^{k+2} x_i$  over  $\mathcal{A}$ . For the sake of simplicity, we write  $n = k+2$ . Clearly,  $f$  is idempotent and symmetric. If we identify  $x_1 = x_2$  in  $f$ , we obtain the polynomial  $f(x_2, x_2, x_3, \dots, x_n)$ .

As  $\mathcal{Q}$  represents  $\langle 0, 0, 1, \dots, 1 \rangle$  and  $f$  is symmetric, we have

$$f(x_2, x_2, x_3, \dots, x_n) = \begin{cases} x_2 \\ \prod_{i=3}^n x_i \\ \prod_{i=2}^n x_i \end{cases}$$

If  $f(x_2, x_2, x_3, \dots, x_n) = x_2$ , setting  $x_3 = x_4$ , we obtain  $x_3 = x_2$ ,

which is impossible. If  $f(x_2, x_2, x_3, \dots, x_n) = \prod_{i=3}^n x_i$ , setting

$x_3 = x_4$ , it follows that  $x_2 \cdot \prod_{i=5}^n x_i = \prod_{i=4}^n x_i$ . Set  $x_4 = \dots = x_n$

We have  $x_2 \cdot x_n = x_n$ , a contradiction. Thus it is necessary that

$$(1) \quad f(x_2, x_2, x_3, \dots, x_n) = \prod_{i=2}^n x_i.$$

From this, it follows immediately that

$$(2) \quad f(x_1 x_2, x_1, x_2, x_4, \dots, x_n) = x_1 x_2 \prod_{i=4}^n x_i.$$

Let us now consider the polynomial  $p = f(x_1 x_2, x_2, x_3, \dots, x_n)$ .

Setting  $x_1 = x_2$ , we obtain by (1) that  $p = \prod_{i=2}^n x_i$ . Thus,

$p$  depends on  $x_i$ , for each  $i = 3, \dots, n$ . Setting  $x_3 = x_4$ , we

have  $p = x_1 x_2 \prod_{i=4}^n x_i$ . Thus  $p$  depends on  $x_1$  and  $x_2$ . Hence  $p$  is

essentially  $n$ -ary.

$$\text{If } f(x_1 x_2, x_2, \dots, x_n) = f(x_1, \dots, x_n) \quad \text{--- (A),}$$

$$\text{then } \prod_{i=1}^n x_i = f\left(\prod_{i=1}^n x_i, \prod_{i=2}^n x_i, x_1 \prod_{i=3}^n x_i, x_1 x_2 \prod_{i=4}^n x_i, \dots, \prod_{i=1}^{n-2} x_i \cdot x_n\right) \quad (2)$$

$$= f\left(x_1, \prod_{i=2}^n x_i, x_1 \prod_{i=3}^n x_i, x_1 x_2 \prod_{i=4}^n x_i, \dots, \prod_{i=1}^{n-2} x_i \cdot x_n\right) \quad (A)$$

$$= f(x_1, x_2, \dots, x_i x_j, \dots, x_n) \quad (A)$$

$$= f(x_1, x_2, \dots, x_j, \dots, x_n), \quad (A)$$

which is a contradiction. Thus, we obtain

$$(3) \quad f(x_1 x_2, x_2, \dots, x_n) = \prod_{i=1}^n x_i.$$

Let us write  $\prod_j x_i = x_1 x_2 \dots x_{j-1} x_{j+1} \dots x_n$  for each  $j = 1, \dots, n$ . The polynomial  $f(\prod_1 x_i, \prod_2 x_i, \dots, \prod_n x_i)$  is clearly essentially  $n$ -ary.

$$\text{If} \quad f(\prod_1 x_i, \prod_2 x_i, \dots, \prod_n x_i) = f(x_1, x_2, \dots, x_n) \quad (B)$$

$$\text{then} \quad \prod_{i=1}^n x_i = f\left(\prod_{i=1}^n x_i, \dots, \prod_{i=1}^n x_i\right)$$

$$= f\left(\prod_2 x_i \cdot \prod_3 x_i \cdot \dots \cdot \prod_n x_i, \dots, \prod_1 x_i \cdot \prod_2 x_i \cdot \dots \cdot \prod_{n-1} x_i\right)$$

$$= f\left(\prod_1 x_i, \prod_2 x_i, \dots, \prod_n x_i\right)$$

$$= f(x_1, x_2, \dots, x_n), \quad (B)$$

a contradiction. Thus, it follows that

$$(4) \quad f\left(\prod_1 x_i, \prod_2 x_i, \dots, \prod_n x_i\right) = \prod_{i=1}^n x_i.$$

Because of what we have now proved, it is evident that we can apply arguments similar to those of Chapter 1 of Part I to derive several identities that hold in  $\mathcal{A}$ . In this way it can be shown that there are two types of algebras satisfying the identities

$$f(x_1, \dots, x_n) x_1 = f(x_1, \dots, x_n) \quad \text{I}$$

$$f(x_1, \dots, x_n) x_1 = \prod_{i=1}^n x_i \quad \text{II}$$

respectively.

In conclusion, we arrive at the following result.

Theorem 1.1.

For each positive integer  $k = 1, 2, \dots$ , there exist two equational classes of algebra  $\underline{K}_{1k}$  and  $\underline{K}_{2k}$  such that an algebra  $\mathcal{A}$  represents the sequence  $\langle 0, 0, \overline{1, \dots, k}, \overline{1, 2} \rangle$  if and only if  $\mathcal{A}$  can be represented as an algebra of type  $\langle 2, k+2 \rangle$  belonging to either  $\underline{K}_{1k}$  or  $\underline{K}_{2k}$ .

If  $k = 2$ , the following examples can easily be checked to represent the sequence  $\langle 0, 0, 1, 1, 2 \rangle$ . Let us note that the free semilattice on four generators consists of 15 elements, and that therefore, there are eleven non-isomorphic semilattices generated by four elements, the homomorphic images of the free one. Hence, we have the following twenty two algebras, (see Fig. 30)

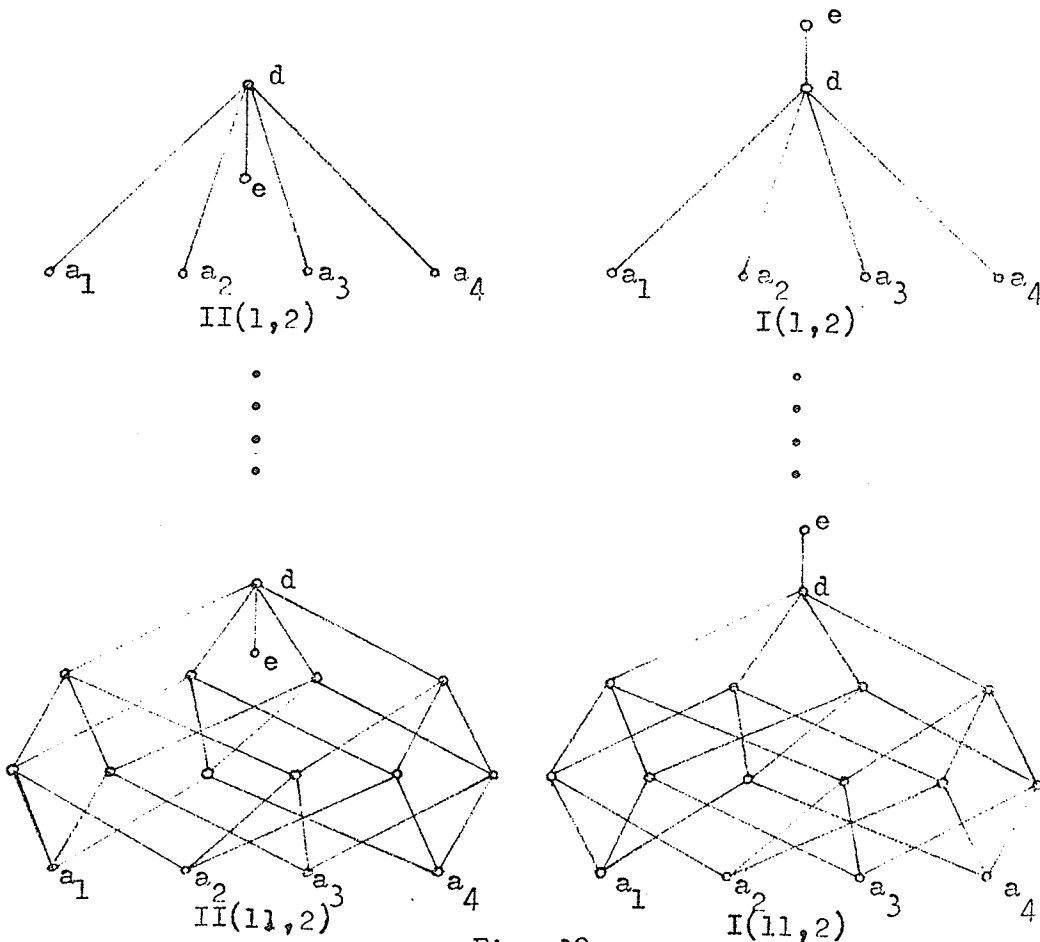


Fig. 30

where each algebra  $I(j,2)$ ,  $II(j,2)$  is a semilattice shown as above and the 4-ary operation  $f$  is defined by

$$f(x_1, x_2, x_3, x_4) = \begin{cases} e & \text{if } \{x_1, x_2, x_3, x_4\} = \{a_1, a_2, a_3, a_4\} , \\ \prod_{i=1}^4 x_i & \text{otherwise .} \end{cases}$$

In general, for each positive integer  $k$ , the free semilattice on  $k+2$  generators consists of  $2^{k+2} - 1$  elements. Hence, there are  $2^{k+2} - (k+3)$  non-isomorphic semilattices generated by  $k+2$  elements which are the homomorphic images of the free one. Therefore, by using the same idea, we can construct  $2^{k+3} - 2(k+3)$  algebras i.e.,  $I(j,k)$ ,  $II(j,k)$ ,  $j = 1, 2, \dots, 2^{k+2} - (k+3)$  each of which represents the sequence  $\langle 0, 0, \overline{1, \dots, 1}, 2 \rangle$ .

The following result can similarly be shown.

Theorem 1.2.

Let  $\mathcal{A}$  be an algebra representing the sequence  $\langle 0, 0, \overline{1, \dots, 1}, 2 \rangle$  where  $k$  is an arbitrary positive integer. Then  $\mathcal{A}$  contains one of the algebras  $I(j,k)$ ,  $II(j,k)$ ,  $j = 1, 2, \dots, 2^{k+2} - (k+3)$  as a subalgebra.  
can be represented as an algebra of type  $\langle 2, k+2 \rangle$  which

Let  $\mathcal{A}$  be an algebra in  $\underline{K}_{1k}$  or  $\underline{K}_{2k}$ . Let  $p, q$  be two essentially  $n$ -ary polynomials over  $\mathcal{A}$ . By Theorem 1.2, it follows that if  $p \neq q$  over  $I(1,k)$  and  $II(1,k)$ , then  $p \neq q$  over  $\mathcal{A}$ . In view of this fact, we can now provide a lower bound for the sequence  $\langle p_n(\mathcal{A}) \rangle$ .

Theorem 1.3.

Let  $\mathcal{A}$  be an algebra representing the sequence  $\langle 0, 0, \overline{1, \dots, 1}, 2 \rangle$  for  $k \geq 1$ . Then  $p_n(\mathcal{A}) \geq \left[ 1 + \frac{1}{2}(k+3)(k+4) \right] (k+3)^{n-(k+3)}$ , for each  $n \geq k+3$ .

Proof : It is not difficult to check that the following  $1 + \frac{1}{2}(k+3)(k+4)$  polynomials are distinct and essential over the algebras  $I(1, k)$  and  $II(1, k)$ . Thus, by the above observation,  $\mathcal{A}$  has at least  $1 + \frac{1}{2}(k+3)(k+4)$  essentially  $(k+3)$ -ary polynomials, namely :

$$\prod_{i=1}^{k+3} x_i, \\ f(x_1, \dots, x_{k+2})x_{k+3}, \quad f(x_2, \dots, x_{k+3})x_1, \quad \dots, \quad f(x_{k+3}, x_1, \dots, x_{k+1})x_{k+2}, \\ f(x_1, \dots, x_{k+1}, x_{k+2}, x_{k+3}), \dots, \quad f(x_2, \dots, x_{k+2}, x_1 x_{k+3})$$

We shall now construct new polynomials inductively from the above polynomials in the following manner :

Let  $p(x_1, \dots, x_n)$  be an  $n$ -ary polynomial obtained by induction. With respect to  $p$ , we produce the following  $(n+1)$ -ary polynomials by using three types of constructions :

- (A)  $p(x_1, \dots, x_n) \cdot x_{\underline{n+1}}$ ;
- (B) If there is a factor  $f(A_1, \dots, A_{k+2})$  in  $p$ , we then set
  - 1)  $f(A_1 x_{\underline{n+1}}, A_2, \dots, A_{k+2})$ ,
  - 2)  $f(A_1, A_2 x_{\underline{n+1}}, \dots, A_{k+2})$ ,
  - ⋮
  - ⋮
  - ⋮
  - $k+2$ )  $f(A_1, A_2, \dots, A_{k+2} x_{\underline{n+1}})$  ;

(C) If there is a product  $\prod_{j \in J} A_j$  in  $p$  where  $J$  is maximal ( in the sense of Theorem 4.1(I,2) ), we set

$f(\prod_{j \in J(1)} A_j, \dots, \prod_{j \in J(k+1)} A_j, \underline{x}_{n+1})$  in  $p$  ( if this is possible ) where  $\{ J(1), \dots, J(k+1) \}$  is a partition of the index set  $J$ .

Now, observe that if there is an occurrence of "f" in  $p$ , then by constructions (A) and (B), we have at least  $k + 3$   $(n+1)$ -ary polynomials. If there is no occurrence of "f" in  $p(x_1, \dots, x_n)$ , then since  $n \geq k + 3$ , we obtain at least  $k + 3$   $(n+1)$ -ary polynomials by using the constructions (A) and (C). Applying the same argument as in Theorem 4.1(I,2), it follows that all such  $(n+1)$ -ary polynomials are essential and distinct over  $\mathcal{A}$ . Therefore, the theorem is proved.

For each positive interger  $k$ , let  $\underline{K}_{1k}, \underline{K}_{2k}$  be the two equational classes of algebras as shown in Theorem 1.1. We have the following

Theorem 1.4.

There is a one-to-one butnot onto mapping from  $\underline{K}_{is}$  to  $\underline{K}_{it}$  for each  $i = 1, 2$ , if  $s < t$ .

Proof : We consider the case  $i = 1$ . The other case is similar.

For simplicity, we replace  $s, t$  by  $s - 2$  and  $t - 2$  respectively. Let  $\mathcal{A} \in \underline{K}_{1(s-2)}$ . Set

$$D = \left\{ (a_1, \dots, a_s) / a_i \in A, f(a_1, \dots, a_s) \neq \prod_{i=1}^s a_i \right\}.$$

Since  $f(x_1 x_2, x_2, \dots, x_s) = \prod_{i=1}^s x_i$  and



$f(x_1, x_1, x_3, \dots, x_s) = x_1 \prod_{i=3}^s x_i$  hold in  $\mathcal{A}$ , it follows that for each  $(a_1, \dots, a_s) \in D$ , the elements  $a_i$ 's are pairwise distinct and incomparable.

To each  $(a_1, \dots, a_s) \in D$ , let us adjoin a set of new elements  $\{a_{s+1}, \dots, a_t\}$  in such a way that

- 1) The set  $\{a_1, \dots, a_s, a_{s+1}, \dots, a_t\}$  is pairwise distinct and incomparable;
- 2)  $a_j \prec \prod_{i=1}^s a_i$ , for each  $j = s+1, \dots, t$ ;
- 3)  $a_j < c$  iff  $\prod_{i=1}^s a_i \leq c$  for  $c \in A$ , where  $j = s+1, \dots, t$ ;
- 4)  $a_j \parallel b$ , for each  $b \neq \prod_{i=1}^s a_i$  in  $A$ ,  $j = s+1, \dots, t$ ;
- 5) If  $(a_1, \dots, a_s), (b_1, \dots, b_s) \in D$  and  $\{a_1, \dots, a_s\} \neq \{b_1, \dots, b_s\}$ , then  $a_j \parallel b_r$  for  $j, r = s+1, \dots, t$ .

Let  $A^* = A \cup \bigcup \left\{ \{a_{s+1}, \dots, a_t\} / (a_1, \dots, a_s) \in D \right\}$ . Then

$A^*$  is a semilattice. Define a  $t$ -ary operation  $g$  in  $A^*$  as follows :

$$g(x_1, x_2, \dots, x_t) = \begin{cases} f(a_1, \dots, a_s) & \text{if } \{x_1, \dots, x_t\} = \{a_1, \dots, a_s, a_{s+1}, \dots, a_t\} \\ & \text{and } (a_1, \dots, a_s) \in D ; \\ \prod_{i=1}^t x_i & \text{otherwise .} \end{cases}$$

From the fact that  $\mathcal{A} \in \mathcal{K}_{1(s-2)}$ , it is easy to show that the algebra  $\mathcal{A}^* = \langle A^*; \cdot, g \rangle$  belongs to  $\mathcal{K}_{1(t-2)}$ . Moreover, it follows from our construction that if  $\mathcal{A} \neq \mathcal{B}$ , then  $\mathcal{A}^* \neq \mathcal{B}^*$ . Thus, the mapping  $\varphi : \mathcal{K}_{1(s-2)} \longrightarrow \mathcal{K}_{1(t-2)}$  defined by  $\varphi(\mathcal{A}) = \mathcal{A}^*$  for  $\mathcal{A}$  in  $\mathcal{K}_{1(s-2)}$  is one-to-one. Clearly,  $\varphi$  is not onto since there does not exist an algebra  $\mathcal{A}$  in  $\mathcal{K}_{1(s-2)}$  such that  $\mathcal{A}^* = I(2^{t-(t+1)}, t-2)$ .

Corollary 1.5.

For each  $i = 1, 2$  and  $k \geq 1$ , we have  $|\widetilde{K}_{ik}| \leq |\widetilde{K}_{i(k+1)}|$ .

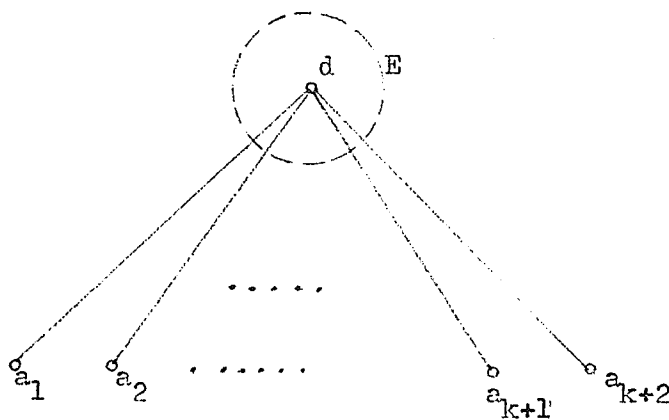
2. The Representability Theorem.

In this section, our purpose is to extend the results of Chapter 3 of Part I to the much more general situation. We will observe that by expanding those algebras representing  $\langle 0, 0, 1, m \rangle$  in a suitable way, we can construct algebras representing the sequence  $\langle 0, 0, \overline{1, \dots, 1}^k, m \rangle$  for a given pair of positive integers  $k, m$ .

One remark should be mentioned is that for  $k > 1$ , if  $\mathcal{A}$  is an algebra representing  $\langle 0, 0, \overline{1, 1, \dots, 1}^k, m \rangle$ , then  $\mathcal{A}$  has a unique semilattice operation. For  $k = 1$ , this is, however, not necessary in general.

Construction.

For each pair of positive integers  $\langle m, k \rangle$ , let us consider the following types of semilattice  $\langle A; \cdot \rangle$  ( see Fig. 31 )



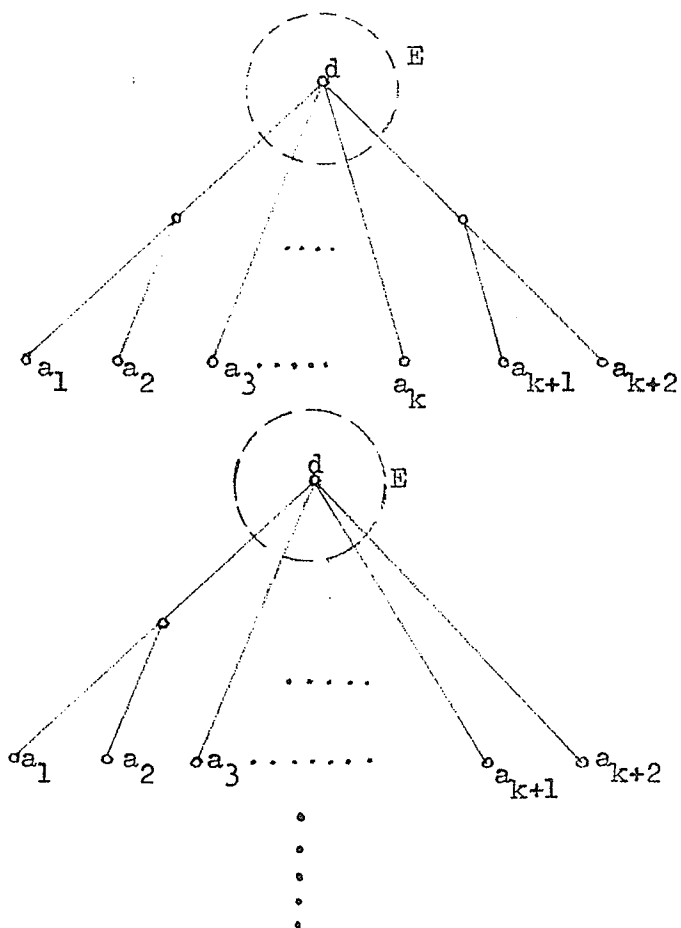


Fig. 31

where  $E = \{d, e_1, e_2, \dots, e_{m-1}\}$  is a  $m$ -element sub-semilattice of  $\langle A; \cdot \rangle$  and for each type,  $\langle (A-E) \cup \{d\}; \cdot \rangle$ , considered as a semilattice, is a homomorphic image of the free semilattice generated by  $\{a_1, a_2, \dots, a_{k+2}\}$ . The elements of  $E$  satisfy the following conditions: for each  $i = 1, 2, \dots, m-1$ ,

either (a)  $e_i > d$ ;

or (b)  $e_i < d$  and  $e_i$  is incomparable with every element of  $A - E$ ;

or (c)  $e_i$  is incomparable with every element of  $(A - E) \cup \{d\}$ .

Given any one of the semilattices  $\langle A; \cdot \rangle$ , we shall define,

for each  $i = 1, 2, \dots, m-1$ , a  $(k+2)$ -ary operation  $f_i$  on  $A$  as follows:

$$f_i(x_1, x_2, \dots, x_{k+2}) = \begin{cases} e_i & \text{if } \{x_1, \dots, x_{k+2}\} = \{a_1, \dots, a_{k+2}\} \\ \prod_{j=1}^{k+2} x_j & \text{otherwise .} \end{cases}$$

Note that for each  $i = 1, 2, \dots, m-1$

- (1)  $f_i(x_1, \dots, x_{k+2})$  is essentially  $(k+2)$ -ary;
- (2)  $f_i(x_1, \dots, x_{k+2}) \neq \prod_{j=1}^{k+2} x_j$ ;
- (3)  $f_i(x_1, \dots, x_{k+2})$  is idempotent and symmetric;
- (4)  $f_i(x_1, \dots, x_{k+2}) \neq f_j(x_1, \dots, x_{k+2})$  if  $i \neq j$ .

Thus, each of the semilattices  $\langle A; \cdot \rangle$ , is associated with an idempotent algebra  $\mathcal{A} = \langle A; F \rangle$  where  $F = \{\cdot, f_1, \dots, f_{m-1}\}$  consists of a semilattice operation and  $m-1$   $(k+2)$ -ary operations defined as above.

Theorem 2.1.

For each pair of positive integers  $\langle m, k \rangle$ , any one of the algebras with  $|E| = m$  constructed above has exactly  $m$  essentially  $(k+2)$ -ary polynomials.

Corollary 2.2.

For each pair of positive integers  $\langle m, k \rangle$ , the sequence  $\langle 0, 0, \overline{1, \dots, k}, \overline{1, m} \rangle$  is representable in  $\underline{K}$  where  $\underline{K}$  is the class of all idempotent algebras with a semilattice operation.

The proof of Theorem 2.1 can be carried out by modifying

that of Theorem 2.1(I,3).

### 3. The Characterization Theorem.

Consider the following sequence

$$(*) \quad \langle 0, 0, \overbrace{1, \dots, 1}^k, m, n \rangle$$

where  $k$  is a natural number,  $m$  and  $n$  are positive integers.

Theorem 2.1 says that if  $m=1$  then the sequence  $(*)$  is representable for each  $n = 1, 2, \dots$ . In this section, we are going to prove the converse of this result; namely, we show that if the sequence  $(*)$  is representable for each  $n = 1, 2, \dots$ , then it is necessary that  $m = 1$ .

First of all, we shall state the following Lemma which is a slight generalization of the Lemma 3 in J. Płonka [34]. Its proof is in fact identical with the proof of that Lemma.

#### Lemma 3.1.

Let  $\mathcal{A}$  be an algebra such that  $p_0(\mathcal{A}) = p_1(\mathcal{A}) = 0$ ,  $p_2(\mathcal{A}) > 1$ . If  $p_3(\mathcal{A}) > 0$  then  $p_3(\mathcal{A}) \geq 3$ .

#### Remark.

If, in addition, we assume  $p_2(\mathcal{A}) = 2$ , then this is Płonka's Lemma.

#### Lemma 3.2.

Suppose that the sequence

$$(*) : \quad \langle 0, 0, \overbrace{1, \dots, 1}^k, m, n \rangle$$

is representable. Then any one of the following conditions implies that  $m = 1$ .

- (1)  $k > 1$  and  $n \leq m + \max. \{ m, k+3 \}$  ;  
 (2)  $k = 1$  and  $n \leq 4$  ;  
 (3)  $k = 0, m > 0$  and  $0 < n \leq 2$ .

Proof : Assume (1) holds. Since  $k > 1$ ;  $p_2 = p_3 = 1$  and hence it follows that the only binary polynomial is a semilattice polynomial. If  $m \neq 1$ , then we can apply a result of G. Grätzer and J. Plonka to yield the following:

$$\begin{aligned} n = p_{k+3} &\geq p_k + 1 + \max. \{ p_{k+2}, k+3 \} \\ &= m + 1 + \max. \{ m, k+3 \} \\ &> m + \max. \{ m, k+3 \}, \end{aligned}$$

which contradicts our assumption. Thus,  $m = 1$ , as required.

Assume (2) holds. As  $k = 1$ , the sequence becomes

$$(*) : \langle 0, 0, 1, m, n \rangle.$$

If  $p_4 = n \leq 4 < \frac{1}{3}(2^4 - (-1)^4)$ , then by Lemma 1.2(I,1), it follows that there is a semilattice polynomial over any algebra  $\mathcal{A}$  representing the sequence (\*). Thus, if  $m > 1$ , we obtain

$$\begin{aligned} n = p_4(\mathcal{A}) &\geq p_3(\mathcal{A}) + 1 + \max. \{ p_3(\mathcal{A}), 4 \} \\ &= m + 1 + \max. \{ m, 4 \} \\ &> 1 + 4 = 5, \end{aligned}$$

which is a contradiction. Hence  $m = 1$ .

Finally, suppose that the condition (3) holds. In this case the sequence becomes (\*) :  $\langle 0, 0, m, n \rangle$ .

If  $m \neq 1$  then by (3),  $p_2 = m > 1$ . Since  $p_3 = n > 0$ , by invoking Lemma 3.1, it follows that  $p_3 = n \geq 3$ . But this

contradicts our assumption that  $n \leq 2$ . Hence we must have  $m = 1$ , completing the proof of Lemma 3.2.

In view of this Lemma and Corollary 2.2, we arrive at the following result.

Theorem 3.3.

Let  $k$  be a non-negative integer and  $m$ , a positive integer. The sequence  $\langle 0, 0, \overbrace{1, \dots, 1}^k, m, n \rangle$  is representable for every positive integer  $n$  if and only if  $m = 1$ .

## CHAPTER 2

### THE FUNCTION $F(n,k)$

In this chapter, we are devoted to the following problem :  
Given a positive integer  $n$ , what is the minimum value of  $m$  so that the sequence  $\langle 0,0,1,n,m \rangle$  is representable ? In other words, our object is to search for the number  $m^*$  such that

- (1) the sequence  $\langle 0,0,1,n,m^* \rangle$  is representable;
- (2) if the sequence  $\langle 0,0,1,n,m \rangle$  is representable, then  $m \geq m^*$ .

The result we obtain is the following : Let  $\underline{K}(1)$  be the class of all idempotent algebras with a semilattice operation such that all the essentially ternary polynomials are symmetric. For each  $n$ , let  $F(n)$  be the smallest integer such that the sequence  $\langle 0,0,1,n,F(n) \rangle$  is representable by algebras in  $\underline{K}(1)$ . Then  $F(n) = 10n - 9$ .

Furthermore, by applying similar techniques, we are able to extend the above result to a more general situation and obtain the following result : For each pair of positive integers  $\langle n,k \rangle$ ,  $k > 1$ , let  $F(n,k)$  be the smallest value such that the sequence  $\langle 0,0,\overbrace{1,\dots,1}^k,n,F(n,k) \rangle$  is representable in  $\underline{K}(k)$  where  $\underline{K}(k)$  is the class of all idempotent algebras such that all the essentially  $(k+2)$ -ary polynomials are symmetric. Then

$$F(n,k) = 1 + \frac{1}{2}(n-1)(k+3)(k+4).$$



1. Some Preliminary Results.

Throughout this chapter, we shall adopt the following notation.

Let  $\mathcal{A}(1) = \langle A_1; \cdot \rangle$  be the four-element semilattice ( see Fig. 32 )

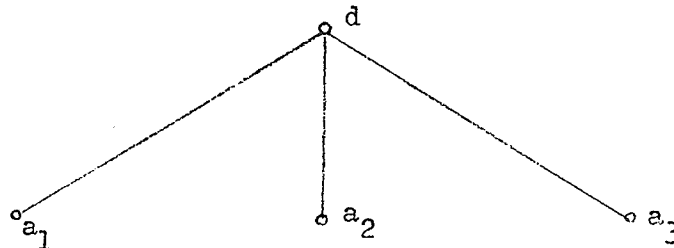


Fig. 32

Let  $\mathcal{A}(2) = \langle A_2; \cdot, f_1 \rangle$  be the algebra  $\text{II}(1)$  ( see Fig. 5 ).

Inductively, for each positive integer  $n$ , let  $\mathcal{A}(n) =$

$\langle A_n; \cdot, f_1, \dots, f_{n-1} \rangle$  be the algebra ( see Fig. 33 )

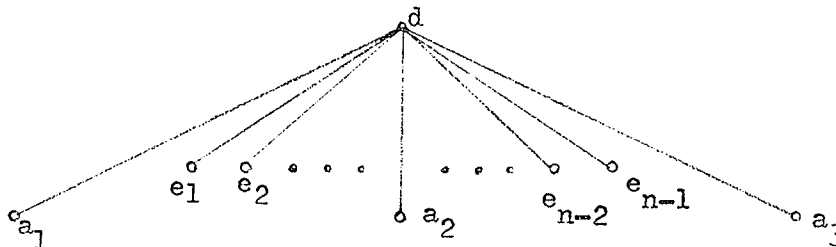


Fig. 33

where  $|A_n| = n + 3$ ; " $\cdot$ " is the join semilattice operation and

for each  $i = 1, 2, \dots, n-1$ ,  $f_i$  is a ternary operation defined as

follows :

$$f_i(x, y, z) = \begin{cases} e_i & \text{if } \{x, y, z\} = \{a_1, a_2, a_3\} \\ xyz & \text{otherwise.} \end{cases}$$

According to the results of Chapter 1, it is known that

for each positive integer  $n$ , the algebra  $\mathcal{A}(n)$  represents the

sequence  $\langle 0, 0, 1, n \rangle$ . We shall show that  $p_4(\mathcal{A}(n)) = 10n - 9$ .

To this end, we prove the following lemmas. For simplicity, if

$p = q$  is an identity, we write  $L$  for  $p$  and  $R$  for  $q$  ( $L, R$  stand

for left and right respectively ).

Lemma 1.1.

In the algebra  $\mathcal{A}(n)$ ,  $f_i(x_1x_2, x_3, x_4) = f_i(x_1, x_3, x_4)f_i(x_2, x_3, x_4)$ ,  
for each  $i = 1, 2, \dots, n-1$ .

Proof : It suffices to consider the case  $i = 1$ .

Let  $S$  be a substitution such that  $L(S) = e_1$ , i.e.,

$$f_1(x_1x_2, x_3, x_4)(S) = e_1.$$

This can happen only if either (1)  $(x_1x_2)(S) = x_3(S) = x_4(S) = e_1$

or (2)  $(x_1x_2)(S) = a_1$ ,  $x_3(S) = a_2$ ,  $x_4(S) = a_3$  ( or symmetrically ).

Assume (1) holds. We obtain  $x_i(S) = e_1$ , for each  $i = 1, 2, 3, 4$ .

Thus,  $R(S) = e_1$ .

If (2) is the case, then  $x_1(S) = x_2(S) = a_1$ ,  $x_3(S) = a_2$ ,

$x_4(S) = a_3$ . Thus,  $R(S) = f_1(a_1, a_2, a_3)f_1(a_1, a_2, a_3) = e_1$ .

Hence,  $L(S) = e_1$  implies that  $R(S) = e_1$ .

Let  $S$  be a substitution such that  $L(S) = e_k$ ,  $k = 2, \dots, n-1$ ,

i.e.,  $f_1(x_1x_2, x_3, x_4)(S) = e_k$ . Since  $k \neq 1$ , we have  $(x_1x_2)(S)$

$= x_3(S) = x_4(S) = e_k$ , by definition of  $f_1$ . Thus,  $x_i(S) = e_k$ ,

for each  $i = 1, 2, 3, 4$  and hence  $R(S) = e_k$ . Thus,  $L(S) = e_k$  implies

$R(S) = e_k$ .

Conversely, let  $S$  be a substitution such that  $R(S) = e_1$ .

Then  $f_1(x_1, x_3, x_4)(S) = f_1(x_2, x_3, x_4)(S) = e_1$ . We have either

(1)  $x_i(S) = e_1$ , for each  $i = 1, 2, 3, 4$  or (2)  $x_1(S) = x_2(S) = a_1$ ,

$x_3(S) = a_2$ ,  $x_4(S) = a_3$  ( or symmetrically ). Clearly, in each case,

$L(S) = e_1$ .

Let  $S$  be a substitution such that  $R(S) = e_k$ ,  $k = 2, \dots, n-1$ ,

i.e.,  $f_1(x_1, x_3, x_4)(S) = f_1(x_2, x_3, x_4)(S) = e_k$ . Since  $k \neq 1$ , it follows by definition that  $x_i(S) = e_k$ , for each  $i = 1, 2, 3, 4$ . Thus,  $L(S) = e_k$ . Hence, we prove that  $R(S) = e_i$  implies  $L(S) = e_i$ , for each  $i = 1, 2, \dots, n-1$ .

Therefore, we conclude that  $L = R$ , as required.

The following Lemma can be proved easily.

Lemma 1.2.

In the algebra  $\mathcal{A}(n)$ ,  $f_i(x_1, x_2, x_3)x_1 = x_1x_2x_3$ , for each  $i = 1, 2, \dots, n-1$ .

Lemma 1.3.

In  $\mathcal{A}(n)$ ,  $f_i(f_i(x_1, x_2, x_3), x_4, x_1) = \prod_{j=1}^4 x_j$ , for each  $i = 1, \dots, n-1$ .

Proof : We need only prove the lemma for  $i = 1$ .

Let  $S$  be a substitution such that  $L(S) = e_1$ . Then either (1)  $f_1(x_1, x_2, x_3)(S) = x_4(S) = x_1(S) = e_1$  or (2)  $f_1(x_1, x_2, x_3)(S) = a_1$ ,  $x_4(S) = a_2$ ,  $x_1(S) = a_3$  (or symmetrically). Evidently, (2) is impossible as  $f_1(a_3, x_2, x_3)(S) \neq a_1$ . Thus, we have (1). But then it follows that  $x_j(S) = e_1$ , for each  $j = 1, 2, 3, 4$ . Therefore,  $R(S) = \left( \prod_{j=1}^4 x_j \right)(S) = \prod e_1 = e_1$ .

Let  $S$  be a substitution such that  $L(S) = e_k$ ,  $k = 2, \dots, n-1$ . Then  $f_1(f_1(x_1, x_2, x_3), x_4, x_1)(S) = e_k$ . As  $k \neq 1$ , we obtain  $f_1(x_1, x_2, x_3)(S) = x_4(S) = x_1(S) = e_k$ . Thus,  $x_j(S) = e_k$  for each  $j = 1, 2, 3, 4$  and so  $R(S) = e_k$ . Hence,  $L(S) = e_j$  implies  $R(S) = e_j$ , for each  $j = 1, 2, \dots, n-1$ .

Conversely, it is clear that if  $S$  is a substitution and  $R(S) = e_j$ , then  $L(S) = e_j$ , for each  $j = 1, \dots, n-1$ .

From these, we thus conclude that  $L = R$  over  $\mathcal{C}(n)$ .

Lemma 1.4.

In  $\mathcal{C}(n)$ ,  $f_i(f_j(x_1, x_2, x_3), x_4, x_1) = \prod_{t=1}^4 x_t$ , where  $i, j = 1, 2, \dots, n-1$ , and  $i \neq j$ .

Proof : Without loss of generality, we may assume  $i = 1, j = 2$ .

Let  $S$  be a substitution such that  $L(S) = e_1$ , i.e.,

$$f_1(f_2(x_1, x_2, x_3), x_4, x_1)(S) = e_1$$

Then either (1)  $f_2(x_1, x_2, x_3)(S) = x_4(S) = x_1(S) = e_1$  or (2)  $f_2(x_1, x_2, x_3)(S) = a_1, x_4(S) = a_2, x_1(S) = a_3$  ( or symmetrically )

Observe that (2) is impossible as  $f_2(a_3, x_2, x_3)(S) \neq a_1$ .

Hence, we have (1). But this implies that  $x_t(S) = e_1$ , for each  $t = 1, 2, 3, 4$ . Thus,  $R(S) = e_1$ .

Let  $S$  be such that  $L(S) = e_2$ . Then we get

$$f_2(x_1, x_2, x_3)(S) = x_4(S) = x_1(S) = e_2.$$

Since  $f_2(e_2, x_2, x_3)(S) = e_2$  implies  $x_t(S) = e_2, t = 1, 2, 3, 4$ , it follows that  $R(S) = e_2$ .

Let  $S$  be such that  $L(S) = e_k, k = 3, \dots, n-1$ . Then we get  $x_t(S) = e_k, t = 1, 2, 3, 4$ . Thus,  $R(S) = e_k$ .

Conversely, if  $S$  is a substitution such that  $R(S) = e_k, k = 1, 2, \dots, n-1$ , then it follows immediately that  $L(S) = e_k$ .

Hence, we have  $L(S) = e_k$  if, and only if  $R(S) = e_k, k = 1, 2, \dots, n-1$ , in  $\mathcal{C}(n)$ . Therefore,  $L = R$ , which completes the proof of Lemma 4.

Lemma 1.5.

Let  $f_i(p, q, r)$  be an essentially 4-ary polynomial over  $\mathcal{K}(n)$  where  $p$ ,  $q$  and  $r$  are pairwise distinct polynomials over  $\mathcal{K}(n)$  which contain no sub-polynomials of the product form  $x_s x_t$ . Then

$$f_i(p, q, r) = \prod_{j=1}^4 x_j \text{ in } \mathcal{K}(n).$$

Proof : Let  $i = 1$ , and let  $S$  be a substitution such that  $L(S) = e_1$ . Then we have either (1)  $p(S) = a_1$ ,  $q(S) = a_2$ ,  $r(S) = a_3$  ( or symmetrically ) or (2)  $p(S) = q(S) = r(S) = e_1$ .

Claim : (1) is impossible.

Observe that since  $f_1(p, q, r)$  is essentially 4-ary, it consists of four distinct variables. As  $f_j(x_1, x_2, x_2) = x_1 x_2$  holds in  $\mathcal{K}(n)$  and there is no product of the form  $x_s x_t$  occurring in  $p$ ,  $q$  and  $r$ , it follows that at least one of the  $\{p, q, r\}$  must contain a factor  $f_j( , , )$  consisting of at least three distinct variables. Therefore, at least two of  $\{p, q, r\}$  have a variable in common. Suppose that (1) is the case. Then all variables in  $p$ ,  $q$  and  $r$  must be substituted by  $a_1, a_2$  and  $a_3$  respectively ( or symmetrically ). This can happen only if any two of  $\{p, q, r\}$  have no variables in common, a contradiction. Thus, (1) is impossible, as required.

Hence, we have (2). It then follows immediately from the assumption that  $x_i(S) = e_1$ , for each  $i = 1, 2, 3, 4$ . Thus,  $R(S) = \prod e_1 = e_1$ .

Let  $S$  be such that  $f_1(p, q, r)(S) = e_k$ ,  $k = 2, \dots, n-1$ . Then,  $p(S) = q(S) = r(S) = e_k$  by definition. From this, it follows

that  $x_i(S) = e_k$ , for each  $i = 1, 2, 3, 4$ . Thus,  $R(S) = \prod e_k = e_k$ .

Conversely, it is clear that  $R(S) = e_j$  implies  $L(S) = e_j$ , for each  $j = 1, 2, \dots, n-1$ . Hence we conclude that  $L = R$ , as was to be shown.

Remark.

Let us note that Lemmas 1.3 and 1.4 are indeed special cases of Lemma 1.5.

Lemma 1.6.

Let  $\prod_{\alpha \in \Lambda} f_i(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)})$  be an essentially 4-ary polynomial over  $\mathcal{C}(n)$ , where  $i = 1, 2, \dots, n-1$  is a fixed index.

Then  $\prod_{\alpha \in \Lambda} f_i(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}) = \prod_{j=1}^4 x_j$  in  $\mathcal{C}(n)$  if  $|\Lambda| \geq 3$ .

Proof : Let  $i = 1$ . We may assume that for each  $\alpha \in \Lambda$ , the set

$\{x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}\}$  of variables is pairwise distinct. For otherwise, using the identities

$$f_1(x_1, x_2, x_2) = x_1 x_2$$

$$\text{and } f_1(x_1, x_2, x_3) x_1 = x_1 x_2 x_3$$

that hold in  $\mathcal{C}(n)$ , it is easy to check that  $\prod_{\alpha \in \Lambda} f_1(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}) = \prod_{j=1}^4 x_j$ .

If  $|\Lambda| = 3$ , then  $\prod_{\alpha \in \Lambda} f_1(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}) = f_1(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}) f_1(x_{\beta(1)}, x_{\beta(2)}, x_{\beta(3)}) f_1(x_{\gamma(1)}, x_{\gamma(2)}, x_{\gamma(3)})$ , say.

In order that the variables in each factor  $f_1(, , )$  are pairwise distinct, the number of occurrences of each of the four variables  $x_1, x_2, x_3, x_4$  is at least two and at most three in the

polynomial. Thus the partition of nine positions should be

$$2 : 2 : 2 : 3$$

Without loss of generality, we may assume the following distribution (See Fig. 34)

$x_1$	$x_2$	$x_3$	$x_4$
3	2	2	2

Fig. 34

$$\begin{aligned} \text{Thus, we have } \prod_{\alpha \in \Lambda} f_1(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}) \\ = f_1(x_1, x_2, x_3) f_1(x_1, x_2, x_4) f_1(x_1, x_3, x_4). \end{aligned}$$

To prove that  $f_1(x_1, x_2, x_3) f_1(x_1, x_2, x_4) f_1(x_1, x_3, x_4) = \prod_{j=1}^4 x_j$  in  $\mathcal{L}(n)$ , let  $S$  be a substitution such that  $L(S) = e_1$ . This implies

$$f_1(x_1, x_2, x_3)(S) = f_1(x_1, x_2, x_4)(S) = f_1(x_1, x_3, x_4)(S) = e_1.$$

From this, it follows that  $x_i(S) = e_1$ , for each  $i = 1, 2, 3, 4$ .

Thus  $R(S) = e_1$ . If  $L(S) = e_k$ ,  $k = 2, \dots, n-1$  then  $x_i(S) = e_k$ , for each  $i = 1, 2, 3, 4$ . Hence,  $R(S) = e_k$ .

Conversely, it is clear that  $R(S) = e_j$  implies  $L(S) = e_j$ , for each  $j = 1, \dots, n-1$ . Hence we conclude that for  $|\Lambda| = 3$

$$\prod_{\alpha \in \Lambda} f_1(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}) = \prod_{j=1}^4 x_j.$$

If  $|\Lambda| > 3$ , then

$$\begin{aligned} \prod_{\alpha \in \Lambda} f_1(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}) &= f_1(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}) f_1(x_{\beta(1)}, x_{\beta(2)}, x_{\beta(3)}) \\ &\quad f_1(x_{\gamma(1)}, x_{\gamma(2)}, x_{\gamma(3)}) \prod_{\delta} f_1(x_{\delta(1)}, x_{\delta(2)}, x_{\delta(3)}) \\ &= x_1 x_2 x_3 x_4 \prod_{\delta} f_1(x_{\delta(1)}, x_{\delta(2)}, x_{\delta(3)}) \\ &= \prod_{j=1}^4 x_j. \end{aligned}$$

Hence, the proof of Lemma 16 is complete.

Lemma 1.7.

In  $\mathcal{A}(n)$ ,  $f_i(x_1, x_2, x_3) f_j(x_2, x_3, x_4) = \prod_{t=1}^4 x_t$  where  $i, j = 1, 2, \dots, n-1$  and  $i \neq j$ .

Proof : We may assume  $i = 1, j = 2$ .

Let  $S$  be a substitution such that  $L(S) = e_1$ . Then we get  $f_1(x_1, x_2, x_3)(S) = f_2(x_2, x_3, x_4)(S) = e_1$ . Hence,  $x_i(S) = e_1$ , for each  $i = 1, 2, 3, 4$ . This implies  $R(S) = e_1$ . Symmetrically,  $L(S) = e_2$  implies  $R(S) = e_2$ .

Let  $S$  be such that  $L(S) = e_k, k = 3, 4, \dots, n-1$ . Clearly, we obtain  $x_i(S) = e_k$ . Thus,  $R(S) = e_k$ .

The converse is trivial. Hence Lemma 7 follows.

We are now in a position to establish the following

Proposition 1.8.

There are exactly  $10n-9$  distinct essentially 4-ary polynomials over  $\mathcal{A}(n)$ . They are :

$$(*) \left[ \begin{array}{l} x_1 x_2 x_3 x_4 ; \\ f_i(x_1, x_2, x_3) x_4, f_i(x_2, x_3, x_4) x_1, f_i(x_3, x_4, x_1) x_2, f_i(x_4, x_1, x_2) x_3 ; \\ f_i(x_1, x_2, x_3) f_i(x_1, x_2, x_4), f_i(x_1, x_3, x_2) f_i(x_1, x_3, x_4), \\ f_i(x_1, x_4, x_2) f_i(x_1, x_4, x_3), f_i(x_2, x_3, x_1) f_i(x_2, x_3, x_4), \\ f_i(x_2, x_4, x_1) f_i(x_2, x_4, x_3), f_i(x_3, x_4, x_1) f_i(x_3, x_4, x_2), \end{array} \right.$$

For each  $i = 1, 2, \dots, n-1$ .

Proof : First of all, it is routine to check that the 4-ary



polynomials in (\*) are distinct and essential over  $\mathcal{K}(n)$ .

Now, let  $p(x_1, x_2, x_3, x_4)$  be an essentially 4-ary polynomial over  $\mathcal{K}(n)$ . By Lemma 1.1,  $p$  can be written as

$$p(x_1, x_2, x_3, x_4) = \prod f_1(A_{\alpha 1}, B_{\alpha 1}, C_{\alpha 1}) \prod f_2(A_{\alpha 2}, B_{\alpha 2}, C_{\alpha 2}) \dots \dots \prod f_{n-1}(A_{\alpha(n-1)}, B_{\alpha(n-1)}, C_{\alpha(n-1)}) \prod x_k,$$

where, for each  $\alpha$  and  $i$ ,  $A_{\alpha i}$ ,  $B_{\alpha i}$ ,  $C_{\alpha i}$  are polynomials which consist of no sub-polynomials of the product of the form  $x_i x_j$ .

We may assume that  $p$  cannot be reduced to a simpler form.

By Lemmas 1.3, 1.4, 1.5, we have

$$p(x_1, x_2, x_3, x_4) = \prod f_1(x_{\alpha 1}, x_{\beta 1}, x_{\gamma 1}) \prod f_2(x_{\alpha 2}, x_{\beta 2}, x_{\gamma 2}) \dots \dots \prod f_{n-1}(x_{\alpha(n-1)}, x_{\beta(n-1)}, x_{\gamma(n-1)}) \prod x_k.$$

If  $p$  has no "f<sub>i</sub>" factors, for each  $i = 1, \dots, n-1$ , then

$$p = \prod_{j=1}^4 x_j.$$

If  $p$  has a factor "f<sub>i</sub>" for some  $i$ , then by Lemma 1.7, it follows that

$$p = \prod_{\Lambda} f_i(x_{\alpha i}, x_{\beta i}, x_{\gamma i}) \prod x_k.$$

Case 1.  $|\Lambda| = 1$ .

In this case,  $p = f_i(x_{\alpha}, x_{\beta}, x_{\gamma}) x_{\delta}$ , where  $\{\alpha, \beta, \gamma, \delta\} = \{1, 2, 3, 4\}$  by Lemma 1.2.

Case 2.  $|\Lambda| = 2$ .

$$\begin{aligned} \text{In this case, } p &= f_i(x_{\alpha}, x_{\beta}, x_{\gamma}) f_i(x_{\alpha}, x_{\beta}, x_{\delta}) \prod x_k \\ &= f_i(x_{\alpha}, x_{\beta}, x_{\gamma}) f_i(x_{\alpha}, x_{\beta}, x_{\delta}), \end{aligned}$$

where  $\{\alpha, \beta, \gamma, \delta\} = \{1, 2, 3, 4\}$ , by Lemma 1.2.

Case 3.  $|\Lambda| \geq 3$ .

If this is the case, then by Lemma 1.6,  $p = \prod_{j=1}^4 x_j$ .

Hence, any essentially 4-ary polynomial over  $\mathcal{A}(n)$  must be equal to one of the polynomials in (\*). The number of the polynomials in (\*) is  $1 + (n-1)\binom{4}{1} + (n-1)\binom{4}{1} = 10n - 9$ . This completes the proof of Proposition 1.8.

## 2. The Value of $F(n,1)$ .

Let  $\underline{K}(1)$  be the class of all idempotent algebras with a semilattice operation such that all the essentially ternary polynomials are symmetric. Thus, for instance,  $\mathcal{A}(n)$  is an element in  $\underline{K}(1)$ , for  $n = 1, 2, \dots$ . For each positive integer  $n$ , let  $\mathcal{A}$  be an algebra in  $\underline{K}(1)$  representing  $\langle 0, 0, 1, n \rangle$ . Since  $\mathcal{A}$  has one semilattice operation "."; we have one essentially ternary polynomial  $x_1 x_2 x_3$ . As  $p_3(\mathcal{A}) = n$ , let

$$\left\{ \begin{array}{l} g_1(x_1, x_2, x_3) \\ g_2(x_1, x_2, x_3) \\ \vdots \\ g_{n-1}(x_1, x_2, x_3) \end{array} \right.$$

denote the other  $n-1$  essentially ternary polynomials over  $\mathcal{A}$ .

Note that as  $\mathcal{A} \in \underline{K}(1)$ ,  $g_i(x_1, x_2, x_3)$  is symmetric, for each  $i = 1, 2, \dots, n-1$ .

In this section, our purpose is to prove that, corresponding to the  $10n - 9$  distinct essentially 4-ary polynomials of  $\mathcal{A}(n)$  which were described in Proposition 1.8, the following  $10n - 9$  4-ary polynomials are distinct and essential over  $\mathcal{A}$  :

$$\begin{aligned}
 & x_1 x_2 x_3 x_4, \\
 & g_i(x_1, x_2, x_3) x_4, g_i(x_2, x_3, x_4) x_1, g_i(x_3, x_4, x_1) x_2, g_i(x_4, x_1, x_2) x_3, \\
 (\Delta) \quad & g_i(x_1, x_2, x_3) g_i(x_1, x_2, x_4), g_i(x_1, x_3, x_2) g_i(x_1, x_3, x_4), \\
 & g_i(x_1, x_4, x_2) g_i(x_1, x_4, x_3), g_i(x_2, x_3, x_4) g_i(x_2, x_3, x_1), \\
 & g_i(x_2, x_4, x_1) g_i(x_2, x_4, x_3), g_i(x_3, x_4, x_1) g_i(x_3, x_4, x_2),
 \end{aligned}$$

for each  $i = 1, 2, \dots, n-1$ .

If  $n = 1$ , we have nothing to prove.

If  $n = 2$ , then  $\mathcal{A}$ , being an algebra representing  $\langle 0, 0, 1, 2 \rangle$ , must belong to one of the equational classes  $\underline{K}_1$  and  $\underline{K}_2$ . If  $\mathcal{A} \in \underline{K}_1$ , then  $p_4(\mathcal{A}) \geq p_4(I(1))$ . If  $\mathcal{A} \in \underline{K}_2$ , then  $p_4(\mathcal{A}) \geq p_4(II(1)) = p_4(\mathcal{A}(2))$ . Thus, it suffices to show that the above eleven polynomials are distinct and essential over  $I(1)$ . However, it is clear that this is, indeed, the case. Hence, from now on, we may assume that  $n \geq 3$ .

We need the following Lemmas :

Lemma 2.1.

In  $\mathcal{A}$ ,  $g_i(x_1, x_2, x_2) = x_1 x_2$ , for each  $i = 1, 2, \dots, n-1$ .

Proof : We need only prove the lemma for  $i = 1$ . Since  $\mathcal{A}$  represents  $\langle 0, 0, 1 \rangle$ ; we have only three possible cases :

$$g_1(x_1, x_2, x_2) = \begin{cases} x_1 \\ x_2 \\ x_1 x_2 \end{cases}$$

Case 1.  $g_1(x_1, x_2, x_2) = x_1$ .

We first claim that the following polynomials are pairwise distinct and essentially ternary.

$$\begin{bmatrix} g_1(x_1, x_2, x_3)x_1, g_1(x_1, x_2, x_3)x_2, g_1(x_1, x_2, x_3)x_3 \\ g_2(x_1, x_2, x_3) \\ \vdots \\ g_{n-1}(x_1, x_2, x_3) \end{bmatrix}$$

Observe that if  $g_1(x_1, x_2, x_3)x_1 = g_1(x_1, x_2, x_3)x_2$ , then setting  $x_2 = x_3$ , we obtain  $x_1 = x_1x_2$ , a contradiction. Hence by symmetry, the polynomials  $g_1(x_1, x_2, x_3)x_1, g_1(x_1, x_2, x_3)x_2, g_1(x_1, x_2, x_3)x_3$  are pairwise distinct.

If  $g_1(x_1, x_2, x_3)x_1 = g_i(x_1, x_2, x_3)$  for some  $i = 2, \dots, n-1$ , then  $g_i(x_2, x_3, x_1) = g_1(x_2, x_3, x_1)x_2$ . From this, it follows that

$$\begin{aligned} g_1(x_1, x_2, x_3)x_1 &= g_i(x_2, x_3, x_1) \\ &= g_1(x_2, x_3, x_1)x_2 \\ &= g_1(x_1, x_2, x_3)x_2, \end{aligned}$$

which is a contradiction. Hence we conclude that the above  $n+1$  ternary polynomials are pairwise distinct.

Consider  $p = g_1(x_1, x_2, x_3)x_1$ . Set  $x_2 = x_3$ . We have

$$p = g_1(x_1, x_2, x_2)x_1 = x_1.$$

Hence,  $p$  depends on  $x_1$ . Setting  $x_1 = x_3$ , it follows that

$p = x_2x_1$ . Thus,  $p$  depends on  $x_2$ . As  $p$  is symmetric with respect

to  $x_2$  and  $x_3$ ,  $p$  also depends on  $x_3$ . Thus  $p$  is essentially ternary.

Similarly,  $g_1(x_1, x_2, x_3)x_2$  and  $g_1(x_1, x_2, x_3)x_3$  are essential.

Hence, if we assume  $g_1(x_1, x_2, x_2) = x_1$ , we have  $p_3(\mathcal{U}) \geq n+1$ .

which is a contradiction. Thus case 1 is impossible.

Case 2.  $g_1(x_1, x_2, x_2) = x_2$ .

Again, we claim that the polynomials as shown in case 1 are distinct and essential.

Note that if  $g_1(x_1, x_2, x_3)x_1 = g_1(x_1, x_2, x_3)x_2$ , then setting  $x_1 = x_3$ , it follows that  $x_1 = x_1x_2$ , a contradiction. If  $g_1(x_1, x_2, x_3)x_1 = g_i(x_1, x_2, x_3)$  for some  $i = 2, 3, \dots, n-1$ , then, as in case 1, we would have

$$g_1(x_1, x_2, x_3)x_1 = g_1(x_1, x_2, x_3)x_2,$$

which is impossible. Hence we have  $n + 1$  distinct polynomials.

Now, observe that if one of the polynomials  $g_1(x_1, x_2, x_3)x_i$ ,  $i = 1, 2, 3$  is essentially ternary, then, by the symmetry of  $g_1(x_1, x_2, x_3)$ , so are the other two. In this situation, we would have  $p_3(\mathcal{R}) \geq n + 1$ , which is a contradiction. Hence,  $g_1(x_1, x_2, x_3)x_1$  cannot be essentially ternary. As  $\mathcal{R}$  represents  $\langle 0, 0, 1 \rangle$ ; we have the following possibilities :

$$g_1(x_1, x_2, x_3)x_1 = \begin{cases} x_1 \\ x_2 \\ x_3 \\ x_1x_2 \\ x_1x_3 \\ x_2x_3 \end{cases}$$

Since  $g_1(x_1, x_2, x_3)x_1$  is symmetric with respect to  $x_2$  and  $x_3$ ; it follows that

$$g_1(x_1, x_2, x_3)x_1 = \begin{cases} x_1 \\ x_2x_3 \end{cases}$$

If  $g_1(x_1, x_2, x_3)x_1 = x_1$ , then setting  $x_2 = x_3$ , we obtain  $x_2x_1 = x_1$ , a contradiction. If  $g_1(x_1, x_2, x_3)x_1 = x_2x_3$ , setting  $x_2 = x_3$ , it follows that  $x_2x_1 = x_2$ , which is a contradiction.

These arguments show that the assumption  $g_1(x_1, x_2, x_2) = x_2$  is impossible. Therefore, we must have

$$g_1(x_1, x_2, x_2) = x_1x_2,$$

as was to be shown.

Lemma 2.2.

In  $\mathcal{R}$ ,  $g_i(x_1x_2, x_1, x_2) = x_1x_2$ , for each  $i = 1, 2, \dots, n-1$ .

Proof : Since  $p_0(\mathcal{R}) = 0$ ,  $g_i(x_1x_2, x_1, x_2)$  is not a constant.

However, if  $g_i(x_1x_2, x_1, x_2)$  depends on  $x_1$ , it depends on  $x_2$  by symmetry. Thus,  $g_i(x_1x_2, x_1, x_2)$  is essentially binary and hence

$$g_i(x_1x_2, x_1, x_2) = x_1x_2,$$

as  $p_2(\mathcal{R}) = 1$ .

Lemma 2.3.

In  $\mathcal{R}$ ,  $g_i(x_1x_2, x_2, x_3) = x_1x_2x_3$ , for each  $i = 1, 2, \dots, n-1$ .

Proof : Assume  $i = 1$ . Observe that the polynomial  $p = g_1(x_1x_2, x_2, x_3)$  is essentially ternary. For, if we set  $x_1 = x_2$ , we obtain  $p = x_1x_3$ , by Lemma 2.1. Thus,  $p$  depends on  $x_3$ . Setting  $x_1 = x_3$ , we have  $p = x_1x_2$ , by Lemma 2.2. Thus,  $p$  depends on  $x_2$ . Setting  $x_2 = x_3$ , it follows that  $p = x_1x_2$ , by Lemma 2.1. Thus,  $p$  depends on  $x_1$ . Hence,  $p$  is an essentially ternary polynomial.

Since  $p_3(\mathcal{R}) = n$ , it follows that

$$g_1(x_1x_2, x_2, x_3) = \begin{cases} g_1(x_1, x_2, x_3) \\ g_k(x_1, x_2, x_3), & k \neq 1 \\ x_1x_2x_3 \end{cases}$$

$$\text{If } g_1(x_1x_2, x_2, x_3) = g_1(x_1, x_2, x_3) \text{ ————— (1)}$$

then observe that

$$\begin{aligned} x_1x_2x_3 &= g_1(x_1x_2x_3, x_1x_2, x_1x_3) && \text{(Lemma 2.2)} \\ &= g_1(x_3(x_1x_2), x_1x_2, x_1x_3) \\ &= g_1(x_3, x_1x_2, x_1x_3) && \text{(by (1))} \\ &= g_1(x_1, x_3, x_1x_2) && \text{(by (1))} \\ &= g_1(x_1, x_2, x_3) && \text{(by (1))} \end{aligned}$$

which is a contradiction. Hence (1) is impossible.

$$\text{If } g_1(x_1x_2, x_2, x_3) = g_k(x_1, x_2, x_3), \text{ for some } k = 2, 3, \dots, n-1 \text{ — (2)}$$

$$\begin{aligned} \text{then note that } g_k(x_1x_2, x_2, x_3) &= g_1((x_1x_2)x_2, x_2, x_3) && \text{(by (2))} \\ &= g_1(x_1x_2, x_2, x_3) \\ &= g_k(x_1, x_2, x_3) && \text{(by (2))} \end{aligned}$$

$$\text{Thus, we have } g_k(x_1x_2, x_2, x_3) = g_k(x_1, x_2, x_3) \text{ ————— (3)}$$

Now, observe that

$$\begin{aligned} x_1x_2x_3 &= g_1(x_1x_2x_3, x_1x_2, x_1x_3) && \text{(Lemma 2.2)} \\ &= g_k(x_3, x_1x_2, x_1x_3) && \text{(by (2))} \\ &= g_k(x_1, x_3, x_1x_2) && \text{(by (3))} \\ &= g_k(x_2, x_1, x_3) && \text{(by (3))} \\ &= g_k(x_1, x_2, x_3), \end{aligned}$$

a contradiction. Hence, (2) is impossible. It therefore

$$\text{follows that } g_1(x_1x_2, x_2, x_3) = x_1x_2x_3,$$

proving Lemma 2.3.

With the aid of these Lemmas, we are now in a position to prove the following result.

Proposition 2.4.

Let  $\mathcal{A}$  be an algebra in  $\underline{K}(1)$  representing  $\langle 0,0,1,n \rangle$ .  
Then  $p_4(\mathcal{A}) \geq 10n-9$ .

Proof : It suffices to consider  $n \geq 3$ .

By Lemma 2.1, it is easy to prove that the polynomials in  $(\Delta)$  are essentially 4-ary Over  $\mathcal{A}$ . For instance, take

$$p = g_i(x_1, x_2, x_3)x_4.$$

Clearly,  $p$  depends on  $x_4$ . Setting  $x_1 = x_2$ , we have  $p = x_1x_3x_4$ .

Thus,  $p$  depends on  $x_3$ . By symmetry,  $p$  depends on every variable.

Hence,  $p$  is essentially 4-ary. Next, consider

$$p = g_i(x_1, x_2, x_3)g_i(x_1, x_2, x_4).$$

Set  $x_3 = x_4$ . We obtain  $p = g_i(x_1, x_2, x_3)$  which is essentially ternary. Thus,  $p$  depends on  $x_1$  and  $x_2$ . Set  $x_1 = x_2$ . It follows that  $p = g_i(x_1, x_1, x_3)g_i(x_1, x_1, x_4) = x_1x_3x_4$ , which depends on  $x_3$  and  $x_4$ . Hence  $p$  is an essentially 4-ary polynomial.

It remains to prove that all the polynomials in  $(\Delta)$  are distinct. By symmetry, we have only to prove that the twenty-one essential polynomials with  $i = 1, 2$  are distinct.

In preparation, we make the following observations.

Since  $x_1x_2x_3 \neq g_1(x_1, x_2, x_3)$ ; there exist  $c_1, c_2, c_3 \in A$  such that  $c_1c_2c_3 \neq g_1(c_1, c_2, c_3)$ . Let

$$c^* = g_1(c_1, c_2, c_3)$$

$$\text{and } \bar{c} = c_1c_2c_3.$$



Observe that if  $c_2 = c_3$  (or symmetrically), then by Lemma 2.1,

$$g_1(c_1, c_2, c_3) = g_1(c_1, c_2, c_2) = c_1 c_2 = c_1 c_2 c_3, \text{ a contradiction.}$$

If  $c_2 \geq c_3$  then  $c_2 c_3 = c_2$ . It follows by Lemma 2.3 that  $c_1 c_2 c_3$

$$= g_1(c_1, c_3, c_2 c_3) = g_1(c_1, c_3, c_2), \text{ which is impossible. Hence}$$

we conclude that (1)  $c_1, c_2, c_3$  are pairwise distinct;

(2)  $c_1, c_2, c_3$  are pairwise incomparable (Thus,

$$\bar{c} > c_i, i = 1, 2, 3).$$

As  $g_1(x_1, x_2, x_3) \neq g_2(x_1, x_2, x_3)$ ; there exist  $b_1, b_2, b_3 \in A$  such that  $g_1(b_1, b_2, b_3) \neq g_2(b_1, b_2, b_3)$ . Let

$$b^* = g_1(b_1, b_2, b_3)$$

$$\text{and } b' = g_2(b_1, b_2, b_3).$$

Note that if  $b_1 \geq b_2$  then  $b_1 b_2 = b_1$ . Thus, by Lemma 2.3, we have

$$g_1(b_1, b_2, b_3) = g_1(b_1 b_2, b_2, b_3) = b_1 b_2 b_3 = g_2(b_1 b_2, b_2, b_3)$$

$= g_2(b_1, b_2, b_3)$ , a contradiction. Hence we conclude that the

elements  $b_1, b_2$  and  $b_3$  are pairwise distinct and incomparable.

We are now ready to prove that the polynomials are distinct.

$$(1) x_1 x_2 x_3 x_4 \neq g_1(x_1, x_2, x_3) x_4.$$

There are several cases to consider :

$$(a) \bar{c} > c^*.$$

Let  $S$  be a substitution such that  $x_i(S) = c_i, i = 1, 2, 3,$   
 $x_4(S) = c^*$ . Then  $L(S) = \bar{c} c^* = \bar{c}$  while  $R(S) = c^* c^* = c^*$ . Thus,  
 $L(S) \neq R(S)$ .

$$(b) \bar{c} < c^*.$$

Let  $S$  be such that  $x_i(S) = c_i, i = 1, 2, 3, x_4(S) = c_1$ . Then  
 $R(S) = c^* c_1 = (c^* \bar{c}) c_1 = c^* (\bar{c} c_1) = c^* \bar{c} = c^*$  while  $L(S) = c_1 c_2 c_3 c_1$

$= \bar{c}$ . Thus,  $L(S) \neq R(S)$ .

(c)  $\bar{c} \parallel c^*$ .

Let  $S$  be the same as that in (a). We have  $R(S) = c^*$  while  $L(S) = \bar{c}c^* > c^*$ . Thus,  $L(S) \neq R(S)$ .

$$(2) \quad x_1x_2x_3x_4 \neq g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4).$$

If (2) is not the case, then setting  $x_3 = x_4$ , we obtain  $x_1x_2x_3 = g_1(x_1, x_2, x_3)$ , a contradiction. Thus, (2) follows.

Before carrying on, let us prove the following assertion.

(W) In  $\mathcal{R}$ , if  $c^* < \bar{c}$ , then  $c_i \not\leq c^*$  for each  $i = 1, 2, 3$ .

To this end, we need only prove that  $c_1 \not\leq c^*$ .

Assume that  $c^* < \bar{c}$ . Consider the polynomial  $g_1(x_1, x_2, x_3)x_1$ .

Evidently, it is essential. Thus we have

$$g_1(x_1, x_2, x_3)x_1 = \begin{cases} g_1(x_1, x_2, x_3) \\ x_1x_2x_3 \\ g_k(x_1, x_2, x_3), k \neq 1. \end{cases}$$

The case  $g_1(x_1, x_2, x_3)x_1 = g_1(x_1, x_2, x_3)$  is impossible. For equality implies, by symmetry, that  $g_1(x_1, x_2, x_3)x_1x_2x_3 = g_1(x_1, x_2, x_3)$ . Thus, we obtain  $c^*\bar{c} = c^*$ , i.e.,  $c^* \geq \bar{c}$ , a contradiction.

If  $g_1(x_1, x_2, x_3)x_1 = x_1x_2x_3$ , we have  $c^*c_1 = \bar{c} > c^*$ . Thus, if  $c_1 \leq c^*$ , it follows that  $c^* = c^*c_1 > c^*$ , a contradiction. Hence  $c^* \not\leq c_1$ , as required.

If  $g_1(x_1, x_2, x_3)x_1 = g_k(x_1, x_2, x_3)$ ,  $k \neq 1$ , then by symmetry,  $g_1(x_1, x_2, x_3)x_1x_2x_3 = g_k(x_1, x_2, x_3)$ . Thus,  $g_k(c_1, c_2, c_3) = c^*\bar{c}$

$= \bar{c} > c^*$ . If  $c_1 \leq c^*$ , then  $c^* = c^*c_1 = g_1(c_1, c_2, c_3)c_1 = g_k(c_1, c_2, c_3) > c^*$ , a contradiction. Hence,  $c_1 \not\leq c^*$ , as required. This proves (W).

$$(3) \quad g_1(x_1, x_2, x_3)x_4 \neq g_1(x_2, x_3, x_4)x_1.$$

There are three cases to consider :

$$(a) \quad c^* < \bar{c}.$$

Let  $S$  be such that  $x_i(S) = c_i$ ,  $i = 1, 2, 3$ ,  $x_4(S) = c^*$ . Then  $L(S) = c^*c^* = c^*$ ,  $R(S) = g_1(c_2, c_3, c^*)c_1$ . Note that  $R(S) \neq c^*$ . For, if it were, then  $g_1(c_2, c_3, c^*)c_1 = c^*$  implies  $c^* \geq c_1$ . This, however, contradicts the assertion (W).

$$(b) \quad c^* > \bar{c}.$$

Let  $S$  be such that  $x_i(S) = c_i$ ,  $i = 1, 2, 3$ ,  $x_4(S) = c_3$ . Then  $L(S) = c^*c_3 = c^*$  while  $R(S) = \bar{c}$  by Lemma 2.1.

$$(c) \quad c^* \parallel \bar{c}.$$

Let  $S$  be that of (b). Then  $L(S) \geq c^*$ ,  $R(S) = \bar{c}$ .

$$(4) \quad g_1(x_1, x_2, x_3)x_4 \neq g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4).$$

$$(a) \quad c^* < \bar{c}.$$

Let  $S$  be such that  $x_i(S) = c_i$ ,  $i = 1, 2, 3$ ,  $x_4(S) = c_3$ . Then  $R(S) = c^*$ ,  $L(S) = c^*c_3$ . If  $R(S) = L(S)$ , we would have  $c^* = c^*c_3$  which implies that  $c^* \geq c_3$ . This, however, contradicts the assertion (W). Thus,  $L(S) \neq R(S)$ .

$$(b) \quad c^* > \bar{c}.$$

Let  $S$  be such that  $x_i(S) = c_i$ ,  $i = 1, 2$ ,  $x_3(S) = c_1$ ,  $x_4(S) = c_3$ . Then by Lemma 2.1,  $R(S) = c_1c_2c^* = c^*$ ,  $L(S) = \bar{c}$ .

$$(c) \quad c^* \parallel \bar{c}.$$

Apply the same argument as in (b).

$$(5) \quad g_1(x_1, x_2, x_3)x_4 \neq g_1(x_1, x_4, x_2)g_1(x_1, x_4, x_3).$$

If (5) is not the case, then setting  $x_2 = x_3$ , we have by Lemma 2.1,  $x_1x_2x_4 = g_1(x_1, x_4, x_2)$ , a contradiction.

$$(6) \quad g_1(x_1, x_2, x_3)x_4 \neq g_2(x_1, x_2, x_3)x_4.$$

If (6) is not the case, setting  $x_4 = g_1(x_1, x_2, x_3)$ , we obtain  $g_1(x_1, x_2, x_3) = g_2(x_1, x_2, x_3)g_1(x_1, x_2, x_3)$ . Thus, by symmetry, it follows that  $g_1(x_1, x_2, x_3) = g_2(x_1, x_2, x_3)$ , a contradiction.

$$(7) \quad g_1(x_1, x_2, x_3)x_4 \neq g_2(x_2, x_3, x_4)x_1.$$

$$(a) \quad c^* < \bar{c}.$$

Let  $S$  be such that  $x_i(S) = c_i$ ,  $i = 1, 2, 3$ ,  $x_4(S) = c^*$ . Then  $L(S) = c^*$ ,  $R(S) = g_2(c_2, c_3, c^*)c_1$ . If  $L(S) = R(S)$ , then  $c^* \geq c_1$ , contradicting the assertion (W).

$$(b) \quad c^* \neq \bar{c}.$$

Let  $S$  be such that  $x_i(S) = c_i$ ,  $i = 1, 2, 3$ ,  $x_4(S) = c_2$ . Then  $L(S) = c^*c_2 \geq c^*$ ,  $R(S) = \bar{c}$  by Lemma 2.1. If  $L(S) = R(S)$ , we would have  $\bar{c} > c^*$ , a contradiction.

$$(8) \quad g_1(x_1, x_2, x_3)x_4 \neq g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4).$$

$$(a) \quad b^* < b'.$$

Let  $S$  be such that  $x_i(S) = b_i$ ,  $i = 1, 2, 3$ ,  $x_4(S) = b^*$ . Then  $L(S) = b^*$  while  $R(S) = b'g_2(b_1, b_2, b^*) \geq b'$ . Thus,  $R(S) \geq b' > b^* = L(S)$ .

$$(b) \quad b^* \neq b'$$

Let  $S$  be such that  $x_i(S) = b_i$ ,  $i = 1, 2, 3$ ,  $x_4(S) = b_3$ . Then

$R(S) = b'$ ,  $L(S) = b*b_3 \geq b^*$ . Thus, if  $L(S) = R(S)$ , we would have  $b' > b^*$ , a contradiction.

$$(9) \quad g_1(x_1, x_2, x_3)x_4 \neq g_2(x_3, x_4, x_1)g_2(x_3, x_4, x_2).$$

If  $g_1(x_1, x_2, x_3)x_4 = g_2(x_3, x_4, x_1)g_2(x_3, x_4, x_2)$ , setting  $x_1 = x_2$ , we would have  $x_1x_3x_4 = g_2(x_3, x_4, x_1)$ , a contradiction.

$$(10) \quad g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) \neq g_1(x_2, x_4, x_1)g_1(x_2, x_4, x_3).$$

$$(a) \quad c^* < \bar{c}.$$

Let  $S$  be such that  $x_i(S) = c_i$ ,  $i = 1, 2$ ,  $x_i(S) = c_3$ ,  $i = 3, 4$ . Then  $L(S) = c^*$  and  $R(S) = c^*c_2c_3$ . If  $L(S) = R(S)$ , we would have  $c^* \geq c_2c_3 > c_2$ , which contradicts the assertion (W).

$$(b) \quad c^* \not\leq \bar{c}.$$

Let  $S$  be such that  $x_i(S) = c_i$ ,  $i = 1, 2, 3$ ,  $x_4(S) = c_2$ . Then  $L(S) = c^*c_1c_2 \geq c^*$  and  $R(S) = \bar{c}$  by Lemma 2.1. If  $L(S) = R(S)$ , it follows that  $\bar{c} > c^*$ , a contradiction.

$$(11) \quad g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) \neq g_1(x_3, x_4, x_1)g_1(x_3, x_4, x_2).$$

If (11) is not the case, setting  $x_3 = x_4$ , we obtain

$$g_1(x_1, x_2, x_3) = x_1x_2x_3, \text{ a contradiction.}$$

$$(12) \quad g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) \neq g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4).$$

If (12) is not the case, setting  $x_3 = x_4$ , we have

$$g_1(x_1, x_2, x_3) = g_2(x_1, x_2, x_3), \text{ a contradiction.}$$

$$(13) \quad g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) \neq g_2(x_1, x_3, x_2)g_2(x_1, x_3, x_4).$$

$$(a) \quad b^* > b'.$$

Let  $S$  be such that  $x_1(S) = b_1$ ,  $x_2(S) = x_4(S) = b_3$ ,  $x_3(S) = b_2$ . Then  $R(S) = b'$  while  $L(S) = b*b_1b_3$ . Thus,  $L(S) \geq b^* > b' = R(S)$ .

(b)  $b^* \not\geq b'$ .

Let  $S$  be such that  $x_i(S) = b_i$ ,  $i = 1, 2, 3$ ,  $x_4(S) = b_3$ . Then  $L(S) = b^*$  and  $R(S) = b'_1 b_3 \geq b'$ . If  $L(S) = R(S)$ , it follows that  $b^* > b'$ , a contradiction.

$$(14) \quad g_1(x_1, x_2, x_3) g_1(x_1, x_2, x_4) \neq g_2(x_3, x_4, x_1) g_2(x_3, x_4, x_2).$$

If (14) is not the case, setting  $x_3 = x_4$ , we obtain  $x_1 x_2 x_3 = g_1(x_1, x_2, x_3)$  by Lemma 2.1, which is impossible.

Now, it is a simple matter to check that all the other possible cases are just permutations of the above fourteen cases. Thus, a similar argument can be applied for them.

It therefore follows that the polynomials in  $(\Delta)$  are essential and distinct over  $\mathcal{R}$ . Hence, we have  $p_4(\mathcal{R}) \geq 10n-9$ . This completes the proof of Proposition 2.4.

We shall now establish the following main result.

Theorem 2.5.

For each positive integer  $n$ , let  $F(n, 1)$  be the smallest value such that the sequence  $\langle 0, 0, 1, n, F(n, 1) \rangle$  is representable in  $\underline{K}(1)$ . Then  $F(n, 1) = 10n-9$ .

Proof : By Proposition 1.8 and the above result.

3.  $F(n, k)$  Described.

We shall extend the results in the previous sections to a more general case in this section. Instead of the sequence  $\langle 0, 0, 1, n, F(n, 1) \rangle$ , we shall consider the sequence

$\langle 0, 0, \overbrace{1, \dots, 1}^k, n, F(n, k) \rangle$  where  $k > 1$ .

Throughout this section, the following notations will be adopted.

For each positive integer  $k > 1$ , let  $\mathcal{A}(1, k) = \langle A(1, k); \cdot \rangle$  be the  $(k+3)$ -element semilattice ( see Fig. 35 ).

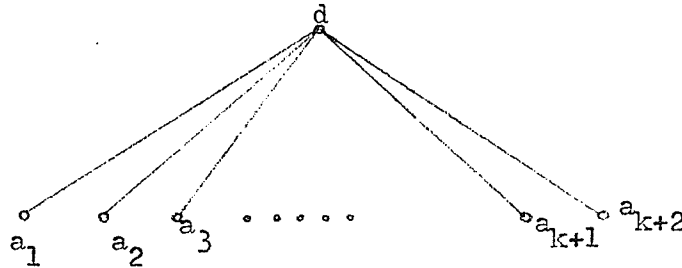


Fig. 35

For each pair of positive integers  $n, k$ , let  $(n, k) = \langle A(n, k); F \rangle$  be the algebra ( see Fig. 36 )

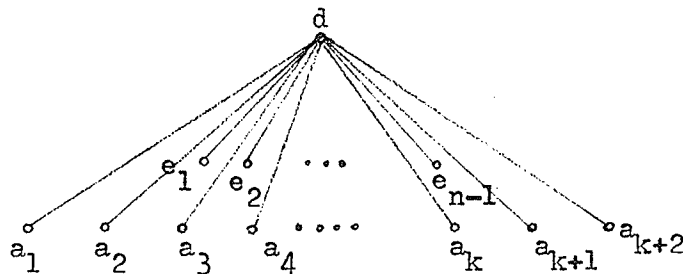


Fig. 36

where  $|A(n, k)| = n+k+2$  and the set of the operations  $F = \{ \cdot, f_1, \dots, f_{n-1} \}$  consists of one join semilattice operation and  $n-1$   $(k+2)$ -ary operation  $f_i$  such that for each  $i = 1, \dots, n-1$ ,  $f_i$  is defined by the following rule :

$$f_i(x_1, \dots, x_{k+2}) = \begin{cases} e_i & \text{if } \{x_1, \dots, x_{k+2}\} = \{a_1, \dots, a_{k+2}\} \\ \prod_{j=1}^{k+2} x_j & \text{otherwise .} \end{cases}$$

According to a previous result, it is known that the algebra  $\mathcal{A}(n,k)$  represents the sequence  $\langle 0, 0, \overbrace{1, \dots, 1}^k, n \rangle$ . Furthermore, applying an argument similar to one used in the previous sections, we can prove that the following identities hold in  $\mathcal{A}(n,k)$ .

- (1)  $f_i(x_1, \dots, x_{k+1}, x_{k+2}, x_{k+3})$   
 $= f_i(x_1, \dots, x_{k+1}, x_{k+2}) f_i(x_1, \dots, x_{k+1}, x_{k+3});$
- (2)  $f_i(x_1, p_1, \dots, p_{k+2}) x_1 = x_1 \prod_{j=1}^{k+2} p_j$  where  $p_j$  is a polynomial over  $\mathcal{A}(n,k)$ ;
- (3)  $f_i(f_j(x_1, \dots, x_{k+1}, x_{k+2}), x_{k+3}, x_1, \dots, x_k) = \prod_{j=1}^{k+3} x_j,$   
 $i, j = 1, 2, \dots, n-1;$
- (4)  $f_i(p_1, \dots, p_{k+2}) = \prod_{j=1}^{k+3} x_j$ , where  $f_i(p_1, \dots, p_{k+2})$  is an essentially  $(k+3)$ -ary polynomial; the  $p_j$ 's are pairwise distinct polynomials over  $\mathcal{A}(n,k)$  containing no sub-polynomials of the product of the form  $x_t x_s$ .
- (5)  $f_i(x_1, \dots, x_{k+2}) f_j(x_2, \dots, x_{k+2}, x_{k+3}) = \prod_{t=1}^{k+3} x_t$ , where  
 $i \neq j;$
- (6)  $\prod_{\Lambda} f_i(x_{\alpha(1)}, \dots, x_{\alpha(k+2)}) = \prod_{j=1}^{k+3} x_j$  if  $|\Lambda| \geq 3$ .

With the aid of the identities (1)---(6), we are able to arrive at the following

Proposition 3.1.

In the algebra  $\mathcal{A}(n,k)$ , only the following  $(k+3)$ -ary polynomials are distinct and essential :



$$\begin{aligned}
 & \prod_{j=1}^{k+3} x_j \\
 & f_i(x_1, \dots, x_{k+2})x_{k+3}, f_i(x_2, \dots, x_{k+3})x_1, \dots, \\
 & \dots, \dots, f_i(x_{k+3}, x_1, \dots, x_{k+1})x_{k+2}, \\
 & f_i(x_1, \dots, x_{k+1}, x_{k+2})f_i(x_1, \dots, x_{k+1}, x_{k+3}), \\
 & \quad \cdot \\
 & \quad \cdot \\
 & \quad \cdot \\
 & \quad \cdot \\
 & f_i(x_2, \dots, x_{k+2}, x_1)f_i(x_2, \dots, x_{k+2}, x_{k+3}), \\
 & \text{for } i = 1, 2, \dots, n-1.
 \end{aligned}
 \tag{\#}$$

Thus,  $p_{k+3}(\mathcal{L}(n, k)) = 1 + \frac{1}{2}(n-1)(k+3)(k+4)$ .

Proof : It is trivial that all the  $(k+3)$ -ary polynomials in (#) are distinct and essential over  $\mathcal{L}(n, k)$ .

On the other hand, let  $p$  be an essentially  $(k+3)$ -ary polynomial over  $\mathcal{L}(n, k)$ . By (1),  $p$  can be expressed as

$$p = \prod f_1(A_1, \dots, A_{k+2}) \prod f_2(B_1, \dots, B_{k+2}) \dots \prod f_{n-1}(M_1, \dots, M_{k+2}) \prod x_k$$

where  $A_i, \dots, M_i$  are polynomials which contain no sub-polynomials of the product of the form  $x_i x_j$  and the  $A_i$ 's are pairwise distinct over  $\mathcal{L}(n, k)$ . The same is true for  $B_i$ 's, ..., and  $M_i$ 's.

By (3) and (4), we have

$$p = \prod f_1(x_{\alpha(1)}, \dots, x_{\alpha(k+2)}) \dots \prod f_{n-1}(x_{\beta(1)}, \dots, x_{\beta(k+2)}) \prod x_j.$$

If there is no occurrence of  $f_i$  in  $p$ , then

$$p = \prod_{j=1}^{k+3} x_j.$$

If  $p$  has a factor " $f_i$ ", for some  $i = 1, \dots, n-1$ , then by (5),

$$p = \prod_{r \in \Lambda} f_i(x_{r(1)}, \dots, x_{r(k+2)}) \prod x_j.$$

By (6), the number of occurrences of the symbol " $f_i$ " is either one or two. If the number is one, then by (2),

$$p = f_i(x_1, \dots, x_{k+2})x_{k+3}, \text{ or symmetrically.}$$

If the number is two, then, again by (2), we have

$$p = f_i(x_1, \dots, x_{k+1}, x_{k+2})f_i(x_1, \dots, x_{k+1}, x_{k+3}), \text{ and so on.}$$

Thus, the polynomials in (#) are the only distinct and essentially  $(k+3)$ -ary polynomials over  $\mathcal{A}(n, k)$ . From this, it follows that

$$\begin{aligned} p_{k+3}(\mathcal{A}(n, k)) &= 1 + (n-1)\binom{k+3}{1} + (n-1)\binom{k+3}{2} \\ &= 1 + \frac{1}{2}(n-1)(k+3)(k+4), \end{aligned}$$

which was to be shown.

Now, for each  $k > 1$ , let us denote by  $\underline{K}(k)$  the class of all idempotent algebras such that all the essentially  $(k+2)$ -ary polynomials are symmetric. Evidently,  $\mathcal{A}(n, k) \in \underline{K}(k)$  for each positive integer  $n$ . Let  $\mathcal{A}$  be an algebra in  $\underline{K}(k)$  representing the sequence  $\langle 0, 0, \overline{1, \dots, 1}, n \rangle$ . Since  $k > 1$ ; it is easily seen that the only essentially binary polynomial over  $\mathcal{A}$  must be a semilattice polynomial. Thus, we have already one essentially  $(k+2)$ -ary polynomial, namely,  $\prod_{j=1}^{k+2} x_j$ . As  $p_{k+2}(\mathcal{A}) = n$ , let  $g_i$ ,  $i = 1, \dots, n-1$  denote the remaining  $n-1$  essentially  $(k+2)$ -ary polynomials. Since  $\mathcal{A} \in \underline{K}(k)$ ,  $g_i$  is symmetric for each  $i = 1, 2, \dots, n-1$ .

In what follows, we are going to prove that corresponding to the  $1 + \frac{1}{2}(n-1)(k+3)(k+4)$  distinct essentially  $(k+3)$ -ary polynomials of  $\mathcal{A}(n, k)$  which were described in Proposition 3.1,

the following  $(k+3)$ -ary polynomials are distinct and essential over  $\mathcal{A}$  :

$$(\theta) \left\{ \begin{array}{l} \prod_{j=1}^{k+3} x_j; \\ g_i(x_1, \dots, x_{k+2})x_{k+3}, g_i(x_2, \dots, x_{k+3})x_1, \dots \dots \dots \\ \dots \dots \dots, g_i(x_{k+3}, x_1, \dots, x_{k+1})x_{k+2} ; \\ g_i(x_1, \dots, x_{k+1}, x_{k+2})g_i(x_1, \dots, x_{k+1}, x_{k+3}) \\ \vdots \\ g_i(x_2, \dots, x_{k+2}, x_1)g_i(x_2, \dots, x_{k+2}, x_{k+3}) \end{array} \right.$$

where  $i = 1, 2, \dots, n-1$

By generalizing the technique which was applied in proving Lemmas 2.1, 2.2 and 2.3 in a suitable way, we have the following basic Lemmas:

Lemma 3.2.

In  $\mathcal{A}$  ,  $g_i(x_1, \dots, x_{k+1}, x_{k+1}) = \prod_{j=1}^{k+1} x_j$  , for each  $i = 1, 2, \dots, n-1$ .

Lemma 3.3.

In  $\mathcal{A}$  ,  $g_i(x_1, x_2, \dots, x_{k-1}, x_k, x_{k+1}, x_k x_{k+1}) = \prod_{j=1}^{k+1} x_j$  , for each  $i = 1, 2, \dots, n-1$ .

Lemma 3.4.

In  $\mathcal{A}$  ,  $g_i(x_1, x_2, \dots, x_k, x_{k+1}, x_{k+1} x_{k+2}) = \prod_{j=1}^{k+2} x_j$  , for each  $i = 1, 2, \dots, n-1$ .

With the help of these Lemmas, we have

Proposition 3.5.

Let  $\mathcal{A}$  be algebra in  $K(k)$  representing  $\langle 0, 0, \overline{1, \dots, k}, 1, n \rangle$ .

Then  $p_{k+3}(\mathcal{A}) \geq 1 + \frac{1}{2}(n-1)(k+3)(k+4)$ .

Proof: Lemma 3.2 implies that the polynomials in  $(\theta)$  are essentially  $(k+3)$ -ary over  $\mathcal{U}$ . Our proof will be complete if we can show that the polynomials in  $(\theta)$  are distinct. By making use of Lemmas 3.2, 3.3 and 3.4, and following the same arguments as in the proof of Proposition 2.4, it can be shown that this is, indeed, the case. For instance, to check that

$$\prod_{j=1}^{k+3} x_j \neq g_1(x_1, \dots, x_{k+2})x_{k+3},$$

choose a subset  $\{c_1, \dots, c_{k+2}\}$  of  $A$  such that

$$\prod_{j=1}^{k+2} c_j = \bar{c} \neq c^* = g_1(c_1, \dots, c_{k+2}).$$

By Lemmas 3.2 and 3.4, it follows that  $c_1, \dots, c_{k+2}$  are pairwise distinct and incomparable.

Case 1.  $\bar{c} < c^*$

Let  $S$  be a substitution such that  $x_i(S) = c_i$ ,  $i = 1, 2, \dots, k+2$ , and  $x_{k+3}(S) = c_1$ . Then we have  $L(S) = \prod_{j=1}^{k+2} c_j = \bar{c}$  while  $R(S) = g_1(c_1, \dots, c_{k+2})c_1 = c^*c_1 = (c^*\bar{c})c_1 = c^*(\bar{c}c_1) = c^*\bar{c} = c^*$ . Thus  $R(S) \neq L(S)$ .

Case 2.  $\bar{c} \not< c^*$

Let  $S$  be such that  $x_i(S) = c_i$ ,  $i = 1, 2, \dots, k+2$  and  $x_{k+3}(S) = c^*$ . Then  $L(S) = \bar{c}c^* \geq \bar{c}$ ,  $R(S) = c^*c^* = c^*$ . If  $L(S) = R(S)$ , it follows that  $c^* > \bar{c}$ , which is impossible. Thus,  $L(S) \neq R(S)$ .

Hence, it follows that  $\prod_{j=1}^{k+3} x_j \neq g_1(x_1, \dots, x_{k+2})x_{k+3}$ , as required.

By combining this with Proposition 3.1, we obtain the following main result.

Theorem 3.6.

For each integer  $k > 1$ , let  $\underline{K}(k)$  be the class of all idempotent algebras such that all the essentially  $(k+2)$ -ary polynomials are symmetric. For each positive integer  $n$ , let  $F(n,k)$  be the smallest value such that the sequence

$\langle 0, 0, \overbrace{1, \dots, 1}^k, n, F(n,k) \rangle$  is representable in  $\underline{K}(k)$ . Then

$$F(n,k) = 1 + \frac{1}{2}(n-1)(k+3)(k+4).$$

----- PART III -----

IDEMPOTENT ALGEBRAS WITH TWO

OR THREE ESSENTIALLY BINARY POLYNOMIALS

## CHAPTER I

### THE SEQUENCE $\langle 0,0,2,m \rangle$

Idempotent algebras with exactly one essentially binary polynomial have been discussed in Part I and Part II. In this chapter, we start to consider idempotent algebras with two essentially binary polynomials, i.e., algebras representing  $\langle 0,0,2 \rangle$ .

It is known that the sequence  $\langle 0,0,2,0 \rangle$  is representable. For instance, if  $\mathcal{A}$  is a diagonal semigroup, then  $p_2(\mathcal{A}) = 2$  and  $p_n(\mathcal{A}) = 0$ , for each  $n \neq 2$ . J. Płonka proved in [34] that if the sequence  $\langle 0,0,2,k \rangle$  is representable and  $k > 0$ , then  $k \geq 3$ . The case  $k = 3$  is possible. It turns out that algebras representing  $\langle 0,0,2,3 \rangle$  can be classified into four equational classes of algebras which are described in [35]. It has been pointed out by J. Gerhard in [5] that the sequence  $\langle 0,0,2,6 \rangle$  is representable. In fact, if  $\mathcal{A}$  is an idempotent semigroup satisfying  $aba = ab$ , then  $p_n(\mathcal{A}) = n!$  for each  $n \geq 2$ .

It is, perhaps, important to note that there is a common feature in all the algebras shown above. That is, each of them has one and only one essentially binary polynomial which is non-commutative. Thus, in order to continue their investigations, it is natural to study the opposite case; namely, algebras with two distinct commutative essentially binary polynomials. The main object of this chapter is to deal with this case.

In section 1, the binary polynomials are studied in detail. The results are summarized in Proposition 1.4 which will be of great use in developing the Characterization Theorems in Chapter 1 of Part IV. Some results of Płonka are mentioned in section 2 which are applied to prove our main results in sections 3 and 4.

### 1. Binary Polynomials.

Throughout the remaining chapters, let  $\underline{K}$  be the equational classes of algebras defined by the following two identities:

$$\left[ \begin{array}{l} x + y = y + x \\ xy = yx \end{array} \right.$$

where  $x + y$ ,  $xy$  are two distinct essentially binary polynomials over algebras in  $\underline{K}$ . ~~Thus, if  $\mathcal{A}$  is an algebra of type  $\langle 2, 2 \rangle$  and  $p_3(\mathcal{A}) = 2$ , then  $\mathcal{A} \in \underline{K}$ .~~

Let  $\mathcal{A} \in \underline{K}$  represent the sequence  $\langle 0, 0, 2 \rangle$ . Then clearly,  $x + y$  and  $xy$  are the only two idempotent, commutative essentially binary polynomials over  $\mathcal{A}$ .

Consider the binary polynomials  $x + xy$ ,  $x(x + y)$ . As  $p_3(\mathcal{A}) = 2$ , we have the following possibilities :

$$x + xy = \left\{ \begin{array}{l} x \\ y \\ xy \\ x + y \end{array} \right. \quad x(x + y) = \left\{ \begin{array}{l} x \\ y \\ xy \\ x + y \end{array} \right.$$



Thus, there are sixteen cases for  $x + xy$  and  $x(x + y)$  in general. However, we shall show in Proposition 1.4 that there are, in fact, only four. To this end, we first establish the following Lemmas.

Lemma 1.1.

Let  $\mathcal{N} \in \underline{K}$  represent the sequence  $\langle 0, 0, 2 \rangle$ . Then

$$x + xy \neq y \quad \text{and} \quad x(x + y) \neq y.$$

Proof : Assume that  $x(x + y) = y$  \_\_\_\_\_ (a)

We have the following four possible cases.

Case 1.  $x + xy = x$  \_\_\_\_\_ (b)

Observe that  $x + y = x + x(x + y)$  ( by (a) )

$$= x \quad \text{( by (b) )}$$

which is a contradiction.

Case 2.  $x + xy = y$  \_\_\_\_\_ (c)

We have  $x + y = y(y + (x + y))$  ( by (a) )

$$= y((x + y) + y)$$

$$= y((x + y) + x(x + y)) \quad \text{( by (a) )}$$

$$= y((x + y) + (x + y)x)$$

$$= yx \quad \text{( by (c) )}$$

which is a contradiction.

Case 3.  $x + xy = xy$  \_\_\_\_\_ (d)

Observe that  $x = (xy)((xy) + x)$  ( by (a) )

$$= (xy)(xy) \quad \text{( by (d) )}$$

$$= xy,$$

which is a contradiction.

Case 4.  $x + xy = x + y$  \_\_\_\_\_ (e)

Note that  $xy = x(x + xy)$  ( by (a) )

$= x(x + y)$  ( by (e) )

$= y$  ( by (a) ),

which is a contradiction.

Hence, if (a) holds in  $\mathcal{N}$ , we have no choice for  $x + xy$ .

Therefore,  $x(x + y) \neq y$ , as required. Similarly,  $x + xy \neq y$ .

Lemma 1.2.

Let  $\mathcal{N} \in \underline{\mathbb{K}}$  represent the sequence  $\langle 0, 0, 2 \rangle$ . Then

$x + xy = x$  if and only if  $x(x + y) = x$ .

Proof: By symmetry, it suffices to prove that  $x + xy = x$

implies  $x(x + y) = x$ . Thus, assume

$x + xy = x$  \_\_\_\_\_ (a)

holds in  $\mathcal{N}$ . By Lemma 1.1, we have three possible cases:

$$x(x + y) = \begin{cases} x \\ xy \\ x + y \end{cases}$$

Case 1.  $x(x + y) = xy$  \_\_\_\_\_ (b)

Observe that  $x + y = (x + y) + (x + y)y$  ( by (a) )

$= (x + y) + xy$  ( by (b) )

Thus, we have  $x + y = (x + y) + xy$  \_\_\_\_\_ (c)

Moreover,  $x + y = (x + y)(x + y)$

$= ((x + y) + xy)(x + y)$  ( by (c) )

$= (x + y)(xy)$  ( by (b) )

i.e.,  $x + y = (x + y)(xy)$  \_\_\_\_\_ (d).

From these, we obtain

$$\begin{aligned}x + y &= (x + y) + xy && \text{( by (c) )} \\ &= (xy) + (xy)(x + y) && \text{( by (d) )} \\ &= xy && \text{( by (a) )}\end{aligned}$$

a contradiction.

Case 2.  $x(x + y) = x + y$ . \_\_\_\_\_ (e)

If this is the case, we would have

$$\begin{aligned}x &= x + xy && \text{( by (a) )} \\ &= (xy)(xy + x) && \text{( by (e) )} \\ &= (xy)x && \text{( by (a) )}\end{aligned}$$

i.e.,  $x = (xy)x$  \_\_\_\_\_ (f)

Now, it follows that  $x + y = x(x + y)$  ( by (e) )  
 $= (x(x + y))x$  ( by (e) )  
 $= x$  ( by (f) )

which is impossible.

Thus, we must have  $x(x + y) = x$ , completing the proof of Lemma 1.2.

Lemma 1.3.

Let  $\mathcal{R} \in K$  represent the sequence  $\langle 0, 0, 2 \rangle$ . Then

- 1)  $x + xy = x + y$  implies  $x(x + y) = x + y$ ;
- 2)  $x(x + y) = xy$  implies  $x + xy = xy$ .

Proof: By symmetry, it suffices to prove 1).

Thus, assume that  $x + xy = x + y$  \_\_\_\_\_ (a)

By Lemmas 1.1 and 1.2, we have

$$x(x + y) = \begin{cases} xy \\ x + y. \end{cases}$$

If  $x(x + y) = xy$ , \_\_\_\_\_ (b)

we would have  $x(xy) = x(x + xy)$  ( by (b) )  
 $= x(x + y)$  ( by (a) )  
 $= xy$ , ( by (b) )

i.e.,  $x(xy) = xy$  \_\_\_\_\_ (c).

Similarly,  $x + (x + y) = x + x(x + y)$  ( by (a) )  
 $= x + xy$  ( by (b) )  
 $= x + y$  ( by (a) ),

i.e.,  $x + (x + y) = x + y$  \_\_\_\_\_ (d)

Furthermore,  $(xy)(x + y) = (xy)(x + xy)$  ( by (a) )  
 $= (xy)x$  ( by (b) )  
 $= xy$  ( by (c) )

i.e.,  $(xy)(x + y) = xy$  \_\_\_\_\_ (e)

Similarly, by (a), (b), (d), we have

$$(x + y) + xy = x + y \quad \text{_____ (f)}$$

It thus follows that  $x + y = xy + (x + y)$  ( by (f) )  
 $= xy + (xy)(x + y)$  ( by (e) )  
 $= xy + xy$  ( by (e) )  
 $= xy$ ,

which is a contradiction.

which is (1). Moreover, it follows that

$$\begin{aligned} (x + y) + x &= x(x + y) + x && \text{( by (a) )} \\ &= x(x + y) && \text{( by (b) )} \\ &= x + y && \text{( by (a) )} \end{aligned}$$

which is (2).

To prove (3), consider  $(x + y) + xy$  and  $(x + y)(xy)$ .

Clearly, both of them are symmetric with respect to  $x$  and  $y$ .

Thus, if they depend on  $x$ , they must depend on  $y$  simultaneously

and vice versa. As  $p_0(\mathcal{A}) = p_1(\mathcal{A}) = 0$ , we have

$$(x + y) + xy = \begin{cases} x + y \\ xy \end{cases}, \quad (x + y)(xy) = \begin{cases} x + y \\ xy \end{cases}$$

If  $(x + y) + xy = x + y$  ----- (c)

$$\begin{aligned} \text{then } (x + y)(xy) &= ((x + y) + xy)(xy) && \text{( by (c) )} \\ &= xy + (x + y) && \text{( by (a) )} \\ &= x + y && \text{( by (c) ).} \end{aligned}$$

Thus,  $(x + y) + xy = x + y$  implies that  $(x + y)(xy) = x + y$ .

On the other hand, if  $(x + y) + xy = xy$  ----- (d)

$$\begin{aligned} \text{then } (x + y)(xy) &= (x + y)((x + y) + xy) && \text{( by (d) )} \\ &= (x + y) + xy && \text{( by (a) )} \\ &= xy && \text{( by (d) ).} \end{aligned}$$

Thus,  $(x + y) + xy = xy$  implies  $(x + y)(xy) = xy$ .

From these, (3) follows.

## 2. Płonka's Basic Lemmas.

Let  $\mathcal{A}$  be an algebra having two distinct binary polynomials

denoted by  $x + y$  and  $xy$ . We shall introduce the following notation :

$$\left[ \begin{array}{ll} f_{11} = (x + y) + z & f_{21} = (xy)z \\ f_{12} = (y + z) + x & f_{22} = (yz)x \\ f_{13} = (z + x) + y & f_{23} = (zx)y \\ \\ f_{31} = (x + y)z & f_{41} = xy + z \\ f_{32} = (y + z)x & f_{42} = yz + x \\ f_{33} = (z + x)y & f_{43} = zx + y \end{array} \right.$$

The following Lemmas are stated without proofs. In fact, they are included in J. Płonka [ 29 ] .

Lemma 2.1.

If  $x + y$  and  $xy$  are both commutative and idempotent, then  $f_{ik}$  is an essentially ternary polynomials for each  $i, k = 1, 2, 3, 4$ .

Lemma 2.2.

If  $x + y$  and  $xy$  are both idempotent and commutative, then  $f_{ik}$  and  $f_{jt}$  are distinct for  $i \neq j$ .

The next Lemma says that if we have two idempotent and commutative binary polynomials in an algebra we then have at least eight distinct essentially ternary polynomials.

Lemma 2.3.

Let  $x + y$  and  $xy$  be both idempotent and commutative.

(1) If  $x + y$  and  $xy$  are not associative, then  $f_{1k}, f_{2k}, f_{3l},$

$f_{41}$ ,  $k = 1, 2, 3$  are distinct;

(2) If  $x + y$  is associative but  $xy$  not, then  $f_{11}$ ,  $f_{2k}$ ,  $f_{3k}$ ,

$f_{41}$ ,  $k = 1, 2, 3$  are distinct;

(3) If  $xy$  is associative but  $x + y$  not, then  $f_{1k}$ ,  $f_{21}$ ,  $f_{31}$ ,

$f_{4k}$ ,  $k = 1, 2, 3$  are distinct.

Let  $\mathcal{A} = \langle A; +, \cdot \rangle$  be an algebra where "+" and " $\cdot$ " satisfy the idempotent, commutative and associative laws. It is known that  $\mathcal{A}$  is not a lattice in general since the absorption laws are independent of those mentioned above. However, if we assume that  $|A| \geq 2$  and  $p_2(\mathcal{A}) = 2$ , then we can prove that  $\mathcal{A}$  is, indeed, a lattice. This can be seen from the following.

Lemma 2.4.

Let  $\mathcal{A} = \langle A; +, \cdot \rangle$  be an algebra such that (1)  $|A| \geq 2$ ;  
(2)  $x + y$  and  $xy$  are the two distinct idempotent, commutative, associative essentially binary polynomials in  $\mathcal{A}$ ; (3) no other essentially  $j$ -ary polynomials for  $j = 1, 2$  except  $x$ ,  $y$ ,  $x + y$ ,  $xy$ . Then the absorption laws  $x(x + y) = x + xy = x$  hold in  $\mathcal{A}$ .

Lemma 2.5.

If  $x + y$  and  $xy$  are distinct idempotent, commutative essentially binary polynomials in  $\mathcal{A}$  with  $|A| \geq 2$  and the polynomial  $f_{31} = (x + y)z$  is symmetric, then there exists in  $\mathcal{A}$  an essentially binary polynomial which is different from  $x + y$  and  $xy$ . The same statement is true if we replace the polynomial  $f_{31}$  by  $f_{41} = xy + z$ .

### 3. The Smallest Value of $m$ .

In this section, to begin with, we bring forward the following problem : Find the greatest integer  $m^*$  such that if a sequence  $\langle 0,0,2,m \rangle$  is representable in  $\underline{K}$ , then  $m \geq m^*$ . Our first result reveals that the required integer  $m^*$  is "9". Thus, to proceed, it is natural to investigate whether there is any algebra representing  $\langle 0,0,2,9 \rangle$ . Our next result provides a positive answer to this.

#### Theorem 3.1.

If the sequence  $\langle 0,0,2,m \rangle$  is representable in  $\underline{K}$ , then  $m \geq 9$ .

Proof : Let  $\mathcal{A}$  be an algebra in  $\underline{K}$  representing the sequence

$\langle 0,0,2,m \rangle$ . Let  $x + y$  and  $xy$  be the only two distinct idempotent, commutative essentially binary polynomials. Since  $x + y \neq xy$ ,  $|A| \geq 2$ . The following are the only four possible cases :

- (1)  $x + y$  and  $xy$  are non-associative ;
- (2)  $x + y$  is associative but  $xy$  not ;
- (3)  $xy$  is associative but  $x + y$  not ;
- (4)  $x + y$  and  $xy$  are associative.

If (1) is the case, then according to Lemmas 2.1 and 2.3(1), the following  $f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{31}, f_{41}, f_{23}$  are distinct essentially ternary polynomials over  $\mathcal{A}$ .

Consider the polynomial  $f_{31} = (x + y)z$ . If it is symmetric, then by Lemma 2.5, we obtain  $p_2(\mathcal{A}) > 2$ , a contradiction.



The case for  $f_{41} = xy + z$  is similar. Thus,  $f_{31}$  and  $f_{41}$  are not symmetric. Hence the following essentially ternary polynomials

$$\begin{array}{ccc} f_{31} & f_{32} & f_{33} \\ f_{41} & f_{42} & f_{43} \end{array}$$

are pairwise distinct. For example, if  $f_{31} = f_{32}$ , then  $f_{33} = (z + x)y = (x + z)y = (z + y)x = (y + z)x = f_{32}$ ; i.e.,  $f_{31} = f_{32} = f_{33}$ , a contradiction.

By Lemmas 2.1 and 2.2, it follows that  $m = p_3(\mathcal{R}) \geq 12 > 9$ .

If (2) is the case, then according to Lemmas 2.1 and 2.3(2), we have the following distinct essentially ternary polynomials :

$$f_{11}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33}, f_{41} .$$

If  $f_{41}$  is symmetric, then by Lemma 2.5, we would have  $p_2(\mathcal{R}) > 2$ , a contradiction. So  $f_{41}, f_{42}, f_{43}$  are pairwise distinct. By Lemmas 2.1 and 2.2, we have

$$m = p_3(\mathcal{R}) \geq 10 > 9.$$

The case (3) is symmetric to (2). The proof is similar.

Thus, it remains to consider case (4). In this situation, observe that as  $\mathcal{R}$  represents  $\langle 0, 0, 2 \rangle$ , all the three conditions of Lemma 2.4 are fulfilled. Accordingly, the two absorption laws hold in  $\mathcal{R}$ . In other words,  $\mathcal{R}$  is a lattice. Now, let us look at the following polynomials :

$xyz$	$x + y + z$
$xy + z$	$(x + y)z$
$yz + x$	$(y + z)x$
$zx + y$	$(z + x)y$
$xy + yz + zx$	

It is a simple matter to check that they are distinct and essentially ternary over  $\mathcal{A}$ . For instance, if  $xy + z = xy + yz + zx$ , setting  $x = y$ , we have  $x + z = x + xz = x$ , a contradiction. Therefore, we obtain  $m = p_3(\mathcal{A}) \geq 9$ . The proof of Theorem 3.1 is thus complete.

The following result shows that the case  $m = 9$  is possible.

Theorem 3.2.

Every distributive lattice with more than one element represents the sequence  $\langle 0, 0, 2, 9 \rangle$ .

Proof : Let  $\mathcal{A}$  be a given distributive lattice with more than one element. Let us consider  $P^{(n)}(\mathcal{A})$ , the set of  $n$ -ary polynomials over  $\mathcal{A}$ . Being a lattice, it is well-known that  $P^{(n)}(\mathcal{A})$  is a free distributive lattice on  $n$  generators. Clearly,  $P^{(0)}(\mathcal{A}) = \emptyset$ ;  $P^{(1)}(\mathcal{A})$  is one element lattice;  $P^{(2)}(\mathcal{A})$  is the four-element lattice  $C_2^2$ . Moreover, the lattice  $P^{(3)}(\mathcal{A})$  consists of eighteen elements, and its diagram is given below ( see Fig. 37 ).

Let  $\mathcal{C}$  be an equational class of algebras, and let  $F_n$  denote the cardinality of the free algebra over  $\mathcal{C}$  on  $n$  generators. Let  $\mathcal{B}$  be the free algebra over  $\mathcal{C}$  on  $\omega$  generators,  $p_k = p_k(\mathcal{B})$ .

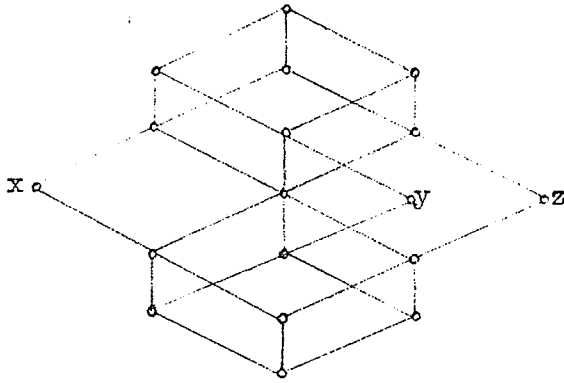


Fig. 37

Then we have the following nice formula ( see [ 21 ] )

$$F_n = n + \sum_{k=0}^n \binom{n}{k} p_k .$$

Invoking this, a direct computation shows that  $\mathcal{R}$  represents the sequence  $\langle 0,0,2,9 \rangle$  .

Summarizing all the results about the sequence  $\langle 0,0,2,m \rangle$  , we have

Corollary 3.3.

Let  $\mathcal{R}$  be an algebra representing the sequence  $\langle 0,0,2,m \rangle$  .

The following are the only two possible cases :

(1) If  $\mathcal{R}$  has a non-commutative essentially binary polynomial, then  $m = 0$  or  $m \geq 3$  .

(2) If  $\mathcal{R}$  has two distinct commutative essentially binary polynomials, then  $m \geq 9$  .

4. Algebras Representing  $\langle 0,0,2,10 \rangle$  .

It was shown in section 3 that the sequence  $\langle 0,0,2,9 \rangle$  is representable. In this section, we study the sequence

$\langle 0,0,2,10 \rangle$  .

Let us consider the following algebra  $\mathcal{L}(4) = \langle L(4); +, \cdot \rangle$  where  $\langle L(4); \cdot \rangle$  is the following four-element join semilattice ( see Fig. 38 )

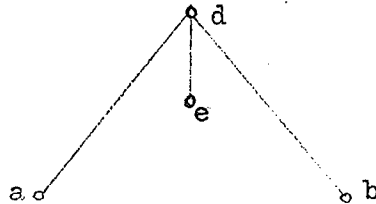


Fig. 38

and the binary operation "+" is defined as follows :

$$x + y = \begin{cases} e & \text{if } \{x,y\} = \{a,b\} \\ xy & \text{otherwise .} \end{cases}$$

For the sake of convenience, let us denote  $x + y$  by  $f(x,y)$ .

It is a simple matter to prove that the following identities hold in  $\mathcal{L}(4)$ .

- (1)  $f(x,y)x = xy$
- (2)  $f(xy,z) = f(x,z)f(y,z)$
- (3)  $f(f(x,y),f(x,z)) = f(x,y)f(x,z)$
- (4)  $f(f(f(x,y),z),x) = xyz$
- (5)  $f(f(f(x,y),z),z) = f(x,y)z$
- (6)  $f(f(f(x,y),z),f(x,y)) = f(x,y)z$
- (7)  $f(f(f(x,y),z),f(x,z)) = f(x,z)f(y,z)$
- (8)  $f(f(x,y),z)x = xyz$
- (9)  $f(f(x,y),z)f(x,z) = f(x,z)f(y,z)$
- (10)  $f(f(x,y),z)f(f(x,z),y) = xyz$
- (11)  $f(x,y)f(y,z)f(z,x) = xyz$

It follows from (1) and (2) that the polynomials  $f(x,y)$ ,  $xy$  are the only two essentially binary polynomials over  $\mathcal{L}(4)$ .

Let  $p$  be an arbitrary essentially ternary polynomial over  $\mathcal{L}(4)$ . By (2),  $p$  can be written as

$$p = \prod_{i \in I} f(q_i, r_i) \prod x_j$$

where  $q_i, r_i$  are polynomials containing no sub-polynomials of the form  $A.B$  and  $x_j \in \{x, y, z\}$ .

Assume that  $p$  cannot be reduced to any simpler form. Then according to the above identities, the case  $|I| \geq 3$  is impossible. If  $|I| = 2$ , we have  $f(x,y)f(x,z)$  and the polynomials formed from it by symmetry. If  $|I| = 1$ , we have  $f(x,y)z$ ,  $f(f(x,y),z)$  and the polynomials formed from them by symmetry. If  $|I| = 0$ , we obtain  $xyz$ .

Thus, the following are the ten and only ten distinct essentially ternary polynomials over  $\mathcal{L}(4)$  :

$$\left[ \begin{array}{l} xyz \\ (x + y) + z, \quad (y + z) + x, \quad (z + x) + y, \\ (z + x)(z + y) \quad (= z + xy), \\ (x + y)(x + z), \\ (y + x)(y + z), \\ (x + y)z, \quad (y + z)x, \quad (z + x)y. \end{array} \right.$$

Consequently, we have the following

Theorem 4.1.

The algebra  $\mathcal{L}(4)$  represents the sequence  $\langle 0, 0, 2, 10 \rangle$ .

As a matter of fact, the algebra  $\mathcal{C}(4)$  plays a central role in the class of algebras representing  $\langle 0,0,2,10 \rangle$ . This is a consequence of the following Theorem.

Theorem 4.2

Let  $\mathcal{A}$  be an algebra of  $\mathbb{K}$  representing  $\langle 0,0,2,10 \rangle$ . Then  $\mathcal{A}$  can be represented as an algebra which contains  $\mathcal{C}(4)$  as a subalgebra.

Proof : Let  $x + y, xy$  be the two commutative essentially binary polynomials over  $\mathcal{A}$ .

Case 1.  $x + y$  and  $xy$  are non-associative.

In this case, as in the proof of Theorem 3.1, we have  $p_3(\mathcal{A}) \geq 12$ , a contradiction.

Case 2.  $x + y$  and  $xy$  are associative.

In this case, it follows that  $\langle A; +, \cdot \rangle$  is a lattice. Thus, by Theorem 3.3 (IV,1), either  $p_3(\mathcal{A}) = 9$  or  $p_3(\mathcal{A}) \geq 19$ , which is impossible.

Accordingly, one of them,  $x + y$  say is associative and the other is not. In this situation, as in the proof of Theorem 3.1, we have at least the following ten distinct essentially ternary polynomials :

$$(*) \left[ \begin{array}{l} xyz \\ (x + y) + z, (y + z) + x, (z + x) + y, \\ x + yz, y + zx, z + xy, \\ (x + y)z, (y + z)x, (z + x)y \end{array} \right.$$

In virtue of Proposition 1.4, we have the following four possible cases :