The degree sequence problem for 3-hypergraphs

by

Yangsheng Zou

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Department of Computer Science
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Thesis advisor
Kocay William

Author
Yangsheng Zou

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Abstract

Currently the degree sequence problem for 3-hypergraphs is still unsolved efficiently. This paper researches the 3-hypergraphic problem in terms of edge switching and exchanges in the sequence to implement Dewdney's reduction. It proposes the idea of an irreducible decomposition and makes use of it to find some sufficient conditions for a 3-hypergraphic sequence. In addition, this paper explores a related problem: intersection preserving mappings.

Contents

List of Figures

List of Tables

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Chapter 1

Introduction

A graph, or 2-hypergraph $G = (V, T)$ consists of a vertex set V and an edge set T, where each edge is a pair of distinct vertices in V. The degree of a vertex $v \in V$, denoted by d_i, is the number of edges incident with v. Let $|V| = 1, 2, ..., n$, the n-tuple $(d_1, d_2, ..., d_n)$ is called the degree sequence of G, see [Diestel, 2005]. A 2-hypergraph with degree sequence D is called a realization of D. Determining whether a realization exists for a specific sequence is called the graphic or 2-hypergraphic problem.

The definition of "degree sequence" can be extended to a k-hypergraph where $k \geq 3$. In a 3-hypergraph (also known as a 3-uniform hypergraph), every edge contains a triple of vertices rather than a pair. A non-increasing sequence is called a 3-hypergraphic sequence when it has a 3-hypergraph realization $H = (V, T)$, and the problem whether a sequence is 3-hypergraphic is called the 3-hypergraphic problem.

According to Frosini, Picouleau and Rinaldi's work [Frosini et al., 2013], the k-hypergraphic problem can contribute to discrete tomography as the construction of a hypergraph with a specific degree sequence can be used to construct the binary matrix with specific projection vectors with distinct rows. The reader is referred to their work for more information on discrete tomography.

For the 2-hypergraphic problem, there are some efficient solutions shown in the Chapter 2, but for the k-hypergraphic problem, where $k \geq 3$, even its complexity status remains open. Dewdney [1975] proposed a reduction to solve the k-hypergraphic problem, but this has not progressed much so far.

Therefore, trying to implement this reduction to the 3-hypergraphic problem could be a good start for solving the case when $k \geq 3$. In this thesis, we investigate the degree sequences of 3-hypergraphs, and their transformations. Chapter 2 is a review of related work by other authors. Chapter 3 to 6 contain new work by this author.

Chapter 2

Literature Review

2.1 Introduction

Havel [1955] and Hakimi [1962] proposed a greedy algorithm to solve the graphic sequence problem for 2-hypergraphs. In their work, they proved that a non-increasing degree sequence $(d_1, d_2, ..., d_n)$ is 2-hypergraphic if and only if $(0, d_2-1, d_3-1..., d_{d_1+1}-1, d_{d_1+2}, ..., d_n)$ is graphic. Therefore the 2-hypergraphic problem with n vertices can be reduced to that with $n-1$ vertices, and after at most n reductions the problem can be solved.

To determine whether a sequence is 2-hypergraphic, Erdős and Gallai [1960] presented a system of inequalities which consider the sum of degrees from d_1 to d_k for any k, where $1 \leq k \leq n$. Their system considers the number of edges involving vertices in $\{1, 2, ..., k\}$ and how many degrees these edges can contribute to these vertices. Tripathi and Vijay [2003] noted that in this inequality system the number of values k that needs to be checked in the sequence H is the number of distinct degrees in

H exactly. Tripathi et al. [2010] gave a constructive proof for this theorem. In the proof, they introduced the idea of *subrealization* such that the degree of every vertex is smaller than that in the original sequence H, to approach H gradually and finally showed that H has a realization.

Dewdney [1975] considered k-hypergraphic degree sequences and he separated the edges into two parts, those edges containing a fixed vertex and those not containing it. His theorem showed that the k-hypergraphic sequence problem with n vertices can be reduced to that with $n-1$ vertices. At the end of his paper, he considered whether the theorem by Erdős and Gallai [1960] can be extended to k-hypergraphs.

Billington [1988] and Choudum [1991] tried to make such an extension. Their works are quite similar: the edges containing vertices from $k+1$ to n are divided into two parts, those containing two vertices from vertex set $\{1, 2, ..., k\}$ and those containing only one. In this way, they proposed a necessary condition, but they remarked that the upper bounds given by their inequalities might not be tight. A non-hypergraphic sequence that satisfies their conditions has been found by Achuthan, Achuthan and Simanihuruk [Achuthan et al., 1993].

Achuthan et al. [1993] extended the works of Billington [1988] and Choudum [1991]. They found a similar necessary condition by combining the conditions of Choudum [1991] and Billington [1988] and discovered a case that showed that it was not sufficient. By making a further edge classification, they designed an integer linear program to check whether a degree sequence is 3-hypergraphic.

Behrens et al. [2013] found some sufficient conditions for a k-hypergraphic degree sequence where $k \geq 3$. One condition is that the beginning of a sequence is nearly regular, i.e., the gap between degrees of vertices in the head of the sequence is small.

Billington [1988] implemented Dewdney's reduction and he found four special classes of degree sequences whose reduction is unique. In his conclusion, when the largest degree in the sequence is exactly or close to $\binom{n-1}{2}$, where n is the number of vertices, the reduction can be easily done.

Kocay and Li [2007] considered the transformation between two distinct 3-hypergraphic realizations with identical degree sequences. They found that such transformations can be implemented through a sequence of edge exchanges that do not change the degree sequence and involve at most five or six vertices for each exchange.

Colbourn et al. [1986] found that some problems on k-hypergraphs are NP complete when $k \geq 3$ although the corresponding problem on 2-hypegraphs have solutions on polynomial time. However, the complexity status of the 3-hypergraphic problem is still unknown.

2.2 Discussion

For the k-hypergraphic problem, Havel [1955] and Hakimi [1962] proved a reduction for the case when $k = 2$.

Theorem 2.2.1. *([Havel, 1955; Hakimi, 1962])*

A non-increasing degree sequence $D = (d_1, d_2, ..., d_n)$ *is* 2*-hypergraphic if and only if* $D' = (0, d_2 - 1, ..., d_{d_1+1} - 1, d_{d_1+2}, ..., d_n)$ *is* 2*-hypergraphic.*

The sequence D' generated by a reduction is called the *residual sequence*.

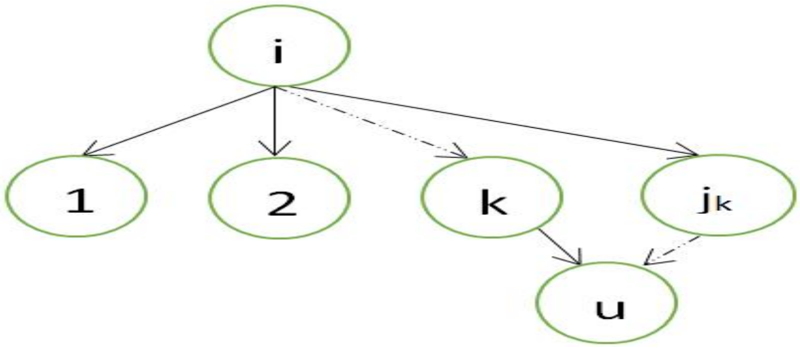

Figure 2.1: The illustration of the edge switching in the proof of Remark 1.

Remark 1. ([West, 2001]) This reduction can start with any vertex i rather than the first vertex.

Proof. If D' is graphic, then after connecting vertex i to the first i vertices on the realization of D', the new graph will be a realization of D and thus D is graphic.

Assume D has a realization in which the vertex i is adjacent to vertices $j_1 < j_2 < ... < j_{d_i}$. If $m = j_m$ for each m, then after removing these edges containing vertex i, it would be a realization of D' and thus D' would be graphic. Otherwise assume j_k is the first value such that $j_k \neq k$. Then $k < j_k$ and $d_k \geq d_{j_k}$. As vertex k does not connect to vertex i but vertex j_k does, there must be one vertex u that k connects to but j_k does not. Therefore we can add edges (j_k, u) and (i, k), and drop edges (i, j_k) and (k, u) (see graph 2.1). The sequence remains the same but k connects to i now. By repeating this process of edge switching, finally $m = j_m$ for each m and thus D' would be graphic. $\square$

Theorem 2.2.1 reduces the n-tuple graphic problem to an $(n-1)$-tuple. To see whether a sequence is 2-hypergraphic, we only need to check the $(n-1)$-tuple generated by this theorem and after at most $n-1$ reductions the problem can be solved. For example, let $D_1 = (4,3,3,2,2)$, The progress of transformation with reductions is, $(4,3,3,2,2) \rightarrow (0,2,2,1,1) \rightarrow (0,0,1,0,1) \rightarrow (0,0,0,0,0)$ and a realization of D_1 with edges $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,3\}$, $\{2,4\}$, $\{3,5\}$ can be found from the progress. Another example is for $D_2 = (4,4,3,2,1)$. After one reduction, the residual sequence is $(0,3,2,1,0)$. As $d_2 = 3$ but only $d_4, d_5 > 0$, no further reduction can be done and thus D_2 is 2-hypergraphic.

Dewdney [1975] extends such reduction to the case for the k-hypergraphic problem with any k.

Theorem 2.2.2 ([Dewdney, 1975]). *A degree sequence $H = (h_1, h_2, ..., h_n)$ is k-hypergraphic if and only if there exists a sequence $E = (0, e_2, e_3, ..., e_n)$ such that*

(1) E is $(k-1)$-hypergraphic,

(2) $\sum_{i=2}^{n} e_i = (k-1)h_1$, and

(3) $D = (0, h_2 - e_2, h_3 - e_3, ..., h_n - e_n)$ is k-hypergraphic.

This theorem separates a k-hypergraph into two parts, one part with any edge involving the first vertex that can be transformed to a $(k-1)$-hypergraph, and the other part with no edges containing the first vertex that can be viewed as a k-hypergraph with $n-1$ vertices. If such a separation is feasible, then the degree sequence is k-hypergraphic. A hypergraph with vertex set V, and k-sets $T \subseteq \binom{V}{k}$ can be denoted by (V,T). Usually $k = 3$ so T is a set of triples.

For example, for $H = (3,3,2,2,2)$ in the 3-hypergraphic problem, $E = (0,3,1,1,1)$ and $D = (0,0,1,1,1)$ meet dewdney's conditions. Clearly E and D can be realized by $\{\{2,3\},\{2,4\},\{2,5\}\}$ and $\{\{3,4,5\}\}$ respectively. Therefore H can be realized by $\{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{3,4,5\}\}$.

Remark 2. Dewdney's reduction can start with any vertex rather than the one with the largest degree.

Corollary 2.2.1. *A degree sequence $H = (h_1, h_2, ..., h_n)$ is k-hypergraphic if and only if for any j where $1 \leq j \leq n$, there exists a sequence $E = (e_1, e_2, ..., e_{j-1}, 0, e_{j+1}, ..., e_n)$ such that*

(1) E is $(k-1)$-hypergraphic,

(2) $\sum_{i=1, i\neq j}^{n} e_i = (k-1)h_j$, and

(3) $D = (h_1 - e_1, h_2 - e_2, ..., h_{j-1} - e_{j-1}, 0, h_{j+1} - e_{j+1}, ..., h_n - e_n)$ is k-hypergraphic.

Proof. If E is $(k-1)$-hypergraphic and D is k-hypergraphic, then E has a realization (V', T') with every edge containing vertex i due to (2), and D has a realization (V'', T'') without any edge containing vertex i. Therefore there exists a k-hypergraph (V, T) where $V = V' \cup V''$ and $T = T' \cup T''$ and its degree sequence is $(h_1, h_2, ..., h_n)$.

If H is k-hypergraphic, for any j its realization (V, T) can be divided into two subgraphs (V', T') and (V'', T'') such that every edge in T contains vertex j while no edge in T'' contains it. Then let the degree sequence for (V', T'), be $(e_1, e_2, ...e_n)$ and the degree sequence for (V'', T'') will be $(h_1 - e_1, h_2 - e_2, ..., h_n - e_n)$. Since no edge contains j in (V'', T''), $h_j - e_j = 0$. Since every edge in (V', T') contains vertex j, $\sum_{i=1}^{n} e_i = \sum_{i=1, i\neq j}^{n} d_i' = (k-1)h_j$ and $E = (e_1, e_2, ..., e_{j-1}, e_{j+1}, ..., e_n)$ should be $(k-1)$-hypergraphic. So E and D satisfy conditions (1), (2) and (3). $\square$

However how to find the E and D satisfying the above conditions for such a reduction [Behrens et al., 2013] is unknown. And there has not been much research in finding them and it is even unclear, for the case when $k \geq 3$, whether there exists a $(E, D)^*$ similar to the case for $k = 2$.

Billington [1988] solved such a problem with some special conditions for which the reduction is simple to see. In Chapter 4 we will discuss this reduction.

Another theorem dealing with the case when $k = 2$ is the Erdős–Gallai inequality systems [Erdős and Gallai, 1960].

Theorem 2.2.3 ([Erdős and Gallai, 1960])**.** *A non-increasing sequence* $(d_1, d_2, ..., d_n)$ *is graphic if and only if it satisfies*

$$\sum_{i=1}^{n} d_i = 0(mod\ 2); \tag{2.1}$$

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} min\left\{d_i, k\right\},\ \ k = 1, 2, ..., n, \tag{2.2}$$

where (2.2) is called a partial sum for the degree sequence [Billington, 1988]. Its main idea is to separate a non-increasing degree sequence into two parts with a break point k and then assess whether the sum of the degrees from vertex 1 to k is too large to realize.

This theorem can be extended to 3-hypergraphs. The difference is that the vertex in the right hand of break point can contribute one or two degrees (which may be difficult to track).

Theorem 2.2.4 ([Billington, 1988])**.** *A non-increasing sequence* $(d_1, d_2, ..., d_n)$ *is* 3-*hypergraphic only if it satisfies*

$$\sum_{i=1}^{n} d_i = 0(mod\ 3) \tag{2.3}$$

$$\sum_{i=1}^{k} d_i \leq 3\binom{k}{3} + 2\sum_{i=k+1}^{n} min\left\{d_i, \binom{k}{2}\right\} + \sum_{i=k+1}^{n} min\left\{d_i, (i-k-1)k\right\},\ k = 1, 2, ..., n. \tag{2.4}$$

To explain the inequality systems, here we introduce the notation in Achuthan et al.'s work [Achuthan et al., 1993]. Assume $H = (V, T)$ is a realization of $(d_1, d_2, ..., d_n)$. The vertex set V is divided into $S_k = \{1, 2, ..., k\}$ and $T_k = \{k+1, k+2...., n\}$. Then let A_k,B_k and C_k be the number of edges which contributes three, two and one degree to S_k respectively:

$$\begin{aligned} A_k &= \{e \in T : |e \cap S_k| = 3\}, \\ B_k &= \{e \in T : |e \cap S_k| = 2\}, \\ C_k &= \{e \in T : |e \cap S_k| = 1\}, \\ D_k &= \{e \in T : |e \cap S_k| = 0\}. \end{aligned} \tag{2.5}$$

Therefore

$$\sum_{i=1}^{k} d_i = 3|A_k| + 2|B_k| + |C_k|; \tag{2.6}$$

Equation (2.4) provides the upper bounds for A_k,B_k and C_k respectively. For A_k, the maximum possible value is $\binom{k}{3}$ when all vertices in S_k connect to each other with hyper-edges. To each vertex $i \in T_k$, the number of edges in B_k is bounded by its degree, as well as by the number of possible pairs in S_k. To evaluate C_k, equation (2.4) considers the number of possible pairs (u, v) where $u \in S_k$ and $v \in T_k \setminus \{i\}$. To avoid repeated accumulation one is counted only when $i > v$ in the edge in C_k.

Let degree sequence $D = (3, 3, 3, 3)$ and $k = 4$. In Equation 2.4 $RHS = 12$, then A_4 reaches the upper bound $3\binom{3}{3}$ when $\{1, 2, 3\}$ is included in the realization of D. As three 2-tuple can be generated from the set of vetices $\{1, 2, 3\}$ and $d_4 = 3$, the maximum value of B_4 is 6. To reach the upper bound, vertex 4 should be connected to all three 2-tuples from $\{1,2,3\}$. Therefore $RHS = LHS$ in 2.4 with the realization of D with edge set $\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$.

Corollary 2.2.2. *A non-increasing sequence* $(d_1, d_2, ..., d_n)$ *is* 3*-hypergraphic only if it satisfies*

$$\sum_{i=1}^{n} d_i = 0(mod\ 3) \tag{2.7}$$

$$\begin{aligned}\sum_{i=1}^{k} d_i \leq 3\binom{k}{3} + 2\sum_{i=k+1}^{n} min\left\{d_i, \binom{k}{2}\right\} + \sum_{i=k+1}^{n} min \\ \left\{(d_i - \binom{k}{2})^+, (i-k-1)k\right\}, k = 1,2,...,n\end{aligned} \tag{2.8}$$

where x^+ denotes $max\{x, 0\}$.

Proof. Equation (2.8) can replace (2.4) because for any vertex $i \in T_k$ when i is assumed to contributes $min\{d_i, k\}$ edges in B_k (only under this condition the RHS in (2.6) can reach the upper bound), vertex i will be involved with at most $d_i - \binom{k}{2}$ or zero edges in C_k. □

Choudum [1991] defined $(x_1, x_2, ..., x_n)^+$ to mean $(x_1^+, x_2^+, ..., x_n^+)$, and then replaced (2.4) with:

$$\begin{aligned}\sum_{i=1}^{k} d_i \leq 3\binom{k}{3} + 2\sum_{i=k+1}^{n} min\left\{d_i, \binom{k}{2}\right\} + M_k((d_{k+1} - \binom{k}{2}, \\ d_{k+2} - \binom{k}{2}, ..., d_n - \binom{k}{2})^+),\ k = 1,2,...,n\end{aligned} \tag{2.9}$$

where $M_k(D)$ denotes the maximum number of edges in the class of 2-hypergraphs in which its degree sequence's length equals the length of D and any vertex in its degree sequence is not larger than that in D when at most k repeated edges are allowed.

Achuthan et al. [1993] found an example, $D = (7, 5, 5, 3, 3, 3, 1)$, which shows that all inequality systems mentioned above are not sufficient. The key is in the case when $k = 3$. Although when the last three vertices only involve the edges in the class of B_k, $d_1 + d_2 + d_3 = 3 + 2(d_4 + d_5 + d_6)$, this cannot be realizable. Here we give an analysis. The first three vertices can connect with each other and $(6, 4, 4)$ remains. Note that there are only three vertex pairs for the first three vertices, as $d_4 = d_5 = 3$, both vertex 4 and 5 have to connect with the three pairs and thus the remaining degree sequence is $(2, 0, 0, 0, 0, 1)$, which is not realizable. All the necessary conditions above use $min(d_i, \binom{k}{2})$ to estimate $|B_k|$, and this is why they cannot work for this example. It seems that when the edge involves more than two vertices, counting will not be enough. Achuthan et al. used more constraints to avoid this counter example, but it is difficult to assess the complexity of this integer linear programming in his solution and the sufficiency of his conditions is unknown.

In addition, the inequality systems can be extended for any $k > 2$ by considering the maximum number of edges e where $|e \cap S_k| = k - 1$:

Theorem 2.2.5 (Behrens et al. [2013]). *A non-increasing sequence $(d_1, d_2, ..., d_n)$ is k-hypergraphic only if it satisfies*

$$\sum_{i=1}^{n} d_i = 0(mod\ k) \tag{2.10}$$

$$\sum_{i=1}^{r} d_i \leq k\binom{n}{k} + (k-1)\sum_{i=r+1}^{n} d_i \quad for\ r = 1, 2, ..., n. \tag{2.11}$$

Chapter 3

Edge Switching and Exchanges

The Havel-Hakimi theorem shows a reduction method for 2-hypergraphs and we would like to find something similar for the 3-hypergraphic problem. Given a 3-hypergraphic sequence H, Dewdney's decomposition (E, D) reduces it to a shorter 3-hypergraphic sequence D and a graphic sequence E. In this chapter we investigate the sequences D and E, and their transformations using edge switching.

3.1 Edge Switching

Definition 3.1.1. Let i be a vertex of a k-hypergraph H, and let φ be a $(k-1)$-tuple. Then vertex i is said to be *adjacent* to φ if the k-tuple $i \cup \varphi$ is an edge of H.

Let $D = (d_1, d_2, ..., d_n)$ be a non-increasing k-hypergraphic degree sequence. For any pair (i, j) such that $d_i > d_j$, we can conclude that there must exist a $(k-1)$-tuple φ which is adjacent to vertex i but not j. Then we can replace the edge $i \cup \varphi$ with $j \cup \varphi$ and therefore the new sequence, where vertex i's degree decreases by one and j's degree increases by one, would be k-hypergraphic as well. This is called an *edge switching*.

When we perform an edge switching on (i, j) such that $d_i = d_j + 1$, in the new sequence $(d'_i, d'_j) = (d_i - 1, d_j + 1) = (d_j, d_i)$. Therefore the new sequence remains the same after we put it in non-increasing order. We call this an *ineffective edge switching*. If $d_i > d_j + 1$, the edge switching is called an *effective edge switching*. In the rest of this section, we only discuss effective edge switching.

Sometimes it is necessary to keep D in non-increasing order after an effective edge switching. If $d_i < d_{i+1}$ after edge switching, we find k such that $d_i = d_{i+k} > d_{i+k+1}$ and involve vertex $i + k$ in this edge switching instead of vertex i; if $d_j > d_{j-1}$ after edge switching, we find the k such that $d_j = d_{j-k} < d_{j-k-1}$ and involve vertex $j - k$ in this edge switching instead of j. For example, let $D = (5, 5, 4, 3, 3, 3, 2)$, the edge switching between vertex 1 and 6 will break the non-increasing order, so we perform $(2, 4)$ instead.

For the case when $k = 2$, every sequence obtained by reduction can be transformed to a sequence D^* by Havel-Hakimi's algorithm [Havel, 1955; Hakimi, 1962]. Therefore D^* will be the only sequence in consideration for the next reduction. The task in this research is to try to extend such transformations based on edge switching to reduce a sequence for the 3-hypergraphic problem.

In Billington's terminology [Billington, 1986], if sequence D' is obtained from another sequence D by a sequence of edge (k-tuple) switches, D is said to be *steeper* than D', and D' is said to be *flatter* than D. When D is flattened to D, we write this as $D \Rightarrow D'$.

Definition 3.1.2. The non-increasing sequence $D = (d_1, d_2, ..., d_n)$ is said to be $near - regular$ if $d_1 = d_n + 1$.

A simple example of a near-regular sequence is $(5, 5, 4, 4, 4, 4)$. A sequence that is regular or near-regular cannot be flattened further.

3.2 Sequence Comparison

Definition 3.2.1. Let $D = (d_1, d_2, ..., d_n)$ and $D' = (d'_1, d'_2, ..., d'_n)$ be two non-increasing sequences such that $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} d'_i$. Assume j is the first vertex such that $d_j \neq d'_j$. If $d_j > d'_j$, then we say $D \succ D'$ or D is *larger* than D'; if $d_j < d'_j$, we say $D \prec D'$ or D is *smaller* than D'; if no such j exists, $D = D'$.

Lemma 3.2.1. *If $D \succ D'$ and $D' \succ D''$, then $D \succ D''$.*

Proof. Let i be the first vertex such that $d_i \neq d'_i$ and j be the first vertex such that $d'_j \neq d''_j$. Then we have $d_i > d'_i$ and $d'_j > d''_j$. if $i \leq j$, then we have $d_i > d'_i = d''_i$ and i will be the first vertex such that $d_i \neq d''_i$. Otherwise, we have $d_j = d'_j > d''_j$ and j will be the first vertex such that $d_j \neq d''_j$. In both cases, $D \succ D''$. □

Lemma 3.2.2. *If $D \Rightarrow D'$, then $D \succ D'$.*

Proof. We assume the sequence of transformations from D to D' is $D \Rightarrow D_1 \Rightarrow D_2 \Rightarrow ... \Rightarrow D'$ where in each transformation only one edge switching occurs. $D \succ D_1$, $D_1 \succ D_2, ..., \succ D'$ Based on Lemma 3.2.1, this lemma is true. □

From Lemma 3.2.1, we can see that comparison is a linear order. If D is not regular or near-regular, there exists a smaller sequence D' such that there is no sequence D''

with $D \succ D'' \succ D'$. In this case D is said to *cover* D'. Let i be the first index such that $d'_i < d_i$, then there should be no degree transfer between j and k where $i < j < k$ which steepens D' and keeps D' in non-increasing order, otherwise, after the transfer, the new sequence D'' satisfies $D \succ D'' \succ D'$.

To find D', let i be the last vertex such that $d_i > d_n + 1$. The subsequence $(d_{i+1}, d_{i+2}, ..., d_n)$ is regular or near-regular. Transfer one degree from d_i to this subsequence and then redistribute the degrees $d_{i+1} + d_{i+2} + ... + d_n + 1$ in this subsequence to make it as large as possible while keeping the whole sequence in non-increasing order. For example let $D = (4, 1, 1, 1, 1)$, then $i = 1$. So we get $(3, 1, 1, 1, 1)$ and then redistribute $(1 + 1 + 1 + 1 + 1) = 5$ degrees to the last four vertices. Finally we have $D' = (3, 3, 2, 0, 0)$. Algorithm 1 (on the next page) shows the process.

Lemma 3.2.3. *If D is not regular or near regular, then D' obtained by Algorithm 1 is the largest sequence which is smaller than D.*

Proof. Assume there exists a $D'' = (d''_1, d''_2, ..., d''_n)$ such that $D \succ D'' \succ D'$. Let i be the last vertex such that $d_i > d_n + 1$, then $d'_i = d_i - 1$, and for any r where $r < i$, $d_r = d'_r$. Let j be the first vertex such that $d''_j < d'_j$ and k be the first index such that $d''_k > d'_k$. Clearly $j, k \geq i$, otherwise D'' would be larger or smaller than both D and D'. As $d_i \neq d''_i$, then $j = i$ or $k = i$.

1. If $j = i$, as $D'' \succ D'$, $d''_i = d_i - 1 = d'_i$, but in D' the subsequence including vertex from $i + 1$ to n, is the largest. As $d''_r = d'_r$ for any $r \leq i$, $D' \succ D''$.

2. If $k = i$, as $D \succ D''$, $d''_i = d_i$, but in D the subsequence including vertex from $i+1$ to n, is regular or near regular (i.e., smallest). As $d''_r = d_r$ for any $r \leq i$, $D'' \succ D$.

In conclusion, D'' does not exist. □

Algorithm 1 Find the largest sequence which is smaller than D.

Input: D: the given degree sequence which is not regular or near regular $(d_1, d_2, ...d_n)$.

```
for k = n − 1 to 1 do
  if d_k > d_n + 1 then
    i = k; break;
  end if
end for{find the last vertex i with d_i > d_n + 1}
S = d_{i+1} + d_{i+2} + ... + d_n + 1;
j = i + 1; d_i = d_i − 1;
while j < n do
  if S > d_i then
    d_j = d_i; S = S − d_i; j + +;
  else
    d_j = S; break;
  end if
end while
return D {The sequence covered by the input sequence}
```

Also consider the example $D = (4, 1, 1, 1, 1)$, which covers $D' = (3, 3, 2, 0, 0)$. Obviously D is graphic but D' is not. Therefore if $D \succ D'$ and D is k-hypergraphic, it is unknown whether D' is k-hypergraphic, even though D' is obtained by algorithm 1.

Definition 3.2.2. $S = (s_1, s_2, ..., s_n)$ is said to be the *cumulative sequence* of $D = (d_1, d_2, ..., d_n)$ when $s_k = \sum_{i=1}^{k} d_i$ for any k.

Lemma 3.2.4. *Let D and D' be two sequences with cumulative sequences* $(s_1, s_2, ..., s_n)$

and $(s'_1, s'_2, ..., s'_n)$ *respectively. Then* $D \Rightarrow D'$ *or* $D = D'$*, if and only if* $s_i \geq s'_i$ *for* $i = 1, 2, .., n-1$*, and* $s_n = s'_n$*.*

Proof. $D = D'$ if and only if $s_k = s'_k$ for $k = 1, 2, ..., n$. Now we discuss the condition when $D \Rightarrow D'$.

If $D \Rightarrow D'$, then in each edge switching that involves vertices u and v where $u < v$, $s_u, s_{i+1}, ...s_{v-1}$ will decrease by one while other elements in S will remain the same. So $s_k \geq s'_k$ for any k.

If $s_k \geq s'_k$ for any k, let u be the first index such that $d_u \neq d'_u$, then we have $d_u > d'_u$. Let v be the first index such that $d'_v < d_v$. We do an edge switching involving (u, v) (replace u or v with the vertex with the same degree if necessary to keep D in non-increasing order) until $d_u = d'_u$ or $d_v = d'_v$. In each switch we keep $s_k > s'_k$ for any k as $d_i \geq d'_i$ when $u < i < v$. Therefore, by repeating the process we can transform D to D'. □

Lemma 3.2.4 shows a progression of edge switches transforming D to D'. Following this progression the degree sequence always gets flatter and remains in non-increasing order. Therefore, in this progression, all degree sequences obtained must be k-hypergraphic if D is. For example, let $D = (6, 5, 3, 2, 2, 2, 2)$ with cumulative sequence $S = (6, 11, 14, 16, 18, 20, 22)$ and $D' = (4, 4, 4, 3, 3, 2, 2)$ with $S' = (4, 8, 12, 15, 18, 20, 22)$. One progression is $(6, 5, 3, 2, 2, 2, 2) \overset{(1,3)}{\rightarrow} (5, 5, 4, 2, 2, 2, 2) \overset{(2,4)}{\rightarrow} (5, 4, 4, 3, 2, 2, 2) \overset{(1,5)}{\rightarrow} (4, 4, 4, 3, 3, 2, 2)$ and all degree sequences in it are graphic and in non-increasing order.

Corollary 3.2.1. *Let* L_C *be a set of degree sequences whose sum of degrees equals the constant* C*, the sequence* $D^* = (d^*_1, d^*_2, ...d^*_n)$ *is the flattest if and only if for any* *other sequence* $D = (d_1, d_2, ..., d_n) \in L_C$*,* $s_k \leq s_k^*$ *for any* k*.*

Corollary 3.2.2 ([Aigner and Triesch, 1994])**.** *A degree sequence* $D' = (d'_1, d'_2, ..., d'_n)$ *is* k*-hypergraphic if and only if there exists a* k*-hypergraphic sequence* $D = (d_1, d_2, ...d_n)$ *such that for their cumulative sequences* S *and* S'*,* $s_k \geq s'_k$*, for* $k = 1, 2, ..., n-1$ *and* $s_n = s'_n$*.*

Remark 3. In the 2-hypergraphic problem, D^* is the flattest residual sequence among the set $\{D' = (d'_2, d'_3, ...d'_n) : \sum_{i=2}^{n} d'_i = 2d_1 \textit{ and } (d_2 - d'_2, d_3 - d'_3, ..., d_n - d'_n) \textit{ is graphic}\}$ for the sequence $D = (d_1, d_2, ..., d_n)$.

It should be noted that D is not necessarily flatter than D' when $D \succ D'$. An example is D=(4,2,1,1,1,1) and D' = (3,3,2,1,1,0) even though both are 2-hypergraphic.

3.3 Edge Exchanges

Now we try to implement Dewdney's reduction with decomposition (E, D) for the 3-hypergraphic problem. Let the sequence $H = (h_1, h_2, ..., h_n)$, and let $E = (0, e_2, e_3, ...e_n)$ be the graphic degree sequence such that $\sum_{i=2}^{n} e_i = 2h_1$. Then $D = (0, d_2..., d_n) = (0, h_2 - e_2, h_3 - e_3, ..., h_n - e_n)$ is called the *residual sequence* after the reduction. We call (E, D) a decomposition of H. Basically, for a decomposition (E, D), we require that E should be graphic and D should be non-negative. In this section we perform a reduction on the first vertex.

Definition 3.3.1. In the 3-hypergraphic problem, in a decomposition (E, D), an (i, j) edge switching in D which simultaneously transfers one degree from vertex j to i in E is called an *edge exchange* on (i, j).

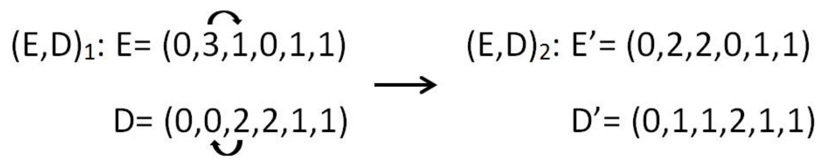

Figure 3.1: Transformation to a decomposition with a flatter residual sequence by an edge exchange

Let's compare the following two decompositions for $H = (3, 3, 3, 2, 2, 2)$: in $(E, D)_1$, $E = (0, 3, 1, 0, 1, 1)$, $D = (0, 0, 2, 2, 1, 1)$ and in $(E, D)_2$: $E' = (0, 2, 2, 0, 1, 1)$, $D' = (0, 1, 1, 2, 1, 1)$. We will show that $(E, D)_1$ can be transformed to $(E, D)_2$ with an edge exchange. We assume in $(E, D)_1$, E is graphic and D is 3-hypergraphic. As $e_2 > e_3$ and $d_2 < d_3$, in any realization of E we can find a vertex u which is adjacent to vertex 2 but not 3, and in a realization of D a pair $\{v, w\}$ such that $\{v, w\}$ is adjacent to vertex 3 but not 2. By adding the triples $\{1, 3, u\}$ and $\{2, v, w\}$, and dropping triples $\{1, 2, u\}$ and $\{3, v, w\}$, $(E, D)_2$ can be obtained. Such a transformation is illustrated by Figure 3.1.

We conclude that if D is 3-hypergraphic then D' is as well. Therefore, to see whether H is 3-hypergraphic, it is not necessary to check whether D is 3-hypergraphic.

There is a kind of less obvious edge exchange. For $H = (3, 3, 3, 1, 1, 1)$, there is an (E, D) decomposition where $E = (0, 1, 2, 1, 1, 1)$ and $D = (0, 2, 1, 0, 0, 0)$. Although $e_2 = e_4$, we can still make an edge exchange $(2, 4)$ to flatten D while the new E is still graphic. However in such a transformation, E gets steeper and the changes in the realization of E cannot be described by edge switching. We can just replace the realization of the old E with the new (these two realizations can be found with Havel and Hakimi's reduction). In conclusion, for a decomposition (E, D), we can flatten D through steepening E as long as the new E is graphic. With the discussion above, the following definition is introduced.

Definition 3.3.2. For an (E, D) decomposition, an (i, j) exchange is said to be *effective* if after its implementation:

1. E becomes flatter while D does not become flatter or steeper, or
2. D becomes flatter while E is still graphic.

After implementing the first type of exchange, a new edge exchange of the second type may appear. This is because keeping E graphic is a significant constraint while flattening D. Sometimes we may need to steepen E. If E can be flattened without steepening D, it provides more room for the second type of exchange. For example, let $E = (5, 4, 2, 3, 3, 1)$ and $D = (5, 3, 4, 1, 1, 1)$ (we discard zero in the first place for D and E). We cannot do a $(2, 4)$ exchange as $(5, 5, 2, 2, 3, 1)$ is not graphic. But if we do a $(3, 2)$ exchange first, after which $E = (5, 3, 3, 3, 3, 1)$ and $D = (5, 4, 3, 1, 1, 1)$, a $(2, 4)$ exchange can be done, as $(5, 4, 3, 2, 3, 1)$ is graphic, and then D becomes flatter.

An edge exchange should not make E non-graphic or D steeper. We want each transformation to lead to a new decomposition whose residual sequence would be 3-hypergraphic if the former one is. Therefore we don't need to check the former one for a further reduction.

There is also a type of edge exchange which does not flatten or steepen D or E. For example, in the (E, D) where $E = (0, 2, 2, 0, 1, 1)$ and $D = (0, 1, 1, 1, 0, 0)$ we can implement a $(4, 5)$ exchange and obtain $E = (0, 2, 2, 1, 0, 1)$ and $D = (0, 1, 1, 0, 1, 0)$. However, this is just an isomorphism to the original decomposition whose mapping is

$4 \to 5$ and $5 \to 4$. This type of exchange is said to be an *ineffective edge exchange*.

So how to define a "best" decomposition? if we implement an effective edge exchange on (E, D) and get a new decomposition (E', D'), then D' would be 3-hypergraphic if D is. Then in the 3-hypergraphic problem it is not necessary to consider D' for the next reduction. Therefore (E', D') is a better decomposition.

Definition 3.3.3. An (E, D) decomposition is said to be *irreducible* when no effective edge exchange can be done.

Does an irreducible (E, D) decomposition on some vertex always exist? The answer is true when an (E, D) decomposition, in which all elements are non-negative and E is graphic, exists. We can search for an effective (i, j) exchange on this decomposition. Implementing it will make E or D flatter, and for a sequence, the flattest condition is when it is regular or near-regular. So the number of exchanges will be finite.

Corollary 3.3.1. *A degree sequence is* 3*-hypergraphic if and only if the residual sequence in some irreducible decomposition is* 3*-hypergraphic.*

3.4 Standard Form

For the k-hypergraphic problem the sequence is required to be in non-increasing order. Therefore, in the decomposition (E, D), it is expected that the residual sequence D be kept in this order for further reduction. For convenience, the vertex of reduction in the decomposition will be removed in the rest of this chapter.

Definition 3.4.1. An (E, D) decomposition is said to be in D-standard form when the residual sequence D is in non-increasing order. It is said to be in E-standard form if E is in non-increasing order.

When is a D-standard decomposition (E, D) irreducible? Firstly consider the first type of effective edge exchanges in Definition 3.3.2. Assume $d_i = d_j + 1$, as D is in non-increasing order, $i < j$. Therefore $d_i + e_i \geq d_j + e_j$ and thus $e_i + 1 \geq e_j$. After an (i, j) exchange $(e_i, e_j) \to (e_i + 1, e_j - 1)$, as $e_i + 1 \geq max\{e_j, e_i\}$, so such an exchange cannot flatten E. So in a D-standard decomposition, the first type of exchange does not exist.

Then consider the second type of effective exchange. When $d_i > d_j + 1$ for any (i, j) where $i < j$, we must check if $(e_1, e_2, ..., e_{i-1}, e_i + 1, e_{i+1}, ..., e_{j-1}, e_j - 1, e_{j+1}, ..., e_n)$ is graphic. This gives:

Proposition 3.4.1. *the D-standard (E, D) decomposition of a non-increasing sequence H is irreducible, if and only if for any i and j where $1 \leq i < j \leq n$, $(e_1, e_2, ..., e_{i-1}, e_i + 1, e_{i+1}, ..., e_{j-1}, e_j - 1, e_{j+1}, ..., e_n)$ is non-graphic, when $d_i > d_j + 1$.*

We can transform a D-standard decomposition to an irreducible one by only checking the second type of exchanges. A problem is that right after we make an edge exchange (i, j), an edge exchange (j, k) may be available, where $i < j < k$, which indicates that initially an (i, k) exchange is available. For efficiency, we should check whether such an edge exchange is feasible in advance. An easy solution is that for a fixed i, we scan from n to $j - 1$.

Possibly we can keep D in non-increasing order after each edge exchange. Assume we can make an edge exchange between vertices i and j where $i < j$. If the new residual sequence $(d_1, d_2, ..., d_{i-1}, d_i - 1, d_{i+1}, ..., d_{j-1}, d_j + 1, d_{j+1}, ...d_n)$ is not in non-increasing order, as $(d_1, d_2, ..., d_n)$ is, there are two possibilities:

1. $d_i - 1 < d_{i+1}$. As $d_i \geq d_{i+1}$, $d_i = d_{i+1}$. As $d_i + e_i \geq d_{i+1} + e_{i+1}$, $e_i \geq e_j$. If after $(e_i, e_j) \to (e_i + 1, e_j - 1)$ by an (i, j) exchange, the new E is still graphic, then $(e_{i+1}, e_j) \to (e_{i+1}+1, e_j - 1)$ holds too, as $(e_i+1, e_{i+1}, e_j - 1)$ is equivalent to or steeper than $(e_i, e_{i+1} + 1, e_j - 1)$. We can avoid this condition simply by scanning i from 1 to $n - 1$.

2. $d_j + 1 > d_{j-1}$. The discussion is similar and (i, j) exchange can be replaced by an $(i, j - 1)$ exchange. Note that we decide to scan j from n to $i - 1$, so we should check whether $d_j = d_{j-1}$.

Algorithm 2, based on the discussion above, transforms a D-standard decomposition to an irreducible decomposition.

To find an effective exchange, we need to search for a possible (i, j) pair, and its complexity is $O(n^2)$. The complexity of sorting a sequence and then determining whether it is graphic is $O(n^2)$. Any exchange itself only needs a constant number of steps. Now consider how many exchanges are needed to transform a decomposition to the irreducible form with Algorithm 2. Recall that each exchange will flatten D and the flattest D is the case when $d_i - d_j \in \{0, 1\}$ for any (i, j) where $i < j$. Since $\sum_{i=1}^{n} e_i = 2d_R$, where d_R is the degree of the vertex on the reduction in the original sequence, the largest possible $D = (\sum_{i=1}^{n} h_i - 2d_R, 0, 0, ..., 0)$ where $h_i \in O(n^2)$. If it can be transformed to the case mentioned above with effective edge exchanges (if it cannot, fewer exchanges are needed), then in the final residual sequence D^*, $d_i^* \approx d_1/n$. To reach it, $(n - 1)d_1/n$ exchanges are needed. Therefore the complexity

Algorithm 2 Transform a D-standard decomposition to an irreducible one

Input: d_R : the degree of the vertex on decomposition (this vertex is omitted in (E, D)),

E: a graphic degree sequence $(e_1, e_2, ..., e_n)$ such that $E = 2d_R$,

D: the non-increasing residual sequence $(d_1, d_2, ...d_n)$.

while true **do**
  for $i = 1$ to $n - 1$ **do**
    for $j = i + 1$ to n **do**
      if $d_i > d_j + 1$ and $d_{j-1} > d_j$ **then**
        if $(e_1, e_2, ...e_i + 1, e_{i+1}, ..., e_j - 1, e_{j+1}, ..., e_n)$ is graphic **then**
          $d_i = d_i - 1, d_j = d_j + 1, e_i = e_i + 1, e_j = e_j - 1$
          continue {restart the search}
        end if
      end if
    end for
  end for
  break; {no edge exchange is found}
end while
return (E, D)

of Algorithm 2 is at most $O(n^6)$.

Lemma 3.4.2. *Any irreducible (E, D) decomposition can be transformed to a D-standard irreducible (E, D) decomposition by ineffective edge exchanges.*

Proof. Assume (E, D) is irreducible and $d_1 \geq d_2 \geq ... \geq d_k$ and $d_k < d_{k+1}$. if

$d_k < d_{k+1} - 1$, as $d_k + e_k \geq d_{k+1} + e_{k+1}$, we have $e_k > e_{k+1} + 1$. Then the exchange $(k, k+1)$ will reduce D and flatten E, which shows that (E, D) is reducible. So $d_k = d_{k+1} - 1$. Let k' be the first index from 1 to k such that $d_{k'} = d_k$. We make a $(k+1, k')$ ineffective exchange and now $d_2 \geq d_3 \geq ... \geq d_k \geq d_{k+1}$. By repeating this process, we can make a D-standard irreducible (E, D) decomposition. □

For example let $D = (2, 2, 2, 0, 0, 1, 0)$ and $E = (3, 1, 1, 1, 1, 0, 1)$. Now $d_5 < d_6$, the first index k' such that $d'_k = d_5$ is 4, then we make the ineffective $(4, 6)$ exchange and now $D = (2, 2, 2, 1, 0, 0, 0)$ and $E = (3, 1, 1, 0, 1, 1, 1)$.

We deduce a property for a D-standard, irreducible decomposition (E, D):

Lemma 3.4.3. *If (E, D) is a D-standard irreducible decomposition, then for any i and j where $i < j$, $e_i \geq e_j - 1$. Moreover, $e_i = e_j - 1$ only when $e_i + d_i = e_j + d_j$.*

Proof. If $e_i < e_j - 1$, as $e_i + d_i \geq e_j + d_j$, we have $d_i > d_j + 1$. Then the (i, j) exchange will flatten both E and D and it is not irreducible. When $e_i = e_j - 1$, if $d_i > d_j + 1$, then the (i, j) exchange will reduce D and does not affect E (in terms of non-increasing form) and then this decomposition is not irreducible. So $d_i \leq d_j + 1$. In this case when $e_i = e_j - 1$ we have $e_i + d_i \leq e_j + d_j$ while $e_i + d_i \geq d_j + d_j$. So $e_i + d_i = e_j + d_j$. □

It should be noted that we can also require E to be in the non-increasing order (here it is called E-standard) and this is what Havel and Hakimi's reduction does for the 2-hypergraphic problem. For example, for $D = (3, 2, 2, 2, 2, 1)$, their first reduction should be $(1, 1, 1, 2, 1)$ by joining vertex 1 to vertices 2, 3 and 4. But if we make vertex 1 adjacent to vertices 3, 4 and 5 instead, it can be $(2, 1, 1, 1, 1)$. In a

3-hypergraphic reduction these two versions can be easily transformed to each other as well and the transformation is similar to that of Lemma 3.4.2.

Corollary 3.4.1. *If* (E, D) *is an* E*-standard, irreducible decomposition, then for any* i *and* j *where* $i < j$, $d_i \geq d_j - 1$. *Moreover,* $d_i = d_j - 1$ *only when* $d_i + e_i = d_j + e_j$.

If we want to transform an E-standard decomposition to an irreducible one, we must consider the first type of edge exchange in Definition 3.3.2. For example, let (E, D) be an E-standard decomposition and $(e_i, e_j) = (5, 3)$, $(d_i, d_j) = (2, 3)$ where $i < j$, we can perform a (j, i) exchange, which does not flatten or steepen D but flattens E.

The two standard and irreducible forms can be mutually transformed. If we make an E-standard and irreducible decomposition, we may need to sort the residual sequence D for the next decomposition. The advantage of this form is that it is easier to find some property of an irreducible decomposition, as a graphic sequence is better understood than a 3-hypergraphic sequence. Some significant conclusions made in Chapter 5 are based on E-standard, irreducible decompositions. As to D-standard form, it facilitates the search of potential edge exchanges with Proposition 3.4.1 and thus it is better for decomposition transformation.

Chapter 4

Sequence Reduction

For the 3-hypergraphic problem, effective edge exchanges can transform an (E, D) decomposition into an irreducible decomposition. In this chapter we investigate irreducible decompositions and related reductions.

4.1 An Algorithm for Irreducible decompositions

We begin with finding an initial decomposition (but not necessarily irreducible) and a decomposition with the flattest E is built. For convenience, in this chapter when we discuss a reduction, the vertex of the reduction will be omitted, in H, E and D and we label this vertex as R.

Algorithm 3 shows how to construct such a decomposition. The main idea of this algorithm is to distribute the degrees evenly in E under the restriction that D should be non-negative until $\sum_{i=1}^{n} e_i = 2d_R$. Let $H = (6, 6, 6, 5, 5, 2)$, when $d_R = 8$ the decomposition generated by Algorithm 3 is $E : (3, 3, 3, 3, 2, 2)$ and $D : (3, 3, 3, 2, 3, 0)$.

Algorithm 3 Build a decomposition with the flattest E.

Input: d_R: the degree of the vertex on reduction.

H: $(h_1, h_2, ..., h_n)$ the original sequence without the vertex of reduction

E: a graphic degree sequence $(e_1, e_2, ..., e_n)$, initially it is a zero sequence.

D: the residual sequence $(d_1, d_2, ..., d_n)$ and initially $D = H$

$k = 1$; {k indicates which vertex in E should add one degree}

for $i = 1; i < 2d_R; i++$ **do**

  if $d_k > 0$ **then**

    $e_k = e_k + 1$; $d_k = d_k - 1$; $k = k + 1$;

  else if $d_1 > 0$ **then**

    $k = 1$; $e_k = e_k + 1$; $d_k = d_k - 1$; $k = k + 1$ {we cannot decrease d_k any more, try d_1}

  else

    return null; {$2d_R > \sum_{i=1}^{n} h_i$, no valid decomposition exists}

  end if

  if $k > n$ **then**

    $k = 1$; {the augment restarts from the first vertex}

  end if

end for

return (E, D)

In E for any (i, j) where $i < j$, $e_i - e_j \in \{0, 1\}$. But as $d_6 = 0$, when $d_R = 9$, e_5 and e_1, instead of e_5 and e_6, should increase by one. Thus the new decomposition is $E : (4, 3, 3, 3, 3, 2)$ and $D : (2, 3, 3, 2, 2, 0)$ and we can see $e_1 - e_6 > 1$. The following conclusion can be made for this algorithm.

Lemma 4.1.1. *Assume* $E = (e_1, e_2, ..., e_n)$ *is generated by the Algorithm 3. Then for any* (i, j) *where* $i < j$*, if* $e_j \neq h_j$*, then* $e_i = e_j$ *or* $e_i = e_j + 1$*.*

Proof. In Algorithm 3, when $k = j$ and $d_j > 0$, e_j will increase by one such that $e_i = e_j$ for any $i < j$. So $e_i > e_j + 1$ occurs only when $h_j - e_j = d_j = 0$. □

Lemma 4.1.2. *Any decomposition* (E, D) *in which* E *is not in non-increasing order, can be transformed to a decomposition* (E', D') *in which* E' *is in non-increasing order, with edge exchanges, each of which will not steepen either* E *or* D*.*

Proof. Let j be the first vertex such that $e_{j-1} < e_j$ in E. Find the first vertex i such that $e_i < e_j$. As $e_i + d_i \geq e_j + d_j$, $d_i > d_j$. Therefore an (i, j) exchange will not steepen either E or D. Note that $e_i < e_{i-1}$ (if $i > 1$), so the subsequence $(e_1, e_2, ..., e_{j-1})$ will be kept in non-increasing order. Repeat finding such an i and making edge exchanges (i, j) until $e_{j-1} \geq e_j$. Then $(e_1, e_2, ..., e_j)$ will be in non-increasing order. Repeat the process until E is in non-increasing order. □

Lemma 4.1.3. *The decomposition* (E, D) *generated by Algorithm 3 includes the flattest* E *among all decompositions.*

Proof. Lemma 4.1.2 shows that if in a decomposition (E, D), E is not in non-increasing order, then there exists another decomposition (E', D'), where E' is in non-increasing order and is flatter than (or equivalent to) E after sorting E to non-increasing order. Therefore, to prove this lemma, we only need to prove the decomposition generated by Algorithm 3 includes the flattest among all other decompositions (E', D') where E' is in non-increasing order. Assume the cumulative sequence (in

Definition 3.2.2) for $E = (e_1, e_2, ..., e_n)$ is $S = (s_1, s_2, ..., s_n)$ and that E is not the flattest one. Then there exists a sequence $E' = (e'_1, e'_2, ..., e'_n)$ with cumulative sequence $S' = (s_1, s_2, ..., s_n)$ with at least one vertex i such that $s'_i < s_i$. Let i be the first. If $i > 1$, as $s_{i-1} \leq s'_{i-1}$ and $s_i > s'_i$, $e'_i < e_i$; if $i = 1$, $e'_i < e_i$ holds as well. Since $s_n = s'_n$, there exists $j > i$ such that $e'_j > e_j$. As $e'_j \geq e'_i$ we have $e_i > e_j + 1$ and thus $e_j = h_j$ with Lemma 4.1.1. However $e_j < e'_j \leq h_j$ and this is a contradiction. □

We can easily convert this decomposition to D-standard form. For any (i, j) where $i < j$,

(1) if $e_j = h_j$, then $d_j = 0$ and $d_i \geq d_j$;

(2) if $e_j < h_j$, then $e_i - e_j \leq 1$. As $h_i \geq h_j$, $d_i \geq d_j - 1$. Therefore $d_i < d_j$ only when $h_i = h_j$ and $e_i = e_j - 1$. By the ineffective exchange (j, i), we can make $d_i > d_j$. Also consider the example in which $d_R = 9$ and $H = (6, 6, 6, 5, 5, 2)$ with the decomposition generated by Algorithm 3, $E = (4, 3, 3, 3, 3, 2)$ and $D = (2, 3, 3, 2, 2, 0)$. By performing a $(3, 1)$ exchange we can get the new $D = (3, 3, 2, 2, 2, 0)$, which is in non-increasing order.

Algorithm 4 (on the next page) can be used to find an irreducible decomposition. It begins with finding the decomposition (E, D) with the flattest E, and then it converts this decomposition to D-standard form. At last, it is made irreducible by Algorithm 2. Regarding the complexity of this algorithm, the complexity of constructing an initial decomposition by Algorithm 3 is $\Theta(d_R)$. When we convert it to D-standard form, we only need to perform an edge exchange when $d_i = d_j - 1$ where $i < j$, so its complexity is $O(n)$. The last step, as mentioned in Chapter 3, has a complexity of $O(n^6)$.

Algorithm 4 Build an irreducible decomposition.

Input: d_R: the degree of the vertex on reduction(it has been discarded in (E, D)) of H,

H: $(h_1, h_2, ..., h_n)$ the original sequence without the vertex of reduction

E: a graphic degree sequence $(e_1, e_2, ..., e_n)$, initially it is a zero sequence.

D: the residual sequence $(d_1, d_2, ..., d_n)$ and initially $D = (h_1, h_2, ..., h_n)$

find the decomposition (E, D) with flattest E by Algorithm 3.

if (E, D) is null **then**

  return null {the original sequence is not 3-hypergraphic}

else

  convert (E, D) to D-standard from.

  transform (E, D) to an irreducible decomposition (E^*, D^*) by Algorithm 2.

  return (E^*, D^*)

end if

4.2 The Uniqueness of the Irreducible decomposition

For any H and d_R, is the irreducible decomposition unique? (if two irreducible decompositions can be transformed to each other with ineffective exchanges, they are isomorphic and we regard them as identical) Billington [1988] gives an answer for $d_R = \binom{n}{2}$ or $d_R = \binom{n}{2} - 1$. For $d_R = \binom{n}{2}$, the decomposition is unique as all pairs (i, j) ,where i and j are two vertices in H, should be adjacent to R. For the case $d_R = \binom{n}{2} - 1$, only one pair (i, j) is not adjacent to R. As any irreducible decomposition can be transformed to E-standard form, we consider an irreducible and E-standard decomposition (E^*, D^*). Clearly E^* must be $(n-1, n-1, ..., n-1, n-2, n-2)$. Since E^* is unique, the irreducible decomposition must be also unique. By considering the idea of irreducible decomposition, we can make a further conclusion.

Lemma 4.2.1. *For any H and d_R, if $d_R = \binom{n}{2} - 2$, the irreducible decomposition is unique.*

Proof. Only two pairs of vertices in H are not adjacent to R and the intersection of these two pairs can be zero or one vertex. Consider an E-standard irreducible decomposition on R, $E = E' = (n-1, n-1, ..., n-1, n-1, n-2, n-2, n-3)$ or $E = E'' = (n-1, n-1, ..., n-1, n-2, n-2, n-2, n-2)$. Suppose both (E', D') and (E'', D'') are irreducible decompositions. Clearly we have $e'_{n-4} > e''_{n-4}$ and $e'_n < e''_n$, and thus $d'_{n-4} < d''_{n-4}$ and $d'_n > d''_n$. As $e''_{n-4} \geq e''_n$, we have $e'_{n-4} > e'_n + 1$. With Corollary 3.4.1, we have $d'_{n-4} \geq d'_n$ and thus $d''_{n-4} > d''_n + 1$. Therefore by an effective edge exchange $(n-4, n)$ which flattens D'', we can transform (E'', D'') and (E', D'). So (E'', D'') is not irreducible. □

Corollary 4.2.1. *For $H = (h_1, h_2, ..., h_n)$ with d_R, when $d_R \in \{0, 1, 2\}$ the irreducible decomposition is unique.*

However when $d_R \notin \left\{0, 1, 2, \binom{n}{2} - 2, \binom{n}{2} - 1, \binom{n}{2}\right\}$, the decomposition may be not unique. To convert one irreducible decomposition to another, the transformations involves an edge exchange which steepens D or makes E non-graphic. For example let $d_R = 3$, $E = (2, 2, 2, 0, 0, 0)$, $E' = (3, 1, 1, 1, 0, 0)$, and the corresponding $D = (4, 2, 2, 2, 1, 1)$ and $D' = (3, 3, 3, 1, 1, 1)$. If we try to transform D to D'. we have to perform a $(1, 2)$ exchange after which the new $D = (3, 1, 2, 0, 0, 0)$ is non-graphic, or a $(4, 3)$ exchange which steepens D. Both of these two exchanges are invalid and thus we cannot tell which residual sequence is more likely to to 3-hypergraphic (although both of them are). The condition is similar if we try to transform D to D'.

Can we find a decomposition as a starting point, to reach all irreducible decompositions by edge exchanges? Let's review the decomposition generated by Algorithm 3.

Theorem 4.2.2. *All irreducible decompositions can be derived from the decomposition generated by Algorithm 3 with edge exchanges.*

Proof. Let (E^*, D^*) be an irreducible decomposition and (E, D) be the decomposition generated by Algorithm 3. As mentioned before, all irreducible decompositions can be transformed to E-standard form with ineffective exchanges, here we suppose (E^*, D^*) to be E-standard. Lemma 3.2.4 shows a progression of edge switching transforming E^* to E in which all nodes are graphic. So we can start with E and follow this progression to E^* and after each exchange, the new E will remain graphic. The remaining work is to prove that the exchanges in our progression will flatten D.

Assume the exchange is between vertex i and j where $i < j$ (the exchange will steepen E but keep it graphic). Clearly $e_i < e_i^*$ and $e_j > e_j^*$. Therefore $d_i > d_i^*$ and $d_j < d_j^*$. As $e_i \geq e_j$, $e_i^* > e_j^* + 1$. Because (E^*, D^*) is E-standard and $i < j$, with Corollary 3.4.1 we have $d_i^* \geq d_j^*$. So $d_i \geq d_j^* + 1 > d_j + 1$ and thus this exchange will flatten D. □

For example, let $d_R = 9$ and $H = (6, 6, 6, 5, 5, 2)$. The decomposition generated by Algorithm 3 is $E = (4, 3, 3, 3, 3, 2)$ and $D = (2, 3, 3, 2, 2, 0)$. The irreducible decomposition in E-standard form is $E^* = (4,4,4,3,3,0)$ and $D^* = (2,2,2,2,2,2)$. A path from E^* to E is $(4,4,4,3,3,0) \overset{(3,6)}{\to} (4,4,3,3,3,1) \overset{(2,6)}{\to} (4,3,3,3,3,2)$. Since both E and E^* are graphic, it can be guaranteed that all intermediary sequences are graphic too. Now we implement the exchange following the reversed direction: $(2,3,3,2,2,0) \overset{(2,6)}{\to} (2,2,3,2,2,1) \overset{(3,6)}{\to} (2,2,2,2,2,2)$. As in each transformation, D gets flatter, each exchange is effective. This is the process of transforming (E, D) to (E^*, D^*).

4.3 Comparsion of Different Irreducible decompositions

Since in general, an irreducible decomposition may not be unique, can we know which residual sequence is likely to be 3-hypergraphic? Although the residual sequences in different irreducible decompositions shown in the Chapter 3 cannot be transformed to each other by edge switching, intuitively we can guess that the smallest one may be the most likely to be 3-hypergraphic, because in the 2-hypergraphic problem, the smallest residual sequence (but it is also the flattest one) is the only one in the consideration for the further reductions.

Now consider $d_R = 3$ and this is an easy case because when d_R is small, the number of irreducible decompositions will be small too. Therefore we can find a decomposition with the smallest D by enumerating all D-standard irreducible decompositions. There are following two examples to compare.

Let $H = (13, 9, 9, 7, 7, 6)$ with $d_R = 3$. Two irreducible decompositions are (E, D)

in which $E = (3, 1, 1, 0, 1, 0)$ and $D = (10, 8, 8, 7, 6, 6)$, and (E', D') in which $E' = (2, 2, 2, 0, 0, 0)$ and $D' = (11, 7, 7, 7, 7, 6)$. The accumulative sum sequences for D and D' are $S = (10, 18, 26, 33, 39, 45)$ and $S' = (11, 18, 25, 32, 39, 45)$. Clearly $D \Rightarrow D'$ or $D' \Rightarrow D$ does not hold with Lemma 3.2.4. Therefore if we do a transformation between D and D', some edge exchange which steepens D or D' must be implemented. D is 3- hypergraphic with the following realization: $\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,2,6\}$, $\{1,3,4\}$, $\{1,3,5\}, \{1,3,6\}, \{1,4,5\}, \{1,4,6\}, \{1,5,6\}, \{2,3,4\}, \{2,3,5\}, \{2,3,6\}$, $\{2,4,5\}$, $\{3,4,6\}$, while D' is not because $d'_1 = 11 > \binom{5}{2}$.

However for $H = (9, 7, 7, 3, 2, 2)$ with $d_R = 3$, two irreducible decompositions are (E, D) in which $E = (3, 1, 1, 1, 0, 0)$ and $D = (6, 6, 6, 2, 2, 2)$, and (E', D') in which $E' = (2, 2, 2, 0, 0, 0)$ and $D' = (7, 5, 5, 3, 2, 2)$. D is not 3-hypergraphic as it does not meet the inequality system (2.8) when $k = 3$ ($LHS = 18$ but $RHS = 15 < LHS$). But D' is 3-hypergraphic with realization: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,2,6\}$, $\{1,3,4\}$, $\{1,3,5\}$, $\{1,3,6\}$, $\{2,3,4\}$.

Therefore, in the 3-hypergraphic problem, every residual sequence in the irreducible decomposition may need to be checked in the next reduction.

Chapter 5

Other applications of irreducible decompositions

Although the irreducible decompositions of a degree sequence H may not be unique, and it is difficult to determine the one containing the 3-hypergraphic residual sequence, we can make some significant conclusions for a 3-hypergraphic degree sequence with the property of an irreducible decomposition. That is, assume we have a realization of a degree sequence. With the conclusion in Chapter 3 we can perform edge exchanges to make the decomposition irreducible and E-standard. In this way we can obtain a new realization of this degree sequence with some special properties. In this chapter, to avoid index chaos, in the decomposition vertex on the reduction will not be omitted.

5.1 Sequence Augmentation

Assume H is known to be 3-hypergraphic, we can find a class of 3-hypergraphic sequences based on H. We begin with a result of Choudum [1986].

Lemma 5.1.1 (Choudum [1986])**.** *In a graphic degree sequence $D = (d_1, d_2, ..., d_n)$ with a realization G, if $d_i < n-1$ and $i \neq n$, then $D' = (d_1, d_2, ..., d_{i-1}, d_i + 1, d_{i+1} ..., d_n + 1)$ is graphic.*

Proof. If $\{i, n\}$ is not included in G, it is true. Otherwise, as $d_i < n-1$, at least one vertex in G is not adjacent to vertex i. Let such one vertex be j where $j \neq n$. Since $d_j \geq d_n$ and vertex i is adjacent to n but not j, there exists a vertex $m \neq i$ where $\{m, j\} \in G$ but $\{m, n\} \notin G$. Therefore we can construct G' by adding edges $\{i, j\}$ and $\{m, n\}$, and dropping $\{i, n\}$ and $\{m, j\}$. Now $\{i, n\} \notin G'$ and the sequence D does not change. Adding edge $\{i, n\}$ in G' will generate a realization with degree sequence D'. □

This lemma can be extended to 3-hypergraphic sequences: assume we have a non-increasing 3-hypergraphic sequence $H = (h_1, h_2, ..., h_n)$, then for any pair $\{i, j\}$ in a realization G of H, if there exist a vertex k such that edge $\{i, j, k\}$ is not included G, then $(h_1, h_2 ... h_i + 1 ..., h_j + 1, ..., h_k + 1, ... h_n)$ is 3-hypergraphic. The proof is also similar: if $\{i, j, n\} \notin G$, then the work is done; Otherwise, $k \neq n$. Since $h_k \geq h_n$ and $\{i, j\}$ is adjacent to n but not k, there exists a pair $\{u, v\} \neq \{i, j\}$ such that $\{u, v, k\} \in G$ but $\{u, v, n\} \notin G$. Therefore we can construct G' by adding edges $\{u, v, n\}$ and $\{i, j, k\}$ and dropping edges $\{i, j, n\}$ and $\{u, v, k\}$. Now $\{i, j, n\} \notin G'$ and the sequence H does not change.

To implement such an augmentation, we need a clear condition when the pair $\{i, j\}$ is not "saturated" in a realization of H. An easy case meeting this condition is $min\{d_i, d_j\} < n-2$, because it will guarantee that not all other $n-2$ vertices are adjacent to $\{i, j\}$ in any realization. Here I propose a looser condition by constraining the decomposition (E, D) to be irreducible and E-standard form.

Lemma 5.1.2. *In a 3-hypergraphic sequence $H = (h_1, h_2, ..., h_n)$, there exists a realization G of H excluding edge $\{i, j, n\}$ where $i < j < n$, if*

$$h_i < (2n-2-j)(j-1)/2 \quad or \tag{5.1}$$

$$h_j < (2n-3-i)i/2 \tag{5.2}$$

Proof. According to Corollary 2.2.1, we can begin the reduction with any vertex rather than just h_1. Now we have an (E, D) decomposition on vertex i based on a realization of H. Perform edge exchanges until the new decomposition (E', D') is irreducible and E-standard. As H is known to be 3-hypergraphic, E' must be graphic and D' must be 3-hypergraphic. Now if there exists a realization G' of $E' = (e_1, e_2, ..., e_{i-1}, 0, e_{i+1}, ..., e_n)$ excluding edge $\{j, n\}$, then edge $\{i, j, n\}$ will not be included in the set of edges involving vertex i, and thus it will not be included in this realization of H.

Assume there does not exist such a G', based on Lemma 5.1.1, $e_j = n-1$. As E' is in non-increasing order, $e_k = n-1$ for any $k \leq j$ and $k \neq i$. These vertices will be adjacent to any vertex k where $k > j$, so $e_k \geq j-1$. Then $\sum_{u=1,u\neq i}^{n} e_u \geq (j-1)(n-2)+(n-j)(j-1) = (2n-2-j)(j-1)$. Therefore $h_i \geq (2n-2-j)(j-1)/2$ and we have the sufficient condition (5.1). We can repeat the process for the reduction on vertex j and then get (5.2). □

Corollary 5.1.1. *In a* 3*-hypergraphic sequence* $H = (h_1, h_2, ..., h_n)$*, if* $d_i < \binom{n}{2}$ *where* $i < n-1$*, then there exists a realization excluding edge* $\{i, n-1, n\}$*.*

Proof. Set $j = n-1$ in (5.1) and the proof is done. □

Can we extend our conclusion to a realization without edge $\{i, j, k\}$ where k is arbitrary rather than n? We still start from the graphic sequences.

Lemma 5.1.3. *Let* $D = (d_1, d_2, ..., d_n)$ *be a graphic sequence and* $i < j$*. If* $d_j < i$ *or* $d_i < j-1$*, then there exists a realization of* D *excluding edge* $\{i, j\}$*.*

Proof. According to remark 1 in Chapter 2, the reduction can be done on vertex j. Since D is graphic, we have a graphic reduction in which j is adjacent to the first d_j vertices. Then if $i > d_j$, vertices i and j are not adjacent in this realization. If we do a reduction on vertex i, then we can find another sufficient condition $d_i < j-1$. □

Corollary 5.1.2. *Let* $D = (d_1, d_2, ..., d_n)$ *be a graphic sequence. If* $d_j \geq i$ *or* $d_i \geq j-1$ *where* $i < j$*, then there exists a realization of* D *including edge* $\{i, j\}$*.*

The proof of Corollary 5.1.2 is easy to see when reductions on vertex i and j are considered. However, we can prove it by considering $\overline{D}$.

Proof. Consider the complementary sequence of D, $\overline{D} = (d'_1, d'_2, ..., d'_n) = (n-1-d_n, n-1-d_{n-1}, ..., n-1-d_1)$, a realization G of D including edge $\{i, j\}$ exists if and only if the $\overline{G}$ with sequence $\overline{D}$ excluding edge $\{n+1-j, n+1-i\}$ exists. According to Lemma 5.1.3, we have $n-1-d_i = d'_{n+1-j} < n+1-j$ or $n-1-d_j = d'_{n+1-i} < n+1-j-1$. Therefore, two sufficient conditions, $d_j \geq i$ and $d_i \geq j-1$, can be concluded. □

Corollary 5.1.3. *If $D = (d_1, d_2, ..., d_n)$ and $D^* = (d_1, d_2, ..., d_i+1, ..., d_j+1, ..., d_n)$ are both graphic, there exists a realization G of D excluding edge $\{i, j\}$ and a realization G^* of D^* including it.*

Proof. Assume all realizations of D include edge $\{i, j\}$, then we have $d_i \geq j - 1$ and $d_j \geq i$. And no realization of D^* includes edge $\{i, j\}$, otherwise by dropping edge $\{i, j\}$ we get a realization of D excluding it. So we have $d_i + 1 < j - 1$ and $d_j + 1 < i$. There conditions are mutually exclusively, so there exists such a G excluding edge $\{i, j\}$ and by adding this edge, we get G^*. □

As before, by making use of the conclusion for 2-hypergraphs, we can extend it to 3-hypergraphs.

Lemma 5.1.4. *In a 3-hypergraphic sequence $H = (h_1, h_2, ..., h_n)$, there exists a realization of H excluding edge $\{i, j, k\}$ where $i < j < k$, if*

$$h_i < (2k - j - 2)(j - 1)/2 \quad or \tag{5.3}$$

$$h_j < (2k - i - 3)i/2 \quad or \tag{5.4}$$

$$h_k < (2j - i - 1)i/2 \tag{5.5}$$

Proof. We do a reduction on vertex i for H, and perform edge exchanges to make the decomposition (E', D') irreducible and E-standard. If some realization G' of E' excludes edge $\{j, k\}$, then there should exist a realization G of H excluding edge $\{i, j, k\}$.

If every realization of E' includes edge $\{j, k\}$, based on Lemma 5.1.3, we have $e_k \geq j - 1$ (note that $e_i = 0$ and we need to skip it) and $e_j \geq k - 2$. As E is in non-increasing order, we have $e_1 \geq e_2 \geq ... \geq e_j \geq k - 2$ and $e_{j+1} \geq e_{j+2} \geq ... \geq e_k \geq j - 1$.

Therefore $\sum_{u=1,u\neq i}^{n} e_u \geq (j-1)(k-2)+(k-j)(j-1) = (j-1)(2k-j-2)$. So $h_i = \sum_{u=1,u\neq i}^{n} e_u/2 < (j-1)(2k-j-2)/2$ is a sufficient condition we need. Similarly after a reduction on vertices j and k for H, The other two sufficient conditions in this lemma can be obtained. □

Here we show an example of the use of Lemma 5.1.4. Let $H = (6,3,3,3,2,2,2)$ and $\{i,j,k\} = \{2,3,5\}$. One realization G of H includes edges $\{1,2,3\}, \{1,2,4\}, \{1,3,4\}$, $\{1,4,6\}, \{1,5,7\}, \{1,6,7\}$ and $\{2,3,5\}$. Although $\{2,3,5\}$ is included in G, $(2k-i-3)i/2 = 7 > h_j$, which indicates that there should exist another realization excluding edge $\{2,3,5\}$. Now we check the reduction on vertex j for the realization G. It should be $E = (2,2,0,1,1,0,0)$ and $D = (4,1,0,2,1,2,2)$. We can perform an edge exchange $(1,2)$ and get the E-standard, irreducible decomposition in which $E = (3,1,0,1,1,0,0)$ and $D = (3,2,0,2,1,2,2)$. To implement this switching in D we can drop edge $\{1,6,7\}$ and add $\{2,6,7\}$. In E we replace the edges $\{2,5\}, \{1,4\}$ and $\{1,2\}$ by a realization of $(3,1,0,1,1,0,0)$, which includes edges $\{1,2\}, \{1,4\}$ and $\{1,5\}$. Therefore $\{2,3,5\}$ is not included in the new realization.

Lemma 5.1.5. *In a 3-hypergraphic sequence $H = (h_1, h_2, ..., h_n)$, there exists a realization G including edge $\{i,j,k\}$ where $i < j < k$, if*

$$h_i > \binom{n-1}{2} - (n-2j+k)(n+1-k)/2 \quad \text{or} \tag{5.6}$$

$$h_j > \binom{n-1}{2} - (n-2i+k-2)(n+1-k)/2 \quad \text{or} \tag{5.7}$$

$$h_k > \binom{n-1}{2} - (n-2i+j-1)(n-j)/2 \tag{5.8}$$

Proof. Consider the complementary sequence of H, $\overline{H} = (d'_1, d'_2, ..., d'_n) = (\binom{n-1}{2} - d_n, \binom{n-1}{2} - d_{n-1}, ..., \binom{n-1}{2} - d_1)$, a realization G of H including edge $\{i,j,k\}$ exists if and only if $\overline{G}$ with the sequence $\overline{D}$ excluding this edge exists. Using the equations in Lemma 5.1.4 for the edge $\{n+1-k, n+1-j, n+1-i\}$ in $\overline{G}$, we have the sufficient conditions in this lemma. □

Unfortunately, we cannot make a conclusion like Corollary 5.1.3 due to the gap between the bounds of Lemmas 5.1.4 and 5.1.5.

5.2 Sequence Approach

From the discussion above, given a 3-hypergraphic sequence, in some situations we can find three vertices such that when their degrees increase by one respectively the sequence is still realizable . For example Corollary 5.1.1 shows that the augmentation can be done as long as any vertex is not adjacent to all vertex pairs. Each time this augmentation is made, the sequence will approach the complete 3-hypergraph sequence further. Can the augmentation be utilized to approach an arbitrary one? Assume we know a 3-hypergraphic sequence $H^* = (h_1^*, h_2^*, ..., h_n^*)$, and another 3-hypergraphic sequence $H = (h_1, h_2, ..., h_n)$ such that $h_i \leq h_i^*$ for any i and $H \neq H^*$ (we indicate these two conditions by $H < H^*$ for short), does there exist a 3-hypergraphic degree sequence $H' = (h_1', h_2', ..., h_n')$ such that

$$H < H' \leq H^* \quad and \tag{5.9}$$

$$\sum_{i=1}^{n} h_i + 3 = \sum_{i=1}^{n} h_i' \tag{5.10}$$

(5.10 means H's realization has one more edge than H)? If H' always exists and we find how to construct its realization from H's, the 3-hypergraphic problem can be solved from a zero sequence whose realization is obvious: if finally we make $H' = H^*$

and get its realization, then H^* is 3-hypergraphic; Otherwise it fails in some step and then we can see H^* is not realizable.

One problem we should note is that the H' we find may not be in non-increasing order. Assume that in H, $d_i = d_{i-1}$. If $d_i < d_i^*$, then $d_{i-1} = d_i < d_i^* \leq d_{i-1}^*$. Therefore if a realization of G does not include edge $\{i, j, k\}$, we can find an isomorphism of G which does not include edge $\{i', j, k\}$ to construct H'. By repeating the process, we can transform H' to non-increasing order.

We can only augment the degree of vertex i such that $h_i < h_i^*$. However there are some difficulties we may face. For example, let $H^* = (6, 6, 6, 6, 6, 6, 6)$ and $H = (6, 6, 6, 6, 6, 6, 3)$ in which only two such vertices i exist. We may need to add an edge involving one or two vertices whose degrees are equal in H and H^*, and then make some edge switches. Here we construct a sequence $I(H, H^*)$, where $H \leq H^*$, by putting vertex i into the sequence $|h_i^* - h_i|$ times and keeping it in non-decreasing order. For example, $I(H, H^*) = (1, 1, 1, 2, 3, 6)$ for $H = (3, 3, 3, 3, 2, 1)$ and $H^* = (6, 4, 4, 3, 2, 2)$. Vertex 1 is triple in $I(H, H^*)$ as $h_1^* - h_1 = 3$, and vertices 4 and 5 are not put into $I(H, H^*)$ as $h_4^* = h_4$ and $h_5^* = h_5$. Obviously when we choose the last three vertices in I for augmentation, then this constructed H' is likely to be 3-hypergraphic. For the example shown in this paragraph, suppose G is one realization of H. We can add any edge $\{i, j, k\}$ excluded in G and then perform the edge switching $\{i, n\}$,$\{j, n\}$, and $\{k, n\}$.

Definition 5.2.1. In a k-hypergraph with vertex set V, let $V' \subseteq V$. If H contains every edge containing any three vertices of V', then V' is said to be a *clique* in this hypergraph.

Lemma 5.2.1. *Let $H < H^*$ and u be the third last element in $I(H, H^*)$. If some realization of H does not include the clique $\{1, 2, ..., u\}$, then a sequence H' meeting* (5.9) *and* (5.10) *exists.*

Proof. Let the last three vertices in the sequence i be v_1, v_2, v_3. Assume a realization G of H does not include edge $\{i, j, k\}$ where $i < j < k \leq u$. Add this edge to G and do edge switches (i, v_1), (i, v_2) and (i, v_3). Then $H < H' \leq H^*$ is guaranteed. □

Corollary 5.2.1. *Let $H < H^*$ and let u, v and r be the last three elements in $I(H, H^*)$. If $u \geq v - 1$, $v \geq r - 1$ and a realization G of H includes the clique $\{1, 2, ..., r\}$, there exists an edge e such that $|e \cap \{r + 1, r + 2, ...n\}| \geq 2$.*

Lemma 5.2.2. *Let $H < H^*$ and r be the last element in $I(H, H^*)$. If a realization G of H includes the clique $\{1, 2, ..., r\}$, there exists an edge e such that $|e \cap \{r + 1, r + 2, ...n\}| \geq 2$.*

Proof. Let $T_r = \{r + 1, r + 2, ...n\}$. For any $i \in T_r$, $h_i = h_i^*$. Assume any edge e in G satisfies $|e \cap T_r| \in \{0, 1\}$. Then in G, $|A_r| = \binom{r}{3}$, $|C_r| = |D_r| = 0$, and $|B_r| = \sum_{i=r+1}^{n} h_i$ where the notations refer to (2.5). For any realization $G^*$of H^*, $|A_r^*|+|B_r^*|+|C_r^*|+|D_r^*| > |A_r|+|B_r|+|C_r|+|D_r| = \binom{r}{3}+\sum_{i=r+1}^{n} h_i = \binom{r}{3}+\sum_{i=r+1}^{n} h_i^*$ and $|B_r^*|+|C_r^*|+|D_r^*| \leq |B_r^*|+2|C_r^*|+3|D_r^*| = \sum_{i=r+1}^{n} h_i^*$. Therefore $|A_r^*|+\sum_{i=r+1}^{n} h_i^* > \binom{r}{3} + \sum_{i=r+1}^{n} h_i^*$, but this is contradicted by the fact that $|A_r^*| \leq \binom{r}{3}$. □

We can make edge switches in the realization of H to connect these conclusions but how to utilize the condition in Lemma 5.2.2 is unclear yet. One possible solution is to find a construction for H that avoids this condition and we can construct H'

further. Another is to find a contradiction to force any realization of H to follow the condition in Lemma 5.2.1.

Another problem is that we may need a realization of H^* to guide the construction of H' from H. However in the 3-hypergraphic problem, H^* itself is unknown to be 3-hypergraphic.

Chapter 6

Intersection Preserving Mappings

Let U be a set with n elements. A subset of U with size k is called a k-set of U. Let $\binom{U}{k}$ denote the set of all k-sets of U. A k-hypergraph is a subset of $\binom{U}{k}$. The investigation into the k-hypergraphic problem leads to an interesting problem involving the mappings of $\binom{U}{k}$, which was introduced by Czimmermann [2015]. It is not related to the degree sequence, but is an interesting problem for k-hypergraphs.

6.1 Intersection Preserving Property

Definition 6.1.1. Let U and V be two sets each with n elements and f be a bijection from $\binom{U}{k}$ to $\binom{V}{k}$. If $|X \cap Y| = |f(X) \cap f(Y)|$ for all $X,\ Y \in \binom{U}{k}$, then f is an *intersection preserving* mapping.

It should be noted that V can be an arbitrary set with fixed size and in this problem only the size of the intersection between k-sets of V is considered, so V can be mapped to any set of the same size. For convenience, here we set $V = U$. For example,

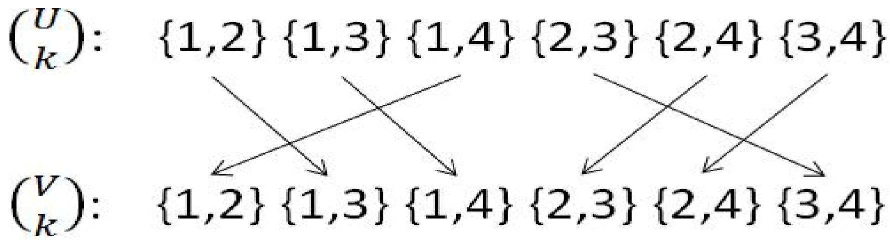

Figure 6.1: An example of intersection preserving mapping

$\binom{U}{k} = \binom{V}{k} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$, when $U = V = \{1,2,3,4\}$ and $k = 2$. Then Figure 6.1 shows a bijection from $\binom{U}{k}$ to $\binom{V}{k}$ and it can be verified that this bijection is intersection preserving.

We can loosen the condition in Definition 6.1.1 with the following lemma:

Lemma 6.1.1. *Let U and V be two sets each with n elements and f be a bijection from $\binom{U}{k}$ to $\binom{V}{k}$. If $|X \cap Y| = |f(X) \cap f(Y)|$ for all $X,\ Y \in \binom{U}{k}$ such that $|X \cap Y| = k-1$, then f is an intersection preserving mapping.*

Proof. We will prove that when $|X \cap Y| < k-1$, $|X \cap Y| = |f(X) \cap f(Y)|$ by mathematical induction. Assume it holds for $|X \cap Y| > r$ where $r < k-1$. Now let's consider $|X \cap Y| = r$ and assume it does not hold.

Without loss of generality, let $X = \{x_1, x_2, ..., x_k\}$ and $Y = \{y_1, y_2, ..., y_k\}$ in which for any $i \leq r$, $x_i = y_i$ and for i and j where $r < i, j \leq k$, $x_i \neq y_j$. Assume $|f(X) \cap f(Y)| = r'$ where $r \neq r'$. We have two conditions to consider:

1. $r < r'$, let $P = \left\{p \in \binom{U}{k} : |p \cap X| = r'\right\}$ and $Q = \left\{q \in \binom{U}{k} : |q \cap Y| = r'\right\}$. We have $|P| = |Q|$. Note that $r' > r$, according to our assumption, every $p \in P$ should

be mapped to some $q \in Q$ by f. So $f(Y) \in Q$ and $Y \in P$, this is contradicted by the assumption that $X \cap Y = r \neq r'$ indicating $Y \notin P$.

2. $r > r'$, let $Z = \{x_1, x_2, ..., x_{k-1}, y_k\}$. Clearly $|Z \cap X| = k-1$ and $|Z \cap Y| = r+1$. Then based on our assumption $f(Z) = f(X) \cup \{x\} \setminus \{y\}$ where $y \in f(X)$ and $x \notin f(X)$. As $|(f(X) \setminus \{y\}) \cap f(Y)| \leq r' < r$, then $|f(Z) \cap f(Y)| < r+1$, which is contradicted by our assumption that $|f(Z) \cap f(Y)| = |Z \cap Y| = r+1$. □

6.2 Induced Bijections

A permutation of U induces a permutation of $\binom{U}{k}$. Refer to the example of an intersection preserving mapping shown in Figure 6.1 in the last section. This mapping is induced by a bijection g given by Table 6.1. Every bijection induces a mapping of k-sets. There are $n!$ bijections where n is $|U|$ and every bijection induces an intersection preserving f. Another easy example is the case $|U| = 2$ and $k = 1, 2$ and thus from now on we assume that $|U| = n > 2$ by default.

Table 6.1: The bijection g

i	1	2	3	4
g(i)	1	3	4	2

Definition 6.2.1 ([Gross and Yellen, 2005]). A *line graph* of a 2-hypergraph G is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge if and only if the corresponding edges of G have a vertex in common.

Are all intersection preserving mappings f induced by bijections? For $k = 2$,

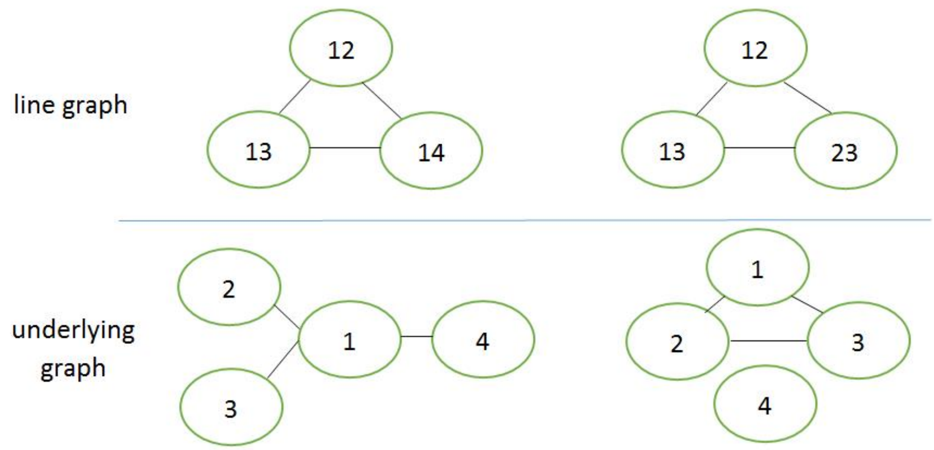

Figure 6.2: The exception in line graph isomorphism theorem

the answer is given by Whitney's isomorphism theorem on line graphs [Whitney, 1932]. His work shows that if two connected line graphs are isomorphic, then so are their underlying graphs, except for the case, K_3 and $K_{1,3}$. The Figure 6.2 shows the exception. Here K_3 is extended by an isolated vertex so that it also has 4 vertices.

Based on $\binom{U}{2}$, we can construct a line graph G_U, in which each 2-set is viewed as a vertex and one vertex is adjacent to another when their intersection is not empty. Similarly we can construct a line graph G_V for $\binom{V}{2}$. If f is intersection preserving, for any 2-set $\{u_1, u_2\} \in \binom{U}{2}$, they are adjacent in G_U, if and only if $f(u_1)$ and $f(u_2)$ are adjacent in G_V. Therefore G_U and G_V are isomorphic and their underlying graphs should be isomorphic, except for the counter-example mentioned before. So there exists a bijection g inducing f.

Therefore for $k = 2$, it is only when $n = 4$, that it is possible to find an intersection preserving mapping not induced by a bijection. Without loss of generality let $U =$

$V = \{1, 2, 3, 4\}$, and let $f(X) = \overline{X}$ for any $X \in \binom{U}{2}$. Clearly this f is intersection preserving. Note that $f(\{1, 2\}) = \{3, 4\}$, $f(\{1, 3\}) = \{2, 4\}$ and $f(\{1, 4\}) = \{2, 3\}$. If f is induced by a bijection g, then $g(1) \in f(\{1, 2\}) \cap f(\{1, 3\}) \cap f(\{1, 4\}) = \emptyset$. Therefore such a g does not exist. We can extend the ideas above to any size of U.

Lemma 6.2.1. *Let f be the bijection of k-sets such that $f(X) = \overline{X}$ for any k-set $X \subset U$, where $|U| \geq 4$ and $k = |U|/2$. Then f is intersection preserving, but not induced by a bijection of U and V.*

Proof. Let X and Y be two k-sets of U, then their mapped sets are $\overline{X}$ and $\overline{Y}$ respectively. As $\overline{X} \cap \overline{Y} = (U \setminus X) \cap (U \setminus Y) = U \setminus (X \cup Y)$ and $|X| = |Y| = |U|/2$, $|\overline{X} \cap \overline{Y}| = |U| - |X| - |Y| + |X \cap Y| = |X \cap Y|$. Therefore f is an intersection preserving mapping.

Let $i \in U$. Let $U_i = \left\{I \in \binom{U}{k} : i \in I\right\}$ (i.e., U_i is the set of all k-sets which contains i). If f is induced by a bijection g, for any $I \in U_i$, $g(i) \in f(I) = \overline{I}$. Let $U'_i = \left\{I' \in \binom{U}{k} : i, g(i) \in I\right\}$ (i.e., U'_i is the set of all k-sets which contain both i and $g(i)$). Clearly for any $I' \in U'_i$, $g(i) \notin \overline{I'} = f(I')$. But this is contradicted to $U'_i \subset U_i$. So f is not induced by any bijection. □

Now try to find another mapping which is not induced by a bijection. Based on Lemma 6.1.1, we only need to consider how to map a subset which has a $(k-1)$ intersection with a mapped set.

Let $U = V = \{1, 2, ..., n\}$ and begin with a k-set $S_1 = \{1, 2, ..., k\}$. Without loss of generality, assume it is mapped to $\{1, 2, ..., k\}$. Consider the k-sets which have a $(k-1)$ intersection with S_1 and classify them according to their intersections.

For example, We try to map all k-sets S such that $S \cap S_1 = \{1, 2, ..., k-1\}$, i.e., $\{1, 2, ..., k-1, k+1\}$, $\{1, 2, ..., k-1, k+2\}$,...,$\{1, 2, ..., k-1, n\}$.

There are two possible ways of mapping this group of subsets. The first way is to make mapped sets share the same intersection.

Without loss of generality, let the intersection of their mapped sets be $\{1, 2, ..., k-1\}$ and $f(\{1, 2, ..., k-1, i\}) = \{1, 2, ..., k-1, i\}$ for $i = k+1, k+2, ..., n$. Now let's consider another group of S such that $S \cap S_1 = \{2, 3, ..., k\}$. To map $\{2, 3, ..., k, i\}$, as $|S_1 \cap \{2, 3, ..., k, i\}| = k-1$, $f(\{2, 3, ..., k, i\}) = \{1, 2, ..., k\} \cup \{x\} \setminus \{y\}$, where $y \in \{1, 2, ..., k\}$ but x does not. $y \neq k$ because any set containing all of $1, 2, ..., k-1$ has been used to map the subsets in the first group. As $f(\{1, 2, ..., k-1, i\}) = \{1, 2, ..., k-1, i\}$, with intersection preserving property $x = i$ and $y \in \{1, 2, 3, ..., k-1\}$. Without loss of generality, we set $y = 1$.

The discussion above for other groups of subsets S such that $S \cap S_1 = \{1, 2, 3, ..., k\} \setminus \{z\}$ will be applied. Note that as f is an one-to-one mapping we cannot choose the same y for their mapped sets. So here without loss of generality we choose $y = z$ and so far the mapping will be induced by a bijection $g : i \to i$ for $i = 1, 2, ..., n$ (if we choose $y = z' \neq z$, then by the automorphism $z' \to z$, we can still get this bijection).

Now we have shown that following the first way of mapping the initial group of subsets, $f(S)$ follows a bijection for any S such that $|S \cap S_1| \geq k-1$. We will show that f will follow this bijection for any other k-set as well.

Lemma 6.2.2. *If f is intersection preserving and $f(X)$ follows a bijection for any X such that $|X \cap S_1| \geq k-1$. Then f is induced by this bijection.*

Proof. Let the bijection be $g : i \to i$ for $i = 1, 2, ..., n$. Assume $f(X)$ follows by this bijection when $|X \cap S_1| > r$. Now consider the condition that $|X \cap S_1| = r$.

For any such X, we can write $X = \{1, 2, 3, ..., k\} \setminus \{x_1, x_2, ...x_{k-r}\} \cup \{y_1, y_2, ...y_{k-r}\}$ where $x_1, x_2, ...x_{k-r} \in S_1$ and $y_1, y_2, ...y_{k-r} \notin S_1$. Consider $X_{ij} = X \cup \{x_i\} \setminus \{y_j\}$. Clearly $|X_{ij} \cap S_1| = r + 1$ and $|X_{ij} \cap X| = k - 1$. According to our assumption $f(X_{ij}) = X_{ij}$. Therefore $f(X) = X_{ij} \cup \{u\} \setminus \{v\} = X \cup \{x_i\} \setminus \{y_j\} \cup \{u\} \setminus \{v\}$ for any i and j, where $u \in f(X_{ij})$ and $v \notin f(X_{ij})$. As $f(X)$ should be unique and independent of i and j, then $u = y_j$, $v = x_i$ and thus we have $f(X) = X$. Therefore $f(X)$ follows the bijection g when $|X \cap S_1| = r$.

By mathematical induction, $f(X)$ follows the bijection g for any size of $X \cap S_1$. So f is induced by this bijection. □

So when we map the group of k-sets, in which each two share the same intersection $\{1, 2, ..., k-1\}$, their mapped sets should not share the same $(k-1)$ intersection. Without loss of generality, let $f(S_1) = S_1$ and $f(\{1, 2, ..., k-1, k+1\}) = \{1, 2, ..., k-1, k+1\}$. As $|\{1, 2, ..., k-1, k+2\} \cap S_1| = k-1$, $f(\{1, 2, ..., k-1, k+2\}) = \{1, 2, ..., k\} \cup \{x\} \setminus \{y\}$ where $y \in \{1, 2, 3, ..., k\}$ while $x \notin \{1, 2, 3, ..., k\}$. If $x \notin \{1, 2, ..., k+1\}$ (i.e., $x = k+1$), as $\{1, 2, ..., k\} \setminus \{y\}$ will share a $(k-2)$ intersection with one of previous two mapped sets. Therefore we cannot introduce any new element for further mapping. In this mapping method, we can choose k elements from $\{1, 2, ..., k+1\}$ to make a mapped set and thus each pair of such mapped sets share a $(k-1)$ intersection.

Theorem 6.2.3. *There is an intersection preserving mapping from* $\binom{U}{k}$ *to* $\binom{V}{k}$ *not induced by a bijection, where* $|U| = |V| = n$*, if and only if* $k = n/2$.

Proof. The number of k-sets S such that $S \cap S_1 = \{1, 2, ..., k-1\}$ is $n - k + 1$. Based on the discussion above, to map these k-sets, we must introduce exactly $k+1$ elements to construct k-sets. Thus we have at most $k+1$ sets to use for mapping all of S. Therefore the number of sets S should not be larger than $k+1$ and thus $k \geq n/2$. Assume $k+1 > n-k+1$, there exists a k-set R among those $k+1$ sets, which is not used to map any one of S. Let S' be the k-set such that $f(S') = R$. Then S' should share a $(k-1)$ intersection with any S. Recall that all S share the same intersection $\{1, 2, ..., k-1\}$, due to $|S'| = k$ and $\{1, 2, ..., k-1\} \subset S'$. Thus S' should be one of S, whole this is contradicted, as R is not used to map any one of S. Therefore $n-k+1 \geq k+1$ and $k = n/2$.

When $k = n/2$, we have shown that the complementary mapping is a feasible solution. Therefore this theorem holds. □

The remaining question is whether there exists another intersection preserving mapping which is not induced by a bijection but not the same as the complementary mapping.

Now let's map all sets S such that $\{1, 2, ..., k-1\} \subset S$. Based on the discussion above, for each set we need to choose k elements from a $(k+1)$-subset for mapping. Without loss of generality, let the $k+1$-subset be $\{k, k+1, ..., n\}$ (note that $n = 2k$). Therefore $f(\{1, 2, 3, ..., k-1, i\}) = \{k, k+1, ..., n\} \backslash \{j\}$, where $i, j \in \{k, k+1, ..., n\}$. Without loss of generality let $i = j$.

Then we map S such that $\{2, 3, ..., k\} \subset S$. Note that $\{2, 3, ..., k, i\}$, where $i \in \{k, k+1, ..., n\}$, shares a (k-1) intersection with both $\{1, 2, ..., k\}$ and $\{1, 2, ..., k-1, i\}$. As $|f(S) \cap f(S_1)| = k-1$, $f(S) = \{k+1, k+2, ..., n\} \cup \{x\} \setminus \{y\}$ where $y \in \{k+1, k+2, ..., n\}$ but x does not. Remember all $\{k, k+1, ..., n\} \backslash \{j\}$ have been used for mapping before, therefore $x \neq k$. As $f(\{1, 2, ..., k-1, i\}) = \{k, k+1, ..., n\} \setminus \{i\}$, to keep intersection preserving property, we must have $y = i$. As x can be any element in $\{1, 2, ..., k-1\}$, without loss of generality we set $x = 1$.

We repeat the discussion above for other sets S such that $S \cap S_1 = \{1, 2, ..., k\} \setminus \{z\}$. $f(S) = \{k+1, k+2, ..., n\} \cup \{x\} \setminus \{y\}$ where $y \in \{k+1, k+2, ..., n\}$ but $x \notin \{k+1, k+2, ..., n\}$. As f is a one-to one mapping, x must be different for each group and without loss of generality let $x = z$ for mapping every such group of sets. So far the f can be induced by complementary mapping (after permutation if necessary).

Lemma 6.2.4. *Let f be an intersection preserving and let $f(X)$ be induced by a complementary mapping for any X such that $|X \cap S_1| \geq k-1$, where S_1 is any k-set of U. Then f will be induced by the complementary mapping for all subsets.*

Proof. We set $S_1 = \{1, 2, ...k\}$. Assume $f(X)$ can be induced by the complementary mapping when $|X \cap S_1| > r$. Now consider the condition when $|X \cap S_1| = r$.

For any such X, we can write $X = \{1, 2, ..., k\} \setminus \{x_1, x_2, ..., x_{k-r}\} \cup \{y_1, y_2, ..., y_{k-r}\}$ where $x_1, x_2, ..., x_{k-r} \in S_1$ and $y_1, y_2, ..., y_{k-r} \notin S_1$. Consider $X_{ij} = X \cup \{x_i\} \setminus \{y_j\}$. Clearly $|X_{ij} \cap S_1| = r+1$ and $|X_{ij} \cap X| = k-1$. According to our assumption $f(X_{ij}) = \overline{X_{ij}}$. Therefore $f(X) = \overline{X_{ij}} \cup \{u\} \setminus \{v\} = \overline{X} \cup \{y_j\} \setminus \{x_i\} \cup \{u\} \setminus \{v\}$ for any i and j, where $v \in \overline{X_{ij}}$ and $u \notin \overline{X_{ij}}$. As $f(X)$ should be unique and independent of i and j, then $v = y_j$, $u = x_i$ and thus we have $f(X) = \overline{X}$.

By mathematical induction, f is induced by the complementary mapping for all k-sets. □

In conclusion, the complementary mapping is the unique intersection preserving mapping which is not induced by any bijection, and it is feasible only when $k = n/2$.

Chapter 7

Conclusion and Future Work

In this thesis we have discussed how to implement Dewdney's reduction with edge exchanges for the 3-hypergraphic problem. We would like to have a theorem of the form "Let H be a non-increasing degree sequence, then H is 3-hypergraphic if and only if H' is 3-hypergraphic", where H' is a reduced sequence that can be readily calculated. This is the purpose of the irreducible decompositions. Although the irreducible decompositions may not be unique, we have eliminated many reducible decompositions in which the residual sequence needs to be considered in the next reduction. Moreover, with the property of irreducible decompositions, we find some helpful conditions to construct a new 3-hypergraphic sequence, starting from a known one. We also point out a potential constructive method to determine whether a sequence is 3-hypergraphic.

We have to admit that the difficulty increases considerably, when a vertex is adjacent to a pair of vertices, as in a 3-hypergraph, instead of one as in a 2-hypergraph. This is why many methods and theory cannot be applied to 3-hypergraphs although they work for 2-hypergraphs. In this thesis, we don't provide a complete solution for the 3-hypergraphic problem. To research the 3-hypergraphic problem further, we can start from three points.

One point is to find another criterion to judge a "good" decomposition. The criterion in this thesis is induced from the reduction of 2-hypergraphs. If a better one exists, it may be found from some properties of 2-hypergraphs too. Therefore the key is how to choose the correct decomposition in the 2-hypergraphic problem.

The second point is to find a better edge exchange method. Currently what is relied on is based on edge switching, which is sufficiently strong for 2-hypergraphs. For the 3-hypergraphic reduction, we may need something stronger. For example, can we find a transformation which does two edge exchanges simultaneously?

The last point is to extend the work in Chapter 5. Can we finish the sequence approach theory in the arbitrary case? For 2-hypergraphs, one solution is balanced flow networks of Kocay and Stone [1995, 1993]. For k-hypergraphs, Berge [1973] built the flow network while the augmentation algorithm has not be found.

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