Equivariant Projection Morphisms of Specht Modules

by

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ABSTRACT

This thesis is devoted to a problem in the representation theory of the symmetric group over $\mathbb{C}$ (the field of complex numbers). Let d be a positive integer, and let $\mathfrak{S}_d$ denote the symmetric group on d letters. Given a partition λ of d, the Specht module V_λ is a finite dimensional vector space over $\mathbb{C}$ which admits a natural basis indexed by all standard tableaux of shape λ with entries in $\{1, 2, \ldots, d\}$. It affords an irreducible representation of the symmetric group $\mathfrak{S}_d$, and conversely every irreducible representation of $\mathfrak{S}_d$ is isomorphic to V_λ for some partition λ. Given two Specht modules V_λ, V_μ, their tensor product representation $V_\lambda \otimes V_\mu$ is in general reducible, and hence it splits into a direct sum $\bigoplus\limits_{\nu} V_\nu^{m_\nu}$ of irreducibles. This raises the problem of describing the $\mathfrak{S}_d$-equivariant projection morphisms (alternately called $\mathfrak{S}_d$-homomorphisms) of the form $V_\lambda \otimes V_\mu \longrightarrow V_\nu$ in terms of the standard tableaux bases. In this work we give explicit formulae describing this morphism in the following cases:

- $V_{(d-1,1)} \otimes V_{(d-1,1)} \longrightarrow V_{(d-1,1)}$,
- $V_{(1^d)} \otimes V_\lambda \longrightarrow V_{\tilde{\lambda}}$, where $\lambda = (d-1, 1)$ or $(2, 1^{d-2})$ and $\tilde{\lambda}$ is its conjugate.

The isomorphism $V_\lambda \simeq V_\lambda^*$ induces an equivariant projection morphism (which we call a q-morphism)

$$V_\lambda \otimes V_\lambda \longrightarrow V_{(d)} \cong \mathbb{C}.$$

We have found explicit formulae for this morphism in the following cases:

$$\lambda = (d-1, 1),\ (d-2, 1, 1),\ (2, 1, \ldots, 1).$$

Finally, we present a conjectural formula for the q-morphism in the case

$$\lambda = (d-r, 1, \ldots, 1).$$

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Contents

Chapter 1

Introduction

From the general theory of representations and characters of finite groups, we focus on the representation theory of the symmetric group $\mathfrak{S}_d$ over $\mathbb{C}$. We use the Specht modules associated to the various partitions of d as the realizations of the irreducible representations of $\mathfrak{S}_d$. In this thesis we exhibit explicit formulae for certain equivariant projection morphisms of Specht modules.

If G is a finite group, then (for us) a representation of G will be a finite dimensional $\mathbb{C}$-vector space V together with a group homomorphism $\rho : G \longrightarrow GL(V)$. A subrepresentation W of V is a subspace such that $\rho(g)$ stabilizes W for each $g \in G$. The representation V is said to be irreducible if it has no subrepresentations except $W = \{\mathbf{0}\}, V$. It is a fundamental result due to Maschke that each representation is a direct sum of irreducible representations. Moreover, up to isomorphism G has only finitely many irreducible representations, and their number equals the number of conjugacy classes in G (see [21]). A summary of the general theory of representations of finite groups is given in Chapter 2.

In general, there is no explicitly known bijection between conjugacy classes and irreducible representations for an arbitrary G, but it is known when $G = \mathfrak{S}_d$. We briefly describe this below. There is a bijection between the conjugacy classes in $\mathfrak{S}_d$ and the partitions of d (see [9, §4]). A **partition** λ of d is a non-increasing sequence of non-negative integers, whose sum equals d. A partition is associated to a Young diagram, e.g., the partition $(3, 2)$ of $d = 5$ has Young diagram

$$\ydiagram{3,2}\,.$$

A standard λ-tableau is a Young diagram of shape λ where the boxes contain entries from the set $\{1, 2, \ldots, d\}$, such that the entries are strictly increasing along the rows and columns. Let $\mathcal{B}_\lambda$ denote the set of all standard λ-tableaux. The **Specht module** V_λ (see §3.2 below) is a $\mathbb{C}$-vector space with a basis naturally indexed by the elements in $\mathcal{B}_\lambda$. (For simplicity, henceforth we will identify this basis with $\mathcal{B}_\lambda$.)

For notational convenience, we will write a tableau as an array. Then, for instance, a basis of $V_{(3,2)}$ is given by:

$$\mathcal{B}_{(3,2)} = \{\begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & \end{bmatrix}, \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & \end{bmatrix}, \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & \end{bmatrix}, \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & \end{bmatrix}\}.$$

In general, the dimension of V_λ can be found by using a combinatorial formula called the hook length formula. Each V_λ carries the structure of an irreducible representation of $\mathfrak{S}_d$, and conversely each irreducible representation of $\mathfrak{S}_d$ is isomorphic to some V_λ. We describe the construction of Specht modules in Chapter 3.

If λ, μ are two partitions of d, then in general the tensor product representation $V_\lambda \otimes V_\mu$ is not irreducible, and hence decomposes as a direct sum of irreducibles. For example, $V_{(3,2)} \otimes V_{(2,2,1)}$ decomposes as

$$V_{(4,1)} \oplus V_{(3,2)} \oplus V_{(3,1,1)} \oplus V_{(2,2,1)} \oplus V_{(2,1,1,1)} \oplus V_{(1,1,1,1,1)}.$$

(However, no such general formula is known for arbitrary λ and μ.) If V_ν appears as a summand in $V_\lambda \otimes V_\mu$ with multiplicity one, then this raises the problem of describing the projection morphism

$$\pi : V_\lambda \otimes V_\mu \longrightarrow V_\nu$$

in terms of the bases $\mathcal{B}_\lambda, \mathcal{B}_\mu, \mathcal{B}_\nu$. Of course, π is assumed to be $\mathfrak{S}_d$-equivariant, i.e., it should be compatible with the action of $\mathfrak{S}_d$ in the natural sense. In this thesis, we give a solution to this problem in a few special cases.

For $d \geq 4$, we have a general formula

$$V_{(d-1,1)} \otimes V_{(d-1,1)} \cong V_{(d)} \oplus V_{(d-1,1)} \oplus V_{(d-2,2)} \oplus V_{(d-2,1,1)}.$$

In Chapter 4, we give an explicit description of the projection morphism

$$V_{(d-1,1)} \otimes V_{(d-1,1)} \longrightarrow V_{(d-1,1)}$$

in terms of the basis $\mathcal{B}_{(d-1,1)}$ (see Proposition 4.2.3).

For each partition λ, there is a conjugate partition, denoted by $\widetilde{\lambda}$, whose Young diagram is the transpose of the Young diagram of λ. If $T \in \mathcal{B}_\lambda$, then its transpose $T^t \in \mathcal{B}_{\widetilde{\lambda}}$.

In general, there is an isomorphism $V_{(1^d)} \otimes V_\lambda \cong V_{\widetilde{\lambda}}$ for any λ (see [8, chapter 7]). In Propositions 4.2.8 and 4.2.10 respectively, we give explicit formulae for this isomorphism in the following two cases:

$$\lambda = (d-1, 1),\ (2, 1, \ldots, 1).$$

For any λ, the one-dimensional trivial representation $V_{(d)}$ always appears with multiplicity one in the tensor product $V_\lambda \otimes V_\lambda$. By identifying $V_{(d)}$ with $\mathbb{C}$, this defines a $\mathfrak{S}_d$-equivariant projection morphism (which we call a q-morphism)

$$\theta_\lambda : V_\lambda \otimes V_\lambda \longrightarrow \mathbb{C}.$$

In general, it appears difficult to describe θ_λ in terms of $\mathcal{B}_\lambda$. In Chapter 5, we give such a description in the cases

$$\lambda = (d-1, 1),\ (2, 1, \ldots, 1),\ (d-2, 1, 1).$$

The results appear in Propositions 5.2.1, 5.2.2 and Theorem 5.2.5 respectively. Finally, we present a general conjectural formula for θ_λ in the case

$$\lambda = (d-1, \underbrace{1, \ldots, 1}_{r \text{ times}}).$$

All the results above were obtained as follows. We wrote a set of MAPLE routines to calculate all equivariant projection morphisms of the form

$$V_\lambda \otimes V_\mu \longrightarrow V_\nu$$

for arbitrary partitions λ, μ, ν of d, in terms of the tableaux bases. Then the formulae were conjectured based upon several computational examples, and finally proved using the straightening rules for tableaux.

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Chapter 2

Representations and characters of finite groups

2.1 History

Since the discovery of group characters by Frobenius at the end of the 19th century, the development of group representation theory has been spectacular, and the theory has shown powerful connections to other branches of mathematics. It makes an appearance in areas of pure mathematics such as invariant theory and the theory of symmetric functions (see [17]), and in areas of applied mathematics such as quantum theory and nuclear physics (see [18]). The fundamental theory of complex (or ordinary) representations of finite groups was almost completed by Frobenius and Burnside. Schur later simplified the rather complicated theory of Frobenius to a considerable extent by using a lemma now called Schur's lemma.

2.2 Notations and Definitions

The main resources for this chapter are [2, Chapter 2], [9, Chapter 1], [10, §41], [14, Chapter 3], [15, Chapter VII], [20, Chapter 1], [21, Chapter 1].

Throughout this chapter, G denotes a finite group. Throughout this work, all vector spaces are over $\mathbb{C}$ (the field of complex numbers). The **general linear group**, $GL(V)$, is the set of all invertible linear transformations from V to itself.

Definition 2.2.1. *A* **representation of** G **in** V *is a group homomorphism*

$$\rho : G \to GL(V).$$

If ρ is understood from the context, then we say that V is a representation of G or that V is a G-representation. If dim $(V) = n$, then V is said to be a representation of degree n. We will frequently use the notation $g\,v$ instead of $\rho\,(g)\,(v)$. For $w, v \in V$, $g, h \in G$ and $c, d \in \mathbb{C}$ the definition above implies that

1. $g\,(c\,v + d\,w) \;=\; c\,(g\,v) + d\,(g\,w)$,
2. $(g\,h)\,v \;=\; g\,(h\,v)$,
3. $e\,v = v$, where e is the identity of the group G.

Since $\rho\,(g)$ is a linear transformation in $GL(V)$, the corresponding matrix of this transformation, denoted by $A_g^V \;=\; (a_{ij})$, is simply the action of $\rho(g)$ on an ordered basis $\{v_1, \ldots, v_n\}$ of V. According to our conventions,

$$g\,v_i = \sum_{j=1}^{n} a_{ij}\,v_j,$$

i.e., the image of v_i via g is given by the i-th row of A_g^V. This leads us to an equivalent definition for a linear representation of G.

Let GL_n stand for the group of invertible matrices of size $n \times n$ with entries from $\mathbb{C}$.

Definition 2.2.2. *A* **matrix representation** *of G is a group homomorphism*

$$\mu : G \to GL_n.$$

This means that, to each $g \in G$ is assigned a matrix $\mu(g) = M_g \in GL_n$ such that:

1. *$M_e = I$,*
2. *$M_{gh} = M_g\,M_h$ for any $g, h \in G$.*

In the notation above, $A_g^V\,A_h^V = A_{hg}^V$, hence

$$G \longrightarrow GL_n, \quad g \longrightarrow A_{g^{-1}}^V$$

is a matrix representation. Conversely, given a matrix representation $g \longrightarrow M_g$, one can define a linear representation $G \longrightarrow GL(\mathbb{C}^n)$ by letting $A_g^{\mathbb{C}^n} = M_g^{-1}$ with respect to the standard basis of $\mathbb{C}^n$. Hence the two concepts are equivalent.

Example 2.2.3. *All groups have the* **trivial representation***, which sends every $g \in G$ to the 1×1 identity matrix.*

Example 2.2.4. Let G be the dihedral group D_8, which is a finite group generated by two elements a and b where $a^4 = b^2 = e$, and $b^{-1}\,ab = a^{-1}$. Define the matrices X and Y in GL_2 by

$$X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Notice that $X^4 = Y^2 = I, Y^{-1}\,XY = X^{-1}$. Hence we have a representation

$$\mu : D_8 \longrightarrow GL_2$$

defined by

$$\mu\,(a^i b^j) = X^i\,Y^j \quad \text{where,} \;\; 1 \le i \le 4 \text{ and } 1 \le j \le 2.$$

If $1 \le r, s \le 4,$ and $1 \le u, v \le 2,$ then

$$\mu(a^r b^u\, a^s b^v) = \mu(a^i b^j) = X^i\,Y^j = X^r\,Y^u\,X^s\,Y^v = \mu(a^r b^u)\,\mu(a^s b^v)$$

for some $1 \le i \le 4$, and $1 \le j \le 2$. The corresponding matrices for each $g \in D_8$ are as follows:

$$\mu\,(e) = \mu\,(a^4 b^2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mu\,(a) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mu\,(a^2) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

and,

$$\mu\,(a^3) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mu\,(b) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mu\,(ab) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

$$\mu\,(a^2 b) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mu\,(a^3 b) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Definition 2.2.5. *Let G be a finite group, and let V denote the free vector space on the set of variables $\{x_g : g \in G\}$. Then dim(V) equals the order of G. Now the* **regular representation** *of G can be defined as follows: for any $h \in G$ and $x_g \in V$,*

$$h\,x_g = x_{hg}.$$

Example 2.2.6. Consider the cyclic group $C_4 = \{e, g, g^2, g^3\}$ where e is the identity element. Let V be the free vector space with the ordered basis

$$\{x_e, x_g, x_{g^2}, x_{g^3}\}.$$

Then the regular representation can be defined via the following action:

$$g\,x_e = x_g,\ g\,x_g = x_{g^2},\ g\,x_{g^2} = x_{g^3},\ g\,x_{g^3} = x_e,$$

$$g^2\,x_e = x_{g^2},\ g^2\,x_g = x_{g^3},\ g^2\,x_{g^2} = x_e,\ g^2\,x_{g^3} = x_g,$$

and similarly for other elements in G. The corresponding matrices are

$$A_g^V = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad A_{g^2}^V = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad A_{g^3}^V = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Definition 2.2.7. *Let V and W be two representations of G. A linear transformation $\theta : V \longrightarrow W$ is called a G-**equivariant morphism** if the diagram*

$$\begin{array}{ccc} V & \xrightarrow{\theta} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{\theta} & W \end{array}$$

commutes for all $g \in G$.

In terms of matrices, suppose that V and W are of dimensions n and m respectively. Let M be the corresponding $n \times m$ matrix of the linear transformation. Also let the matrices A_g^V and A_g^W represent the actions of g with respect to ordered bases of V and W respectively. Then θ is a G**-equivariant morphism** if the corresponding matrices satisfy the equality $A_g^V\,M = M\,A_g^W$ for every $g \in G$.

A G-equivariant morphism is also frequently referred to as a G-homomorphism in the literature.

Definition 2.2.8. *Let V and W denote two representations of G. We say that V and W are* **isomorphic**, *denoted $V \cong W$, if there is an invertible G-equivariant morphism $\theta : V \longrightarrow W$.*

Henceforth we will sometimes drop the explicit reference to G if it is understood from the context.

2.3 Subrepresentations and Reducibility

Definition 2.3.1. *Let V be a representation of G, and W a subspace of V. We say that W is a* **subrepresentation** *of V, if W is invariant (or stable) under the action of G. In other words, for every $w \in W$ and $g \in G$, $g\,w \in W$.*

Example 2.3.2. Each representation V has two trivial subrepresentations, namely $W = V$, and $W = \{\mathbf{0}\}$. Others are called nontrivial subrepresentations.

As we know, if θ is a linear transformation between the vector spaces V and W, then $\theta(V)$ and $\ker\,\theta$ are vector subspaces of W and V respectively.

Lemma 2.3.3. *For any equivariant morphism θ between the representations V and W, the subspaces $\theta(V)$ and $\ker\theta$ are subrepresentations of W and V respectively.*

Proof. Let $w \in \theta(V)$. Then there is a vector $v \in V$ such that $w = \theta(v)$. Let $g \in G$. Then $g\,w = g\,\theta(v) = \theta\,(g\,v)$, hence $g\,w \in \theta(V)$. Also, since $\theta\,(v) = 0$ for any $v \in \ker\,\theta$, then

$$0 = g\,\theta(v) = \theta\,(g\,v), \quad \text{which implies that } g\,v \in \ker\theta.$$

So, $\theta(V)$ and $\ker\theta$ are a subrepresentations of W and V respectively. □

Example 2.3.4. Let $G = \mathfrak{S}_3$ be the symmetric group on the set $\{1, 2, 3\}$,

$$G = \{e,\,(1\,2),\,(1\,3),\,(2\,3),\,(1\,2\,3),\,(1\,3\,2)\}.$$

Also, let V denote the regular representation with basis $\{x_\alpha \,:\, \alpha \in G\}$. Clearly, $\dim(V) = |G| = 6$. Let W be the one-dimensional subspace of V spanned by the element $\sum\limits_{\sigma\in G} x_\sigma$. Then W is a subrepresentation of V, since for any $\alpha \in G$,

$$\alpha \sum_{\sigma\in G} x_\sigma = \sum_{\sigma\in G} x_{\alpha\,\sigma} = \sum_{\sigma\in G} x_\sigma.$$

This example will be used later. Of course, the same construction will work for any finite group G.

Definition 2.3.5. *Let V be a representation of G. If V has a nontrivial subrepresentation, then we say that V is* **reducible**. *Otherwise V is called an* **irreducible** *representation.*

Thus V in Example 2.3.4 is reducible.

Lemma 2.3.6. *(Schur's Lemma) Let V and W denote two irreducible representations of G, and $\theta : V \longrightarrow W$ an equivariant morphism. Then*

1. *either θ is an isomorphism, or $\theta = 0$;*
2. *if $V = W$, then $\theta = c \cdot I$ for some $c \in \mathbb{C}$ (where I is the identity map).*

PROOF. Part (1) is clear from Lemma 2.3.3. As to part (2), let c be any eigenvalue of θ. Then $\theta - c\,I$ is an equivariant morphism from V to V with a nonzero kernel. Hence $\theta - c\,I = 0$. □

Definition 2.3.7. *Let V be a vector space with subspaces U and W. Then V is the* **direct sum** *of U and W, written $V = U \oplus W$, if every $v \in V$ can be written uniquely as a sum*

$$v = u + w, \quad u \in U, w \in W.$$

In this case, U and W are called **complements** *of each other.*

The question now is, if V is a reducible representation and W is a subrepresentation, is there a complement of W in V, which is a subrepresentation of V as well? The next proposition answers this question.

If $W \subseteq V$ is a subspace, then a linear map $\pi : V \longrightarrow W$ is said to be a **projection morphism** if $\pi(w) = w$ for all $w \in W$. Such a morphism always exists. It is sometimes called a 'projection' in the literature.

Proposition 2.3.8. *Let V be a G-representation, and let W be a subrepresentation of V. Then there exists a subrepresentation of V, which is a complement of W.*

PROOF. Let $\pi : V \longrightarrow W$ be any projection morphism. Define an endomorphism $\pi^0 : V \longrightarrow V$ by

$$\pi^0(v) = \frac{1}{|G|} \sum_{g \in G} g\,\pi\,g^{-1}\,v.$$

Claim 1: π^0 is a projection on W. To show this, we have to prove that

1. $\forall v \in V$, then $\pi^0\,(v) \in W$, and
2. $\forall w \in W,\ \pi^0\,(w) = w.$

Let $v \in V$. Then $g^{-1}\,v \in V$ gives $\pi\,(g^{-1}\,v\,) \in W$, and therefore $g\,\pi\,(g^{-1}\,v) \in W$, since W is invariant under the action of G.

Now, for any $w \in W$, $\pi(w) = w$ since π is a projection morphism. So, $\pi(g^{-1} w) = g^{-1} w$, which implies that $g\,\pi(g^{-1} w) = w$. This proves Claim 1.
<u>Claim 2</u>: π^0 is an equivariant morphism, i. e., $h\,\pi^0 = \pi^0\,h$ for each $h \in G$.
Let $h \in G$. Then

$$\begin{aligned} h\,\pi^0 &= \frac{1}{|G|}\sum_{g\in G} h\,g\,\pi\,g^{-1} = \frac{1}{|G|}\sum_{g\in G} h\,g\,\pi\,g^{-1}\,(h^{-1}\,h) \\ &= \frac{1}{|G|}\sum_{g\in G}(hg)\,\pi\,(h\,g)^{-1}\,h \quad \text{(let } hg = b\text{)} \\ &= \frac{1}{|G|}\sum_{b\in G} b\,\pi\,b^{-1}\,h = \pi^0\,h. \end{aligned}$$

Now, by Lemma 2.3.3, $\ker\pi^0$ is a subrepresentation of V. Hence

$$V = \pi^0(W) \oplus \ker\pi^0 = W \oplus \ker\pi^0. \quad \square$$

In the next example we will try to write the given representation as a direct sum of irreducible representations.

Example 2.3.9. In Example 2.3.4, we have a subrepresentation W of V generated by $z = \sum_{\sigma\in G} x_\sigma$. To write V as a decomposition of irreducible representations, let W' be the subspace generated by $z' = \sum_{\sigma\in G}(\text{sign}\,\sigma)\,x_\sigma$. To see that W' is invariant under G, let $\alpha \in G$. Then,

$$\begin{aligned} \alpha\,z' &= \sum_{\sigma\in G}(\text{sign}\,\sigma)\,\alpha\,x_\sigma = \sum_{\sigma\in G}(\text{sign}\,\sigma)\,x_{\alpha\,\sigma} \\ &= (\text{sign}\,\alpha)\sum_{\sigma\in G}(\text{sign}\,\alpha)\,(\text{sign}\,\sigma)\,x_{\alpha\,\sigma} = (\text{sign}\,\alpha)\,z' \in W'. \end{aligned}$$

Now define the projection morphism

$$\Phi : V \longrightarrow W \oplus W', \quad \Phi(x_\sigma) = (z,\ (\text{sign}\,\sigma)\,z').$$

If $\alpha \in G$, then

$$\begin{aligned} \alpha\,\Phi(x_\sigma) &= (\alpha\,z,\ \alpha\,(\text{sign}\,\sigma)\,z') = (z,\ (\text{sign}\,\sigma)(\text{sign}\,\alpha)\,z') \\ &= (z,\ (\text{sign}\,\alpha\,\sigma)\,z') = \Phi(x_{\alpha\,\sigma}) = \Phi(\alpha\,x_\sigma), \end{aligned}$$

hence Φ is an equivariant morphism. Let Y be the kernel of Φ. By Lemma 2.3.3, Y is a subrepresentation of G. It is easy to check that the set

$$\{\underbrace{x_{(1\,2)} - x_{(1\,3)}}_{y_1},\ \underbrace{x_e - x_{(1\,2\,3)}}_{y_2},\ \underbrace{x_{(2\,3)} - x_{(1\,3)}}_{y_3},\ \underbrace{x_{(1\,3\,2)} - x_{(1\,2\,3)}}_{y_4}\}$$

forms a basis for Y. The next table describes the action of G on Y.

$\alpha \in G$	y_1	y_2	y_3	y_4
e	y_1	y_2	y_3	y_4
$(1\,2)$	$y_2 - y_4$	$y_1 - y_3$	$-y_4$	$-y_3$
$(1\,3)$	$-y_2$	$-y_1$	$y_4 - y_2$	$y_3 - y_1$
$(2\,3)$	y_4	y_3	y_2	y_1
$(1\,2\,3)$	$-y_3$	$-y_4$	$y_1 - y_3$	$y_2 - y_4$
$(1\,3\,2)$	$y_3 - y_1$	$y_4 - y_2$	$-y_1$	$-y_2$

Let U be the subspace of Y generated by the set $\{u_1,\, u_2\}$, where

$$u_1 = y_1 + y_2, \quad \text{and} \quad u_2 = y_3 + y_4,$$

and similarly $\widetilde{U}$ is generated by

$$\widetilde{u}_1 = y_1 + y_2 - y_4, \quad \text{and} \quad \widetilde{u}_2 = y_3 + y_4 - y_1.$$

The next two tables describe the action of G on $u_1, u_2, \widetilde{u}_1, \widetilde{u}_2$. It follows that the subspaces $U, \widetilde{U}$ are invariant under this action.

$\alpha \in G$	u_1	u_2
e	u_1	u_2
$(1\,2)$	$u_1 - u_2$	$-u_2$
$(1\,3)$	$-u_1$	$u_2 - u_1$
$(2\,3)$	u_2	u_1
$(1\,2\,3)$	$-u_2$	$u_1 - u_2$
$(1\,3\,2)$	$u_2 - u_1$	$-u_1$

$\alpha \in G$	$\widetilde{u}_1$	$\widetilde{u}_2$
e	$\widetilde{u}_1$	$\widetilde{u}_2$
$(1\,2)$	$\widetilde{u}_1$	$-\widetilde{u}_1 - \widetilde{u}_2$
$(1\,3)$	$-\widetilde{u}_1 - \widetilde{u}_2$	$\widetilde{u}_2$
$(2\,3)$	$\widetilde{u}_2$	$\widetilde{u}_1$
$(1\,2\,3)$	$-\widetilde{u}_1 - \widetilde{u}_2$	$\widetilde{u}_1$
$(1\,3\,2)$	$\widetilde{u}_2$	$-\widetilde{u}_1 - \widetilde{u}_2$

These subspaces have the following properties:

1. $U \cap \widetilde{U} = \{\mathbf{0}\}$,
2. Each $y \in Y$ can be written as a sum of an element in U and an element in $\widetilde{U}$. For example, $y_3 = (u_2 - u_1) + \widetilde{u}_1$.

Hence, $Y = U \oplus \widetilde{U}$. In fact, both U and $\widetilde{U}$ are irreducible G-representations because neither one of them has a subspace invariant under the action of G. To see this, suppose that F is a nontrivial subrepresentation of U. Say F is generated by the element f which is a linear combination of u_1 and u_2. This means that

$$f = p\,u_1 + q\,u_2, \text{ for some } p, q \in \mathbb{C}.$$

Now since F is invariant under the action of G, we have $\alpha f \in F$ for any $\alpha \in \mathfrak{S}_3$. Let $\alpha = (1\,2)$, and assume $(1\,2)\,f = t\,f$ for some $t \in \mathbb{C}$. By using the table of the action of G on U, we have

$$(1\,2)\,f - t\,f = p(u_1 - u_2) + q(-u_2) - tp\,u_1 - tq\,u_2 = 0.$$

Since u_1 and u_2 are linearly independent we have $t = 1$ and $p = -2q$. Doing the same calculation for $\alpha = (1\,3)$, we get $q = -2p$, which implies $p = q = 0$. A similar proof shows that $\widetilde{U}$ is irreducible. So, U and $\widetilde{U}$ are irreducible subrepresentations. Altogether, we have written the regular representation V as a sum of irreducible G-representations, as follows:

$$V = W \oplus W' \oplus U \oplus \widetilde{U}.$$

In the next chapter, we will discuss a more direct and shorter method of finding the irreducible subrepresentations of a symmetric group representation.

Theorem 2.3.10. *(Maschke's Theorem) Let G denote a finite group. Then every finite-dimensional representation V of G is isomorphic to a direct sum of irreducible representations.*

PROOF. (By induction on the dimension of V). Note that a one-dimensional representation must be irreducible. If V is irreducible, then there is nothing to prove. Otherwise, by Proposition 2.3.8, V can be written as

$$V = W \oplus W'.$$

Since $\dim(W), \dim(W') < \dim(V)$, by our induction hypothesis each subrepresentation is isomorphic to a direct sum of irreducible representations, and hence so is V. □

Example 2.3.11. Maschke's theorem may not hold if the characteristic of the ground field is positive. For example, let V denote the regular representation of the cyclic group $C_2 = \{e, g\}$ in characteristic two. It is easy to verify that $W = \langle x_e + x_g \rangle$ is the only one-dimensional subrepresentation of V. In particular V cannot be written as a direct sum of irreducible subrepresentations.

The following section gives a method of constructing new representations from old ones.

2.4 Tensor Product and Kronecker Product

The main resources for the following are [2, Ch. 2], [14, chapter 19], and [16, §3.5].

Let M, N and P be vector spaces (over $\mathbb{C}$). A mapping $f : M \times N \to P$ is called bilinear if for a fixed $m \in M$, the function $n \to f(m, n)$ is linear on N, and similarly for the other argument.

We will construct a vector space T (depending on M, N) such that for all P, the bilinear mappings $f : M \times N \to P$ are in a natural one-to-one correspondence with linear mappings $f' : T \to P$. More precisely there is a bilinear mapping $g : M \times N \to T$ such that the following diagram commutes:

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "g"'] & P \\ T \arrow[ur, "f'"'] & \end{tikzcd}$$

$$f = f' \circ g.$$

Let $\mathcal{F}(M, N)$ denote the free vector space on pairs (x, y) where $x \in M, y \in N$. Let D be the subspace of $\mathcal{F}(M, N)$ spanned by elements of the form

$$(x + x', y) - (x, y) - (x', y), \quad (x, y + y') - (x, y) - (x, y'),$$

and

$$(ax, y) - a\,(x, y), \quad (x, ay) - a\,(x, y), \quad a \in \mathbb{C}.$$

Now, let T denote the quotient space $\mathcal{F}(M, N)/D$. Let $x \otimes y$ denote the image of (x, y) in T. Then T is spanned by the elements of the form $x \otimes y$, and from our definitions,

$$(x + x') \otimes y = x \otimes y + x' \otimes y, \quad x \otimes (y + y') = x \otimes y + x \otimes y',$$

and

$$ax \otimes y = x \otimes ay = a\,(x \otimes y).$$

Equivalently, the mapping

$$g : M \times N \to T \quad \text{defined by } g(x, y) = x \otimes y$$

is bilinear. Given a bilinear map $f : M \times N \to P$, it gives a linear map

$$\tilde{f} : \mathcal{F}(M, N) \longrightarrow P, \quad (x, y) \longrightarrow f(x, y).$$

We have an inclusion $D \subseteq \ker \tilde{f}$ by bilinearity. Hence this induces a map $f' : T \longrightarrow P$.

The vector space T is called the tensor product of M and N, and is denoted by $M \otimes N$. If $(x_i)_{i\in I}$, $(y_j)_{j\in J}$ are bases of M, N respectively, then the elements $x_i \otimes y_j$ form a basis of $M \otimes N$. Also, if M, N are finite-dimensional vector spaces, then so is $M \otimes N$, and clearly, dim $(M \otimes N) =$ (dim M)(dim N).

Now, if V and W are representations of G, we can define a new representation $V \otimes W$ by the formula: $g\,(v \otimes w) = g\,v \otimes g\,w$.

When a representation is written in the language of matrices, the notion of the tensor product translates into the Kronecker product of matrices.

Definition 2.4.1. Let $A = [a_{i\,j}]$ and $B = [b_{k\,l}]$ be two matrices of size $n \times m$ and $p \times q$, respectively. Then their **Kronecker product** is defined as

$$A \boxtimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix} = [a_{i\,j}\,B].$$

Notice that the size of the new matrix is $n\,p \times m\,q$. The standard properties of the Kronecker product can be found in [10, §52].

Lemma 2.4.2. *Let A, X be square matrices of size n and B, Y of size m. Then*

$$(A \boxtimes B)\,(X \boxtimes Y) = A\,X \boxtimes B\,Y.$$

PROOF.

$$\begin{aligned}(A \boxtimes B)\,(X \boxtimes Y) &= [a_{i\,j}\,B]\,[x_{i\,j}\,Y] = [\sum_k (a_{i\,k}\,B)\,(x_{k\,j}\,Y)] \\ &= [(\sum_k a_{i\,k}\,x_{k\,j})\,B\,Y] = A\,X \boxtimes B\,Y.\end{aligned}$$

(Also see [20, §1.7] and [15].) $\square$

Let $G \longrightarrow GL_m$ and $G \longrightarrow GL_n$ be matrix representations, with $M_g \in GL_m$ and $N_g \in GL_n$ the images of $g \in G$. Then their tensor product representation $G \longrightarrow GL_{mn}$ is given by the formula $g \longrightarrow M_g \boxtimes N_g$. Indeed, by the lemma,

$$(M_g \boxtimes N_g)(M_h \boxtimes N_h) = (M_{gh} \boxtimes N_{gh}).$$

Example 2.4.3. Consider U and $\widetilde{U}$ from Example 2.3.9. Let $(1\,2) \in \mathfrak{S}_3$, from the tables of the action of $\mathfrak{S}_3$ on U and $\widetilde{U}$, we have:

$$A^U_{(1\,2)} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \quad A^{\widetilde{U}}_{(1\,2)} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

Hence

$$A^U_{(1\,2)} \boxtimes A^{\widetilde{U}}_{(1\,2)} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

which gives the action of $(1\,2)$ on $U \otimes \widetilde{U}$.

2.5 Characters

The concept of a character is a milestone in the theory of group representations. It can be used to obtain information about isomorphisms, reducibility and the decomposition of given representations. The main resources for this section are [3, chapter 1], [9], [10, §55], [14, Chapter 13], [15, Chapter 3] and [20].

Recall that, for any square matrix A, the trace of A (denoted $\operatorname{tr} A$) is the sum of entries on the main diagonal. We recall some properties of the trace. The proofs may be found in [14].

Proposition 2.5.1. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two square matrices of size n. Then*

1. *$\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B$,*

2. *$\operatorname{tr}(AB) = \operatorname{tr}(BA)$,*

3. *$\operatorname{tr}(A \boxtimes B) = (\operatorname{tr} A)\,(\operatorname{tr} B)$,*

4. *for any invertible square matrix T of size n, we have tr$(T\,A\,T^{-1}) =$ tr$\,A$.*

It follows from (4) that a linear endomorphism $V \longrightarrow V$ has a well-defined trace.

Definition 2.5.2. *Let $\rho : G \longrightarrow GL(V)$ be a representation. Then its* **character** *is the function $\chi^\rho : G \longrightarrow \mathbb{C}$, defined by*

$$\chi^\rho\,(g) = \textit{tr}\,\rho(g).$$

We will sometimes write χ^V (or even χ) instead of χ^ρ if confusion is unlikely.

Example 2.5.3. In Example 2.2.4,

$$\chi\,(e) = \text{tr}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2, \quad \chi\,(a) = \text{tr}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = 0, \quad \chi\,(a^2) = \text{tr}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -2.$$

Recall that, in a group G, we say that an element y is conjugate to an element x if $x = t^{-1}\,y\,t$ for some $t \in G$. The set of all elements conjugate to x is called the conjugacy class of x. Conjugacy is an equivalence relation, hence conjugacy classes partition G.

Proposition 2.5.4. *For any finite group G:*

1. *Isomorphic representations have the same characters.*

2. *Conjugate elements of G have the same characters.*

3. *If V and W are two G-representations, then*

$$\chi^{V\otimes W} = \chi^V\,\chi^W, \quad \textit{and} \quad \chi^{V\oplus W} = \chi^V + \chi^W.$$

4. *If V is an n-dimensional representation, then $\chi(e) = n$.*

PROOF. To show (1), let V and W be G-representations, and $g \in G$. Then

$$\chi^V\,(g) = \text{tr}\,A_g^V = \text{tr}\,(M\,A_g^W M^{-1}) = \text{tr}\,A_g^W = \chi^W\,(g),$$

by part (2) of Proposition 2.5.1, and Definition 2.2.8. Now let x and y be conjugate in G, so there is $t \in G$ so that $x = t^{-1}yt$. By part (4) of Proposition 2.5.1;

$$\chi^V\,(x) = \chi^V\,(t^{-1}yt) = \text{tr}\,A_{t^{-1}yt}^V = \text{tr}\,(A_t^V\,A_y^V\,A_{t^{-1}}^V) = \text{tr}\,(A_y^V) = \chi^V\,(y),$$

which proves (2). Now (3) is clear from proposition 2.5.1. Since A_e^V is the identity matrix, we have (4). □

The next corollary is a direct result of (3).

Corollary 2.5.5. *Let V be a G-representation, and let*

$$V = \bigoplus_{i=1}^{k} W_i^{\oplus m_i}, \quad \text{where } m_i \text{ is the multiplicity of } W_i \text{ in } V$$

be the complete irreducible decomposition of V. Then:

$$\chi^V(g) = \sum_{i=1}^{k} m_i \, \chi^{W_i}(g)$$

for any $g \in G$.

Definition 2.5.6. *For any vector space V over the field $\mathbb{C}$, the dual space, V^* is the space of all linear transformations $\phi : V \longrightarrow \mathbb{C}$. In other words, $V^* = \text{Hom}(V, \mathbb{C})$.*

Now suppose that V is a G-representation. Given $g \in G$ and $\phi \in V^*$, define $g\,\phi$ to be the element in V^* which sends $v \in V$ to $\phi(g^{-1}\,v)$. It is easy to verify that this turns V^* into a G-representation. If A_g^V denotes the matrix of the action of g with respect to an ordered basis of V, then

$$A_g^{V^*} = ((A_g^V)^{-1})^t$$

with respect to the dual ordered basis of V^*.

Now we list some standard results in the theory of characters.

Theorem 2.5.7.

1. *Any two G-representations with the same characters are isomorphic.*

2. *The number of irreducible representations of G is equal to the number of conjugacy classes in G.*

3. *If V is the regular G-representation with irreducible decomposition*

$$V \simeq \bigoplus_i W_i^{\oplus m_i},$$

   then $m_i = \dim W_i$. In particular, $|G| = \dim(V) = \sum_{i=1}^{k} m_i^2$, where k is the number of non-isomorphic irreducible representations.

4. *For any G-representation V, and element $g \in G$,*

$$\chi^{V^*}(g) = \overline{\chi^V(g)}.$$

PROOF. The proofs may be found in [9, Ch. 2], [20, p. 37] or [21, Ch. 2]. We include a proof of part (4). Let $A = A_g^V$ with respect to some ordered basis of V. Now $A^{|G|} = I$, hence all the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are roots of unity. Since $||\lambda_i|| = 1$, we have $\frac{1}{\lambda_i} = \overline{\lambda_i}$. Hence

$$\chi^{V^*}(g) = \text{tr}\left((A^{-1})^t\right) = \sum_{i=1}^{n} \frac{1}{\lambda_i} = \sum_{i=1}^{n} \overline{\lambda_i} = \overline{\chi^V(g)}.$$

□

Example 2.5.8. Let us reconsider Example 2.3.9. We have seen that V has irreducible decomposition

$$V = W \oplus W' \oplus U \oplus \widetilde{U}.$$

All the theory above is illustrated by the following character table.

	χ^W	$\chi^{W'}$	χ^U	$\chi^{\widetilde{U}}$
e	1	1	2	2
$(1\,2)$	1	-1	0	0
$(1\,3)$	1	-1	0	0
$(2\,3)$	1	-1	0	0
$(1\,2\,3)$	1	1	-1	-1
$(1\,3\,2)$	1	1	-1	-1

- The rows corresponding to conjugate permutations are identical.
- The representations U and $\widetilde{U}$ have the same characters (notice their columns), so they are isomorphic representations. Hence, the complete decomposition of V may be rewritten as
$$V \cong W \oplus W' \oplus U^{\oplus 2}.$$
- The dimension of each representation is the value of the character at $e \in G$. In accordance with the theorem above,
$$|\mathfrak{S}_3| = \dim(V) = \dim(W) + \dim(W') + (\dim(U))^2 = 1 + 1 + 4 = 6.$$
- There are three different conjugacy classes in $\mathfrak{S}_3$, namely the identity, the 2-cycles and the 3-cycles. We have found three non-isomorphic irreducible representations, and hence there are no others.

Chapter 3

Representations of the Symmetric Group; Specht Modules

In this chapter we use some combinatorial tools (specifically Young tableaux and straightening rules) to describe the irreducible representations of $\mathfrak{S}_d$. We will construct an irreducible representation V_λ (called the Specht module) corresponding to each partition λ. The main resources for this chapter are [8], [9],[13] and [20].

3.1 Young tableaux

Definition 3.1.1. *A non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of non-negative integers is called* **a partition of** d *if $\lambda_1 + \lambda_2 + \cdots + \lambda_k = d$.*

For example, $(5,1,1,1), (2,2,2,1,1)$ are partitions for 8. Sometimes we may write them as $(5,1^3), (2^3,1^2)$ respectively.

Definition 3.1.2. *A* ***Young diagram*** *is a collection of boxes, or cells, arranged in left-justified rows, with a non-increasing number of boxes in each row. Listing the number of boxes in each row gives a partition of the integer d that is the total number of boxes. Conversely, every partition λ of d corresponds to a Young diagram.*

For example, the partition $\lambda = (3,2,2,1) = (3,2^2,1)$ of 8 corresponds to the Young diagram

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} & \\ \cline{1-2} \\ \cline{1-1}\end{array}\ .$$

Definition 3.1.3. *A Young diagram of partition λ with a distinct integer between 1 and d in each box is called a λ-**tableau**, or simply a **tableau**. A **standard tableau** is a tableau which is strictly increasing across each row, and down each column.*

The following is an example of a standard tableau for the partition $\lambda = (5, 2, 1, 1)$.

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 5 & 6 & 7 \\ \hline 2 & 4 \\ \cline{1-2} 8 \\ \cline{1-1} 9 \\ \cline{1-1}\end{array}\ .$$

Let $T(i, j)$ denote the entry in row i and column j of T; e.g., $T(3, 1) = 8$. The number of standard tableaux corresponding to a partition λ is denoted by h^λ. It is calculated by the following formula (see [8, 9]).

3.1.1 Hook length formula

Each box in a Young diagram has a hook length which is the number of boxes strictly to its right or below, with the box itself counted once. Let λ be a partition of d. Now

$$h^\lambda = \frac{d!}{\prod \text{(hook length of each box)}}\ .$$

For example

$$h^{(2,1)} = \frac{3!}{3 \cdot 1 \cdot 1} = 2, \quad \text{for} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$$

and

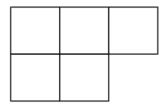
$$h^{(5,2,1,1)} = \frac{9!}{8 \cdot 5 \cdot 3 \cdot 2 \cdot 1 \cdot 4 \cdot 1 \cdot 2 \cdot 1} = 63, \quad \text{for} \quad \begin{array}{|c|c|c|c|c|}\hline 8 & 5 & 3 & 2 & 1 \\ \hline 4 & 1 \\ \cline{1-2} 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}\ .$$

In each case, we have written the hook length for each box in the box itself.

Henceforth, as a notational convenience we will write a tableau as an array. For example,

$$\begin{bmatrix} 1 & 3 & 5 & 6 \\ 2 & 4 & & \\ 7 & & & \\ 8 & & & \end{bmatrix},$$

is a standard tableau for the partition $(4, 2, 1, 1)$.

Example 3.1.4. *Let $d = 5$ and $\lambda = (3, 2)$ with Young diagram*  *. Now*

$$h^\lambda = \frac{5!}{4 \cdot 3 \cdot 1 \cdot 2 \cdot 1} = 5,$$

and the standard tableaux are :

$$\left\{ \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & \end{bmatrix}, \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & \end{bmatrix}, \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & \end{bmatrix}, \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & \end{bmatrix} \right\}.$$

3.2 Specht Modules

Our goal is to describe all the irreducible representations of $\mathfrak{S}_d$. Following [8, Ch. 7], we construct an irreducible representation V_λ (called the Specht module) corresponding to each partition λ of d. In brief, the main idea is the following: we describe a set of rules (called the straightening rules) which convert a tableau of shape λ into an integer linear combination of standard tableaux of the same shape. Then the Specht module is obtained by factoring the free vector space over all λ-tableaux by the subspace generated by relations coming from the straightening rules. These rules have their origin in the so called 'First Fundamental Theorem of Invariant Theory' (see [1, Ch. 2]).

Let W_λ denote the free $\mathbb{C}$-vector space on the set of all (not necessarily standard) Young tableaux of shape λ. For any $g \in \mathfrak{S}_d$ and a tableau $T \in W_\lambda$, the action of g on T is obtained by permuting the entries in T. That is to say, the (i, j)-th entry of the tableau gT is $g\,(T(i, j))$. Thus, W_λ is an $\mathfrak{S}_d$-representation.

Example 3.2.1. *Let $(1\,2) \in \mathfrak{S}_3$, and let $\begin{bmatrix} 1 & 2 \\ 3 & \end{bmatrix} \in W_{(2,1)}$, then*

$$(1\,2) \begin{bmatrix} 1 & 2 \\ 3 & \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & \end{bmatrix}.$$

Now we define rules to change a non-standard tableau into a linear combination of standard tableaux. These rules constitute the straightening algorithm. They will lead to the construction of all irreducible representations of $\mathfrak{S}_d$.

3.2.1 Straightening Algorithm

Rule I. Column interchange:

This rule enables us to interchange two adjacent numbers in a column with a sign introduced. For example [1],

$$\begin{bmatrix} 2 & 5 \\ 3 & 4 \\ 1 & \end{bmatrix} \Rightarrow - \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 3 & \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5 \\ 2 & 4 \\ 3 & \end{bmatrix} \Rightarrow - \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & \end{bmatrix}$$

However this rule might not be enough to get a standard tableau. For example,

$$\begin{bmatrix} 4 & 3 \\ 5 & 2 \\ 1 & \end{bmatrix} \Rightarrow - \begin{bmatrix} 4 & 3 \\ 1 & 2 \\ 5 & \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 \\ 4 & 2 \\ 5 & \end{bmatrix} \Rightarrow - \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 5 & \end{bmatrix}$$

Rule II. Right to left interchange:

Let C_i, C_{i+1} denote two adjacent columns of the tableau T, and assume that they have lengths l_i, l_{i+1} respectively. Fix a positive integer $k \le l_{i+1}$. Given an increasing length k subsequence $a = (a_1, a_2, \dots, a_k)$ of $(1, 2, \dots, l_i)$, let $T(a)$ denote the tableau obtained by interchanging the (a_r, i) and $(r, i+1)$ entries in T for all $1 \le r \le k$. Now the rule allows us to replace T by the linear combination $\sum\limits_a T(a)$, where the sum runs over all such subsequences a of a fixed length k.

Now the straightening algorithm is as follows. By Rule I, one can always ensure that every tableau has increasing columns. Define a *spot* in a tableau T to be a pair (a, b) such that $T(a, b) > T(a, b+1)$. If T has increasing columns but is not standard, then it must contain at least one spot. Choose the unique spot (k, i) in T satisfying the following condition: for any other spot (k', i') in T (if any),

$$i' < i, \quad \text{or} \quad i' = i,\, k' < k.$$

Now use Rule II on T, and continue the same process on each summand in the linear combination. It is proved in [8, chapter 7], that any tableau T may be changed into a unique integer linear combination of standard tableaux by repeatedly applying rules I and II.

[1] Here $\Rightarrow$ stands for the successive replacement of tableaux using the straightening rules.

Example 3.2.2. We would like to straighten the Young tableau $T = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 5 & \end{bmatrix}$. In this tableau the columns are strictly increasing but the second row is not.

$$T = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 5 & \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ 5 & \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 4 & 5 \\ 3 & \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & \end{bmatrix}$$
$$\Rightarrow \underbrace{\begin{bmatrix} 2 & 1 \\ 3 & 4 \\ 5 & \end{bmatrix}}_{S_1} - \underbrace{\begin{bmatrix} 2 & 1 \\ 3 & 5 \\ 4 & \end{bmatrix}}_{S_2} + \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & \end{bmatrix}$$

Clearly, S_1 and S_2 are not standard, so we will work on both of them separately.

$$S_1 = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ 5 & \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 3 & 4 \\ 1 & \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 2 & 1 \\ 3 & 5 \\ 4 & \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 4 & \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 4 & \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & \end{bmatrix}. \text{ So,}$$
$$T \Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 4 & \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 4 & \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & \end{bmatrix}.$$

Given a tableau T, let $St_T = \sum \alpha_i S_i$ be the unique linear combination of standard tableaux obtained by straightening T. Now define Q_λ to be the subspace of W_λ spanned by the elements $T - St_T$ for all T. The proofs of the next two Theorems may be found in [8, Chapter 7] and [20, Chapter 2].

Theorem 3.2.3. *The subspace $Q_\lambda \subseteq W_\lambda$ is invariant under the action of $\mathfrak{S}_d$.*

Hence the quotient vector space $V_\lambda = \frac{W_\lambda}{Q_\lambda}$ is an $\mathfrak{S}_d$-representation. It is called the **Specht module** associated to the partition λ.

Theorem 3.2.4. *1. The following set*

$$\{T + Q_\lambda | \, T \text{ is a standard tableau in } W_\lambda\}$$

is a basis of V_λ.

2. *The Specht module V_λ is an irreducible representation of $\mathfrak{S}_d$.*

3. *If λ and μ are two different partitions of d, then $V_\lambda \ncong V_\mu$ as $\mathfrak{S}_d$-representations.*

Since the conjugacy classes in $\mathfrak{S}_d$ are in bijection with partitions of d, this gives all irreducible representations of $\mathfrak{S}_d$. Henceforth, for simplicity, we identify $T + Q_\lambda$ with T, and regard V_λ as the vector space with basis $\mathcal{B}_\lambda$ consisting of all standard tableaux of shape λ.

Corollary 3.2.5. *Let λ be a partition of d. Then dim $V_\lambda = h^\lambda$.*

PROOF. This follows by part (1) of the previous Theorem, since the number of standard tableaux of shape λ is h^λ. □

The Specht module $V_{(d-1,1)}$ is called the standard representation, and $V_{(1^d)}$ is called the alternating representation. It is easy to check that $V_{(d)}$ is the trivial representation.

3.2.2 An Ordering of $\mathcal{B}_\lambda$

It will be convenient to have a total order on the basis $\mathcal{B}_\lambda$. The following ordering is adapted from [13, §3.1].

Definition 3.2.6. Let T denote a standard tableau. The **row-list** of T, denoted RT is the sequence $(r_1, r_2, \ldots, r_d)$ such that the integer i occurs in row r_i in T.

Now we order the tableaux in the reverse-lexicographic order of their row-lists. Let T, S denote two tableaux. If there is an integer j such that

$$RT_j < RS_j, \quad \text{and} \quad RT_i = RS_i, \quad \text{for all } i > j,$$

then we say that $T < S$.

Example 3.2.7. *Let* $\lambda = (3,\, 2,\, 1,\, 1)$, $\quad T = \begin{bmatrix} 1 & 2 & 3 & 7 \\ 4 & 6 & & \\ 5 & & & \\ 8 & & & \end{bmatrix}$ *and* $S = \begin{bmatrix} 1 & 3 & 5 & 6 \\ 2 & 4 & & \\ 7 & & & \\ 8 & & & \end{bmatrix}$,
then the row-lists are $RT = (1,\, 1,\, 1,\, 2,\, 3,\, 2,\, 1,\, 4)$*, and* $RS = (1,\, 2,\, 1,\, 2,\, 1,\, 1,\, 3,\, 4)$. *Clearly $T < S$, since $RT_7 < RS_7$ and $RT_8 = RS_8$.*

Example 3.2.8. Let $\lambda = (3,2)$. The Specht module $V_{(3,2)}$ is of dimension 5, and its basis is the set

$$\mathcal{B}_{(3,2)} = \{ \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & \end{bmatrix}, \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & \end{bmatrix}, \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & \end{bmatrix}, \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & \end{bmatrix} \}.$$

It is easy to check that they are listed in increasing order.

The next example illustrates the action of a permutation on the basis $\mathcal{B}_\lambda$.

Example 3.2.9. Let $\mathcal{B}_{(3,2)} = \{T_1, \ldots, T_5\}$ be ordered as above, and let $g = (1\,2) \in \mathfrak{S}_5$. Then

$$g\,T_1 = (1\,2) \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & \end{bmatrix} = - \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & \end{bmatrix} = -T_1,$$

by the straightening rules. Using similar calculations, the action of g on $V_{(3,2)}$ is given by the matrix

$$A^{(3,2)}_{(1\,2)} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

There is a similarly defined square matrix A^λ_g of size h^λ for each partition λ, and $g \in \mathfrak{S}_d$.

3.3 The Character of V_λ

Recall that in $\mathfrak{S}_d$ (the symmetric group on d elements) every permutation can be expressed as a product of disjoint cycles in a unique manner. A permutation $\tau \in \mathfrak{S}_d$ is said to have the cycle pattern $(1^{\alpha_1},\, 2^{\alpha_2}, \cdots, d^{\alpha_d})$ if its decomposition contains α_i cycles of length i. Similarly, the sequence $\alpha = [\alpha_1,\, \alpha_2, \cdots, \alpha_d]$ is called its cycle type. Two permutations are conjugate iff they have the same cycle type.

For example, in $\mathfrak{S}_6$ the permutation $(1\,3\,5\,6)$ is a product of one 4-cycle and two 1-cycles, hence its cycle pattern is $(1^2,\, 4)$ and cycle type is $[2,0,0,1,0,0]$. The identity permutation has cycle pattern $= (1^6\,)$ and cycle type $[6,0,0,0,0,0]$. Moreover, since $\sum i\,\alpha_i = d$, each cycle pattern can be identified with a partition of d, where the part α_i is repeated i times.

3.3.1 Frobenius formula

This formula is a tool to compute the characters of all irreducible representations of $\mathfrak{S}_d$. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ be any partition of d. Given a cycle type $\alpha = [\alpha_1, \cdots, \alpha_d]$, let C_α stand for the corresponding conjugacy class. Introduce independent variables $x_1, \cdots, x_k$, and let $x = (x_1, \ldots, x_k)$. Define

$$\Delta(x) = \prod_{1 \leq i < j \leq k} (x_i - x_j), \quad \text{and} \quad P_r(x) = x_1^r + \cdots + x_k^r \quad \text{for all integers } r > 0.$$

Also, if $f(x) = f(x_1, \cdots, x_k)$ is a formal power series and $l = (l_1, \cdots, l_k)$ is a k-tuple of non-negative integers, let

$$[f(x)]_l = \text{coefficient of } x_1^{l_1} \cdots x_k^{l_k} \text{ in } f(x).$$

Now define the specific k-tuple $l = (l_1, \cdots, l_k)$, where

$$l_1 = \lambda_1 + k - 1,\ l_2 = \lambda_2 + k - 2, \cdots, l_k = \lambda_k,$$

which is a strictly increasing sequence of k non-negative integers. Now we can find the character of V_λ for any $g \in C_\alpha$ by using the **Frobenius Formula**,

$$\chi^\lambda(g) = \left[\Delta(x) \prod_{j=1}^{d} P_j(x)^{\alpha_j} \right]_{(l_1, \cdots, l_k)}. \tag{3.1}$$

(See [9, Chapter 4] or [8, Chapter 7, Lemma 4] for a proof.)
In particular, from Proposition 2.5.4, we have $h^\lambda = \dim V_\lambda = \chi^\lambda(C_{[d,0,\ldots,0]})$.

Example 3.3.1. *Let $d = 3, \lambda = (2,1)$. Then $l = (3,1)$. Suppose that C_α is the conjugacy class of the permutation $(1\,2)$. Then $\alpha = [1,1,0]$, and hence*

$$\chi^{(2,1)}(C_\alpha) = \left[(x_1 - x_2)(x_1 + x_2)(x_1^2 + x_2^2)(x_1^3 + x_2^3)^0 \right]_{(3,1)} = 0.$$

Remark 3.3.2. We have written a MAPLE program to find the character table for any d, which works by building the polynomial expression in (3.1) for every λ and extracting the appropriate coefficients. The rows of the table correspond to conjugacy classes and the columns correspond to Specht modules. For instance, the character table for $d = 3$ is

	$\chi^{(3)}$	$\chi^{(2,1)}$	$\chi^{(1,1,1)}$
$C_{[0,0,1]}$	1	-1	1
$C_{[1,1,0]}$	1	0	-1
$C_{[3,0,0]}$	1	2	1

The dimension of each Specht module can be read off from the last row. It follows immediately from the Frobenius formula that $\chi^\lambda(C_\alpha)$ is always an integer.

The next example illustrates the fact that the tensor product of two irreducible representations is in general reducible.

Example 3.3.3. We would like to find the irreducible decomposition of $V_{(2,1,1)} \otimes V_{(3,1)}$. The following is the character table for $\mathfrak{S}_4$. The additional last column is the character of $V_{(2,1,1)} \otimes V_{(3,1)}$, which is given by an entrywise multiplication of the columns for $\chi^{(2,1,1)}$ and $\chi^{(3,1)}$ (see Proposition 2.5.4).

	$\chi^{(4)}$	$\chi^{(3,1)}$	$\chi^{(2,2)}$	$\chi^{(2,1,1)}$	$\chi^{(1,1,1,1)}$	$\chi^{V_{(2,1,1)} \otimes V_{(3,1)}}$
$C_{[0,0,,0,1]}$	1	-1	0	1	-1	-1
$C_{[1,0,1,0]}$	1	0	-1	0	1	0
$C_{[0,2,0,0]}$	1	-1	2	-1	1	1
$C_{[2,1,0,0]}$	1	1	0	-1	-1	-1
$C_{[4,0,0,0]}$	1	3	2	3	1	9

Since each representation of $\mathfrak{S}_d$ is a direct sum of the V_λ, assume that

$$V_{(2,1,1)} \otimes V_{(3,1)} = n_1 \, V_{(4)} \oplus n_2 \, V_{(3,1)} \oplus n_3 \, V_{(2,2)} \oplus n_4 \, V_{(2,1,1)} \oplus n_5 \, V_{(1,1,1,1)},$$

for some non-negative integers $n_1, \ldots, n_5$. By Proposition 2.5.4, for any $g \in C_{[0,0,0,1]}$, we have

$$\chi^{V_{(2,1,1)} \otimes V_{(3,1)}}(g) = n_1 \, \chi^{(4)}(g) + n_2 \, \chi^{(3,1)}(g) + n_3 \, \chi^{(2,2)}(g) + n_4 \, \chi^{(2,1,1)}(g) + n_5 \, \chi^{(1,1,1,1)}(g).$$

This gives an equation $-1 = n_1 - n_2 + n_4 - n_5$. Repeating the procedure for other conjugacy classes, we get a system of five linear equations in the n_i. From their solution, we get the decomposition

$$V_{(2,1,1)} \otimes V_{(3,1)} = V_{(3,1)} \oplus V_{(2,2)} \oplus V_{(2,1,1)} \oplus V_{(1,1,1,1)}.$$

Proposition 3.3.4. *For a partition λ of d,*

$$V_\lambda \cong V_\lambda^*.$$

PROOF. Note that

$$\chi^{V_\lambda^*}(g) = \overline{\chi^{V_\lambda}(g)} = \chi^{V_\lambda}(g).$$

Therefore, by part (1) of Theorem 2.5.7, we have $V_\lambda \cong V_\lambda^*$. □

The next results will be used later in Chapter 4. They can be found in [9, Chapter 4].

Lemma 3.3.5. *For any conjugacy class C_α of $\mathfrak{S}_d$, we have the following:*

1. $\chi^{(d-1,1)}(C_\alpha) = \alpha_1 - 1$;
2. $\chi^{(d-2,1,1)}(C_\alpha) = \frac{1}{2}(\alpha_1 - 1)(\alpha_1 - 2) - \alpha_2$;
3. $\chi^{(d-2,2)}(C_\alpha) = \frac{1}{2}(\alpha_1 - 1)(\alpha_1 - 2) + \alpha_2 - 1$.

PROOF. Recall that for any positive integer n and any complex numbers x, y:

$$(x+y)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^{n-k}\, y^k. \tag{3.2}$$

Now, for part (1), let $\lambda = (d-1,\, 1)$, therefore $l = (d,\, 1)$, and $x = (x_1,\, x_2)$. Hence $\chi^{(d-1,1)}(C_\alpha) =$

$$\left[(x_1 - x_2)\, \prod_j^d P_j\,(x)^{\alpha_j}\right]_{(d,1)} = \left[x_1\, \prod_j^d P_j\,(x)^{\alpha_j}\right]_{(d,1)} - \left[x_2\, \prod_j^d P_j\,(x)^{\alpha_j}\right]_{(d,1)}.$$

We are looking for the coefficient of $x_1^d\, x_2$. Since

$$\begin{aligned} \prod_j^d P_j(x)^{\alpha_j} &= (x_1 + x_2)^{\alpha_1} (x_1^2 + x_2^2)^{\alpha_2} \,\cdots\, (x_1^d + x_2^d)^{\alpha_d} \\ &= (x_1^{\alpha_1} + \alpha_1\, x_1^{\alpha_1 - 1}\, x_2 + \cdots + x_2^{\alpha_1})\, (x_1^{2\,\alpha_2} + \cdots + x_2^{2\,\alpha_d}) \cdots (x_1^{d\,\alpha_d} + \cdots + x_2^{d\,\alpha_d}), \end{aligned}$$

and since $\sum_i i\, \alpha_i = d$, by (3.2) we have:

$$\chi^{(d-1,1)}(C_\alpha) = \alpha_1 - 1 \quad \text{as required.}$$

The proof for parts (2) and (3) are similar, and are omitted. □

Chapter 4

Equivariant Projection Morphisms

4.1 Introduction

Assume that λ, μ are two partitions of d, and V_ν appears as a direct summand of $V_\lambda \otimes V_\mu$ with multiplicity one (see Example 3.3.3). This determines an $\mathfrak{S}_d$-equivariant projection morphism

$$V_\lambda \otimes V_\mu \longrightarrow V_\nu.$$

(Here we have identified V_ν with its isomorphic copy inside $V_\lambda \otimes V_\mu$.) By Schur's lemma, the projection is uniquely determined up to a constant. In this section we explicitly describe such projections in terms of the bases $\mathcal{B}_\lambda$ etc., in the following cases:

- $V_{(d-1,1)} \otimes V_{(d-1,1)} \longrightarrow V_{(d-1,1)}$,
- $V_{(1^d)} \otimes V_{(d-1,1)} \longrightarrow V_{(2,1^{d-2})}$,
- $V_{(1^d)} \otimes V_{(2,1^{d-2})} \longrightarrow V_{(d-1,1)}$.

The results appear in Propositions 4.2.3, 4.2.8, and 4.2.10 respectively. Throughout this chapter, define the elements

$$\sigma = (1\ 2), \quad \text{and} \quad \tau = (1\ 2 \cdots d) \tag{4.1}$$

in $\mathfrak{S}_d$. The next lemma is useful in checking the equivariance of a map of representations.

Lemma 4.1.1. *Let H be the subgroup of $\mathfrak{S}_d$ generated by σ and τ, then $H = \mathfrak{S}_d$.*

PROOF. The permutation group $\mathfrak{S}_d$ is generated by the transpositions

$$(1\ 2), (2\ 3), \cdots, (j\ j+1), \cdots, (d-1\ d) \quad \text{for } 1 \leq\ j\ \leq (d-1)\,.$$

We will show that each of those transpositions is in H. Notice that $\tau\,\sigma\tau^{-1} = (2\ 3)$ and $\tau\,(2\ 3)\tau^{-1} = (3\ 4)$. In general,

$$\tau\,(j,\ j+1)\,\tau^{-1} = (j+1,\ j+2), \quad \text{for any } \ 1 \leq\ j\ \leq (d-2).$$

This implies that $H = \mathfrak{S}_d$. □

It follows that a morphism $\pi : V_\lambda \otimes V_\mu \longrightarrow V_\nu$ is equivariant if the diagram

$$\begin{array}{ccc} V_\lambda \otimes V_\mu & \xrightarrow{\pi} & V_\nu \\ {\scriptstyle g}\Big\downarrow & & \Big\downarrow{\scriptstyle g} \\ V_\lambda \otimes V_\mu & \xrightarrow{\pi} & V_\nu \end{array}$$

commutes for $g = \sigma, \tau$.

Recall that we have a basis $\mathcal{B}_\lambda$ of V_λ, and hence a lexicographically ordered basis $\mathcal{B}_{\lambda,\mu}$ of $V_\lambda \otimes V_\mu$. Using these bases, let M denote the matrix of π and $A_g^{V_\lambda}$ the matrix describing the action of g on V_λ, etc. Then we have an equation

$$(A_g^{V_\lambda} \otimes A_g^{V_\mu})\,M - M\,A_g^{V_\nu} = 0.$$

Example 4.1.2. Let $\lambda = (2,1)$, and suppose that we want to find a projection morphism $\pi : V_\lambda \otimes V_\lambda \longrightarrow V_\lambda$. Since $\dim V_\lambda = 2$, let $\mathcal{B}_\lambda = \{T_1,\ T_2\}$, and

$$\mathcal{B}_{\lambda,\,\lambda} = \{T_1 \otimes T_1,\ T_1 \otimes T_2,\ T_2 \otimes T_1,\ T_2 \otimes T_2\}.$$

Assume that $\pi(T_i \otimes T_j) = \alpha_{i\,j}^{(1)}\,T_1 + \alpha_{i\,j}^{(2)}\,T_2$, hence $M = \begin{bmatrix} \alpha_{1\,1}^{(1)} & \alpha_{1\,1}^{(2)} \\ \alpha_{1\,2}^{(1)} & \alpha_{1\,2}^{(2)} \\ \alpha_{2\,1}^{(1)} & \alpha_{2\,1}^{(2)} \\ \alpha_{2\,2}^{(1)} & \alpha_{2\,2}^{(2)} \end{bmatrix}$. Our problem is to find the coefficients $\alpha_{i\,j}^{(k)}$ for $i, j, k = 1, 2$.

As in Example 3.2.9, we have $A_\sigma^{(2,1)} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$, and

$$A_\sigma^{(2,1)} \boxtimes A_\sigma^{(2,1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

We want M to satisfy the equation

$$(A^{(2,1)}_{\sigma} \boxtimes A^{(2,1)}_{\sigma})\, M - M\, A^{(2,1)}_{\sigma} = 0,$$

which means

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha^{(1)}_{1\,1} & \alpha^{(2)}_{1\,1} \\ \alpha^{(1)}_{1\,2} & \alpha^{(2)}_{1\,2} \\ \alpha^{(1)}_{2\,1} & \alpha^{(2)}_{2\,1} \\ \alpha^{(1)}_{2\,2} & \alpha^{(2)}_{2\,2} \end{bmatrix} - \begin{bmatrix} \alpha^{(1)}_{1\,1} & \alpha^{(2)}_{1\,1} \\ \alpha^{(1)}_{1\,2} & \alpha^{(2)}_{1\,2} \\ \alpha^{(1)}_{2\,1} & \alpha^{(2)}_{2\,1} \\ \alpha^{(1)}_{2\,2} & \alpha^{(2)}_{2\,2} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = 0.$$

The left hand side is a matrix of size 4×2, so we have a linear system of eight equations in $\alpha^{(k)}_{ij}$. We get another eight equations from $\tau = (1\,2\,3)$.

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha^{(1)}_{1\,1} & \alpha^{(2)}_{1\,1} \\ \alpha^{(1)}_{1\,2} & \alpha^{(2)}_{1\,2} \\ \alpha^{(1)}_{2\,1} & \alpha^{(2)}_{2\,1} \\ \alpha^{(1)}_{2\,2} & \alpha^{(2)}_{2\,2} \end{bmatrix} - \begin{bmatrix} \alpha^{(1)}_{1\,1} & \alpha^{(2)}_{1\,1} \\ \alpha^{(1)}_{1\,2} & \alpha^{(2)}_{1\,2} \\ \alpha^{(1)}_{2\,1} & \alpha^{(2)}_{2\,1} \\ \alpha^{(1)}_{2\,2} & \alpha^{(2)}_{2\,2} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = 0.$$

After solving the equations above we have

$$\begin{bmatrix} \alpha^{(1)}_{1\,1} & \alpha^{(2)}_{1\,1} \\ \alpha^{(1)}_{1\,2} & \alpha^{(2)}_{1\,2} \\ \alpha^{(1)}_{2\,1} & \alpha^{(2)}_{2\,1} \\ \alpha^{(1)}_{2\,2} & \alpha^{(2)}_{2\,2} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & -1 \\ -1 & -1 \\ -2 & 1 \end{bmatrix},$$

up to a scalar. This numerical method can always be used for any λ, μ, ν. If there is no non-zero solution, then it implies that V_ν is not a summand in $V_\lambda \otimes V_\mu$.

Remark 4.1.3. We have encoded this method into a set of MAPLE routines in order to calculate all equivariant projection morphisms of the form

$$V_\lambda \otimes V_\mu \longrightarrow V_\nu$$

in terms of the standard tableaux bases. (That is to say, we explicitly calculate the matrices A^λ_g from the straightening rules, and solve a system of linear equations as above.) All the subsequent results were obtained by calculating several examples using these routines, then conjecturing formulae based upon these examples and then giving formal proofs.

4.2 Formulae for projection morphisms

Let $\lambda = (d-1, 1)$. The following lemma is found in [9, Exercise 4.19].

Lemma 4.2.1. *Assume $d \geq 4$. Then we have a decomposition*

$$V_{(d-1,1)} \otimes V_{(d-1,1)} \cong V_{(d)} \oplus V_{(d-1,1)} \oplus V_{(d-2,\,2)} \oplus V_{(d-2,1,1)}.$$

PROOF. By Proposition 2.5.4, it is enough to show that both sides have the same character. Let $\alpha = [\alpha_1, \ldots, \alpha_d]$ be an arbitrary cycle type, and C_α the corresponding conjugacy class. By the Frobenius formula and Lemma 3.3.5, we have

$$\chi^{(d-1,1)}\left(C_\alpha\right) = \alpha_1 - 1, \quad \chi^{(d-2,2)}\left(C_\alpha\right) = \frac{1}{2}\left(\alpha_1 - 1\right)\left(\alpha_1 - 2\right) + \alpha_2 - 1,$$

$$\chi^{(d-2,1,1)}\left(C_\alpha\right) = \frac{1}{2}\left(\alpha_1 - 1\right)\left(\alpha_1 - 2\right) - \alpha_2, \qquad \chi^{(d)}\left(C_\alpha\right) = 1.$$

By Proposition 2.5.4,

$$\chi^{V_{(d-1,1)} \otimes V_{(d-1,1)}}\left(C_\alpha\right) = \chi^{(d-1,1)}(C_\alpha)\, \chi^{(d-1,1)}(C_\alpha) = (\alpha_1 - 1)^2,$$

and

$$\chi^{(d)}(C_\alpha) + \chi^{(d-1,1)}(C_\alpha) + \chi^{(d-2,2)}(C_\alpha) + \chi^{(d-2,1,1)}(C_\alpha) =$$
$$1 + (\alpha_1 - 1) + \frac{1}{2}\left(\alpha_1 - 1\right)\left(\alpha_1 - 2\right) + \alpha_2 - 1 + \frac{1}{2}\left(\alpha_1 - 1\right)\left(\alpha_1 - 2\right) - \alpha_2 =$$
$$\left(\alpha_1 - 1\right)\left(1 + \alpha_1 - 2\right) = (\alpha_1 - 1)^2.$$

This ends the proof. □

For $d = 2, 3$, we have special cases

$$V_{(1,1)} \otimes V_{(1,1)} \cong V_{(2)}, \quad V_{(2,1)} \otimes V_{(2,1)} \cong V_{(3)} \oplus V_{(2,1)} \oplus V_{(1,1,1)}.$$

Note that

$$\mathcal{B}_{(d-1,1)} = \{ \begin{bmatrix} 1 & 3 & 4 & \cdots & d \\ 2 & & & & \end{bmatrix}, \begin{bmatrix} 1 & 2 & 4 & \cdots & d \\ 3 & & & & \end{bmatrix}, \cdots \}.$$

Write

$$T_k = \begin{bmatrix} & 1 & \cdots & \widehat{k+1} & \cdots & d \\ k+1 & & & & & \end{bmatrix}, \text{ for } 1 \leq k \leq d-1.$$

(The notation $\widehat{k+1}$ means that the entry $k+1$ has been omitted.) Then we have an ordering $\{T_1, T_2, \cdots, T_{d-1}\}$ of $\mathcal{B}_{(d-1,1)}$. Let σ and τ be as in (4.1). The following Lemma shows the action of σ and τ on $\mathcal{B}_{(d-1,1)}$.

Lemma 4.2.2. *Assume $d \geq 3$ and $\lambda = (d-1,1)$.*

1. *The action of σ on V_λ with respect to the basis $\mathcal{B}_\lambda$ is given by the matrix,*

$$A^\lambda_\sigma = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

   which is interpreted as follows:
   $\sigma T_1 = -T_1$, *and*
   $\sigma T_i = T_i - T_1$, *where* $T_i \in \mathcal{B}_\lambda$, $2 \leq i \leq d-1$.

2. *Similarly, the action of τ is given by the matrix*

$$A^\lambda_\tau = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ -1 & 0 & 0 & \cdots & 1 \\ -1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

   This means $\tau T_k = T_{k+1} - T_1$, *for* $1 \leq k \leq d-2$, *and* $\tau T_{d-1} = -T_1$.

PROOF. For (1),
$\sigma\, T_1 = \begin{bmatrix} 2 & 3 & \cdots & d \\ 1 & & & \end{bmatrix} = -T_1$, (by straightening). For $2 \leq k \leq d-1$, we have:

$$\begin{aligned}
\sigma\, T_k &= \sigma \begin{bmatrix} 1 & 2 & \cdots & k & \widehat{k+1} & \cdots & d \\ k+1 & & & & & & \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 & \cdots & k & \widehat{k+1} & \cdots & d \\ k+1 & & & & & & \end{bmatrix} \text{(by straightening)} \\
&= \begin{bmatrix} 1 & 2 & \cdots & k & \widehat{k+1} & \cdots & d \\ k+1 & & & & & & \end{bmatrix} + \begin{bmatrix} 2 & k+1 & \cdots & k & \widehat{k+1} & \cdots & d \\ 1 & & & & & & \end{bmatrix} \\
&= T_k - T_1.
\end{aligned}$$

For (2) we just follow the same technique as above.

$$\tau\, T_{d-1} = \tau \begin{bmatrix} 1 & 2 & \cdots & d-1 \\ d & & & \end{bmatrix} \qquad = \begin{bmatrix} 2 & 3 & \cdots & d \\ 1 & & & \end{bmatrix} = -T_1.$$

For $1 \le k \le d-2$ we have;

$$\begin{aligned}\tau\, T_k &= \tau \begin{bmatrix} 1 & \cdots & k & \widehat{k+1} & \cdots & d \\ k+1 & & & & & \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 & \cdots & k+1 & \widehat{k+2} & \cdots & 1 \\ k+2 & & & & & & \end{bmatrix} \\ &= T_{k+1} - T_1. \quad \square\end{aligned}$$

Proposition 4.2.3. *Assume $d \ge 3$. Then the projection*

$$\theta : V_{(d-1,1)} \otimes V_{(d-1,1)} \longrightarrow V_{(d-1,1)}$$

which is defined by

$$\theta(T_i \otimes T_j) = \sum_{k=1}^{d-1} \delta_{i\,j}^{k}\, T_k, \text{ where } \quad \delta_{i\,j}^{k} = \begin{cases} 2-d & \text{if } i = j = k\ , \\ 2 & \text{if } i = j \neq k, \\ 1 & \text{otherwise}, \end{cases}$$

is equivariant.

Notice that the result agrees with Example 4.1.2.

Proof. To show that θ is an equivariant morphism, it suffices to show that $g\,\theta = \theta\, g$ for $g = \sigma, \tau$. Assume $g = \sigma$, and $i = j = 1$,

$$\begin{aligned}\sigma\,\theta\,(T_1 \otimes T_1) &= \sigma\,[(2-d)T_1 + \sum_{k=2}^{d-1} 2\,T_k] \\ &= -(2-d)\,T_1 + \sum_{k=2}^{d-1} 2\,(T_k - T_1) \quad \text{(by part 1 of Lemma 4.2.2)} \\ &= d\,T_1 + (-2)\,T_1 + (-2)(d-2)\,T_1 + \sum_{k=2}^{d-1} 2\,T_k \\ &= (2-d)\,T_1 + \sum_{k=2}^{d-1} 2\,T_k.\end{aligned}$$

The right hand side is:

$$\theta\,\sigma\ (T_1 \otimes T_1) = \theta\,(-T_1 \otimes -T_1) = \theta\,(T_1 \otimes T_1) = (2-d)\,T_1 + \sum_{k=2}^{d-1} 2\,T_k = \sigma\,\theta(T_1 \otimes T_1).$$

Let $g = \sigma$, and let i be an integer such that $2 \leq \ i \ \leq d-1$. Then the left hand side will be:

$$\begin{aligned}
\sigma\,\theta(T_i \otimes T_i) &= \sigma\,[(2-d)\,T_i + \sum_{\substack{k=1\\k\neq i}}^{d-1} 2\,T_k] \\
&= (2-d)\,(T_i - T_1) + (-2)\,T_1 + \sum_{\substack{k=2\\k\neq i}}^{d-1} 2\,(T_k - T_1) \\
&= (2-d)\,T_i - (2-d)\,T_1 + (-2)\,T_1 + (-2)(d-3)\,T_1 + \sum_{\substack{k=2\\k\neq i}}^{d-1} 2\,T_k \\
&= (2-d)\,T_1 + (2-d)\,T_i + \sum_{\substack{k=2\\k\neq i}}^{d-1} 2\,T_k.
\end{aligned}$$

The right hand side is:

$$\begin{aligned}
\theta\,\sigma\,(T_i \otimes T_i) &= \theta\,\left((T_i - T_1) \otimes (T_i - T_1)\right) \\
&= \theta\,(T_i \otimes T_i - T_i \otimes T_1 - T_1 \otimes T_i + T_1 \otimes T_1) \\
&= (2-d)\,T_i + \sum_{\substack{k=1\\k\neq i}}^{d-1} 2\,T_k - \sum_{k=1}^{d-1} T_k - \sum_{k=1}^{d-1} T_k + (2-n)\,T_1 + \sum_{k=2}^{d-1} 2\,T_k \\
&= (2-d)T_1 + (2-d)T_i + \sum_{\substack{k=1\\k\neq i}}^{d-1} 2\,T_k,
\end{aligned}$$

which is the same as above.
The case of $T_i \otimes T_j$ where $i \neq j$ and both are different from 1, remains to be checked.

$$\begin{aligned}
\sigma\,\theta(T_i \otimes T_j) &= \sigma\ \sum_{k=1}^{d-1} T_k = -T_1 + \sum_{k=2}^{d-1}(T_k - T_1) \\
&= -T_1 + \sum_{k=2}^{d-1} T_k + \sum_{k=2}^{d-1}(-T_1) \\
&= (1-d)T_1 + \sum_{k=2}^{d-1} T_k.
\end{aligned}$$

The right hand side equals

$$
\begin{aligned}
\theta\,\sigma\,(T_i \otimes T_j) &= \theta\,(\,(T_i - T_1) \otimes (T_j - T_1)\,) \\
&= \theta\,(T_i \otimes T_j - T_i \otimes T_1 - T_1 \otimes T_j + T_1 \otimes T_1) \\
&= \sum_{k=1}^{d-1} T_k - \sum_{k=1}^{d-1} T_k - \sum_{k=1}^{d-1} T_k + (2-d)\,T_1 + \sum_{k=2}^{d-1} 2\,T_k \\
&= (2-d)\,T_1 + \sum_{k=2}^{d-1} T_k - T_1 \\
&= (1-d)\,T_1 + \sum_{k=2}^{d-1} T_k,
\end{aligned}
$$

which is the same as above. Thus $\sigma\,\theta = \theta\,\sigma$.

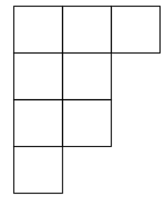
For $g = \tau$, we will show the equality $\tau\,\theta\,(T_i \otimes T_j) = \theta\,\tau\,(T_i \otimes T_j)$, for the case $i = j = d-1$. The other cases are similar. By using part 2 of Lemma 4.2.2, we have:

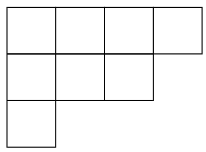
$$
\begin{aligned}
\tau\,\theta\,(T_{d-1} \otimes T_{d-1}) &= \tau\,[\sum_{k=1}^{d-2} 2\,T_k + (2-d)\,T_{d-1}] \\
&= \sum_{k=1}^{d-2} 2\,(T_{k+1} - T_1) - (2-d)\,T_1 \\
&= \sum_{k=1}^{d-2} 2\,T_{k+1} + 2(2-d)\,T_1 - (2-d)\,T_1 \\
&= \sum_{k=1}^{d-2} 2\,T_{k+1} + (2-d)\,T_1,
\end{aligned}
$$

and

$$
\begin{aligned}
\theta\,\tau\,(T_{d-1} \otimes T_{d-1}) &= \theta\,(-T_1 \otimes -T_1) = \theta\,(T_1 \otimes T_1) \\
&= (2-d)\,T_1 + \sum_{k=2}^{d-1} 2\,T_k \\
&= (2-d)\,T_1 + \sum_{k=1}^{d-2} 2\,T_{k+1} \\
&= \tau\,\theta\,(T_{d-1} \otimes T_{d-1}). \quad \square
\end{aligned}
$$

Definition 4.2.4. *For every partition λ of d we can find another partition of d, called the* **conjugate** *of λ, denoted by $\widetilde{\lambda}$. If we look at a λ-tableau as an array, then $\widetilde{\lambda}$ is exactly the transpose of this array.*

For example, let $\lambda = (3, 2, 2, 1)$. The Young diagram of shape λ is  ,

while for $\widetilde{\lambda}$ it will be  . That means $\widetilde{\lambda} = (4, 3, 1)$. Also,

$$\text{if } T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 6 & \\ 4 & 8 & \\ 7 & & \end{bmatrix}, \quad \text{then } T^t = \begin{bmatrix} 1 & 2 & 4 & 7 \\ 3 & 6 & 8 & \\ 5 & & & \end{bmatrix}.$$

The hook length of the (i, j)-th box in T is the same as that of the (j, i)-th box in the transpose of T. Hence $h^\lambda = h^{\widetilde{\lambda}}$.

Lemma 4.2.5. *For any partition λ,*

$$V_{(1^d)} \otimes V_\lambda \cong V_{\widetilde{\lambda}}.$$

PROOF. See [8, §7.3, Corollary to Proposition 3]. □

If $\lambda = (d-1, 1)$, then $\dim V_\lambda = \dim V_{\widetilde{\lambda}} = d - 1$. For $1 \leq k \leq d-1$ define

$$S_k = \begin{bmatrix} 1 & r \\ 2 & \\ \vdots & \\ \widehat{r} & \\ \vdots & \\ d & \end{bmatrix}, \quad \text{where } r = d - k + 1.$$

Then $\mathcal{B}_{(2,\,1^{d-2})}$ is the set of tableaux

$$\left\{ \underbrace{\begin{bmatrix} 1 & d \\ 2 & \\ \vdots & \\ d-1 & \end{bmatrix}}_{S_1}, \underbrace{\begin{bmatrix} 1 & d-1 \\ 2 & \\ \vdots & \\ d & \end{bmatrix}}_{S_2}, \dots, \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & \\ \vdots & \\ d & \end{bmatrix}}_{S_{d-1}} \right\}.$$

Lemma 4.2.6. *Assume $d \geq 3$, and $\lambda = (2,\,1^{d-2})$. Then the action of σ on V_λ with respect to $\mathcal{B}_\lambda$ is given by the matrix*

$$A_\sigma^\lambda = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ (-1)^d & (-1)^{d-1} & (-1)^{d-2} & \cdots & (-1)^2 \end{bmatrix}.$$

More precisely,

$$\sigma\, S_{d-1} = \sum_{k=1}^{d-1} (-1)^{d-k+1}\, S_k, \quad \textit{and } \sigma\, S_k = -S_k \quad \textit{for} \quad 1 \leq k \leq d-2.$$

For $g = \tau$, the matrix is

$$A_\tau^\lambda = \begin{bmatrix} (-1)^d & (-1)^{d-1} & (-1)^{d-2} & \cdots & (-1)^3 & (-1)^2 \\ (-1)^d & 0 & 0 & \cdots & 0 & 0 \\ 0 & (-1)^d & 0 & \cdots & 0 & 0 \\ \vdots & & & & \vdots & \\ 0 & 0 & 0 & \cdots & (-1)^d & 0 \end{bmatrix}.$$

That is, for $S_i \in \mathcal{B}_\lambda$, and $1 \leq i \leq d-1$, we have

$$\tau\, S_1 = \sum_{k=1}^{d-1} (-1)^{d-k+1}\, S_k.$$

Also,

$$\tau\, S_k = (-1)^d\, S_{k-1}, \quad \textit{for} \quad 1 < k \leq d-1.$$

PROOF. It is clear that when $1 \leq k < d-1$ we have $\sigma\, S_k = -S_k$. Let us check the case when $k = d-1$:

$$\begin{aligned}
\sigma\, S_{d-1} &= \begin{bmatrix} 2 & 1 \\ 3 & \\ \vdots & \\ d & \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & \\ \vdots & \\ d & \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & \\ \vdots & \\ d & \end{bmatrix} + \cdots + \begin{bmatrix} 2 & d \\ 3 & \\ \vdots & \\ 1 & \end{bmatrix} \\
&= S_{d-1} + \sum_{k=1}^{d-2} (-1)^{d-k+1}\, S_k \\
&= \sum_{k=1}^{d-1} (-1)^{d-k+1}\, S_k.
\end{aligned}$$

Similarly,

$$\tau\, S_1 = \sum_{k=1}^{d-1} (-1)^{d-k+1}\, S_k.$$

However, when $1 < k \leq d-1$ we have:

$$\begin{aligned}
\tau\, S_k &= \tau \begin{bmatrix} 1 & r \\ 2 & \\ \vdots & \\ \widehat{r} & \\ \vdots & \\ d & \end{bmatrix}, \quad \text{(where } r = d-k+1) \\
&= \begin{bmatrix} 2 & r+1 \\ 3 & \\ \vdots & \\ \widehat{r}+1 & \\ \vdots & \\ 1 & \end{bmatrix} = (-1)^{d-2} \begin{bmatrix} 1 & r+1 \\ 2 & \\ \vdots & \\ \widehat{r}+1 & \\ \vdots & \\ d & \end{bmatrix} \\
&= (-1)^d\, S_{k-1}, \quad \text{where} \quad r+1 = d-(k-1)+1. \qquad \square
\end{aligned}$$

Recall that $V_{(1^d)}$ is called the alternating representation. Let $\mathcal{B}_{(1^d)} = \{U\}$, where $U = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ d \end{bmatrix}$ is the only possible standard tableau of that shape. The next lemma shows the action of the permutations σ and τ on U.

Lemma 4.2.7. *With notation as above,*

1. $\sigma\, U = -U$,

2. $\tau\, U = (-1)^{d+1}\, U$.

PROOF. This is clear from the straightening rules. $\square$

By Lemma 4.2.5, we have an isomorphism

$$\Pi : V_{(1^d)} \otimes V_{(d-1,1)} \longrightarrow V_{(2,1^{d-2})}.$$

Proposition 4.2.8. *With notation as above, the morphism Π is defined by*

$$\Pi(U \otimes T_i) = \sum_{j=1}^{d-1} \alpha_j^i\, S_{d-j},$$

where

$$\alpha_j^i = \begin{cases} (-1)^{j+1}\, 2 & \textit{when} \quad i = j, \\ \\ (-1)^{j+1} & \textit{when}\ i \neq j. \end{cases}$$

PROOF. We want to show $g\,\Pi = \Pi\, g$ for $g = \sigma, \tau$. First, let $g = \sigma$ and $i = 1$. By Lemma 4.2.2, we have:

$$\begin{aligned} \Pi\, \sigma\, (U \otimes T_1) &= \Pi(\sigma\, U \otimes \sigma\, T_1) = \Pi(-U \otimes -T_1) = \Pi(U \otimes T_1) \\ &= \sum_{j=1}^{d-1} \alpha_j^1\, S_{d-j} = 2\, S_{d-1} + \sum_{j=2}^{d-1} (-1)^{j+1}\, S_{d-j}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sigma\, \Pi(U \otimes T_1) &= \sigma \left[2\, S_{d-1} + \sum_{j=2}^{d-1} (-1)^{j+1}\, S_{d-j} \right] \\ &= 2 \sum_{k=1}^{d-1} (-1)^{d-k+1}\, S_k + \sum_{j=2}^{d-1} (-1)^{j+1}\, (-S_{d-j}) \quad \text{(Lemma 4.2.6)} \\ &= 2 \sum_{j=1}^{d-1} (-1)^{j+1}\, S_{d-j} + \sum_{j=2}^{d-1} (-1)^{j}\, S_{d-j} \\ &= 2\, S_{d-1} + \sum_{j=2}^{d-1} (-1)^{j+1}\, S_{d-j} = \Pi\, \sigma\, (U \otimes T_1). \end{aligned}$$

Now, assume $i \neq 1$. By Lemma 4.2.2,

$$\begin{aligned}
\Pi\,\sigma\,(U \otimes T_i) &= \Pi\,[-U \otimes (T_i - T_1)] = \Pi\,[-(U \otimes T_i) + (U \otimes T_1)] \\
&= -\sum_{j=1}^{d-1} \alpha_j^{d-1}\, S_{d-j} + \sum_{j=1}^{i} \alpha_j^{1}\, S_{d-j}, \quad \text{(since } i \neq 1\text{)} \\
&= -\left[S_{d-1} + \sum_{j=2}^{d-1} \alpha_j^{i}\, S_{d-j} \right] + 2\,S_{d-1} + \sum_{j=2}^{d-1} (-1)^{j+1}\, S_{d-j} \\
&= \sum_{j=1}^{d-1} (-1)^{j+1}\, S_{d-j} - \sum_{j=2}^{d-1} \alpha_j^{i}\, S_{d-j},
\end{aligned}$$

and

$$\begin{aligned}
\sigma\,\Pi(U \otimes T_i) &= \sigma \sum_{j=1}^{d-1} \alpha_j^{i}\, S_{d-j} \\
&= \sigma \left[S_{d-1} + \sum_{j=2}^{d-1} \alpha_j^{i}\, S_{d-j} \right] \quad \text{(since } i \neq 1\text{)} \\
&= \sum_{k=1}^{d-1} (-1)^{d-k+1}\, S_k + \sum_{j=2}^{d-1} \alpha_j^{i}\,(-S_{d-j}) \quad \text{(by Lemma 4.2.6)} \\
&= \sum_{j=1}^{d-1} (-1)^{j+1}\, S_{d-j} - \sum_{j=2}^{d-1} \alpha_j^{i}\, S_{d-j}, \quad \text{(by letting } k = d-j\text{)} \\
&= \Pi\,\sigma\,(U \otimes T_i).
\end{aligned}$$

Secondly, when $g = \tau$ and $i = d-1$, then by using Lemma 4.2.2, we have:

$$\begin{aligned}
\Pi\,\tau(U \otimes T_{d-1}) &= \Pi\left[(-1)^{d+1}\, U \otimes (-T_1)\right] = \Pi\,(-1)^{d}(U \otimes T_1) \\
&= (-1)^{d} \sum_{j=1}^{d-1} \alpha_j^{1}\, S_{d-j} = (-1)^{d} \left[2\,S_{d-1} + \sum_{j=2}^{d-1} (-1)^{j+1}\, S_{d-j} \right] \\
&= (-1)^{d}\, S_{d-1} + \sum_{j=1}^{d-1} (-1)^{d+j+1}\, S_{d-j}.
\end{aligned}$$

Now the other side is

$$
\begin{aligned}
\tau\,\Pi(U \otimes T_{d-1}) &= \tau \sum_{j=1}^{d-1} \alpha_j^{d-1}\, S_{d-j} = \tau \left[(-1)^d\, 2\, S_1 + \sum_{j=1}^{d-2} (-1)^{j+1}\, S_{d-j} \right] \\
&= (-1)^d\, 2 \sum_{k=1}^{d-1} (-1)^{d-k+1}\, S_k + \sum_{j=1}^{d-2} (-1)^{j+1}\, (-1)^d\, S_{d-j-1}
\end{aligned}
$$

By setting $k = d - j$ in the first term we obtain

$$
\begin{aligned}
&= \sum_{j=1}^{d-1} 2\, (-1)^{d+j+1}\, S_{d-j} + \sum_{j=1}^{d-2} (-1)^{d+j+1}\, S_{d-j-1} \\
&= 2\, (-1)^d\, S_{d-1} + \sum_{j=2}^{d-1} (-1)^{d+j+1}\, S_{d-j} \\
&= (-1)^d\, S_{d-1} + \sum_{j=1}^{d-1} (-1)^{d+j+1}\, S_{d-j} = \Pi\, \tau\, (U \otimes T_{d-1}).
\end{aligned}
$$

As the last case when $i \neq d-1$, we have

$$
\begin{aligned}
\tau\,\Pi(U \otimes T_i) &= \tau \sum_{j=1}^{d-1} \alpha_j^i\, S_{d-j} = \tau \left[(-1)^d\, S_1 + \sum_{j=1}^{d-2} \alpha_j^i\, S_{d-j} \right] \\
&= (-1)^d \sum_{k=1}^{d-1} (-1)^{d-k+1}\, S_k + \sum_{j=1}^{d-2} \alpha_j^i\, (-1)^d\, S_{d-j-1} \\
&= \sum_{k=1}^{d-1} (-1)^{k+1}\, S_k + (-1)^d \sum_{j=1}^{d-2} \alpha_j^i\, S_{d-j-1},
\end{aligned}
$$

and

$$
\begin{aligned}
\Pi\,\tau(U \otimes T_i) &= \Pi\left[(-1)^{d+1}\, U \otimes (T_{i+1} - T_1) \right] \quad \text{(by Lemma 4.2.7)} \\
&= \Pi\, \left[(-1)^{d+1}\, U \otimes T_{i+1} + (-1)^d\, U \otimes T_1 \right] \\
&= (-1)^{d+1} \sum_{j=1}^{d-1} \alpha_j^{i+1}\, S_{d-j} + (-1)^d \sum_{j=1}^{d-1} \alpha_j^1\, S_{d-j}
\end{aligned}
$$

In the first term, when $j = 1$, the value of $i+1$ can never be equal to 1, hence this expression is

$$
\begin{aligned}
&= (-1)^{d+1} \left[S_{d-1} + \sum_{j=2}^{d-1} \alpha_j^{i+1} \, S_{d-j} \right] + (-1)^d \left[2\, S_{d-1} + \sum_{j=2}^{d-1} (-1)^{j+1} \, S_{d-j} \right] \\
&= (-1)^d \, S_{d-1} + (-1)^{d+1} \sum_{j=2}^{d-1} \alpha_j^{i+1} \, S_{d-j} + (-1)^d \sum_{j=2}^{d-1} (-1)^{j+1} \, S_{d-j} \\
&= (-1)^{d+1} \sum_{j=2}^{d-1} \alpha_j^{i+1} \, S_{d-j} + \sum_{j=1}^{d-1} (-1)^{d+j+1} \, S_{d-j}.
\end{aligned}
$$

By setting $k = j - 1$ we have $\alpha_j^{i+1} = (-1)\, \alpha_k^i$. Letting $t = d - j$, this equals

$$
= (-1)^d \sum_{k=1}^{d-2} \alpha_k^i \, S_{d-(k+1)} + \sum_{t=1}^{d-1} (-1)^{t+1} \, S_t
$$

which completes the proof. □

Example 4.2.9. *Let $d = 5$. The projection $\Pi : V_{(1^5)} \otimes V_{(4,1)} \longrightarrow V_{(2,1^3)}$ can be described by the matrix:*

$$
M = \begin{bmatrix} -1 & 1 & -1 & 2 \\ -1 & 1 & -2 & 1 \\ -1 & 2 & -1 & 1 \\ -2 & 1 & -1 & 1 \end{bmatrix}.
$$

For instance, the first row can be interpreted as

$$
\Pi(U \otimes T_1) = -S_1 + S_2 - S_3 + 2\, S_4.
$$

By Lemma 4.2.5, we have an isomorphism $\Pi : V_{(1^d)} \otimes V_{(2,1^{d-2})} \longrightarrow V_{(d-1,1)}$.

Proposition 4.2.10. *The isomorphism Π is given the following formula:*

$$
\Pi(U \otimes S_i) = \sum_{j=1}^{d-1} \alpha_j^i \, T_{d-j}, \quad \textit{where} \quad \alpha_j^i = \begin{cases} (-1)^i \, (d-1) & \textit{if } i = j, \\ \\ (-1)^{i+1} & \textit{if } i \neq j. \end{cases}
$$

PROOF. We want to show that $g\,\Pi = \Pi\, g$, for $g = \sigma, \tau$. Let $g = \sigma$ and $i = d-1$. Then

$$\begin{aligned}
\sigma\,\Pi(U \otimes S_{d-1}) &= \sigma \sum_{j=1}^{d-1} \alpha_j^{d-1}\, T_{d-j} = \sigma \left[\sum_{j=1}^{d-2} (-1)^d\, T_{d-j} + (-1)^{d-1}(d-1)T_1\right] \\
&= \sum_{j=1}^{d-2} (-1)^d\,(T_{d-j} - T_1) + (-1)^d\,(d-1)\,T_1 \quad \text{(by Lemma 4.2.2)} \\
&= \sum_{j=1}^{d-2} (-1)^d\, T_{d-j} + (-1)^{d+1}\,(d-2)\,T_1 + (-1)^d\,(d-1)\,T_1 \\
&= (-1)^d \sum_{j=1}^{d-1} T_{d-j}.
\end{aligned}$$

On the other hand,

$$\Pi\,\sigma\,(U \otimes S_{d-1}) = \Pi\,(-U \otimes \sum_{k=1}^{d-1} (-1)^{d-k+1}\, S_k).$$

By Lemma 4.2.6, and Lemma 4.2.7, this is the same as

$$\begin{aligned}
&= \sum_{k=1}^{d-1} (-1)^{d-k}\, \Pi(U \otimes S_k) = \sum_{k=1}^{d-1} (-1)^{d-k} \left[\sum_{j=1}^{d-1} \alpha_j^k\, T_{d-j}\right] \\
&= \sum_{k=1}^{d-1} (-1)^{d-k} \left[(-1)^k\,(d-1)\,T_{d-k} + \sum_{\substack{j=1 \\ \neq k}}^{d-1} (-1)^{k+1}\, T_{d-j}\right] \\
&= \sum_{k=1}^{d-1} (-1)^d\,(d-1)\,T_{d-k} + \sum_{k=1}^{d-1} (-1)^{d+1} \sum_{\substack{j=1 \\ \neq k}}^{d-1} T_{d-j} \\
&= (-1)^d\,(d-1) \sum_{k=1}^{d-1} T_{d-k} + (-1)^{d+1}\,(d-2) \sum_{j=1}^{d-1} T_{d-j} \\
&= (-1)^d \sum_{k=1}^{d-1} T_{d-k} = \sigma\,\Pi(U \otimes S_i).
\end{aligned}$$

For $i \neq d-1$, we have

$$\begin{aligned}
\sigma\,\Pi(U \otimes S_i) &= \sigma \sum_{j=1}^{d-1} \alpha_j^i \, T_{d-j} \\
&= \sigma \left[\sum_{j=1}^{d-2} \alpha_j^i \, T_{d-j} + (-1)^{i+1}\, T_1\right] \quad \text{(by Lemma 4.2.2)} \\
&= \sum_{j=1}^{d-2} \alpha_j^i \,(T_{d-j} - T_1) + (-1)^{i+1}\,(-T_1) \\
&= \sum_{j=1}^{d-2} \alpha_j^i \, T_{d-j} - \sum_{j=1}^{d-2} \alpha_j^i \, T_1 + (-1)^i \, T_1 \\
&= \sum_{j=1}^{d-2} \alpha_j^i \, T_{d-j} - T_1 \left[(-1)^i \,(d-1) + (-1)^{i+1}\,(d-3)\right] + (-1)^i \, T_1 \\
&= \sum_{j=1}^{d-2} \alpha_j^i \, T_{d-j} - T_1 \left[(-1)^i \, 2\right] + (-1)^i \, T_1 = \sum_{j=1}^{d-1} \alpha_j^i \, T_{d-j}.
\end{aligned}$$

By using Lemma 4.2.6, and Lemma 4.2.7,

$$\Pi\,\sigma\,(U \otimes S_i) = \Pi\,(-U \otimes -S_i) = \Pi(U \otimes S_i) = \sum_{j=1}^{d-1} \alpha_j^i \, T_{d-j} = \sigma\,\Pi(U \otimes S_i).$$

The argument for $g = \tau$ is similar. First, let $i = 1$.

$$\begin{aligned}
\Pi\,\tau\,(U \otimes S_1) &= \Pi \left[(-1)^{d+1}\, U \otimes \sum_{k=1}^{d-1} (-1)^{d-k+1}\, S_k\right] \quad \text{(by Lemma 4.2.6, and Lemma 4.2.7)} \\
&= \Pi \left[U \otimes \sum_{k=1}^{d-1} (-1)^k \, S_k\right] = \sum_{k=1}^{d-1} (-1)^k \sum_{j=1}^{d-1} \alpha_j^k \, T_{d-j} \\
&= \sum_{k=1}^{d-1} (-1)^k \left[(-1)^k (d-1)\, T_{d-k} + \sum_{\substack{j=1 \\ \neq k}}^{d-1} (-1)^{k+1}\, T_{d-j}\right] \\
&= \sum_{k=1}^{d-1} (d-1)\, T_{d-k} - (d-2) \sum_{k=1}^{d-1} T_{d-k} = \sum_{k=1}^{d-1} T_{d-k} = \sum_{k=1}^{d-1} T_k,
\end{aligned}$$

whereas the other side is

$$
\begin{aligned}
\tau\,\Pi(U \otimes S_1) &= \tau \sum_{j=1}^{d-1} \alpha_j^1\, T_{d-j} \\
&= \tau \left[(-1)(d-1)\, T_{d-1} + \sum_{j=2}^{d-1} T_{d-j} \right] \\
&= \left[(-1)(d-1)\,(-T_1) + \sum_{j=2}^{d-1} (T_{d-j+1} - T_1) \right] \quad \text{(by Lemma 4.2.2)} \\
&= \left[(d-1)\, T_1 + \sum_{j=2}^{d-1} T_{d-j+1} - (d-2)\, T_1) \right] \\
&= \sum_{j=1}^{d-1} T_j = \Pi\, \tau(U \otimes S_1).
\end{aligned}
$$

When $i \neq 1$, we have

$$
\begin{aligned}
\Pi\, \tau\, (U \otimes S_i) &= \Pi\, ((-1)^{d+1}\, U \otimes (-1)^d\, S_{i-1}) \quad \text{(by Lemma 4.2.6, and Lemma 4.2.7)} \\
&= -\Pi\, (U \otimes S_{i-1}) = -\sum_{j=1}^{d-1} \alpha_j^{i-1}\, T_{d-j} \\
&= -\left[(-1)^k\, (d-1)\, T_{d-k} + \sum_{\substack{j=1 \\ \neq k}}^{d-1} (-1)^i\, T_{d-j} \right] \\
&= (-1)^{k+1}\, (d-1)\, T_{d-k} + \sum_{\substack{j=1 \\ \neq k}}^{d-1} (-1)^{i+1}\, T_{d-j} \quad \text{(letting } d-k=t \text{)} \\
&= (-1)^{d-t+1}\, (d-1)\, T_t + \sum_{\substack{j=1 \\ \neq t}}^{d-1} (-1)^{i+1}\, T_j.
\end{aligned}
$$

The other side is

$$\begin{aligned}
\tau\,\Pi\,(U\otimes S_i) &= \tau\left[\sum_{j=1}^{d-1}\alpha_j^i\,T_{d-j}\right]\\
&= \tau\left[\alpha_1^i\,T_{d-1}+\sum_{j=2}^{d-1}\alpha_j^i\,T_{d-j}\right] \quad \text{(since } i\neq 1\text{, and by Lemma 4.2.2)}\\
&= (-1)^{i+1}\,(-T_1)+\sum_{j=2}^{d-1}\alpha_j^i\,(T_{d-j+1}-T_1)\\
&= (-1)^i\,T_1+\sum_{j=2}^{d-1}\alpha_j^i\,T_{d-j+1}-\sum_{j=2}^{d-1}\alpha_j^i\,T_1\\
&= (-1)^i\,T_1+\left[(-1)^k(d-1)\,T_{d-k+1}+\sum_{\substack{j=2\\ \neq k}}^{d-1}(-1)^{i+1}\,T_{d-j+1}\right]\\
&\quad -\left[(-1)^i\,(d-1)\,T_1+(-1)^{i+1}\,(d-3)\,T_1\right] \quad \text{(let } d-k+1=t\text{)}\\
&= (-1)^i\,T_1+\left[(-1)^{d-t+1}\,(d-1)\,T_t+\sum_{\substack{j=2\\ \neq t}}^{d-1}(-1)^{i+1}\,T_j\right]-(-1)^i\,2\,T_1\\
&= (-1)^{d-t+1}\,(d-1)\,T_t+\sum_{\substack{j=1\\ \neq t}}^{d-1}(-1)^{i+1}\,T_j=\Pi\,\tau\,(U\otimes S_i).
\end{aligned}$$

This completes the proof. □

Example 4.2.11. *As in Example 4.2.9, the projection* $\Pi : V_{(1^5)}\otimes V_{(2,1^3)}\longrightarrow V_{(4,1)}$ *is described by the matrix*

$$P=\begin{bmatrix} -1 & -1 & -1 & 4\\ 1 & 1 & -4 & 1\\ -1 & 4 & -1 & -1\\ -4 & 1 & 1 & 1 \end{bmatrix}.$$

For instance, the first row gives the equation

$$\Pi\,(U\otimes S_1)=-T_1-T_2-T_3+4\,T_4.$$

Chapter 5

Equivariant q-Forms on Specht Modules

5.1 Preliminaries

Recall that (see Proposition 3.3.4) we have an equivariant isomorphism

$$\eta : V_\lambda \longrightarrow V_\lambda^*, \quad v \longrightarrow \eta_v.$$

Moreover η is uniquely defined up to a scalar. (If η, η' were two such isomorphisms, then by Schur's lemma $\eta^{-1} \circ \eta' : V_\lambda \longrightarrow V_\lambda$ must be a scalar multiple of the identity.) This defines a morphism

$$\theta_\lambda : V_\lambda \otimes V_\lambda \longrightarrow \mathbb{C},$$

by the formula $\theta_\lambda(v \otimes w) = \eta_v(w)$. We can identify $\mathbb{C}$ with the trivial representation $V_{(d)}$ by mapping $1 \in \mathbb{C}$ to the unique element in $\mathcal{B}_{(d)}$.

Lemma 5.1.1. *With this identification, the morphism θ_λ is $\mathfrak{S}_d$-equivariant.*

PROOF. Note that,

$$\theta_\lambda(g\, v \otimes g\, w) = \eta_{g\, v}(g\, w) = (g\, \eta_v)(g\, w) = \eta_v(g^{-1}\, g\, w) = \eta_v(w) = \theta_\lambda(v \otimes w),$$

hence θ_λ is equivariant. □

We call θ_λ the q-form associated to λ. As explained above, it is uniquely determined up to a multiplicative scalar. If $\mathcal{B}_\lambda = \{T_1, \ldots, T_{h^\lambda}\}$, then θ_λ can be represented by a matrix of size $h^\lambda \times h^\lambda$ whose (i, j)-th element is $\theta_\lambda(T_i \otimes T_j)$.

Example 5.1.2. Let $\lambda = (3,2)$. Then a calculation similar to Example 4.1.2 shows that $\theta_{(3,2)}$ is represented by the matrix $\begin{bmatrix} 4 & 2 & 2 & 1 & -1 \\ 2 & 4 & 1 & 2 & 1 \\ 2 & 1 & 4 & 2 & 1 \\ 1 & 2 & 2 & 4 & 2 \\ -1 & 1 & 1 & 2 & 4 \end{bmatrix}$.

This suggests that θ_λ is represented by a symmetric matrix. We prove this below.

Proposition 5.1.3. *Given $v, w \in V_\lambda$, we have an identity $\theta_\lambda(v \otimes w) = \theta_\lambda(w \otimes v)$.*

PROOF. Define a new map $f_\lambda : V_\lambda \otimes V_\lambda \longrightarrow \mathbb{C}$ as follows. For basis vectors $T, T' \in \mathcal{B}_\lambda$, let

$$f_\lambda(T \otimes T') = \begin{cases} 1 & \text{if } T = T', \\ 0 & \text{if } T \neq T'. \end{cases} \tag{5.1}$$

Although f_λ may not be equivariant, we can define an $\mathfrak{S}_d$-equivariant map by letting

$$h_\lambda(T \otimes T') = \sum_{g \in \mathfrak{S}_d} f_\lambda(g\,T \otimes g\,T').$$

By the uniqueness proved above,

$$\theta_\lambda(v \otimes w) = \xi\, h_\lambda(v \otimes w),$$

for some constant ξ (independent of v, w). But $h_\lambda(v \otimes w) = h_\lambda(w \otimes v)$ by construction, which completes the proof. $\square$

We can thus define a quadratic form $Q_\lambda : V_\lambda \longrightarrow \mathbb{C}$ by the formula

$$Q_\lambda(v) = h_\lambda(v \otimes v).$$

If $g\,T = \sum_{S \in \mathcal{B}_\lambda} a_S\, S$, then $f_\lambda(g\,T \otimes g\,T) = \sum_{S \in \mathcal{B}_\lambda} a_S^2 > 0$ by (5.1). Hence $h_\lambda(v \otimes v) > 0$ for every nonzero $v \in V_\lambda$. This proves that

Proposition 5.1.4. *Q_λ is a positive definite quadratic form.*

In this chapter, we give explicit formulae for θ_λ when

$$\lambda = (d-1, 1),\ (2, 1^{d-2}),\ (d-2, 1^2).$$

Finally we conjecture a formula for the case $\lambda = (d-r, 1^r)$.

Recall that we have defined permutations

$$\sigma = (1\,2), \quad \tau = (1\,2\,\ldots\,d). \tag{5.2}$$

To check that a form $\theta_\lambda : V_\lambda \otimes V_\lambda \longrightarrow \mathbb{C}$ is equivariant, it is enough to verify the equality $\theta(v \otimes w) = \theta(g\,v \otimes g\,w)$ for $g = \sigma, \tau$ and $v, w \in \mathcal{B}_\lambda$.

5.2 Formulae for θ_λ

Proposition 5.2.1. *Assume $d \geq 3$, and $\lambda = (d-1, 1)$. Then the q-form*

$$\theta_\lambda : V_{(d-1,1)} \otimes V_{(d-1,1)} \longrightarrow \mathbb{C}$$

is given by:

$$\theta\,(T_i \otimes T_j) = \begin{cases} 2 & \textit{if } i = j, \\ 1 & \textit{if } i \neq j. \end{cases}$$

PROOF. Let $i = j$ and $g = \sigma$. We want to show that $\theta\,(T_i \otimes T_i) = \theta\,\sigma\,(T_i \otimes T_i)$. By using Lemma 4.2.2,

$$\begin{aligned} \theta\,\sigma\,(T_i \otimes T_i) &= \theta\,[(T_i - T_1) \otimes (T_i - T_1)] \\ &= \theta\,[T_i \otimes T_i - T_i \otimes T_1 - T_1 \otimes T_i + T_1 \otimes T_1] \\ &= 2 - 1 - 1 + 2 = 2 = \theta\,(T_i \otimes T_i). \end{aligned}$$

Now let $i \neq j$ where $i \neq 1$ and $j \neq 1$. Then

$$\begin{aligned} \theta\,\sigma\,(T_i \otimes T_j) &= \theta\,[(T_i - T_1) \otimes (T_j - T_1)] \quad \text{(by Lemma 4.2.2)} \\ &= \theta\,[T_i \otimes T_j - T_i \otimes T_1 - T_1 \otimes T_j + T_1 \otimes T_1] \\ &= 1 - 1 - 1 + 2 = 1 = \theta\,(T_i \otimes T_j) \end{aligned}$$

Now let $i \neq j$, and either $i = 1$ or $j = 1$. Say $i = 1$, then

$$\theta\,\sigma\,(T_1 \otimes T_j) = \theta[-T_1 \otimes (T_j - T_1)] = \theta[-T_1 \otimes T_j + T_1 \otimes T_1] = -1 + 2 = 1 = \theta\,(T_1 \otimes T_j).$$

The calculation is similar for $g = \tau$. $\square$

Proposition 5.2.2. *Assume $d \geq 3$, and $\lambda = (2,\, 1^{d-2})$. Then the q-form $\theta_\lambda : V_\lambda \otimes V_\lambda \longrightarrow \mathbb{C}$ is given by*

$$\theta\,(S_i \otimes S_j) = \begin{cases} (d-1) & \textit{when } i = j, \\ \\ (-1)^{i+j+1} & \textit{when } i \neq j. \end{cases}$$

PROOF. As before, we will explicitly check only some of the cases.

Case (1), when $i \neq 1$, $i \neq d-1$. We want to show $\theta\,(S_i \otimes S_i) = \theta\, g\,(S_i \otimes S_i)$, for $g = \sigma, \tau$. By Lemma 4.2.6,

$$\theta\,\sigma\,(S_i \otimes S_i) = \theta\,(-S_i \otimes -S_i) = \theta\,(S_i \otimes S_i) = (d-1) = \theta\,(S_i \otimes S_i),$$

and

$$\theta\,\tau\,(S_i \otimes S_i) = \theta\,[(-1)^d\, S_{i-1} \otimes (-1)^d\, S_{i-1}] = \theta\,(S_{i-1} \otimes S_{i-1}) = (d-1) = \theta\,(S_i \otimes S_i).$$

Case (2) when $i = j$, $i = 1$. By using Lemma 4.2.6, we have

$$\begin{aligned} \theta\,\tau(S_1 \otimes S_1) &= \theta\,(\sum_{i=1}^{d-1} (-1)^{d-i+1}\, S_i \otimes \sum_{j=1}^{d-1} (-1)^{d-j+1}\, S_j) \\ &= \theta\,(\sum_{j=1}^{d-1} \sum_{i=1}^{d-1} (-1)^{i+j}\, S_i \otimes S_j) \\ &= (d-1)\,(d-1) + \sum_{j=1}^{d-1} \sum_{\substack{i=1 \\ i \neq j}}^{d-1} (-1)^{i+j}\,(-1)^{i+j+1} \\ &= (d-1)^2 - [(d-1)\,(d-2)] \\ &= (d-1) = \theta(S_1 \otimes S_1). \end{aligned}$$

Case (3): $i = j$, $i = d-1$, and $g = \sigma$. The proof is similar, because the action of τ on S_1 is the same as the action of σ on S_{d-1}.
Case (4): $i \neq j$, and both $\neq 1, d-1$. Follow the same technique as in Case (1). $\square$

The partition $\lambda = (d-r,\, 1^r),\ 1 \leq\ r\ <\ d$ is usually called a hook, due to the appearance

$$\begin{array}{|c|c|c|} \hline & \cdots & \\ \hline \vdots & \multicolumn{2}{c}{} \\ \cline{1-1} & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

of its Young diagram. A basis element in $\mathcal{B}_{(d-r,1^r)}$ can be distinguished only by the sequence of numbers in its first column (second row onwards). For $p = (p_1, p_2, \ldots, p_r)$, let T_p denote the basis element whose first column is $(1, p_1, p_2, \ldots, p_r)$. e.g., $\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}$

will be denoted by $T_{3\,4\,6}$.

Lemma 5.2.3. *Let* $\lambda = (d-r,\, 1^r)$*, where* $d \geq 2$ *and* $1 \leq r < d$*. Let* $T_p \in \mathcal{B}_\lambda$*, where* $p = (p_1,\, p_2, \cdots, p_r)$*. Then*

$$\sigma\, T_p = \begin{cases} -T_p & \text{if } p_1 = 2, \\ T_p + \sum\limits_{i=1}^{r} (-1)^i\, T_{2\, p_1 \cdots \hat{p}_i \cdots p_r} & \text{otherwise;} \end{cases}$$

whereas,

$$\tau\, T_p = \begin{cases} (-1)^r\, T_{2\,(p_1+1)\cdots(p_{r-1}+1)} & \text{if } p_r = d, \\ T_{(p_1+1)\,(p_2+1)\cdots(p_r+1)} + \sum\limits_{i=1}^{r} (-1)^i\, T_{2\,(p_1+1)\cdots\widehat{(p_i+1)}\cdots(p_r+1)} & \text{otherwise.} \end{cases}$$

PROOF. Let first $T_p \in \mathcal{B}_\lambda$ such that $p_1 = 2$. That means

$$T_p = \begin{bmatrix} 1 & a_1 & \cdots & a_{d-r-1} \\ 2 & & & \\ p_2 & & & \\ \vdots & & & \\ p_r & & & \end{bmatrix}, \quad \text{where } a_i \in \{3, 4, \cdots, d\}. \text{ Then}$$

$$\sigma\, T_p = \begin{bmatrix} 2 & a_1 & \cdots & a_{d-r-1} \\ 1 & & & \\ p_2 & & & \\ \vdots & & & \\ p_r & & & \end{bmatrix} = - \begin{bmatrix} 1 & a_1 & \cdots & a_{d-r-1} \\ 2 & & & \\ p_2 & & & \\ \vdots & & & \\ p_r & & & \end{bmatrix} = -T_p.$$

However, if $p_1 \neq 2$ then

$$\sigma\, T_p = \sigma \begin{bmatrix} 1 & 2 & a_1 & \cdots & a_{d-r-2} \\ p_1 & & & & \\ \vdots & & & & \\ p_r & & & & \end{bmatrix} = \begin{bmatrix} 2 & 1 & a_1 & \cdots & a_{d-r-2} \\ p_1 & & & & \\ \vdots & & & & \\ p_r & & & & \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & a_1 & \cdots & a_{d-r-2} \\ p_1 & & & & \\ \vdots & & & & \\ p_r & & & & \end{bmatrix} + \begin{bmatrix} 2 & p_1 & a_1 & \cdots & a_{d-r-2} \\ 1 & & & & \\ \vdots & & & & \\ p_r & & & & \end{bmatrix} +$$

$$\cdots + \begin{bmatrix} 2 & p_r & a_1 & \cdots & a_{d-r-2} \\ p_1 & & & & \\ \vdots & & & & \\ 1 & & & & \end{bmatrix},$$

$$= T_p + (-1)\, T_{2\, p_2 \cdots p_r} + (-1)^2\, T_{2\, p_1\, p_3 \cdots p_r} + \cdots + (-1)^r\, T_{2\, p_1\, p_2 \cdots p_r}$$

$$= T_p + \sum_{i=1}^{r} (-1)^i\, T_{2\, p_1 \cdots \widehat{p_i} \cdots p_r}.$$

Now, let $T_p \in \mathcal{B}_\lambda$ be such that $p_r = d$. Then

$$\tau\, T_p = \tau \begin{bmatrix} 1 & a_1 & \cdots & a_{d-r-1} \\ p_1 & & & \\ \vdots & & & \\ p_{r-1} & & & \\ d & & & \end{bmatrix} = \begin{bmatrix} 2 & (a_1+1) & \cdots & (a_{d-r-1}+1) \\ (p_1+1) & & & \\ \vdots & & & \\ (p_{r-1}+1) & & & \\ 1 & & & \end{bmatrix},$$

and if $a_i \in \{2, 3, \cdots, d-1\}$, then $\tau\, T_p = (-1)^r\, T_{2\,(p_1+1)\cdots(p_{r-1}+1)}$.
Finally, when $p_r \neq d$, the last number in the first row is equal to d. So, we have

$$\tau\, T_p = \tau \begin{bmatrix} 1 & \cdots & d \\ p_1 & & \\ \vdots & & \\ p_r & & \end{bmatrix} = \begin{bmatrix} 2 & \cdots & 1 \\ (p_1+1) & & \\ \vdots & & \\ (p_r+1) & & \end{bmatrix} \text{(by straightening)},$$

$$= T_{(p_1+1)\,(p_2+1)\cdots(p_r+1)} + (-1)\, T_{2\,\widehat{(p_1+1)}\cdots(p_r+1)} + (-1)^2\, T_{2\,(p_1+1)\,\widehat{(p_2+1)}\cdots(p_r+1)} +$$

$$\cdots + (-1)^r\, T_{2\,(p_1+1)\,(p_2+1)\cdots\widehat{(p_r+1)}}.$$

This ends the proof. □

Notice that this is a generalization of lemma 4.2.2. The next example explains the action of $\mathfrak{S}_5$ on $\mathcal{B}_{(3,1,1)}$.

Example 5.2.4. *Note that $\mathcal{B}_{(3,1,1)}$ is the ordered set*

$$\left\{\begin{bmatrix} 1 & 4 & 5 \\ 2 & & \\ 3 & & \end{bmatrix}, \begin{bmatrix} 1 & 3 & 5 \\ 2 & & \\ 4 & & \end{bmatrix}, \begin{bmatrix} 1 & 2 & 5 \\ 3 & & \\ 4 & & \end{bmatrix}, \begin{bmatrix} 1 & 3 & 4 \\ 2 & & \\ 5 & & \end{bmatrix}, \begin{bmatrix} 1 & 2 & 4 \\ 3 & & \\ 5 & & \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 4 & & \\ 5 & & \end{bmatrix}\right\}$$
$$= \{T_{2\,3}, T_{2\,4}, T_{3\,4}, T_{2\,5}, T_{3\,5}, T_{4\,5}\}.$$

Then the action of σ on $\mathcal{B}_{(3,1,1)}$ is given by

$$A_\sigma^{(3,1,1)} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Also, the action of $\tau = (1\,2 \cdots 5)$ on $\mathcal{B}_{(3,1,1)}$ is given by

$$A_\tau^{(3,1,1)} = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The next result concerns the partition $\lambda = (d-2, 1^2)$. Given two basis elements $T_{p_1\,p_2}, T_{q_1\,q_2} \in \mathcal{B}_\lambda$, let m be the number of indices i such that $p_i = q_i$. Moreover, define n to be 1 if $p_1 = q_2$ or $q_1 = p_2$, and 0 otherwise. For instance, for the pairs $(2,3), (3,4)$ we have $m = 0, n = 1$. Similarly, for $(2,4), (2,7)$, we have $m = 1, n = 0$.

Theorem 5.2.5. *Assume $d \geq 4$. Then the q-form $\theta : V_{(d-2,1,1)} \otimes V_{(d-2,1,1)} \longrightarrow \mathbb{C}$ is defined by*

$$\theta\left(T_{p_1\,p_2} \otimes T_{q_1\,q_2}\right) = \begin{cases} 3 & \textit{when } m = 2, \\ (-1)^n & \textit{when } m + n = 1, \\ 0 & \textit{when } m = n = 0. \end{cases}$$

PROOF. We have to show that $\theta = \theta\, g$ for every $g = \sigma, \tau$. Let $T_{p_1\, p_2} \otimes T_{q_1\, q_2}$ be an element in the basis of $V_{(d-2,1,1)} \otimes V_{(d-2,1,1)}$. We have the following cases.

(I) Assume $g = \sigma$.

1. $m = 2$. This means $p_i = q_i,\ \forall\, i = 1, 2$. To show that $\theta\, \sigma\, (T_{p_1\, p_2} \otimes T_{p_1\, p_2}) = 3$, we have to look at the following subcases:

   - $p_1 = 2$, then $\theta\, \sigma\, (T_{2\, p_2} \otimes T_{2\, p_2}) = \theta\, (-T_{2\, p_2} \otimes\ -T_{2\, p_2}) = 3.$

   - $p_1 \neq 2$, then

$$\begin{aligned}
\theta\, \sigma\, (T_{p_1\, p_2} \otimes T_{p_1\, p_2}) &= \theta\, (\, (T_{p_1\, p_2} + (-1) T_{2\, p_2} + T_{2\, p_1}) \otimes (T_{p_1\, p_2} + (-1) T_{2\, p_2} + T_{2\, p_1})\,) \\
&= \theta\, (\, T_{p_1\, p_2} \otimes T_{p_1\, p_2} - T_{p_1\, p_2} \otimes T_{2\, p_2} + T_{p_1\, p_2} \otimes T_{2\, p_1} \\
&\quad - T_{2\, p_2} \otimes T_{p_1\, p_2} + T_{2\, p_2} \otimes T_{2\, p_2} - T_{2\, p_2} \otimes T_{2\, p_1} \\
&\quad + T_{2\, p_1} \otimes T_{p_1\, p_2} - T_{2\, p_1} \otimes T_{2\, p_2} + T_{2\, p_1} \otimes T_{2\, p_1}\,) \\
&= 3 - 1 + (-1) - 1 + 3 - 1 + (-1) - 1 + 3 = 3.
\end{aligned}$$

2. $m + n = 1$. In this situation we have the following subcases:

   - $p_1 = q_1 = 2$, that means $m = 1$ and $n = 0$. Then

$$\theta\, (T_{2\, p_2} \otimes T_{2\, q_2}) = (-1)^0 = 1.$$

     The other side is $\theta\, \sigma\, (T_{2\, p_2} \otimes T_{2\, q_2}) = \theta\, (-T_{2\, p_2} \otimes\ -T_{2\, q_2}\,) = 1.$

   - $p_1 = 2$, (or $q_1 = 2$), and $p_1 \neq q_1, p_2 = q_2$. That means $m = 1$ and $n = 0$, so we have

$$\theta(T_{2\, p_2} \otimes T_{q_1\, p_2}) = 1.$$

     Now, the other side will be

$$\begin{aligned}
\theta\, \sigma\, (T_{2\, p_2} \otimes T_{q_1\, p_2}) &= \theta\, (\, -T_{2\, p_2} \otimes (T_{q_1\, p_2} - T_{2\, p_2} + T_{2\, q_1}\,)\,) \\
&= \theta\, (\, -T_{2\, p_2} \otimes T_{q_1\, p_2} + T_{2\, p_2} \otimes T_{2\, p_2} - T_{2\, p_2} \otimes T_{2\, q_1}) \\
&= -1 + 3 - 1 = 1.
\end{aligned}$$

   - $p_1 = q_1 \neq 2$. Then $p_2 \neq q_2$, which implies that $m = 1$ and $n = 0$, which is similar to the second subcase in part (2) above.

   - $p_1 \neq q_1$ and none of them is 2, but $p_2 = q_2$. This case is the same as the third case in part (2) above.

- The last possible subcase is $m = 0$ and $n = 1$. That means $p_1 = q_2$, (or $p_2 = q_1$). In this case,

$$\theta(T_{p_1\, p_2} \otimes T_{q_1\, p_1}) = -1.$$

Then

$$\begin{aligned}\theta\,\sigma\,(T_{p_1\, p_2} \otimes T_{q_1\, p_1}) &= \theta\,(\,(T_{p_1\, p_2} - T_{2\, p_2} + T_{2\, p_1}) \otimes (T_{q_1\, p_1} - T_{2\, p_1} + T_{2\, q_1})\,) \\ &= \theta\,(\,T_{p_1\, p_2} \otimes T_{q_1\, p_1} - T_{p_1\, p_2} \otimes T_{2\, p_1} + T_{p_1\, p_2} \otimes T_{2\, q_1} \\ &\quad - T_{2\, p_2} \otimes T_{q_1\, p_1} + T_{2\, p_2} \otimes T_{2\, p_1} - T_{2\, p_2} \otimes T_{2\, q_1} \\ &\quad + T_{2\, p_1} \otimes T_{q_1\, p_1} - T_{2\, p_1} \otimes T_{2\, p_1} + T_{2\, p_1} \otimes T_{2\, q_1}\,) \\ &= -1 - (-1) + 0 - 0 + 1 - 1 + 1 - 3 + 1 = -1.\end{aligned}$$

3. $m = 0$ and $n = 0$, that means p_1, p_2, q_1, q_2 are all pairwise different numbers. So, we have

$$\theta\,(T_{p_1\, p_2} \otimes T_{q_1\, q_2}) = 0.$$

However, for the other side we have to check the following possibilities:

- $p_1 = 2$, (or $q_1 = 2$), in this case

$$\begin{aligned}\theta\,\sigma(T_{2\, p_2} \otimes T_{q_1\, q_2}) &= \theta\,(-T_{2\, p_2} \otimes (T_{q_1\, q_2} - T_{2\, q_2} + T_{2\, q_1})\,) \\ &= \theta\,(\,-T_{2\, p_2} \otimes T_{q_1\, q_2} + T_{2\, p_2} \otimes T_{2\, q_2} - T_{2\, p_2} \otimes T_{2\, q_1}) \\ &= 0.\end{aligned}$$

- When none of p_1, q_1 is 2, we have

$$\theta\,\sigma(T_{p_1\, p_2} \otimes T_{q_1\, q_2}) = \theta\,(\,(T_{p_1 p_2} - T_{2\, p_2} + T_{2\, p_1}) \otimes (T_{q_1\, q_2} - T_{2\, q_2} + T_{2\, q_1})\,) = 0.$$

(II) Assume $g = \tau$. When $r = 2$, then by Lemma 5.2.3, we have:

$$\tau\, T_p = \begin{cases} T_{2\,(p_1+1)} & \text{if } p_2 = d, \\ T_{(p_1+1)\,(p_2+1)} - T_{2\,(p_2+1)} + T_{2\,(p_1+1)} & \text{otherwise.} \end{cases}$$

1. $m = 2$ then $\theta\,(T_p \otimes T_p) = 3$. For the other side we distinguish the cases $p_2 = d$ and $p_2 \neq d$, and the discussion follows part (1) above.

2. When $m + n = 1$, we have exactly the same cases as above, namely :

- $p_2 = q_2 = d$, that means $m = 1$ and $n = 0$,

- $p_2 = d$, (or $q_2 = d$), and $p_2 \neq q_2, p_1 = q_1$. That means $m = 1$ and $n = 0$.
- $p_2 = q_2 \neq d$. i.e. $p_1 \neq q_1$.
- $p_2 \neq q_2$ and none of them is d, but $p_1 = q_1$.
- $m = 0$ and $n = 1$. That means $p_1 = q_2$, (or $p_2 = q_1$), and none of them is d.

In all those cases the proof follows part (2) above.

3. $m = 0$ and $n = 0$, that means p_1, p_2, q_1, q_2 are all pairwise different numbers. So, we have

$$\theta\left(T_p \otimes T_q\right) = 0.$$

For the other side we have the following cases

- $p_2 = d$, (or $q_2 = d$),
- none of p_2, q_2 is d,

and in each subcase the proof follows part (3) above. □

5.3 A conjecture for the hook diagram

In this section we conjecture a formula for θ_λ when $\lambda = (d-r, 1^r)$ and $2 \leq r \leq d-2$.

Let $p = (p_1, p_2, \ldots, p_r)$ and $q = (q_1, q_2, \ldots, q_r)$ denote two increasing sequences such that $T_p, T_q \in \mathcal{B}_{(d-r,1^r)}$. Define m to be the number of indices i such that $p_i = q_i$. On the other hand, let n denote the number of ordered pairs (i, j) such that

$$i \neq j, \quad p_i = q_j.$$

For example, if

$$p = (2, 4, 6, 8, 9), \quad q = (3, 4, 5, 6, 8),$$

then $m = 1, n = 2$.

Conjecture 5.3.1. *The q-form $\theta : V_{(d-r,1^r)} \otimes V_{(d-r,1^r)} \longrightarrow \mathbb{C}$ is defined by*

$$\theta\left(T_p \otimes T_q\right) = \begin{cases} (r+1) & \text{when } m = r, \\ (-1)^n & \text{when } m \neq r \text{ and } m + n = r - 1, \\ 0 & \text{when } m + n < r - 1. \end{cases}$$

This conjecture would generalize Theorem 5.2.5. In future we will try to find a proof of this conjecture, as well as find similar formulae for other partitions λ.

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