UNIVERSITY OF MANITOBA

Yang-Mills Flow In 1+1 Dimensions Coupled With A Scalar Field

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Abstract

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We define a Yang-Mills model in 1+1 dimensions coupled to a real scalar field and we study the Yang-Mills flow equations for this simple model. Yang-Mills flows have not been thoroughly studied, especially in a physical context, but may be able to provide valuable insight into both particle physics as well as gravity. We study our model using both the Hamiltonian equations and Euler-Lagrange equations, and we calculate the flow numerically using a simple finite difference method for the case of an Abelian Lie group and static fields. We are able to find several analytic solutions to the equations of motion and the numerical calculation of the flow suggests most non-constant solutions are unstable. We also find that the flow depends upon the relative values of the coupling constant and the mass of the scalar field. The results found with this simple model provide a starting point for the study of Yang-Mills flow in the context of more complicated (but more physical) models such as the Abelian Higgs.

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Introduction

1.1 Gauge Field Theories

Yang-Mills theory began as a generalization of Abelian gauge field theories, such as those used to describe electromagnetism, to non-Abelian symmetry groups. We start with the vector potential (or the Yang-Mills connection) A^μ, whose components belong to an n-dimensional non-Abelian Lie algebra. That is, we can write

$$A^\mu = \sum_{a=1}^{n} A_a^\mu T_a \tag{1.1}$$

using Greek letters to label space-time components, and Roman letters for the algebra, where the T_a are the generators of the algebra which is defined by the commutation relation

$$[T_a, T_b] = \imath f_{abc} T_c. \tag{1.2}$$

The f_{abc}, called the structure constants of the algebra, are real valued, anti-symmetric in all of their indices, and can be (in the case of a compact Lie Algebra) chosen such that

$$Tr(T_a T_b) = \delta_{ab}. \tag{1.3}$$

We will use the convention that repeated indices are summed over, however, unlike space-time indices where indices must be raised or lowered by the metric before summing, the Lie algebra indices will always be on the bottom. More information on Lie algebras can be found in for example [1].

The Yang-Mills connection is used to define the covariant derivative

$$D_\mu = \partial_\mu \ + \imath q[A_\mu, \ ] \tag{1.4}$$

where q is a coupling constant, and now we can obtain the curvature

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \imath q[A_\mu, A_\nu] \tag{1.5}$$

We define the Yang-Mills action as a functional of the connection A^μ:

$$S[A^\mu] = -\int dx^n \ \frac{\sqrt{|g|}}{4} \, \mathrm{Tr}(F^{\mu\nu} F_{\mu\nu}) \tag{1.6}$$

where g denotes the determinant of the metric. The Yang-Mills action has, with various choices of the gauge group, been used to describe electromagnetism, weak interactions, and also quantum chromodynamics. For example, electromagnetism corresponds to an Abelian gauge group, and $F^{\mu\nu}$ is the electromagnetic field strength tensor in four dimensions, with

$$E_i = F^{i0} \qquad B_i = -\frac{1}{2}\epsilon_{ijk} F^{jk} \tag{1.7}$$

The action (1.6) in this case becomes:

$$S[A^\mu] = \int dx^n \; \frac{1}{2} \left(B^2 - E^2 \right) \tag{1.8}$$

Which is the usual action of classical electromagnetism. To obtain the action for something like chromodynamics however, one needs to include a non trivial Lie algebra, in this case SU(3), which has eight different components.

1.1.1 Mass Generation in Yang-Mills

One of the early difficulties with Yang-Mills theory was that if we include the typical mass terms for the gauge fields of the form $m^2 A^\mu A_\mu$, we find that for a gauge transformation with an arbitrary function, f,

$$A^\mu \to A^\mu + \partial^\mu f \tag{1.9}$$

$$m^2 A^\mu A_\mu \to m^2 A^\mu A_\mu + m^2 \left(\partial^\mu f + 2A^\mu \right) \partial_\mu f \tag{1.10}$$

the action is no longer gauge invariant for any non-zero mass m. As a result, we need to generate mass for the gauge particles in some other way. One such mechanism for generating mass is by coupling the gauge fields to a complex scalar field with a potential that allows for symmetry breaking such as the Mexican-hat potential. The additional terms in the Lagrangian due to the scalar field,

$$(D^\mu \phi)(D_\mu \phi)^\dagger + \left(m^2 - \lambda \phi^2 \right) \phi^2, \tag{1.11}$$

are gauge invariant and the spontaneous symmetry breaking of the potential gives rise to a mass term in the equations of motion for the gauge fields. This is the basis of the Higgs mechanism in the standard model of particle physics.

We can also generate mass by introducing 'topological' terms that do not depend on the metric to the action. The most notable example being Chern-Simons theory in three dimensions, named for the topological term known as the Chern-Simons density,

$$\kappa\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(\frac{1}{2}A_\mu\partial_\nu A_\rho+\frac{1}{3}A_\mu A_\nu A_\rho\right), \tag{1.12}$$

where κ is a constant and Tr is just the usual matrix trace. Including the Chern-Simons density in our action gives rise to massive gauge fields [2].

In two space-time dimensions we can consider the Schwinger model, where the gauge field is coupled with massless Dirac fields ψ whose contribution to the Lagrangian is given by

$$\imath\bar{\psi}\gamma_\mu\left(\partial^\mu+\imath qA^\mu\right)\psi. \tag{1.13}$$

In two dimensions the gamma matrices, γ_μ, can be written in terms of the familiar Pauli matrices:

$$\gamma_0=\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{1.14}$$

$$\gamma_1=-\imath\sigma_2=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{1.15}$$

From this we can find the γ^5 matrix,

$$\gamma^5=\gamma^0\gamma^1=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}=-\sigma_3. \tag{1.16}$$

The resulting Lagrangian density can be reformulated in terms of the (two dimensional) Chern-Pontryagin density,

$$\mathcal{P}_2=\frac{1}{2}\epsilon^{\mu\nu}F_{\mu\nu}, \tag{1.17}$$

and the Chern-Simons current,

$$\mathcal{C}^{\alpha} = \epsilon^{\alpha\mu} A_{\mu}, \tag{1.18}$$

Along with the axial vector current,

$$\mathcal{J}^5_{\alpha} = \epsilon_{\alpha\mu}\bar{\psi}\gamma^{\mu}\psi, \tag{1.19}$$

to obtain

$$\mathcal{L} = \frac{1}{2}\mathcal{P}_2^2 + q\mathcal{C}^{\alpha}\mathcal{J}^5_{\alpha}. \tag{1.20}$$

This theory also leads to the generation of massive fields without breaking the gauge invariance, and can be generalized into higher dimensions using the natural generalizations of $\mathcal{P}, \mathcal{C}$ and $\mathcal{J}$ [3, 4]. This generated mass can be viewed in terms of a scalar field η related to the vector current by

$$\partial_{\alpha}\eta = \mathcal{J}^5_{\alpha}, \tag{1.21}$$

and if we include a dynamical term for η we can obtain a Lagrangian for a generalized version of the Schwinger model in two dimensions similar to the hybrid model in [3] given by

$$\mathcal{L} = \frac{1}{2}\mathcal{P}_2^2 - q\eta\mathcal{P}_2 + \frac{1}{2}\partial_{\mu}\eta\partial^{\mu}\eta, \tag{1.22}$$

where we have used the fact that $\partial_{\alpha}\mathcal{C}^{\alpha} = \mathcal{P}_2$ and ignored a total derivative term.

1.1.2 Yang-Mills and Gravity

In addition to being used in the standard model to describe particle physics, field theories can also be used to study theories of gravity. This has been studied mainly using two different approaches, the first one being the anti-deSitter (AdS) / conformal field theory (CFT) correspondence [5]. The first formulation of the AdS/CFT correspondence showed that a string-gravity theory in five dimensional anti-deSitter space is equivalent to a four dimensional Yang-Mills field theory defined on the boundary of the space. This equivalence has since been extended to a number of different cases where a gravity theory in some n dimensional bulk space is equivalent to a field theory defined in $n-1$ dimensions on the boundary of the bulk space. The major benefit of the AdS/CFT correspondence is that weakly coupled interactions in one theory (which tend to be easier to solve), are equivalent to strongly coupled interactions in the other theory (which tend to be more difficult).

The AdS/CFT correspondence allows us to look at the Yang-Mills flow (which will be introduced in more detail in the next section) in the context of a gravity theory as well as the field theory in which it is originally defined. Through the AdS/CFT correspondence, we can compare the Yang-Mills flow directly to flows defined in the gravity theory on the bulk space such as Ricci flow which has been more extensively studied.

The other main approach to studying gravity through gauge-field theories is by viewing the usual Christoffel connection of general relativity as a gauge connection. In this context the Chern-Simons term (1.12), without being coupled with the Yang-Mills term, is equivalent to the Einstein-Hilbert term in general relativity as discussed in [6]. So in this way we have an equivalence between a gauge theory and a gravitational theory in three dimensions, which further motivates our study of the Yang-Mills flow equations.

1.2 Yang-Mills Flow

The Yang-Mills flow was first introduced by Atiyah and Bott [7] as the gradient flow of the Yang-Mills action on Riemannian manifolds in order to study Riemann surfaces. Given the Yang-Mills action (1.6) we define the Yang-Mills flow with respect to a flow parameter τ as:

$$\frac{\partial A^\mu}{\partial \tau} = D_\nu(\sqrt{|g|}F^{\mu\nu}) + \partial^\mu \chi. \tag{1.23}$$

Since A^ν and its flow are only determined up to a choice of gauge, we include a 'deTurck' term $\partial^\mu \chi$ in order to ensure our initial gauge choice is preserved under the flow.

For a more complicated action, we can consider its gradient flow as a generalization of the Yang-Mills flow equations. Consider, for example, a Yang-Mills theory coupled with a scalar field ϕ:

$$S[A^\mu, \phi] = \int dx^n \mathcal{L}[A^\mu, \phi] \tag{1.24}$$

We define the gradient flow of the action in terms of flow equations for each of our fields as

$$\frac{\partial \phi}{\partial \tau} = \frac{\delta \mathcal{L}[A^\mu, \phi]}{\delta \phi} + X, \tag{1.25}$$

and

$$\frac{\partial A^\mu}{\partial \tau} = \frac{\delta \mathcal{L}[A^\mu, \Phi_a]}{\delta A^\mu} + \partial^\mu \chi. \tag{1.26}$$

Here X denotes any additional deTurck terms we may need to include to ensure the flow equations are gauge invariant. The deTurck terms will depend on how the ϕ field changes under a gauge transformation, in a case such as the Higgs (1.11) the transformation is related to the transformation of A^μ and the deTurck terms will be related. In other cases, such as for the scalar field η in (1.22), the scalar field is gauge invariant, and no additional terms are needed.

Yang-Mills flow was initially introduced on Riemannian manifolds and although nothing in the definition requires this to be the case, there are some advantages to working with a Riemannian background metric. For a Riemannian manifold, the inner product defined by the metric is positive definite, as is the inner product in our Lie algebra (1.3). Combining these, we can define a positive definite inner product for two gauge vectors as

$$A_a^\mu \cdot B_b^\nu = \delta_{ab} A_a^\mu g_{\mu\nu} B_b^\nu. \tag{1.27}$$

Now we can consider how the action itself changes with respect to our flow parameter τ,

$$\frac{dS}{d\tau} = \int dx^n \left[\frac{\delta \mathcal{L}[A^\mu, \phi]}{\delta A^\mu} \cdot \frac{\partial A^\mu}{\partial \tau} + \frac{\delta \mathcal{L}[A^\mu, \phi]}{\delta \phi} \cdot \frac{\partial \phi}{\partial \tau} \right]. \tag{1.28}$$

Since the action is gauge invariant, including a deTurck term in our flow equations will not alter the flow of the action. Now we can see that, since the inner product is positive definite, equations (1.25) and (1.26) mean that the action will be a strictly increasing function of the flow parameter τ. However, in physics, the manifolds of interest are Lorentzian manifolds so we no longer have a positive definite metric, and we can no longer predict the behaviour of the action as the flow progresses. In flat space-times (i.e. Minkowski space) we can avoid this difficulty by performing a Wick rotation (i.e. making the substitution $x^0 \to \imath x^0$) to transform the Lorentzian manifold into a Riemannian one.

Since the Yang-Mills flow is a gradient flow, we can expect it to behave in a way similar to other gradient flows, such as the heat equation from classical physics. The similarity between Yang-Mills flow and heat flow becomes obvious when we consider Abelian Yang-Mills with a flat background metric. We can rewrite the flow equation for A^μ as an equation for the gauge invariant quantity $F^{\mu\nu}$ as

$$\frac{\partial F^{\mu\nu}}{\partial \tau} = -\partial^\alpha \partial_\alpha F^{\mu\nu}. \tag{1.29}$$

If we perform a Wick rotation into Euclidean space, this gives a heat equation for each component of $F^{\mu\nu}$. Another flow equation similar to the heat equation is the Ricci flow which is defined on the metric in general relativity, given by:

$$\frac{\partial g_{\mu\nu}}{\partial \tau} = -2R_{\mu\nu}, \tag{1.30}$$

where $R_{\mu\nu}$ is the Ricci tensor. The Ricci flow has been much more extensively studied than the Yang-Mills flow and it may be useful to compare the two flows. Two promising areas where we can attempt such a comparison were outlined in the previous section, the AdS/CFT correspondence, and the equivalence between the Chern-Simons action and the Einstein-Hilbert action.

In this thesis we consider Yang-Mills flows for a particular model of a Yang-Mills action coupled with a scalar field in two dimensions. Of particular relevance is the work of Rade [8] which establishes that of the Yang-Mills flow in two and three dimensions converges as $\tau \to \infty$, as well as estimates the rate at which the flow converges. Rade's work deals with pure Yang-Mills flow, however, and it is not clear that the results will still hold for Yang-Mills coupled with a scalar field.

1.2.1 Current Applications of Yang-Mills Flow

In lattice QCD the Yang-Mills flow (or Wilson flow), is used to study non-Abelian gauge field theories without the limitation that the coupling be small as in perturbation theory. In this context, the Yang-Mills flow has been shown to renormalize the gauge fields, see [9] and references therein. Presently, they do not consider any modifications to the Yang-Mills flow due to the inclusion of additional fields in the action.

Yang-Mills flow has also be studied in the context of general relativity where, in three dimensions, gravity can be formulated in terms of the Yang-Mills connection. In this context the Yang-Mills flow can be compared directly to Ricci flow [10, 11]. Since the

Yang-Mills action does depend on the metric, coupling the Yang-Mills flow to the Ricci flow directly has also been considered by Streets [12] and some of the references within.

In this thesis we apply the flow to a two dimensional Yang-Mills model coupled with a scalar field through a topological term. We define the flow as the gradient flow of the new action which leads to scalar field terms in the flow of the gauge fields in addition to the gauge field terms present in the scalar field flow equations.

Chapter 2

A 1+1 Dimensional Model

We start with the following action:

$$S[A,\phi] = \int dx^2 \left[\mathrm{Tr}\left(-\frac{\sqrt{|g|}}{4} F^{\mu\nu} F_{\mu\nu} \,+\, \frac{\sqrt{|g|}}{2} D^{\mu}\phi D_{\mu}\phi \,+\, \beta\epsilon^{\mu\nu} F_{\mu\nu}\phi \right) - V(|\phi|) \right] \quad (2.1)$$

- β is a constant parameter
- g is the determinant of a fixed background metric $g_{\mu\nu}$ whose components depend on x^{μ} and that we take to have a Lorentzian (+,-) signature so that $|g| = -g$.
- We consider an n dimensional compact Lie algebra with generators T_a (a goes from 1 to n):
  - $A_{\mu}(x) = \sum_{a=1}^{n} A_{\mu\, a}(x) T_a$ is the Yang-Mills potential
  - $\phi(x) = \sum_{a=1}^{n} \phi_a(x) T_a$ is a real scalar field
- $V(|\phi|)$ is a potential function that depends on $\mathrm{Tr}(\phi\phi)$

2.1 The Equations of Motion

In order to find the equations of motion for our system, we extremize the action with respect to variations of $A_{\mu\,a}$ and ϕ_a. We find that:

$$\delta S = \int dx^2 \left[-\frac{\sqrt{|g|}}{2} F_a^{\mu\nu} \delta F_{\mu\nu\,a} + \sqrt{|g|} (D^\mu \phi)_a \, \delta (D_\mu \phi)_a + \beta \epsilon^{\mu\nu} \delta F_{\mu\nu\,a} \phi_a \right.$$
$$\left. + \beta \epsilon^{\mu\nu} F_{\mu\nu\,a} \delta\phi_a - V'(\phi_a) \delta\phi_a \right], \quad (2.2)$$

where we sum over repeated Lie Algebra indices, and we define:

$$V'(\phi_a) = \frac{\partial V}{\partial \phi_a}. \quad (2.3)$$

We can express the variations of $F_{\mu\nu}$ and $D_\mu \phi$ in terms of variations in A_μ and ϕ in the following way:

$$\delta F_{\mu\nu} = D_\mu(\delta A_\nu) - D_\nu(\delta A_\mu) \quad (2.4)$$

$$\delta(D_\mu \phi) = D_\mu(\delta\phi) + \imath q [\delta A_\mu, \phi] \quad (2.5)$$

The variation of the action with respect to arbitrary variations $\delta A_{\mu\,a}$ and $\delta\phi_a$ can be expressed as:

$$\delta S = \int dx^2 \left[(D_\mu(-\sqrt{|g|} F^{\mu\nu}))_a + \imath q \sqrt{|g|} [D^\nu \phi, \phi]_a + 2\beta \epsilon^{\mu\nu} (D_\mu \phi)_a \right] \delta A_{\nu\,a}$$
$$+ \left[-(D_\mu(\sqrt{|g|} D^\mu \phi))_a + \beta \epsilon^{\mu\nu} F_{\mu\nu\,a} - V'(\phi_a) \right] \delta\phi_a \quad (2.6)$$

The extrema of the action are found by taking $\delta S = 0$ which leads to the following equations of motion for A^ν and ϕ:

$$D_\mu(-\sqrt{|g|}F^{\mu\nu} + 2\beta\epsilon^{\mu\nu}\phi) + \imath q\sqrt{|g|}[D^\nu\phi, \phi] = 0 \tag{2.7}$$

$$\beta\epsilon^{\mu\nu}F_{\mu\nu} - D_\mu(\sqrt{|g|}D^\mu\phi) - V'(\phi) = 0 \tag{2.8}$$

Here, although we have left off the Lie algebra indices, we have $2n$ equations for A^ν_a and n equations for ϕ_a, that are coupled through the Lie bracket appearing in (2.7). If we assume our background metric is flat (Minkowski) space-time, then $|g| = 1$ and our equations of motion become:

$$D_\mu(-F^{\mu\nu} + 2\beta\epsilon^{\mu\nu}\phi) + \imath q[D^\nu\phi, \phi] = 0 \tag{2.9}$$

$$\beta\epsilon^{\mu\nu}F_{\mu\nu} - D_\mu(D^\mu\phi) - V'(\phi) = 0 \tag{2.10}$$

2.2 Yang-Mills Flow Equations

The Yang-Mills Flow equations can be defined for this model as in (1.25) and (1.26) so that solutions to the equations of motion are the stationary points of the flow. Although we do not work with a Riemannian metric explicitly, if we perform a Wick rotation into Euclidean space, the resulting flow equation is a heat equation in 2+1 dimensions. The resulting system of partial differential equations is as follows:

$$\frac{\partial A^\nu}{\partial \tau} = -D_\mu(\sqrt{|g|}F^{\mu\nu} - 2\beta\epsilon^{\mu\nu}\phi) + \imath q\sqrt{|g|}[D^\nu(\phi), \phi] + \partial^\nu\chi \tag{2.11}$$

$$\frac{\partial \phi}{\partial \tau} = -D_\mu(\sqrt{|g|}D^\mu\phi) + \beta\epsilon^{\mu\nu}F_{\mu\nu} - V'(\phi) \tag{2.12}$$

2.2.1 The Hessian of the Action

The stability of the stationary points of the Yang-Mills flow is determined by the stability of the corresponding solutions to the equations of motion, and the stability of these solutions under small perturbations is determined by the Hessian of the action which is defined as the matrix of second functional derivatives of the action. The Hessian is an infinite dimensional matrix operator with respect to the space-time variable x^μ, that can be viewed as a 3×3 matrix acting on the fields $A^\nu(x^\mu)$ and $\phi(x^\mu)$. Each of these entries are $n \times n$ matrices that act on the components of the fields in the Lie algebra.

We will denote the Hessian $\hat{H}$ to avoid confusion with the Hamiltonian H which we define later. In order to calculate the Hessian we need to compute the second functional derivative of the action. We start by writing (2.1) as:

$$\delta S = \int dx^2 \; \left[\delta A_{\nu\, a} S^\nu_{A\, a} + \delta\phi_a S_{\phi\, a}\right]. \tag{2.13}$$

We now vary $S^\nu_{A\,a}$ and $S_{\phi\, a}$ with respect to $A_{\mu\, b}$ and ϕ_b to obtain

$$\delta^2 S = \int dx^2 \; \left[\delta A_{\nu\, a}(\hat{H}^{\nu\mu}_{ab})\delta A_{\mu\, b} + \delta A_{\nu\, a}(\hat{H}^{\nu\phi}_{ab})\delta\phi_b + \delta\phi_a(\hat{H}^{\phi\phi}_{ab})\delta\phi_b + \delta\phi_a(\hat{H}^{\nu\phi}_{ab})\delta A_{\nu\, b}\right]. \tag{2.14}$$

Each component of the Hessian is given by:

$$\hat{H}^{\phi\,\phi} = - D_\alpha\sqrt{|g|}D^\alpha - V''(\phi)\mathbf{I}, \tag{2.15}$$

$$\hat{H}^{\phi\mu} = - 2\beta\epsilon^{\mu\alpha}D_\alpha + q\sqrt{|g|}\phi D_\alpha g^{\alpha\mu} + qD^\mu(\sqrt{|g|}\phi) + q\sqrt{|g|}D^\mu(\phi), \tag{2.16}$$

$$\hat{H}^{\mu,\phi} = - q\sqrt{|g|}D^\mu(\phi) + q\sqrt{|g|}\phi D^\mu + 2\beta\epsilon^{\mu\alpha}D_\alpha, \tag{2.17}$$

$$\hat{H}^{\nu\mu} = D_\alpha \sqrt{|g|} D^\alpha g^{\nu\mu} - D_\alpha D^\nu g^{\alpha\mu} + qF^{\nu\mu} - q\phi^2 g^{\nu\mu} + 2\beta q \epsilon^{\nu\mu}\phi. \tag{2.18}$$

Here we have represented ϕ, and A^μ as $n \times n$ matrices in the adjoint representation, $\mathbf{I}$ is the $n \times n$ identity matrix, and D_μ is a matrix operator defined as

$$D_\mu = \mathbf{I}\partial_\mu + A_\mu, \tag{2.19}$$

and unless indicated by brackets, D acts on everything to its right. In particular, if we consider an Abelian gauge group and a flat background metric (with (+,-) signature) we can write the Hessian as a 3×3 matrix operator:

$$\hat{H} = \begin{pmatrix} -\partial_\alpha \partial^\alpha - V''(\phi) & -2\beta\partial_1 & +2\beta\partial_0 \\ +2\beta\partial_1 & \partial_1\partial^1 & \partial_1\partial^0 \\ -2\beta\partial_0 & -\partial_0\partial^1 & \partial_0\partial^0 \end{pmatrix}. \tag{2.20}$$

The Hessian acts on a perturbation of our original fields δW_b given by

$$\delta W_b = \begin{pmatrix} \delta\phi_b \\ \delta A_{0\,b} \\ \delta A_{1\,b} \end{pmatrix}, \tag{2.21}$$

which defines a perturbation for each Lie algebra component of each of our fields. In order to determine the stability of a solution using the Hessian, we must look at the eigenvalues of $\hat{H}$ at the solution. A solution will be stable under perturbations that are linear combinations of eigenfunctions of $\hat{H}$ that correspond to negative eigenvalues, and unstable for positive eigenvalues. More generally, when $\hat{H}$ acts on a perturbation, δW, we can find

$$\delta W^T \hat{H} \delta W = c|\delta W|^2, \tag{2.22}$$

and the sign of c indicates the stability of the solution under the given perturbation. In general, although we consider only small perturbations, we can not find all eigenvalues and eigenfunctions since the Hessian is infinite dimensional, and eigenfunctions will be found as solutions to a system of partial differential equations. The Hessian will also have zero eigenvalues corresponding to the gauge invariance of the action, this can be easily seen in the flat Abelian case above if we take δW to correspond to the usual gauge transformation $\partial_\mu f$ for an arbitrary function f,

$$\begin{pmatrix} -\partial_\alpha\partial^\alpha - V''(\phi) & -2\beta\partial_1 & +2\beta\partial_0 \\ +2\beta\partial_1 & \partial_1\partial^1 & \partial_1\partial^0 \\ -2\beta\partial_0 & -\partial_0\partial^1 & \partial_0\partial^0 \end{pmatrix} \begin{pmatrix} 0 \\ \partial_0 f \\ \partial_1 f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.23}$$

This allows us to classify the stationary points of our action, and predict the initial behaviour of the Yang-Mills flow. For a stable stationary point, the eigenvalues of $\hat{H}$ will, apart from the zero eigenvalues associated with gauge invariance, all be negative. For initial values close to a stable stationary point the Yang-Mills flow will move towards the stationary point. If the eigenvalues are positive, the stationary point is unstable, and the Yang-Mills flow will move away from the stationary point. In general we can have a mix of positive and negative eigenvalues, which allow for more complex flow behaviour which may include stable oscillations around the stationary point.

2.3 Hamiltonian Approach

In addition to the equations of motion derived above, which are represented as a system of differential equations that are second order in both our time and space variables, we derive an equivalent system of equations of motion that are first order in time using the Hamiltonian. There are several advantages to having the equations of motion in this form, in particular numerical methods for solving the equations typically require that they are first order. Additionally, the Hamiltonian plays an important role in quantum field theories.

In order to formulate a Hamiltonian for the system we need to label one of our coordinates as a time coordinate t, we choose x^0. In 2 dimensions $F_{\mu\nu}$ has only 1 independent component, so we can write

$$F^{\mu\nu}F_{\mu\nu} = 2F^{01}F_{01} = \frac{-2}{|g|}(F_{01})^2 \tag{2.24}$$

and

$$\epsilon^{\mu\nu}F_{\mu\nu} = 2F_{01}, \tag{2.25}$$

noticing that F_{01} is the Chern-Pontryagin density defined in (1.17). We also define

$$F = \frac{F_{01}}{|g|} \tag{2.26}$$

and we will also use the fact that

$$D^\mu\phi D_\mu\phi = g^{00}(D_0\phi)^2 + 2g^{10}D_1\phi D_0\phi + g^{11}(D_1\phi)^2. \tag{2.27}$$

Now we write down the Lagrangian:

$$L = \int dx \left[\frac{(F_{01\,a})^2}{2\sqrt{|g|}} + \frac{\sqrt{|g|}}{2}\left(g^{00}(D_0\phi)_a^2 + 2g^{10}(D_1\phi)_a(D_0\phi)_a + g^{11}(D_1\phi)_a^2\right) \right. \\ \left. +2\beta\phi_a F_{01\,a} - V(|\phi|) \right]. \tag{2.28}$$

We will now write derivatives in the following way:

$$\partial_0 A := \dot{A}, \tag{2.29}$$

$$\partial_1 A := \partial A. \tag{2.30}$$

So, for example:

$$D_0\phi := \dot{\phi} + \imath q[A_0, \phi]. \tag{2.31}$$

We have three Lie Algebra valued coordinate functions, A_0, A_1, and ϕ for a total of $3n$ coordinates (where n is the dimension of the Lie algebra) We define the conjugate momentum for each coordinate as a functional derivative of the Lagrangian:

$$\Pi_{0\,a} := \frac{\delta L}{\delta \dot{A}_{0\,a}} = 0 \tag{2.32}$$

$$\Pi_{1\,a} := \frac{\delta L}{\delta \dot{A}_{1\,a}} = \frac{F_{01\,a}}{\sqrt{|g|}} + 2\beta\phi_a \tag{2.33}$$

$$\Pi_{\phi\,a} := \frac{\delta L}{\delta \dot{\phi}_a} = \sqrt{|g|}D^0_a\phi \tag{2.34}$$

Since the momenta conjugate to $A_{0\,a}$ are 0, we get primary constraints.

The Hamiltonian, H, including the primary constraints is defined as:

$$H = \int dx \left(\Pi_{1\,a}\dot{A}_{1\,a} + \Pi_{\phi\,a}\dot{\phi}_a + u_a\Pi_{0\,a}\right) - L, \tag{2.35}$$

and the u_a are Lagrange multipliers.

By inverting our expressions for the momenta we can find equations for the time derivatives of our coordinates, and write the Hamiltonian in terms of the coordinates and their momenta only with no explicit time dependence. Using the Hamiltonian density $\mathcal{H}$ defined as $H = \int dx\, \mathcal{H}$,

$$\mathcal{H} = \frac{\sqrt{|g|}}{2}(\Pi_{1\,a})^2 - 2\beta\sqrt{|g|}\Pi_{1\,a}\phi_a + \Pi_{1\,a}(D_1 A_0)_a + \frac{(\Pi_{\phi\,a})^2}{2g^{00}\sqrt{|g|}} - \frac{g^{10}}{g^{00}}\Pi_{\phi\,a}(D_1\phi)_a$$
$$-\imath q\Pi_{\phi\,a}[A_0,\phi] + 2\beta^2\sqrt{|g|}\phi_a^2 + \frac{1}{2g^{00}\sqrt{|g|}}(D_1\phi)_a^2 + V(|\phi|) + u_a\Pi_{0\,a}. \tag{2.36}$$

We can now obtain the equations of motion for our fields using

$$\dot{X} = \{X, H\}, \tag{2.37}$$

here we use { , } to denote the Poisson bracket. First we need to look at the time derivative of our primary constraints, we find it is not identically zero so we define secondary constraints Ψ_a as

$$\Psi_a = \dot{\Pi}_{0\,a} = \{\Pi_{0\,a}, H\} = (D_1\Pi_1)_a - \imath q[\Pi_\phi, \phi]_a. \tag{2.38}$$

Now we need to check $\dot{\Psi}_a = \{\Psi_a, H\}$ and we find

$$\dot{\Psi}_a = -\imath q[A_0, \Psi]_a, \tag{2.39}$$

which is a multiple of existing constraints, therefore no new constraints are needed. We can now write down the total Hamiltonian,

$$H_T = H + u_{2\,a}\Psi_a$$

Where $u_{2\,a}$ are Lagrange multipliers and we can simplify by making the substitution $u_{2\,a} \rightarrow u_{2\,a} + A_{0\,a}$ We find the total Hamiltonian density to be :

$$\begin{aligned}\mathcal{H} = \quad & \frac{\sqrt{|g|}}{2}(\Pi_{1\,a})^2 - 2\beta\sqrt{|g|}\Pi_{1\,a}\phi_a + \frac{(\Pi_{\phi\,a})^2}{2g^{00}\sqrt{|g|}} \\ & -\frac{g^{10}}{g^{00}}\Pi_{\phi\,a}(D_1\phi)_a + 2\beta^2\sqrt{|g|}\phi_a^2 + \frac{(D_1\phi)_a^2}{2g^{00}\sqrt{|g|}} + V(|\phi|) \\ & + u_a\Pi_{0\,a} + u_{2\,a}\left(D_1\Pi_1 + \imath q[\phi, \Pi_\phi]\right)_a\end{aligned} \tag{2.40}$$

Now we can obtain equations of motion for our fields and their conjugate momenta:

$$\dot{\phi}_a = \frac{\Pi_{\phi\,a}}{g^{00}\sqrt{|g|}} - \frac{g^{10}}{g^{00}}(D_1\phi)_a - \imath q[\phi, u_2]_a \tag{2.41}$$

$$\dot{A}_{1\,a} = \sqrt{|g|}(\Pi_{1\,a} - 2\beta\phi_a) - (D_1 u_2)_a \tag{2.42}$$

$$\dot{A}_{0\,a} = u_a \tag{2.43}$$

$$\begin{aligned}\dot{\Pi}_{\phi\,a} = \; & D_1\left(\frac{1}{g^{00}\sqrt{|g|}}D_1\phi - \frac{g^{10}}{g^{00}}\Pi_\phi\right)_a - 2\beta\sqrt{|g|}(2\beta\phi_a - \Pi_{1\,a}) \\ & -V'(\phi) + \imath q[u_2, \Pi_\phi]_a\end{aligned} \tag{2.44}$$

$$\dot{\Pi}_{1\,a} = \imath q\left([u_2, \Pi_1]_a + \frac{g^{10}}{g^{00}}[\phi, \Pi_\phi]_a - \frac{1}{g^{00}\sqrt{|g|}}[\phi, D_1\phi]_a\right) \tag{2.45}$$

$$\dot{\Pi}_{0\,a} = 0 \tag{2.46}$$

The equations (2.43) and (2.46) are simply the requirement that our constraint equations are satisfied, and (2.45) becomes redundant once we choose a gauge so only (2.41), (2.42) and (2.44) along with the constraint equation (2.38) are necessary.

If we work in flat space-time with $g_{00} = -g_{11} = 1$ the Hamiltonian density simplifies to:

$$\mathcal{H} = \frac{(\Pi_{1\,a})^2}{2} - 2\beta\Pi_{1\,a}\phi_a + \frac{(\Pi_{\phi\,a})^2}{2} + 2\beta^2\phi_a^2 + \frac{(D_1\phi)_a^2}{2} \tag{2.47}$$

$$+u_a\Pi_{0\,a} + u_{2\,a}\left(D_1\Pi_1 + \imath q[\phi, \Pi_\phi]\right)_a + V(\phi) \tag{2.48}$$

And the relevant equations of motion become:

$$\dot{\phi}_a = \Pi_{\phi\, a} - \imath q[\phi, u_2]_a, \tag{2.49}$$

$$\dot{A}_{1\, a} = (\Pi_{1\, a} - 2\beta\phi_a) - (D_1 u_2)_a, \tag{2.50}$$

$$\dot{\Pi}_{\phi\, a} = (D_1 D_1 \phi)_a - 2\beta(2\beta\phi_a - \Pi_{1\, a}) + \imath q[u_2, \Pi_\phi]_a - V'(\phi). \tag{2.51}$$

This system of equations determines ϕ_a, $A_{1\, a}$ and $\Pi_{\phi\, a}$ in terms of u_2 which is determined by our choice of gauge, and $\Pi_{1\, a}$ which is determined by the constraint (2.38). Solving this system is equivalent to solving the system given by (2.9) and (2.10), both will give us the same results for $A^\mu(t, x)$ and $\phi(t, x)$.

Even with the simplest non-Abelian Lie algebra, $n = 3$ and we have a system of 12 coupled partial differential equations to solve the equations of motion, as well as the 9 equations that define the Yang-Mills flow: 3 for each component of A^ν from (2.11) and 3 for ϕ from (2.12). In the next chapter, we simplify things further by considering the Abelian case, where we can find solutions to the equations of motion, and the flow can be simulated numerically.

Chapter 3

The Abelian Case

3.1 Solving the Equations of Motion

In order to analyse the gradient flow equations for this model, we first would like to find the stationary points of the flow, which are given by the solutions to the equations of motion. In order to solve the equations of motion, we first need to specify the potential function $V(|\phi|)$ which we define as:

$$V(|\phi|) = -\frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{3.1}$$

The negative m^2 term in the potential leads to a double-well potential with two distinct vacuum states which allows symmetry breaking of the ϕ^4 potential. If we consider only the case where we have an Abelian Lie algebra, and our background metric is flat, our equations of motion (2.9) and (2.10) using the potential defined above, simplify to the following:

$$-\partial_\nu F^{\mu\nu} + 2\beta\epsilon^{\mu\nu}\partial_\nu\phi = 0, \tag{3.2}$$

$$-\partial^\mu\partial_\mu\phi + \beta F_{\mu\nu}\epsilon^{\mu\nu} + m^2\phi - \frac{\lambda}{6}\phi^3 = 0. \tag{3.3}$$

This gives the gradient flow equations as

$$\frac{\partial A^\mu}{\partial\tau} = -\partial_\nu F^{\mu\nu} + 2\beta\epsilon^{\mu\nu}\partial_\nu\phi + \partial^\mu\chi \tag{3.4}$$

and

$$\frac{\partial\phi}{\partial\tau} = -\partial^\mu\partial_\mu\phi + \beta F_{\mu\nu}\epsilon^{\mu\nu} + m^2\phi - \frac{\lambda}{6}\phi^3. \tag{3.5}$$

In two dimensions, $F^{\mu\nu}$ has only one independent component so we can use F from (2.26) which in flat space is equivalent to the Chern-Pontryagin density (1.17). Using this we can rewrite our equations of motion:

$$-\epsilon^{\mu\nu}\partial_\nu(F + 2\beta\phi) = 0, \tag{3.6}$$

$$-\partial^\mu\partial_\mu\phi - 2\beta F + m^2\phi - \frac{\lambda}{6}\phi^3 = 0. \tag{3.7}$$

Now that our equations of motion depend on F and ϕ only, we would like to write down the flow equation for F instead of the equations for A^μ. We start by rewriting (3.5) as

$$\frac{\partial\phi}{\partial\tau} = -(\partial_0^2 - \partial_1^2)\phi - 2\beta F + m^2\phi - \frac{\lambda}{6}\phi^3. \tag{3.8}$$

From the definition for F we can find that

$$\frac{\partial F}{\partial\tau} = -\frac{\partial(\partial^0 A^1 - \partial^1 A^0)}{\partial\tau} \tag{3.9}$$

Since derivatives with respect to space-time variables x and t will commute with derivatives with respect to our flow parameter τ we can obtain the following flow equation for F,

$$\frac{\partial F}{\partial \tau} = -(\partial_0^2 - \partial_1^2)(F - 2\beta\phi). \tag{3.10}$$

Now that the flow equation is now written in terms of gauge invariant quantities, the deTurck term appearing in (3.4) has dropped out of our equations, and we will only be required to choose a gauge to find the values of A^μ from F.

For the remainder of this chapter we will consider only the static case, where F and ϕ are independent of t, this simplifies our equations of motion to ordinary differential equations, and allows us to work with only the Riemannian part of the metric, without performing a Wick rotation. Now our system of partial differential equations for the flow has simplified to only two equations, one for F and one for ϕ, which are functions of only the spatial variable x as well as the flow parameter τ. The stationary points of the flow will be determined by solutions to the equations of motion which are now ordinary differential equations in terms of x only.

Now we can easily solve for F by integrating (3.6) to find

$$F = 2\beta\phi + b_0, \tag{3.11}$$

where b_0 is a constant.

First we want to consider the case where $m = \lambda = 0$ or $V = 0$, in this case our action corresponds to the modified Schwinger model in (1.22). We have a general solution,

$$\phi = c_1 e^{2\beta x} + c_2 e^{-2\beta x} - \frac{b_0}{2\beta}, \tag{3.12}$$

where c_1 and c_2 are constants. The only solutions that are finite for all x are the constant solutions,

$$\phi = -\frac{b_0}{2\beta} \quad \text{and} \quad F = 0. \tag{3.13}$$

However we can attempt to construct a finite solution by patching together two solutions at the origin.

$$\phi = c_2 e^{-2\beta|x|} - \frac{b_0}{2\beta} \tag{3.14}$$

is a solution everywhere except at $x = 0$ where the derivative is not defined. If we consider the Yang-Mills flow near this solution, it is clear that the flow will move away from the solution towards the constant $-\frac{b_0}{2\beta}$. The flow equations in this model appear to be trivial, but the connection with the Schwinger model suggests that the 2β term acts like a topological mass term. We expect it to still act like a mass term when we include an explicit mass term in the potential.

Now we consider the the full potential (3.1). We substitute this along with the static condition into (3.7) to obtain a second order differential equation that we can solve for $\phi(x)$:

$$\partial_x^2\phi = -M^2\phi + \frac{\lambda}{6}\phi^3 + 2\beta b_0, \tag{3.15}$$

where we define a new quantity

$$M^2 = m^2 - 4\beta^2, \tag{3.16}$$

which, as we can see from (3.15), acts like a mass term for ϕ. We will require

$$M^2 > 0 \Leftrightarrow m^2 > 4\beta^2 \tag{3.17}$$

to ensure symmetry breaking of the potential.

We start by looking for constant solutions to (3.15), which are given by the real roots of the cubic equation on the right hand side. We will have either one, two or three distinct constant solutions depending on the value of b_0, the possibilities are summarized in the following table:

Value of b_0	number of solutions	Value of b_0	number of solutions
$b_0 > \frac{M^3}{3\beta}\sqrt{\frac{2}{\lambda}}$	one negative solution	$b_0 < \frac{-M^3}{3\beta}\sqrt{\frac{2}{\lambda}}$	one positive solution
$\lvert b_0 \rvert = \frac{M^3}{3\beta}\sqrt{\frac{2}{\lambda}}$	two solutions	$\lvert b_0 \rvert < \frac{M^3}{3\beta}\sqrt{\frac{2}{\lambda}}$	three solutions

TABLE 3.1: Number of constant solutions to the equations of motion for ϕ

We are mainly interested in the case where we have 3 constant solutions, and we will label them as

$$\phi_- = \text{the smallest solution,} \tag{3.18}$$

$$\phi_+ = \text{the largest solution, and} \tag{3.19}$$

$$\phi_0 = \phi_- < \phi_0 < \phi_+. \tag{3.20}$$

For cases with only one solution, we denote it with either ϕ_- or ϕ_+ depending on its sign, and for cases with two solutions we exclude ϕ_0.

In order to find non-constant solutions we look at (3.15) which is a second order equation that does not depend explicitly on our variable x, so we can integrate to obtain a first order equation,

$$dx = \frac{d\phi}{\sqrt{\frac{\lambda}{12}\phi^4 - M^2\phi^2 + 4\beta b_0\phi + b_1}}, \tag{3.21}$$

where b_1 is a constant. Integrating both sides of the equation will give x as a function of ϕ which we will then need to invert to find $\phi(x)$. For certain values of the constants b_0 and b_1 we can find $\phi(x)$ analytically.

If $b_0 = 0$, the equations of motion simplify to the equations of motion for a scalar field in a double well potential with mass given by M, that is not coupled to a gauge field. Therefore we expect to find the typical kink solution that moves between two of our constant solutions. By setting $b_1 = \frac{3M^4}{\lambda}$ we can integrate (3.21) and, as expected we find

$$\phi_{odd}(x) = \pm\sqrt{\frac{6M^2}{\lambda}}\tanh\left(\frac{M}{\sqrt{2}}x + c\right). \tag{3.22}$$

Noting that for $c = 0$, ϕ_{odd} is an odd function of x.

If $b_0 = \pm\frac{M^3}{3\beta\sqrt{\lambda}}$, we can find additional solutions by setting $b_1 = 0$ and integrating, which gives

$$\phi_{even}(x) = \pm\frac{\frac{4M}{\sqrt{\lambda}}\tanh^2\left(\frac{Mx+c}{2}\right)}{3 - \tanh^2\left(\frac{Mx+c}{2}\right)}. \tag{3.23}$$

Here we have for $c = 0$, ϕ_{even} is an even function of x.

In both (3.22) and (3.23) we have a constant of integration c that allows translations along the x-axis. We will consider only the positive solution from (3.23), any results for the negative solution will differ only by the sign change. Both of the above solutions are bounded and approach constant solutions as $x \to \infty$,

$$\lim_{x\to\pm\infty}\phi_{odd}(x) = \pm\sqrt{\frac{6M^2}{\lambda}}, \tag{3.24}$$

and,

$$\lim_{x\to\pm\infty}\phi_{even}(x) = 2\sqrt{\frac{M^2}{\lambda}}. \tag{3.25}$$

We also note that

$$|0| < \left|\frac{M^3}{3\beta\sqrt{\lambda}}\right| < \left|\frac{M^3}{3\beta}\sqrt{\frac{2}{\lambda}}\right|, \tag{3.26}$$

therefore, for both choices of b_0 that result in a non-constant solution we have three constant solutions as well.

3.2 The Yang-Mills Flow

In order to examine the flow equations we first consider the case when both F and ϕ are constant in x as well as t. Here (3.10) is trivial, so $F(\tau)$ is a constant. The behaviour of $\phi(\tau)$ will be determined by its initial value relative to the stationary points. The stationary points are the constant solutions (3.18),(3.19) and (3.20). In the case where there are three stationary points they divide possible choices of $\phi(0)$ into four intervals

$$\lim_{\tau \to \infty} \phi(\tau) = \begin{cases} \phi_- \text{ if } \phi(0) < \phi_- \\ \phi_- \text{ if } \phi_- < \phi(0) < \phi_0 \\ \phi_+ \text{ if } \phi_0 < \phi(0) < \phi_+ \\ \phi_+ \text{ if } \phi(0) > \phi_+ \end{cases} \tag{3.27}$$

The case where we have two stationary points is similar to the case where we have three stationary points with the exception that if $b_0 < 0$, $\phi_0 = \phi_-$ and the second interval vanishes or if $b_0 > 0$, $\phi_0 = \phi_+$ and the third interval vanishes. For cases with only one stationary point, all flows move toward the stationary point, which has the opposite sign of b_0.

The flow can be visualized as trajectories in a (ϕ, F) phase space, as in figure 3.1. Points lying on the curve are stationary points of the flow. For any initial point $(\phi(0), F(0))$ that lies below the curve will move along a horizontal trajectory to the right (i.e. constant F, increasing ϕ). Initial points that lie above the curve will move along horizontal trajectories to the left (constant F, decreasing ϕ). All trajectories will approach a stationary point as τ approaches infinity.

Although we can predict the behaviour of the flow for any initial condition if we know the values of the stationary points, this involves solving a cubic equation which would require us to choose specific values for b_0, m and β. To determine whether or not a given constant

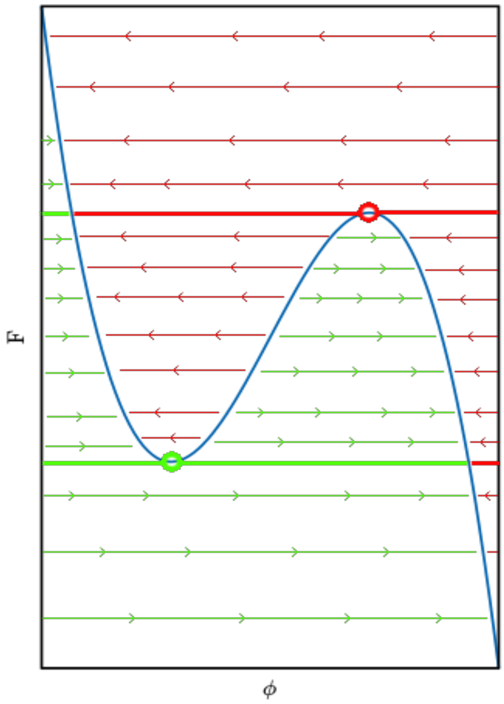

FIGURE 3.1: The blue curve indicates the location of stationary points (ϕ, F) determined from (3.8). The flow moves any initial points to the left (decreasing ϕ) along horizontal trajectories shown in red, or to the right (increasing ϕ) along trajectories shown in green. The two circles on the graph indicate saddle points, and the area between the two bold lines is the main area of interest where we have three stationary points. Notice the portion of the curve with ϕ between the two saddle points corresponds to the unstable stationary points. The exact location of the saddle points will depend on our choice of parameters, but the overall shape seen here is unchanged.

solution is stable or not we can look at the Hessian, (2.20). If we are only considering constant perturbations of ϕ then only one element of the Hessian is relevant,

$$\hat{H}^{\phi\phi} = m^2 - \frac{\lambda}{2}\phi^2. \tag{3.28}$$

As discussed in Chapter 2, the stability of a stationary point is determined by the signs of the eigenvalues of the Hessian, in this case we have only one component, so the only eigenvalue will be the component itself. If we find that for a solution ϕ (3.28) is negative, then that solution will be stable, if it is positive then the solution will be unstable.

Two cases of particular interest are the constant solutions that our non-constant solutions approach as $x \to \infty$, the results are summarized in Table 3.2:

We can see from table 3.2 that the stability of our solutions can depend on the relative values of our symmetry breaking mass term m and our topological mass term β. We

	$\phi_{even}(x \to \infty) = \frac{2M}{\sqrt{\lambda}}$ (3.23)	$\phi_{odd}(x \to \infty) = \sqrt{\frac{6M^2}{\lambda}}$ (3.22)
Stable ($\hat{H} < 0$)	$m^2 > 8\beta^2$	$m^2 > 6\beta^2$
Unstable ($\hat{H} > 0$)	$m^2 < 8\beta^2$	$m^2 < 6\beta^2$

Table 3.2: Stability of Constant Solutions

consider two main cases, the 'large M' case where $m^2 > 8\beta^2$ and the 'small M' case where $m^2 < 6\beta^2$, We notice that in the large M case the asymptotic solutions are stable constant solutions but in the small M case the asymptotic solutions are unstable constant solutions. So far we have only considered the stability of constant perturbations of constant solutions, but we would like to extend what we know about the constant case to non-constant solutions. Although we can't apply the stability conditions from the simplified Hessian directly to the rest of our non-constant solutions, it does suggest that we will find different flow behaviour in each of the two cases. In order to solve the system of partial differential equations, (3.8) and (3.10), we proceed with a numerical calculation.

3.3 The Numerical Method

Systems of non-linear partial differential equations like we have here, are difficult to solve in general, but we can attempt to learn about their behaviour numerically. In order to do this we employ a simple explicit method for solving diffusion problems. First, in order to integrate over $-\infty < x < \infty$ we change coordinates using

$$x = \frac{y}{1 - y^2}, \tag{3.29}$$

which changes our derivatives,

$$\partial_x^2 = \left(\frac{dy}{dx}\right)^2 \left[\partial_y^2 + \left(\frac{2y^3 + 6y}{y^4 - 1}\right)\partial_y\right], \tag{3.30}$$

where

$$\frac{dy}{dx} = \frac{(1-y^2)^2}{1+y^2}. \tag{3.31}$$

Now our range of integration is finite, $-1 < y < 1$. Derivatives with respect to our flow parameter τ (which takes the place of the time variable in a typical diffusion problem) are approximated using a forward finite difference, and derivatives with respect to x are approximated using a central finite difference as follows:

$$\frac{\partial F(\tau, y)}{\partial y} = \frac{F(\tau, y+dy) - F(\tau, y-dy)}{2dy}, \tag{3.32}$$

$$\frac{\partial F(\tau, y)}{\partial \tau} = \frac{F(\tau+d\tau, y) - F(\tau, y)}{d\tau}. \tag{3.33}$$

In order to complete the integration from $\tau = 0$ to $\tau = 10$ in a reasonable number of steps, we have a lower limit on the step size $d\tau$. Additionally, this method will be stable as long as $\frac{d\tau}{(2dx)^2} \leq \frac{1}{2}$[13], which limits how accurately the derivatives can be approximated by placing a lower limit on our spatial grid size dx.

Before we apply the numerical approximation, we can simplify our equations to eliminate the interaction strength λ by making the following substitutions for ϕ and F:

$$\phi \rightarrow m\sqrt{\frac{6}{\lambda}}\phi, \tag{3.34}$$

$$F \rightarrow 2\beta m\sqrt{\frac{6}{\lambda}}F. \tag{3.35}$$

Which changes our system of equations for $F(\tau, x)$ and $\phi(\tau, x)$ from (3.10) and (3.8) to :

$$\frac{\partial F}{\partial \tau} = \partial_x^2(F - \phi), \tag{3.36}$$

and

$$\frac{\partial \phi}{\partial \tau} = \partial_x^2\phi - 4\beta^2 F + m^2(\phi - \phi^3), \tag{3.37}$$

Once we use (3.30) to express the ∂_x^2 in terms of ∂_y, we are ready to implement our numerical procedure. The Fortran code can be found in Appendix A.

3.4 Summary of Numerical Results

We need to specify specific values for our constants m and β. We will look at two main cases in order to perform calculations:

- $m = 2$ and $\beta = 0.5$ corresponding to the large M case in Table 3.2
- $m = 2$ and $\beta = 0.9$ corresponding to the small M case in Table 3.2

In order to solve the equations numerically we need to specify initial functions $\phi(x, 0)$ and $F(x, 0)$. Since we are mainly interested in the flow near stationary points, we consider initial functions given by

$$\phi(x, 0) = \bar{\phi}(x) + h(x) \tag{3.38}$$

$$F(x, 0) = \bar{F}(x) \tag{3.39}$$

where $\bar{\phi}(x)$ is either ϕ_{odd} or ϕ_{even}, $\bar{F}(x) = \phi_{odd}$ or $\bar{F} = \phi_{even} + b_0$ respectively (after we have applied the transformations in (3.34) and (3.35) to (3.11), and we choose $h(x)$ so that $h(x) \to 0$ as $|x| \to \infty$.

We start by looking at the flow near the ϕ_{even} (3.23) in the large M case, where $h(x)$ is chosen to be a small gaussian centred at $x = 0$. We find that the flow has different behaviour depending on the overall sign of $h(x)$ but is not sensitive to the magnitude or the width of the gaussian perturbation.

For a negative gaussian, the flow moves ϕ downwards towards the negative constant solution, starting near the origin and propagating outwards. F flattens out as the flow progresses, moving towards the constant solution given by the initial asymptotic values of $F(x)$, with some structure that propagates along with ϕ outwards toward infinity. The figure 3.2 shows $F(x)$ and $\phi(x)$ for several values of τ as the flow progresses.

For a positive gaussian, the flow moves both ϕ and F upwards towards the positive constant solution determined by their initial endpoints. ϕ initially overshoots the constant solution, but will approach the constant solution from above rather than below, see Figure 3.3

As seen in the figures 3.2 and 3.3, the ϕ_{even} solution, (3.23), is unstable, and it remains unstable regardless of our choice of m and β. We also note that the negative constant solution is the preferred state, that is, even though the asymptotic limit of ϕ is a stable constant solution, the flow will move towards the negative solution. This results in the flow behaviour seen in figure 3.2 where ϕ initially moves downward where it was perturbed at $x = 0$. As the flow progresses the perturbation propagates outwards as $\phi(x)$ moves towards the negative constant solution even though it already lies on a constant solution.

Now we consider flow starting near the ϕ_{odd} (3.22). In the large M case, we find that for anti-symmetric perturbations, the flow moves back toward the solution (see figure B.1). For the large M case, we can consider ϕ_{odd} to be stable under anti-symmetric perturbations. However, if we consider a symmetric perturbation (shown in figure B.2), then the flow moves away from the solution, towards the asymptotic constant solution in the direction of the perturbation. As the flow progresses, the effect of the perturbation propagates outward moving ϕ towards a stable constant solution.

In the small M case, we find that the solution is not stable under even anti-symmetric perturbations. Although in this case we once again find that the flow moves in the

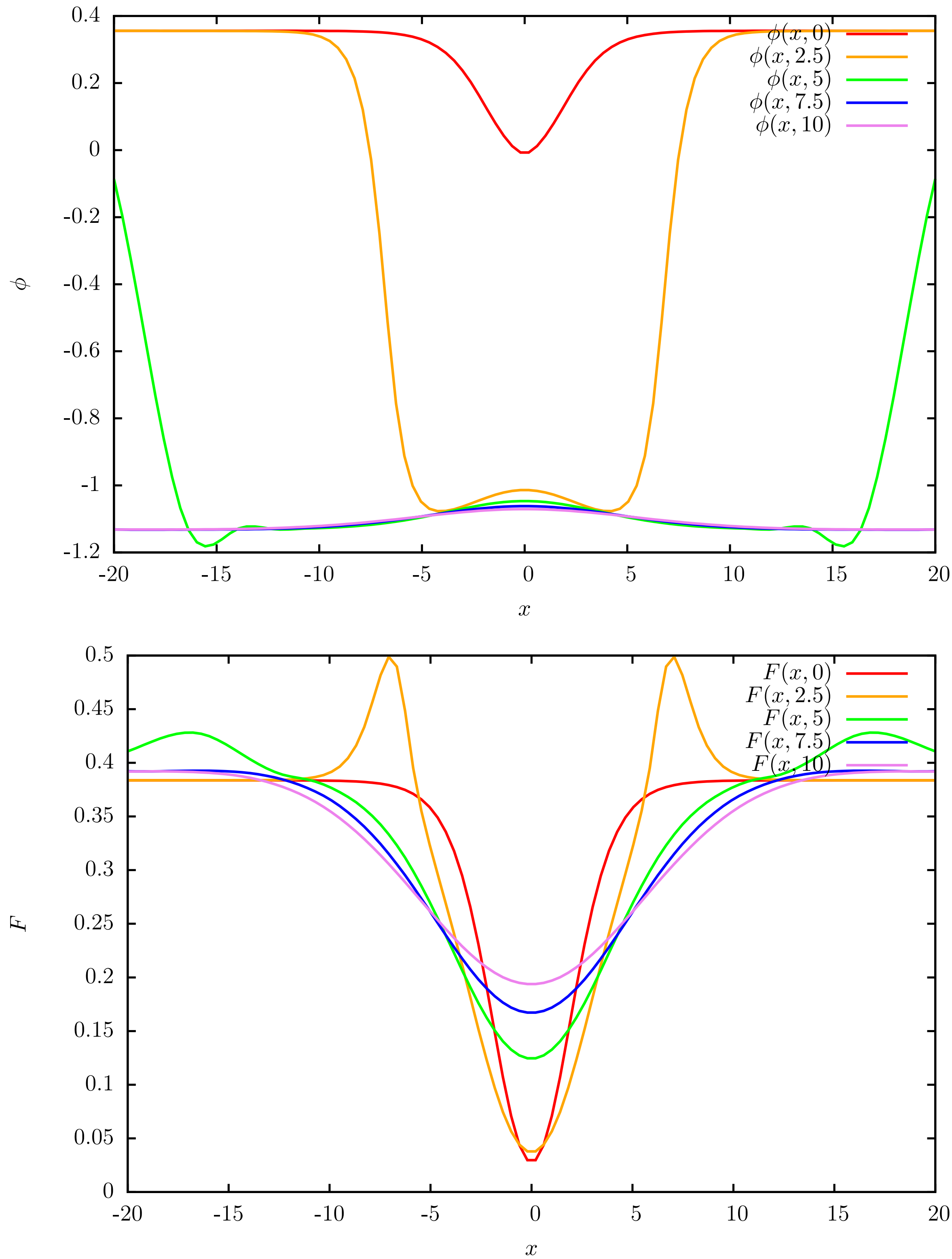

FIGURE 3.2: The Flow for initial data given by $\phi(0) = \phi_{even} - h(x)$,where $h(x) = 0.01e^{\frac{-x^2}{2}}$ and $m = 2$, $\beta = 0.9$ (small M case). $F(0) = \phi_{even} + \frac{M^3}{6\beta^2 m\sqrt{6}}$

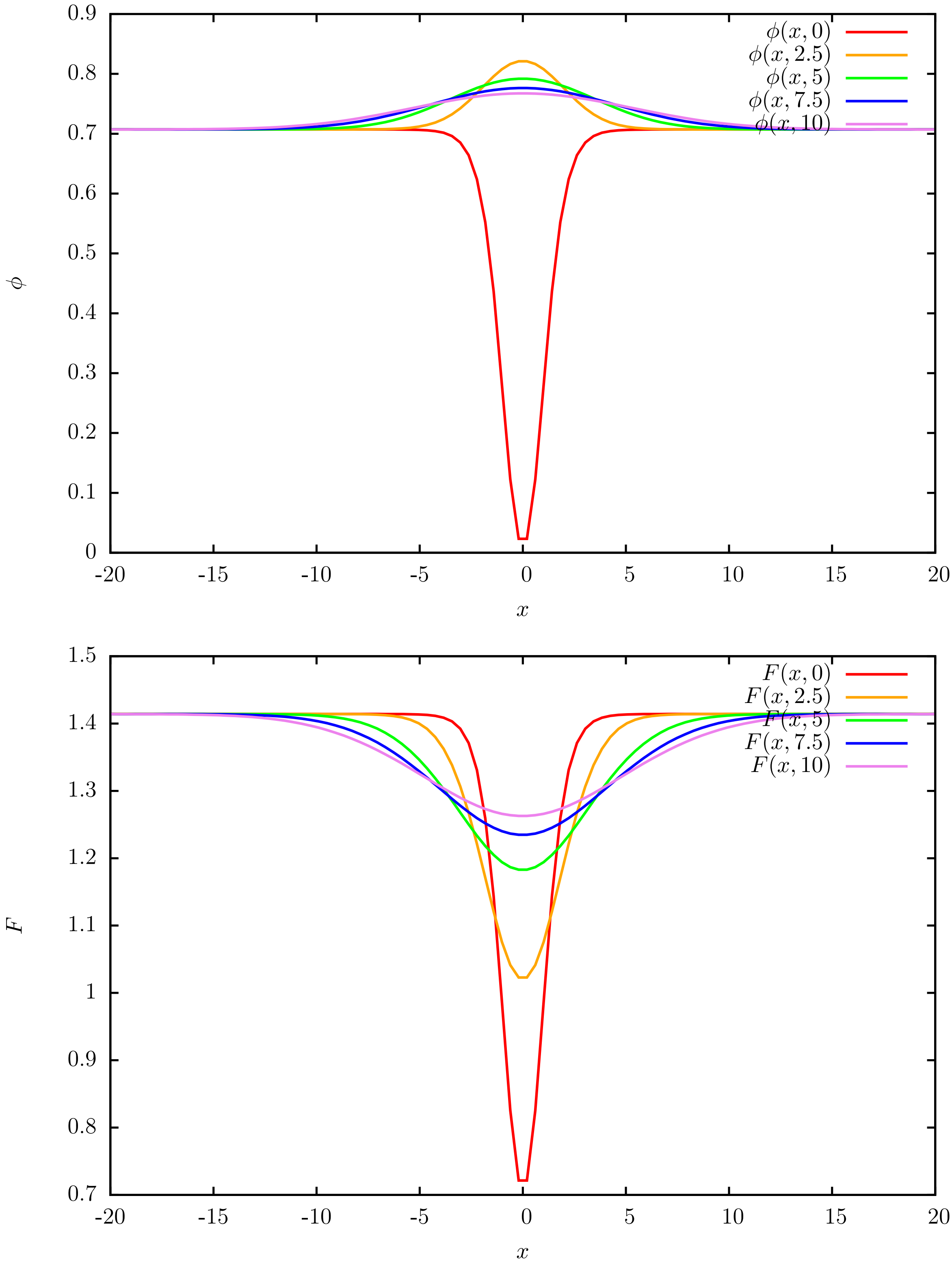

FIGURE 3.3: The Flow for initial data given by $\phi(0) = \phi_{even} + h(x)$,where $h(x) = 0.01e^{\frac{-x^2}{2}}$ and $m = 2$, $\beta = 0.9$ (large M case). $F(0) = \phi_{even} + \frac{M^3}{6\beta^2 m\sqrt{6}}$

direction of the initial perturbation, the flow does not appear to approach either of the initial constant solutions. The plots of this can be seen in Figure 3.4. Additional plots of ϕ and F for various initial functions can be found in Appendix B and the results are summarized in the following table.

Initial Value	Parameter Choice	End Result	Figure
$\phi_{even} + h(x)$	large M	asymptotic constant	fig. 3.3
$\phi_{even} + h(x)$	small M	positive constant	fig. B.4
$\phi_{even} - h(x)$	large M	negative constant	fig. B.3
$\phi_{even} - h(x)$	small M	negative constant	fig. 3.2
$\phi_{odd} + h_{odd}(x)$	large M	ϕ_{odd}	fig. B.1
$\phi_{odd} + h_{odd}(x)$	small M	undetermined	fig. 3.4

Table 3.3: Summary of Numerical Results

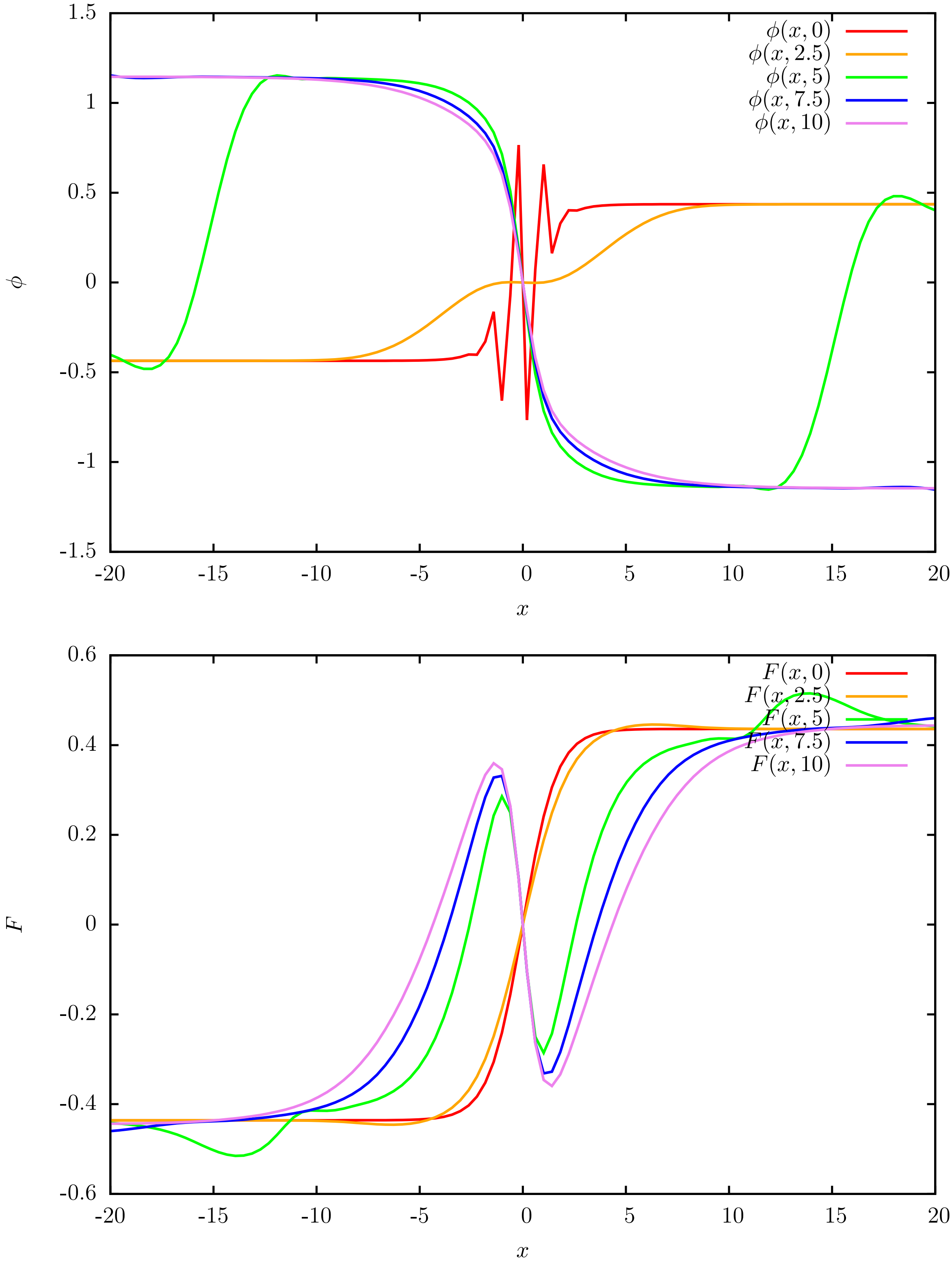

FIGURE 3.4: The Flow for initial data given by $\phi(0) = \phi_{odd} - h(x)$, with $h(x) = \sin(5x)e^{-x^2}$ and $m = 2$, $\beta = 0.9$ (small M case). $F(0) = \phi_{odd}$

Chapter 4

Conclusions

Our model has a physical interpretation as a modified Schwinger model outlined in [4] for the choice that $V(|\phi|) = 0$. As mentioned in Chapter 3 the only finite, static solutions to the equations of motion are the constant solutions, where ϕ can have any value and $F = 0$. If we look at the flow in this case we find that F is not a function of τ, and ϕ is a linear function of τ with slope $-2\beta F$. For any constant, non-zero value of F we find that

$$\lim_{\tau \to \infty} \phi(\tau) = \pm\infty \tag{4.1}$$

Therefore without a potential term in the action, the flow can result in our field ϕ becoming infinite, although only as the flow parameter τ becomes infinite as well. With the inclusion of the potential (3.1) the flow will remain finite for initial functions that are bounded. The potential included is the typical double well potential of classical physics which, for complex valued ϕ becomes the 'Mexican hat' potential used to allow for symmetry breaking in the Abelian Higgs model.

With a non-zero potential term we found several solutions to the equations of motion, and numerically studied the flow in the neighbourhood of these solutions in order to study the stability of the solutions. We found that, for symmetric solutions $\phi_{even}(x)$ and $F_{even}(x)$, the Yang-Mills flow moved towards one of the constant solutions. The overall direction of the flow was determined by the direction of the initial perturbation from the solution. The ϕ_{even} solutions are found to be unstable solutions, and the end result of the flow did not depend on the relative values of the mass of the scalar field and the coupling between the scalar and gauge fields. We also find that the constant solution furthest from 0 (in the cases we considered this is the negative constant solution) is the preferred final state. That is, even if only part of ϕ is flowing towards the negative constant solution, eventually all of ϕ will flow towards the solution as can be seen in figure 3.2.

For the ϕ_{odd} solutions, which are similar to solutions that appear in the Abelian Higgs model or models without any coupling between ϕ and A^μ, we found that the flow was substantially different for different choices of mass and coupling. In particular, the solution becomes unstable if the coupling is chosen to be large enough while still allowing for symmetry breaking, this corresponds to lowering the mass of the scalar field in a model with no coupling. In contrast, without the coupling, the solution would remain stable for any value of the mass. For values of the mass and coupling where the functions are not completely unstable, we find that the solutions are generally stable under anti-symmetric perturbations, while symmetric perturbations move the functions towards one of the constant solutions.

In this thesis we have only considered initial functions that differ from a solution by a perturbation in ϕ, in general we should also consider perturbations of F generated by perturbations of A^μ. However, our results suggest that the flow will smooth out any perturbations of F. Therefore, we expect that, for an initial perturbation $F(x) \to F(x) + g(x)$, subject to the condtion

$$\lim_{x \to \pm\infty} g(x) = 0, \tag{4.2}$$

will not alter the end result of the flow for either F or ϕ. Additionally, if we consider leaving ϕ unchanged, the initial direction of the flow for ϕ will be opposite the direction of the perturbation $g(x)$. We still expect that the initial direction of the flow of ϕ will determine the behaviour of the flow for large values of τ.

There are several interesting extensions to the work in this thesis. So far we have only considered static solutions, including time dependence in our equations may result in bounded solutions without including a potential term for ϕ. Additionally an extra dimension may allow for more complex flow behaviour around the stationary points. Considering the non-static case is interesting for this model whether or not we include the potential term in the action. In order to perform numerical calculations in this case it may be necessary to use a more advanced numerical method to compensate for the greater number of grid points we would need to include in our calculations.

In this thesis we compute the flow only in the case of a flat background metric. There may be interesting results from considering curved space-times although it may require that we consider non-static equations of motion as well. In chapter 2 we derived equations of motion for an arbitrary background metric before simplifying to the flat case, and we can see that a curved metric could significantly alter our solutions, as well as the flow equations. Alternatively we could consider an arbitrary metric that is not fixed, so that when we compute the variation of the action we allow the metric to vary as well. This will result in new equations of motion to compute the metric components $g_{\mu\nu}$ and might require the inclusion of an Einstein-Hilbert term in the action. Additional flow equations would arise for the metric that will be directly related to Ricci flow, and our system will be similar to the Ricci Yang Mills flow considered in [12] coupled with an additional scalar field.

Another obvious extension of this work is to consider the non-Abelian case, the simplest case being $SO(3)$ where our fields can be represented as 3×3 anti-symmetric matrices.

In this case our two coupled flow equations for F and ϕ become a system of six coupled partial differential equations, which can allow for much more complex flow behaviour even if we consider the constant case only. It may also be of interest to consider more complicated Lie Algebras such as $SU(3)$ which forms the basis for chromodynamics.

Finally, the simplest extension would be to consider the flow in the Abelian Higgs model. In order to do this we need to consider a complex scalar field ϕ instead of a real one, and instead of including an explicit coupling term between ϕ and A^μ we couple them through the covariant derivative. The Abelian Higgs model has a solution similar to our ϕ_{odd} solution, although it is not clear if its stability will be affected by the choice of the corresponding parameters in the action similar to what we found for our model. Of course the Abelian Higgs model itself is a starting point for looking at a non-Abelian Higgs model, as well as higher dimensional models which may be of more physical significance in terms of actual particle physics.

Appendix A

Code for Numerical Procedure

```
program YMFLOWSOLVE
implicit none
double precision y, x, dy, t, dt, m, B , q1, q2, ee, a, a2, bo
integer i, j, l, ii, iii, q3(1), q4(1)
integer, parameter :: k = 400 !the number of space steps
double precision, dimension(k-1) :: dp, du
double precision, dimension(k-1) :: p, u
double precision, dimension(21,k-1) :: plowout, flowout

dy = 2.0d+0/k  !space step size
dt = 2.0d-0*dy**2.0   !time step size
m =2.0d+0  !specify the parameter m in the action
B = 0.5d+0 ! specify the parameter beta in the action
l = 0
j = 0
ii = 1
iii = 0
ee = 5.0d-5

  open(101,FILE='phi.dat')
  open(102,FILE='F.dat')
  open(103,FILE='tauspace.dat')

call init(p,u,dy,k,m,B)

plowout(1,:) = p(:)
flowout(1,:) = u(:)

do while (l.le.10)
```

```
 j=j+1
 call slope(p,u,dp,du,dy,k,m,B)
 p = dp*dt+p
 u = du*dt+u
 q1 = Maxval(abs(dp))
 q2 = Maxval(abs(du))
   q3 = Maxloc(abs(dp))
 q4 = Maxloc(abs(du))

 write(*,*) q1 , q3, q2, q4, j*dt

 if (q1.le.ee.and.q2.le.ee) then
  l=15
   ii=ii+1
   plowout(ii,:) = p(:)
   flowout(ii,:) = u(:)
     write(103, "(e20.14)") j*dt
 endif

 if (q1.ge.(1000./ee)) then
  l=25
 endif

 if(j.eq.200000) then
  l=35
 endif

 if((mod(j,10000).eq.0)) then
   ii=ii+1
   plowout(ii,:) = p(:)
   flowout(ii,:) = u(:)
   write(103, "(e20.14)") j*dt
   endif
end do

write(103, *) q1, q2
write(103, *) dp(200), du(200)
write(*,*) 'dy=',dy,'dt=',dt
write(*,*) 'end:', l, 'steps:',j

do l=1,k-1,1 !writes output to files
  y = -1.0d+0+l*dy
  x = y/(1.0d+0-y**2.0)
  write(101, *) x,y,(plowout(ii,l),ii=1,21),dp(l)
```

```
  write(102, *) x,y,(flowout(ii,l),ii=1,21),du(l)
end do

end program YMFLOWSOLVE
!~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
subroutine init(p,u,dy,k,m,2B) !sets initial values for phi, F
integer k,n
double precision, dimension(k-1) :: p, u
double precision dy,y,m,B,a,x,2B
integer l
B=2.0d+0*2B !replace beta with 2*beta
a=sqrt(m**2.-B**2.)/2.d+0
a2=4.0d+0*sqrt(m**2.-B**2.)/(m*sqrt(6.0d+0))
bo=2.0d+0*(sqrt(m**2.-B**2.)**3.)/(((B)**2.)*3.0d+0*m*sqrt(6.0d+0))

do l=1,k-1
y = -1.0d+0+l*dy
x = y/(1.0d+0-y**2.)
p(l)= 1.00d+0*(a2*((tanh(a*x))**2.d+0)/(3.d+0-(tanh(a*x))**2.d+0))
      + 1.0d-2*exp(-25.0d-1*x**2.)
u(l)= 1.00d+0*(a2*((tanh(a*x))**2.d+0)/(3.d+0-(tanh(a*x))**2.d+0))
      + bo
end do

end subroutine init
!~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
subroutine slope(p,u,dp,du,dy,k,m2,B2)
integer k
double precision, dimension(k-1) :: dp, du , F
double precision, dimension(k-1) :: p, u
double precision dy,y,J1,J2,m,B,d,m2,B2
integer l

F(:) = -(u(:)-p(:))

m = m2*m2 ! sets m=m^2
B = B2*B2 ! sets B = beta^2
d = 2.0d+0*dy

do l = 1,k-1
y = -1.0d+0 + l*dy
J1 = ((1.0d+0-y**2.)**2.)/(1.0d+0+y**2.)
J2 = 2.0d+0*y*(y**2.0+3.0d+0)/((y**2.-1.0d+0)*(1.0d+0+y**2.))
```

```
if (l.eq.1) then
du(l)=-J1**2.*(J2*(F(l+1)-F(l))/d + (F(l)+F(l+2)-2.0d+0*F(l))/d**2.)
dp(l)=(-4.0d+0*B*u(l) + J1**2.*(J2*(p(l+1)-p(l))/d +
(p(l)+p(l+2)-2.0d+0*p(l))/d**2.)) + m*(p(l)-(p(l))**3.)

else if (l.eq.2) then
du(l)=-J1**2.*(J2*(F(l+1)-F(l-1))/d + (F(l-1)+F(l+2)-2.0d+0*F(l))/d**2.)
dp(l)=(-4.0d+0*B*u(l) + J1**2.*(J2*(p(l+1)-p(l-1))/d +
(p(l-1)+p(l+2)-2.0d+0*p(l))/d**2.) + m*(p(l)-(p(l))**3.))

else if (l.eq.(k-2)) then
du(l)=-J1**2.*(J2*(F(l+1)-F(l-1))/d + (F(l-2)+F(l+1)-2.0d+0*F(l))/d**2.)
dp(l)=(-4.0d+0*B*u(l) + J1**2.*(J2*(p(l+1)-p(l-1))/d +
(p(l-2)+p(l+1)-2.0d+0*p(l))/d**2.) + m*(p(l)-(p(l))**3.))

else if (l.eq.(k-1)) then
du(l)=-J1**2.*(J2*(F(l)-F(l-1))/d + (F(l-2)+F(l)-2.0d+0*F(l))/d**2.)
dp(l)=(-4.0d+0*B*u(l) + J1**2.*(J2*(p(l)-p(l-1))/d +
(p(l-2)+p(l)-2.0d+0*p(l))/d**2.) + m*(p(l)-(p(l))**3.))

else
du(l)=-J1**2.*(J2*(F(l+1)-F(l-1))/d + (F(l-2)+F(l+2)-2.0d+0*F(l))/d**2.)
dp(l)=J1**2.*(J2*(p(l+1)-p(l-1))/d  + (p(l-2)+p(l+2)-2.0d+0*p(l))/d**2.)
+(-4.0d+0*B*u(l) + m*(p(l)-(p(l))**3.))
end if
end do

end subroutine slope
```

Appendix B

Additional Diagrams of Numerical Results

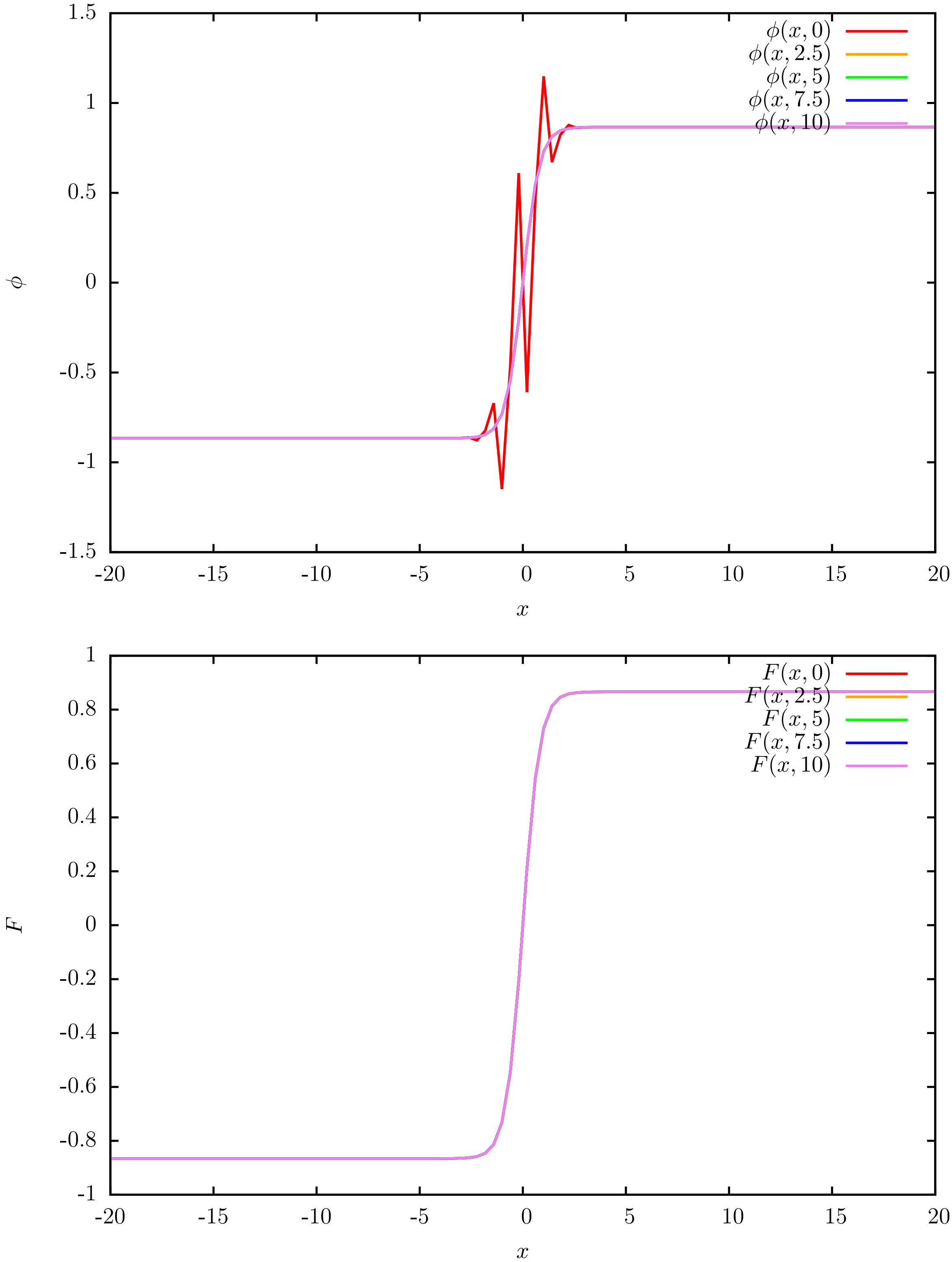

FIGURE B.1: The Flow for initial data given by $\phi(0) = \phi_{odd} - h(x)$, with $h(x) = \sin(5x)e^{-x^2}$ and $m = 2$, $\beta = 0.5$ (large M case). $F(0) = \phi_{odd}$

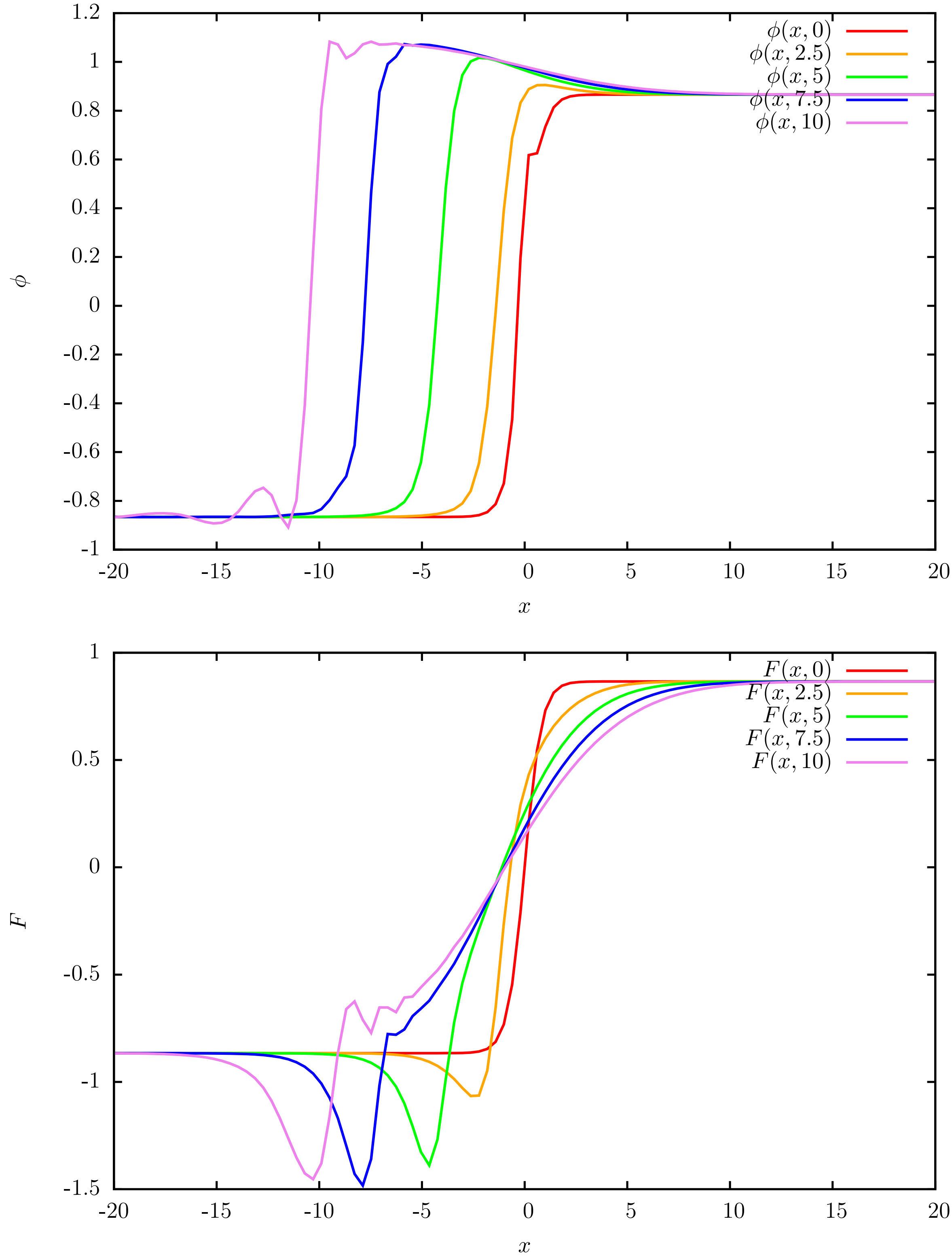

FIGURE B.2: The Flow for initial data given by $\phi(0) = \phi_{odd}+h(x)$, with $h(x) = 0.01e^{\frac{x^2}{2}}$ and $m = 2$, $\beta = 0.5$ (large M case). $F(0) = \phi_{odd}$

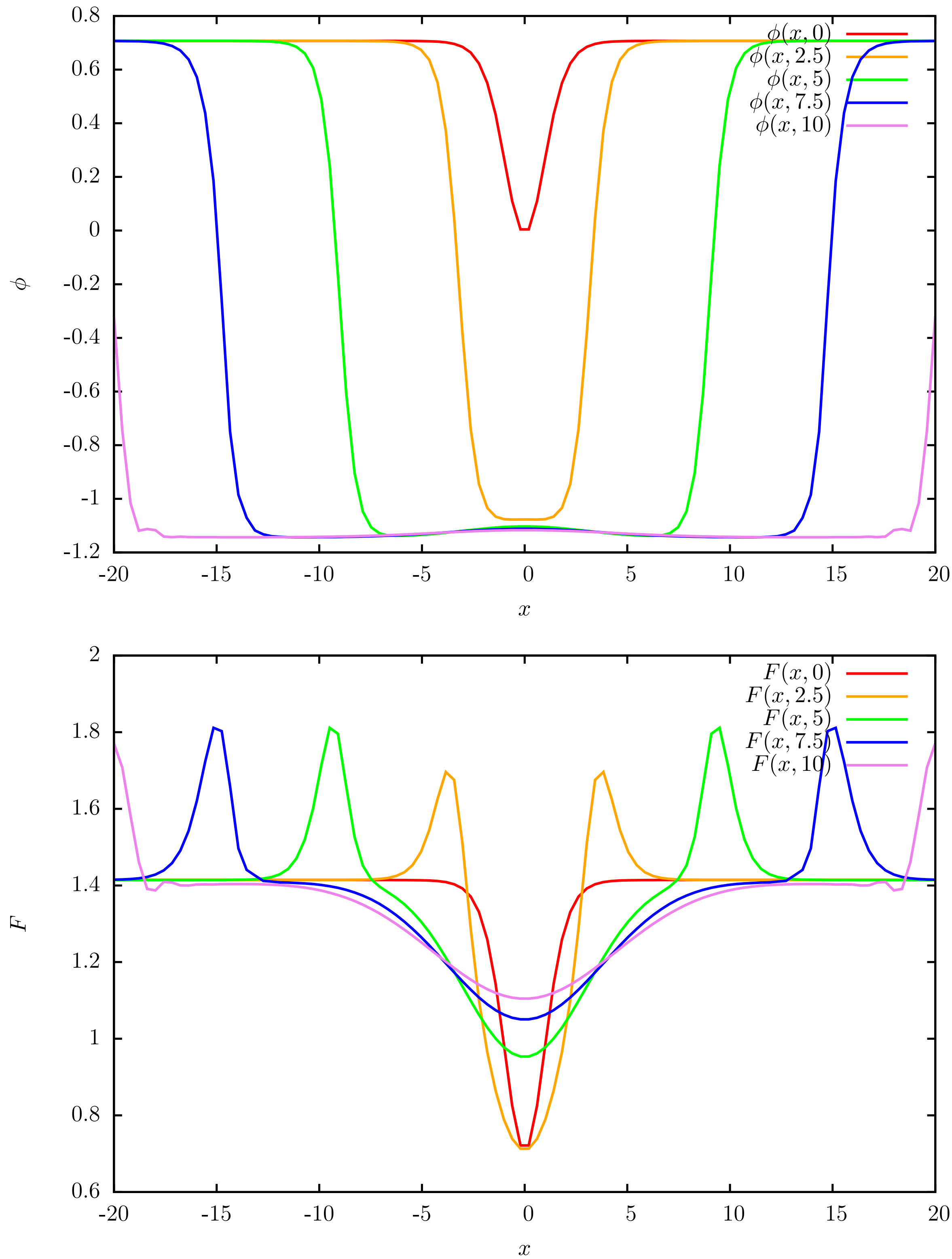

FIGURE B.3: The Flow for initial data given by $\phi(0) = \phi_{even} - h(x)$, with $h(x) = 0.01e^{\frac{x^2}{2}}$ and $m = 2$, $\beta = 0.5$ (large M case).
$F(0) = \phi_{even} + \frac{M^3}{6\beta^2 m\sqrt{6}}$

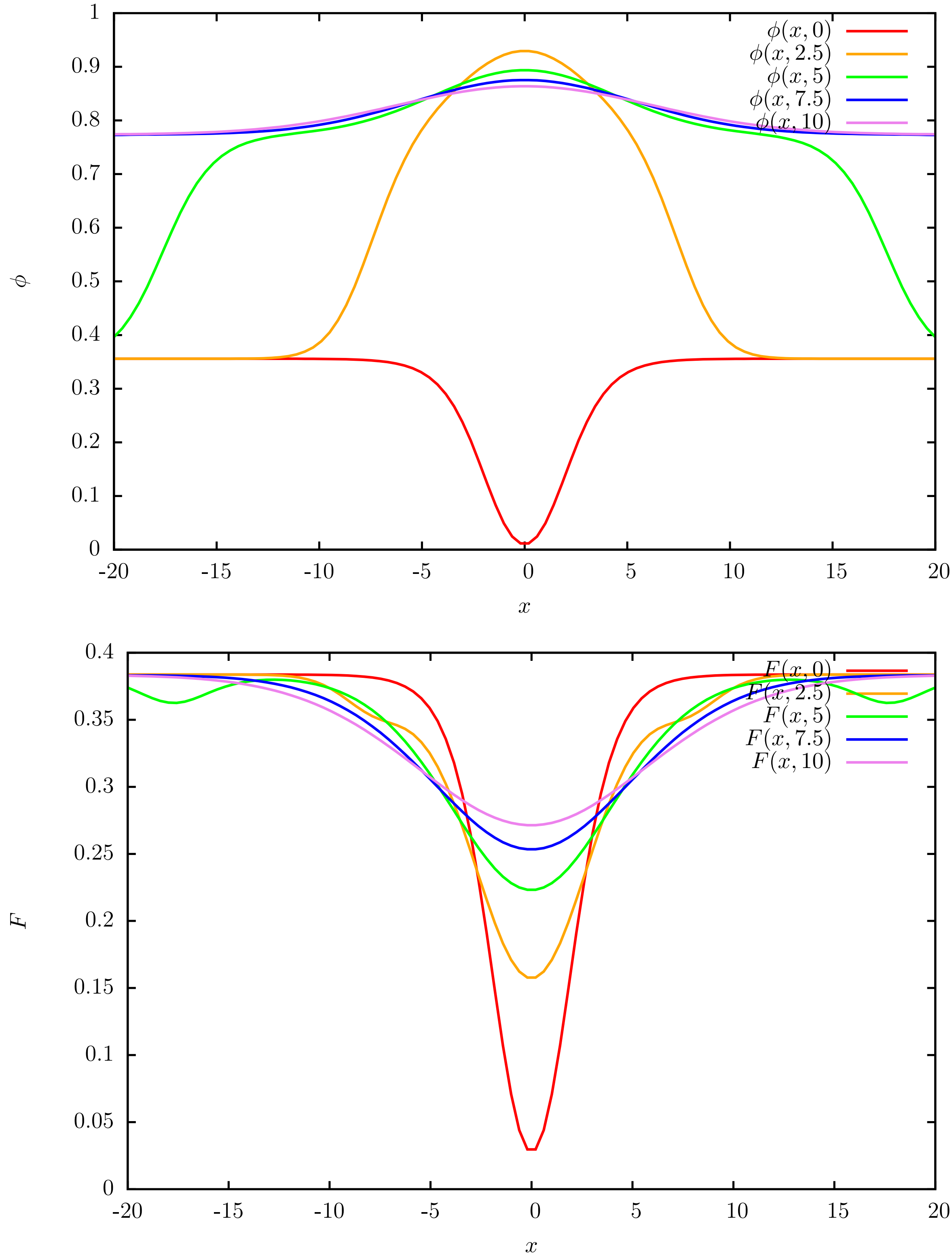

Figure B.4: The Flow for initial data given by $\phi(0) = \phi_{even} + h(x)$, with $h(x) = 0.01e^{\frac{x^2}{2}}$ and $m = 2$, $\beta = 0.9$ (small M case).
$F(0) = \phi_{even} + \frac{M^3}{6\beta^2 m\sqrt{6}}$

Bibliography

[1] H. Georgi. *Lie Algebras In Particle Physics: from Isospin To Unified Theories.* Frontiers in Physics. Perseus Books, Advanced Book Program, 1999. ISBN 9780738202334.

[2] Stanley Deser, R. Jackiw, and S. Templeton. Three-Dimensional Massive Gauge Theories. *Phys.Rev.Lett.*, 48:975–978, 1982. doi: 10.1103/PhysRevLett.48.975.

[3] Shinichi Deguchi. Canonical Formalism for a 2n-Dimensional Model with Topological Mass Generation. *Phys.Rev.*, D78:125014, 2008.

[4] Gia Dvali, R. Jackiw, and So-Young Pi. Topological mass generation in four dimensions. *Phys.Rev.Lett.*, 96:081602, 2006.

[5] Juan Martin Maldacena. The Large N limit of superconformal field theories and supergravity. *Int.J.Theor.Phys.*, 38:1113–1133, 1999.

[6] Edward Witten. (2+1)-Dimensional Gravity as an Exactly Soluble System. *Nucl.Phys.*, B311:46, 1988.

[7] M. F. Atiyah and R. Bott. The Yang-Mills Equations over Riemann Surfaces. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 308:523–615, 1983.

[8] Johan Rade. On the Yang-Mills heat equation in two and three dimensions. *Journal fr die reine und angewandte Mathematik (Crelles Journal)*, 1992:123164, 2009. doi: 10.1515/crll.1992.431.123.

[9] Martin Luscher. Chiral symmetry and the Yang–Mills gradient flow. *JHEP*, 1304: 123, 2013.

[10] Jack Gegenberg, Andrew C. Day, Haitao Liu, and Sanjeev S. Seahra. An instability of hyperbolic space under the Yang-Mills flow. *J.Math.Phys.*, 55:042501, 2014.

[11] S.P. Braham and J. Gegenberg. Yang-Mills flow and uniformization theorems. *J.Math.Phys.*, 39:2242–2253, 1998.

[12] Jeffrey Streets. Ricci Yang-Mills flow on surfaces. *Adv. Math.*, 223:454–475, 2010.

[13] J. Crank. *The Mathematics of Diffusion.* Clarendon Press, 1975.