THE UNIVERSITY OF MANITOBA

CONTRIBUTIONS TO THE THEORY OF HYPERSPACES

by

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ABSTRACT

One of the most natural and most interesting objects associated with a topological space X is its space of closed subsets 2^X. Of the various topologies with which 2^X may be endowed, the one that concerns us here is the so-called finite topology introduced by Vietoris in [48]. We shall refer to the space of closed subsets of X, endowed with the finite topology, as the hyperspace of X. Hyperspaces have been studied by several authors from several points of view. 2^X has been studied in the context of set-valued mappings, fixed-point theorems, and selections. This approach is illustrated in the collected papers in [46]. The study of 2^X when X is a continuum or metric continuum has occupied the interest of many topologists. (A few examples of this are [8], [31], [50]). The comprehensive work of E. Michael [36] is the standard reference for the fundamental properties of 2^X. In [36], Michael describes various topologies and uniformities on spaces of subsets and examines such basic topics as separation axioms, countability, compactness, continuous functions, connectedness and selections. Further basic properties of 2^X are examined in [32], where one may find a treatment of such topics as set-valued mappings and decomposition spaces. The relation of 2^X to lattices and Brouwerian algebras, and the role of 2^X as a topological semilattice are also elucidated in [32].

In this work, we are primarily concerned with properties related to compactness in 2^X. Such properties are of great interest and have received considerable attention. One of the earliest and most elegant results on hyperspaces is the fundamental compactness theorem, established by Vietoris, asserting that 2^X is compact when X is. This result is basic in the study of several compactness-related properties of 2^X. Important progress in the study of compactness-related properties of 2^X has recently been made by J. Keesling, who, in a series of papers, [27], [28], [29], [30], obtained many significant results, including the fascinating result that normality and compactness are equivalent in hyperspaces [28]. Keesling's results have motivated much of the present work.

The first chapter is devoted to a study of pseudocompact and countably compact spaces, the emphasis being on powers and products.

In Chapter 2, we apply the results of the first chapter in examining the countable compactness and pseudocompactness of 2^X.

In the third chapter, our attention is focused on the Stone-Cech compactification of 2^X, and particularly on the validity of the relation $\beta(2^X) = 2^{\beta X}$. The results of Chapter 2 provide us with a fairly large class of spaces for which this relation is valid.

The role of $2^{\beta X}$ as a compactification of 2^X is further examined in Chapter 4, where we describe the G_δ-closure of 2^X

in $2^{\beta X}$. This description enables us to obtain information on the realcompactness of 2^X.

In the final chapter, in a somewhat different, though not unrelated vein, we examine some of the cardinal invariants of 2^X, including weight, character and cellularity.

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Chapter 0

PRELIMINARIES

Our topological terminology and notation are well established, and follow the standard texts on point-set topology. For background material on rings of continuous functions and compactifications, we refer the reader to the Gillman and Jerison text [15]. Following [15], the ring of continuous real-valued functions on a topological space X is denoted by C(X), and its subring of bounded members by $C^*(X)$. For $f \in C(X)$, the set $\{x \in X: f(x) = 0\}$ is called the zero-set of f, and is denoted by Z(f). A cozero-set is the complement of a zero-set. The set of all zero-sets of functions in C(X) is denoted by Z(X). The Stone-Cech compactification of a completely regular, Hausdorff space X is denoted by βX. It is characterized as the compactification of X to which all bounded continuous real-valued functions on X may be continuously extended. The set (and discrete space) of positive integers is denoted by $\underset{\sim}{N}$. The points of $\beta\underset{\sim}{N}-\underset{\sim}{N}$ are the free ultrafilters on $\underset{\sim}{N}$, and they play a dual role in this thesis, as points of the space $\beta\underset{\sim}{N}$, and as ultrafilters. The space of real numbers is denoted by $\underset{\sim}{R}$, and the cardinality of a set S is denoted by $|S|$.

The notions from set theory that we shall employ are standard. An ordinal is thought of as the set of its predecessors, and a cardinal as an initial ordinal. The symbol ω_α is used to denote the α'th infinite cardinal. For a discussion of the cardinal invariants discussed in this thesis, we refer the reader to [4] and [25].

In the present work, the main object of our study is the space of closed subsets of a topological space. We now recall the definition of the space of closed sets, and state several basic facts concerning this space.

Let X be a topological space. Let 2^X denote the set of all non-empty closed subsets of X. For a subset A of X, we let $2^A = \{F \in 2^X: F \subseteq A\}$. We generate a topology on 2^X by taking all sets of the form 2^G and all sets of the form $2^X - 2^{X-G}$, for G open in X, as a sub-basis. This topology on 2^X is known as the finite topology, and 2^X, endowed with this topology, is called the <u>hyperspace of X</u>.

Our basic references for the fundamental properties of 2^X are [32] and [36].

Following [32], we make the following notational convention. For subsets $A_0, A_1, \ldots, A_n$ of X, we let $B(A_0; A_1, \ldots, A_n) = 2^{A_0} \cap \bigcap_{i=1}^{n} (2^X - 2^{X-A_i}) = \{F \in 2^X: F \subseteq A_0 \text{ and } F \cap A_i \neq \phi \text{ for all } i = 1,2 \ldots, n\}$. Using this notation we see that the sets $B(G_0; G_1, \ldots, G_n)$ where $G_0, G_1, \ldots, G_n$ are open and $\bigcup_{i=1}^{n} G_i \subseteq G_0$, form a basis for the open subsets of 2^X.

We now state several basic facts about hyperspaces which we will need in the course of our discussion.

0.1. If X is T_1, the singletons of 2^X form a subspace homeomorphic to X. ([36])

0.2. For each positive integer n, we set $F_n(X) = \{F \in 2^X: |F| \leq n\}$, and we set $F(X) = \bigcup_{n \in \mathbb{N}} F_n(X)$. If X is T_1, then F(X) is dense in 2^X. If X is Hausdorff, then $F_n(X)$ is closed in 2^X for each $n \in \mathbb{N}$. (See 2.4 in [36].)

0.3. The operation of set-theoretic union, $(A,B) \to A \cup B$, is a continuous map from $2^X \times 2^X$ into 2^X. (See page 166 of [32].)

0.4 2^X is compact Hausdorff if, and only if, X is compact Hausdorff. (See 4.9 in [36].)

0.5. 2^X is completely regular (and Hausdorff) if, and only if, X is normal and Hausdorff. (See 4.9 in [36].)

0.6. If X is normal and Hausdorff, the natural mapping $i: 2^X \to 2^{\beta X}$ defined by $i(F) = cl_{\beta X}F$ is an embedding of 2^X onto a dense subspace of $2^{\beta X}$. (See [27].)

0.7. Let f be a bounded, real-valued function on X. We define real-valued functions f^s and f^i on 2^X by $f^s(F) = \sup\{f(x): x \in F\}$ and $f^i(F) = \inf\{f(x): x \in F\}$. Then if f is continuous, so are f^i and f^s. We have for $f \geq 0$, $Z(f^s) = 2^{Z(f)}$ and, if X is countably compact, $Z(f^i) = B(X: Z(f))$. Identifying X with the singletons in 2^X, we see that, for a T_1 space X, X is C^*-embedded in 2^X. (See 4.7 and 4.8 in [36].)

0.8. If X is normal and T_1, the sets of the form $B(X; Z_0) \cup 2^{Z_1} \cup \ldots \cup 2^{Z_n}$, where $Z_0, Z_1, \ldots, Z_n \in Z(X)$, form a base for the closed sets in 2^X. This can be verified in a straightforward manner.

From Chapter 2 on, we will assume that all spaces under consideration are T_1, and this assumption will be used without explicit mention in some cases. These spaces are not consistently assumed to satisfy separation axioms other than T_1. Higher separation axioms do enter in certain of our results and arguments in a significant and essential way, and in such situations we are explicit as to what separation axioms are assumed. But we repeat that the assumption that all topological spaces discussed are T_1 is tacit from Chapter 2 on. One further word on separation axioms: the term completely regular, even when unmodified, implies Hausdorff throughout this thesis.

Theorems are referred to by number. "Theorem 2.6 of Chapter 1" indicates the sixth theorem in the second section of Chapter 1. When the number of the chapter is not indicated, it is to be understood that the reference is to the present chapter.

Chapter 1

COUNTABLY COMPACT AND PSEUDOCOMPACT SPACES

1. In this chapter we are concerned with certain aspects of the theory of countable compactness and pseudocompactness. Several of the ideas and results of this chapter will subsequently be applied to the countable compactness and pseudocompactness of hyperspaces; however, the main interest and significance of these results lie in their contribution to the general theory of countable compactness and pseudocompactness. The material presented in this chapter is part of joint work by the author and Victor Saks, whose contribution the author gratefully acknowledges. This work appears in [20].

We characterize spaces all of whose powers are countably compact, and obtain partial results on the corresponding question for pseudocompactness. The basic tool in this work is A. R. Bernstein's concept of $\mathcal{D}$-compactness ([1]). The maximal $\mathcal{D}$-compact extension of a completely regular space is constructed. Additional product theorems for pseudocompact spaces are proved, imposing conditions closely related to $\mathcal{D}$-compactness on the factors, which imply the pseudocompactness of the product. In the last section of the chapter, we prove several theorems which provide new examples of non-trivial pseudocompact spaces. In particular, we exhibit a homogeneous space, all of whose powers are pseudocompact, in which no discrete countable set has a cluster point.

2. Countably Compact Powers. Let us recall several definitions of compactness-like conditions which depend on the behaviour of countable sets.

Let X be a topological space.

X is said to be countably compact, if every countably infinite subset of X has a cluster point.

A subset A of X is relatively countably compact in X, if every countably infinite subset of A has a cluster point in X.

X is sequentially compact, if every sequence in X has a convergent subsequence.

X is called strongly ω_0-compact, if every infinite subset of X meets some compact subset of X in an infinite set.

Finally, we call X ω_0-bounded, if every countable subset of X is contained in a compact subset of X.

Our first result characterizes those spaces X such that every power of X is countably compact. The main tool in this investigation is Bernstein's concept of $\mathcal{D}$-compactness. In [1] the concept was introduced, and some of the basic theory of $\mathcal{D}$-compact spaces was developed. We now give his definition of $\mathcal{D}$-compactness, and quote the major results in [1], including a proof of his result that $\mathcal{D}$-compactness is a productive property.

2.1 Definition. Let $\mathcal{D}$ be a free ultrafilter on $\underset{\sim}{N}$. Let X be a topological space, and let $(x_n: n \in \underset{\sim}{N})$ be a sequence in X. A point $z \in X$ is said to be a <u>$\mathcal{D}$-limit point</u> of the sequence $(x_n: n \in \underset{\sim}{N})$ if, for every neighbourhood W of z, $\{n: x_n \in W\} \in \mathcal{D}$. We shall express this by writing $z \in \mathcal{D}\text{-}\lim_{n\to\infty} x_n$. In Hausdorff spaces, $\mathcal{D}$-limit points, when they exist, are unique, in which case we write $z = \mathcal{D}\text{-}\lim_{n\to\infty} x_n$. A space X is said to be <u>$\mathcal{D}$-compact</u> if every sequence in X has a $\mathcal{D}$-limit point.

Observe that a $\mathcal{D}$-limit point of a sequence of distinct points $(x_n: n \in \underset{\sim}{N})$ is, in particular, a cluster point of the set $\{x_n: n \in \underset{\sim}{N}\}$. Therefore, a $\mathcal{D}$-compact space is countably compact.

2.2 Lemma. <u>Let</u> $\{x_n: n \in \underset{\sim}{N}\} \subseteq X$ <u>and let</u> $z \in X$ <u>be a cluster point of</u> $\{x_n: n \in \underset{\sim}{N}\}$. <u>Then there exists</u> $\mathcal{D}$ <u>in</u> $\beta N - \underset{\sim}{N}$ <u>such that</u> $z \in \mathcal{D}\text{-}\lim_{n\to\infty} x_n$.

Proof. Let $G(z)$ denote the family of all neighbourhoods of z in X. For $W \in G(z)$, let $s(W) = \{n: x_n \in W\}$. The family $F = \{s(W) - \{k\}: W \in G(z), k \in N\}$ has the finite intersection property, and so there is an ultrafilter $\mathcal{D}$ on $\underset{\sim}{N}$ such that $F \subseteq \mathcal{D}$. Obviously $\mathcal{D}$ is free and $z \in \mathcal{D}\text{-}\lim_{n\to\infty} x_n$.

2.3 Lemma. <u>Let $f: X \to Y$ be a continuous map. Let</u> $(x_n: n \in \underset{\sim}{N})$ <u>be a sequence in X, and let $z \in X$ such that</u> $z \in \mathcal{D}\text{-}\lim_{n\to\infty} x_n$. <u>Then</u> $f(z) \in \mathcal{D}\text{-}\lim_{n\to\infty} f(x_n)$.

Proof. For every neighbourhood W of $f(z)$ in Y, $f^{-1}(W)$ is a neighbourhood of z in X. Since $\{n: x_n \in f^{-1}(W)\} = \{n: f(x_n) \in W\}$, the result follows.

2.4 Theorem (Bernstein). <u>$\mathcal{D}$-compactness is closed hereditary and productive. A completely regular space is ω_0-bounded if, and only if, it is $\mathcal{D}$-compact for every $\mathcal{D}$ in $\beta\mathbb{N}-\mathbb{N}$.</u>

Proof. Obviously a closed subset of a $\mathcal{D}$-compact space is $\mathcal{D}$-compact. We will prove the statement concerning products, and refer the reader to Theorems 3.4 and 3.5 of [1] for the last statement.

Thus, let $\{X_\alpha : \alpha \in I\}$ be a family of $\mathcal{D}$-compact spaces, and let $X = \prod_{\alpha\in I} X_\alpha$. We will show that X is $\mathcal{D}$-compact. Let $(x^{(n)} : n \in \mathbb{N})$ be a sequence in X. Then, for each α in I, $(x_\alpha^{(n)} : n \in \mathbb{N})$ has a $\mathcal{D}$-limit point z_α in X_α. This defines a point $z = (z_\alpha)_{\alpha\in I}$ in X. We claim that $z \in \mathcal{D}\text{-}\lim_{n\to\infty} x^{(n)}$. For, let W be any neighbourhood of z in X. There is a finite subset F of I, and open sets W_α in X_α, for each α in F, such that

$$z \in \prod_{\alpha\in F} W_\alpha \times \prod_{\alpha\notin F} X_\alpha \subseteq W.$$

But $\{n : x^{(n)} \in W\} \supseteq \bigcap_{\alpha\in F} \{n : x_\alpha^{(n)} \in W_\alpha\}$, and therefore $\{n : x^{(n)} \in W\} \in \mathcal{D}$. This proves that $z \in \mathcal{D}\text{-}\lim_{n\to\infty} x^{(n)}$ and thus X is $\mathcal{D}$-compact.

2.5 Corollary. <u>Any product of $\mathcal{D}$-compact spaces is countably compact.</u>

Proof. Immediate.

We are now in a position to characterize spaces all of whose powers are countably compact.

2.6 Theorem. Let X be a topological space. The following statements are equivalent:

(i) Every power of X is countably compact;

(ii) X^{2^c} is countably compact;

(iii) $X^{|X|^{\omega_0}}$ is countably compact;

(iv) There exists $\mathcal{D}$ in $\beta\mathbb{N}-\mathbb{N}$ such that X is $\mathcal{D}$-compact.

Proof. (i) => (ii). This is trivial.

(ii) => (iv). We show that if (iv) fails, so does (ii). Thus, suppose X is not $\mathcal{D}$-compact for any $\mathcal{D}$ in $\beta\mathbb{N}-\mathbb{N}$. Then, for each $\mathcal{D}$ in $\beta\mathbb{N}-\mathbb{N}$, there is a sequence $(x_n^{(\mathcal{D})}: n \in \mathbb{N})$ in X which has no $\mathcal{D}$-limit point in X. Define a sequence $(y^{(n)}: n \in \mathbb{N})$ in $X^{\beta\mathbb{N}-\mathbb{N}}$ as follows: $y_{\mathcal{D}}^{(n)} = x_n^{(\mathcal{D})}$.

For the sake of contradiction, assume (ii) holds. Then $X^{\beta\mathbb{N}-\mathbb{N}}$ is countably compact, and therefore the sequence $(y^{(n)}: n \in \mathbb{N})$ has a cluster point z in $X^{\beta\mathbb{N}-\mathbb{N}}$. By Lemma 2.2, there exists $\mathcal{E}$ in $\beta\mathbb{N}-\mathbb{N}$ such that $z \in \mathcal{E}\text{-}\lim_{n\to\infty} y^{(n)}$. But this implies, by Lemma 2.3, that

$$\Pi_{\mathcal{E}}(z) \in \mathcal{E}\text{-}\lim_{n\to\infty} \Pi_{\mathcal{E}}(y^{(n)}) = \mathcal{E}\text{-}\lim_{n\to\infty} x_n^{(\mathcal{E})}.$$

But this is ridiculous, since $(x_n^{(\mathcal{E})}: n \in \mathbb{N})$ has no $\mathcal{E}$-limit point. Thus (ii) must also fail.

(iv) => (i). This follows immediately from 2.5.

(i) => (iii). This is trivial.

(iii) => (iv). Let Σ be the set of all sequences in X. We write $\sigma \in \Sigma$ as $\sigma = (x_n^{(\sigma)} : n \in \mathbb{N})$. Now $|\Sigma| = |X|^{\omega_0}$, and so (iii) implies that X^Σ is countably compact. Define a sequence $(z^{(n)} : n \in \mathbb{N})$ in X^Σ as follows: $z_\sigma^{(m)} = x_m^{(\sigma)}$. Let $p \in X^\Sigma$ be a cluster point of $(z^{(n)} : n \in \mathbb{N})$. By 2.2, there exists $\mathcal{D}$ in $\beta\mathbb{N}-\mathbb{N}$ such that $p \in \mathcal{D}\text{-}\lim_{n\to\infty} z^{(n)}$. We claim that, for this $\mathcal{D}$, X is $\mathcal{D}$-compact. For, if $\sigma = (x_n^{(\sigma)} : n \in \mathbb{N})$ is any sequence in X, Lemma 2.3 implies that

$$\Pi_\sigma(p) \in \mathcal{D}\text{-}\lim_{n\to\infty} \Pi_\sigma(z^{(n)}) = \mathcal{D}\text{-}\lim_{n\to\infty} x_n^{(\sigma)}.$$

Thus every sequence in X has a $\mathcal{D}$-limit point, and so X is $\mathcal{D}$-compact.

2.7 Remark. In [45], Scarborough and Stone have shown that, if $X = \prod_{\alpha\in I} X_\alpha$, then X is countably compact if, and only if, every subproduct of 2^{2^c} factors is countably compact. Thus the conditions (ii) and (iii) in Theorem 2.6 may be regarded as an improvement of their result in the case where all the factors are the same.

2.8 Corollary. If $|X| \leq c$, then X^c is countably compact if, and only if, there exists $\mathcal{D}$ in $\beta\mathbb{N}-\mathbb{N}$ such that X is $\mathcal{D}$-compact.

In [44] the following theorem is proved.

2.9 Theorem. (Saks-Stephenson). _The product of not more than_ ω_1 _strongly_ ω_0_-compact spaces is countably compact._

Assuming the continuum hypothesis [CH], we obtain the following corollary, which gives natural examples of $\mathcal{D}$-compact spaces.

2.10 Corollary. [CH]. If $|X| \leq c$, and if X is strongly ω_0-compact, then there exists $\mathcal{D}$ in $\beta\underset{\sim}{N}-\underset{\sim}{N}$ such that X is $\mathcal{D}$-compact. In particular, every countably compact k-space of cardinality $\leq c$ is $\mathcal{D}$-compact for some $\mathcal{D}$ in $\beta\underset{\sim}{N}-\underset{\sim}{N}$.

Proof. The first assertion is obvious from 2.8 and 2.9, while the second is a special case, by Theorem 1.2 in [39].

2.11 Remark. Since every sequentially compact space is strongly ω_0-compact, the conclusion of 2.10 holds for sequentially compact spaces, of cardinal $\leq c$. This special case of 2.10 also follows directly from Theorem 5.8 in [45], together with our Theorem 2.6.

For non-trivial examples of the spaces hypothesized in 2.10, the reader is referred to [10].

We now turn to another aspect of $\mathcal{D}$-compactness. It follows from the corollary to Theorem 1 in [23], that every completely regular space has a maximal $\mathcal{D}$-compact extension. That is, for every completely regular space X, there is a completely regular $\mathcal{D}$-compact space $\mathcal{D}(X)$ containing X as a dense subspace, such that every continuous map of X into any (completely regular) $\mathcal{D}$-compact space extends continuously to $\mathcal{D}(X)$. From the final section of [23], it follows that, in fact, we may take $X \subseteq \mathcal{D}(X) \subseteq \beta X$ where $\mathcal{D}(X)$ is the intersection of all $\mathcal{D}$-compact subspaces of βX

containing X.

We now show how $\mathcal{D}(X)$ is built up from X. The construction is an exact analogue of Example 4 in [1]. In this example, Bernstein is constructing a $\mathcal{D}$-compact space which is not ω_0-bounded. His construction, when slightly modified, gives the maximal $\mathcal{D}$-compact extension of an arbitrary completely regular space. R. G. Woods independently characterized $\mathcal{D}(X)$ by the same method as given here, in [52].

Let X be a completely regular space. We first construct a transfinite sequence $(X_\alpha : \alpha < \omega_1)$ of subspaces of βX containing X.

Let $X_0 = X$. Assume we have constructed the spaces X_α, for $\alpha < \beta$ such that

(i) $\alpha_1 \leq \alpha_2 < \beta \Rightarrow X_{\alpha_1} \subseteq X_{\alpha_2} \subseteq \beta X$

(ii) $\alpha_1 < \alpha_2 < \beta \Rightarrow$ every sequence in X_{α_1} has a $\mathcal{D}$-limit point in X_{α_2}.

We now construct X_β. Let Σ_β be the set of all sequences in $\bigcup_{\alpha<\beta} X_\alpha$. For each $\sigma \in \Sigma_\beta$, let x_σ be a $\mathcal{D}$-limit point of σ in βX. Finally, let $X_\beta = (\bigcup_{\alpha<\beta} X_\alpha) \cup \{x_\sigma : \sigma \in \Sigma_\beta\}$. This completes the induction step, and gives a sequence $(X_\alpha : \alpha < \omega_1)$ satisfying (i) and (ii) for all $\alpha_1 < \alpha_2 < \omega_1$.

2.12 Theorem. $\mathcal{D}(X) = \bigcup_{\alpha<\omega_1} X_\alpha$.

Proof. Obviously $X \subseteq \bigcup_{\alpha<\omega_1} X_\alpha \subseteq \beta X$. If $(x_n : n \in \mathbb{N})$ is a sequence in $\bigcup_{\alpha<\omega_1} X_\alpha$, it lies entirely within one X_β, and thus has a $\mathcal{D}$-limit point in $X_{\beta+1}$. Therefore $\bigcup_{\alpha<\omega_1} X_\alpha$ is $\mathcal{D}$-compact. A straightforward induction shows that any $\mathcal{D}$-compact subspace of βX containing X must contain every X_α, that is, must contain $\bigcup_{\alpha<\omega_1} X_\alpha$. From the result we have quoted from the Herrlich and van der Slot paper [23], it now follows that $\mathcal{D}(X) = \bigcup_{\alpha<\omega_1} X_\alpha$.

2.13 Corollary. $|\mathcal{D}(X)| \leq |X|^{\omega_0}$.

Proof. This is obvious from the construction described above.

Information on the role of $\mathcal{D}$-compactness as an extension property of topological spaces can be found in [52].

3. Powers and Products of Pseudocompact Spaces. Recall that a space X is pseudocompact if every continuous real-valued function on X is bounded. There is an obvious modification of $\mathcal{D}$-compactness which is suited to the study of pseudocompactness in completely regular spaces. This is because, as Glicksberg observed in [21], a completely regular space X is pseudocompact if, and only if, every sequence of non-empty open subsets of X has a cluster point. (A cluster point of a sequence of sets is a point such that each of its neighbourhoods meets infinitely many sets in the sequence.)

In fact, as Glicksberg shows, it is necessary and sufficient that every sequence of pairwise disjoint, non-empty open sets have a cluster point. This condition in general, (that is, for non-completely regular spaces) is stronger than pseudocompactness. (See [45].)

3.1 Definition. Let $\mathcal{D}$ be a free ultrafilter on $\mathbb{N}$. Let $(S_n : n \in \mathbb{N})$ be a sequence of subsets of a topological space X. A point $p \in X$ is called a $\mathcal{D}$-limit point of the sequence $(S_n : n \in \mathbb{N})$ if, for every neighbourhood W of p, $\{n : S_n \cap W \neq \phi\} \in \mathcal{D}$. A space X is called $\mathcal{D}$-pseudocompact if every sequence of non-empty open subsets of X has a $\mathcal{D}$-limit point.

Making use of arguments similar to those in 2.4, we can readily establish the following facts.

3.2 Theorem. Every $\mathcal{D}$-pseudocompact space is pseudocompact.

3.3 Theorem. Every product of $\mathcal{D}$-pseudocompact spaces is $\mathcal{D}$-pseudocompact.

A corollary of these two theorems is that every power of a $\mathcal{D}$-pseudocompact space is pseudocompact. Now, it follows from Theorem 4 of [21], that any product of pseudocompact, locally compact spaces is pseudocompact, and that any product of pseudocompact, first countable space is pseudocompact. Since there is no reason, in general, to expect such products to be $\mathcal{D}$-pseudocompact, one cannot hope for a result analogous to Theorem 2.6 for pseudocompact powers of completely regular spaces. This can be seen in another way. In [21], Glicksberg shows that a product of completely regular spaces $\prod_{\alpha\in I} X_\alpha$ is pseudocompact if, and only if, every countable subproduct is pseudocompact. Now, for a sequence of sets $\sigma = (S_n : n \in \underset{\sim}{N})$ in X, let $L(\sigma) = \{\mathcal{D} \in \beta\underset{\sim}{N}-\underset{\sim}{N} : \sigma$ has a $\mathcal{D}$-limit point$\}$. Let Σ be the set of sequences (of points) in X, and let Σ_G be the set of all sequences of non-empty open subsets of X. The proof of 2.6 really shows that every power of X is countably compact if, and only if, for <u>every</u> subset T of Σ, $\bigcap_{\sigma\in T} L(\sigma) \neq \phi$. Since every power of a completely regular space X is pseudocompact if, and only if, X^{ω_0} is pseudocompact, we can, in a similar way, conclude that every power of X is pseudocompact if, and only if, for every <u>countable</u> subset T of Σ_G, $\bigcap_{\sigma\in T} L(\sigma) \neq \phi$.

3.4 Example. <u>A completely regular space, all of whose powers are pseudocompact, which is not $\mathcal{D}$-pseudocompact for any $\mathcal{D}$ in $\beta\underset{\sim}{N}-\underset{\sim}{N}$.</u>

For each p in $\beta\underset{\sim}{N}-\underset{\sim}{N}$, let $X_p = \beta\underset{\sim}{N}-\{p\}$. Let $X = \prod_{p\in\beta\underset{\sim}{N}-\underset{\sim}{N}} X_p$.

Since every power of X is a product of locally compact, pseudocompact spaces, every power of X is pseudocompact. But the factor $X_{\mathcal{D}}$ of X is not $\mathcal{D}$-pseudocompact, and so X is not $\mathcal{D}$-pseudocompact for any $\mathcal{D}$ in $\beta\underset{\sim}{N}-\underset{\sim}{N}$.

3.5 If a space X has a dense subset D such that every sequence in D has an accumulation point in X, then obviously X is pseudocompact. Many of the familiar examples of pseudocompact spaces have this property, and this criterion for pseudocompactness has been used profitably in many instances. We refer the reader to [3] and [13] for excellent examples of this.

With this in mind, another natural application of $\mathcal{D}$-compactness to the study of pseudocompactness arises. Let us consider spaces X which have a dense subset A such that every sequence in A has a $\mathcal{D}$-limit point in X. Calling such spaces densely-$\mathcal{D}$-compact, we can establish the following theorem.

3.6 Theorem. Every product of densely-$\mathcal{D}$-compact spaces is densely-$\mathcal{D}$-compact. Every densely-$\mathcal{D}$-compact space is $\mathcal{D}$-pseudocompact.

Proof. The first assertion follows in a straightforward manner, using an argument similar to that in 2.4. To prove the second statement, let X be densely-$\mathcal{D}$-compact. Let A be a dense subset of X such that every sequence in A has a $\mathcal{D}$-limit point in X. Now, let $(G_n : n \in \mathbb{N})$ be any sequence of non-empty open sets in X. For each n, there exists a point $a_n \in G_n \cap A$. Let $p \in X$ be a $\mathcal{D}$-limit point of the sequence $(a_n : n \in \mathbb{N})$. Then, clearly p is a $\mathcal{D}$-limit point of the sequence $(G_n : n \in \mathbb{N})$. Therefore, X is $\mathcal{D}$-pseudocompact.

4. Examples of Pseudocompact Spaces. In this section we prove several theorems which provide new examples of non-trivial pseudocompact spaces.

Let us first recall the notion of type in $\beta N - N$. The equivalence relation ~ defined on $\beta N - N$ by $x \sim y$ if there exists a homeomorphism of βN onto itself taking x to y, decomposes $\beta N - N$ into equivalence classes called types. For $p \in \beta N - N$, $T(p)$ denotes the type of p. Recall that, for any $p \in \beta N - N$, $T(p)$ is dense in $\beta N - N$. (See 6S in [15], and [12].) Note also that every type is a homogeneous space.

We are indebted to Z. Frolik for communicating the following lemma.

4.1 Lemma. (Frolik). Let T be a type of $\beta N - N$. Then no countable discrete subset of T has a cluster point in T.

Proof. Suppose the statement is false. We shall derive a contradiction. Thus, let $(x_n : n \in N)$ be a discrete subset of a type T which has a cluster point in T, say x. Find pairwise disjoint, infinite subsets $\{A_n : n \in N\}$ of N such that $N = \bigcup_{n \in N} A_n$ and $x_n \in cl_{\beta N} A_n$ for each n.

Now, for each n, x_n and x are of the same type, so we can find, for each n, a homeomorphism $f_n : \beta N \to \beta N$ such that $f_n(x_n) = x$. Let g_n denote the restriction of f_n to A_n. Define

$F\colon \mathbb{N} \to \mathbb{N}$ by $F = \bigcup_{n \in N} g_n$. Let F^β denote the Stone extension of F to $\beta\mathbb{N}$. Continuity implies that $F^\beta(x_n) = x$ for each n, and therefore implies that $F^\beta(cl_{\beta\mathbb{N}}\{x_n : n \in \mathbb{N}\}) = \{x\}$. Thus $F^\beta(x) = x$.

We now appeal to a result of Katetov, in [26], which implies that the fixed points of F^β are precisely the points in the $\beta\mathbb{N}$-closure of the set of fixed points of F. (For a detailed proof, see Lemma 9.1 in [6].). Thus, letting $U = \{p \in \beta\mathbb{N} : F^\beta(p) = p\}$, we have $U = cl_{\beta\mathbb{N}}(U \cap \mathbb{N})$. In particular, U is open in $\beta\mathbb{N}$. Since $x \in U$, and since x is a cluster point of $\{x_n : n \in \mathbb{N}\}$, there is an integer k such that $x_k \in U$. For such an integer k, we then have to conclude that $x_k = F^\beta(x_k) = x$.

But this is ridiculous, since x is a cluster point of $\{x_n : n \in \mathbb{N}\}$ and $\{x_n : n \in \mathbb{N}\}$ is discrete.

As was remarked in 3.5, the pseudocompactness of many familiar spaces can be deduced by the presence of a relatively countably compact dense subspace. One of the first examples of a pseudocompact space which has no dense countably compact subspace appears in [35]. The following Theorem 4.2, together with Lemma 4.1, shows there are pseudocompact spaces in which no countable discrete set has a cluster point. Assuming the continuum hypothesis, in 4.3 below, we exhibit a pseudocompact space in which no countable set has a cluster point. It follows that, in all of these examples, there is no dense relatively countably compact subspace. In Theorem 4.5 we show that these spaces have all of their powers pseudocompact. These results show that pseudocompact spaces can be as far from countably compact as is imaginable.

We will show that, if q is a non-P-point of $\beta N - N$, then T(q) is pseudocompact. (For the definition and basic properties of P-points, see 4K and 4L of [15].) We use the fact that, if q is a non-P-point of $\beta N - N$, there exists a partition $\{B_n : n \in N\}$ of N into infinite sets, such that for each $A \in q$, we have $\{n: A \cap B_n$ is infinite$\}$ is infinite. This can be shown directly, as in Lemma 9.14 of [6].

Our original theorem on the pseudocompactness of types held for a more restricted class of types. We are grateful to W. W. Comfort for pointing out that our construction works for all non-P-point types.

4.2 Theorem. <u>If q is a non-P-point of $\beta N - N$, then T(q) is pseudocompact</u>.

Proof. By the result of Glicksberg's quoted earlier, it is sufficient to prove that every sequence of pairwise disjoint, non-empty open subsets of T(q) has a cluster point in T(q).

Thus, let $(G_n : n \in N)$ be a sequence of pairwise disjoint, non-empty open subsets of T(q). For each n, there is an infinite subset A_n of N such that $(\mathrm{cl}_{\beta N} A_n) \cap T(q) \subseteq G_n$. We claim that, for $n \neq m$, $A_n \cap A_m$ is finite. For, if $A_m \cap A_n$ was infinite, then $cl_{\beta N} A_m \cap cl_{\beta N} A_n$ would be an open subset of βN that meets $\beta N - N$. The density of $T(q)$ would imply that $(cl_{\beta N} A_m) \cap (cl_{\beta N} A_n) \cap T(q) \neq \phi$, which contradicts the disjointness of G_n and G_m. Thus $n \neq m$ implies that $A_n \cap A_m$ is finite. For each n, let $A_n' = A_n - \bigcup_{i<n} A_i$. Then $\{A_n' : n \in N\}$ is a family of pairwise disjoint infinite subsets of N such that $(cl_{\beta N} A_n') \cap T(q) \subseteq G_n$ for each n. Let $C_1 = A_1' \cup (N - \bigcup_{n \in N} A_n')$ and let $C_n = A_n'$ for $n > 1$. To find a cluster point of $(G_n : n \in N)$ it clearly suffices to find a cluster point of the sequence $((cl_{\beta N} C_n) \cap T(q) : n \in N)$.

We have thus reduced the task of showing $T(q)$ is pseudo-compact to the following: We must show that, for every partition of N into infinite sets $\{A_n : n \in N\}$, the sequence $((cl_{\beta N} A_n) \cap T(q) : n \in N)$ has a cluster point in $T(q)$. To this end, let $N = \bigcup_{n \in N} A_n$ be such a partition. Since q is not a P-point of $\beta N - N$, there is a partition $N = \bigcup_{n \in N} B_n$ of N into infinite sets, such that, for each $A \in q$, $\{n : A \cap B_n \text{ is infinite}\}$ is infinite. Let $f: N \to N$ be a bijection taking B_n onto A_n for every n. Let f^β denote its Stone-extension to βN, and let $p = f^\beta(q)$. Then $p \in T(q)$. We claim that p is a cluster point of the sequence $((cl_{\beta N} A_n) \cap T(q) : n \in N)$. To prove this, let $(cl_{\beta N} A) \cap T(q)$ be any basic neighbourhood of p in T(q). Then $A \in p$, and so $f^{-1}(A) \in q$. The set $\{n: f^{-1}(A) \cap B_n \text{ is infinite}\}$ is infinite. Since f is a bijection, for infinitely many n, $A \cap A_n$ is an infinite set. But for any such n, $cl_{\beta \underset{\sim}{N}} A \cap cl_{\beta \underset{\sim}{N}} A_n$ is an open subset of $\beta\underset{\sim}{N}$ that meets $\beta\underset{\sim}{N}-\underset{\sim}{N}$. Since T(q) is dense in $\beta\underset{\sim}{N}-\underset{\sim}{N}$, for any such n, $[(cl_{\beta \underset{\sim}{N}} A) \cap T(q)] \cap [(cl_{\beta \underset{\sim}{N}} A_n) \cap T(q)] = (cl_{\beta \underset{\sim}{N}} A) \cap (cl_{\beta \underset{\sim}{N}} A_n) \cap T(q) \neq \phi$. Thus every neighbourhood of p in T(q) meets infinitely many of the sets $(cl_{\beta \underset{\sim}{N}} A_n) \cap T(q)$. That is, p is a cluster point of $((cl_{\beta \underset{\sim}{N}} A_n) \cap T(q): n \in \underset{\sim}{N})$. As we observed at the beginning of the proof, this enables us to conclude that T(q) is pseudocompact.

4.3 Remark. In [42], assuming the continuum hypothesis, M. E. Rudin shows there exists a non-P-point q in $\beta\underset{\sim}{N}-\underset{\sim}{N}$ such that q is not in the closure of any countable subset of $\beta\underset{\sim}{N}-\underset{\sim}{N}$. By our Theorem 4.2, for such q, T(q) is a pseudocompact space in which no countable subset has a limit point. The assumption that q is not a P-point in 4.2 is essential, since a pseudocompact P-space is finite, and therefore no P-point type is pseudocompact.

The following theorem shows that the non-P-point types are not only pseudocompact, they are, in fact $\mathcal{D}$-pseudocompact. We thus see that $\mathcal{D}$-pseudocompactness arises in very natural and fundamental spaces. As the reader will observe, our proof of the $\mathcal{D}$-pseudocompactness of types exploits the "homogeneity" of $\beta\underset{\sim}{N}$.

4.4 Theorem. <u>Let q be a non-P-point of $\beta\mathbb{N}-\mathbb{N}$. Then there exists $\mathcal{D}$ in $\beta\mathbb{N}-\mathbb{N}$ such that T(q) is $\mathcal{D}$-pseudocompact.</u>

Proof. Let $\{A_n : n \in \mathbb{N}\}$ be an infinite collection of pairwise disjoint infinite subsets of N with $N - \bigcup_{n\in N} A_n$ infinite. Since T(q) is pseudocompact, the sequence $((\mathrm{cl}_{\beta\mathbb{N}} A_n) \cap T(q) : n \in \mathbb{N})$ has a cluster point $p \in T(q)$. A proof completely analogous to Lemma 2.2 shows there exists a free ultrafilter $\mathcal{D}$ in $\beta\mathbb{N}-\mathbb{N}$ such that p is a $\mathcal{D}$-limit point of $((\mathrm{cl}_{\beta\mathbb{N}} A_n) \cap T(q) : n \in \mathbb{N})$. We will show that for this $\mathcal{D}$, T(q) is $\mathcal{D}$-pseudocompact.

Thus, let $(G_n : n \in \mathbb{N})$ be any sequence of non-empty open subsets of T(q). For each n, find an infinite subset B_n of $\mathbb{N}$ such that $(\mathrm{cl}_{\beta\mathbb{N}} B_n) \cap T(q) \subseteq G_n$. Using the Disjoint Refinement Lemma 7.5 of [6], we can find a pairwise disjoint sequence $(C_n : n \in \mathbb{N})$ of infinite subsets of $\mathbb{N}$, such that $\mathbb{N} - \bigcup_{n\in\mathbb{N}} C_n$ is infinite and $C_n \subseteq B_n$ for each n. It clearly suffices to show that $((\mathrm{cl}_{\beta\mathbb{N}} C_n) \cap T(q) : n \in \mathbb{N})$ has a $\mathcal{D}$-limit point in T(q), for such a point will be a $\mathcal{D}$-limit point of $(G_n : n \in \mathbb{N})$. Now, let $f : \mathbb{N} \to \mathbb{N}$ be a bijection taking A_n onto C_n for every n. Let $r = f^{\beta}(p)$. Then $r \in T(q)$. It follows easily that r is a $\mathcal{D}$-limit point of $((\mathrm{cl}_{\beta\mathbb{N}} C_n) \cap T(q) : n \in \mathbb{N})$ in T(q). Thus every sequence of open sets in T(q) has a $\mathcal{D}$-limit point. That is, T(q) is $\mathcal{D}$-pseudocompact.