Application of Translational Addition Theorems to Electric and Magnetic Field Analysis in Many-Sphere Systems

By

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Abstract

The purpose of this study is to find analytical solutions to Laplacian field problems relative to arbitrary configurations of spheres based on novel translational addition theorems derived specifically for scalar Laplacian functions. These theorems are used to express in analytic form the fields due to individual spheres in system of coordinates attached to other spheres, thus allowing for the exact boundary conditions to be imposed.

In the literature, translational addition theorems are available for scalar cylindrical and spherical wave functions. Such theorems are not directly available for the general solution of the Laplace equation.

This thesis presents the derivation of the required translational addition theorems for the general solution of Laplace equation in spherical coordinates and then the application of these theorems to find analytical solutions to some electrostatic and magnetostatic field problems relative to arbitrarily located spheres. Computation results for electric and magnetic spheres have been generated and numerical results are compared with the results obtained by other methods available in the literature for two sphere systems. Such numerical data, of known accuracy, are also useful for validating various approximate numerical methods.

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S.C. Kumara Mudun Kotuwage

Dedication

To my Parents, Ariyapala and Kamala.

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List of Symbols

a_p	Radius of the sphere p
c	Semi-focal distance in bispherical coordinate system
c_{rr}	Self capacitance coefficients
c_{sr}	Mutual capacitance coefficients
d	Separation distance between two spheres in bispherical coordinate system
d_{pq}	Separation distance between the p^{th} and q^{th} spheres
E	Electric field intensity vector
H	Magnetic field intensity vector
j	$\sqrt{-1}$
$j_n(x)$	Spherical Bessel function of order n and argument x
J	Current density vector
k	Wave number or propagation constant
$P_n^m(x)$	Associated Legendre function of degree n, order m and argument x
Q_p^t	Total electric charge on sphere p
u	Any Scalar function satisfying the Laplace equation
$u_{mn}(r,\theta,\phi)$	Solutions to the Laplace equation in spherical coordinates
V_p	Electric potential at the surface of sphere p
$y_n(x)$	Spherical Neumann function of order n and argument x
$Y_n^m(\theta_1,\phi_1)$	Spherical harmonics

$z_n(x)$	Spherical Bessel, Neumann, or Hankel function order n and argument x
(x_p, y_p, z_p)	Cartesian coordinate system of the p^{th} sphere
(r_p, θ_p, ϕ_p)	Spherical coordinate system of the p^{th} sphere
(α, β, ϕ)	Bispherical coordinate system
Φ	Electric potential
$\Phi_p(r_p, \theta_p, \phi_p)$	Electric potential of the p^{th} sphere with respect to coordinate system of the sphere p
$\Phi_p^{(q)}(r_q, \theta_q, \phi_q)$	Translated electric potential of the p^{th} sphere into the coordinate system of the sphere q
$\Phi^t(r_p, \theta_p, \phi_p)$	Total electric potential with respect to the coordinate system of the sphere p
Φ_{im}^t	Total electric potential find by image method
Φ_{bi}^t	Total potential find by using bispherical coordinate system
Φ_m	Magnetic scalar potential
θ_{pq}	Zenith angle of the center of the q^{th} sphere with respect to the coordinate system of the p^{th} sphere
ϕ_{pq}	Azimuthal angle of the center of the q^{th} sphere with respect to the coordinate system of the p^{th} sphere
ρ_{s_q}	Surface charge density on surface of sphere q
ε	Permittivity
μ	Permeability

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Chapter 1

Introduction

Analytical solutions for boundary value field problems relative to multi-object systems can be obtained only in some special cases. To derive solutions to static and stationary field problems, the Laplace equation must be solved subject to the boundary conditions at the surface of each body in the system and to obtain exact analytic solution the surfaces of all the bodies involved must be coordinate surfaces in orthogonal systems of coordinates.

The difficulties encountered when boundaries do not coincide with coordinate surfaces are overcome by applying various numerical methods. In this thesis, a new approach for finding analytical solutions to static and stationary field problems in the presence of many body systems is presented. Available classical methods for systems with one or at most two canonical objects, such as the method of separation variable and method of images cannot be employed for system of three or more objects.

To be able to impose the boundary conditions at the surface of each body, the fields due to all other bodies have to be expressed in term of the coordinates of the system attached to each individual body. Appropriate translational or translational-rotational addition theorems are to be applied to "translate" the field produce by a particular body expressed initially in the coordinate system attached to that body into the coordinate system attached to another body. For particular geometric configurations, it is possible to derive exact analytical solutions which constitute benchmark solutions, useful for determining the accuracy of various approximate techniques.

This thesis is dealing with static electric and magnetic field problems relative to many-sphere systems, such as electrostatic field problems when spheres are kept at known potentials or in an external electric field, while in the case of magnetic fields in the presence of perfect conductor spheres, for instance, the boundary condition at their surface requires that the normal derivative of the scalar potential be equal to zero.

First, the necessary translational addition theorems are derived and, then, those theorems are applied to find the solution to some benchmark problems, relative to two sphere systems, with the results evaluated by comparison with the results obtained by other exact methods. Secondly, the method developed is applied to the analysis of electric and magnetic fields in the presence of a few new configurations of more than two spheres.

In practical engineering applications multi-sphere models are useful for determination of forces on particles and field intensification in colloidal suspensions, computation of fields in material structures with embedded arrays of small bodies and to study the response of nanostructures to electromagnetic fields etc.

1.1 Solution to Laplace Equation

In many electrostatic problems which involve a set of conducting bodies, the charge distributions over each of the metallic surfaces are to be determined when the potentials of all the conducting bodies are given. For homogeneous media outside the conducting bodies, the electrostatic potential satisfies the Laplace equation. The applications of the Laplace equation are not confined to electrostatics. It is widely used in many branches of science and engineering, notably for static and stationary magnetic fields, for direct current fields in conducting media, in astronomy, fluid dynamics, etc.

A general form of the solution of the Laplace equation

$$\nabla^2 u = 0 \tag{1.1.1}$$

has the following expression in spherical coordinates (r, θ, ϕ):

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (A_{nm} u_{nm}^{(1)} + B_{nm} u_{nm}^{(2)}) \tag{1.1.2}$$

where A_{nm} and B_{nm} are constants of integration,

$$u_{nm}^{(1)}(r,\theta,\phi) = r^n P_n^m(\cos\theta)\exp(\mathrm{j}m\phi), \tag{1.1.3}$$

$$u_{nm}^{(2)}(r,\theta,\phi) = r^{-(n+1)} P_n^m(\cos\theta)\exp(\mathrm{j}m\phi). \tag{1.1.4}$$

n, *m* are integers, $\mathrm{j} \equiv \sqrt{-1}$, and P_n^m are associated Legendre functions of the first kind. The above general solution for the Laplace equation is uniquely determined if the value of the function is specified on all boundaries (Dirichlet boundary conditions) or the normal derivative of the function is specified on all boundaries (Neumann boundary conditions).

1.2 Translational Addition Theorems for Spherical Scalar Wave Functions

A first derivation of translational additions theorems for spherical scalar wave functions was presented by Friedman and Russek [1] and later in a more exact form, by Stein [2]. Translational addition theorems for spherical vector wave functions were derived by Cruzan [3].

The scalar Helmholtz equation used to describe time-harmonic scalar waves is

$$\nabla^2 u + k^2 u = 0\,, \tag{1.2.1}$$

where k is the wave number. Its solution in spherical coordinates can be written in the form [2]

$$u_{mn}(r,\theta,\phi) = z_n(kr)P_n^m(\cos\theta)\exp(\mathrm{j}m\phi)\,, \qquad 0 \le n < \infty\,, \quad -n \le m \le n\,. \tag{1.2.2}$$

The symbol z_n stands for either the spherical Bessel function j_n, the spherical Neumann function y_n, or the spherical Hankel functions. Then, for the case of the translation of the original coordinate system (r,θ,ϕ) to the system (r',θ',ϕ') as illustrated in Fig.1, the addition theorems are [3]

$$\begin{aligned} z_n(kr)P_n^m(\cos\theta)\exp(jm\phi) &= \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{p}(-1)^{\mu}\mathrm{j}^{\nu+p-n}(2\nu+1)a(m,n\,|-\mu,\nu\,|\,p) \\ &\cdot j_\nu(kr')z_p(kr_0)P_\nu^\mu(\cos\theta')P_p^{m-\mu}(\cos\theta_0)\exp[\mathrm{j}(m-\mu)\phi_0]\exp(\mathrm{j}\mu\phi'), \quad r' \le r_0, \end{aligned} \tag{1.2.3}$$

$$\begin{aligned} z_n(kr)P_n^m(\cos\theta)\exp(jm\phi) &= \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{p}(-1)^{\mu}\mathrm{j}^{\nu+p-n}(2\nu+1)a(m,n\,|-\mu,\nu\,|\,p) \\ &\cdot j_\nu(kr_0)z_p(kr')P_\nu^\mu(\cos\theta_0)P_p^{m-\mu}(\cos\theta')\exp[\mathrm{j}(m-\mu)\phi']\exp(\mathrm{j}\mu\phi_0), \quad r' \ge r_0. \end{aligned} \tag{1.2.4}$$

where

$$a(m,n\,|\,\mu,\nu\,|\,p) = (-1)^{m+\mu}(2p+1)\left[\frac{(n+m)!(\nu+\mu)!(p-m-\mu)!}{(n-m)!(\nu-\mu)!(p+m+\mu)!}\right]^{1/2}\begin{bmatrix} n & \nu & p \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} n & \nu & p \\ m & \mu & -(m+\mu) \end{bmatrix}, \quad (1.2.5)$$

and $\begin{bmatrix} J_1 & J_2 & J_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}$ is the Wigner 3-J symbol [see Appendix A].

The $\sum_p$ represents the sum over the following p values [2]

$$p = \nu + n, \quad \nu + n - 2, \quad \nu + n - 4, \; \nu + n - 6, \; \cdots\cdots\cdots, |n-\nu|, \qquad (1.2.6)$$

with $|n-\nu| \le p \le n+\nu$. The z_n and z_p functions in (1.2.3), (1.2.4) are of the same type [2]. When z_n is selected to be a spherical Bessel function j_n, then either (1.2.3) or (1.2.4) can be used without restriction on the relative size of r' and r_0 [3].

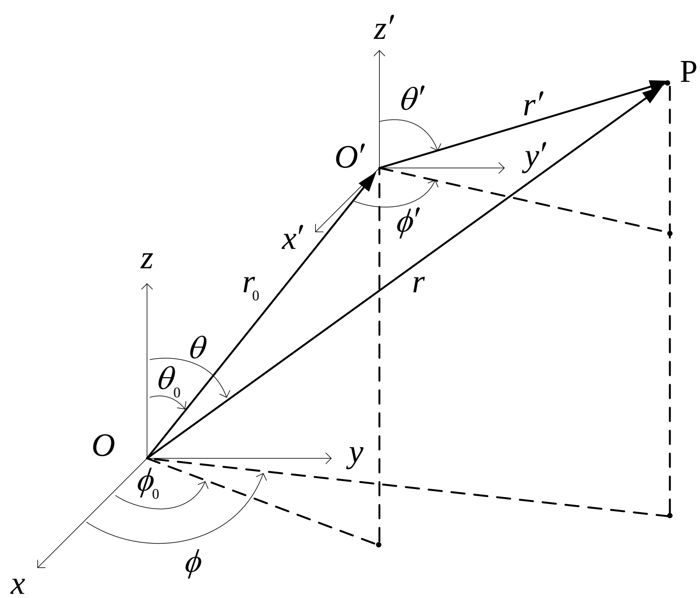

Fig. 1: Translation of the coordinate system

Chapter 2

Translational Addition Theorems for Static and Stationary Fields

To obtain analytical solutions to various static and stationary field problems, for instance problems involving system of arbitrarily located spheres, we need translational addition theorems relative to $u_{nm}^{(1)}$ and $u_{nm}^{(2)}$ in (1.1.2) - (1.1.4). To the best of our knowledge, such theorems are not directly available in the literature. Instead of deriving these theorems from scratch, we can particularize the existing translational addition theorems for spherical scalar waves functions [1]-[3] in the limiting case of a vanishing wave number.

2.1 Limits for Spherical Bessel and Neumann Functions

In the case when the wave number vanishes, $k \to 0$, the Helmholtz equation (1.2.1) becomes the Laplace equation (1.1.1). We derive the translational addition theorems for

(1.1.3) and (1.1.4) from (1.2.3) and (1.2.4) by taking the limits, when $k \to 0$. The expression of spherical Bessel and Neumann functions for vanishing arguments are [4]

$$j_n(kr) \underset{(kr)\to 0}{\to} \frac{(kr)^n}{(2n+1)!!}, \tag{2.1.1}$$

$$y_n(kr) \underset{(kr)\to 0}{\to} -\frac{(2n-1)!!}{(kr)^{(n+1)}}, \tag{2.1.2}$$

where the double factorial notation is used, i.e.,

$$(2n+1)!! \equiv 1\cdot 3\cdot 5\ldots(2n+1),$$
$$(2n-1)!! \equiv 1\cdot 3\cdot 5\ldots(2n-1).$$

2.2 Derivation of the Translational Addition Theorem for $u_{nm}^{(1)}$ in (1.1.3)

To obtain the required result, z_n and z_p in (1.2.3) need to be selected to be spherical Bessel functions j_n and j_p, respectively. Since z_n is selected to be a spherical Bessel function j_n, then either (1.2.3) or (1.2.4) can be used without restriction on the relative size of r' and r_0 [3]. Therefore only (1.2.3) is used to derive the required theorem. Then, (1.2.3) can be written as

$$\begin{aligned} j_n(kr)P_n^m(\cos\theta)\exp(\mathrm{j}m\phi) &= \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{p}(-1)^{\mu}\mathrm{j}^{\nu+p-n}(2\nu+1)a(m,n\,|\,-\mu,\nu\,|\,p) \\ &.j_\nu(kr')j_p(kr_0)P_\nu^{\mu}(\cos\theta')P_p^{m-\mu}(\cos\theta_0)\exp[\mathrm{j}(m-\mu)\phi_0]\exp(\mathrm{j}\mu\phi'). \end{aligned} \tag{2.2.1}$$

For vanishing arguments, kr, kr' and kr_0, (2.2.1) can be written by using (2.1.1) as

$$\frac{(kr)^n}{(2n+1)!!}P_n^m(\cos\theta)\exp(jm\phi)=\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{p}(-1)^{\mu}\mathrm{j}^{\nu+p-n}(2\nu+1)a(m,n\,|-\mu,\nu\,|\,p)$$
$$\cdot\frac{(kr')^{\nu}}{(2\nu+1)!!}\frac{(kr_0)^p}{(2p+1)!!}P_p^{m-\mu}(\cos\theta_0)P_\nu^{\mu}(\cos\theta')\exp[\mathrm{j}(m-\mu)\phi_0]\exp(\mathrm{j}\mu\phi')\,.$$

This equation can be regrouped as

$$r^nP_n^m(\cos\theta)\exp(jm\phi)=\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{p}(-1)^{\mu}\mathrm{j}^{\nu-n}\frac{(2n+1)!!(r')^{\nu}k^{\nu-n}}{(2\nu-1)!!}a(m,n\,|-\mu,\nu\,|\,p)$$
$$\cdot\frac{k^p(\mathrm{j}r_0)^p}{(2p+1)!!}P_p^{m-\mu}(\cos\theta_0)P_\nu^{\mu}(\cos\theta')\exp[\mathrm{j}(m-\mu)\phi_0]\exp(\mathrm{j}\mu\phi'). \tag{2.2.2}$$

For the convenience of the rest of the derivation we can introduce the following notation:

$$f(m,n\,|-\mu,\nu\,|\,p)\equiv a(m,n\,|-\mu,\nu\,|\,p)\frac{(\mathrm{j}r_0)^p}{(2p+1)!!}P_p^{m-\mu}(\cos\theta_0)\;. \tag{2.2.3}$$

According to (1.2.6), the upper and lower limits of p are determined by ν and n. Thus, the summation should be extended over all the possible values of p before taking the limits in (2.2.2). Using (1.2.6) the summation can be written in an expanded form as

$$r^nP_n^m(\cos\theta)\exp(\mathrm{j}m\phi)=\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}(-1)^{\mu}\mathrm{j}^{\nu-n}\frac{(2n+1)!!(r')^{\nu}k^{\nu-n}}{(2\nu-1)!!}[f(m,n\,|-\mu,\nu\,|\,\nu+n)k^{\nu+n}$$
$$+f(m,n\,|-\mu,\nu\,|\,\nu+n-2)k^{\nu+n-2}+f(m,n\,|-\mu,\nu\,|\,\nu+n-4)k^{\nu+n-4}$$
$$+\cdots+f(m,n\,|-\mu,\nu\,|\,|n-\nu|)k^{|n-\nu|}]P_\nu^{\mu}(\cos\theta')\exp[\mathrm{j}(m-\mu)\phi_0]\exp(\mathrm{j}\mu\phi')$$

$$=\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}(-1)^{\mu}\mathrm{j}^{\nu-n}\frac{(2n+1)!!(r')^{\nu}}{(2\nu-1)!!}\Big[f(m,n\,|-\mu,\nu\,|\,\nu+n)\,k^{2\nu}$$
$$+f(m,n\,|-\mu,\nu\,|\,\nu+n-2)k^{2\nu-2}+f(m,n\,|-\mu,\nu\,|\,\nu+n-4)k^{2\nu-4}$$
$$+\ldots+f(m,n\,|-\mu,\nu\,|\,|n-\nu|)k^{|n-\nu|}k^{\nu-n}\Big]P_\nu^{\mu}(\cos\theta')\exp[\mathrm{j}(m-\mu)\phi_0]\exp(\mathrm{j}\mu\phi')\,. \tag{2.2.4}$$

Taking now the limit $k \to 0$, (2.2.4) becomes

$$r^n P_n^m(\cos\theta)\exp(\mathrm{j}m\phi) = \sum_{\nu=0}^{n}\sum_{\mu=-\nu}^{\nu}(-1)^\mu \mathrm{j}^{\nu-n}\frac{(2n+1)!!(r')^\nu}{(2\nu-1)!!} f(m,n|-\mu,\nu|n-\nu) \cdot P_\nu^\mu(\cos\theta')\exp[\mathrm{j}(m-\mu)\phi_0]\exp(\mathrm{j}\mu\phi'). \quad (2.2.5)$$

Note that, when $k \to 0$, the $k^{|n-\nu|}k^{\nu-n} = 0$ for all the values of ν with $\nu > n$ and $k^{|n-\nu|}k^{\nu-n} = 1$ when $\nu \le n$. Hence p is replaced by $(n-\nu)$ in (2.2.3), (2.2.5) . Substituting f from (2.2.3) and regrouping, yields

$$r^n P_n^m(\cos\theta)\exp(\mathrm{j}m\phi) = \sum_{\nu=0}^{n}\sum_{\mu=-\nu}^{\nu}(-1)^\mu \frac{(2n+1)!!}{(2\nu-1)!![2(n-\nu)+1]!!} a(m,n|-\mu,\nu|n-\nu) \cdot r_0^n\left(\frac{r'}{r_0}\right)^\nu P_{n-\nu}^{m-\mu}(\cos\theta_0)P_\nu^\mu(\cos\theta')\exp[\mathrm{j}(m-\mu)\phi_0]\exp(\mathrm{j}\mu\phi'). \quad (2.2.6)$$

The (2.3.6) can be further simplified by substituting for the $a(m,n|-\mu,\nu|n-\nu)$ (see Appendix A-10) and after doing some algebraic manipulations as

$$r^n P_n^m(\cos\theta)\exp(\mathrm{j}m\phi) = \sum_{\nu=0}^{n}\sum_{\mu=-\nu}^{\nu}\frac{(n+m)!}{(\nu+\mu)!(n+m-\nu-\mu)!}\left(\frac{r'}{r_0}\right)^\nu r_0^n \cdot P_{n-\nu}^{m-\mu}(\cos\theta_0)P_\nu^\mu(\cos\theta')\exp[\mathrm{j}(m-\mu)\phi_0]\exp(\mathrm{j}\mu\phi'), \qquad r' \le r_0. \quad (2.2.7)$$

2.3 Derivation of the Translational Addition Theorem for $u_{nm}^{(2)}$ in (1.1.4)

In this case, in order to obtain the required theorem, z_n and z_p in (1.2.3) and (1.2.4) have to be taken to be spherical Neumann functions y_n and y_p, respectively, with their limiting values given by (2.1.2). In the case of $r' \le r_0$, the (1.2.3) can be written as

$$y_n(kr)P_n^m(\cos\theta)\exp(\mathrm{j}m\phi) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{p}(-1)^{\mu}\mathrm{j}^{\nu+p-n}(2\nu+1)a(m,n\,|-\mu,\nu\,|\,p) \cdot j_\nu(kr')y_p(kr_0)P_\nu^\mu(\cos\theta')P_p^{m-\mu}(\cos\theta_0)\exp[\mathrm{j}(m-\mu)\phi_0]\exp(\mathrm{j}\mu\phi')\,. \tag{2.3.1}$$

For vanishing arguments kr, kr' and kr_0 by using (2.1.1) and (2.1.2), (2.3.1) becomes

$$-\frac{(2n-1)!!}{(kr)^{(n+1)}}P_n^m(\cos\theta)\exp(\mathrm{j}m\phi) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{p}(-1)^{\mu}\mathrm{j}^{\nu+p-n}a(m,n\,|-\mu,\nu\,|\,p) \cdot\frac{(kr')^{\nu}}{(2\nu-1)!!}\left[-\frac{(2p-1)!!}{(kr_0)^{(p+1)}}\right]P_\nu^\mu(\cos\theta')P_p^{m-\mu}(\cos\theta_0)\exp[\mathrm{j}(m-\mu)\phi_0]\exp(\mathrm{j}\mu\phi')\,.$$

This equation can be regrouped as

$$r^{-(n+1)}P_n^m(\cos\theta)\exp(\mathrm{j}m\phi) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{p}(-1)^{\mu}\mathrm{j}^{(-n+\nu)}\frac{(r')^{\nu}k^{(\nu+n+1)}}{(2n-1)!!(2\nu-1)!!} \cdot\mathrm{j}^{p}a(m,n\,|-\mu,\nu\,|\,p)\frac{(2p-1)!!}{(kr_0)^{(p+1)}}P_p^{m-\mu}(\cos\theta_0)P_\nu^\mu(\cos\theta')\exp[\mathrm{j}(m-\mu)\phi_0]\exp[\mathrm{j}\mu\phi']. \tag{2.3.2}$$

For the convenience of the rest of the derivation let's introduce the notation:

$$g(m,n\,|-\mu,\nu\,|\,p) \equiv \mathrm{j}^{p}a(m,n\,|-\mu,\nu\,|\,p)\frac{(2p-1)!!}{(r_0)^{(p+1)}}P_p^{m-\mu}(\cos\theta_0)\,. \tag{2.3.3}$$

Before taking the limit for $k \to 0$, the summation in (2.3.2) is expanded over all possible values of p in (1.2.6), i.e.,

$$r^{-(n+1)} P_n^m(\cos\theta)\exp(\mathrm{j}m\phi) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} (-1)^{\mu}\, \mathrm{j}^{(-n+\nu)} \frac{(r')^{\nu}}{(2n-1)!!(2\nu-1)!!} k^{(\nu+n+1)}$$
$$\cdot\left[\frac{g(m,n\,|-\mu,\nu\,|\,n+\nu)}{k^{n+\nu+1}} + \frac{g(m,n\,|-\mu,\nu\,|\,n+\nu-2)}{k^{n+\nu-1}} + \frac{g(m,n\,|-\mu,\nu\,|\,n+\nu-4)}{k^{n+\nu-3}}\right.$$
$$\left. + \ldots + \frac{g(m,n\,|-\mu,\nu\,|\,|n-\nu|)}{k^{|n-\nu|+1}}\right] P_\nu^\mu(\cos\theta')\exp[\mathrm{j}(m-\mu)\phi_0]\exp[\mathrm{j}\mu\phi']$$

$$= \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} (-1)^{\mu}\, \mathrm{j}^{(-n+\nu)} \frac{(r')^{\nu}}{(2n-1)!!(2\nu-1)!!}$$
$$\cdot\left[\frac{g(m,n\,|-\mu,\nu\,|\,n+\nu)}{1} + k^2\,\frac{g(m,n\,|-\mu,\nu\,|\,n+\nu-2)}{1} + k^4\,\frac{g(m,n\,|-\mu,\nu\,|\,n+\nu-4)}{1}\right. \tag{2.3.4}$$
$$\left. + \ldots + \frac{k^{n+\nu+1}k^{|n-\nu|+1} g(m,n\,|-\mu,\nu\,|\,|n-\nu|)}{1}\right] P_\nu^\mu(\cos\theta')\exp[\mathrm{j}(m-\mu)\phi_0]\exp[\mathrm{j}\mu\phi'].$$

For $k \to 0$, (2.3.4) becomes

$$r^{-(n+1)} P_n^m(\cos\theta)\exp(\mathrm{j}m\phi) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} (-1)^{\mu}\, \mathrm{j}^{(-n+\nu)} \frac{(r')^{\nu}}{(2n-1)!!(2\nu-1)!!}$$
$$\cdot\frac{g(m,n\,|-\mu,\nu\,|\,n+\nu)}{1} P_\nu^\mu(\cos\theta')\exp[\mathrm{j}(m-\mu)\phi_0]\exp[\mathrm{j}\mu\phi']. \tag{2.3.5}$$

Substituting g from (2.3.3) and regrouping, yields

$$r^{-(n+1)} P_n^m(\cos\theta)\exp(\mathrm{j}m\phi) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} (-1)^{\mu+\nu} \frac{[2(n+\nu)-1]!!}{(2n-1)!!(2\nu-1)!!} a(m,n\,|-\mu,\nu\,|\,n+\nu)$$
$$\cdot\left(\frac{r'}{r_0}\right)^{\nu} r_0^{-(n+1)} P_{n+\nu}^{m-\mu}(\cos\theta_0) P_\nu^\mu(\cos\theta')\exp[\mathrm{j}(m-\mu)\phi_0]\exp[\mathrm{j}\mu\phi'], \qquad r' \le r_0. \tag{2.3.6}$$

The (2.3.6) can be further simplified by substituting for the $a(m,n\,|-\mu,\nu\,|\,n+\nu)$ (see Appendix A-6) and after doing some algebraic manipulations as

$$r^{-(n+1)} P_n^m(\cos\theta)\exp(\mathrm{j}m\phi) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}(-1)^{\nu+\mu}\frac{(n-m+\nu+\mu)!}{(n-m)!(\nu+\mu)!}\left(\frac{r'}{r_0}\right)^{\nu} r_0^{-(n+1)} \cdot P_{n+\nu}^{m-\mu}(\cos\theta_0)P_\nu^\mu(\cos\theta')\exp[\mathrm{j}(m-\mu)\phi_0]\exp[\mathrm{j}\mu\phi'], \quad r' \le r_0. \tag{2.3.7}$$

Equation (2.3.7) gives the required translational theorem corresponding to (1.1.4), for $r' \le r_0$.

Now consider the case for $r' \ge r_0$. Then, (1.2.4) can be written as

$$y_n(kr)P_n^m(\cos\theta)\exp(\mathrm{j}m\phi) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{p}(-1)^{\mu}\mathrm{j}^{\nu+p-n}(2\nu+1)a(m,n\,|\,-\mu,\nu\,|\,p) \cdot j_\nu(kr_0)y_p(kr')P_\nu^\mu(\cos\theta_0)P_p^{m-\mu}(\cos\theta')\exp[\mathrm{j}(m-\mu)\phi']\exp(\mathrm{j}\mu\phi_0). \tag{2.3.8}$$

For vanishing arguments kr, kr' and kr_0, by using (2.1.1) and (2.1.2), (2.3.8) becomes

$$r^{-(n+1)} P_n^m(\cos\theta)\exp(\mathrm{j}m\phi) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{p}(-1)^{\mu}\mathrm{j}^{\nu-n}\frac{r_0^{\nu}}{(2n-1)!!(2\nu-1)!!}k^{n+\nu+1}\mathrm{j}^{p} \cdot a(m,n\,|\,-\mu,\nu\,|\,p)\frac{(2p-1)!!}{(kr')^{(p+1)}}P_p^{m-\mu}(\cos\theta')P_\nu^\mu(\cos\theta_0)\exp[\mathrm{j}(m-\mu)\phi']\exp(\mathrm{j}\mu\phi_0). \tag{2.3.9}$$

As in the previous cases, all the possible values of p are considered and, then, the limit $k \to 0$ is taken. Finally, we get after some algebraic manipulations the following translational addition theorem for (1.1.4), when $r' \ge r_0$, as

$$r^{-(n+1)} P_n^m(\cos\theta)\exp(\mathrm{j}m\phi) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}(-1)^{\nu+\mu}\frac{(n-m+\nu+\mu)!}{(n-m)!(\nu+\mu)!}\left(\frac{r_0}{r'}\right)^{\nu}(r')^{-(n+1)} \cdot P_{n+\nu}^{m-\mu}(\cos\theta')P_\nu^\mu(\cos\theta_0)\exp[\mathrm{j}(m-\mu)\phi']\exp[\mathrm{j}\mu\phi_0], \quad r' \ge r_0. \tag{2.3.10}$$

2.4 Numerical Evaluation of Series Involved

The convergence of the series in the translational addition theorems given in (2.2.6), (2.3.7) and (2.3.10) can be tested for given *n* and *m* by using numerical values for various variables in both sides of the respective equations. In this section, the numerical testing for (2.3.7) is presented since it is widely used in the next two chapters.

2.4.1 Azimuthal Symmetric Case

First, consider an azimuthal symmetric situation, where the z-axis of the coordinate systems (r,θ,ϕ) and (r',θ',ϕ') are on the same line. Then, (2.3.7) can be simplified as

$$r^{-(n+1)}P_n(\cos\theta)=\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}(-1)^{\nu+\mu}\frac{(n+\nu+\mu)!}{(n)!(\nu+\mu)!}\left(\frac{r'}{r_0}\right)^{\nu}r_0^{-(n+1)}P_{n+\nu}^{-\mu}(\cos\theta_0)P_\nu^{\mu}(\cos\theta'),\ r'\le r_0. \tag{2.4.1}$$

This equation can be normalized as

$$\left(\frac{r_0}{r}\right)^{n+1}P_n(\cos\theta)=\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}(-1)^{\mu+\nu}\frac{(n+\nu+\mu)!}{(n)!(\nu+\mu)!}\left(\frac{r'}{r_0}\right)^{\nu}P_{n+\nu}^{-\mu}(\cos\theta_0)P_\nu^{\mu}(\cos\theta'). \tag{2.4.2}$$

Let's denote the normalized left hand side of the (2.4.2) by

$$f(r,\theta)=\left(\frac{r_0}{r}\right)^{n+1}P_n(\cos\theta),$$

and its normalized right hand side by

$$g(r',\theta')=\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}(-1)^{\mu+\nu}\frac{(n+\nu+\mu)!}{(n)!(\nu+\mu)!}\left(\frac{r'}{r_0}\right)^{\nu}P_{n+\nu}^{-\mu}(\cos\theta_0)P_\nu^{\mu}(\cos\theta'),\quad r'\le r_0\ .$$

Consider Fig. 2 where the point P moves along a circle of radius $r'=$ constant. The numerical values of the functions f and g are calculated at several discrete locations on this circle. Theoretically, f and g should have the same results. In order to find numerical results for *g*, the infinite series is truncated to a finite number of terms. Tables 1-6 show the errors when ν is truncated to *M*.

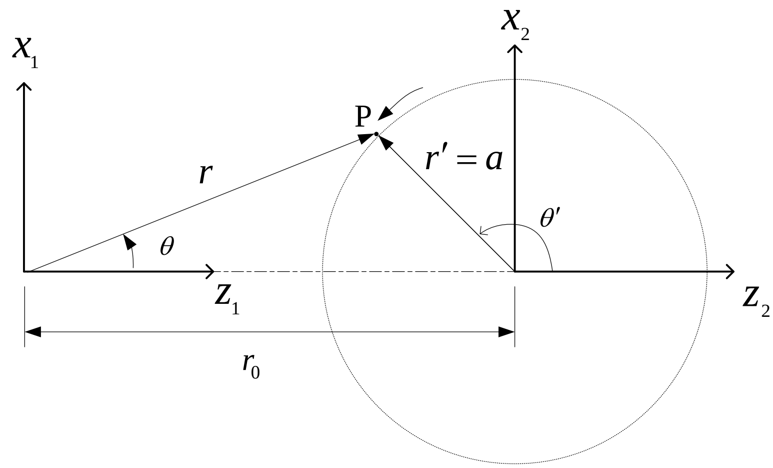

Fig. 2: Point P move along a circle of radius a

Now consider $r_0 = 10$ cm and $a = 5$ cm, then r and θ' are vary when P moves along the surface of the circle. Ten discrete test locations are taken by varying θ' from 0 to 180 degree. In a first case, ν is truncated to 30.

Table 1. Truncation errors of g for $n = 3$, $M = 30$, $\theta_0 = 0$, $r'/r_0 = 0.5$

Test No	r_0/r	θ[deg]	θ'[deg]	$g(r',\theta')$	$f(r,\theta)$	Error [%]
1	0.6667	0.00	0	0.1975	0.1975	0.9116×10^{-03}
2	0.6740	5.98	18	0.1997	0.1997	-0.2124×10^{-03}
3	0.6969	11.82	36	0.2067	0.2067	0.1697×10^{-03}
4	0.7377	17.36	54	0.2197	0.2197	-0.1379×10^{-03}
5	0.8009	22.39	72	0.2425	0.2425	0.1019×10^{-03}
6	0.8944	26.57	90	0.2862	0.2862	-0.0578×10^{-03}
7	1.2289	29.81	126	0.7564	0.7564	0.0265×10^{-03}
8	1.5059	26.27	144	2.3533	2.3533	-0.0268×10^{-03}
9	1.8290	16.41	162	8.5908	8.5908	0.0161×10^{-03}
10	2.0000	0.00	180	16.0000	16.0000	-0.0383×10^{-03}

By increasing the truncation of ν up to the 50, the percentage errors can be reduced as given in Table 2.

Table 2. Truncation errors of g for $n=3$, $M=50$, $\theta_0=0$, $r'/r_0=0.5$

Test No	r_0/r	θ[deg]	θ'[deg]	$g(r',\theta')$	$f(r,\theta)$	Error [%]
1	0.6667	0.00	0	0.1975	0.1975	0.9116×10^{-08}
2	0.6740	5.98	18	0.1997	0.1997	-0.2124×10^{-08}
3	0.6969	11.82	36	0.2067	0.2067	0.1697×10^{-08}
4	0.7377	17.36	54	0.2197	0.2197	-0.1379×10^{-08}
5	0.8009	22.39	72	0.2425	0.2425	0.1019×10^{-08}
6	0.8944	26.57	90	0.2862	0.2862	-0.0578×10^{-08}
7	1.2289	29.81	126	0.7564	0.7564	0.0265×10^{-08}
8	1.5059	26.27	144	2.3533	2.3533	-0.0268×10^{-08}
9	1.8290	16.41	162	8.5908	8.5908	0.0161×10^{-08}
10	2.0000	0.00	180	16.0000	16.0000	-0.0383×10^{-08}

In the third case, n (2.4.2) is increased to 10 and ν is truncated to 50. The results are given Table 3.

Table 3. Truncation errors of g for $n=10$, $M=50$, $\theta_0=0$, $r'/r_0=0.5$

Test No	r_0/r	θ[deg]	θ'[deg]	$g(r',\theta')$	$f(r,\theta)$	Error [%]
1	0.6667	0.00	0	0.0116	0.0116	0.2169
2	0.6740	5.98	18	0.0094	0.0094	-0.0489
3	0.6969	11.82	36	0.0024	0.0024	0.1565
4	0.7377	17.36	54	-0.0112	-0.0112	0.0297
5	0.8009	22.39	72	-0.0342	-0.0342	-0.0081
6	0.8944	26.57	90	-0.0654	-0.0654	0.0030
7	1.2289	29.81	126	-0.1874	-0.1874	-0.0010
8	1.5059	26.27	144	-21.664	-21.664	0.0000
9	1.8290	16.41	162	-203.01	-203.01	0.0000
10	2.0000	0.00	180	2047.9	2048.0	0.0000

The Table 2 and Table 3 show that the percentage error increases with the increase of *n*, if truncation is unchanged. Table 4 further emphasizes the fact that by increasing the truncation, more accurate results can be obtained for higher values for *n*.

Table 4. Truncation errors of g for $n=15$, $M=80$, $\theta_0=0$, $r'/r_0=0.5$

Test No	r_0/r	θ[deg]	θ'[deg]	$g(r',\theta')$	$f(r,\theta)$	Error [%]
1	0.6667	0.00	0	0.0015	0.0015	0.0022331110
2	0.6740	5.98	18	0.0008	0.0008	0.0006443268
3	0.6969	11.82	36	-0.0010	-0.0010	-0.0004170036
4	0.7377	17.36	54	-0.0021	-0.0021	-0.0001741779
5	0.8009	22.39	72	0.0048	0.0048	0.0000627303
6	0.8944	26.57	90	0.0505	0.0505	0.0000040776
7	1.2289	29.81	126	4.3182	4.3182	-0.0000000474
8	1.5059	26.27	144	212.69	212.69	-0.0000000035
9	1.8290	16.41	162	-5266.2	-5266.2	0.0000000003
10	2.0000	0.00	180	65536	65536	-0.0000000002

2.4.2 General Case

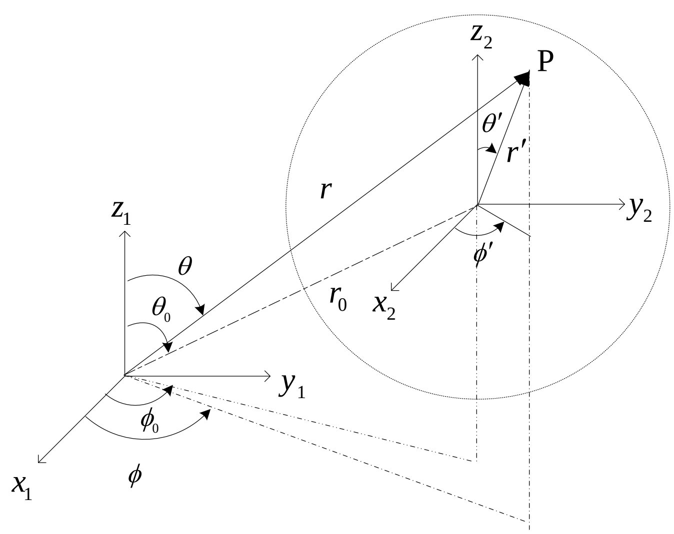

Fig. 3: Point P move along a sphere of radius a

Consider the situation where the azimuthal symmetry is not present in the system. Then f and g should be taken as

$$f(r,\theta,\phi)=\left(\frac{r_0}{r}\right)^{n+1} P_n^m(\cos\theta)\exp(\mathrm{j}m\phi)\,,$$

$$g(r',\theta',\phi')=\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}(-1)^{\mu+\nu}\frac{(n-m+\nu+\mu)!}{(n-m)!(\nu+\mu)!}\left(\frac{r'}{r_0}\right)^{\nu}$$
$$P_{n+\nu}^{m-\mu}(\cos\theta_0)P_\nu^\mu(\cos\theta')\exp[\mathrm{j}(m-\mu)\phi_0]\exp[\mathrm{j}\mu\phi'],\qquad r'\le r_0.$$

When point P moves along a sphere of radius a as in Fig. 3, the numerical values of the function *f* and *g* is calculated at several discrete locations, where $n=10$, $m=-7$, $M=60$, $r_0=10$ cm, $\theta_0=\pi/4$, $\phi_0=\pi/3$, $r'=a=5$ cm and $\phi'=\pi/5$. The real and imaginary parts of f and *g* are given in Table 5 and Table 6, respectively.

Table 5. Truncation errors of the real part of *g*, when $n=10$, $m=-7$, $M=60$, $\theta_0=\pi/4$, $\phi_0=\pi/3$, $r'/r_0=0.5$ and $\phi'=\pi/5$

Test No	r_0/r	θ [deg]	ϕ [deg]	θ' [deg]	Re[$g(r',\theta',\phi')$]	Re[$f(r,\theta,\phi)$]	Error [%]
1	0.7148	30.36	60.00	0	9.7852×10^{-11}	9.7852×10^{-11}	0.1018×10^{-3}
2	0.6865	35.72	55.76	18	2.3855×10^{-10}	2.3855×10^{-10}	0.0061×10^{-3}
3	0.6739	41.48	53.01	36	3.9777×10^{-10}	3.9777×10^{-10}	0.3498×10^{-3}
4	0.6760	47.41	51.31	54	5.9050×10^{-10}	5.9050×10^{-10}	-0.2972×10^{-3}
5	0.6929	53.34	50.38	72	8.1792×10^{-10}	8.1792×10^{-10}	0.0633×10^{-3}
6	0.7262	59.10	50.09	90	1.0020×10^{-09}	1.0020×10^{-09}	-0.0317×10^{-3}
7	0.8584	69.22	51.31	126	-2.4397×10^{-09}	-2.4397×10^{-09}	0.0132×10^{-3}
8	0.9724	72.89	53.01	144	-2.0286×10^{-08}	-2.0286×10^{-08}	0.0016×10^{-3}
9	1.1344	74.77	55.76	162	-1.1559×10^{-07}	-1.1559×10^{-07}	0.0003×10^{-3}
10	1.3572	73.68	60.00	180	-4.4032×10^{-07}	-4.4032×10^{-07}	0.0000×10^{-3}

Table 6. Truncation errors of the imaginary part of *g*, when $n=10$, $m=-7$, $M=60$, $\theta_0=\pi/4$, $\phi_0=\pi/3$, $r'/r_0=0.5$ and $\phi'=\pi/5$

Test No	r_0/r	θ [deg]	ϕ [deg]	θ' [deg]	Im[$g(r',\theta',\phi')$]	Im[$f(r,\theta,\phi)$]	Error [%]
1	0.7148	30.36	60.00	0	-1.6931×10^{-10}	-1.6931×10^{-10}	0.1018×10^{-3}
2	0.6865	35.72	55.76	18	-1.3962×10^{-10}	-1.3962×10^{-10}	-0.2775×10^{-3}
3	0.6739	41.48	53.01	36	-7.8050×10^{-10}	-7.8050×10^{-10}	0.3831×10^{-3}
4	0.6760	47.41	51.31	54	8.4463×10^{-12}	8.4463×10^{-12}	-0.1152×10^{-2}
5	0.6929	53.34	50.38	72	1.0502×10^{-10}	1.0502×10^{-10}	0.7431×10^{-3}
6	0.7262	59.10	50.09	90	1.6552×10^{-10}	1.6552×10^{-10}	0.2753×10^{-3}
7	0.8584	69.22	51.31	126	-3.4896×10^{-11}	-3.4896×10^{-11}	-0.2134×10^{-3}
8	0.9724	72.89	53.01	144	3.9806×10^{-09}	3.9806×10^{-09}	0.7863×10^{-6}
9	1.1344	74.77	55.76	162	6.7657×10^{-08}	6.7657×10^{-08}	0.7353×10^{-7}
10	1.3572	73.68	60.00	180	7.6265×10^{-07}	7.6265×10^{-07}	0.1216×10^{-7}

Chapter 3

Application of Translational Addition Theorems to the Solution of Field Problems with Axisymmetry

3.1 Azimuthally Symmetric Geometries

In the case of azimuthal symmetry, the solution of Laplace equation in spherical coordinates considered in section 1.1 is obtained with $m = 0$, as [5]

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty} [A_n r^n + B_n r^{-(n+1)}] P_n(\cos\theta) \ . \tag{3.1.1}$$

For regions extended to infinity, with $u \to 0$ for $r \to \infty$, we have

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(\cos\theta) \, . \tag{3.1.2}$$

3.2 Point Charge in the Presence of a Conducting Sphere

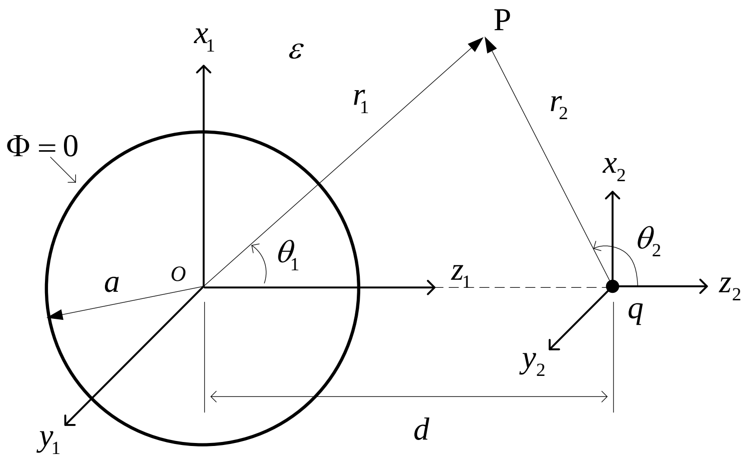

Fig. 4: Conducting sphere in the vicinity of a point charge

Consider a metallic sphere of radius a and a point charge q at a distance d from its centre, as shown in Fig. 4. The sphere has a zero potential and the medium outside the sphere is homogeneous of permittivity ε. The potential due to the presence of the sphere is expanded in (r_1, θ_1, ϕ_1) coordinates as

$$\Phi_1(r_1, \theta_1, \phi_1) = \sum_{n=0}^{\infty} B_n \, r_1^{-(n+1)} P_n(\cos\theta_1) \ , \tag{3.2.1}$$

while the potential of the point charge in (r_2, θ_2, ϕ_2) coordinates is

$$\Phi_2(r_2, \theta_2, \phi_2) = \frac{q}{4\pi\varepsilon r_2} = \frac{K}{r_2} = K r_2^{-1} , \qquad K = \frac{q}{4\pi\varepsilon} . \tag{3.2.2}$$

Φ_2 is translated into the coordinate system (r_1, θ_1, ϕ_1) and the boundary condition is to be imposed to the total potential at the surface of the sphere. Since the points on the sphere have $r_1 = a$, we use equation (2.3.7), corresponding to $r \equiv r_2$, $r' \equiv r_1$, $r_0 \equiv d$ and $r' < r_0$.

Due to the azimuthal symmetry we can use the simplified version of (2.3.7) given in (2.4.1). For this particular translation of the Φ_2, $n = 0$, $\theta_0 = \pi$, $r_0 = d$ and (2.4.1) yields

$$r_2^{-1} = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} (-1)^{\mu+\nu} \left(\frac{r_1}{d}\right)^{\nu} \frac{1}{d} P_\nu^{-\mu}(\cos\pi) P_\nu^{\mu}(\cos\theta_1). \tag{3.2.3}$$

Since $P_\nu^{-\mu}(\pm 1) = 0$, when $\mu \neq 0$, the (3.2.3) can be written in simplified from as

$$r_2^{-1} = \sum_{n=0}^{\infty} \left(\frac{r_1}{d}\right)^{n} \frac{1}{d} P_n(\cos\theta_1), \tag{3.2.4}$$

where ν has been replaced by n. Now we get the expression for the potential Φ_2 translated in (r_1, θ_1, ϕ_1) in the form

$$\Phi_2^{(1)}(r_1, \theta_1, \phi_1) = K \sum_{n=0}^{\infty} \left(\frac{r_1}{d}\right)^{n} \frac{1}{d} P_n(\cos\theta_1). \tag{3.2.5}$$

Note that, since $r_1 \leq d$, the series in this expression is always convergent for all r_1 values.

The total potential Φ^t at any point where $a \leq r_1 \leq d$ is , thus,

$$\Phi^t(r_1, \theta_1, \phi_1) = \Phi_1(r_1, \theta_1, \phi_1) + \Phi_2^{(1)}(r_1, \theta_1, \phi_1). \tag{3.2.6}$$

This expression only contains the coordinates in the sphere system and, therefore, the boundary condition at the sphere surface can easily be imposed to determine all the constants of integration B_n. At $r_1 = a$, $\Phi^t(a,\theta_1,\phi_1) = 0$. Thus, we have

$$\Phi_1(a,\theta_1,\phi_1) + \Phi_2^{(1)}(a,\theta_1,\phi_1) = 0\,,$$

i.e.,

$$\sum_{n=0}^{\infty} B_n a^{-(n+1)} P_n(\cos\theta_1) + K\sum_{n=0}^{\infty}\left(\frac{a}{d}\right)^n \frac{1}{d} P_n(\cos\theta_1) = 0,$$

$$\sum_{n=0}^{\infty} B_n a^{-(n+1)} P_n(\cos\theta_1) = -K\sum_{n=0}^{\infty}\left(\frac{a}{d}\right)^n \frac{1}{d} P_n(\cos\theta_1)\,,$$

which yields

$$B_n a^{-(n+1)} = -K\left(\frac{a}{d}\right)^n \frac{1}{d}\,,$$

$$B_n = -K\,\frac{1}{d^{n+1}}\,a^{(2n+1)}. \qquad (3.2.7)$$

Substituting B_n and K in (3.2.6) yields

$$\Phi^t(r_1,\theta_1,\phi_1) = \frac{q}{4\pi\varepsilon}\sum_{n=0}^{\infty}\frac{r_1^n}{d^{n+1}}\left[1-\left(\frac{a}{r_1}\right)^{2n+1}\right]P_n(\cos\theta_1)\,, \qquad a \le r_1 \le d\,. \qquad (3.2.8)$$

Similarly, using (2.3.10) , we have

$$\Phi_2^{(1)}(r_1,\theta_1,\phi_1) = K\,r_2^{-1} = \sum_{n=0}^{\infty}\frac{d^n}{r_1^{n+1}}P_n(\cos\theta_1)\,, \qquad r_1 > d\,,$$

and the total potential

$$\Phi^t(r_1,\theta_1,\phi_1) = \frac{q}{4\pi\varepsilon}\sum_{n=0}^{\infty}\frac{d^n}{r_1^{n+1}}\left[1-\left(\frac{a}{d}\right)^{2n+1}\right]P_n(\cos\theta_1) \quad , \qquad r_1 > d\,. \qquad (3.2.9)$$

On the other hand, the potential solution for this elementary problem can be obtained by using the image method. The image charge q' with respect to the grounded sphere is placed inside the sphere, as shown in Fig. 5, with

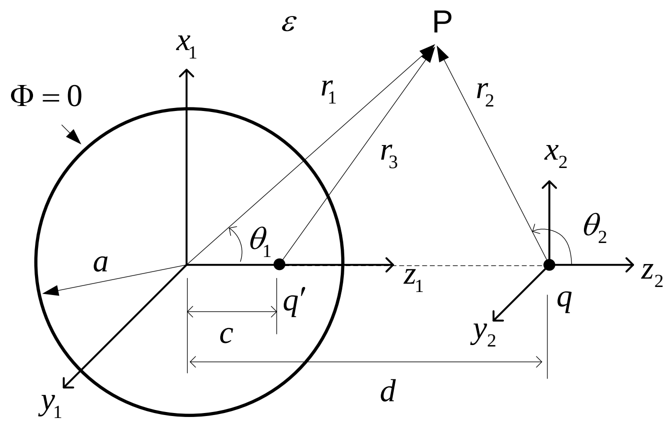

Fig. 5: Image of a point charge with respect to a grounded sphere

The total potential Φ_{im}^t at any point outside the sphere, can be written as

$$\Phi_{im}^t(r_1, \theta_1, \phi_1) = \frac{q}{4\pi\varepsilon r_2} + \frac{q'}{4\pi\varepsilon r_1},$$

i.e.,

$$\Phi_{im}^t(r_1, \theta_1, \phi_1) = \frac{q}{4\pi\varepsilon}\left[\frac{1}{\sqrt{(r_1^2 + d^2 - 2r_1 d\cos\theta_1)}} - \frac{a}{d\sqrt{(r_1^2 + c^2 - 2r_1 c\cos\theta_1)}}\right]. \qquad (3.2.10)$$

It should be noticed that, using the expansion [7]

$$\frac{1}{\sqrt{(r_1^2 + c^2 - 2r_1 c\cos\theta_1)}} = \frac{1}{r_1}\sum_{n=0}^{\infty}\frac{c^n}{r_1^{n+1}}P_n(\cos\theta_1)$$

$$= \sum_{n=0}^{\infty}\left(\frac{a^2}{r_1 d}\right)^n \frac{1}{r_1}P_n(\cos\theta_1), \qquad r_1 \ge a \qquad (3.2.11)$$

and

$$\frac{1}{\sqrt{(r_1^2 + d^2 - 2r_1 d\cos\theta_1)}} = \sum_{n=0}^{\infty}\frac{r_1^n}{d^{n+1}}P_n(\cos\theta_1), \quad a \le r_1 \le d, \qquad (3.2.12)$$

$$\frac{1}{\sqrt{(r_1^2 + d^2 - 2r_1 d\cos\theta_1)}} = \sum_{n=0}^{\infty} \frac{d^n}{r_1^{n+1}} P_n(\cos\theta_1), \qquad r_1 > d, \tag{3.2.13}$$

yields the same expansion (3.2.8) and (3.2.9) for the total potential in terms of Legendre polynomials.

3.2.1 Calculation of the Total Charge on the Sphere

The total charge on the sphere can be found by integrating the charge density over the surface of the sphere, i.e.,

$$Q^t = \int_s \rho_s ds .$$

The surface charge density on the conducting sphere can be obtained by

$$\rho_s(\theta_1, \phi_1) = -\varepsilon \frac{\partial \Phi^t}{\partial r_1}\bigg|_{r_1 = a} .$$

Due to the azimuthal symmetry, the surface charge density does not depend on ϕ. Therefore ρ_s for this case can be expressed as

$$\rho_s(\theta_1) = -\frac{q}{4\pi} \sum_{n=0}^{\infty} \left[\frac{(2n+1)a^{n-1}}{d^{n+1}} \right] P_n(\cos\theta_1).$$

The total charge can be calculated as

$$\begin{aligned} Q^t &= 2\pi a^2 \int_0^{\pi} \rho_s(\theta_1) \sin\theta_1 d\theta_1 \\ &= -a^2 \frac{q}{2} \sum_{n=0}^{\infty} \left[\frac{(2n+1)a^{n-1}}{d^{n+1}} \right] \int_0^{\pi} P_n(\cos\theta_1) \sin\theta_1 d\theta . \end{aligned} \tag{3.2.14}$$

The integral of the Legendre polynomials is [see Appendix B]

$$\int_0^\pi P_n(\cos\theta_1)\sin\theta_1 d\theta_1 = \int_{-1}^{1} P_n(x)dx = \begin{cases} 2, & n=0, \\ 0, & n\neq 0, \end{cases} \qquad \text{with } x=\cos\theta_1. \tag{3.2.15}$$

Thus,

$$Q^t = -a^2 \frac{q}{2}\frac{a^{-1}}{d} 2 = -\frac{a}{d} q \, .$$

This is just the image charge q' given by image method.

3.3 Two Sphere System

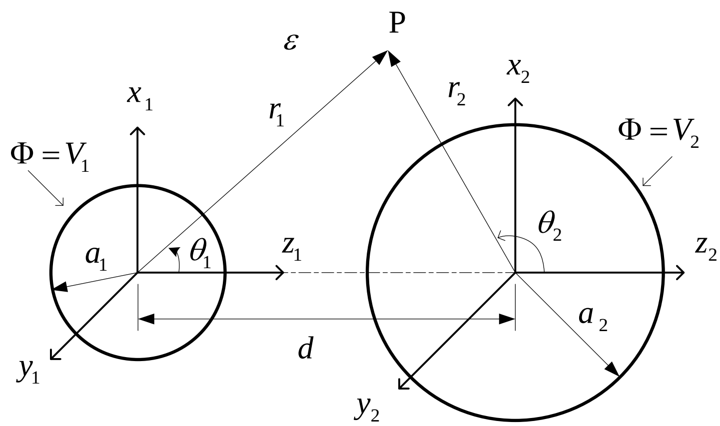

Fig. 6: System of two conducting spheres

Consider now two metallic spheres of radii a_1 and a_2 with a distance d between their centers, as shown in Fig. 6. The spheres are kept at the potentials V_1 and V_2, respectively. The medium outside the spheres is homogeneous of permittivity ε.

The potential produced by the each of the two spheres is first expressed in the coordinate system attached to the respective sphere as [see (3.1.2)]

$$\Phi_1(r_1,\theta_1,\phi_1) = \sum_{n=0}^{\infty} A_n r_1^{-(n+1)} P_n(\cos\theta_1), \qquad r_1 \geq a_1 , \tag{3.3.1}$$

$$\Phi_2(r_2,\theta_2,\phi_2) = \sum_{m=0}^{\infty} B_m r_2^{-(m+1)} P_m(\cos\theta_2), \quad r_2 \geq a_2 . \tag{3.3.2}$$

Then, Φ_2 is translated into the coordinate system (r_1,θ_1,ϕ_1). To impose boundary condition at $r_1 = a_1$, we have to use (2.3.7) corresponding to $r \equiv r_2$, $r' \equiv r_1 = a$, $r_0 \equiv d$ and $\theta_0 = \pi$. Thus,

$$r_2^{-(m+1)} P_m(\cos\theta_2) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} (-1)^{\mu+\nu} \frac{(m+\nu+\mu)!}{(m)!(\nu+\mu)!} \left(\frac{r_1}{d}\right)^{\nu} d^{-(m+1)} P_{m+\nu}^{-\mu}(\cos\pi) P_\nu^{\mu}(\cos\theta_1). \tag{3.3.3}$$

Since $P_{\nu+m}^{-\mu}(\pm 1) = 0$ for all $\mu \neq 0$, we obtain

$$r_2^{-(m+1)} P_m(\cos\theta_2) = \sum_{\nu=0}^{\infty} (-1)^m \frac{(m+\nu)!}{(m)!(\nu)!} \left(\frac{r_1}{d}\right)^{\nu} d^{-(m+1)} P_\nu(\cos\theta_1). \tag{3.3.4}$$

The potential $\Phi_2^{(1)}$ translated in coordinates (r_1,θ_1,ϕ_1) in the form

$$\Phi_2^{(1)}(r_1,\theta_1,\phi_1) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^m B_m \frac{(m+n)!}{(m)!(n)!} \left(\frac{r_1}{d}\right)^{n} d^{-(m+1)} P_n(\cos\theta_1), \tag{3.3.5}$$

where ν has been replaced by n.

The total potential at P is

$$\begin{aligned} \Phi^t(r_1,\theta_1,\phi_1) &= \Phi_1(r_1,\theta_1,\phi_1) + \Phi_2^{(1)}(r_1,\theta_1,\phi_1) \\ &= \sum_{n=0}^{\infty} A_n r_1^{-(n+1)} P_n(\cos\theta_1) + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^m B_m \frac{(m+n)!}{(m)!(n)!} \left(\frac{r_1}{d}\right)^{n} d^{-(m+1)} P_n(\cos\theta_1). \end{aligned} \tag{3.3.6}$$

Now let's apply boundary condition at $r_1 = a_1$, i.e., $\Phi^t(a_1,\theta_1,\phi_1) = V_1$. We get

$$V_1 = \sum_{n=0}^{\infty} A_n a_1^{-(n+1)} P_n(\cos\theta_1) + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^m B_m \frac{(m+n)!}{(m)!(n)!} \left(\frac{a_1}{d}\right)^{n} d^{-(m+1)} P_n(\cos\theta_1),$$

$$\sum_{n=0}^{\infty} A_n a_1^{-(n+1)} P_n(\cos\theta_1) = V_1 - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^m B_m \frac{(m+n)!}{(m)!(n)!} \left(\frac{a_1}{d}\right)^{n} d^{-(m+1)} P_n(\cos\theta_1).$$

Applying the orthogonality properties of the Legendre polynomials [Appendix B] yields

$$A_n a_1^{-(n+1)} = \frac{(2n+1)}{2}\int_{-1}^{1}\left[V_1 - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m B_m \frac{(m+n)!}{(m)!(n)!}\left(\frac{a_1}{d}\right)^n d^{-(m+1)} P_n(x_1)\right] P_n(x_1) dx_1 , \qquad (3.3.7)$$

$$A_0 + \sum_{m=0}^{\infty}(-1)^m B_m \frac{a_1}{d^{m+1}} = V_1 a_1 , \qquad n = 0, \qquad (3.3.8)$$

$$A_n + \sum_{m=0}^{\infty}(-1)^m B_m \frac{(m+n)!}{(m)!(n)!}\frac{a_1^{2n+1}}{d^{n+m+1}} = 0, \qquad n = 1, 2, 3, \cdots. \qquad (3.3.9)$$

In the same way, Φ_1 is translated into the coordinate system (r_2, θ_2, ϕ_2) and the boundary condition is applied at the surface of sphere 2 to obtain the following equations:

$$B_0 + \sum_{n=0}^{\infty}(-1)^m A_n \frac{a_2}{d^{n+1}} = V_2 a_2 , \qquad m = 0, \qquad (3.3.10)$$

$$B_m + \sum_{n=0}^{\infty}(-1)^m A_n \frac{(m+n)!}{(m)!(n)!}\frac{a_2^{2m+1}}{d^{m+n+1}} = 0, \qquad m = 1, 2, 3, \cdots. \qquad (3.3.11)$$

Equations (3.3.8) to (3.3.11) constitute an infinite set of liner algebraic equations and that is to be solved simultaneously in order to find the unknown constants of integrations. To obtain numerical results, the infinite series must be truncated to a finite number of terms $n = m = M$.

Let's denote

$$\xi_1(n, m) = (-1)^m \frac{(m+n)!}{(m)!(n)!}\frac{a_1^{2n+1}}{d^{n+m+1}}$$

$$\xi_2(m, n) = (-1)^m \frac{(m+n)!}{(m)!(n)!}\frac{a_2^{2m+1}}{d^{m+n+1}}$$

The truncated system can be written in a matrix form as

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \xi_1(0,0) & \xi_1(0,1) & \cdots & \xi_1(0,M) \\ 0 & 1 & \cdots & \vdots & \xi_1(1,0) & \xi_1(1,1) & \cdots & \xi_1(1,M) \\ 0 & 0 & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & 0 & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \cdots & 1 & \xi_1(M,0) & \xi_1(M,1) & \cdots & \xi_1(M,M) \\ \xi_2(0,0) & \xi_2(0,1) & \cdots & \xi_2(0,M) & 1 & 0 & \cdots & 0 \\ \xi_2(1,0) & \xi_2(1,1) & \cdots & \xi_2(1,M) & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & 0 \\ \xi_2(M,0) & \xi_2(M,1) & \cdots & \xi_2(M,M) & 0 & \cdots & \cdots & 1 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ \vdots \\ A_M \\ B_0 \\ B_1 \\ \vdots \\ \vdots \\ B_M \end{bmatrix} = \begin{bmatrix} V_1 a_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ V_2 a_2 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix},$$

Once the constants of integrations $A_0, A_1 \ldots\ldots A_M$ and $B_0, B_1 \ldots\ldots B_M$ are found, the total potential Φ^t at any point outside the spheres ($r_1 \geq a_1$, $r_2 \geq a_2$) can be computed as

$$\Phi^t(r_1,\theta_1,\phi_1 \,|\, r_2,\theta_2,\phi_2) \approx \sum_{n=0}^{M} A_n r_1^{-(n+1)} P_n(\cos\theta_1) + \sum_{m=0}^{M} B_m r_2^{-(m+1)} P_m(\cos\theta_2) \,. \qquad (3.3.12)$$

3.3.1 Solution to the Two Sphere Problem in Bispherical Coordinates

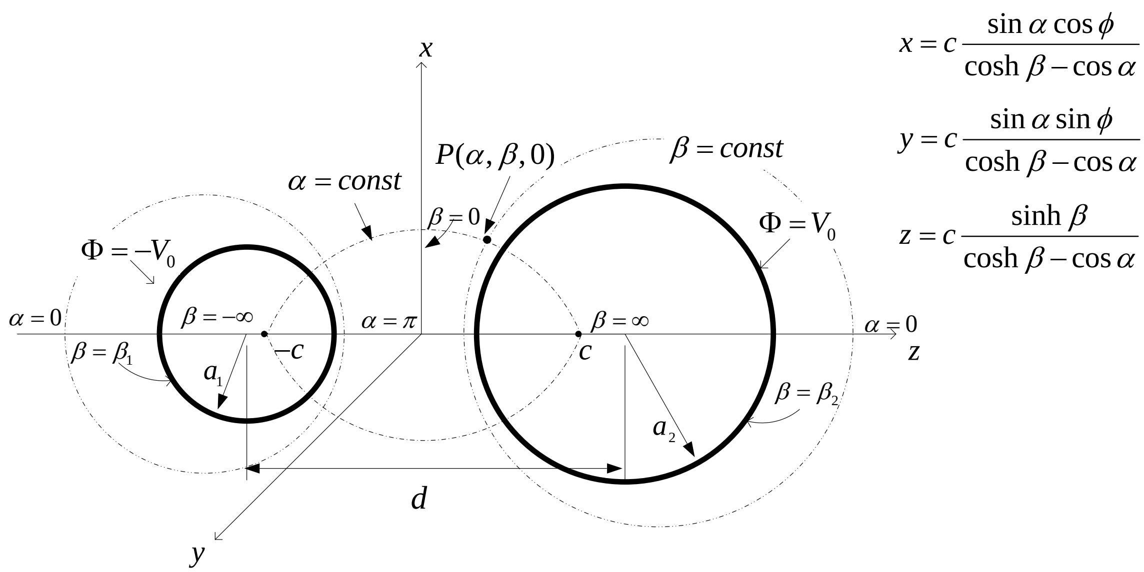

Fig. 7: Two spheres in bispherical coordinate system

On the other hand, in the case of two spheres, an exact analytical solution can be found by using the bispherical coordinate system [13]. For $V_1 = -V_0$ and $V_2 = V_0$, for example, (see Fig.7), the total potential outside the spheres is obtained in the bispherical coordinates (α,β,ϕ) defined in Fig.7 as [6]

$$\Phi^t_{bi}(\alpha,\beta,\phi) = \sqrt{2}V_0(\cosh\beta - \cosh\alpha)^{1/2} \cdot \sum_{n=0}^{\infty}\left[\frac{\left(\cosh[(n+1/2)\beta_1]e^{-(n+1/2)(\beta_2-\beta)} - \cosh[(n+1/2)\beta_2]e^{-(n+1/2)(\beta-\beta_1)}\right)}{\sinh[(n+1/2)(\beta_2-\beta_1)]}P_n(\cos\alpha)\right], \quad (3.3.13)$$

where $\beta_1 = \sinh^{-1}(-c/a_1)$, $\beta_2 = \sinh^{-1}(c/a_2)$, $\sqrt{c^2 + a_1^{\,2}} + \sqrt{c^2 + a_2^{\,2}} = d$, c being the semi focal distance (as shown in Fig. 7). The potential at various points is computed by using

(3.3.12) and the numerical results are compared with those given by (3.3.13) as shown in Table 7 & 8.

Table 7. Accuracy of the numerical results obtained for the potential Φ^t in (3.3.12) truncated to $M = 20$ with respect to values given by (3.3.13) when $V_1 = -1\text{V}$, $V_2 = 1\text{V}$, $a_1 = 3\,\text{cm}$, $a_2 = 5\,\text{cm}$, $d = 10\,\text{cm}$

Test Point	w.r.t the coordinates system at Sphere-1		w.r.t the coordinates system at Sphere-2		Φ^t_{tr} [V]	Φ^t_{bi} [V]	Error [%]
	$r_1\,[cm]$	$\theta_1\,[\text{deg}]$	$r_2\,[cm]$	$\theta_2\,[\text{deg}]$			
1	4.2001	0.00	5.7999	180.00	0.238961	0.240836	-0.778497
2	4.2065	3.15	5.8045	177.71	0.239190	0.240981	-0.743204
3	4.2258	6.33	5.8186	175.41	0.239866	0.241418	-0.642602
4	4.2590	9.54	5.8427	173.06	0.240952	0.242143	-0.491836
5	4.3073	12.81	5.8780	170.65	0.242386	0.243149	-0.313858
6	4.3730	16.17	5.9263	168.15	0.244086	0.244419	-0.135980
7	5.4301	39.33	6.7442	149.31	0.251773	0.251637	0.054054
8	5.8318	43.93	7.0716	145.10	0.249734	0.249698	0.014518
9	6.3758	48.79	7.5265	140.41	0.244761	0.244765	-0.001521
10	7.1360	53.94	8.1805	135.15	0.235300	0.235302	-0.000893

Table 8. Accuracy of the numerical results obtained for the potential Φ^t in (3.3.12) truncated to $M = 40$ with respect to values given by (3.3.13) when $V_1 = -1\text{V}$, $V_2 = 1\text{V}$, $a_1 = 3\,\text{cm}$, $a_2 = 5\,\text{cm}$, $d = 10\,\text{cm}$

Test Point	w.r.t the coordinates sys-tem at Sphere-1		w.r.t the coordinates system at Sphere-2		Φ^t_{tr} [V]	Φ^t_{bi} [V]	Error [%]
	$r_1\,[cm]$	$\theta_1\,[\text{deg}]$	$r_2\,[cm]$	$\theta_2\,[\text{deg}]$			
1	4.2001	0.00	5.7999	180.00	0.240906	0.240836	0.029052
2	4.2065	3.15	5.8045	177.71	0.241052	0.240981	0.029391
3	4.2258	6.33	5.8186	175.41	0.241491	0.241418	0.030181
4	4.2590	9.54	5.8427	173.06	0.242218	0.242143	0.030846
5	4.3073	12.81	5.8780	170.65	0.243224	0.243149	0.030745
6	4.3730	16.17	5.9263	168.15	0.244491	0.244419	0.029488
7	5.4301	39.33	6.7442	149.31	0.251670	0.251637	0.013239
8	5.8318	43.93	7.0716	145.10	0.249725	0.249698	0.010839
9	6.3758	48.79	7.5265	140.41	0.244763	0.244765	-0.001521
10	7.1360	53.94	8.1805	135.15	0.235301	0.235302	-0.000893

3.3.2 Total Charge on the Spheres and Capacitances

As shown in section 3.2.1, the total charge on sphere 1 can be found as

$$Q_1^t = \int_0^{\pi} \rho_{s_1} ds_1 = 2\pi a_1^2 \int_0^{\pi} \rho_{s_1}(\theta_1) \sin\theta_1 d\theta_1 ,$$

where $\rho_{s_1}(\theta_1) = -\varepsilon \left. \frac{\partial \Phi^t}{\partial r_1} \right|_{r_1 = a_1}$,

i.e.,

$$\rho_{s_1}(\theta_1) = -\varepsilon \left[\sum_{n=0}^{\infty} -A_n (n+1) a_1^{-(n+2)} P_n(\cos\theta_1) + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^m B_m \frac{(m+n)!}{(m)!(n)!} \frac{n a_1^{(n-1)}}{d^{(m+n+1)}} P_n(\cos\theta_1) \right] .$$

Therefore the total charged becomes

$$Q_1^t = 2\pi\varepsilon \int_0^{\pi} \left[\sum_{n=0}^{\infty} A_n (n+1) a_1^{-n} P_n(\cos\theta_1) - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^m B_m \frac{(m+n)!}{(m)!(n)!} \frac{n a_1^{(n+1)}}{d^{(m+n+1)}} P_n(\cos\theta_1) \right] \sin\theta_1 d\theta_1 \cdot$$

Using the property (B.2) of Legendre polynomials give [see Appendix B] yields

$$Q_1^t = 4\pi\varepsilon A_0 . \tag{3.3.14}$$

Similarly the charge on sphere 2 is obtained as

$$Q_2^t = 4\pi\varepsilon B_0 . \tag{3.3.15}$$

The self and mutual capacitances of system of the conductors are defined by [7]

$$Q_1^t = c_{11} V_1 + c_{21} V_2 , \tag{3.3.16}$$

$$Q_2^t = c_{12} V_1 + c_{22} V_2 , \tag{3.3.17}$$

where $c_{12} = c_{21}$. The c_{11} and c_{22} are called the coefficients of capacitance or self capacitance, c_{21} and c_{12} are called the coefficients of induction or the mutual capacitance. If we know the potentials and the total charges of the each sphere, then we can calculate the values of capacitances by using (3.3.16) and (3.3.17).

Normally capacitances are computed by making the potential of one sphere at a time is non zero and making potential of all other spheres are zero. Then we can find the capacitances as

$$c_{11} = \left.\frac{Q_1^t}{V_1}\right|_{V_2=0} = \left.\frac{4\pi\varepsilon A_0}{V_1}\right|_{V_2=0}, \quad c_{21} = \left.\frac{Q_1^t}{V_2}\right|_{V_1=0} = \left.\frac{4\pi\varepsilon A_0}{V_2}\right|_{V_1=0}, \quad c_{22} = \left.\frac{Q_2^t}{V_2}\right|_{V_1=0} = \left.\frac{4\pi\varepsilon B_0}{V_2}\right|_{V_1=0}.$$

In Table 9 the results for a two spheres system are given where $a_1 = 3\,\text{cm}$, $a_2 = 5\,\text{cm}$ and $d = 10\,\text{cm}$. For the comparison, the capacitance values of the same system are calculated by using the method of successive images, as presented in [8], where

$$c_{11} = 4\pi\varepsilon\, a_1 a_2 \sinh\alpha \sum_{n=1}^{\infty}\left[\left|a_2 \sinh(n\alpha) + a_1 \sin[(n-1)\alpha]\right|\right]^{-1}, \tag{3.3.18}$$

$$c_{12} = c_{21} = -4\pi\varepsilon \frac{a_1 a_2}{d} \sinh\alpha \sum_{n=1}^{\infty}\left[\left|\sinh(n\alpha)\right|\right]^{-1}, \tag{3.3.19}$$

with $\cosh(\alpha) \equiv \dfrac{(d^2 - a_1^2 - a_2^2)}{2a_1 a_2}$.

Table 9. Comparison of numerical results obtained for self and mutual capacitances by translational addition method (with $M = 20$) and method of successive images (with $n = 300$), when $a_1 = 3\,\text{cm}$, $a_2 = 5\,\text{cm}$ and $d = 10\,\text{cm}$

Symbols of Coefficients	Capacitance [pF] *(by translational method)*	Capacitance [pF] *(by method of successive images)*	Error [%]
c_{11}	4.2089	4.2089	-1.5247×10^{-07}
c_{12}	-2.1650	-2.1650	-3.4058×10^{-07}
c_{22}	6.7559	6.7559	-1.3219×10^{-07}
c_{21}	-2.1650	-2.1650	-3.4058×10^{-07}

3.4 Two Sphere System in External Electric Field

The two spheres system in section 3.3 is placed in an external electric field oriented along the common axis of symmetry of the spheres $\mathbf{E} = E_0\hat{z}$ (see Fig.8). The potential due to the external field in the coordinate system of each is, respectively,

$$\Phi_{m1}^{ex} = -E_0 z_1 = -E_0 r_1 P_1(\cos\theta_1) + \mathrm{C}_1 , \quad (3.4.1)$$

$$\Phi_{m2}^{ex} = -E_0 z_2 = -E_0 r_2 P_2(\cos\theta_2) + \mathrm{C}_2 , \quad (3.4.2)$$

where C_1 and C_2 are constants of reference. Let's consider the potential produced by external field at $z_1 = 0$ is zero, then $\mathrm{C}_1 = 0$. Thus (3.4.1) and (3.4.2) become

$$\Phi_{m1}^{ex} = -E_0 z_1 = -E_0 r_1 P_1(\cos\theta_1) , \quad (3.4.3)$$

$$\Phi_{m2}^{ex} = -E_0 r_2 P_2(\cos\theta_2) - E_0 d . \quad (3.4.4)$$

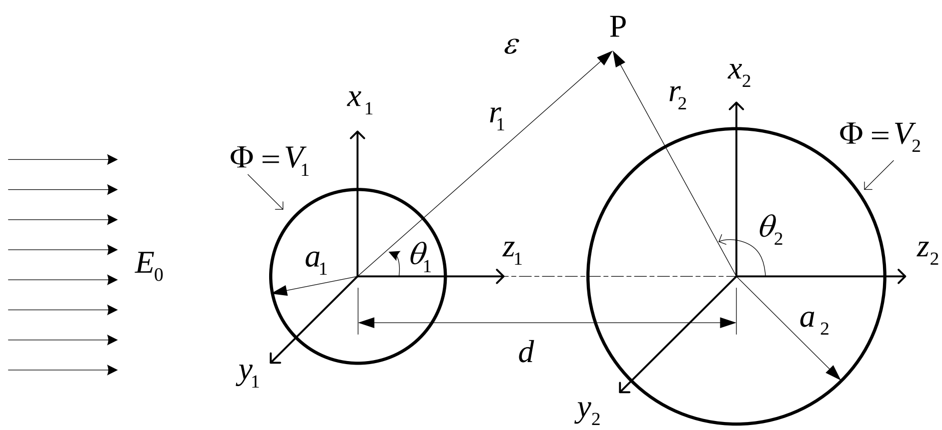

Fig. 8: System of two conducting spheres placed in initially uniform electric field

The Laplacian potential outside the spheres due to the presence of each of them is expressed by (3.3.1) and (3.3.2). Φ_2 is translated into the coordinate system (r_1, θ_1, ϕ_1), then the total potential at P becomes

$$\Phi^t = \Phi_1^{ex} + \Phi_1 + \Phi_2^{(1)} \tag{3.4.5}$$

$$\begin{aligned}\Phi^t(r_1,\theta_1,\phi_1) &= \Phi_1^{ex}(r_1,\theta_1,\phi_1) + \Phi_1(r_1,\theta_1,\phi_1) + \Phi_2^{(1)}(r_1,\theta_1,\phi_1) \\ &= -E_0 r_1 P_1(\cos\theta_1) + \sum_{n=0}^{\infty} A_n r_1^{-(n+1)} P_n(\cos\theta_1) + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m B_m \frac{(m+n)!}{(m)!(n)!}\left(\frac{r_1}{d}\right)^n \\ &\quad \cdot d^{-(m+1)} P_n(\cos\theta_1).\end{aligned}$$

As before, we can apply the boundary condition at surface of sphere 1 at $r_1 = a_1$ and then use orthogonality properties of the Legendre polynomials to obtain

$$A_n a_1^{-(n+1)} = \frac{(2n+1)}{2}\int_{-1}^{1}\left[V_1 + E_0 a_1 P_1(x_1) - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m B_m \frac{(m+n)!}{(m)!(n)!}\left(\frac{a_1}{d}\right)^n d^{-(m+1)} P_n(x_1)\right] P_n(x_1)\,dx_1, \tag{3.4.6}$$

$$A_0 + \sum_{m=0}^{\infty}(-1)^m B_m a_1 d^{-(m+1)} = V_1 a_1, \qquad n = 0, \tag{3.4.7}$$

$$A_1 + \sum_{m=0}^{\infty}(-1)^m B_m (m+1) a_1^3 d^{-(m+2)} = E_0 a_1^3, \quad n = 1, \tag{3.4.8}$$

$$A_n + \sum_{m=0}^{\infty}(-1)^m B_m \frac{(m+n)!}{(m)!(n)!}\frac{a_1^{2n+1}}{d^{n+m+1}} = 0, \qquad n = 1, 2, 3, \ldots. \tag{3.4.9}$$

In the same way, Φ_1 is translated into the coordinate system (r_2, θ_2, ϕ_2) and the boundary condition is applied at the surface of sphere 2 to obtain the following equations:

$$B_0 + \sum_{n=0}^{\infty}(-1)^m A_n \frac{a_2}{d^{n+1}} = V_2 a_2 + E_0 a_2 d, \qquad m = 0, \tag{3.4.10}$$

$$B_1 + \sum_{n=0}^{\infty}(-1)^m A_n (n+1)\, a_2^3 d^{-(n+2)} = E_0 a_2^3, \quad m = 1, \tag{3.4.11}$$

$$B_m + \sum_{n=0}^{\infty} (-1)^m A_n \frac{(m+n)!}{(m)!(n)!} \frac{a_2^{\,2m+1}}{d^{m+n+1}} = 0, \qquad m = 1, 2, 3, \ldots. \tag{3.4.12}$$

Unknown constants of integrations are to be found by solving equations (3.4.7) to (3.4.12) simultaneously after infinite series are truncated to a finite number of terms $n = m = M$.

For the comparison, the potential values of the same system is calculated by using the bispherical coordinates system, as presented in [15], where

$$\Phi_{bi}^{t}(\alpha, \beta, \phi) = (\cosh\beta - \cosh\alpha)^{1/2} \sum_{n=0}^{\infty} \left[\left(A_n e^{(n+1/2)\beta} + B_n e^{-(n+1/2)\beta} \right) P_n(\cos\alpha) \right] - E_0 z\,, \tag{3.4.13}$$

with $A_n = \dfrac{\sqrt{2}\left[cE_0(2n+1)[e^{-(2n+1)\beta_1} + 1] + [V_2 e^{-(2n+1)\beta_1} - V_1] \right]}{e^{(2n+1)(\beta_2 - \beta_2)} - 1}$,

$$B_n = \frac{\sqrt{2}\left[-cE_0(2n+1)[e^{-(2n+1)\beta_2} + 1] + [V_1 e^{-(2n+1)\beta_2} - V_2] \right]}{e^{(2n+1)(\beta_2 - \beta_2)} - 1}.$$

The symbols in (3.4.13) are same as the symbols in (3.3.13) for the bispherical coordinates system and external electric field is parallel to the z-axis as in Fig. 8.

Table 10. Accuracy of numerical results obtained for the potential Φ^t in (3.4.5) after truncated to $M = 20$ with respect to values given by (3.4.13) when $V_1 = -1\text{V}$, $V_2 = 1\text{V}$, $E_0 = 10\,\text{V/m}$, $a_1 = 3\,\text{cm}$, $a_2 = 5\,\text{cm}$, $d = 10\,\text{cm}$

Test Point	w.r.t the coordinates system at Sphere-1		w.r.t the coordinates system at Sphere-2		Φ_{tr}^{t} [V]	Φ_{bi}^{t} [V]	Error [%]
	$r_1\,[cm]$	θ_1 [deg]	$r_2\,[cm]$	θ_2 [deg]			
1	4.2001	0.00	5.7999	180.00	0.247870	0.247883	-0.005241
2	4.2065	3.15	5.8045	177.71	0.248217	0.248229	-0.004652
3	4.2258	6.33	5.8186	175.41	0.249271	0.249279	-0.003136
4	4.2590	9.54	5.8427	173.06	0.251074	0.251077	-0.001314
5	4.3073	12.81	5.8780	170.65	0.253696	0.253695	0.000127
6	4.3730	16.17	5.9263	168.15	0.257231	0.257229	0.000765
7	5.4301	39.33	6.7442	149.31	0.302689	0.302689	0.000202
8	5.8318	43.93	7.0716	145.10	0.312997	0.312996	0.000349
9	6.3758	48.79	7.5265	140.41	0.321583	0.321582	0.000426
10	7.1360	53.94	8.1805	135.15	0.235300	0.235302	-0.000893

3.5 Three Sphere System

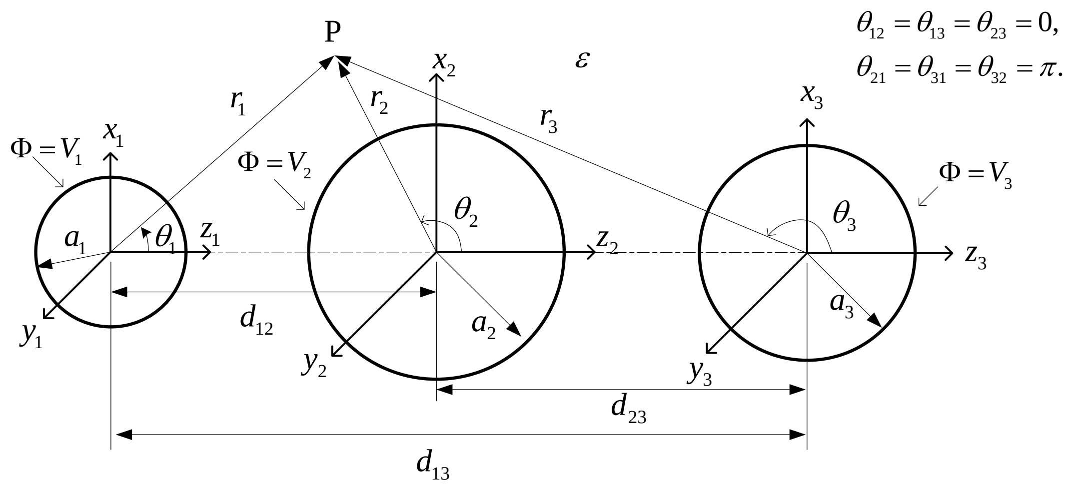

Fig. 9: System of three conducting spheres

Consider three metallic spheres of radii a_1, a_2 and a_3 with the separation distances between their centers, as shown in Fig. 9. The spheres are kept at the potentials V_1, V_2 and V_3 respectively. The medium outside the spheres is homogeneous of permittivity ε.

The potential produce by the each of three spheres is first expressed in coordinate system attached to the respective sphere as

$$\Phi_1(r_1, \theta_1, \phi_1) = \sum_{n=0}^{\infty} A_n r_1^{-(n+1)} P_n(\cos\theta_1), \qquad r_1 \geq a_1, \tag{3.5.1}$$

$$\Phi_2(r_2, \theta_2, \phi_2) = \sum_{m=0}^{\infty} B_m r_2^{-(m+1)} P_m(\cos\theta_2), \qquad r_2 \geq a_2, \tag{3.5.2}$$

$$\Phi_3(r_3, \theta_3, \phi_3) = \sum_{l=0}^{\infty} C_l r_3^{-(l+1)} P_l(\cos\theta_3), \qquad r_3 \geq a_3. \tag{3.5.3}$$

As already done in the section 3.3 for two spheres, the required coupled set of linier algebraic equations for three sphere case can be written directly by looking at (3.3.8), (3.3.9), (3.3.10) and (3.3.11) as

$$A_0+\sum_{m=0}^{\infty}(-1)^m B_m\,\frac{a_1}{d_{21}^{m+1}}+\sum_{l=0}^{\infty}(-1)^l C_l\,\frac{a_1}{d_{31}^{l+1}}=V_1a_1\,,\qquad n=0, \qquad (3.5.4)$$

$$A_n+\sum_{m=0}^{\infty}(-1)^m B_m\,\frac{(m+n)!}{(m)!(n)!}\frac{a_1^{\,2n+1}}{d_{21}^{m+n+1}}+\sum_{l=0}^{\infty}(-1)^l C_l\,\frac{(l+n)!}{(l)!(n)!}\frac{a_1^{\,2n+1}}{d_{31}^{l+n+1}}=0,\quad n=1,2,3\ldots, \qquad (3.5.5)$$

$$B_0+\sum_{n=0}^{\infty}A_n\,\frac{a_2}{d_{12}^{n+1}}+\sum_{l=0}^{\infty}(-1)^l C_l\,\frac{a_2}{d_{32}^{l+1}}=V_2a_2\,,\qquad m=0, \qquad (3.5.6)$$

$$B_m+(-1)^m\sum_{n=0}^{\infty}A_n\,\frac{(n+m)!}{(n)!(m)!}\frac{a_2^{\,2m+1}}{d_{12}^{n+m+1}}+\sum_{l=0}^{\infty}(-1)^l C_l\,\frac{(l+m)!}{(l)!(m)!}\frac{a_2^{\,2m+1}}{d_{32}^{l+m+1}}=0,\quad m=1.2.3\ldots, \qquad (3.5.7)$$

$$C_0+\sum_{n=0}^{\infty}A_n\,\frac{a_3}{d_{13}^{n+1}}+\sum_{m=0}^{\infty}B_m\,\frac{a_3}{d_{23}^{m+1}}=V_3a_3\,,\qquad l=0, \qquad (3.5.8)$$

$$C_l+(-1)^l\sum_{n=0}^{\infty}A_n\,\frac{(n+l)!}{(n)!(l)!}\frac{a_3^{\,2l+1}}{d_{13}^{n+l+1}}+(-1)^l\sum_{m=0}^{\infty}B_m\,\frac{(m+l)!}{(m)!(l)!}\frac{a_3^{\,2l+1}}{d_{23}^{m+l+1}}=0,\qquad l=1,2,3\ldots. \qquad (3.5.9)$$

Equations (3.5.4) to (3.5.9), constitute an infinite set of liner algebraic equations and that is to be solved simultaneously in order to find the unknown constants of integrations. To obtain numerical results, the infinite series must be truncated to finite number of terms $n=m=l=M$. Then the total charge on each sphere are found as

$$Q_1^t=4\pi\varepsilon A_0\,,\quad Q_2^t=4\pi\varepsilon B_0\,,\quad Q_3^t=4\pi\varepsilon C_0.$$

For this three spheres system the self and mutual capacitances are can be defied as

$$Q_1^t=c_{11}V_1+c_{12}V_2+c_{13}V_3,$$
$$Q_2^t=c_{21}V_1+c_{22}V_2+c_{23}V_3,$$
$$Q_3^t=c_{31}V_1+c_{32}V_2+c_{33}V_3.$$

Numerical results for a three spheres system are given where $a_1=3\,\text{cm}$, $a_2=5\,\text{cm}$, $a_3=7\text{cm}$, $d_{12}=10\,\text{cm}$, $d_{13}=25\text{cm}$, $d_{23}=15\text{cm}$ and $M=25$.

Table 11. Numerical results obtained for capacitances, when $a_1 = 3\,\text{cm}$, $a_2 = 5\,\text{cm}$, $a_3 = 7\,\text{cm}$, $d_{12} = 10\,\text{cm}$, $d_{13} = 25\,\text{cm}$, $d_{23} = 15\,\text{cm}$

	Coefficients Symbols	Capacitance [pF]
1	c_{11}	4.2261
2	$c_{12} = c_{21}$	-2.0367
3	$c_{31} = c_{13}$	-4.0188
4	c_{22}	8.0936
5	$c_{23} = c_{32}$	-3.2090
6	c_{33}	9.6151

Since the solution for three sphere system is not found in literature the method is verified indirectly by taking the limit $a_3 \to 0$ where capacitances values should approach to the values obtained in two spheres case.

When $a_3 \to 0$

$c_{11} \to 4.2089\,\text{pF}$, $c_{22} \to 6.7559\,\text{pF}$, $c_{12} \to \text{-}2.1650\,\text{pF}$.

3.5.1 Three Spheres in External Electric Field

Consider the three sphere system in Fig. 9 is placed in an external electric field oriented along the common axis of symmetry of the spheres $\mathbf{E} = E_0 \hat{z}$ and the potentials on the surfaces of spheres are unknown. But the total charge on each of spheres are considered as zero, i.e., $Q_1^t = Q_2^t = Q_3^t = 0$.

As before the Laplacian potentials can be expressed by (3.5.1)-(3.5.3) and are translated into the coordinate system (r_1, θ_1, ϕ_1) as

$$\Phi^t(r_1,\theta_1,\phi_1) = \Phi_1^{ex}(r_1,\theta_1,\phi_1) + \Phi_1(r_1,\theta_1,\phi_1) + \Phi_2^{(1)}(r_1,\theta_1,\phi_1) + \Phi_3^{(1)}(r_1,\theta_1,\phi_1)$$
$$= -E_0 r_1 P_1(\cos\theta_1) + \sum_{n=0}^{\infty} A_n r_1^{-(n+1)} P_n(\cos\theta_1) + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m B_m \frac{(m+n)!}{(m)!(n)!}\left(\frac{r_1}{d_{21}}\right)^n$$
$$\cdot d_{21}{}^{-(m+1)} P_n(\cos\theta_1) + \sum_{l=0}^{\infty}\sum_{n=0}^{\infty}(-1)^l C_l \frac{(l+n)!}{(l)!(n)!}\left(\frac{r_1}{d_{31}}\right)^n d_{31}{}^{-(l+1)} P_n(\cos\theta_1)\,.$$

The total charge on the sphere 1 can be found as

$$Q_1^t = -\varepsilon \oint_{s_1} \left.\frac{\partial \Phi^t}{\partial r_1}\right|_{r_1 = a_1} ds_1$$

$$Q_1^t = 2\pi\varepsilon \int_0^{\pi} \Bigg[E_0 a^2 P_1(\cos\theta_1) + \sum_{n=0}^{\infty} A_n (n+1) a_1^{-n} P_n(\cos\theta_1) - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (-1)^m B_m \frac{(m+n)!}{(m)!(n)!}\frac{n a_1^{(n+1)}}{d^{n+m+1}} P_n(\cos\theta_1)$$
$$- \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} (-1)^l C_l \frac{(l+n)!}{(l)!(n)!}\frac{n a_1^{(n+1)}}{d^{n+l+1}} P_n(\cos\theta_1) \Bigg] \sin\theta_1 d\theta_1$$
$$= 4\pi\varepsilon A_0\,.$$

The boundary condition at sphere 1, i.e., $Q_1^t = 0$, yield $A_0 = 0$. Then, using the orthogonality properties of the Legendre polynomials we can obtain

$$A_1 + \sum_{m=1}^{\infty} (-1)^m B_m (m+1) a_1^3 d_{21}{}^{-(m+2)} + \sum_{l=1}^{\infty} (-1)^l C_l (l+1) a_1^3 d_{31}{}^{-(l+2)} = E_0 a_1^3, \quad n = 1, \tag{3.5.10}$$

$$A_n + \sum_{m=1}^{\infty} (-1)^m B_m \frac{(m+n)!}{(m)!(n)!}\frac{a_1^{2n+1}}{d_{12}{}^{m+n+1}} + \sum_{l=1}^{\infty} (-1)^l C_l \frac{(l+n)!}{(l)!(n)!}\frac{a_1^{2n+1}}{d_{32}{}^{l+n+1}} = 0, \qquad n = 2,3,\ldots. \tag{3.5.11}$$

Similarly by applying the boundary condition at the surface of sphere 2 and 3, we can show $B_0 = C_0 = 0$ and can obtain

$$B_1 - \sum_{n=1}^{\infty} A_n (n+1)\, a_2^3 d_{12}{}^{-(n+2)} + \sum_{l=1}^{\infty} (-1)^l C_l (l+1) a_1^3 d_{32}{}^{-(l+2)} = E_0 a_2^3, \qquad m = 1, \tag{3.5.12}$$

$$B_m + (-1)^m \sum_{n=1}^{\infty} A_n \frac{(n+m)!}{(n)!(m)!}\frac{a_2^{2m+1}}{d_{12}{}^{n+m+1}} + \sum_{l=1}^{\infty} (-1)^l C_l \frac{(l+m)!}{(l)!(m)!}\frac{a_1^{2m+1}}{d_{32}{}^{l+m+1}} = 0, \quad m = 2,3,\ldots, \tag{3.5.13}$$

$$C_1 - \sum_{n=1}^{\infty} A_n (n+1)\, a_3^3 d_{13}^{-(n+2)} - \sum_{m=1}^{\infty} B_m (m+1) a_3^3 d_{23}^{-(m+2)} = E_0 a_3^3, \qquad l = 1, \tag{3.5.14}$$

$$C_l + (-1)^l \sum_{n=1}^{\infty} A_n \frac{(n+l)!}{(n)!(l)!} \frac{a_3^{2l+1}}{d_{13}^{n+l+1}} + (-1)^l \sum_{m=1}^{\infty} B_m \frac{(m+l)!}{(m)!(l)!} \frac{a_3^{2l+1}}{d_{23}^{m+l+1}} = 0, \quad l = 2, 3, \ldots \tag{3.5.15}$$

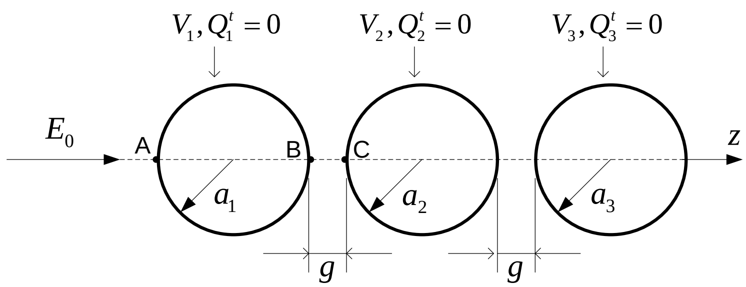

Fig. 10: System of three conducting spheres placed in uniform electric field

Numerical results are generated for the three spheres system in Fig.10 with $a_1 = a_2 = a_3 = 5\,\text{cm}$, $E_0 = 1\,\text{V/m}$ and for the various g/a_1 ratios, the electric filed at points A, B and C are calculated and given in Table 12 with 5 digits accuracy.

Table 12. . Relative values of field components at selected points on the sphere in Fig. 10 for various gaps g/a_1 when $\mathbf{E_0} = E_0 \hat{x}(\psi = 0)$

Point	Fields \ g/a_1	1.00	0.50	0.10	0.05	0.02	0.01	0.005
A	E_x/E_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	E_z/E_0	3.1513	3.2701	3.5946	3.7270	3.8832	3.9862	4.0771
B	E_x/E_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	E_z/E_0	3.8067	5.2258	16.111	28.391	62.076	113.92	210.91
C	E_x/E_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	E_z/E_0	3.8549	5.2568	16.111	28.391	62.076	113.92	210.91

Chapter 4

Application of Translational Addition Theorems to the Solution of Field Problems with Arbitrarily Located Spheres

4.1 General Solution for Spherical Bodies

In previous chapter, the applications of transitional additional theorems to solve azimuthal symmetric electrostatic problems are discussed. In this section we are going to extend the applications of those theorems to solve problems which do not have azimuthal symmetry. In presence of non azimuthal symmetric geometries the solution of Laplace equation in spherical coordinates considered in (1.1.2) is obtained as [5]

$$u(r,\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}[A_{nm}r^{n}+B_{nm}r^{-(n+1)}]P_{n}^{m}(\cos\theta)\exp(jm\phi) \tag{4.1.1}$$

For regions extended to infinity, with $u \to 0$ for $r \to \infty$, we have

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} B_{nm} r^{-(n+1)} P_n^m(\cos\theta)\exp(\mathrm{j}m\phi) \tag{4.1.2}$$

4.2 Two Spheres at Arbitrary Locations

Consider the two spheres case discussed in section 3.3 is changed to a non azimuthal symmetry problem by moving the center of the sphere 2 away from z-axis of sphere 1 as given in Fig. 11.

Let's take the potential produced by two spheres at P as

$$\Phi_1(r_1,\theta_1,\phi_1) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm} r_1^{-(n+1)} P_n^{\ m}(\cos\theta_1)\exp(\mathrm{j}m\phi_1), \quad r_1 \ge a_1, \tag{4.2.1}$$

$$\Phi_2(r_2,\theta_2,\phi_2) = \sum_{q=0}^{\infty}\sum_{p=-q}^{q} B_{qp} r_2^{-(q+1)} P_q^{\ p}(\cos\theta_2)\exp(\mathrm{j}p\phi_2), \quad r_2 \ge a_2. \tag{4.2.2}$$

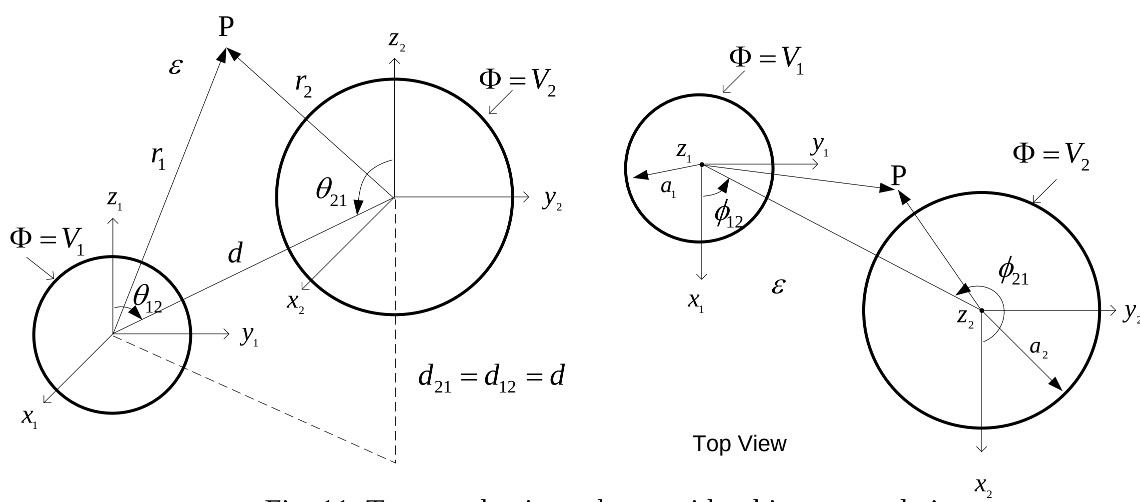

Fig. 11: Two conducting spheres with arbitrary translation

The Φ_2 can be translated into the coordinate system (r_1,θ_1,ϕ_1) and then the total potential when $d_{12}\ge r_1\ge a_1$ can be denoted as

$$\begin{aligned}\Phi^t(r_1,\theta_1,\phi_1)&=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}A_{nm}r_1^{-(n+1)}P_n^m(\cos\theta_1)\exp(\mathrm{j}m\phi_1)+\sum_{q=0}^{\infty}\sum_{p=-q}^{q}\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}B_{qp}\\&\cdot\beta_2^{(1)}(p,q\,|\,\mu,\nu\,|\,d_{21},\theta_{21},\phi_{21})r_1^{\nu}P_\nu^\mu(\cos\theta_1)\exp(\mathrm{j}\mu\phi_1)\,,\end{aligned}\tag{4.2.3}$$

where

$$\beta_2^{(1)}(p,q\,|\,\mu,\nu\,|\,d_{21},\theta_{21},\phi_{21})\equiv(-1)^{\mu+\nu}\frac{(q-p+\nu+\mu)!}{(q-p)!(\nu+\mu)!}\frac{1}{d_{21}^{\nu+q+1}}P_{\nu+q}^{p-\mu}(\cos\theta_{21})\exp[\mathrm{j}(p-\mu)\phi_{21}]\ .$$

Boundary conditions on the surface of sphere 1, i.e,

$$\begin{aligned}&\Phi^t(r_1,\theta_1,\phi_1)=V_1\\&V_1=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}A_{nm}a_1^{-(n+1)}P_n^m(\cos\theta_1)\exp(\mathrm{j}m\phi_1)+\sum_{q=0}^{\infty}\sum_{p=-q}^{q}\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}B_{qp}a_1^{\nu}\\&\quad\cdot\beta_2^{(1)}(p,q\,|\,\mu,\nu\,|\,d_{21},\theta_{21},\phi_{21})P_\nu^\mu(\cos\theta_1)\exp(\mathrm{j}\mu\phi_1)\ ,\end{aligned}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}A_{nm}a_1^{-(n+1)}P_n^m(\cos\theta_1)\exp(\mathrm{j}m\phi_1)&=V_1-\sum_{q=0}^{\infty}\sum_{p=-q}^{q}\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}B_{qp}a_1^{\nu}\\&\cdot\beta_2^{(1)}(p,q\,|\,\mu,\nu\,|\,d_{21},\theta_{21},\phi_{21})P_\nu^\mu(\cos\theta_1)\exp(\mathrm{j}\mu\phi_1)\,.\end{aligned}\tag{4.2.4}$$

Now (4.2.4) can be expressed by using spherical harmonics as [see Appendix C]

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}A_{nm}a_1^{-(n+1)}\frac{Y_n^m(\theta_1,\phi_1)}{\sqrt{\frac{(2n+1)}{4\pi}\frac{(n-m)!}{(n+m)!}}}&=V_1-\sum_{q=0}^{\infty}\sum_{p=-q}^{q}\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}B_{qp}\\&\cdot\beta_2^{(1)}(p,q\,|\,\mu,\nu\,|\,d_{21},\theta_{21},\phi_{21})a_1^{\nu}\frac{Y_\nu^\mu(\theta_1,\phi_1)}{\sqrt{\frac{(2\nu+1)}{4\pi}\frac{(\nu-\mu)!}{(\nu+\mu)!}}}\,.\end{aligned}\tag{4.2.5}$$

Spherical harmonics expansion [Appendix C] yield

$$\frac{A_{nm}a_1^{-(n+1)}}{\sqrt{\frac{(2n+1)}{4\pi}\frac{(n-m)!}{(n+m)!}}} = V_1\int_\Omega [Y_n^m(\theta_1,\phi_1)]^* d\Omega - \sum_{q=0}^{\infty}\sum_{p=-q}^{q}\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} B_{qp}$$
$$.\beta_2^{(1)}(p,q\,|\,\mu,\nu\,|\,d_{21},\theta_{21},\phi_{21})\frac{a^\nu}{\sqrt{\frac{(2\nu+1)}{4\pi}\frac{(\nu-\mu)!}{(\nu+\mu)!}}}\int_\Omega Y_\nu^\mu(\theta_1,\phi_1)[Y_n^m(\theta_1,\phi_1)]^* d\Omega \tag{4.2.6}$$

Above integration of (4.2.6) can be solved as

$$A_{00} + \sum_{q=0}^{\infty}\sum_{p=-q}^{q} B_{qp}\beta_2^{(1)}(p,q\,|\,0,0\,|\,d_{21},\theta_{21},\phi_{21})a_1 = V_1 a_1\,, \qquad n=0, \tag{4.2.7}$$

$$A_{nm} + \sum_{q=0}^{\infty}\sum_{p=-q}^{q} B_{qp}\beta_2^{(1)}(p,q\,|\,m,n\,|\,d_{21},\theta_{21},\phi_{21})a_1^{2n+1} = 0\,, \qquad n=1,2,3,\ldots, \tag{4.2.8}$$

Same way by imposing boundary condition at sphere 2 we can obtained

$$B_{00} + \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm}\beta_1^{(2)}(m,n\,|\,0,0\,|\,d_{12},\theta_{12},\phi_{12})a_2 = V_2 a_2, \quad q=0, \tag{4.2.9}$$

$$B_{qp} + \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm}\beta_1^{(2)}(n,m\,|\,p,q\,|\,d_{12},\theta_{12},\phi_{12})a_2^{(2q+1)} = 0, \quad q=1,2,3,\ldots, \tag{4.2.10}$$

Above (4.2.7)-(4.2.10) are coupled set of linier algebraic equations and we can solve it numerically as before by taking $n=q=M$.

Exact numerical results are generated for the case in Fig. 11, where $a_1 = 3\,\text{cm}$, $a_2 = 5\,\text{cm}$, $d = 10\,\text{cm}$, $\theta_{12} = \pi/3$, $\theta_{21} = 2\pi/3$, $\phi_{12} = \pi/3$, $\phi_{21} = 4\pi/3$. The potentials of the spheres are taken as $V_1 = -1\text{V}$ and $V_2 = 1\text{V}$. The infinite series are truncated as $M = 10$. Six arbitrary points $P_1 \ldots P_6$ are selected along the line between the centers of spheres. The potentials at these points are first calculated by using translational approach discussed above. The potential at the very same points were also calculated by using the bispherical coordinates (see Section 3.31).

According to the Fig. 11 and Fig. 12, the electrostatic potential at P_1 and P_1' should be identical as long as distance between spheres and radii are same and P_1 and P_1' are located same distance from the centers of spheres. The potentials values at the points $P_1, P_2 \ldots P_6$ are tabulated in Table 13.

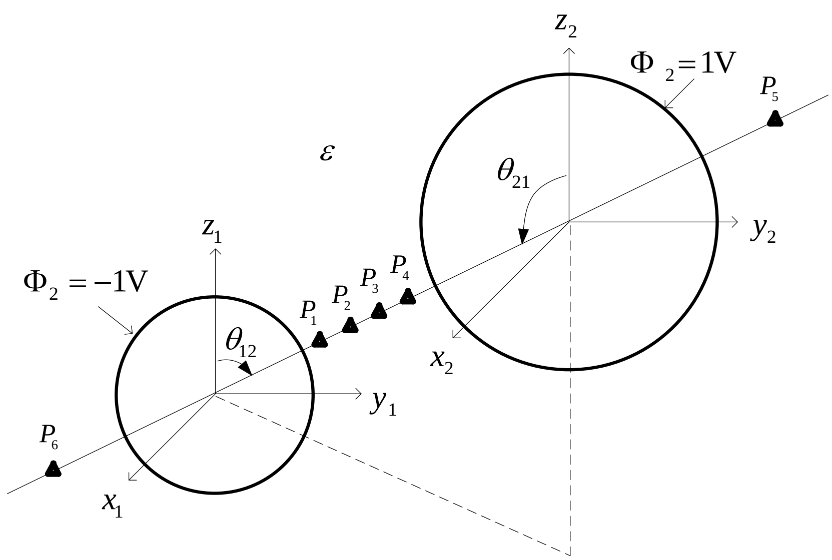

Fig. 12: Two conducting spheres at arbitrary position

Table 13. Comparison between numerical results obtained for potential by translational method (Φ_{tr}^{t}) and bispherical method (Φ_{bi}^{t}), when $V_1 = -1\text{V}$, $V_2 = 1\text{V}$, $a_1 = 3\text{cm}$, $a_2 = 5\text{cm}$, $d = 10\,\text{cm}$, $\theta_{12} = \pi/3$, $\theta_{21} = 2\pi/3$, $\phi_{12} = \pi/3$, $\phi_{21} = 4\pi/3$ and $M = 10$ for translational method

Test Point	w.r.t the coordinate system of Sphere-1			Φ_{tr}^{t} [V]	Φ_{bi}^{t} [V]	Error [%]
	r_1 [cm]	θ_1 [rad]	ϕ_1 [rad]			
P1	3.2113	π/3	π/3	-0.7368	-0.7368	-0.0043
P2	3.6199	π/3	π/3	-0.3001	-0.3000	0.0234
P3	4.2001	π/3	π/3	0.2405	0.2408	-0.1491
P4	4.3469	π/3	π/3	0.3731	0.3736	-0.1370
P5	17.484	π/3	π/3	0.6290	0.6290	-0.0040
P6	4.5387	2 π/3	4 π/3	-0.5526	-0.5526	-0.0000

4.2.1 Total Charge on the Spheres and Capacitances

$$\rho_s(a_1,\theta_1,\phi_1)=\varepsilon\Bigg[\sum_{n=0}^{\infty}\sum_{m=-n}^{n}A_{nm}(1+n)a_1^{-(n+2)}P_n^m(\cos\theta_1)\exp(\mathrm{j}m\phi_1)-\sum_{q=0}^{\infty}\sum_{p=-q}^{q}\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}B_{qp}$$
$$\cdot\beta_2^{(1)}(p,q\,|\,m,n\,|\,d_{21},\theta_{21},\phi_{21})\nu a_1^{\nu}P_\nu^{\mu}(\cos\theta_1)\exp(\mathrm{j}\mu\phi_1)\Bigg],$$

$$Q_1^t=2\pi\varepsilon\int_0^{\pi}\Bigg[\sum_{n=0}^{\infty}A_{n0}(1+n)a_1^{-n}P_n^0(\cos\theta_1)-\sum_{q=0}^{\infty}\sum_{\nu=0}^{\infty}B_{q0}\nu a_1^{\nu+1}\beta_2^{(1)}(0,q\,|\,0,\nu\,|\,d_{21},\theta_{21},\phi_{21})$$
$$\cdot P_\nu^0(\cos\theta_1)\Bigg]\sin(\theta_1)d\theta_1,$$

$$Q_1^t=4\pi\varepsilon A_{00}\,. \tag{4.2.11}$$

Same way the total charge on sphere 2 can be found as

$$Q_2^t=4\pi\varepsilon B_{00}\,. \tag{4.2.12}$$

4.3 Three Spheres at Arbitrary Locations

Consider three metallic spheres with radii a_1, a_2 and a_3 with the separation distances are d_{12}, d_{13} and d_{32} have been kept at potentials V_1, V_2 and V_3 as in Fig. 13. The medium outside the spheres is homogeneous of permittivity ε.

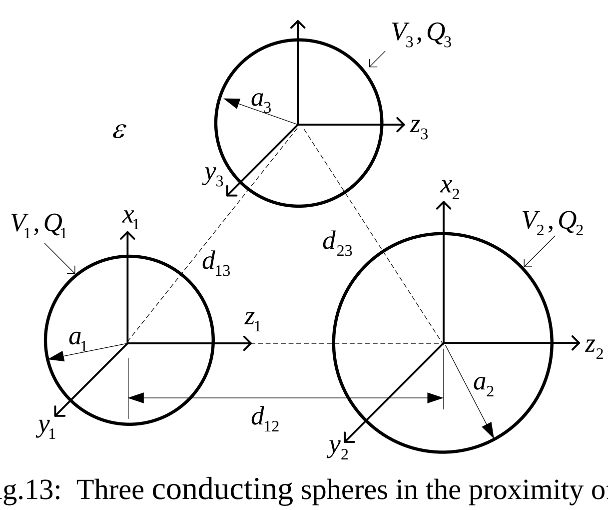

Fig.13: Three conducting spheres in the proximity of each other

The Laplacian potential outside the spheres due to the presence of each of them is first expressed in the spherical coordinate system attached to the respective sphere in the form

$$\Phi_1(r_1,\theta_1,\phi_1)=\sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1}A^1_{n_1m_1}r_1^{-(n_1+1)}P_{n_1}^{m_1}(\cos\theta_1)\exp(\mathrm{j}m_1\phi_1),\qquad r_1\geq a_1, \tag{4.3.1}$$

$$\Phi_1(r_2,\theta_2,\phi_2)=\sum_{n_2=0}^{\infty}\sum_{m_2=-n_2}^{n_2}A^2_{n_2m_2}r_2^{-(n_2+1)}P_{n_2}^{m_2}(\cos\theta_2)\exp(\mathrm{j}m_2\phi_2),\qquad r_2\geq a_2, \tag{4.3.2}$$

$$\Phi_1(r_3,\theta_3,\phi_3)=\sum_{n_3=0}^{\infty}\sum_{m_3=-n_3}^{n_3}A^3_{n_3m_3}r_3^{-(n_3+1)}P_{n_3}^{m_3}(\cos\theta_3)\exp(\mathrm{j}m_3\phi_3),\qquad r_3\geq a_3. \tag{4.3.3}$$

Now without starting from beginning we can write required equation by looking at (4.2.7) - (4.2.10) as

$$A^1_{001}+\sum_{n_2=0}^{\infty}\sum_{m_2=-n_2}^{n_2}A^2_{n_2m_2}\beta_2^{(1)}(m_2,n_2\,|\,0,0\,|\,d_{21},\theta_{21},\phi_{21})a_1+\sum_{n_3=0}^{\infty}\sum_{m_3=-n_3}^{n_3}A^3_{n_3m_3}$$
$$\cdot\beta_3^{(1)}(m_3,n_3\,|\,0,0\,|\,d_{31},\theta_{31},\phi_{31})a_1=V_1a,\quad n_1=0, \tag{4.3.4}$$

$$A^1_{n_1m_1}+\sum_{n_2=0}^{\infty}\sum_{m_2=-n_2}^{n_2}A^2_{n_2m_2}\beta_2^{(1)}(m_2,n_2\,|\,m_1,n_1\,|\,d_{21},\theta_{21},\phi_{21})a_1^{2n_1+1}+\sum_{n_3=0}^{\infty}\sum_{m_3=-n_3}^{n_3}A^3_{n_3m_3}$$
$$\cdot\beta_3^{(1)}(m_3,n_3\,|\,m_1,n_1\,|\,d_{31},\theta_{31},\phi_{31})a_1^{2n_1+1}=0,\quad n_1=1,2,3,\ldots, \tag{4.3.5}$$

$$A^2_{00}+\sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1}A^1_{n_1m_1}\beta_1^{(2)}(m_1,n_1\,|\,0,0\,|\,d_{12},\theta_{12},\phi_{12})a_2+\sum_{n_3=0}^{\infty}\sum_{m_3=-n_3}^{n_3}A^3_{n_3m_3}$$
$$\beta_3^{(2)}(m_3,n_3\,|\,0,0\,|\,d_{32},\theta_{32},\phi_{32})a_2=V_2a_2,\quad n_2=0, \tag{4.3.6}$$

$$A^2_{n_2m_2}+\sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1}A^1_{n_1m_1}\beta_1^{(2)}(m_1,n_1\,|\,m_2,n_2\,|\,d_{12},\theta_{12},\phi_{12})a_2^{2n_2+1}+\sum_{n_3=0}^{\infty}\sum_{m_3=-n_3}^{n_3}A^3_{n_3m_3}$$
$$\cdot\beta_3^{(2)}(m_3,n_3\,|\,n_2,m_2\,|\,d_{32},\theta_{32},\phi_{32})a_2^{2n_2+1}=0,\quad n_2=1,2,3,\ldots, \tag{4.3.7}$$

$$A^3_{00}+\sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1}A^1_{n_1m_1}\beta_1^{(3)}(m_1,n_1\,|\,0,0\,|\,d_{13},\theta_{13},\phi_{13})a_3+\sum_{n_2=0}^{\infty}\sum_{m_2=-n_2}^{n_2}A^2_{n_2m_2}$$
$$\cdot\beta_2^{(3)}(m_2,n_2\,|\,0,0\,|\,d_{23},\theta_{23},\phi_{23})a_3=V_3a_3,\quad n_3=0, \tag{4.3.8}$$

$$A^3_{n_3m_3} + \sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1} A^1_{n_1m_1}\beta_1^{(3)}(m_1,n_1\,|\,m_3,n_3\,|\,d_{13},\theta_{13},\phi_{13})a_3^{2n_3+1} + \sum_{n_2=0}^{\infty}\sum_{m_2=-n_2}^{n_2} A^2_{n_2m_2} \cdot \beta_2^{(3)}(m_2,n_2\,|\,n_3,m_3\,|\,d_{23},\theta_{23},\phi_{23})a_3^{2n_3+1} = 0, \quad n_3 = 1,2,3,\ldots. \tag{4.3.9}$$

where $\beta_x^{(y)}$ is defined as

$$\beta_x^{(y)}(m_2,n_2\,|\,\mu,\nu\,|\,d_{xy},\theta_{xy},\phi_{xy}) \equiv (-1)^{\mu+\nu}\frac{(m_2-n_2+\nu+\mu)!}{(m_2-n_2)!(\nu+\mu)!}\frac{1}{d_{xy}^{\nu+n_2+1}}P_{\nu+n_2}^{m_2-\mu}(\cos\theta_{xy})\exp[\mathrm{j}(m_2-\mu)\phi_{xy}]\ .$$

Above (4.3.4)-(4.3.9) are a linearly coupled set of equations and we can solve it numerically by truncating $n_1 = n_2 = n_3 = M$.

The total charge on each sphere can be found as

$$Q_1^t = 4\pi\varepsilon A_{00}^1, \qquad Q_2^t = 4\pi\varepsilon A_{00}^2, \qquad Q_3^t = 4\pi\varepsilon A_{00}^3,$$

and the capacitances can be defined as before i.e. for instance

$$c_{11} = \left.\frac{4\pi\varepsilon A_{00}^1}{V_1}\right|_{\substack{V_2=0\\V_3=0}}.$$

Table 14. Numerical results obtained for capacitances, when $\theta_{12}=0$, $\theta_{21}=\pi$, $\phi_{12}=0$, $\phi_{21}=0$, $\theta_{13}=\pi/3$, $\theta_{31}=4\pi/3$, $\phi_0^{13}=0$, $\phi_0^{31}=\pi$, $\theta_{23}=2\pi/3$, $\theta_{32}=5\pi/3$, $\phi_{23}=0$, $\phi_{32}=\pi$, $a_1=3\,\mathrm{cm}$, $a_2=5\,\mathrm{cm}$, $a_3=4\,\mathrm{cm}$, $d_{12}=d_{13}=d_{23}=10\,\mathrm{cm}$ and $M=10$

	Parameter	Capacitance [pF]
1	c_{11}	4.4710
2	$c_{21}=c_{12}$	-1.7474
3	$c_{31}=c_{13}$	-1.0902
4	c_{22}	8.4680
5	$c_{32}=c_{23}$	-3.0661
6	c_{33}	6.6793

Chapter 5

Application of Translational Addition Theorems to the Solution of Magnetic Field Problems

5.1 Magnetic Scalar Potential

The magnetic scalar potential Φ_m is defined in a region where $\mathbf{J} = 0$ and satisfies Laplace equation just as electrostatic potentials. Hence,

$$\nabla^2 \Phi_m = 0, \quad (\mathbf{J} = \mathbf{0}),$$

and magnetic scalar potentials is related to magnetic field intensity $\mathbf{H}$ according to

$$\mathbf{H} = -\nabla \Phi_m .$$

5.2 Two Perfect Conductor Spheres in Uniform Magnetic Field

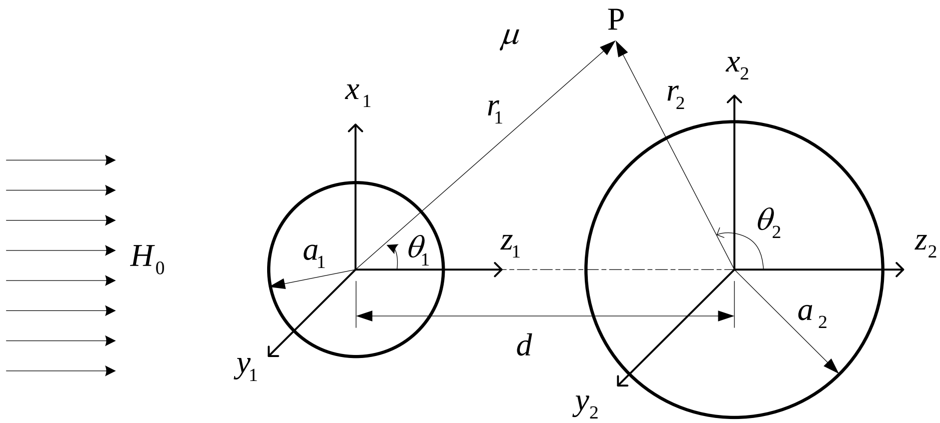

Fig. 14: System of two perfect conductor spheres placed in uniform magnetic field along z -axis

Consider two metallic spheres of radii a_1 and a_2 with a distance d between their centers, as shown in Fig. 14. The spheres are located in an external magnetic field oriented along the common axis of symmetry of the spheres, $\mathbf{H_0} = \hat{z}H_0$. The medium outside the spheres is homogeneous of permeability μ. The magnetic scalar potential outside the spheres satisfies the Laplace equation.

The Laplacian potential outside the spheres due to the presence of each of them is first expressed in the spherical coordinate system attached to the respective sphere in the form (see Fig.14)

$$\Phi_{m1}(r_1,\theta_1,\phi_1) = \sum_{n=0}^{\infty} A_n r_1^{-(n+1)} P_n(\cos\theta_1) \ , \qquad r_1 \geq a_1, \tag{5.1.1}$$

$$\Phi_{m2}(r_2,\theta_2,\phi_2)=\sum_{m=0}^{\infty}B_m r_2^{-(m+1)}P_m(\cos\theta_2)\,, \qquad r_2\ge a_2. \tag{5.1.2}$$

The potential due to the external field $\mathbf{H_0}=H_0\hat{z}$ in the coordinate system of each is, respectively,

$$\Phi_{m1}^{ex}=-H_0 z_1=-H_0 r_1 P_1(\cos\theta_1)+\mathrm{C}_1, \tag{5.1.3}$$

$$\Phi_{m2}^{ex}=-H_0 z_2=-H_0 r_2 P_2(\cos\theta_2)+\mathrm{C}_2, \tag{5.1.4}$$

where C_1 and C_2 are constants of reference. To impose the boundary condition at $r_1=a_1$, we have to use translational addition theorem (2.3.7) corresponding to $r\equiv r_2$, $r'\equiv r_1=a$, $r_0\equiv d$ and $\theta_0=\pi$, in order to translate Φ_{m2} into the coordinate system (r_1,θ_1,ϕ_1). Thus,

$$\begin{aligned}\Phi_{m1}^{t}(r_1,\theta_1,\phi_1)=&\sum_{n=0}^{\infty}A_n r_1^{-(n+1)}P_n(\cos\theta_1)+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m B_m\frac{(m+n)!}{(m)!(n)!}\left(\frac{r_1}{d}\right)^n d^{-(m+1)}P_n(\cos\theta_1)\\ &-H_0 r_1 P_1(\cos\theta_1)+C_1\,,\end{aligned} \tag{5.1.5}$$

Its derivative with respect to r_1 is

$$\frac{\partial\Phi_m^{(1)}}{\partial r_1}=-\sum_{n=0}^{\infty}A_n(n+1)r_1^{-(n+2)}P_n(\cos\theta_1)+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m B_m\frac{(m+n)!}{(m)!(n)!}\frac{n r_1^{n-1}}{d^{n+m+1}}P_n(\cos\theta_1)-H_0P_1(\cos\theta_1).$$

Applying the boundary condition at $r_1=a_1$, i.e., $\dfrac{\partial\Phi_m^{(1)}}{\partial r_1}=0$, yields

$$\sum_{n=0}^{\infty}A_n(n+1)a_1^{-(n+2)}P_n(\cos\theta_1)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m B_m\frac{(m+n)!}{(m)!(n)!}\frac{n a_1^{n-1}}{d^{n+m+1}}P_n(\cos\theta_1)-H_0P_1(\cos\theta_1). \tag{5.1.6}$$

Applying the orthogonality properties of the Legendre polynomials [Appendix B] gives

$$A_n(n+1)a_1^{-(n+2)}=\frac{(2n+1)}{2}\int_{-1}^{1}\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m B_m\frac{(m+n)!}{(m)!(n)!}\frac{n a_1^{n-1}}{d^{n+m+1}}P_n(x)-H_0P_1(x)\right]P_n(x)dx. \tag{5.1.7}$$

We obtain $A_0=0$ for $n=0$,

$$A_1 - \sum_{m=1}^{\infty}(-1)^m B_m (m+1)\frac{a_1^3}{2d^{m+2}} = -\frac{1}{2}H_0 a_1^3 \text{ for } n=1, \tag{5.1.8}$$

$$A_n - \sum_{m=1}^{\infty}(-1)^m B_m \frac{(m+n)!}{(m)!(n)!}\frac{na_1^{2n+1}}{(n+1)d^{n+m+1}} = 0 \quad \text{for } n=2,3\ldots \tag{5.1.9}$$

In the same way, Φ_{m1} is translated into the coordinate system (r_2, θ_2, ϕ_2) and the boundary condition is applied at the surface of sphere 2 to obtain the following equations:

$B_0 = 0$ for $n = 0$,

$$B_m - \sum_{n=1}^{\infty}(-1)^m A_n (n+1)\frac{a_2^3}{2d^{n+2}} = -\frac{1}{2}H_0 a_2^3, \quad \text{for } m=1, \tag{5.1.10}$$

$$B_m - \sum_{n=1}^{\infty}(-1)^m A_n \frac{(m+n)!}{(m)!(n)!}\frac{ma_2^{2m+1}}{(m+1)d^{m+n+1}} = 0, \text{ for } m=2,3\ldots. \tag{5.1.11}$$

As before, equations (5.1.8) to (5.1.11) are solved after truncating $n=m=M$ to find unknown constants of integrations A_n and B_m.

Table 15. Numerical results obtained for the magnetic field intensity with a after truncation to $M=20$ for translational addition method and to $M=200$ when using bispherical coordinates [16], for system with respect to values given by when, $H_0 = 1$ A/m, $a_1 = 3$ cm, $a_2 = 5$ cm, $d = 10$ cm and $\beta = 0.1$ (for bispherical coordinates)

Test Point	α *(rad) for bispherical Coordinates*	w.r.t. the coordinate system of sphere2		Translational Addition Method		Bispherical Coordinates	
		r_2 [*cm*]	θ_2[deg]	Hz [A/m]	Hx[A/m]	Hx[A/m]	Hz[A/m]
1	2.8560	5.6658	175.7332	0.1550	0.0270	0.1550	0.0270
2	2.0944	5.8540	163.2060	0.3266	0.1214	0.3266	0.1214
3	1.5708	6.2355	152.0275	0.5812	0.1953	0.5812	0.1953
4	1.0472	7.2544	135.9844	0.8912	0.1846	0.8912	0.1846
5	0.6283	9.8073	115.9900	1.0272	0.0632	1.0272	0.0632
6	0.3491	15.4470	94.7976	1.0168	0.0016	1.0168	0.0016
7	0.2618	19.5449	84.9091	1.0082	-0.0034	1.0082	-0.0034
8	0.1571	28.8216	67.1253	1.0014	-0.0032	1.0014	-0.0032
9	0.0628	44.9940	36.0078	0.9993	-0.0011	0.9993	-0.0011
10	0.0157	52.4006	9.8946	0.9991	-0.0002	0.9991	-0.0002

5.2.1 The External Magnetic Field along x Axis

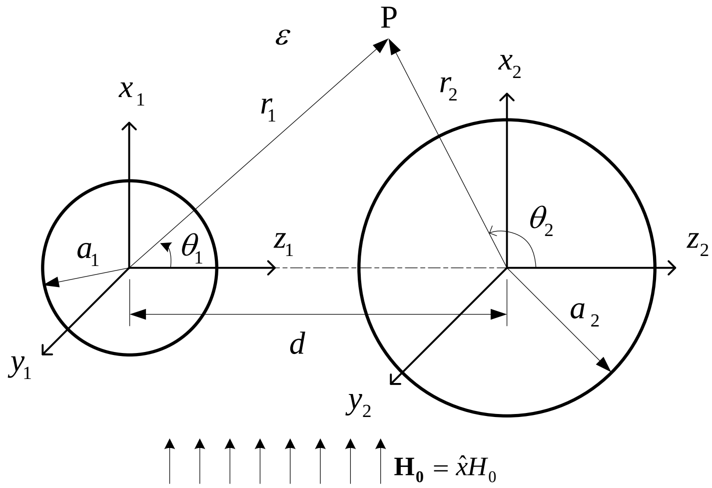

Fig. 15: System of two perfect conductor spheres placed in a uniform magnetic field along x -axis

The potential produce by the each of the two spheres is first expressed in the coordinate system attached to the respective sphere as

$$\Phi_{m1}(r_1,\theta_1,\phi_1) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A_{nm} r_1^{-(n+1)} P_n^m(\cos\theta_1)\cos m\phi_1, \tag{5.1.12}$$

$$\Phi_{m2}(r_2,\theta_2,\phi_2) = \sum_{q=0}^{\infty}\sum_{p=0}^{q} B_{qp} r_2^{-(q+1)} P_q^p(\cos\theta_2)\cos p\phi_2. \tag{5.1.13}$$

The (5.1.12) and (5.1.13) are written considering only the real part of (1.1.4) since the external magnetic field is along x axis.

The potential due to the external field $\mathbf{H_0} = H_0\hat{x}$ in the coordinate system of each is, respectively,

$$\Phi_{m1}^{ex} = -H_0 x_1 = -H_0 r_1 \sin\theta_1 \cos\phi_1 = -H_0 r_1 P_1^1(\cos\theta_1)\cos\phi_1 + \mathrm{C}_1, \qquad r_1 \ge a_1, \tag{5.1.14}$$

$$\Phi_{m2}^{ex} = -H_0 x_2 = -H_0 r_2 \sin\theta_2 \cos\phi_2 = -H_0 r_2 P_1^1(\cos\theta_2)\cos\phi_2 + \mathrm{C}_2, \qquad r_2 \ge a_2, \tag{5.1.15}$$

where C_1 and C_2 are constants of reference. To impose the boundary condition at $r_1 = a_1$, we have to use real part of translational addition theorem (2.3.7) corresponding to $r \equiv r_2$, $r' \equiv r_1 = a$, $r_0 \equiv d$ and $\theta_0 = \pi$, in order to translate Φ_{m2} into the coordinate system (r_1, θ_1, ϕ_1). Thus,

$$\begin{aligned}\Phi^t_{m1}(r_1,\theta_1,\phi_1) &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} A_{nm} r_1^{-(n+1)} P_n^m(\cos\theta_1)\cos m\phi_1 - H_0 r_1 P_1^1(\cos\theta_1)\cos\phi_1 + C_1 \\ &+ \sum_{q=0}^{\infty}\sum_{p=0}^{q}\sum_{\nu=0}^{\infty}(-1)^{p+q} B_{qp}\,\frac{(q+\nu)!}{(q-p)!(\nu+p)!}\,\frac{\nu r_1^{\nu-1}}{d^{q+\nu+1}}\,P_\nu^p(\cos\theta_1)\cos(p\phi_1).\end{aligned} \tag{5.1.16}$$

As in previous case, the boundary condition at $r_1 = a_1$ is applied to the derivative of (5.1.16) with respect to r_1. Then orthogonality properties of spherical harmonics yields [see Appendix C].

$$\begin{aligned}A_{nm}(n+1)a_1^{-(n+2)} &= \frac{(2n+1)}{2\pi}\,\frac{(n-m)!}{(n+m)!}\int_0^{\pi}\int_0^{2\pi}\Big[-H_0 P_1^1(\cos\theta_1)\cos\phi_1 P_n^m(\cos\theta_1)\cos m\phi_1 + \sum_{q=0}^{\infty} \\ &\cdot\sum_{p=0}^{q}\sum_{\nu=0}^{\infty}(-1)^{p+q}B_{qp}\,\frac{(q+\nu)!}{(q-p)!(\nu+p)!}\,\frac{\nu r_1^{\nu-1}}{d^{q+\nu+1}}\,P_\nu^p(\cos\theta_1)\cos(p\phi_1)\Big]P_n^m(\cos\theta_1)\cos m\phi_1 d\theta_1 d\phi_1.\end{aligned} \tag{5.1.17}$$

For $m = 1$ we get

$$A_{11} + \sum_{q=1}^{\infty}(-1)^q B_{q1}(q+1)\frac{a_1^3}{4d^{q+2}} = -\frac{1}{2}H_0 a_1^3, \qquad n = 1, \tag{5.1.18}$$

$$A_{n1} + \sum_{q=1}^{\infty}(-1)^q B_{q1}\,\frac{(q+n)!}{(q-1)!(n+1)!}\,\frac{n a_1^{2n+1}}{(n+1)d^{(q+n+1)}} = 0, \qquad n = 2.3... \tag{5.1.19}$$

In the same way, the boundary condition is applied at the surface of sphere 2 to obtain the following equations (for $p = 1$) :

$$B_{11} + \sum_{n=1}^{\infty}(-1)A_{n1}(n+1)\frac{a_2^{\,3}}{4d^{n+2}} = -\frac{1}{2}H_0 a_2^3, \qquad q = 1, \tag{5.1.20}$$

$$B_{q1} + \sum_{n=1}^{\infty} (-1)^q A_{n1} \frac{(q+n)!}{(n-1)!(q+1)!} \frac{q a_2^{2q+1}}{(q+1) d^{(n+q+1)}} = 0, \quad q = 2.3\ldots \tag{5.1.21}$$

In the case of $m \neq 1$ and $p \neq 1$ we obtain homogeneous set of linear equations whose solution is the trivial solution, i.e., $A_{nm} = B_{qp} = 0$ for all $m \neq 1$ and $p \neq 1$.

Table 16. Numerical results obtained for the magnetic field intensity with a after truncation to $M = 20$ for translational addition method and to $M = 200$ when using bispherical coordinates [16], for system with respect to values given by when, $H_0 = 1\text{A/m}$, $a_1 = 3\,\text{cm}$, $a_2 = 5\,\text{cm}$, $d = 10\,\text{cm}$ and $\beta = 0.1$ (for bispherical coordinates)

Test Point	α *(rad)* *for bispherical Coordinates*	w.r.t. the coordinate system of Sphere2		Translational Addition Method		Bispherical Coordinates	
		r_2 [*cm*]	θ_2[deg]	Hz [A/m]	Hx[A/m]	Hx[A/m]	Hz[A/m]
1	2.8560	5.6658	175.7332	0.0382	1.5600	0.0382	1.5600
2	2.0944	5.8540	163.2060	0.1463	1.3264	0.1463	1.3264
3	1.5708	6.2355	152.0275	0.2105	1.0837	0.2105	1.0837
4	1.0472	7.2544	135.9844	0.1867	0.8906	0.1867	0.8906
5	0.6283	9.8073	115.9900	0.0617	0.8887	0.0617	0.8887
6	0.3491	15.4470	94.7976	0.0008	0.9626	0.0008	0.9626
7	0.2618	19.5449	84.9091	-0.0039	0.9818	-0.0039	0.9818
8	0.1571	28.8216	67.1253	-0.0034	0.9956	-0.0034	0.9956
9	0.0628	44.9940	36.0078	-0.0011	1.0000	-0.0011	1.0000
10	0.0157	52.4006	9.8946	-0.0002	1.0005	-0.0002	1.0005

Chapter 6

Conclusions and Suggestions for Future Works

6.1 Summary and Conclusions

In order to get a physical insight and quantitative relationships for engineering problems, various real world shapes can mathematically be modeled at a very first approximation by using systems of spheres.

The translational addition theorems for static and stationary fields in spherical coordinates presented in this thesis can be used to solve boundary value field problems relative to many-sphere structures. In the case of axisymmetric geometries, the general expressions (2.2.7, 2.3.7 and 2.3.10) can be further simplified, leading to much simpler matrices. When dealing with arbitrarily located systems of spheres, spherical harmonics need to be employed.

The formulations presented in this thesis particularized for a system of two spheres constitute an alternative to the classical formulations using the method of images in the case of Dirichlet boundary conditions [7] or employing the bispherical coordinates [13-16].

Translational addition theorems presented in this thesis are also applicable to multi sphere systems involving scalar Laplacian fields such as in fluid dynamics, heat flow studies etc.

6.2 Future Research Direction

The research presented in this thesis is confined to field problems involving systems of conducting spherical bodies. This work can be extended to field-penetrable spheres, di-electric or magnetic, where the boundary conditions are more complex. Obviously, the extension to systems of prolate or oblate spheroids yields results of more practical impor-tance, since the spheroidal shapes approximates much better real world shapes.

Appendices

Appendix A

Wigner 3-J Symbols

1) Some specialized formulas for Wigner **3-J** symbol [9] are given below.

i) $\begin{bmatrix} J_1 & J_2 & J_3 \\ 0 & 0 & 0 \end{bmatrix} = 0,$ if $J_1+J_2+J_3$ is odd. A-1

ii) $$\begin{bmatrix} J_1 & J_2 & J_3 \\ 0 & 0 & 0 \end{bmatrix} = (-1)^{J/2} \left[\frac{(J-2J_1)!(J-2J_2)!(J-2J_3)}{(J+1)} \right]^{1/2} \quad \text{A-2}$$

$$\cdot \frac{(J/2)!}{(J/2-J_1)!(J/2-J_3)!(J/2-J_3)!}, \qquad \text{If } J=J_1+J_2+J_3 \text{ is even.}$$

iii) $$\begin{bmatrix} J_1 & J_2 & (J_1+J_2) \\ m_1 & m_2 & -M \end{bmatrix} = (-1)^{(J_1+J_2-M)} \sqrt{\frac{(2J_1)!(2J_2)!(J_1+J_2+M)!(J_1+J_2-M)!}{(2J_1+2J_2+1)!(J_1+m_1)!(J_1-m_1)!(J_2-m_2)!(J_2+m_2)!}} \quad \text{A-3}$$

A-2 gives

$$\begin{bmatrix} n & \nu & (n+\nu) \\ 0 & 0 & 0 \end{bmatrix} = (\text{-}1)^{n-\nu} \frac{(n+\nu)!}{n!\nu!} \sqrt{\frac{(2n)!(2\nu)!}{(2n+2\nu+1)!}} \quad \text{A-4}$$

A-3 gives

$$\begin{bmatrix} n & \nu & (n+\nu) \\ m & -\mu & -(m-\mu) \end{bmatrix} = (-1)^{(n+m-\nu-\mu)} \sqrt{\frac{(2n)!(2\nu)!(n+\nu+m-\mu)!(n+\nu-m+\mu)!}{(2n+2\nu+1)!(n+m)!(n-m)!(\nu-\mu)!(\nu+\mu)!}} \quad \text{A-5}$$

The A-4 and A-3 yields to write simplified version of (1.2.5) when $p = n+\nu$ as

$$a(m,n|-\mu,\nu|n+\nu) = (-1)^{2(n+m-\nu-\mu)}[2(n+\nu)+1] \frac{(n+\nu-m+\mu)!}{(n-m)!(\nu+\mu)!} \frac{(2n)!(2\nu)!}{(2n+2\nu+1)!} \frac{(n+\nu)!}{n!\nu!} \quad \text{A-6}$$

Similarly when $p = n - \nu \geq 0$, we obtain

$$\begin{bmatrix} n & \nu & n-\nu \\ m & -\mu & -m+\mu \end{bmatrix} = (-1)^{n+m} \left[\frac{(n+m)!(n-m)!(2\nu)![2(n-\nu)]!}{(2n+1)!(\nu+\mu)!(\nu-\mu)!(n+m-\nu-\mu)!(n-m-\nu+\mu)!} \right]^{1/2} \quad \text{(A-7)}$$

and, for $m = \mu = 0$,

$$\begin{bmatrix} n & \nu & n-\nu \\ 0 & 0 & 0 \end{bmatrix} = (-1)^{n} \frac{n!}{\nu!(n-\nu)!} \left[\frac{(2\nu)![2(n-\nu)]!}{(2n+1)!} \right]^{1/2} . \quad \text{(A-9)}$$

Thus,

$$a(m,n\,|-\mu,\nu\,|\,n-\nu) = (-1)^{\mu} \frac{n!(n+m)!(2\nu)![2(n-\nu)+1]!}{(2n+1)!\nu!(\nu+\mu)!(n-\nu)!(n+m-\nu-\mu)!}, \quad |m-\mu| \leq n-\nu. \quad \text{(A-10)}$$

Appendix B

Associate Legendre Functions

1) Some useful properties of Associate Legendre function [10]

i) Orthogonality property of the associate Legendre Function

$$\int_{-1}^{1} P_n^m(x)P_{n'}^m(x)dx = \frac{2}{(2n+1)}\frac{(n+m)!}{(n-m)!}\delta_{nn'}\ , \qquad \text{for } x\in[-1,+1]\ , \qquad \text{B-1}$$

where $\left.\begin{matrix} n=0,1,2,3,\cdots \\ m=0,\pm1,\pm2,\pm3,\cdots \end{matrix}\right\}$ integers and $\delta_{nn'}$ is the Kronecker delta.

For $m=0\,, n'=0$ we have

$$\int_{-1}^{1} P_n(x)P_0(x)dx = \frac{2}{(2n+1)}\delta_{n0},$$

$$\int_{-1}^{1} P_n(x)dx = \begin{cases} 2, & n=0, \\ 0, & n\neq 0. \end{cases} \qquad \text{B-2}$$

ii) $P_n^0(x) = P_n(x)$ B-3

iii) $P_n^m(x) = 0,\ \text{ if } \ |m|>n$ B-4

iv) $P_n^{-m}(x) = (-1)^m \dfrac{(n-m)!}{(n+m)!}P_n^m(x)$ B-5

2) Legendre series expansion [6]

For any $f(x)$ defined in [-1,1] and satisfying the Dirichlet conditions can be expand as

$$f(x) = \sum_{n=0}^{\infty} a_n P_n^m(x) \ \text{ with } \ a_n = \frac{(2n+1)}{2}\frac{(n-m)!}{(n+m)!}\int_{-1}^{1} f(\xi)P_n^m(\xi)d\xi\ . \qquad \text{B-6}$$

Appendix C

Spherical Harmonics

1) The spherical harmonics is defined as [10]

$$Y_n^m(\theta,\phi) = \sqrt{\frac{(2n+1)}{4\pi}\frac{(n-m)!}{(n+m)!}}P_n^{\,m}(\cos\theta)\exp(jm\phi). \qquad \text{C-1}$$

2) Orthogonality property of the spherical harmonics [10]

$$\begin{aligned}\int_\Omega Y_n^m(\theta,\phi)Y_l^k(\theta,\phi)^* d\Omega &= \int_0^{2\pi}\int_0^{\pi} Y_n^m(\theta,\phi)Y_l^k(\theta,\phi)^* \sin\theta d\theta d\phi \\ &= \delta_{mk}\delta_{nl}\ ,\end{aligned} \qquad \text{C-2}$$

where $Y_l^k(\theta,\phi)^* = (-1)^k Y_l^{-k}(\theta,\phi)$ and $d\Omega$ is an element of solid angle.

3) Spherical harmonics expansion [11]

For any square integrable function $f(\theta,\phi)$ can be expanded as,

$$f(\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_{nm}Y_n^m(\theta,\phi),$$

with

$$f_{nm} = \int_\Omega f(\theta,\phi)Y_n^m(\theta,\phi)^* d\Omega = \int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)Y_n^m(\theta,\phi)^* \sin\theta d\theta d\phi\ . \qquad \text{C-3}$$

4) Another form of spherical harmonics can be defined as [13] odd and even functions:

$$Y_{nm}^e = P_n^m(\cos\theta)\cos(m\phi),\quad Y_{nm}^o = P_n^m(\cos\theta)\sin(m\phi),\quad 0 \le m \le n,\ n = 0,1,2\ldots. \qquad \text{C-4}$$

These harmonics functions form a complete system of orthogonal functions on the surface of a sphere, thus

$$\int_0^{2\pi}\int_0^{\pi}\left[Y_{nm}(\theta,\phi)\right]^2 \sin\theta d\theta d\phi = \frac{4\pi}{\lambda_m(2n+1)}\frac{(n+m)!}{(n-m)!} \quad , \qquad \text{C-5}$$

where the superscript of the Y_{nm} can be either e (even) or o (odd) (except that Y_{n0}^{o} does not exist) and where $\lambda_0 = 1$, $\lambda_m = 2$ ($n = 1.2\ldots$).

Any function $f(\theta,\phi)$, specified over the surface of a sphere, may be expressed in terms of the series

$$f(\theta,\phi) = \sum_{n=0}^{n}\sum_{m=0}^{n}\left(A_{nm}Y_{nm}^{e}(\theta,\phi) + B_{nm}Y_{nm}^{0}(\theta,\phi)\right),$$

where $$A_{nm} = \frac{(2n+1)\lambda_m}{4\pi}\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)Y_{nm}^{e}(\theta,\phi)\sin\theta d\theta d\phi \, , \qquad \text{C-6}$$

with the integral for B_{nm} similar to that for A_{nm}, except that Y_{nm}^{o} is substitute for Y_{nm}^{e} and the terms for $m = 0$ are omitted.

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