ALGORITHM FOR THE VERIFICATION OF A COMBINATORIAL CONJECTURE ON A FINITE ABELIAN GROUP

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ABSTRACT

An ALGOL 60 algorithm has been developed to verify the following conjecture for the case where $p=7$: any sequence, with length $3p-3$ (p being a prime) of elements for $C_p \otimes C_p$ and containing no non empty subsequences of length less than or equal to p , such that the sum of its elements is null, is always comprised of three distinct elements, each one repeated $p-1$ times.

This thesis essentially presents a FORTRAN IV version of the same algorithm and investigates the case where $p=11$. The thesis ends with considerations on the general case (p is any prime number).

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CHAPTER I

INTRODUCTION

1.1 The Central Problem

We shall first recall a well-known result: if A is a finite Abelian group of order $w(A)$ and if C_{a_i} denotes a cyclic group of order a_i then A can be expressed as a direct product of the type

$$A = C_{a_1} \otimes C_{a_2} \otimes \dots \otimes C_{a_n}$$

where $a_1 > 1$ and $a_i | a_{i+1}$ for $i = 1, n-1$. Furthermore, this decomposition of A is unique.

Then, let us define $M(A)$ to be $(\sum_{i=1}^{n} a_i) - (n-1)$ and define $\mu(A)$ to be the smallest positive integer such that any sequence $\alpha_1, \alpha_2, \dots \alpha_{\mu(A)}$ of elements of A contains a non empty subsequence with sum zero. Hence, note that, as a consequence of the unicity of the decomposition of an Abelian group into a direct product of cyclic groups, $M(A)$ is an invariant of A. In addition $\mu(A)$ will also be an invariant of A.

The central problem is to define an expression of $\mu(A)$ in terms of some other invariants of A. In 1965 P.C. Baayen conjectured that for all finite Abelian groups the following relation should exist:

$$M(A) = \mu(A) \quad . \qquad (1)$$

He later had to reject his hypotheses when he found a counter-example with the group $C_2 \boxtimes C_2 \boxtimes C_2 \boxtimes C_2 \boxtimes C_6$, for which $\mu(A) > M(A)$.

1.2 Particular Cases

However, although the above equality (1) is not generally true, the research done in this area by some mathematicians has shown that it is valid in many cases. In a paper published in 1969 [1], P. van Emde Boas presented the following families of groups for which we have $M(A) = \mu(A)$:

I Any Abelian p-group. A group A is called an Abelian p-group if its decomposition into a direct product of cyclic groups contains only factors of order equal to the power of some prime.

II Any group $A = C_a \boxtimes C_{a_1 b}$. $a_1\ b \in N$

III Any group $A = H \boxtimes C_{q^n m}$ where H is a p-group of order q^j such that $q^n \geq M(H)$

IV Any $A = C_{2p^{n_1}} \boxtimes C_{2p^{n_2}} \boxtimes C_{2p^{n_3}}$ where p is a prime

V Any $A = C_2 \boxtimes C_{2nm_1} \boxtimes C_{2nm_2}$ where $n = 2^{k_1} 3^{k_2} 5^{k_3} 7^{k_4}$ and either $m_{-1} = 1$, m_2 arbitrary and $n = 1$, or $m_1 = p^r$, $m_2 = p^s$ and p prime.

In the same year, P. C. Baayens [2] concluded that:

VI Any $A = C_2 \otimes C_2 \otimes C_2 \otimes C_{2m}$ for m odd.

Subsequently, the list was extended by the following cases [3] :

VII Any $A = C_3 \otimes C_3 \otimes C_{6m}$ m is not a multiple of 3

VIII Any $A = C_{3.2^{n_1}} \otimes C_{3.2^{n_2}} \otimes C_{3.2^{n_3}}$

IX Any $A = C_3 \otimes C_{6nm_1} \otimes C_{6nm_2}$ with n, m, and m_2 as in V.

1.3 The Conjecture

The proof presented for cases V and IX employs the following property.

P : Any sequence, with length $3p-3$ of elements from $C_p \otimes C_p$ and containing no (non empty) subsequences of length less than or equal to p , such that the sum of its elements is null, is always comprised of three distinct elements, each one repeated $p-1$ times.

This property, however, has never been established. It is assumed true for all prime p on the basis that it is valid for $p = 2,3,5$ and 7 . For the case $p=5$ an Algol-60 programme was written to carry out an exhaustive search for the sequences of $C_5 \otimes C_5$ satisfying P . The same algorithm was later modified to treat the case $p=7$.

This thesis is involved mainly in checking the results obtained for this last case, using a FORTRAN version of the algorithm. An investigation of the procedure to determine $p=11$, will illustrate an adaption of the algorithm treating a part of the case $p=11$ and finally concluding with a prediction of the "minimum" output of this algorithm for the general case where p is any prime greater than two.

CHAPTER II

DEFINITIONS AND CASE $C_7 \otimes C_7$

2.1 Definitions

Although our problem can be stated quite easily without specialized vocabulary, it will be useful in the following pages to have the following consistant definitions.

A sequence of elements taken in a given set E will be called an E-sequence.

If A is an additive Abelian group, we shall call a null-sequence (or null-A-sequence) a non-empty sequence of elements of A having the sum of its elements equal to 0 .

An A-sequence is said to be primitive if it does not contain any null-sequence.

If S is an A-sequence and C_{a_n} is the group of highest order in the decomposition of A as a direct product of cyclic groups, then S is said to be a short null-subsequence provided it contains a null-sequence and provided its length is no greater than a_n .

It will be convenient to talk about the "multiplication" of an element of A by a positive integer n. If $a \in A$, then $n.a$ means

$$a + a + \ldots + a \quad \text{(n times)}.$$

We shall also add that $0.a = 0$ (0 on the right hand side of the equal sign is element of A). Then we shall say that two sequences S and S' of elements from a group A are equivalent if S' can be derived from S by any of the following methods; the permutation of the elements of S, the multiplication of S by a non-zero integer n , or the combination of these two operations.

The $C_n \boxtimes C_n$ - sequence

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \cdots \begin{pmatrix} a_m \\ b_m \end{pmatrix}$$

is called the <u>mirror image</u> of the $C_n \boxtimes C_n$ - sequence

$$\begin{pmatrix} b_1 \\ a_1 \end{pmatrix} \begin{pmatrix} b_2 \\ a_2 \end{pmatrix} \cdots \begin{pmatrix} b_m \\ a_m \end{pmatrix} .$$

2.2 Elimination of Sequences

Since the number of sequences with length $3p-3$ taken among p^2 elements (p^2 being the order of $C_p \boxtimes C_p$) increases markedly as p increases, it is necessary to eliminate as many sequences, and elements, as possible from consideration. In the algorithm used, this will be done mainly by eliminating the equivalent sequences and by eliminating mirror-image sequences. Later it will be demonstrated how the primitive sequences of C_p are divided into two groups to be defined as A and B for the present. Accordingly we add to our list. Sequences that can be derived from primitive sequences of group A will also be eliminated as well as some elements as possible new members of the sequences. This last elimination is made according to a set of rules determined by the sequences of group B .

The next section of this chapter will provide the information necessary to determine the sequences of groups A and B as well as indicating the use of these simplifications in the case $C_7 \boxtimes C_7$.

2.3 Application to $C_7 \boxtimes C_7$

In addition to the development of the points outlined in section 2.2, we include in this section the other major ideas completing the algorithm.

2.3.1 Equivalent Sequences

In order to reference easily any element of $C_7 \boxtimes C_7$ we shall adopt an enumeration of these elements as suggested by diagrams 1 and 2.

j ↑							
6	0 6	1 6	2 6	3 6	4 6	5 6	6 6
5	0 5	1 5	2 5	3 5	4 5	5 5	6 5
4	0 4	1 4	2 4	3 4	4 4	5 4	6 4
3	0 3	1 3	2 3	3 3	4 3	5 3	6 3
2	0 2	1 2	2 2	3 2	4 2	5 2	6 2
1	0 1	1 1	2 1	3 1	4 1	5 1	6 1
0	0 0	1 0	2 0	3 0	4 0	5 0	6 0
	0	1	2	3	4	5	6

Diagram 1

6		29	30	31	32		
5		25	26	27	28	33	34
4		22	23	24	4	14	18
3	36	20	21	3	10	13	17
2		19	2	7	9	12	16
1	35	1	5	6	8	11	15
0							
	0	1	2	3	4	5	6

i →

Diagram 2

The number in square (i,j) in diagram 2 will be a new representation for the element $\binom{i}{j}$ in square (i,j) of diagram 1. The choice of this particular order to enumerate the elements of $C_7 \boxtimes C_7$ and the fact that some of them are not represented in diagram 2 will be justified later.

An immediate result arising from the choice of an order on the elements of $C_7 \boxtimes C_7$ is the possibility of looking at all the sequences made from these ordered elements as well as eliminating all the permutations of a given sequence. This can be done easily if the sequences are considered sequentially, following a lexicographical order. A second step, that involves keeping the first element of all sequences equal to 1, will prevent any sequence from being a multiple of some other. Thus we have a way of looking at only one sequence taken out of each class of equivalent sequences. In addition, the sequence obtained this way shall precede lexicographically all the other sequences in its equivalence class.

2.3.2 Primitive C_7 - sequence and Lines of $C_7 \boxtimes C_7$

Let $\sum_p$ be the set of all equivalence classes of primitive sequences of C_p . First we are interested in having a representative sequence for each element of $\sum_p$. Table Ia (page 9) provides an exhaustive list of these representatives for $\sum_7$. From now on, we shall identify the elements of $\sum_7$ with their representative sequences in Table Ia, unless a distinction is explicitly stated.

TABLE I

Type I	Group	Type II
11	A	11
12	A	11
13	B	
111	A	111
112	A	111
113	A	111
114	A	444
123	A	111
1111	A	1111
1112	A	1111
1113	A	1111
1122	A	1111
11111	A	11111
11112	A	11111
111111	A	111111

Secondly, we shall try to find for each sequence s of length ℓ listed under type I, a corresponding primitive C_7 - sequence s' made out of one single C_7 element repeated ℓ times. Note that the sum of any subsequence of s' must be the sum of a subsequence of its corresponding s sequence. Moreover, the length of the subsequence of s shall be at most the length of the subsequence of s' under consideration.

Each element of $\sum_7$ having such a corresponding sequence has that sequence listed beside it in Table I, underneath "type II". According to this, we shall divide the elements of $\sum_7$ into two groups; the group A consisting of elements for which we have a corresponding type II sequence, and group B consisting of the other elements. From Table I, we conclude that group B is composed of one single sequence 1,3 , and the larger group A is comprised of the other fourteen sequences.

Before we can utilize this information, the following lemma about the C_7 - sequences must be proven.

Lemma:

Let S be a C_7 - sequence containing a subsequence s of group A and let S' be the new sequence obtained by replacing s by s' , where s' is the sequence of type II corresponding to s . Then, if S' contains a short null-subsequence, S must contain one as well.

Proof: Let us designate by t' a particular short null-subsequence contained in S' and let us define by u' the sequence of elements common to t' and s'. As a consequence of the properties of the type II sequences, we know that we can find a subsequence of s called u, such that the sum of its elements is equal to the sum of the elements of u' and such that its length is not greater than the length of u'. It is then sufficient to remark that $t' \setminus u'$ is comprised of elements of S exclusively, thus illustrating that $(t' \setminus u') \cap u$ is a short null-sequence of S.

Finally, since our real interest lies in studying $C_7 \boxtimes C_7$ - sequences (as a particular case of $C_p \boxtimes C_p$), the following remarks provide access to the information gathered about the primitive C_7 - sequences will allow us to discard a very large number of sequences in the forthcoming subsection.

A proper cyclic group of $C_p \boxtimes C_p$ is called a <u>line</u> of $C_p \boxtimes C_p$. Any two elements belonging to two different lines are sufficient to span the whole group; in this case those two lines are called <u>base lines</u>. There are eight lines in the group. The generators of two different lines can be denoted by $(1,0)$ and $(0,1)$. The elements of the base lines are called base elements.

2.3.3 Simplifications: Case A

Since we are interested in relatively long $C_7 \boxtimes C_7$

sequences, we shall first ask ourself if there would not be any subsequence that would be common to all these $C_7 \boxtimes C_7$ - sequences of length 3p-3 = 18 containing no short null-subsequences. In order to answer this question, let us consider first that there are exactly eight different lines in $C_7 \boxtimes C_7$. Suppose that we are dealing with a sequence S such that all eight lines of $C_7 \boxtimes C_7$ are represented by at least two elements in S , then we know that there are at least 16 elements of S . The two remaining elements can be in any line (1°), and obviously either the two elements are in different lines, or they are in the same line (2°) .

We shall consider S to be made out of the two independent sequences σ_1 and σ_2 , i.e. the sequence made out of the top (or first) componenents of S and the sequence made out of the bottom (or second) components of S respectively. If the two elements are in different lines, we shall choose the two lines represented by three elements as base lines and we have a sequence $\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{1}\binom{0}{1}$. Note that the fact that two particular lines are chosen will not prevent the algorithm from looking at all the sequences that should be looked at. On the other hand, we shall use repetition of the same base element since if we find a $C_7 \boxtimes C_7$ - sequence S containing no short null-subsequence and consisting of repetitions of elements we can derive for each σ_1 and σ_2 all the C_7 - sequences containing no short null-subsequence and thus trace back the $C_7 \boxtimes C_7$ - sequences which interest us. In addition, we are sure to cover all the possibilities for this first case since all the elements of length 3 (or more) of $\sum_7$ have a corresponding sequence of type II.

If the two elements are in the same line, using the same approach, we find a sequence

$$\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{1}$$

which will cover all the cases where the elements of length two of $\sum_7$ have a corresponding sequence of type II, i.e. for 1 1 and 1 2 .

Those two cases cover all the sequences of group A and we shall consider them to be two subcases of a main CASE A . The sequence

$$\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{1}$$

being common to 1° and 2° will then be used as one single sequence to treat the whole case A .

2.3.4 Simplification: Case B

Since 2° of the preceding subsection was not completely treated there, we shall elaborate on it now. The only element constituting group B is 1 3 . A starting sequence as

$$\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{3}$$

would complete the part of 2° left by case A and, moreover, would take care of all the cases where there is one line represented in S by at least four elements and where the other lines are represented by one or more elements.

For the purposes of clarification, it should be sufficient to exemplify the particular case where seven lines are represented by at least two elements and one line by one single element. We then have three elements left and accordingly, when the three are in the same line we have a subsequence

$$\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{1}$$

or

$$\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{3} \quad ;$$

when two are in a same line and the third in another one,

$$\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{1}\binom{0}{1}$$

and when no two are in the same line,

$$\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{1}\binom{0}{1}$$

If we add that no $C_7 \boxtimes C_7$ - sequences without a short null-subsequence may contain more than six elements of the same line, it then appears that this case, and any other, will always give a starting sequence containing one of the sequences of CASE A or CASE B . Those are contained in any $C_7 \boxtimes C_7$ - sequence containing no short null-subsequence.

2.3.5 Mirror-image Sequences

To eliminate the mirror-image sequences encountered in treating CASE A , the algorithm uses the following ideas.

As can be seen in diagram 1, there is symmetry with respect to mirroring the main diagonal. The order in which the elements of $C_7 \boxtimes C_7$ are enumerated in diagram 2 will allow us, given any element $\binom{a}{b}$ represented by a number N $(5 \leq N \leq 18)$, to easily find the number N' $(19 \leq N' \leq 32)$ associated with its mirror-image. More explicitly, if N is associated with the element $\binom{a}{b}$ then $N' = N + 14$ will represent $\binom{b}{a}$. The diagonal elements are numbered from 1 to 4 . We shall see subsequently why, when giving the "starting positions" for CASE A and CASE B , 5 is not used to denote $\binom{5}{5}$ as we would expect.

If weights of 1, -1 and 0 are attributed respectively to the elements on the right side of the diagonal, on its left side and on the diagonal itself, it is possible to gather information about a $C_7 \boxtimes C_7$ - sequence simply by looking at the sum of the weights of the elements constituting it. Let us look at all sequences for which this sum is positive, eliminating the cases where the sum is negative since there exists a one-to-one mapping between the set of sequences having the sum of the weights of their elements less than 0 and the set of sequences having the sum of the weights of their elements greater than 0 . If the sum is null, there are two possibilities to be considered. If all the elements of S are on the diagonal, then we must check the sequence as we do when the sum is positive. The second possibility is an equal number of elements coming from each side of the diagonal, and in this case, we compare lexicographically the sequence with its mirror image and check only the first one.

This type of simplification is not used for CASE B, but the amount of work can be greatly reduced if we exclude all repetitions of elements - because those cases are done in CASE A - and also note that once a sequence has been incremented by an element x, we do not have to check with the elements $2x$ (i.e. $x + x$) or $4x$ (i.e. $x + x + x + x$). The reason for this is that the sequences covered by CASE A are 1 1, 1 2 and all the sequences in the two equivalence classes represented by those sequences.

2.3.6 Addition of a New Element to a Sequence

We are concerned here with the logical structure of the algorithm ignoring temporarily the simplifications proposed in 2.2 .

The Array K

Let us consider a sequence S of length $\ell < 18$ and containing no short null-subsequence. If a new element e is added to the sequence, a way of checking whether the addition of this element creates some short null-subsequence would be to verify all the short sequences of S containing e ; the others having presumably been checked before. This, however, would involve a great deal of time-consuming repetition. The following procedure avoids this problem and, as well, it explains how sequences are extended by non-base elements (in the case of $C_7 \otimes C_7$, those elements are the ones numbered from 1 to 30 inclusively).

We assume that we have a sequence S :

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \quad \begin{pmatrix} a_j \\ b_j \end{pmatrix}, \quad j < 18$$

and we want to increase it with the element $\begin{pmatrix} x \\ y \end{pmatrix}$, which, if it is accepted (i.e. if it does not create a short null-subsequence), will be denoted $\begin{pmatrix} a_{j+1} \\ b_{j+1} \end{pmatrix}$. If we could determine quickly whether S contains a subsequence of length 6 or less with sum $-\begin{pmatrix} x \\ y \end{pmatrix}$ (or $\begin{pmatrix} 7-x \\ 7-y \end{pmatrix}$), we could speedily decide if $\begin{pmatrix} x \\ y \end{pmatrix}$ can be used to increase S . We shall use an array $K(\alpha,\beta,j)$, $0 \le \alpha \le 6$, $0 \le \beta \le 6$ and $1 \le j \le 18$, whose elements have the value $7-v$; v being the length of the shortest subsequence of S with sum $-\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ and where $K[\alpha,\beta,j]$ is 0 when $v > 6$ or when there is no subsequence of S with sum $-\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Of course, for a particular $\begin{pmatrix} x \\ y \end{pmatrix}$, as soon as $K(x,y,j) \neq 0$ it becomes impossible to add the element $\begin{pmatrix} x \\ y \end{pmatrix}$ to S without creating a short null-subsequence. To complete the description of K we add that $K(0,0,j) = 7$ for all values of j. It is easy to see that the array

$$K(\alpha,\beta,j+1) = \max(K(\alpha,\beta,j),\ K(\alpha+a_{j+1},\ \beta+b_{j+1},j)-1)\ ,$$

$$0 \le \alpha \le 6,\quad 0 \le \beta \le 6$$

will provide us with all the information we could need, at any time.

Addition of Non-base Elements

With reference to the above remarks the procedure used to increase a sequence by a non-base element can now be outlined:

CASE A : Let S be of length j (j is not yet 18). Suppose that the next candidate element $\binom{x}{y}$ has been numbered N (diagram 2).

1°) If $K(x,y,j) = 0$, $\binom{x}{y}$ becomes a new member of S and the length of S becomes $j+1$. Then $K(\alpha,\beta,j+1)$ is updated and we are ready to repeat the same process with a new candidate element. Being in CASE A , this new candidate element will still be N , since in this case, we consider repetition of elements. We keep increasing S by element N until one of the following three situations arises:

a) $K(x,y,j+r)$, where r is the number of times the element N has been repeated, ceases to be 0 . This situation is treated in 2° .

b) The length of S has reached the value 18 . This leads to the print-out of the sequence.

c) All non-base elements have been tried and the length S still has not reached 18. We then try to extend S by base elements.

Note here that in CASE A the last element is numbered 32 .

2°) If $K(x,y,j) \neq 0$ then N cannot be added to S without creating a short null-subsequence. We must pass to the next element, i.e. N is replaced by $N+1$. From this point we start again from 1°.

In the case where neither the addition of a non-base element nor the expansion by base elements can increase the length of S to 18 , we must reject the sequence. This is done by shortening the length of S by one and trying to develop a new sequence from that point. More precisely if the last element of S after it has been shortened is N , then the next candidate element we use is $N+1$ starting at 1°) . Note that this way we look at all sequences in a lexicographical order. If the stage is reached where S has been shortened to a length of four (the length of the starting sequence), we have exhausted all the possibilities and we are now ready to start CASE B .

Addition of Base Elements

Whenever it is necessary to expand a sequence by base elements, only $\binom{1}{0}$ and $\binom{0}{1}$ will be used. We shall see by the results that this restriction is legitimate. Since all the base elements appearing in the accepted sequences are repeated exactly six times, we know from Table I that all primitive C_7 - sequences of length six are in the same equivalence class. In other words, the expansion by other base elements is useless.

At this point, it is possible to add one more simplification to our algorithm. As a consequence of 2.3.3 , the maximum number of base elements by which a sequence can be increased is eight since the longest primitive C_7 - sequence is six. Thus, if a $C_7 \boxtimes C_7$ - sequence containing no short null-subsequence is of length less than ten when we have exhausted all the possible non-base elements, we can reject this sequence as we know that it will never be possible to extend it up to length 18 (CASE A) .

In the same way, we obtain from 2.3.4 that the maximum number of base elements by which a sequence can be extended is two i.e. if a sequence is of length less than 16 before expansion by base elements, it can be rejected (CASE B) .

Starting Positions

We have pointed out in subsections 2.3.3 and 2.3.4 that some subsequences can be assumed to be contained in S at the beginning of the algorithm. The sequences were

$$\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{1} \quad \text{for CASE A and}$$

$\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{3}$ for CASE B .

Accordingly we shall start CASE A and CASE B with $K(\alpha,\beta,4)$ and $K(\alpha,\beta,6)$ (respectively) $0 \le \alpha \le 6$, $0 \le \beta \le 6$ initialized as in diagrams 3 and 4 .

6	6	0	0	0	0	4	5
5	5	0	0	0	0	3	4
4	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
6	7	0	0	0	0	5	6
↗	0	1	2	3	4	5	6

K(0,0,4)

Diagram 3
Initialization of K for CASE A

6	6	0	0	2	3	4	5
5	0	0	0	0	0	0	0
4	6	0	0	2	3	4	5
3	5	0	0	1	2	3	4
2	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
6	7	0	0	3	4	5	6
↗	0	1	2	3	4	5	6

K(0,0,6)

Diagram 4
Initialization of K for CASE B

At this point it is possible to explain why the enumeration of the elements of $C_7 \boxtimes C_7$ done in diagram 1 and 2 is not exactly what one would expect initially. For example, the element $\binom{5}{5}$ is not associated with 5 simply because $\binom{5}{5}$ is already excluded (diagram 3) as a possible choice for a new member of S at the beginning of the algorithm.

Flowchart

Flowchart I is a synthesis of subsection 2.3.6 . The variables that are functions of the parameter p are expressed in terms of p in order to make it easier to follow the algorithm for values of p greater than seven. As in 2.3.6 no mention is made of any simplification.

At 1 we have a sequence at a given stage of development and we want to know if we can increase it by element $\binom{x}{y}$ which is also represented by N .

$\ell(S)$ is the length of S .

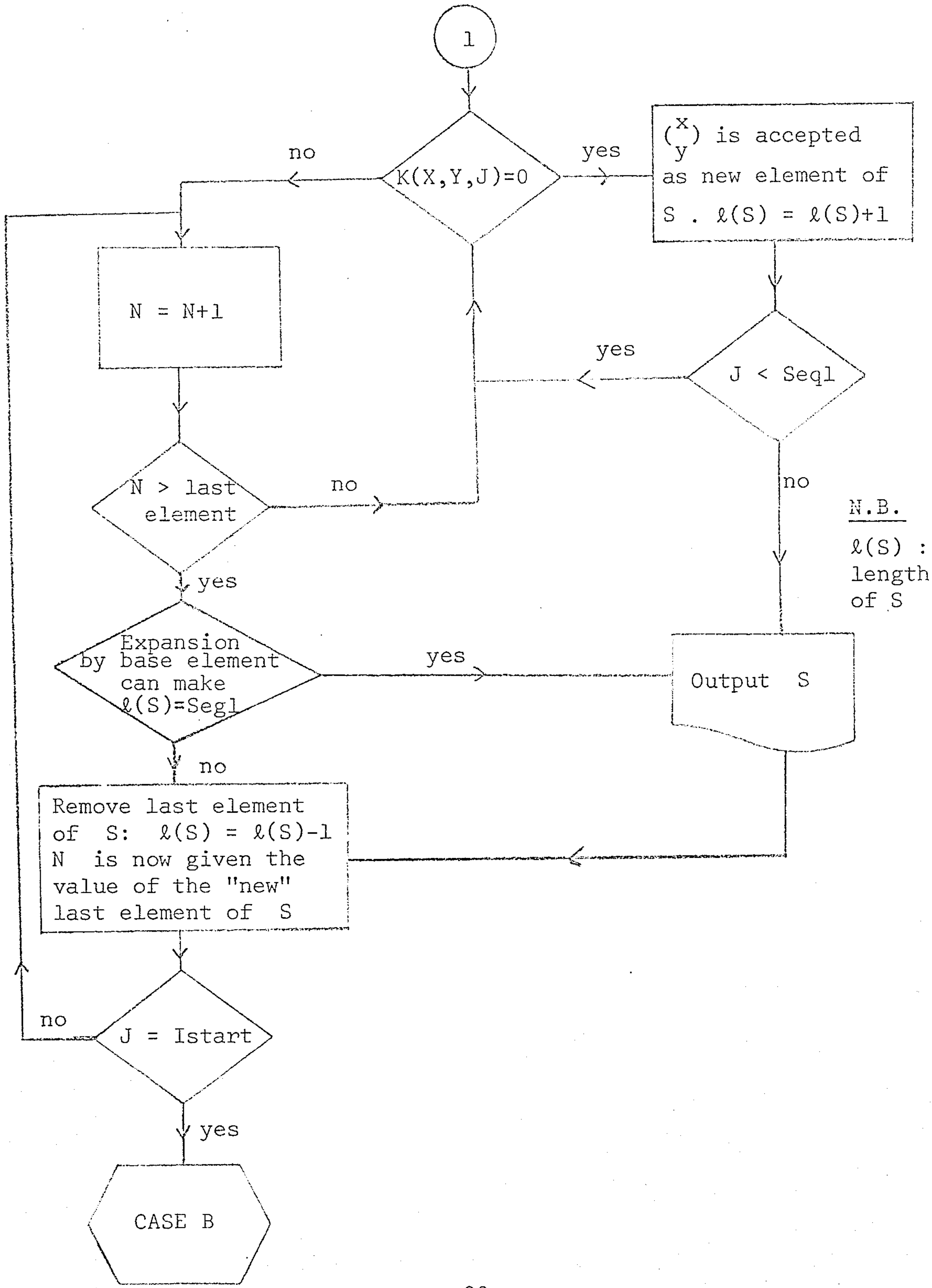
SEQL is 3P-3 i.e. 18 when p = 7 and 30 when p = 11 .

ISTART is the length of S when starting the algorithm.

FLOWCHART 1 : CASE A, P = 7 or 11

BASIC MECHANISM OF THE ALGORITHM

ALL "SIMPLIFICATIONS" HAVE BEEN OMITTED



CHAPTER III

CASE $C_{11} \boxtimes C_{11}$

We shall, in this chapter, go through the same steps as we did in the previous one, but this time studying the case $C_{11} \boxtimes C_{11}$. As is to be expected, we are not going to give nearly as many details as we did for $C_7 \boxtimes C_7$ since it would be a repetition of chapter two. Sufficient information is provided, however, to ensure readability and comprehension.

3.1 Primitive Sequences of C_{11}

We will now provide a list of the elements of $\sum_{11}$. As for C_7 , each representing sequence given precedes alphabetically all the members of its class. Beside each one is listed its corresponding type II sequence, whenever there is one.

TABLE II

Type I	Group	Type II
11	A	11
12	A	11
13	B	
15	B	
17	B	
111	A	111
112	A	111
113	A	111
114	B	
115	B	
116	A	666
117	B	
118	B	
123	A	111
124	A	222
125	B	
127	A	777
134	B	
135	A	333
145	A	555
157	B	
167	A	666
169	A	666
1111	A	1111
1112	A	1111
1113	A	1111
1114	A	1111
1115	B	
1116	A	1116
1117	B	
1122	A	1111
1123	A	1111
1124	A	1111
1125	A	2222
1126	A	2222
1133	A	1111
1134	A	1111
1135	A	3333
1148	A	4444
1157	B	
1158	A	8888
1167	A	6666
1177	B	
1178	B	
1234	A	1111
1257	A	7777
1345	A	4444

Type I	Group B	Type II
11111	A	11111
11112	A	11111
11113	A	11111
11114	A	11111
11115	A	11111
11116	A	66666
11122	A	11111
11123	A	11111
11124	A	11111
11125	A	11111
11133	A	11111
11134	A	11111
11157	B	
11166	A	66666
11167	A	66666
11177	B	
11223	A	11111
11224	A	11111
11233	A	11111
11577		
111111	A	111111
111112	A	111111
111113	A	111111
111114	A	111111
111115	A	111111
111122	A	111111
111123	A	111111
111124	A	111111
111133	A	111111
111166	A	111111
111222	A	111111
111223	A	111111
1111111	A	1111111
1111112	A	1111111
1111113	A	1111111
1111114	A	1111111
1111122	A	1111111
1111123	A	1111111
1111222	A	1111111
11111111	A	11111111
11111112	A	11111111
11111113	A	11111111
11111122	A	11111111
111111111	A	111111111
111111112	A	111111111
1111111111	A	1111111111

We notice that, in this case, 19 of them are without corresponding Type II sequences. $C_{11} \boxtimes C_{11}$ contains 12 distinct lines and again, two of them are required to span the whole group. As we did for $C_7 \boxtimes C_7$ we shall use $\binom{1}{0}$ and $\binom{0}{1}$.

3.2 Starting Sequence

3.2.1 CASE A

Since $C_{11} \boxtimes C_{11}$ contains 12 distinct lines and we are dealing with sequences of length 30 , if we are going to have all 12 lines represented in the sequence we shall have to have each line represented at least twice. The six elements left can be in any line according to the cases illustrated in Table III.

For CASE A the corresponding starting sequence to the first line of Table III is

$$\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{1}\binom{0}{1} \quad . \tag{1}$$

TABLE III

N.B. ℓ_i represents any line of $C_{11} \boxtimes C_{11}$ but $\ell_i \neq \ell_j$ for $i+j$

Lines		ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6
1.		1	1	1	1	1	1
2.		1	1	1	1	2	
3.	NUMBER OF	1	1	1	3		
4.	REPRESENTANTS	1	1	2	2		
5.	OF THE LINE IN	1	1	4			
6.	THE SEQUENCE	1	2	3			
7.		1	5				
8.		2	2	2			
9.		2	4				
10.		3	3				
11.		6					

It then appears that this sequence is a subsequence of each of the other starting sequences determined by the lines two to ten (incl.) of Table III while the eleventh line gives the sequence

$$\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{1}\binom{0}{1}\binom{0}{1}\binom{0}{1}\binom{0}{1}\binom{0}{1}\binom{0}{1} \quad . \qquad (2)$$

This gives us the choice of either treating CASE A in two subcases using (1) and (2) or to treating in its entirety by using the starting sequence

$$\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{1} \quad . \qquad (3)$$

3.2.2 CASE B

As for $C_7 \boxtimes C_7$, the cases not treated by (1) and (2) (or (3)) are the cases where the sequences σ_1 (C_{11} - sequence made of the first components of a $C_{11} \boxtimes C_{11}$ sequence) and σ_2 (second components ..) constituting S will contain some primitive sequences of C_{11} having no corresponding type II sequences in Table II. The cases left by (1) are

CASE B1: a) $\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{1}\binom{0}{1}$

b) $\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{1}\binom{0}{7}$

c) $\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{1}\binom{0}{8}$

d) $\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{2}\binom{0}{5}$