THE STABILITY OF LIMIT CYCLES IN TIME-LAG RELAY CONTROL SYSTEMS

A Thesis

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by

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ABSTRACT

The failure of Loeb's Rule in a particular example leads to a general limit cycle stability analysis for time-lag relay control systems and ultimately, to the reason for this failure.

ACKNOWLEDGEMENT

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CHAPTER I

INTRODUCTION

The describing function, supplemented by Loeb's Criterion,[1] is a powerful, simple, and widely used stability analysis for non-linear feedback control systems. In this technique, the describing function locates possible limit cycles and Loeb's Criterion (or Loeb's Rule as it is often called) indicates whether each limit cycle is stable or unstable. This latter information is essential because only a stable limit cycle will appear as a physical oscillation. Thus, Loeb's Criterion plays a crucial role in the analysis.

Unfortunately, both the describing function method and Loeb's Criterion are approximate and in both cases the error is very difficult to estimate. Nevertheless, Loeb's Criterion does give correct results in a large number of practical cases. However, for at least one control system, Loeb's Criterion gives erroneous results.

1. THE PROBLEM

Origin of the Problem

The genesis of the problem was the failure of Loeb's Criterion for the system in Figure 1.

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[1]J. Loeb, "Phénomènes Héréditaires dans les Servomécanismes; un Critérium Général de Stabilité", Annales des Télécommunications, 6(12): 346-356(1951).

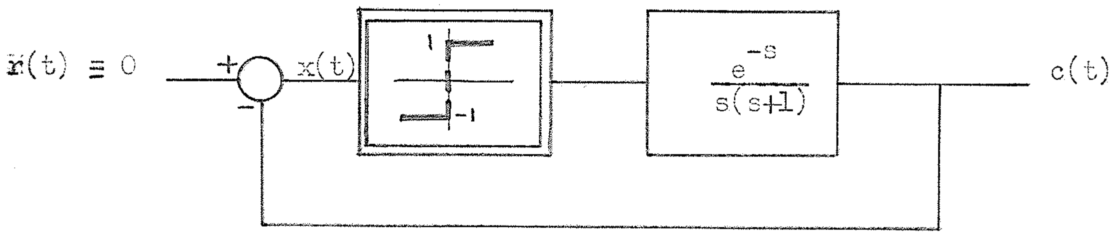

FIGURE 1

A SYSTEM FOR WHICH LOEB'S CRITERION FAILS

The conventional describing function analysis, in which the critical locus is superimposed on the Nyquist Diagram, is shown in Figure 2, for this system.[2]

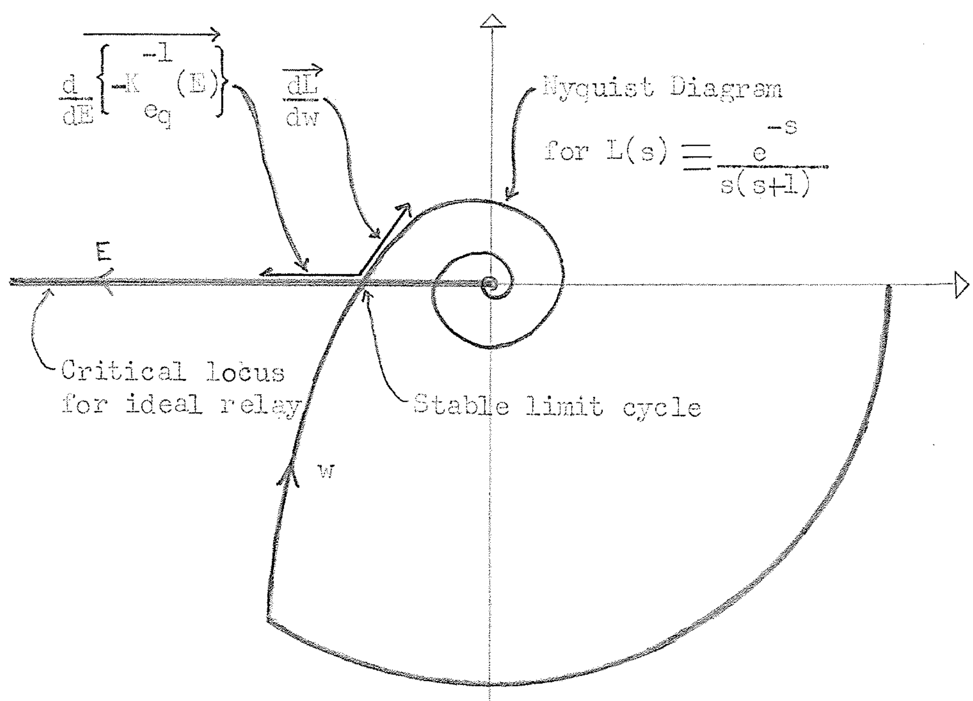

FIGURE 2

DESCRIBING FUNCTION – LOEB ANALYSIS FOR THE SYSTEM SHOWN IN FIGURE 1

[2] The symbolism is that of John E. Gibson, Nonlinear Automatic Control, pp. 344-50.

To each intersection of the Nyquist Diagram with the critical locus, there corresponds a limit cycle whose frequency is that of the Nyquist diagram at the intersection and whose amplitude (E) may be read from the scaled critical locus at the intersection. Loeb's Criterion predicts a stable limit cycle if the vectorial cross product, denoted symbolically by $\overrightarrow{\frac{dL}{dw}(jw) \text{ X } \frac{d}{dE}\left\{-K_{eq}^{-1}(E)\right\}}$, is positive at the intersection; an unstable limit cycle if the product is negative.[3] Therefore Loeb's Criterion predicts that all limit cycles in this system (Figure 1) are stable. This is a flagrant contradiction of the experimental fact that only the lowest frequency oscillation (which corresponds to the first intersection, counting from left to right in Figure 2) is stable.[4]

Statement of Thesis Objectives

A major objective of this thesis is to explain why Loeb's Criterion fails in the example just cited. But, the general discussion on limit cycle stability in Chapter II and the exact stability analysis of Chapter III, both of which are required to resolve this problem are each as significant as the explanation itself.

Significance of the Study

The failure of Loeb's Criterion in the previous example casts doubt upon its validity in all control systems. Moreover, by exposing the reason for this failure, this study further undermines confidence in the Criterion in general and in the positive vector product in particular. Furthermore it

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[3] J. C. Gille, M. J. Pelegrin, and P. Decaulne, Feedback Control Systems, pp. 419-21.

[4] This was established independently on the analogue computer by two summer students at the University of Manitoba: Temple in 1965 and Carson in 1966.

shows that a reliable alternative analysis to the describing function - Loeb technique exists, at least for a restricted class of systems.

2. DEFINITIONS

The following definitions are deemed sufficiently precise for the purposes of this thesis:

Limit Cycles

An isolated periodic oscillation in a system will be called a limit cycle.[5] All periodic oscillations discussed herein are isolated; therefore they are all limit cycles.

Stability of Limit Cycles

A system will possess a stable (unstable) limit cycle if the linear variational equation about the limit cycle is stable (unstable). This definition of a stable limit cycle is similar, though not as mathematically rigorous, as that given by Hayashi for orbital stability of a trajectory.[6]

3. REVIEW OF THE LITERATURE

Loeb first propounded his criterion for testing the stability of limit cycles in 1951.[7] Gille et al. formulated this criterion in terms of the convenient cross product rule stated earlier.[8] In a later paper, Loeb developed a more complicated test which in first approximation reduced to the vector rule, but even this more complicated rule

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[5] N. Minorsky, Nonlinear Oscillations, p. 71.
[6] C. Hayashi, Nonlinear Oscillations in Physical Systems, pp. 70-71.
[7] Loeb, loc. cit.
[8] Gille et al., loc. cit.

was not exact.[9] Furthermore, by assuming a differential equation throughout (see Equation (7) therein) he excluded time-lag systems (because they require a differential-difference equation for their representation). Grensted, in three interesting papers, developed a stability test for limit cycles using the operational calculus.[10] His work, however, suffered from the same shortcomings as Loeb's, namely it was approximate and it excluded time-lag systems.[11] This thesis surmounts these difficulties, but at present it is restricted to ideal relay control systems.

4. PLAN OF THE THESIS

First of all, by definition a limit cycle is stable or unstable accordingly as the linear variational equation about the limit cycle is stable or unstable. Consequently, in Chapter II a method is developed to derive this variational equation in a general single loop time-lag nonlinear feedback control system. In Chapter III, the specialization of this equation to the ideal relay system and the subsequent deduction of its stability properties brings the thesis objectives close to consummation. Indeed, when this knowledge is coupled with the extended Hamel

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[9] J. Loeb, "Recent Advances in Nonlinear Servo Theory", (1953) In Frequency Response ed. R. Oldenburger, pp.260-67.

[10] P. E. W. Grensted, "The Frequency Response Analysis of Non-Linear Systems," Proc. I.E.E., Vol. 102, part C, 1955, pp.244-55.

P. E. W. Grensted, "Analysis of the Transient Response of Non-Linear Control Systems," A.S.M.E. Trans., Vol. 80, 1958, pp.427-32.

P. E. W. Grensted, "Frequency Response Methods Applied to Non-Linear Systems" In Progress in Control Engineering, Vol. 1, pp. 105-139.

[11] By assuming a rational operator form throughout, Grensted effectively excluded time-lag.

Locus the reason for the failure of Loeb's Rule is readily uncovered. In the fourth and final chapter, the work is summarized; conclusions are stated, and areas of further research are outlined.

CHAPTER II

THE LINEAR VARIATIONAL EQUATION

In order to determine the stability of a limit cycle it is necessary to find the linear variation (or equivalently the first variation) from the limit cycle. This may be found by substitution of the periodic solution plus a small perturbation (of unspecified form) in the system equation(s) in the traditional way. After the periodic solution is cancelled and all higher order terms in the variation are discarded the linear variational equation(s) remains (remain). Henceforth this (these) equation(s) will simply be referred to as the variational equation(s). Now if the system is specified by N state equations there will be N variational equations also. However, the presence of time-lag implies that the state (and hence the variational) equations are infinite in number.[1] Therefore, a more convenient method for deriving the variational equation for time-lag systems is formulated in this chapter. In particular, the explicit form of the variational equation for a saturating system is deduced.

Next, a connection between the state approach and operational approach for getting the variational equation is discussed for the zero

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[1] R. A. Johnson, "State Space and Systems Incorporating Delay," Electronics Letters, Vol. 2, No. 7, July 1966, pp. 277-78.

time-lag case. This is followed by some remarks on the inadequacies of existing stability theorems.

For completeness, the nonautonomous case is also discussed, although the results are not used in the remainder of the thesis.

1. THE OPERATIONAL APPROACH

Cosgriff has given the following convenient method for finding the variational equation for the general system shown in Figure 3.[2]

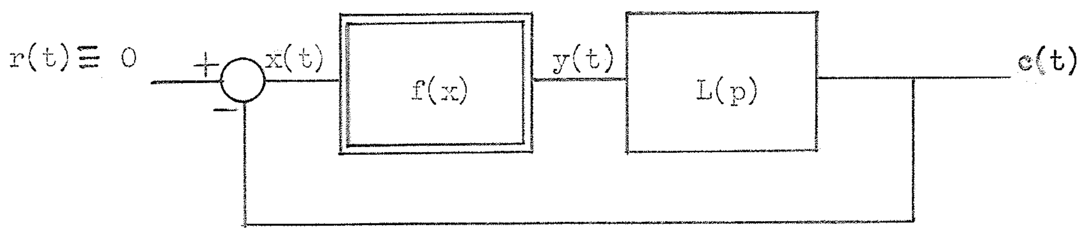

FIGURE 3

THE GENERAL SINGLE LOOP NONLINEAR FEEDBACK CONTROL SYSTEM

The operational differential equation for the autonomous system (Figure 3) is given by

$$L(p)\left\{f(x)\right\} + x = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 1.$$

If x' is a periodic solution of Equation (1) then the substitution of the perturbed solution, $x' + u$, into Equation (1) yields

$$L(p)\left\{f(x'+u)\right\} + x' + u = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 2$$

[2] R. L. Cosgriff, "Application of Linear Differential Equations with Periodic Coefficients in the Study of Non-linear Phenomena", Proc. of First International Congress of the I.F.A.C., Vol. 2, 1960, pp. 883-87.

which, if f(x) has **a Taylor series** with respect to x, becomes

$$L(p)\left\{f(x') + u\frac{df}{dx}(x') + \frac{u^2}{2!}\frac{d^2}{dx^2}f(x') + \cdots\right\} + x' + u = 0 \quad \ldots\ldots 3.$$

When the limit cycle solution,

$$L(p)\left\{f(x')\right\} + x' = 0,$$

is cancelled and the higher order terms in the perturbation, u, are discarded Equation (3) reduces to

$$L(p)\left\{u\frac{df}{dx}(x')\right\} + u = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 4.$$

Since $x'(t)$ is a periodic function of time it follows that $\frac{df}{dx}(x')$ is also periodic and of the same period. Therefore Equation (4) is a linear differential equation with (time) periodic coefficients. Thus the nonlinear problem of limit cycle stability has been reduced to an equivalent (time-variable) linear problem. This is not surprising in view of the fact that only linear variations from the periodic solution were considered.

Cosgriff then proceeded to deduce u(t) by setting $x'(t+\Delta t)$ equal to $x'(t)+u(t)$. This, however, is a questionable tactic since $x'(t+\Delta t)$ represents a perturbation along the limit cycle whereas $x'(t)+u(t)$ is a perturbation off the limit cycle. If these two perturbed solutions are to be identically equal for all time, then, contrary to the original hypothesis, u(t) cannot be a perturbation off the limit cycle. The correct way to find either u(t) or any of its properties is by a consideration of the variational equation (Equation (4)).

The method for finding the variational equation given by Cosgriff may easily be extended to the general nonlinear system with time-lag by replacement of L(p) in Figure 3 by $G(p)\ e^{-\tau p}$, in which G(p) is a rational, stable, and minimum phase function in p and $e^{-\tau p}$ represents a pure time-delay of τ seconds. For the autonomous case, the delay may be either in the forward or in the feedback path; the analysis is the same in both cases. Substitution of $G(p)\ e^{-\tau p}$ for L(p) in Equation (4), yields

$$G(p)\ e^{-\tau p}\left(u\frac{df}{dx}(x')\right) + u = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 5$$

which is the explicit form of the variational equation for the time-lag case. This equation, which is basic for later work, is equivalent to the system in Figure 4 below.

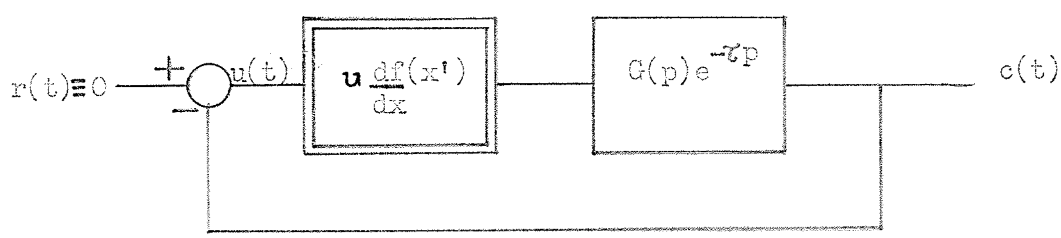

FIGURE 4

VARIATIONAL SYSTEM FOR THE GENERAL TIME-LAG NONLINEAR SYSTEM

Notes: 1. The function, $\frac{df}{dx}(x')$, is essentially a time variable gain factor (it is periodic in time with the same period as the limit cycle). Taken as a whole, this representation is a time-variable linear feedback control system with time-lag.

2. The symbols, r(t) and c(t), in this figure are not related to those in the system representation in Figure 3.

A lemma which will be useful later is that $\dot{x}'$ is always a solution to the variational equation (Equation (4)).[3] This may be established by differentiating the limit cycle solution,

$$L(p)\left\{f(x')\right\} + x' = 0,$$

with respect to time; the result (which proves the lemma) is Equation (4).

The advantage of this operational method is that a single variational equation is obtained rather than infinitely many as in the state approach. This method applies to all systems which can be represented by a single differential or differential-difference equation and, in particular, to all single-loop time-lag nonlinear feedback control systems (see Figure 3).

<u>The Variational Equation for Time-Lag Systems With A Saturation Nonlinearity</u>

The variational equation and its system representation will now be derived for the saturating system illustrated in Figure 5.

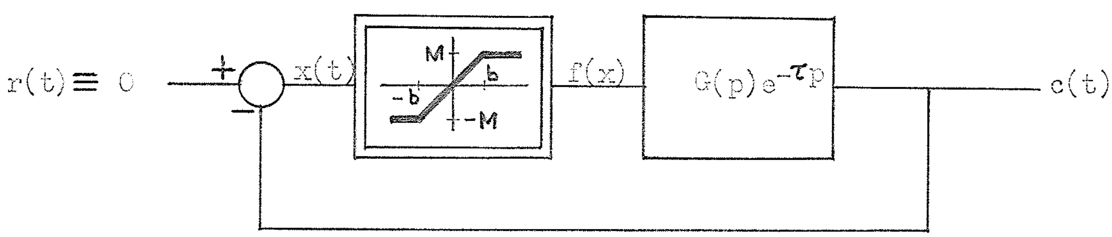

FIGURE 5

THE TIME-LAG SATURATING SYSTEM

Unfortunately, f(x) does not possess continuous derivatives when |x| is equal

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[3] E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, p. 322.

to b. Therefore $f(x'+u)$ cannot be expanded in a valid Taylor series at these points. Hence the following direct analysis is employed.

The saturation function is defined analytically by the relations:

$$f(x) = \frac{M}{b}x \quad \text{when } |x| < b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 6a$$

and

$$f(x) = M\,\mathrm{sgn}(x) \quad \text{when } |x| \geq b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 6b$$

which on perturbation become

$$f(x+u) = \frac{M}{b}(x+u) \quad \text{when } |x+u| < b \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 7a$$

and

$$f(x+u) = M\,\mathrm{sgn}(x+u) \text{ when } |x+u| \geq b \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 7b.$$

Since $u(t)$ may be made arbitrarily small initially, $|x+b|$ will differ from $|x|$ by less than any preassigned number for any finite b. Therefore, the perturbed solution at the saturation output may be written as

$$f(x'+u) = f(x') + \frac{uM}{b}\left\{U(x'+b) - U(x'-b)\right\} \quad \ldots\ldots\ldots\ldots \quad 8$$

in which $U(x)$ is the unit step function.$^{+}$ Now, $\frac{df}{dx}(x')$ is not defined at x' equal to $\pm b$. However, if the derivative is assigned the value zero at these points (without loss of generality) then Equation (8) may be rewritten as

$$f(x'+u) = f(x') + u\frac{df}{dx}(x') \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 9,$$

a result which produces the standard variational equation given by Equation (5).

If $x'(t)$ has odd symmetry about its zero crossings and if these zeroes are uniformly spaced

$^{+}$ $U(0) = 1$

then a graphical evaluation of $\frac{df}{dx}(x')$ is given in Figure 6.[4] The time origin has been selected so that $x'(-\tau)$ is zero to agree with later work on the Hamel Locus.

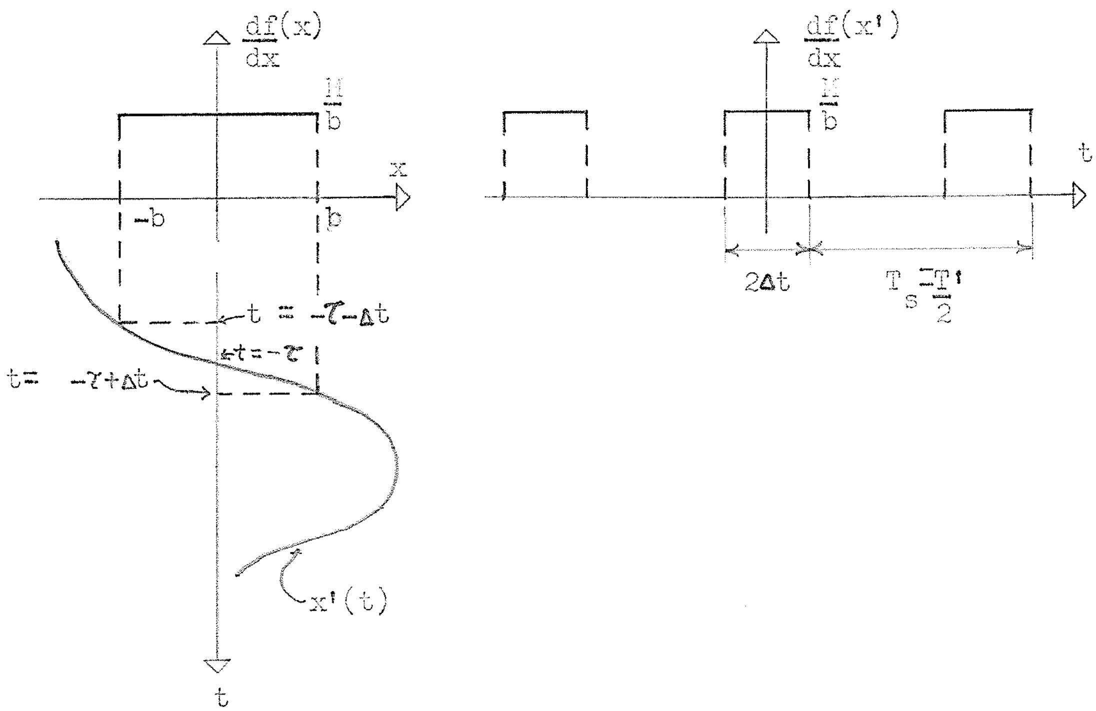

FIGURE 6

GRAPHICAL EVALUATION OF $\frac{df}{dx}(x')$ FOR THE SATURATING SYSTEM (FIGURE 5)

Definitions: T' = period of the limit cycle

T_s = period of $\frac{df}{dx}(x')$

Δt is defined such that:

$$x'(-\tau-\Delta t) = -b$$

The variational equation is equivalent to the non-ideal sampled data system in Figure 7. From Figure 6 it is evident that the sampling

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[4] Similar results have been obtained for the zero time-lag case by Mohammed, "Steady-State Oscillations and Stability of On-off Feedback Systems," Ph.D. Thesis, U.B.C.

duration is $2\Delta t$ seconds and the sampling period, T_s, is exactly half the limit cycle period, T'.

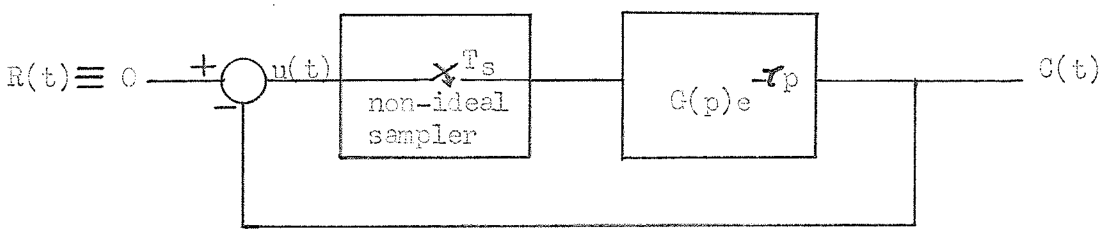

FIGURE 7

SYSTEM REPRESENTATION OF VARIATIONAL EQUATION

FOR SATURATING SYSTEM (FIGURE 5)

Note: The form of the non-ideal sampler is given by $\frac{df}{dx}(x')$ shown in Figure 6.

This system representation is fundamental for the results obtained in Chapter III.

2. THE AUTONOMOUS CASE WITHOUT TIME-LAG

This discussion is not in the main stream of the study since it deals with the case of zero time-lag. Nevertheless, it has merit because it establishes a link between the operational approach of Cosgriff and the traditional state equation approach used by mathematicians.

In the Appendix, the equivalence of these two methods is established for zero time-lag. The significance of this is that the large number of theorems which have been established by mathematicians for the state variational equations, now, by virtue of the Appendix, apply to the operational variational equation as well. Some of these theorems and their shortcomings are now reviewed.

The phase variational equations are, in general, a set of linear differential equations with (time) periodic coefficients. Such equations are difficult to solve though, in principle, they can be transformed to an equivalent set of linear constant coefficient equations by a Theorem of Liapunov.[5] The stability of a limit cycle is thus determined by a set of linear constant coefficient equations which can readily be solved in a variety of ways.

Using a different approach, Floquet gave the form of the general solution in terms of characteristic exponents with periodic coefficients.[6] In this formulation the stability of the limit cycle depends on the sign of the real part of the characteristic exponent. But no general method was given for finding these exponents and the reduction theorems available are of little use in determining the characteristic exponents, except in second order systems.

Both the Floquet and Liapunov methods are inconvenient for feedback systems because the system must first be converted to a state form. Furthermore, neither of these methods apply at all to the time-lag case because the appropriate equations are of the differential-difference variety.

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[5] Pontryagin, L. S. Ordinary Differential Equations. Reading Mass.: Addison-Wesley Publishing Company, Inc., 1962, pp. 146-149.

[6] C. Hayashi. Nonlinear Oscillations in Physical Systems, pp. 82-86.