Evaluation of Credit Value Adjustment with a Random Recovery Rate via a Lévy Default Model

by

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Abstract

Credit value adjustment (CVA), as a quantified measure of counterparty credit risk for financial derivatives, is becoming an increasingly important concept for the financial industry. In this thesis, we evaluate CVA for an interest rate swap via a new structural default model. In our model, the asset value of a company is assumed to follow meromorphic Lévy processes with infinite jumps but finite variation. One important advantage of our model is that we are able to assume a random recovery rate which depends on default severity. Compared with the case with a fixed recovery rate, we show that the effect on CVA with a random recovery rate is significant.

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Chapter 1

Introduction

This chapter will be arranged as follows. In Section 1.1, we will start with an introduction to the historical background of counterparty credit risk, in particular, how the topic becomes important after the 2007 financial crisis. Then, we will review the literatures on credit value adjustment (CVA) and explain the motivation and contribution of this thesis in Section 1.2. Finally, Section 1.3 will give us an outline to demonstrate how we are going to construct this thesis.

1.1 Background

This thesis gives an alternative method for pricing credit value adjustment (CVA) that quantifies the counterparty credit risk for a financial contract. Counterparty credit risk is the risk of the counterparty's failure to perform its financial obligation for a contract. [1] It is important to price and recognize the counterparty

[1] A counterparty refers to the other party that participates in a financial contract.

credit risk, because counterparty credit risk is inherent in financial transcactions since the 2007 financial crisis (Gregory, 2012, p.6). Lack of recognition of the counterparty credit risk, particularly for the over-the-counter (OTC) derivatives could lead to serious failures to the financial market. (See the detail discussion in Chapter 2.)

Prior to the 2007 crisis, institutions and investors tried to control their counterparty credit risk by setting the credit limit for given counterparties based on their financial stability and credit rating (Gregory, 2012, p.34-38). One limitation of this approach is that it makes the decision making process a binary event. The transaction will be approved only if its value does not go beyond the credit limit. As a result, institutions and investors are more likely to trade with those large financial institutions who seem to be financially stable. Eventually, those large financial institutions became the common counterparties to numerous institutions and investors. Another limitation of the credit limit approach is that it ignores the counterparty risk for those transactions that are approved. But we learned, from the crisis, that even the companies with the highest credit ratings could still suddenly go bankruptcy (i.e. they still have counterparty credit risk). And when those financial institutions default, it could create a chain reaction in the entire market since there will be large numbers of investors involved. CVA, on the other hand, allows us to move beyond the binary decision process and to price the counterparty credit risk directly for all the transactions. As a result, CVA has become a major subject for credit risk in the financial markets ever since the 2007 crisis.

1.2 Literature review

CVA is the difference between the risk-free value of a financial contract and its risky value when taking counterparty credit risk into account (Gregory, 2012, p.242). With the assumption of independence, CVA (without considering DVA component) at each future transaction date can be calculated as the product of two separate parts: the credit exposure and the default probability with the recovery rate. The credit exposure can be evaluated by multiplying the expected risk exposure with the corresponding discount factor. With the assumption of a fixed recovery rate, the other part can be obtained by multiplying the marginal default probability with the fixed recovery rate. (See the detail discussion in Chapter 2)

In this thesis, we will focus on the default probability and recovery rate. Generally speaking, in the literature, there are two kinds of models for calculating the default probability: the reduced-form model and the structural model. (See Arora et al. (2005) for detailed literature review on these two models.) The Reduced-form models assume that the default event is an unpredictable random event, and is characterized by a default intensity function. Duffie and Singleton (1999) and Hull and White (2000) presented detailed explanations of several reduced-form models. Instead of modeling the default time with an intensity function, the structural models consider that default happens the first time when the asset value falls below a certain threshold level. Merton (1974) proposed the first structural model by reformulating the problem of a firm's default into the problem of modeling the firm's asset value. His basic assumption was that default happens when the firm's asset value falls below its liabilities. Black and Cox (1976), Vasicek (1984), and

Zhou (2001), among others, proposed different structural models by eliminating some unrealistic assumptions from the Merton's model. In this thesis, we will develop a new structural model in which the asset-value process includes pure jumps with infinite activities and finite variation.

It is worthwhile to point out the importance of infinite activities and finite variation for the processes in modeling asset values. Carr et al. (2002) found evidence that the equity prices are pure jump processes with infinite activities and finite variation. Their findings are summarized by Hao and Li (2015) as follows. (1) A diffusion component is statistically insignificant while jump components consistently account for significant skewness levels from equity prices. (2) The shape of the mean corrected density for asset returns appears to depart from that of a normal distribution. However, the densities of processes with infinite activity and finite variation are consistent with equity prices.

Another important feature is that we are able to assume a random recovery rate in our model. According to Das and Hanouna (2009), the recovery rate is commonly assumed to be a constant number at 40% to 50% range for U.S. corporates and 25% for sovereigns. It might be practically exigent for this assumption due to the difficulty and complexity of including the random recovery rate in the pricing of the financial products. (See comments in Das and Hanouna (2009).) However, it is an unrealistic one since the recovery rate shows a strong fluctuation and large variation over time and sectors. (See Figures 1.1 and Table 1.1) In fact, Altman et al. (2005) studied the default rate and the recovery rate for corporate bonds defaulted between 1982 and 2002, and reported that their empirical result shows the recovery rate has a strong variation and negative correlation with the default rate. Thus, it will be a great advantage for a model that is able to capture a random recovery rate and take the dependence between default probabilities and recovery rate into account. Calabrese and Zenga (2010) analyzed 149,378 recovery rates on the Italian bank loans by regarding the recovery rate as a mixed beta random variable. They applied a kernel estimation on the recovery rates and found that the recovery rate shows a bimodal distribution. Van Damme (2011) introduced a method to take into account a stochastic loss-given-default by incorporating a common dependence of the loss-given-default and the default probability on a latent variable that represents the systemic risk. Ruf and Scherer (2011) provided a Monte Carlo simulation algorithm for computing the bond prices in a structural default model with jumps. They showed that a structural model with jumps is able to capture the stochastic recovery rate. Zhou (2001) studied the defaultable bonds with a Monte Carlo simulation and suggested modeling the logarithm of the asset value as a process with the combination of a diffusion component and a jump component and the assumption that the jumps are normally distributed.

It is reasonable to assume a random recovery rate depending on default severity. Guo et al. (2009) provided a model for a piecewise recovery rate in a reduced form model. In their model, the recovery rate is determined by the severity of default and by whether a firm is insolvent or bankrupt (i.e. the level of a firm's financial distress) when default happens. Tang and Yuan (2013) introduced a static structural model for the loss-given-default with the consideration of the severity of default; and their results show that the model provides an accurate estimate for

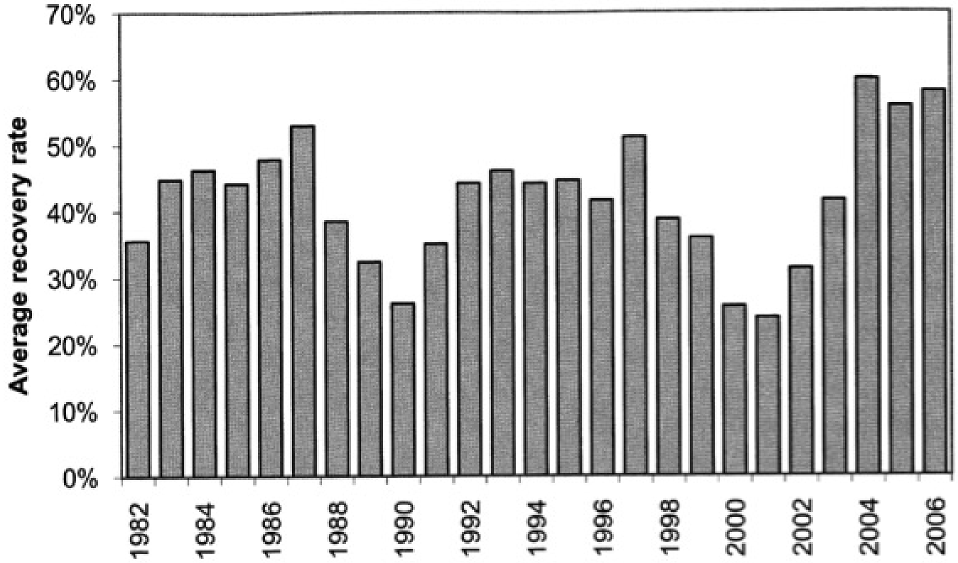

Figure 1.1: Strong Variation of the Recovery Rates Over Time. (Moody's Investors Service, 2009)

the tail probability of the loss-given-default.

1.3 Structure of the thesis

Chapter 2 will start with the discussion on the importance of quantifying counterparty credit risk for OTC derivatives as well as the assumptions and formulas for computing CVA. Then, Chapter 3 will have a detailed discussion of several important structural models as well as a new structural model. In particular, we will introduce a Lévy default model that includes finite variation and infinite activities in the asset value process; and we will discuss the method for computing the expected discounted penalty functions (EDPFs) for particular families of meromorphic Lévy processes. In Chapter 4, we will talk about a numerical experiment on the calculation of CVA for an interest rate swap. We will provide detailed descrip-

Industry	Average recovery rate
Public utilities	70.5%
Chemicals, petroleum, rubber and plastic products	62.7%
Machinery, instruments and related products	48.7%
Services (business and personal)	46.2%
Food and kindred products	45.3%
Wholesale and retail trade	44.0%
Diversified manufacturing	42.3%
Casino, hotel and recreation	40.2%
Building material, metals and fabricated products	38.8%
Transportation and transportation equipment	38.4%
Communication, broadcasting, movie, printing and publishing	37.1%
Financial institutions	35.7%
Construction and real-estate	35.3%
General merchandise stores	33.3%
Mining and petroleum drilling	33.0%
Textile and apparel products	31.7%
Wood, paper and leather products	29.8%
Lodging, hospitals and nursing facilities	26.5%
TOTAL	41.0%

Table 1.1: Strong Variation of the Recovery Rates Over Sectors. (Moody's Investors Service, 2009)

tion on the evaluation processes, including the valuation of the credit exposure as well as the default probabilities. Finally, conclusions will be drawn and summarized in Chapter 5.

Chapter 2

Credit Value Adjustment

This chapter discusses the counterparty credit risk and CVA. In particular, Section 2.1 will talk about the counterparty credit risk and why counterparty credit risk of OTC derivatives should be of the concern. In Section 2.2, we will discuss the general formulas and the major components for calculating CVA with a fixed recovery rate. Then, we are going to propose an equation for the calculation of CVA with a random recovery rate in Section 2.3.

2.1 Counterparty credit risk of the OTC derivatives

Counterparty credit risk is the risk that the counterparty defaults (i.e. is unable to fulfill its financial obligations) before the maturity of a contract. According to Gregory (2012, p.21), the counterparty credit risk for the over-the-counter (OTC) derivatives [2] should be of particular concern, because

[2] An over-the-counter (OTC) derivative is a financial contract settled between two private parties for exchanging a series of cash flows at some specified time in the future.

1. The high profile financial institutions are the common counterparties to a large number of investors in the OTC derivative market. As a result, the failure of those institutions could lead to a contagious effects and perceptional impact to the whole financial market.

2. Those high profile institutions are considered to be ”too big to fail”, thus a very small amount of collateral, or sometimes even no collateral, is required when trading with those high profile institutions. In other words, people tend to underestimate the credit risk for the transactions with those high profile institutions.

2.2 CVA

CVA, by definition, is the difference between the risk-free value of a financial contract and its risky value when taking the counterparty credit risk into consideration (Gregory, 2012, p. 242). In other words, it is the market value of the counterparty credit risk.

$$\text{CVA} = \text{Risk-free value} - \text{Risky value}$$

Gregory (2012) provided us a practical formula for the calculation of CVA, which can be expressed as:

$$\text{CVA} = (1 - R) \sum_{k=1}^{K} DF(t_k) EE(t_k) PD(t_{k-1}, t_k), \tag{2.1}$$

where

- $\{t_0, t_1, ..., t_K\}$ is a set of valuation dates;
- K is the total number of valuation dates;
- R represents the recovery rate, that is, the proportion of the exposure that can be recovered at default;
- $DF(t_k)$ is the discount factor used to discount the future cash flows at time t_k back to the current time;
- $PD(t_{k-1}, t_k)$ is the marginal probability that counterparty's default happens between time t_{k-1} and t_k;
- $EE(k_i)$ refers to the expected credit exposure, which is the expected future value of the contract at default.

Note that (2.1) is based on the following assumptions:

1. The institution itself is considered to be free of credit risk (i.e. it will not default). This means there is no need to consider debt value adjustment (DVA) component.
2. All the components are assumed to be independent to each other. This assumption allows us to separate the responsibilities of calculating the components to different sources, and then combining them together.
3. The recovery rate is assumed to be a fixed ratio.

2.2.1 Credit exposure

The credit exposure of a contract is defined as the amount of loss in case of a counterparty's default. In the event of default, the net amount owing between the two parties will be determined when the contract is closed and future payment is stopped. If this amount has a negative present value, then it is a liability to the default side (i.e. the default party has the obligation to pay the value to the counterparty). If the contract has a positive present value, then it is an asset of the default party that is to be received from the counterparty. Note that the institution will incur a loss if the value is positive and will not have a gain if the value is negative. Thus, the exposure is equal to the net amount if it is positive and zero otherwise. (See Gregory (2012, p. 121) for credit exposure.)

2.2.2 Recovery rate

When a default happen, the default party sometimes may still have the capability to pay off some portion of their liabilities. The recovery rate is the proportional amount that can be recovered in case of a default. It can also be expressed as $1 - \text{LGD}$, where LGD is the loss-given-default defined as the proportional amount that would not be recovered at default. (See Gregory (2012, p.209) about the recovery rate.)

2.2.3 Default probability

According to Gregory (2012, p.197-200), there are two different ways to define the default probabilities: real-world versus risk-neutral. Real-world default probability aims at the estimation of the actual probability of the counterparty's default. Risk-neutral default probability is the counterparty's default probability, which reflects the market information. In this thesis, our concern will be the risk-neutral default probability estimated by a structural model.

As we will discuss in more detail in Chapter 3, structural models estimate the default probabilities based on the information from the equity market. In the framework of structural models, a firm's asset value is considered to follow a stochastic process; and the time of default is considered to be the first time where the firm's asset value is lower than its liabilities (or some predetermined barrier). The major idea is to translate the low-frequency default event into modelling a continuous process of the asset value and calibrating with high-frequency equity market data.

2.3 CVA with random recovery rate

We know that (2.1) is developed based on the assumption of a fixed recovery rate and the independence among risk exposure, recovery rate and default probability. However, as we discussed in Chapter 1, it is not realistic to make such an assumption on the recovery rate. Therefore, we want to remove this assumption from our model. Considering the recovery rate as a random variable in the calculation of the default probabilities, we propose our equation for CVA as follows.

$$\mathrm{CVA} = \sum_{k=1}^{K} \mathrm{DF}(t_k)\mathrm{EE}(t_k)\mathbb{E}\left((1-R_\tau)\mathbf{1}_{\{t_{k-1}<\tau\leq t_k\}}\right), \tag{2.2}$$

where

- τ is the time of default for the counterparty;
- R_τ represents the random recovery rate that is the proportion of the exposure that can be recovered at default;
- $\mathbf{1}_{\{t_{k-1}<\tau\leq t_k\}}$ is an indicator function of the counterparty's default between time t_{k-1} and t_k;

Note that there are three following assumptions for (2.2).

1. The institution itself is free of credit risk (i.e. it will not default). So, again, there is no need to consider debt value adjustment (DVA) component.
2. The expected credit exposure is independent of default probability and recovery rate.
3. The recovery rate is assumed to be a random ratio depending on default severity.

Chapter 3

Lévy Default Model

This chapter introduces a Lévy default model. We will structure the chapter as follows. First, we will review several important structural models and introduce our default model in Section 3.1. Then, in Section 3.2, we will introduce some important definitions and properties of Lévy processes, meromorphic Lévy processes, and the beta and theta families of meromorphic Lévy processes. Next, in Section 3.3, we will discuss the finite-time and the infinite-time expected discounted penalty functions. It is essential for us to be able to apply theorems and properties discussed in this chapter in order to efficiently and accurately perform a numerical experiment in Chapter 4.

3.1 Strucutral models

3.1.1 Merton model

Merton (1974) first introduced the structural models that assume the default occurs when the asset value of a firm (V) falls below the value of its liabilities. In Merton's model, the asset value of a firm (V_t) is assumed to follow a geometric Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$

$$\mathrm{d}V_t = V_t(\mu \mathrm{d}t + \sigma \mathrm{d}W_t)$$

where σ is the volatility of the asset value V and can be inferred from the volatility of the equity value E, which is the sample mean of log returns on equity value.

Merton's model further assumed that the firm has a simple debt structure consisting of a single outstanding bond with a face value D and maturity T. Thus, in Merton's model, a counterparty's default can only happen at maturity T. Since the firm's assets are first used to pay off its debt in case of default, the shareholders only get the payoffs when there is any residual values. If the asset value of a firm is larger or equal to its debt (i.e. $V \geq D$), then the debtor is paid in full amount and the remaining value is distributed among the shareholders. If the asset value is less than its debt (i.e. $V < D$), then the default occurs and the debtor get the liquidation value of the asset while the shareholders get nothing. In other words, in case of a default event, the debtor receives a fraction of the debt value V_T/D, called the recovery rate; the remaining value of the debt $1 - V_T/D$ is called the loss given default or LGD. According to the fundamental accounting principles, the asset value of a firm is equal to its debt plus its equities (i.e. $V = D + E$). So the value of the firm's equities at time t can be written as

$$E_t = \max(0, V_t - D)$$

The payoff to shareholders is equivalent to a call option on the value of the firm's assets. Therefore, the equity value of the firm can be viewed as an European call option with the strike price equaling to the debt value and the maturity equaling to the period of the observation. Then, we can apply the Black-Scholes pricing formula (Black and Scholes, 1973) for European options as follows.

$$E_t = \mathbb{E}_t^{\mathbb{Q}}[e^{-r(T-t)}(V_T - D)_+] = V_t N(d_1) - e^{-r(T-t)} D N(d_2)$$

where $\mathbb{E}_t^{\mathbb{Q}}$ is the expectation taken under the risk-neural probability measure $\mathbb{Q}$ with respect to $\mathcal{F}_t$; $N(d_1)$ and $N(d_2)$ are the standard normal cumulative distribution function; and

$$d_1 = \frac{\ln(V_t/D) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, d_2 = \frac{\ln(V_t/D) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

Therefore, the probability of default at the time of maturity is

$$\mathbb{Q}(\tau = T) = \mathbb{Q}[V_t \leq D] = N(-\frac{\ln(V_t/D) + (\mu - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}})$$

There are three major shortcomings for the Merton model:

1. The Merton model assumes a simplest debt structure with only the bond.

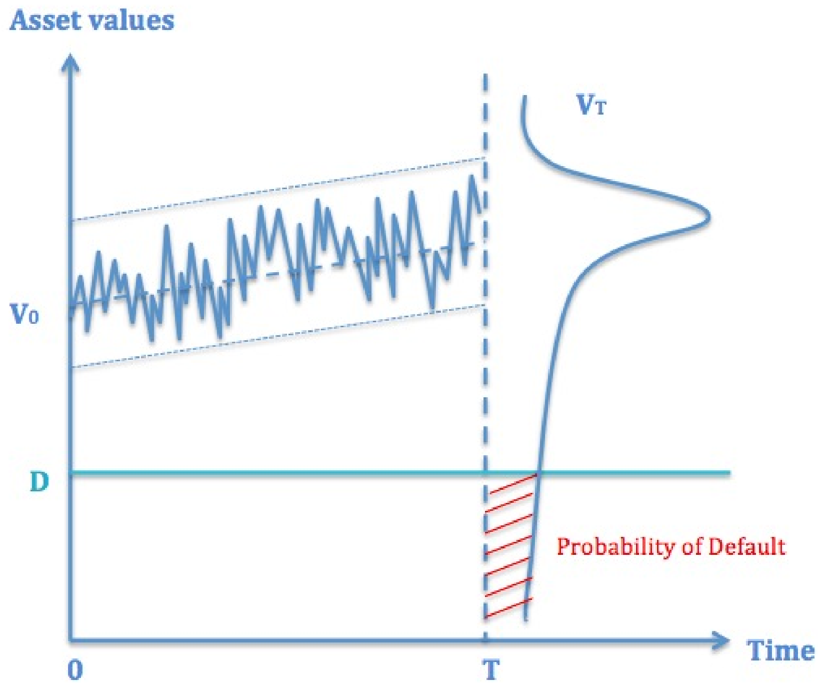

Figure 3.1: Merton's Model

Whilst, in the reality, the debt structure is often much more complicated than a single bond.

2. In the Merton model, the default can only occur at the time of maturity. This allows the asset value to drop below the debt value then rise above the debt value before maturity, without being recognized as a default (Feng and Volkmer, 2012). In practice, a company would have been defaulted before maturity in such a situation.

3. The asset value in the Merton model is continuous as it is assumed to follow a geometric Brownian motion. However, Carr et al. (2002) suggested evidence that the asset value seems to follow jump processes with infinite activities

and finite variation.

3.1.2 Poisson jump-diffusion model

Kou (2002) and Kou and Wang (2003) introduced a jump-diffusion structural model. Suppose that that the asset value of a firm is

$$V_t = e^{X_t},$$

then the logarithmic asset value X_t is defined to be a combination of drifted Brownian motion and compound Poisson process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$.

$$X_t = X_0 + \mu t + \sigma W_t + \sum_{k=1}^{N_t} Y_k$$

with the cumulant generating function

$$k(s) = \ln \mathbb{E}[\exp\{sX1\}] = \frac{1}{2}\sigma^2 s^2 + \mu s + \lambda\Big(\frac{p\eta_1}{\eta_1 - s} + \frac{q\eta_2}{\eta_2 + s} - 1\Big),$$

where μ, σ are the drift and volatility parameters; $\lambda, p, \eta_1, q, \eta_2$ are the parameters from the jump component; and

$$\mu = v - \frac{\sigma^2}{2}.$$

Feng and Volkmer (2012) pointed out that the equity value should be equivalent to the price of a down-and-out barrier option instead of a European call option, thus the equity value is given by

$$E_t = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-r(T-t)}(V_T - D)_+ \mathbf{1}_{\{\inf_{t\le s\le T} V_s \le D\}}\right];$$

and the probability of default by the maturity of debt is

$$\begin{aligned}\mathbb{Q}(\tau \le T) = \mathbb{Q}(\inf_{t\le s\le T} V_s \le D) &= N(\frac{\ln(D/V_t) - (\mu - \sigma_V^2/2)(T-t)}{\sigma_V\sqrt{T-t}}) \\ &+ \exp\{\frac{(2\mu - \sigma_V^2)\ln(D/V_t)}{\sigma_V^2}\} \\ &\times N(-\frac{\ln(D/V_t) + (\mu - \sigma_V^2/2)(T-t)}{\sigma_V\sqrt{T-t}})\end{aligned}$$

There are three shortcomings for the Poisson jump-diffusion model:

1. The Poisson jump-diffusion model assumes a fixed recovery rate when calculating the CVA. However, as we discussed in Section 1.2, this is an unrealistic assumption since the recovery rate shows a strong fluctuation and randomness.

2. There are too many parameters in the Poisson jump-diffusion model. Therefore, Feng and Volkmer (2012) used a two-stage calibration for the model. In the first stage, the drift and volatility parameters are estimated without considering the jump components. Then, in the second stage, the parameters from the jump component are determined by matching the model with the empirical quantiles. The problem of this two-stage model calibration is that it may effect the fitting precision of the model.

3. In addition, the quantile-matching method requires a precise choice of quan-

tile to match. For example, if we choose to match the 90^{th} percentile, we are going to fit the distribution of the tail rather than the center. As a result, we might get a good fit in the tail but a poor overall fit for the calibration. (See Hardy (2003, Chapter 4) for details about the quantile-matching method.)

3.1.3 KMV model

Vasicek (1984) developed the KMV model that regards the event of default as a firm's asset values reach a certain critical level instead of the liabilities. Typically, the critical level is define as the sum of all the short-term debt and half of the long-term debt. Instead of inferring the default probability from a parametric lognormal distribution, KMV model derived default probability from the historical database with the following steps:

1. Estimate the asset value as well as its volatility.
2. Calculate the distance-to-default (DD) which is defined as

$$DD = \frac{\ln(V_t/D) + (\mu - \sigma_V^2/2)(T-t)}{\sigma_V\sqrt{T-t}}$$

3. Scale the distance-to-default to the probability of default with a default database.

3.1.4 Our model

We model the counterparty's asset value by a stochastic process $V = \{V_t, t \geq 0\}$ with

$$V_t = e^{Z_t} \tag{3.1}$$

on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$ where $\mathbb{Q}$ is the pricing measure or risk-neutral measure. The asset value is standardized such that $V_0 = 1$. We assume that the process $Z = \{Z_t, t \geq 0\}$ starts from 0 and takes the form of a positive drift minus a pure-jump subordinator [3]. Specifically,

$$Z_t = \mu t - S_t, \quad t \geq 0, \tag{3.2}$$

whose Laplace exponent is

$$\psi(s) := \ln \mathbb{E}(e^{sZ_1}) = \mu s + \int_{-\infty}^{0} (e^{sx} - 1)\,\Pi(\mathrm{d}x),$$

where $\mu > 0$ and $\Pi(\cdot)$, defined on $(-\infty, 0)$, is the Lévy measure of Z. Furthermore, we require that Z has paths of infinite jumps and bounded variation.

$$\Pi((-\infty, 0)) = \infty \quad \text{and} \quad \int_{-1}^{0} |x|\Pi(\mathrm{d}x) < \infty.$$

Following the intuition of the structural models, we also assume that the default happens at the first time when the counterparty's asset value V_t falls below some predetermined barrier $d \in (0,1)$. Denoting the default time by τ, we have, from (3.1),

$$\tau := \inf\{t : V_t \leq d\} = \inf\{t : -\ln d + Z_t \leq 0\}. \tag{3.3}$$

[3] A pure-jump subordinator is a non-decreasing Lévy process with no drift.

For convenience, we denote

$$X_t := x + Z_t, \quad t \geq 0, \tag{3.4}$$

where $X_0 = x > 0$. Then τ is actually the ruin time of process $X = \{X_t, t \geq 0\}$ with $x = -\ln d$. We use $-X_\tau$, the absolute deficit at ruin of X, to specify the default severity.

One important feature of our modeling is that R_τ, the recovery rate at default, is random. Let $0 = p_0 < p_1 < \cdots < p_n = 1$. We assume the recovery rate as

$$R_\tau = d \sum_{i=1}^{n} l_i \mathbf{1}_{\{p_{i-1} < V_\tau / d \leq p_i\}} = d \sum_{i=1}^{n} l_i \mathbf{1}_{\left\{p_{i-1} < e^{X_\tau} \leq p_i\right\}} \tag{3.5}$$

with constants $l_i \in [p_{i-1}, p_i]$, $i = 1, \ldots, n$. So the recovery rate is a random number from the interval $[0, d]$ depending on the default severity. Precisely, the recovery rate is a non-increasing function of $-X_\tau$, which itself is random. It is indeed a very reasonable assumption from an economic perspective since a lower recovery rate is expected if the firm value is at a lower level at the time of default.

Remark 3.1.1 A similar piecewise fixed recovery rate like the one assumed in (3.5) has been employed by Guo et al. (2009). In that paper, the authors assume the recovery rate on a firm's defaulted bond after the event of default is either R or K, where R and K are two constants such that $R < K \leq 1$. If the firm's asset value falls to a prescribed insolvency barrier and thus bankruptcy occurs before the financial distress is resolved, the recovery rate is R. Otherwise, the firm remains solvent up to the resolution of financial distress and the recovery rate is K.

Remark 3.1.2 In our assumption of the recovery rate in (3.5), when $n = 1$ and $l_1 = 1$ we have $R_\tau = d$, i.e., the recovery rate is fixed. This special case has been extensively studied in the literature of default models. See, for example, Madan and Schoutens (2008) and Hao et al. (2013) in a close context of this thesis. We will show in Chapter 4, via a numerical example of an interest rate swap, that the random recovery rate may significantly affect the CVA.

3.2 Lévy processes

According to Kyprianou (2006), Lévy process is a process on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ for $Z = \{Z_t : t \geq 0\}$ with the following properties:

- **Stationary increments:** For $0 \leq s \leq t$, $Z_t - Z_s$ is equal in distribution to Z_{t-s}.
- **Independence of increments:** For $0 \leq s \leq t$, $Z_t - Z_s$ is independent of $\{Z_u : u \leq s\}$.
- The paths of Z are almost surely right-continuous with left limits.
- $\Pr(Z_0 = 0) = 1$

Supposing that $a \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and Π is a measure concentrated on $\mathbb{R}\backslash\{0\}$ such that $\int_{\mathbb{R}} (\mathbf{1} \wedge \mathrm{x}^2)\Pi(\mathrm{x}) < \infty$, a Lévy process has the following property that for all $t > 0$,

$$\mathbb{E}(e^{\mathrm{i}sZ_t}) = e^{-t\Psi(s)},$$

where the characteristic exponent is given by the Lévy-Khintchine formula

$$\Psi(s) = \mathrm{i}as + \frac{1}{2}\sigma^2 s^2 + \int_{\mathbb{R}} (1 - e^{\mathrm{i}sx} + \mathrm{i}sx\mathbf{1}_{(|x|<1)})\Pi(\mathrm{d}x). \tag{3.6}$$

3.2.1 Meromorphic Lévy processes

We assumed the process Z belongs to a specific class of Lévy processes called meromorphic processes, which has been introduced by Kuznetsov and Morales (2014). A meromorphic process is defined by requiring its Lévy measure $\Pi(\mathrm{d}x) = \pi(x)\mathrm{d}x$ to be a mixture of exponential distributions. Precisely, we assume that the Lévy density of Z is given by

$$\pi(x) = \mathbf{1}_{\{x<0\}} \sum_{m=1}^{\infty} b_m e^{\rho_m x}, \tag{3.7}$$

where the coefficients b_m and ρ_m are positive and ρ_m increases to $+\infty$ as $m \to +\infty$. According to Kuznetsov and Morales (2014), if

$$\int_{-\infty}^{0} \pi(x)\mathrm{d}x = \sum_{m=1}^{\infty} \frac{b_m}{\rho_m} = \infty,$$

then we have jumps with infinite activities; and if

$$\int_{-\infty}^{0} | x | \pi(x)\mathrm{d}x = \sum_{m=1}^{\infty} \frac{b_m}{\rho_m^2} < \infty,$$

then we have a process with finite variation. Therefore, if we want $\pi(x)$ given in (3.7) to be a Lévy density, then the following condition must hold

$$\int_{-\infty}^{0} | x |^2 \pi(x) \mathrm{d}x = \sum_{m=1}^{\infty} \frac{b_m}{\rho_m^3} < \infty.$$

According to Kuznetsov et al. (2012), from (3.6) and (3.7), the Laplace exponent of Z becomes

$$\psi(s) = \left(\mu - \sum_{m=1}^{\infty} \frac{b_m}{\rho_m^2} \right) s + s^2 \sum_{m=1}^{\infty} \frac{b_m}{\rho_m^2 (\rho_m + s)}, \tag{3.8}$$

where

- $\{-\rho_m\}_{m=1,2,\ldots}$, are the simple poles of the $\psi(s)$;
- for $q > 0$, $\Phi(q)$ is the unique positive root of the function $\psi(s) - q$;
- for $q \geq 0$, $\{-\zeta_n(q)\}_{n=1,2,\ldots}$ are the negative roots of the function $\psi(s) - q$;
- for $q \geq 0$, there are no other roots in the entire complex plane for the function $\psi(s) - q$, except for $\Phi(q)$ and $\{-\zeta_n(q)\}_{n=1,2,\ldots}$;
- it is assumed that the $\zeta_n(q)$ are labeled so that they increase in n, thus, we have the following interlacing property

$$0 < \zeta_1(q) < \rho_1 < \zeta_2(q) < \rho_2 < \cdots .$$

3.2.2 Beta and theta families of Lévy processes

According to Kuznetsov and Morales (2014), it is possible to compute the Laplace exponent $\psi(s)$ explicitly for meromorphic processes that belong to beta and theta families by choosing coefficients b_m and ρ_m.

Example 3.2.1 Z is a *θ-process with parameter* $\lambda = 3/2$ if

$$b_m = \frac{2}{\pi} c\beta m^2, \quad \rho_m = \beta(\alpha + m^2), \qquad m = 1, 2, \ldots. \tag{3.9}$$

Now its Laplace exponent becomes

$$\psi(s) = \mu s - c\sqrt{\alpha + s/\beta} \coth\left(\pi\sqrt{\alpha + s/\beta}\right) + c\sqrt{\alpha} \coth(\pi\sqrt{\alpha}). \tag{3.10}$$

Example 3.2.2 Z is a *β-process with parameter* $\lambda \in (1, 2)$ if

$$b_m = c\beta \binom{m + \lambda - 2}{m - 1}, \quad \rho_m = \beta(\alpha + m), \qquad m = 1, 2, \ldots. \tag{3.11}$$

Now its Laplace exponent becomes

$$\psi(s) = \mu s + c\mathrm{B}(1 + \alpha + s/\beta, 1 - \lambda) - c\mathrm{B}(1 + \alpha, 1 - \lambda), \tag{3.12}$$

where $\mathrm{B}(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ is the beta function.

Note that all the parameters in the above examples are positive (i.e. $\mu > 0, c > 0, \alpha > 0$ and $\beta > 0$). In particular, μ is the drift parameter, α is the parameter for the rate of decay of the tail of the Lévy measure $\Pi(\mathrm{x})$, β is the parameter for the shape of the Lévy measure $\Pi(\mathrm{x})$, and c is the parameter describing the overall intensity of the jumps.

3.3 Expected discounted penalty functions

The EDPF, originally introduced by Gerber and Shiu (1998), is a function of a quantity that characterizes the relevant information about the ruin event. The generalized EDPFs is a function of time of ruin (τ), the deficit at ruin ($-X_\tau$), the surplus immediately before ruin ($X_{\tau-}$), and the last minimum surplus level before ruin ($\underline{X}_{\tau-}$). With the discount factor $e^{-r\tau}$, if we imagine that $w(-X_\tau, X_{\tau-}, \underline{X}_{\tau-})$ is a penalty function that reflects the costs to the institution at the time of ruin, then the generalized EDPF is given as follows.

Definition 3.3.1 (Definitions 1 of Kuznetsov and Morales (2014)) *For the process X in (3.4), the generalized EDPF ϕ is*

$$\phi(x;r) := \mathbb{E}\left(e^{-r\tau} w(-X_\tau, X_{\tau-}, \underline{X}_{\tau-}) \mathbf{1}_{\{\tau<\infty\}} \middle| X_0 = x \right), \tag{3.13}$$

where $r \geq 0$ and w a bounded measurable function on $\mathbb{R}^3_+ = [0,\infty)^3$.

Definition 3.3.2 (Definitions 2 of Kuznetsov and Morales (2014)) *For the process X in (3.4), the generalized finite-time EDPF ϕ_t is*

$$\phi_t(x;r) := \mathbb{E}\left(e^{-r\tau} w(-X_\tau, X_{\tau-}, \underline{X}_{\tau-}) \mathbf{1}_{\{\tau\leq t\}} \middle| X_0 = x \right), \tag{3.14}$$

where $r \geq 0$ and w a bounded measurable function on $\mathbb{R}^3_+ = [0,\infty)^3$.

3.3.1 Computing infinite-time EDPFs

Theorem 3.3.1 (Proposition 2 of Kuznetsov and Morales (2014)) *Assume that process Z in (3.2) is a meromorphic Lévy process. Then, for $q > 0$, the density of infimum is a mixture of exponential distributions*

$$\mathbb{Q}(-\underline{Z}_{\mathrm{e}(q)} \in \mathrm{d}x) = \Big[\sum_{n\geq 1} c_n(q)\zeta_n(q)e^{-\zeta_n(q)x}\Big]\mathrm{d}x, \ \ x > 0 \tag{3.15}$$

where

$$c_n(q) := -\Big(\frac{1}{\zeta_n(q)} + \frac{1}{\Phi(q)}\Big)\frac{q}{\psi'(-\zeta_n(q))} \tag{3.16}$$

and $\mathrm{e}(q)$ is an exponential random time with mean q^{-1}.

Theorem 3.3.2 (Theorem 2 of Kuznetsov and Morales (2014)) *Assume that process Z in (3.2) is a meromorphic Lévy process. For $q \geq 0$, $x > 0$, $y > 0$, if $z \geq x$, then*

$$\begin{aligned}
&\mathbb{E}\Big[e^{-r\tau}\mathbf{1}_{\{-X_\tau<y;X_{\tau-}<z\}}|X_0 = x\Big] \\
&= \sum_{n\geq 1} c_n e^{-\zeta_n x} - \frac{e^{-\Phi(q)(z-x)}}{\psi'(\Phi(q))}\sum_{m\geq 1}\frac{b_m e^{-\rho_m z}(1-e^{-\rho_m y})}{\rho_m(\Phi(q)+\rho_m)} \\
&+ \frac{\Phi(q)}{q}\sum_{m,n\geq 1}\frac{c_n\zeta_n b_m e^{-\zeta_n x}}{\rho_m(\Phi(q)+\rho_m)}\Big[\frac{e^{-\rho_m y}}{\zeta_n-\rho_m} + \frac{e^{-(\Phi(q)+\rho_m)z}(1-e^{-\rho_m y})}{\Phi(q)+\zeta_n}\Big],
\end{aligned} \tag{3.17}$$

where, for simplicity, we use ζ_n and c_n to represent $\zeta_n(q)$ and $c_n(q)$ respectively. If X is a beta or theta process, then in both cases all infinite series converge exponentially fast, uniformly on compact subsets of the admissible set of variables (x, y, z).

3.3.2 Computing finite-time EDPFs

Section 3.3.1 discussed the formulas for computing the infinite-time EDPFs. In this section, we are going to look at the method introduced by Kuznetsov and Morales (2014) for computing the finite-time EDPFs which is relevant for calculating CVA.

The Laplace transform in the t-variable of the generalized finite-time EDPF $\phi_t(x;r)$ in (3.14) is given as follows:

$$\begin{aligned}&\int_0^\infty e^{-qt}\phi_t(x;r)\mathrm{d}t = \int_0^\infty e^{-qt}\mathbb{E}\left[\, e^{-r\tau}w(-X_\tau, X_{\tau-}, \underline{X}_{\tau-})\mathbf{1}_{\{\tau\le t\}}\,\middle|\, X_0 = x\right]\mathrm{d}t \\ &= \frac{1}{q}\mathbb{E}\left[\, e^{-r\tau}w(-X_\tau, X_{\tau-}, \underline{X}_{\tau-})e^{-q\tau}\,\middle|\, X_0 = x\right] = \frac{\phi(x;r+q)}{q}.\end{aligned} \tag{3.18}$$

Here, I want to focus on two probabilities that are essential for the calculation of CVA.

Example 3.3.1 *It is clear that the finite-time ruin probability is*

$$\mathbb{Q}(\tau\le t) = \phi_t(x;0) \quad \textit{with} \quad w(-X_\tau, X_{\tau-}, \underline{X}_{\tau-}) = 1.$$

According to Theorem 3.3.1, we can get the Laplace transforms of $\mathbb{Q}(\tau\le t)$ *as*

$$\int_0^\infty e^{-qt}\mathbb{Q}\left(\tau\le t\right)\mathrm{d}t = q^{-1}\mathbb{Q}(\tau\le \mathrm{e}(q)) = q^{-1}\sum_{n\ge1} c_n(q)e^{-\zeta_n(q)x}. \tag{3.19}$$

Example 3.3.2 *In addition, we have*

$$\mathbb{Q}\left(-X_\tau < y, \tau\le t\right) = \phi_t(x;0) \quad \textit{with} \quad w(-X_\tau, X_{\tau-}, \underline{X}_{\tau-}) = \mathbf{1}_{\{-X_\tau<y\}}.$$

If we set $z = +\infty$ in Theorem 3.3.2, then we can get the Laplace transforms of $\mathbb{Q}\left(-X_\tau < y, \tau \leq t\right)$ as

$$\begin{aligned}
&\int_0^\infty e^{-qt}\mathbb{Q}\left(-X_\tau < y, \tau \leq t\right) \mathrm{d}t \\
&= q^{-1}\mathbb{Q}\left(-X_\tau < y, \tau \leq \mathrm{e}(q)\right) \\
&= \frac{1}{q}\sum_{n\geq 1} c_n(q)e^{-\zeta_n(q)x} + \frac{\Phi(q)}{q^2}\sum_{m,n\geq 1}\frac{c_n(q)\zeta_n(q)b_m e^{-\zeta_n(q)x-\rho_m y}}{\rho_m(\Phi(q)+\rho_m)(\zeta_n(q)-\rho_m)}. \qquad (3.20)
\end{aligned}$$

Note that the series in expressions (3.19) and (3.20) converge exponentially fast due to the linear/quadratic growth of ρ_n and ζ_n. This feature is so important that we are able to numerically compute the Laplace transforms of $\mathbb{Q}\left(\tau \leq t\right)$ and $\mathbb{Q}\left(-X_\tau < y, \tau \leq t\right)$ in a very efficient way with high accuracy.

Finally, we need to invert Laplace transforms (3.19) and (3.20) numerically. One possible technique suggested by Kuznetsov and Morales (2014) is Gaver-Stehfest algorithm. Using this algorithm, an approximation of generalized finite-time EDPF $\phi_t(x;r)$ is given as

$$\phi_t^{GS}(x;r;M) = \sum_{n=1}^{2M}\frac{a_n}{n}\phi(x;r+n\ln(2)t^{-1}), \qquad (3.21)$$

where M is a reasonably large integer and a_n is a coefficient given as

$$a_n = (-1)^{M+n}\sum_{j=\frac{n+1}{2}}^{n\wedge M}\frac{j^{M+1}}{M!}\binom{M}{j}\binom{2j}{j}\binom{j}{n-j}. \qquad (3.22)$$

Chapter 4

Numerical Experiment

This chapter will show a numerical experiment on a hypothetical interest rate swap, and it will be constructed as follows. We will start with an illustration of a hypothetical interest rate swap that will be shown in Section 4.1. In particular, we are going to describe the hypothetical interest rate swap as well as the valuation process for this hypothetical interest rate swap. Then, a discussion of the calculation of the CVA for the interest rate swap will be presented in Section 4.2. We are going to focus on the recovery rate and the default probability as well as the calibration result in this section. Finally, the numerical results for CVA will be discussed in Section 4.3.

4.1 Interest rate swap

An interest rate swap is a contractual agreement between two parties for exchanging a series of interest payments over a certain period of time. (See Figure

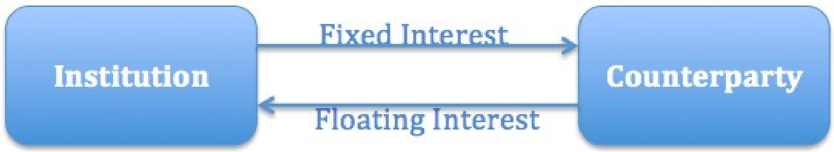

Figure 4.1: Interest Rate Swap

4.1) The most popular interest rate swap is the plain vanilla ("LIBOR-for-fixed") interest rate swap in which the LIBOR rate is exchanged for a fixed rate of interest. (See Hull (2014, p.157) for the definitions.) In this thesis, we are going to show a numerical experiment on a hypothetical plain vanilla interest rate swap.

Suppose that we have a hypothetical interest rate swap between "Institution" and "Counterparty" with a notional principle amount of $100 million in US dollars. The agreement is settled on January 15, 2016 and will be expired on January 15, 2019. In the contract agreement, Institution agrees to pay a fixed interest rate at 1.1985% per annum for receiving a floating interest rate as the 6-month LIBOR quote on the US dollars. The Institution and Counterparty will exchange the payments every 6 months during the 3-year period, starting from July 15, 2016 to and including January 15, 2019. A summary of this hypothetical interest rate swap is given in Table 4.1. Note that the fixed rate can be determined by setting the present value of the fixed-rate payments equal to the present value of the floating-rate payments; and all the rates are quoted continuously.

In order to determine the value of an interest rate swap, we first need to figure out the time value (i.e. the risk-free discount rate) of the cash flows. In this thesis, we will use the LIBOR rate as the risk-free discount rate for the valuation

Effective date	15-Jan-2016
Termination date	15-Jan-2019
Notional principal	USD 100 million
Payment dates	From 15-Jul-2016 to and including 15-Jan-2019
Fixed-rate payer	Institution
Fixed rate	1.1985% per annum
Floating-rate payer	Counterparty
Floating rate	USD 6-month LIBOR

Table 4.1: Hypothetical Interest rate swap confirmation

purposes. Therefore, we will first discuss how to find the LIBOR zero rates in Section 4.1.1 and then we will figure out how to use the LIBOR zero rates to valuate a hypothetical interest rate swap in Section 4.1.2.

4.1.1 LIBOR rates

LIBOR rate, short for London Interbank Offered Rate, is the rate of interest at which the banks and financial institutions will make loans in the capital markets. Since the banks and financial institutions normally have the highest credit rating, LIBOR rates are usually regarded as the risk-free interest rate even though LIBOR rates also carry credit risk. LIBOR rates are available for all major currencies for maturities up to 12 months. (See Table 4.3) According to Ron (2000), the observed market LIBOR rate (r_m) can be converted into the continuously compounded LIBOR rate (r_c) by the following equation:

$$r_c = \frac{t_y}{t_m} \ln(1 + r_m \frac{t_m}{t_y}); \tag{4.1}$$

where t_y represents the total day counts (i.e. 360) for a year, t_m represents the time to maturity.

When valuing the derivatives with longer maturities, we will need to extract the LIBOR rates from the Eurodollar futures market where the maturities are longer than 12 months. The Eurodollar futures is the futures contract on the interest that will be paid on Eurodollars deposited at banks outside the United States. According to Hull (2014, p.140), the futures interest rate can be extracted from the price of Eurodollar futures contract by:

$$\text{Futures rate} = 100 - \text{Futures price}. \tag{4.2}$$

We collect the quotes of the Eurodollar futures from Bloomberg and then we apply the above equation to get the futures rates. Please see Table 4.2 for the quotes and rates of the Eurodollar futures achieved on the January 15, 2016. Note that the futures rates from 4.2 is quarterly compounded (r_{fq}), so we need to compute the continuously compounded futures rates (r_{fc}). (See Table 4.3)

$$r_{fc} = \frac{360}{90} \ln(1 + \frac{r_{fq}}{4}). \tag{4.3}$$

For short maturities, the Eurodollar futures rate is essentially the same as the forward rate. For long maturities, the difference between the two rates is known as the convexity adjustment. Hull (2014, p.143) gives us the formula as follows.

$$\text{Forward rate} = r_{fc} - \frac{1}{2}\sigma^2 T_1 T_2 \tag{4.4}$$

Date	Futures price	Futures rate (quarterly)	Futures rate (continuously)	Convexity adjustment	Forward rate (continuously)
15-Mar-2016	99.3350	0.6650%	0.6644%	0.000003	0.6644%
15-Jun-2016	99.2600	0.7400%	0.7393%	0.000010	0.7393%
15-Sep-2016	99.1800	0.8200%	0.8192%	0.000023	0.8191%
15-Dec-2016	99.0950	0.9050%	0.9040%	0.000040	0.9039%
15-Mar-2017	99.0100	0.9900%	0.9888%	0.000062	0.9887%
15-Jun-2017	98.9150	1.0850%	1.0850%	0.000088	1.0834%
15-Sep-2017	98.8150	1.1850%	1.1850%	0.000120	1.1831%
15-Dec-2017	98.7150	1.2850%	1.2850%	0.000155	1.2828%
15-Mar-2018	98.6300	1.3700%	1.3700%	0.000196	1.3675%
15-Jun-2018	98.5450	1.4550%	1.4525%	0.000241	1.4521%
15-Sep-2018	98.4600	1.5400%	1.5400%	0.000291	1.5368%
15-Dec-2018	98.3750	1.6250%	1.6250%	0.000346	1.6214%

[1] The volatility is 0.00865.
[2] Futures prices and volatility are achieved from Bloomberg on January 15, 2016.

Table 4.2: Forward LIBOR Rates

where r_{fc} is the continuously compounded futures rate, T_1 is the time to maturity of the futures contract, and T_2 is the time to the maturity of the rate underlying the futures contract (usually 90 days), σ is the volatility of the short rate.

Once we have determined the forward rates, we can simply use the bootstrap method (Hull, 2014, p.144) to calculate the zero rates from the forward rates. Suppose that F_i is the forward rate calculated from the ith Eurodollar futures contract and R_i is the zero rate for a maturity T_i, then we can calculate the zero rate R_{i+1} by:

$$R_{i+1} = \frac{F_i(T_{i+1} - T_i) + R_i T_i}{T_{i+1}} \tag{4.5}$$

Following the process described above, we could eventually construct the LIBOR zero curve with the piecewise linear interpolation. (Please see Table 4.3 for

Maturity	LIBOR zero rates
3 m	0.6191%
6 m	0.8473%
9 m	0.9929%
12 m	1.1386%
14 m	1.0502%
15 m	1.0466%
17 m	1.0393%
18 m	1.0415%
20 m	1.0459%
21 m	1.0519%
23 m	1.0638%
24 m	1.0723%
26 m	1.0891%
27 m	1.0998%
29 m	1.1212%
30 m	1.1316%
32 m	1.1522%
33 m	1.1632%
35 m	1.1852%
36 m	1.1967%
38 m	1.2196%

Table 4.3: LIBOR Zero Rates

the LIBOR zero rates.)

4.1.2 Value of an interest rate swap

In the previous section, we discussed how to find the LIBOR zero rates. This section will focus on the valuation of the hypothetical interest rate swap (See Table 4.1) using these LIBOR zero rates. We are going to use the method of valuation in terms of bond prices, discussed in Hull (2014, p.158), to assess the interest rate swap. Our intention is to provide the valuation of the hypothetical interest rate

Date	6 m	12 m	18 m	24 m	30 m	36 m	B_{fixed}	B_{float}	V_{swap}
15-Jan-16	0.8473%	1.1386%	1.0415%	1.0723%	1.1316%	1.1967%	100.0000	100	0.0000
15-Jul-16	1.4429%	1.1386%	1.1472%	1.2026%	1.2665%	-	99.8253	100	0.1747
15-Jan-17	0.8474%	1.0059%	1.1269%	1.2257%	-	-	99.3430	100	0.6570
15-Jul-17	1.1644%	1.2666%	1.3518%	-	-	-	99.7674	100	0.2326
15-Jan-18	1.3688%	1.4455%	-	-	-	-	99.7507	100	0.2493
15-Jul-18	1.5223%	-	-	-	-	-	99.8365	100	0.1635

Table 4.4: Value of the interest rate swap

swap from the perspective of the fixed rate payer (i.e. Institution). The swap can be viewed as a short position in a fixed-rate bond and a long position in a floating-rate bond, so that

$$V_{swap} = B_{float} - B_{fixed} \tag{4.6}$$

where V_{swap} is the value of the interest rate swap, B_{float} is the value of the floating-rate bond and B_{fixed} is the value of the fixed-rate bond. Note that the value of the floating-rate bond is always the notional principal immediately after an interest payment because both the coupon rate and the discount rate for the floating-rate bond are both the LIBOR rate. Now, we can easily calculate the values of two bonds by discounting all the cash flows to the current time values with the above equation.

Table 4.4 lists the results of the valuation of the interest rate swap. In particular, the columns 2-7 show the LIBOR zero rates (for every six months) on the corresponding date. In the last three columns, we show the value of the fixed-rate bond, the value of the floating-rate bond and the value of the swap. Note that the values are in millions of dollars.

4.2 CVA of an interest rate swap

An interest rate swap is a type of the OTC transaction in which two private parties make an agreement to exchange the cash flows. As mentioned in Chapter 2, CVA is needed to reflect the adjustment on the prices due to the counterparty risk of an OTC derivative contract.

To calculate CVA in (2.2), we need to evaluate three major components: credit exposure, recovery rate and the default probability. We have discussed how to evaluate the credit exposure for the hypothetical interest rate swap in the last section. Now, we need to calculate the recovery rate and the default probability, i.e., $\mathbb{E}\left((1-R_\tau)\mathbf{1}_{\{\tau\le t\}}\right)$. Plugging in the random recovery rate R_τ from (3.5), we have

$$\begin{aligned}
&\mathbb{E}\left((1-R_\tau)\mathbf{1}_{\{\tau\le t\}}\right)\\
&=\mathbb{E}\left(\mathbf{1}_{\{\tau\le t\}}\right)-\mathbb{E}\left(R_\tau\mathbf{1}_{\{\tau\le t\}}\right)\\
&=\mathbb{Q}\left(\tau\le t\right)-d\sum_{i=1}^{n}l_i\mathbb{Q}\left(p_{i-1}<e^{X_\tau}\le p_i,\tau\le t\right)\\
&=\mathbb{Q}\left(\tau\le t\right)-d\sum_{i=1}^{n}l_i\left(\mathbb{Q}(-X_\tau<-\ln p_{i-1},\tau\le t)-\mathbb{Q}(-X_\tau<-\ln p_i,\tau\le t)\right).
\end{aligned} \tag{4.7}$$

According to the above derivations, we can see that we essentially need to calculate $\mathbb{Q}\left(\tau\le t\right)$ and $\mathbb{Q}\left(-X_\tau<y,\tau\le t\right)$ for arbitrary $y,t>0$. These two probabilities turn out to be the finite-time EDPFs. As we have discussed in Chapter 3, we are able to numerically compute the finite-time EDPFs by inverting the corresponding Laplace transforms with the Gaver-Stehfest algorithm.

4.2.1 Calibration

The risk-neutral probabilities are obtained from the market spreads of Credit Default Swap (CDS) which is the most common credit derivative designed to shift credit risk for the referenced entity. The market spreads of CDS reflect the investors' perceptions of default probability of the referenced entity. Therefore, our model is calibrated to the market CDS spread curve (data collected from CDX.NA.IG index).

Our target for the calibration is to find the most accurate representation of the market CDS spread curve. The accuracy is captured by the mean absolute error (MAE) defined as the mean of the absolute differences between the observed market CDS par spread and the estimated model CDS par spread Hao and Li (2015).

$$MAE = \frac{\sum |Market\ CDS\ par\ spreads - Model\ CDS\ par\ spreads|}{Number\ of\ CDS\ par\ spreads}. \tag{4.8}$$

In other words, we intend to find out the optimal parameters with minimum MAE for each company. Then, we use the calibrated parameters to calculate term structure of calibrated CDS spread curve.

In order to perform a numerical study for companies with different levels of credit risk, we randomly choose three companies from CDX.NA.IG index with three different Moody's credit rating levels. (See Table 4.2.1 for Moody's rating scales) In particular, United Parcel Service Inc with high grade (Aa3), Home Depot Inc with upper-medium grade (A2) and McDonald's Corp with lower-medium grade (Baa1) are chosen for the calibration.

Rating Scale	Meaning
Aaa	Aaa are judged to be of the highest quality, subject to the lowest level of credit risk.
Aa	Aa are judged to be of high quality, subject to very low credit risk.
A	A are judged to be upper-medium grade and are subject to low credit risk.
Baa	Baa are judged to be medium-grade and subject to moderate credit risk.
Ba	Ba are judged to be speculative, subject to substantial credit risk.
B	B are considered speculative and are subject to high credit risk.

[1] Moody's appends numerical modifiers 1, 2, and 3 to each generic rating classification from Aa through B.
[2] The modifier 1 indicates that the obligation ranks in the higher end of its generic rating category; the modifier 2 indicates a mid-range ranking; the modifier 3 indicates a ranking in the lower end of that generic rating category.

Table 4.5: Moody's credit rating scales (Moody's Investors Service, 2016)

Table 4.6 lists the market CDS spread curve for those chosen companies with six maturities (1 year, 2 years, 3 years, 5 years, 7 years, and 10 years). Note that the spreads are collected from Bloomberg on January 15, 2016, and are quoted in terms of basis points. (One basis point is equal to 0.01%.)

We calibrate our model on a theta process with the parameters $c, \alpha, \beta, \lambda$. (See Equation 3.9) The calibrated results are shown in Table 4.6. For United Parcel Service Inc, we find that the optimal parameters are $c = 4.498$, $\alpha = 3.214$, $\beta = 2.571$ and $\lambda = 1.5$. For Home Depot Inc, we find that the optimal parameters are $c = 2.491$, $\alpha = 2.254$, $\beta = 2.613$ and $\lambda = 1.5$. For McDonald's Corp, we find that the optimal parameters are $c = 12.536$, $\alpha = 1.782$, $\beta = 5.954$ and $\lambda = 1.5$.

In addition, we also show the calibrated CDS spread curve along with the cor-

Company	Moody's		1y	2y	3y	5y	7y	10y	c	α	β	λ	MAE
United Parcel	Aa3	Market	7.11	10.03	14.12	21.98	39.05	50.69					
		Calibrated	3.071	8.895	15.991	29.278	39.051	48.07	4.498	3.214	2.571	1.5	2.827
Home Depot	A2	Market	9.38	11.64	16.40	22.98	34.12	47.91					
		Calibrated	5.624	11.242	16.994	26.932	34.125	40.841	2.491	2.254	2.613	1.5	2.629
McDonald's	Baa1	Market	9.77	11.67	19.59	35.99	58.8	76.87					
		Calibrated	2.359	10.896	22.659	44.188	58.826	71.061	12.536	1.782	5.954	1.5	4.214

Table 4.6: Calibration result

responding MAE in Table 4.6. From the table, we can observe that MAE is quite small (2.827 basis points for United Parcel Service Inc; 2.629 basis points for Home Depot Inc; 4.214 basis points for McDonald's Corp). This means that our model fits the term structures of market CDS well.

4.3 Numerical results

In order to investigate whether there is a significant impact on the result of CVA when we release the assumption of a fixed recovery rate, we calculate CVA in both cases. In Table 4.7, the third column shows the numerical results of CVA with a fixed recovery rate at 40%, and the fourth column shows the results of CVA with a random recovery rate. For United Parcel Service Inc, CVA with a fixed recovery rate is \$739.34 (0.00073934 millions) while CVA with a random recovery rate is \$904.62 (0.00090462 millions), For Home Depot Inc, CVA with a fixed recovery rate is \$901.31 (0.00090131 millions) and CVA with a random recovery rate is \$1105 (0.00110556 millions). For McDonald's Corp, CVA with a fixed recovery rate is \$986.60 (0.00098660 millions) while CVA with a random recovery rate is \$1205.94 (0.00120594 millions). It is easy to observe that company with higher credit rating (i.e. lower credit risk) has lower CVA in both cases. This means that our numerical

Company	Moody's	CVA (fixed RR)	CVA (random RR)	% change
United Parcel Service Inc	Aa3	0.00073934	0.00090462	22.3541%
Home Depot Inc	A2	0.00090131	0.00110556	22.6610%
McDonald's Corp	Baa1	0.00098660	0.00120594	22.2317%

Table 4.7: CVA result

results of CVA is consistent with Moody's credit ratings.

In order to test the effect of a random recovery rate on CVA, we compute the percentage change which expresses the difference between CVA with a fixed recovery rate and CVA with a random recovery rate. From the last column of Table 4.7, we can see that CVA with a random recovery rate is 22.3540%, 22.6593% and 22.2385% higher than CVA with a fixed recovery rate for United Parcel Service Inc, Home Depot Inc and McDonald's Corp respectively. This means that there is a large impact of capturing the randomness of the recovery rate in our model.

Chapter 5

Conclusions

This chapter concludes analysis of this thesis. We will first summarize the results of the thesis in Section 5.1. Then, we will discuss some possibilities for future research.

5.1 Summary of results

In this thesis, we introduce a Lévy default model for pricing CVA. This thesis is inspired by the increasingly importance for CVA and counterparty credit risk since the 2007 crisis. In Chapter 1, the backgrounds for CVA and the motivations of this thesis are introduced. Then, Chapter 2 reviews the formulas for computing CVA.

This thesis focuses on the default probability and recovery rate via a new structural model. In Chapter 3, we review some important structural models. Then, we introduce a new structural model - Lévy default model. One important feature is that we are able to include a random recovery rate in our model. The piecewise random recovery rate is assumed to depend on the default severity which itself is random. Another important feature is that the asset value of a company is assumed to follow meromorphic Lévy processes with infinite jumps but finite variation. Using the theorems and properties of Beta and Theta families of meromorphic Lévy processes, we are able to efficiently and accurately compute the default probability via finite-time EDPFs.

In Chapter 4, a numerical experiment is performed on a hypothetical interest rate swap. In particular, we illustrate the valuation for the credit exposure of the interest rate swap at first. Then, our default model is calibrated to term structure of market CDS spread curve. According to the small mean absolute error, we found that our model fits the market CDS spread curve well. To compare how a random recovery rate and a fixed recovery rate affect CVA, we compute CVA in both cases. The numerical results show the consistency with Moody's credit ratings. The companies with better Moody's rating has a lower CVA. In addition, it is shown that CVA with a random recovery rate is usually 22% higher than CVA with a fixed recovery rate. Therefore, we conclude that there is a significant impact of including a random recovery rate in our model.

5.2 Future research

In this thesis, the Lévy default is calibrated to term structure of market CDS spread curve. One limitation of such calibration is that market CDS par spreads are only available up to ten years. This thesis uses an interest rate swap to show how to price CVA. Since the maturity for an interest rate swap is usually shorter than

10 years, market CDS par spreads is long enough for the calibration. However, our method is not suitable for other common OTC derivatives with longer maturities, such as longevity swap, q-forward, K-forward, etc. Thus, one possible extension is to find a calibration method for OTC derivatives with longer duration. When considering CVA for longevity derivatives, one will need to try to find the risk exposure under risk-neutral probability that will be more complicated than the interest rate swap cases.

In addition, our model assumes a piecewise random recovery rate which depends on default severity. We make such an assumption because it is more efficient and accurate for the numerical experiment. Although it is reasonable to make such an assumption, it will be more realistic if a random recovery rate can be continuously varying. Therefore, another possible extension for the future research is to find an efficient and accurate way to include a continuously varying random recovery rate.

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