Congruence Lattices of Lattices

by

SIONG–KHOW TEO

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Abstract

A congruence relation of a lattice L is an equivalence relation preserving the lattice operations; the set of all congruence relations form a lattice, Con L. The study of the congruence lattices of lattices is one of the fundamental problems in the theory of lattices. In this thesis, we study the relationship between the lattice and its congruence lattice. In Chapter II, we show that if D is a finite distributive lattice with n dual atoms, then there is a lattice L of length 5n such that Con L is isomorphic to D. This answers a problem raised by E. T. Schmidt. We also prove that the bound is best possible in general. Also, we prove that if L is a sectionally complemented lattice, then the length of L is at least $2|\ell(D)| - n$. (Such a lattice was constructed in [Gr, Sc].) If the set of join-irreducibles of L is countable and every element of L is the join of some join-irreducibles, then we construct a planar lattice L such that Con L is isomorphic to D and $|L|$ is of the magnitude of $|J(D)|^2$. In Chapter III, we enumerate all the congruence lattices of lattices of length at most 4. In Chapter IV, we give a simpler proof that the ideal lattice of a countable distributive semilattice with zero is the congruence lattice of some lattice. K. Reuter and R. Wille introduced the notion of complete congruence relation. In Chapter V, we answer a question raised by them. We show that every finite lattice is the complete congruence lattice of a complete lattice. The construction for the finite case can be modified to show that every complete lattice is the complete congruence lattice of a complete lattice. This result was also proved by G. Grätzer [Gr-2].

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Chapter O
Introduction

In this thesis, we study the congruence lattices of lattices. This thesis is divided into six chapters. In this chapter, we mention some basic results and give a brief summary of the results which we obtained. A detail survey on this topic can be found in [Sc-4]. In Chapter I, we introduce the notation which will be used in the thesis. Our results are presented in detail in Chapters II – V.

An equivalence relation Θ of a lattice L is called a *congruence relation* of L if it preserves the lattice operations of L. The lattice of all the congruence relations of a lattice L is denoted by Con L. The following theorem [Fu, Na], is fundamental in the studies of the congruence lattice of lattices: Con L is an algebraic distributive lattice.

The converse of the result of [Fu, Na], that is, whether an algebraic distributive lattice is a congruence lattice of some lattice, is a long standing problem in lattice theory. Also, one can investigate the relationship between the congruence lattices and lattices in terms of some lattice parameters, e.g., the length or the cardinality of the lattice. Some known results are:

In the 40's, R. P. Dilworth stated (unpublished) that every finite distributive lattice is the congruence lattice of some lattice.

In 1962, G. Grätzer and E. T. Schmidt showed that every distributive lattice D in which every element is the join of join-irreducibles, is a congruence lattice of some sectionally complemented lattice L. In particular, if the length of D is n, (in notation $\ell(D) = n$), then L can be constructed such that $\ell(L) \leq 2n-1$.

E. T. Schmidt showed in 1974 that every finite distributive lattice is the congruence lattice of an infinite modular lattice.

J. Berman showed in 1975 that if D is a finite chain, then one can construct a lattice of length 5 such that Con $L \cong D$. This result was later improved by E. T. Schmidt to a finite distributive lattice D having only one dual atom.

In 1981, E. T. Schmidt showed that the ideal lattice of a distributive lattice with O is the congruence lattice of some lattice.

In 1985, Pudlák gave a new proof of Schmidt's result of 1981. His proof uses the concept of representation in category theory, which suggests a new line of attack to the converse of the theorem of Funayama and Nakayama.

In 1986, motivated by Pudlák's result, A. P. Huhn showed that the ideal lattice of every countable distributive join-semilattice with zero is the congruence lattice of some lattice.

In this thesis, we prove the following results:

(i) Let D be a distributive lattice having n dual atoms, then there is a finite lattice L such that $\ell(L) \leq 5n$ and Con L $\cong$ D. Conversely, given any positive integer n, there exists a finite distributive lattice D_n such that if L is a finite lattice and Con L $\cong D_n$, then $\ell(L) \geq 5n$. Hence the bound obtained is best possible.

(ii) If D has n dual atoms, and L is a finite sectionally complemented lattice with Con L $\cong$ D, then $\ell(L) \geq 2|J(D)| - n$.

(iii) Let D be an algebraic distributive lattice such that every element is the join of some join-irreducibles and J(D) is countable. Then there exists a planar lattice L such that Con L $\cong$ D. In particular, if D is finite, then $|L|$ is of order $|J(D)|^2$. (see also [Gr, La-1]).

(iv) We enumerate all congruence lattices of lattices of length at most 4.

(v) By using the approach as proposed in Pudlák's paper [Pu], We show that the ideal lattice of a distributive join-semilattice with zero is the congruence lattice of some lattices. The proof is different from that of A. P. Huhn's.

Results (i), (ii) and (iii) are presented in Chapter II; (iv) is presented in Chapter III, and (v) is presented in Chapter IV.

In a series of papers, K. Reuter, and R. Wille study the concept lattices which lead to the notion of complete congruence relation.

Let L be a complete lattice. A congruence relation Θ of L is called a *complete congruence relation* if and only if $x_i \, \Theta \, y_i$, $i \in I$, implies that $\bigvee x_i \, \Theta \bigvee y_i$ and $\bigwedge x_i \, \Theta \bigwedge y_i$. The lattice of the complete congruence relations of L is a complete lattice and is denoted by Com L. K. Reuter and R. Wille proved the following result: Let D be a complete distributive lattice in which each element is a supremum of join-irreducible elements. Then there exists a complete lattice L, such that Com L $\cong$ D.

In contrast with the congruence lattice, the complete congruence lattice of a complete lattice L is not neccessary algebraic nor distributive. Examples can be found in [Re, Wi].

In Chapter V, we answer a question raised in [Re, Wi] in the affirmative. We show that every finite lattice can be represented as the complete congruence lattice of a complete lattice. The construction can be modified to work for the infinite case. The same result was also proved by G. Grätzer [Gr-2].

Lastly, all the theorems and lemmata are numbered consecutively in each chapter, e.g. Theorem 2.2 refers to Theorem 2 of Chapter II. Similarly, Figure 2.1 refers to Figure 1 of Chapter II. The end of a proof is marked with the symbol ❑.

Chapter I
Notation and Preliminaries

A *lattice*, as an algebra, is written as L = (L; $\vee$, $\wedge$), where $\vee$ and $\wedge$ denote respectively the join and meet operations. In this thesis, a *semilattice* will always mean a join-semilattice. A lattice (or semilattice) can also be considered as a poset (L; $\leq$) where $x \leq y$ if and only if $x \vee y = y$, and for every two elements x and y, there is a least upper bound and a largest lower bound (least upper bound). I use both definitions whichever is convenient. For a subset S of a poset L, the *supremum* and *infimum* of S in L are denoted by $\bigvee S$ and $\bigwedge S$, respectively. The *zero* and the *unit* elements of a lattice L are denoted by O and I, respectively. A lattice is *bounded* if it has both the zero and the unit element.

Let L be a lattice. We use the notation $x \prec y$ to mean that y covers x in L. An element x is called an *atom* if L has a O and $O \prec x$. A lattice L is said to be *atomistic* if every element is the join of some atoms. The interval $\{ x \leq z \leq y \mid z \in L \}$ is denoted by [x, y]. For $x \in L$, the *principal ideal* (*principal dual-ideal*) generated by x is denoted by (x] ([x)). The ideal lattice of L is denoted by $\mathcal{I}(L)$, and the dual-ideal lattice is denoted by $\mathcal{I}^d(L)$. An interval [x, y] is called prime if $x \prec y$. An ideal (dual-ideal) S of L is called prime if $x \wedge y \in S$ ($x \vee y \in S$) implies that $x \in S$ or $y \in S$. The cardinality of L is denoted by $|L|$. The *length* of a finite chain C is $|C| - 1$. The length of a finite lattice L, denoted by $\ell(L)$, is the length of a chain of maximum length. We also use the same notation for similar notions in posets. A lattice is called *discrete* if every interval has finite length. We use the notation $\mathbf{M}_3$ and $\mathbf{N}_5$ for the standard lattices, the Diamond and the

Pentagon, respectively (Figure 1.1). The symbol $\cong$ is used for isomorphism between lattices and between posets.

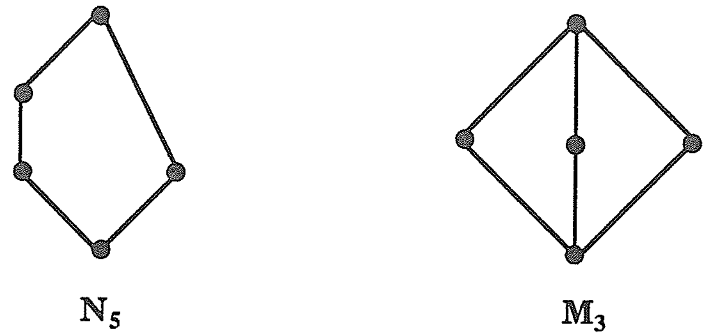


Figure 1.1

An element x of a lattice L is called *compact* if $x \leq \bigvee S$, $S \subseteq L$ implies the existence of a finite set F, $F \subseteq S$ such that $x \leq \bigvee F$. A lattice L is called *algebraic* if and only if it is complete and every element of L is the join of some compact elements of L. The set of all compact elements of L is a join-semilattice and is denoted by L^c. It is well-known that if L is algebraic, then $L \cong \mathcal{I}(L^c)$.

The lattice of all the congruence relations of a lattice L is denoted by Con L. A compact element of Con L is called a *compact congruence*. The *principal congruence* $\theta(x, y)$, $x, y \in L$, is the smallest congruence relation Θ such that $x \equiv y\ (\Theta)$. It is a basic fact that every compact congruence is a finite join of principal congruences. The congruence class of Θ containing x is denoted by $[x]\Theta$. The homomorphic image of a lattice L under the congruence relation Θ is denoted by L / Θ.

One of the basic concepts in the study of congruences of lattices is the notion of *weak-perspectivity* and *weak-projectivity*. We say that [a, b] is weakly perspective into [c, d] if (i) $c \wedge b \geq d$ and $c \vee b = a$, or (ii) $a \wedge d = b$ and $a \vee d \leq c$. Weak-projectivity is the transitive extension of weak-perspectivity. The notation $[a, b] \rightarrow [c, d]$ means [a, b] is weakly projective into [c, d]. The relationship between congruence relation and weak-projectivity is shown in the following theorem.

<u>Theorem 1.1</u> [Di] Let L be a lattice, a, b, c, d $\in$ L, $b \leq a$, and $d \leq c$. Then $c \equiv d$ ($\theta(a, b)$) iff there is a sequence of intervals : $[e_0, e_1], [e_1, e_2], \ldots, [e_k, e_{k+1}]$ from c to d with $c = e_0$ and $e_{k+1} = d$ such that $[e_i, e_{i+1}] \rightarrow [a, b]$ for $i = 0,1,\ldots, k$.

Let L be a discrete lattice and let H_L be the set of all the prime intervals of L. For [a, b], [c, d] $\in H_L$, we say that $[a, b] \sim [c, d]$ if and only if $[a, b] \rightarrow [c, d]$ and $[c, d] \rightarrow [a, b]$. Then $\sim$ is an equivalence relation and $\rightarrow$ induces a partial order relation on $H_L/\sim$.

Let L be a lattice, an element x ($\neq$ O) is called a *join-irreducible element* of L if $x \leq y \vee z$ implies that $x \leq y$ or $x \leq z$. The set of all the join-irreducible elements of L is denoted by J(L). (J(L); $\leq$) inherits the natural partial order of L. If L is a finite distributive lattice, then J(L) is non-empty and $L \cong \mathcal{J}(J(L))$. However, if L is an infinite distributive lattice, we have the following [St] analogous result: every dual-ideal I of a distributive lattice is the intersection of all the prime dual-ideals containing it.

Lemma 1.2 Let L be a discrete lattice. Then $J(\mathrm{Con}\ L) \cong (H_L/\sim, \rightarrow)$.

Proof: Let $\Theta \in \mathrm{Con}\ L$. Since L is discrete, $\Theta = \bigvee (\theta(x, y) \mid x \equiv y\ (\Theta), x \leq y)$ $= \bigvee (\theta(x, y) \mid x \equiv y\ (\Theta), [x, y] \in H_L)$. Since Con L is distributive, $\theta(x, y) \in J(\mathrm{Con}\ L)$ if $[x, y] \in H_L$. Hence $J(\mathrm{Con}\ L) \cong (H_L/\sim, \rightarrow)$. ❑

Let L be a bounded lattice. An element a is a *complement* of b if $a \wedge b = O$ and $a \vee b = I$. A lattice is complemented if every element has a complement. A Boolean lattice is a complemented distributive lattice and 2^n denotes a Boolean lattice generated by n atoms. A generalized Boolean lattice is a relatively complemented distributive lattice with zero. A *sectionally complemented lattice* is a lattice with O and all intervals [O, a] are complemented. In a sectionally complemented lattice L, there is a one to one correspondence between the congruences of L and certain ideals of L. A finite sectionally complemented lattice is also atomistic.

A *context* is a triple (G, M, I) where G, M are sets and $I \subseteq G \times M$ is a binary relation. For all $A \in G$, $B \in M$, the closure of A and B are the sets:

$A^* = \{ m \mid m \in M \text{ and } (g, m) \in I, \forall g \in G \}$,

$B^* = \{ g \mid g \in G \text{ and } (g, m) \in I, \forall m \in M \}$.

An ordered pair (A, B) is called a concept if $A^* = B$ and $B^* = A$. We define a partial order relation on the set of all concepts by the rule: $(A, B) \leq (C, D)$ if and only if $A \subseteq C$. The set of all concepts of (G, M, I) with the given partial order is a complete lattice, denoted by L(G, M, I).

A subcontext (H, N, J) of (G, M, I) is a context such that $H \subseteq G$, $N \subseteq M$ and $J = I \cap (H \times N)$. The subcontext (H, N, J) is said to be compatible if the following conditions are satisfied:

(i) for all $h \in H$, and $m \in M$, $m \in M \setminus \{ h \}^*$ implies that there is an $n \in N \setminus \{ h \}^*$ and $m^* \subseteq n^*$.

(ii) for all $m \in N$, and $g \in G$, $g \in G \setminus \{ m \}^*$ implies that there is an $h \in N \setminus \{ m \}^*$ and $h^* \subseteq g^*$.

The subcontext (H, N, J) is also said to be saturated if

(i) for all $g \in G$, $X \subseteq H$, $\{ g \}^* = X^*$ implies that $g \in H$, and

(ii) for all $m \in M$, $Y \subseteq N$, $\{ m \}^* = Y^*$ implies that $m \in N$.

The set of all compatible and saturated subcontexts of (G, M, I) is denoted by $\Gamma(G, M, I)$. It is given a partial order relation $\leq$ by $(H_1, N_1, J_1) \leq (H_2, N_2, J_2)$ if and only if $H_1 \subseteq H_2$ and $N_1 \subseteq N_2$.

It can be shown that a subcontext (H, N, J) is compatible iff the mapping $A : L(G, M, I) \rightarrow \mathcal{P}(H) \times \mathcal{P}(N)$ given by $(A, B) \rightarrow (A \cap H, B \cap N)$ is a *complete homomorphism*. This gives rise to the definition of complete congruence relations as mentioned in Chapter O, page 4. Our result of Chapter V was motivated by the following theorem:

<u>Theorem 1.3</u> [Re, Wi] Under certain conditions, there is an anti-isomorphism between Com L(G,M,I) and Γ(G,M,I).

Chapter II
Congruence lattices

In this chapter, for a finite distributive lattice D, we construct lattices L with Con L $\cong$ D and L satisfying certain numeric conditions. We prove the followings:

(i) If D is finite and has n dual atoms, then there is a finite lattice L such that $\ell(L) \leq 5n$ and Con L $\cong$ D.

(ii) Given any positive integer n, there exists a finite distributive lattice D_n such that if L is a finite lattice and Con L $\cong D_n$, then $\ell(L) \geq 5n$. Hence the bound obtained in (i) is best possible.

(iii) If D is finite and has n dual atoms, and L is a finite sectionally complemented lattice with Con L $\cong$ D, then $\ell(L) \geq 2|J(D)| - n$.

(iv) If J(D) is finite, then there is a finite planar lattice L such that Con L $\cong$ D and $|L|$ is of order $|J(D)|^2$. The first statement also holds for countable J(D) in which every element of D is the join of elements of J(D).

Statement (i) is presented as Theorem 2.2. This answers the question raised in [Sc-2] in the affirmative. Statement (ii) is Theorem 2.6. Statement (iii) is Theorem 2.8 and Statement (iv) is Theorem 2.9.

1. The Construction of L with $\ell(L) = 5n$

The construction of L is given in two parts. We first present a simplified construction of E. T. Schmidt [Sc-2] when D has only one dual atom. Secondly, we extend this to obtain the required lattice for the general case.

(A) D has only one dual atom.

Let A be the maximal element of J(D). If (A] = {A}, we take L to be any simple lattice of length five. Otherwise, let

$(A) = J(D) - \{A\} = \{ B_1, B_2, \ldots, B_k \}$.

For each i = 1, 2, ..., k, let

$P_i = \{ m_i, n_i, \ell_{ij} \mid j \in \mathcal{J}_i \}$, where $\mathcal{J}_i = \{ s \mid 1 \le s \le k, B_i \succ\!\!- B_s \} \cup \{ k+1, k+2 \}$.

The elements of P_i are ordered as follows:

$I \succ\!\!- \ell_{ij} \succ\!\!- m_i \succ\!\!- n_i \succ\!\!- O$.

For each $B_i \succ\!\!- B_j$, let

$Q_{ij} = \{ u_{ij}, v_{ij} \}$ and let

$u_{ij} \succ\!\!- v_{ij}$, $\ell_{ij} \succ\!\!- u_{ij} \succ\!\!- m_j$ and $v_{ij} \succ\!\!- n_i, n_j$.

Let w_1, w_2 be such that

$I \succ\!\!- w_1, w_2 \succ\!\!- O$ and let L be the set:

$\cup (P_i \mid i = 1, 2, \ldots, k) \cup \cup (Q_{ij} \mid B_i \succ\!\!- B_j, i = 1, \ldots, k) \cup \{I, O, w_1, w_2\}$.

Let the covering relation of the elements of L be precisely those given above, then L is a lattice.

For example, let (A] be the poset as shown in Figure 2.1(a). Then L is the lattice as shown in Figure 2.1(b).

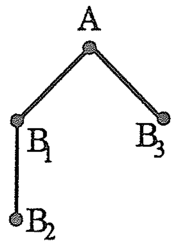


Figure 2.1(a)

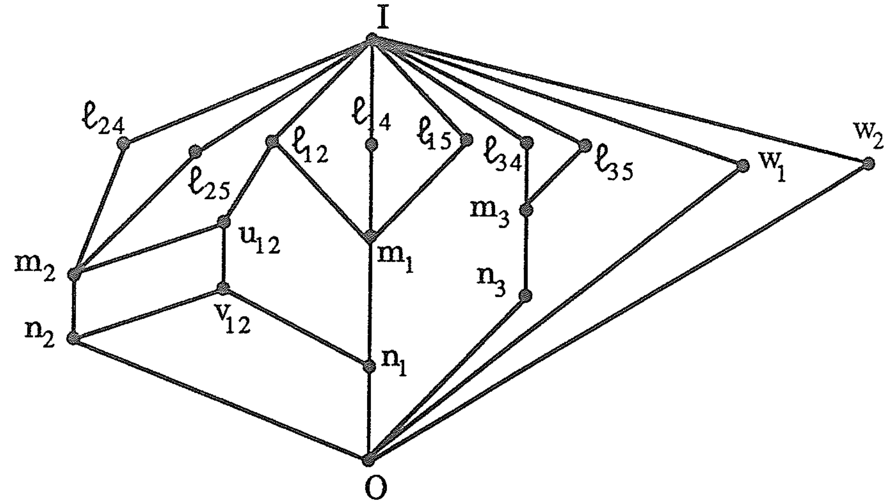


Figure 2.1(b)

It is easy to check that $\theta(O, w_1) = \theta(O, w_2) = \theta(w_1, I) = \theta(w_2, I) = \theta(O, n_i) = \theta(n_i, v_{ij}) = \theta(n_j, v_{ij}) = \theta(\ell_{ij}, I) = \theta(m_i, \ell_{ij}) = \theta(u_{ij}, m_i) = \iota$. The other prime interval congruences are $\theta(m_i, n_i) = \theta(\ell_{ij}, u_{ij})$ and $\theta(u_{ij}, v_{ij}) = \theta(m_j, n_j)$ for $j \in \mathcal{J}_i - \{ k+1, k+2 \}$. The congruence classes of $\theta(m_i, n_i)$ are:

$$\{\{m_r, n_r\} \mid B_r \leq B_i \} \cup \{\{\ell_{rt}, u_{rt}, v_{rt} \} \mid B_t \prec B_r \leq B_i \} \cup \{\{u_{ji}, v_{ji} \} \mid B_i \prec B_j \}.$$

It follows that $\theta(m_i, n_i) \geq \theta(m_j, n_j)$ if and only if $B_i \geq B_j$, and

$$H_L/\!\sim\; = \{ \theta(O, w_1), \theta(m_i, n_i) \mid i = 1, 2, \ldots, k \}.$$

The mapping $\Psi : (A] \to H_L/\!\sim$ given by $\Psi(A) = \theta(O, w_1)$, and $\Psi(B_i) = \theta(m_i, n_i)$ is an isomorphism. Thus $J(\text{Con } L) \cong (A]$ by Lemma 0.4. ❑

<u>(B) D has n dual atoms</u>

Let D be a finite distributive lattice, then D has n dual atoms if and only if J(D) has n maximal elements. Let

$J(D) = \{ A_0, A_1, ..., A_{n-1}, B_0, B_1, ... , B_{k-1} \}$ where $A_0, A_1, ..., A_{n-1}$ are the maximal elements.

We can construct lattices $L_0, L_1, ..., L_{n-1}$ as described in (A) such that $J(\text{Con } L_i) \cong (a_i]$ for $i = 0, 1, ..., n-1$.

Let $L = L_0 \oplus L_1 \oplus \cdots \oplus L_{n-1}$. Then $J(\text{Con } L) = \bigcup ((A_i] \mid i = 0, 1, ..., n-1)$. We shall label the elements of L by attaching a subscript j to each element of L_j, $j = 0, 1, ..., n-1$. For each element $B_i \in J(D)$, let $\mathcal{J}^{(i)}$ be the set $\{ j \mid B_i \in (A_j] \}$ and let $B_i^{(j)}$, $j \in \mathcal{J}^{(i)}$ denote the copy of B_i in $(A_j]$. We shall construct a lattice L^* which precisely identifies all the $B_i^{(j)}$'s, $j \in \mathcal{J}^{(i)}$ to B_i and preserves the ordering relations of the B_i's. Without loss of generality, we can assume that $|\mathcal{J}^{(i)}| \geq 2$ for each i. For each $i = 0, 1, ..., k-1$, let C_i be the chain of L consisting of the set of elements:

$$\bigcup (\{O_t\} \mid t = 0, 1, ..., n-1) \cup \{ I_{n-1} \} \cup \bigcup (\{m_{i,j} , n_{i,j}\} \mid j \in \mathcal{J}^{(i)}) \qquad \textbf{(II-1)}.$$

We first prove a lemma which will be useful in our construction. Let L_1 and L_2 be finite lattices, $L_1 \cap L_2 = \emptyset$. For $i = 1, 2$, let C_i be a $\{0,1\}$–sublattice of L_i. Let $\varphi : C_1 \to C_2$ be an isomorphism between C_1 and C_2. Let L be the set obtained from $L_1 \cup L_2$ by the identification $u \equiv \varphi(u)$ for all $u \in C_1$.

For $x, y \in (L_1 \cup L_2)/\equiv$, we define $x \leq y$ if and only if one of the following conditions holds:

(i) $x \leq_1 y$ for $x, y \in L_1$;

(ii) $x \leq_2 y$ for $x, y \in L_2$;

(iii) $x \leq_1 u \equiv \varphi(u) \leq_2 y$, for $x, u \in L_1$, $y \in L_2$;

(iv) $x \leq_2 u \equiv \varphi^{-1}(u) \leq_1 y$, for $x, u \in L_2$, $y \in L_1$.

Lemma 2.1

(i) $(L, \leq)$ is a poset.

(ii) If C_1 and C_2 are chains, then $(L, \leq)$, also denoted by $G[L_1, L_2, \varphi]$, is a lattice.

We first remark that the condition that C_1 and C_2 be chains cannot be replaced by lattices in general. For example, let $L_1 \equiv L_2 = \mathbf{2}\times\mathbf{3}$, $\varphi(0, 0) = (0, 0)$, $\varphi(1,2) = (1, 2)$, $\varphi(0, 2) = (1, 0)$, and $\varphi(1, 0) = (0, 2)$; but $L_1 \cup L_2/\equiv$ is not a lattice.

Proof:

(i) It is easy to verify that $\leq$ is reflexive, anti-symmetric, and transitive on L.

(ii) Let $x, y \in L_1 \cup L_2$ and let U be the set of upper bounds of x and y in L. Let $U_1 = U \cap L_1$ and $U_2 = U \cap L_2$. Clearly, both U_1 and U_2 are non-empty. By symmetry, we need only consider the join of x and y in the following two cases. The meet of x and y can be proved dually.

Case 1. $x \in L_1$ and $y \in L_2$

Suppose that $p, q \in U_2$. Since $x \leq p, q$, there exist $u, v \in L_1$ such that $x \leq_1 u \equiv \varphi(u) \leq_2 p$, $x \leq_1 v \equiv \varphi(v) \leq_2 q$. Thus $x \leq_1 u \wedge_1 v \equiv \varphi(u) \wedge_2 \varphi(v) \leq p \wedge_2 q$.

Similarly $y \leq p \wedge_2 q$. Therefore U_2 is a dual ideal of L_2. By the same argument, U_1 is a dual ideal of L_1. Let $[r) = U_1$ and $[s) = U_2$. Suppose that neither $r \leq s$ nor $s \leq r$ in L. Since $x \leq s \in L_2$, there exists $r' \in L_1$ such that $x \leq_1 r' \equiv \varphi(r') \leq_2 s$. Similarly, there exists an $s' \in L_2$ such that $y \leq_2 s' \equiv \varphi^{-1}(s')$ $\leq_1 r$. Since C_1 and C_2 are chains, we have $r' < \varphi^{-1}(s')$ or $\varphi^{-1}(s') < r'$. By symmetry, we may assume that $r' < \varphi^{-1}(s') \equiv s' > \varphi(r')$. Since r and s are not comparable in L, $s' \neq s$, we have $s > s' \wedge_2 s \geq \varphi(r')$ and $s' \wedge_2 s \in U_2$. This contradicts the definition of s. Thus $r \leq s$ or $s \leq r$ and the join of x and y exists in L.

<u>Case 2.</u> $x \in L_1$ and $y \in L_1$

Let $z = x \vee_1 y$. Let $s \in L_2$ such that $s \in U_2$, where U_2 is as defined in (i). We show that $z \leq s$ in L, i.e, $x \vee y = z$. Let $x', y' \in L_1$ and $x \leq_1 x' \equiv \varphi(x') \leq_2 s$, $y \leq_1 y' \equiv \varphi(y') \leq_2 s$. We have $z \leq_1 x' \vee_1 y' \equiv \varphi(x') \vee_2 \varphi(y') \leq_2 s$ in L. Thus it remains to show the U_2 is a dual ideal of L_2. This follows from the fact that if s_1 and s_2 are elements of U_2, then $x \vee_1 y \leq s_1$ and s_2, i.e., $x \vee_1 y \leq s_1 \wedge s_2$.

From the above proof, we can describe the join and meet of L. For $x \in L_i$, $i = 1, 2$, let $x^+ \in L_j$, $j \neq i$, be the least element in C_j such that $x^+ \geq x$. For x, y $\in L_1 \cup L_2/\equiv$, by symmetry, we state the join of x and y in following two cases:

(a) If $x, y \in L_i$, $i = 1, 2$, $x \vee y = x \vee_i y$.

(b) If $x \in L_i$, $y \in L_j$, $i \neq j$, and $y^+ \geq x^+$, then $x \vee y = y \vee_j x^+$.

The meet of the elements of L can be obtained dually. ❑

Let $\mathcal{J}^{(i)} = \{ \alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_p^{(i)} \}$ where $0 \le \alpha_1^{(i)} < \alpha_2^{(i)} < \ldots < \alpha_p^{(i)} \le n\text{-}1$. Let $\mathbf{n}$ be the chain $0 \prec 1 \prec 2 \prec \ldots \prec n\text{-}1 \prec n$. Let the interval [i, i+1], i = 0, 1, ..., n-1 be given the colour A_i. We define the following two chains:

P_i : This is the chain obtained from $\mathbf{n}$ by augmenting the interval $\alpha_r^{(i)} \prec \alpha_r^{(i)}+1$ to the interval $\alpha_r^{(i)} \prec \alpha_r^{(i)-} \prec \alpha_r^{(i)+} \prec \alpha_r^{(i)}+1$ for each $\alpha_r^{(i)} \in \mathcal{J}^{(i)} - \{ \alpha_1^{(i)} \}$. The new intervals $[\alpha_r^{(i)}, \alpha_r^{(i)-}]$, $[\alpha_r^{(i)-}, \alpha_r^{(i)+}]$ and $[\alpha_r^{(i)+}, \alpha_r^{(i)}+1]$ are given the colours $A_{\alpha_r(i)}$, B_i and $A_{\alpha_r(i)}$ respectively.

Q_i : This is the chain obtained from $\mathbf{n}$ by augmenting the interval $\alpha_1^{(i)} \prec \alpha_1^{(i)}+1$ to the interval $\alpha_1^{(i)} \prec \alpha_1^{(i)-} \prec \alpha_1^{(i)+} \prec \alpha_1^{(i)}+1$. The new intervals $[\alpha_1^{(i)}, \alpha_1^{(i)-}]$, $[\alpha_1^{(i)-}, \alpha_1^{(i)+}]$ and $[\alpha_1^{(i)+}, \alpha_1^{(i)}+1]$ are given the colours $A_{\alpha_1(i)}$, B_i and $A_{\alpha_1(i)}$ respectively.

Let the elements (. , .) of $P_i \times Q_i$ be labelled by $x_i(.\,,.)$. We add a new element $y_i(u,v)$ to each interval $[x_i(s,t), x_i(u,v)]$ such that $s \prec u$, $t \prec v$, and $[s,u] \in P_i$, $[t,v] \in Q_i$ have the same colour. Let the resulting lattice be denoted by M_i. For example, let n = 5 and $\mathcal{J}^{(i)} = \{1, 2, 4\}$, then M_i is depicted in Figure 2.2.

Let $D_i \subset M_i$ be the chain consisting of the following set of elements:

$$\bigcup \left(\{x_i(r, r)\} \;\middle|\; r = 0, 1, \ldots, n \right) \cup \{x_i(\alpha_1^{(i)}, \alpha_1^{(i)-}), x_i(\alpha_1^{(i)}, \alpha_1^{(i)+})\} \cup$$
$$\bigcup \left(\{x_i(\alpha_r^{(i)-}, \alpha_r^{(i)}), x_i(\alpha_r^{(i)+}, \alpha_r^{(i)})\} \;\middle|\; \alpha_r^{(i)} \in \mathcal{J}^{(i)} - \{\alpha_1^{(i)}\} \right).$$

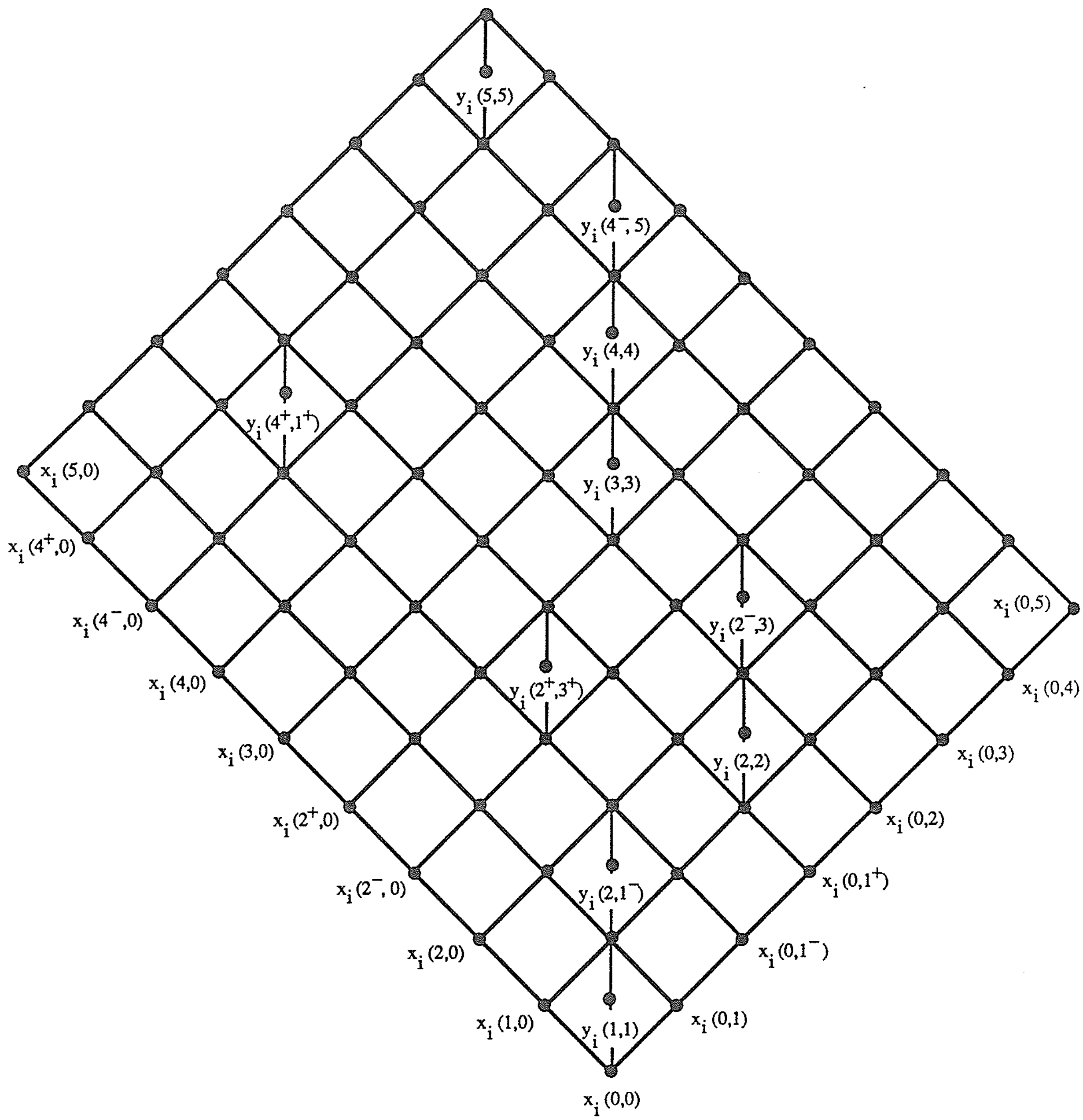

y_i (5,5)
y_i (4^-, 5)
y_i (4,4)
y_i (4^+,1^+)
y_i (3,3)
x_i (5,0)
x_i (4^+,0)
x_i (4^-,0)
x_i (4,0)
x_i (3,0)
x_i (2^+,0)
x_i (2^-, 0)
x_i (2,0)
x_i (1,0)
x_i (0,0)
y_i (1,1)
y_i (2,1^-)
y_i (2^+,3^+)
y_i (2^-,3)
y_i (2,2)
x_i (0,5)
x_i (0,4)
x_i (0,3)
x_i (0,2)
x_i (0,1^+)
x_i (0,1^-)
x_i (0,1)

Figure 2.2

Let $\varphi_i : C_i \to D_i$ be an isomorphism, for i = 0, 1 ,..., k-1; where C_i is the chain of L as stated in (II-1). (see pg. 13.)

The equivalence class of a prime interval of M_i is determined by the colour of its projection on P_i or Q_i. It is not difficult to see that the prime-interval congruence classes $M_i/\sim$ form a totally disconnected poset consisting of the elements A_r, r = 0, 1,..., n-1, and B_i.

In the following, we give the definition of L^* and show that J(Con L^*) is isomorphic to J(D). Let $L^{(0)} = G[L, M_0, \varphi_0]$, and for each integer i = 0, 1, ..., k-2, we define $L^{(i+1)} = G[L^{(i)}, M_{i+1}, \varphi_{i+1}]$. Then $L^{(i+1)}$ is a lattice by Lemma 2. We define $L^* = L^{(k-1)}$.

For i = 0, 1,..., k-1, let $J^{(i)}$ be the set:

$$\{ A_r \mid r = 0, 1,..., n-1 \} \cup \{ B_r \mid r = 0, 1,..., i \} \cup \{ B_r^{(j)} \mid r = i+1, i+2,..., k-1, j \in \mathcal{J}^{(r)} \}.$$

$J^{(i)}$ is an augmented poset of J(D) such that the mapping $J^{(i)} \to J(D)$ given by $A_r \to A_r$, $B_r \to B_r$, $B_r^{(j)} \to B_r$ is order preserving and $B_r^{(j)} \le B_s^{(j')}$ iff $B_r \le B_s$. Clearly, J(Con $L^{(0)}$) $\cong J^{(0)}$ and $J^{(k-1)} \cong$ J(D). We show inductively that for all i = 0, 1,..., k-1, J(Con $L^{(i)}$) $\cong J^{(i)}$. For a lattice K, let $A_r(K)$ denotes the set of all prime intervals which generate the congruence A_r in K. We define inductively on j, j = 0, 1, ..., k-1, (notation: $L^{(-1)} = L$) the followings:

$$A_r(L^{(j)}) = A_r(L^{(j-1)}) \cup A_r(M_j), \text{ for } r = 0, 1,..., n-1;$$
$$B_r(L^{(j)}) = B_r(L^{(j-1)}), \text{ for } r = 0, 1,..., j-1;$$
$$B_j(L^{(j)}) = B_j(M_j) \cup \bigcup (B_j^{(s)}(L^{(j-1)}) \mid s \in \mathcal{J}^{(j)});$$
$$B_r^{(s)}(L^{(j)}) = B_r^{(s)}(L^{(j-1)}), \text{ for } r = j+1, ..., k-1, s \in \mathcal{J}^{(r)}.$$

To show that these form the prime-interval congruence classes of $L^{(j)}$ isomorphic to $J^{(j)}$, we have to verify that under projectivity, each element of a class (-) is projected to an interval whose prime intervals are in the ideal generated by the class (-). By using the join and meet of the elements of $G[L^{(i-1)}, M_i, \varphi_i]$ as described in Lemma 2.1, we summarize the computation of the equivalence classes of the prime-interval congruences under projectivity as follows:

For r = 0, ..., n-1,

$$A_r(L^{(j)}) \to A_r(L^{(j)}) \cup \bigcup (B_t(L^{(j)}) \mid B_t \le A_r) \cup \bigcup (B_t^{(s)}(L^{(j)}) \mid s \in \mathcal{J}^{(t)}, j < t, B_t \le A_r).$$

For r = 0, ..., j,

$$B_r(L^{(j)}) \to B_r(L^{(j)}) \cup \bigcup (B_t(L^{(j)}) \mid B_t \le B_r) \cup \bigcup (B_t^{(s)}(L^{(j)}) \mid s \in \mathcal{J}^{(t)}, j < t, B_t \le B_r).$$

For r = j+1, ...,k-1,

$$B_r^{(s)}(L^{(j)}) \to B_r^{(s)}(L^{(j-1)}) \cup \bigcup (B_t(L^{(j)}) \mid B_t \le B_r) \cup \bigcup (B_t^{(s)}(L^{(j)}) \mid s \in \mathcal{J}^{(t)}, j < t, B_t \le B_r).$$

Therefore $H_{L^{(j)}}/\sim \,\cong J^{(j)}$, for j = 0, 1, ..., k-1; and $H_{L*}/\sim$ is isomorphic to J(D). Hence we have proved:

<u>Theorem 2.2</u> Let D be a finite distributive lattice such that J(D) has n maximal elements. Then there is a finite lattice L of length 5n such that Con L is isomorphic to D.

2. The Lower Bound of $\ell(L)$ and its Congruence Lattice

In this section, we show that in general, the bound on the length of L given by Theorem 1 is best possible. Let A be a partially ordered set, we define

$$\partial(A) = \min \{ \ell(L) \mid J(\mathrm{Con}\, L) \cong A \}.$$

Lemma 2.3 Let L be a finite lattice and let $J(\mathrm{Con}\, L) \cong A_1 \cup A_2 \cup \dots \cup A_k$ where the A_i's are disjoint posets. Let $\Theta_i = \vee (\theta \mid \theta \notin A_i)$ and $L_i = L/\Theta_i$. Then L is a subdirect product of L_i's. The mapping $\pi : L \to L_1 \times \dots \times L_k$ given by $x \to (x_1, \dots, x_k)$ is an embedding, where $x \to x_i$ is the canonical projection $\pi_i : L \to L_i$. Furthermore $\ell(x] = \ell(x_1] + \dots + \ell(x_k]$.

Proof: Since the A_i's are disjoint, we have $\Theta_i = \bigvee (\theta \mid \theta \notin A_i) = A_i^c$, where A_i^c is the ideal generated by the complement of A_i. Thus $\bigwedge \Theta_i = \bigcap A_i^c = \varnothing$ and the mapping $x \to (x_1, \dots, x_k)$ is an embedding. Let $[x, y]$ be a prime interval of L, then $\theta(x, y) \in A_j$ for some unique j and $x_i = y_i$ for all $i \neq j$. Furthermore, $[x_j, y_j]$ is a prime interval in L_j. To prove the second statement, we apply induction on the elements of L. If $x = O$ or x is an atom of L, the second statement is obviously true. Suppose the second statement has been proved for all $x \prec y$. We have

$$\begin{aligned} \ell(y] &= \max \{ \ell(x] + 1 \mid x \prec y \} \\ &= \max \{ \max \{ \ell(x] + 1 \mid x \prec y, \theta(x, y) \in A_i \} \mid i = 1, \dots, k \} \\ &= \ell(x_1] + \dots + (\ell(x_i] + 1) + \dots + \ell(x_k] \\ &= \ell(y_1] + \dots + \ell(y_i] + \dots + \ell(y_k] \end{aligned}$$

□

As a consequence of Lemma 2.3, we have

Lemma 2.4 Let L be a discrete lattice and J(Con L) $\cong A_1 \cup A_2 \cup \ldots \cup A_k$, then $\ell(L) \geq \partial(A_1) + \partial(A_2) + \ldots + \partial(A_k)$.

Lemma 2.5 Let A be the chain $c_0 \prec c_1 \prec c_2 \prec c_3 \prec c_4 \prec c_5$ and let L be a lattice such that J(Con L) $\cong$ A. Then $\ell(L) \geq 5$.

Proof: Suppose that $\ell(L) \leq 4$. There exist prime intervals $[a, b] \in c_1$ and $[c, d] \in c_0$ such that the sublattice generated by them contains an N_5. We can assume that a = 0 since $\ell(L) \leq 4$. We have $\ell(L/\theta(a, b)) \leq 3$ and J(Con $L/\theta(a, b)$) is the chain $c_2 \prec c_3 \prec c_4 \prec c_5$. By using the same argument for $L/\theta(a, b)$, we obtain a lattice of length two whose congruence lattice is a chain of length 3, which is absurd. Thus $\ell(L) \geq 5$. ❑

Theorem 2.6 For any integer n, there exists a finite distributive lattice D_n such that $J(D_n)$ has n maximal elements and any lattice L whose congruence lattice is isomorphic to D_n has length at least 5n.

Proof: This is an easy consequence of Lemma 2.3 and Lemma 2.4. ❑

Given a finite distributive lattice D, a sectionally complemented lattice L with Con L $\cong$ D was constructed in [Gr & Sc]. Such a lattice has length $\leq 2n-1$, where n is the length of D. In Theorem 2.8, we give a lower bound of the length of L. We first prove a Lemma.

__Lemma 2.7__ Let L be a finite lattice and $\Theta \in$ Con L. Let $(a] = [0]\Theta$, $[b) = [1]\Theta$ and $L' = L/\Theta$ then

(i) if $a \leq b$, then $\ell(L') \leq \ell[a, b] \leq \ell(L) - \ell(a] - \ell[b)$.

(ii) if a and b are not comparable, and $\ell(a] \geq \ell[b)$, then $\ell(L') \leq \ell[a)$.

(iii) if Θ is isolated in J(Con L), then $\ell(L') \leq \ell(L) - 1$.

Proof:

(i) For each congruence class of Θ, we can choose a representative $x \in L$ such that $a \leq x \leq b$. Thus $\ell(L') \leq \ell[a, b] \leq \ell(L) - \ell(a] - \ell[b)$.

(ii) Suppose $\ell(a] = \max \{ \ell(a], \ell[b) \}$. For each congruence class of Θ, we choose a representative x which is the maximal element of the class, then $x \geq a$. Thus $\ell(L') \leq \ell[a) \leq \ell(L) - \ell(a]$.

(iii) Let S be any maximal chain of L such that $\ell(S) = \ell(L)$. Then S must contain a prime interval $p \in p^* = \Theta$, for otherwise $\Theta \leq \bigvee(\Theta_i \mid \Theta_i \neq \Theta)$ which is impossible by distributivity. ❑

__Theorem 2.8__ Let D be a finite distributive lattice having n dual atoms, and L be a finite sectionally complemented lattice such that Con L $\cong$ D. Then $\ell(L) \geq 2|J(D)| - n$.

Proof: Since the homomorphic image of a sectionally complemented lattice is also sectionally complemented, we can apply induction on $|J(D)|$. The theorem is clearly true if $|J(D)| \leq 2$. So we assume that $|J(D)| \geq 3$. We may further assume that J(D) has no isolated element by Lemma 2.7(iii). Let L be a finite sectionally complemented lattice such that Con L $\cong$ D. We can partition the set of atoms of L into equivalence classes according to the congruence relations that they represent. Let u be a minimal element of

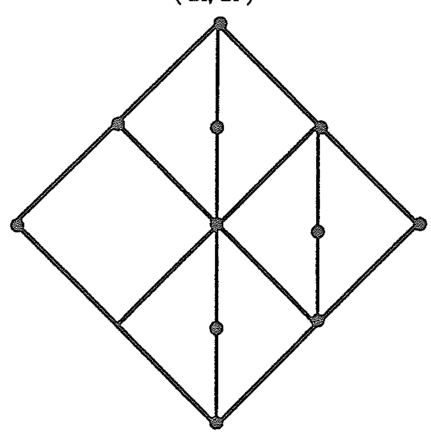
J(D) and let the class of atoms represented by u be C_u. If $|C_u| \geq 2$, we are done by Lemma 2.7 and the induction hypothesis on J(D) − {u}. So we assume that $|C_u| = 1$. Let $C_u = \{a\}$ and let v be the join of all atoms of L other than a. Suppose that v > a. The congruence θ(O, a) cannot collapse any prime interval of (v], for otherwise θ(O, a) collapses some [O, b], b < v which is not the case. Also, for any two distinct elements c, d ∈ (v], we have $a \vee c \neq a \vee d$ by the same reason. Thus L is isomorphic to a direct product of (v] × [O, a], which is not the case. Hence v > a. Let v' be a maximal element in (v] such the v' is not greater then a. Then there is an atom b < v such that $b \vee v' = v'' = v' \vee a > a$. This implies θ(O, a) = θ(O, b), a contradiction to the assumption that $|C_u| = 1$. Hence $|C_u| \geq 2$ and the proof of the theorem is complete. ❑

Instead of considering the length of a lattice, one can also ask the same problem about the cardinality of L. The sectionally complemented lattice constructed in [Gr, Sc] has exponential order. In the following theorem, we give a planar lattice whose cardinality has polynomial bound.

Theorem 2.9 Let D be an algebraic distributive lattice such that J(D) is countable and every element of D is the join of join-irreducibles. Then there exists a countable planar lattice L such that Con L ≅ D. In particular, if D is finite, then $|L|$ is of order $O(|J(D)|^2)$.

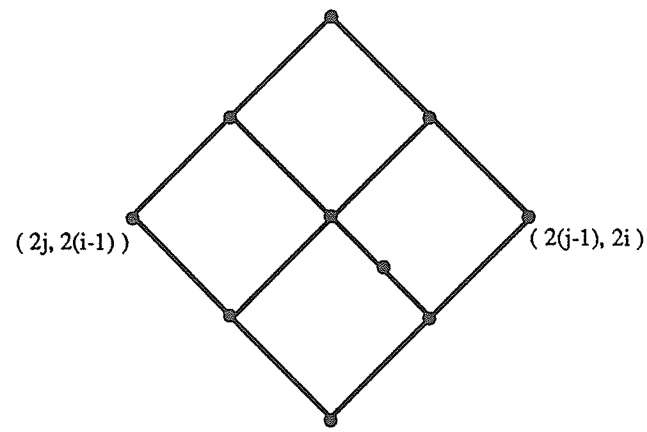
Proof: Let J(D) = { $a_1, a_2, \ldots$ }. For each i = 1, 2, ..., let S_i be the interval [(2(i-1), 2(i-1)), (2(i), 2(i))] of ω×ω. For each $a_j \prec a_i$, let T_{ij} be the interval [(2(j-1), 2(i-1)), (2(j), 2(i))]. We extend S_i and T_{ij} to S_i^+ and T_{ij}^+ as shown in Figure 2.3(a) and Figure 2.3(b).

Let L be the resulting lattice as shown in Figure 2.3(c). Then L is planar and countable. If D is finite, then $|L|$ is of order $O(|J(D)|^2)$. The mapping $\Psi : (J(D), \leq) \to (H_L/\sim, \to)$, with $a_i \to [(2(i-1), 2(i-1)), (2(i-1), 2i-1)]^*$, is an isomorphism. Hence $\mathrm{Con}\, L \cong D$. □



S_i^+

Figure 2.3(a)



T_{ij}^+

Figure 2.3(b)

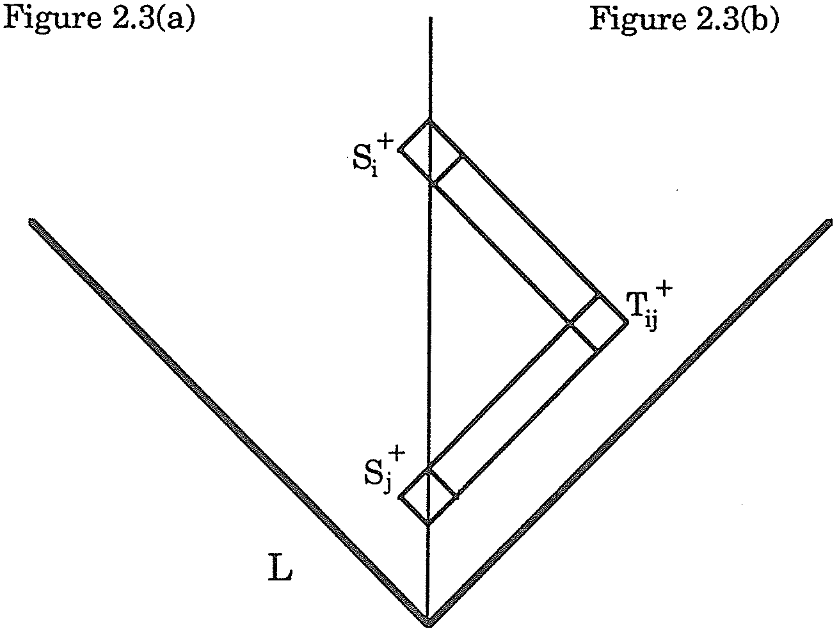


Figure 2.3(c)

Chapter III
Congruence Lattices of Lattices of Length ≤ 4

By applying the results of Chapter II, in particular, Lemma 2.4 and Lemma 2.7, we enumerate all the congruence lattices of lattices of length at most 4. The following lemma follows immediately from Lemma 2.4.

Lemma 3.1 Let L be a lattice of length n, $n \geq 2$; then
(i) Con L $\cong \mathbf{2}^n$ if and only if L is distributive,
(ii) Con L $\cong \mathbf{2}^{n-1}$ implies that L is modular.

For the purpose of the following discussion, we call a prime interval p = [a, b] *exterior* if either a = 0, or b = 1; otherwise it is called *interior*. The congruence class containing the prime interval p is denoted by p*. A class p* is called exterior if it contains some exterior prime interval, otherwise it is called interior. Let $f : L \to L'$ be an onto lattice homomorphism, it would be helpful to note that if p* is interior in L', then its preimages are also interior. By a cycle C of a lattice (or poset), we mean a sublattice (induced subposet) $C = \{ a, b, a_i, b_j \mid a_1 < \dots < a_n; b_1 < \dots < b_m; \sup(a_i, b_j) = b; \inf(a_i, b_j) = a;$ for $i = 1, \dots, n$ and $j = 1, \dots, m \}$.

Lemma 3.2 Let L be a lattice and $\ell(L) \leq 4$. Suppose that Con L $\neq \mathbf{2}^n$, $n \leq 4$. Then there exists prime intervals p and q such that p* $\succ$ q* and a cycle C (containing $\mathbf{N}_5$) containing p and q. Let L' = L/p*. Then
(i) p* is exterior;
(ii) $\ell(L') \leq \ell(L) - 1$ and if $\ell(L) = 3$, then $\ell(L') = 1$;
(iii) J(Con L') $\cong$ J(Con L) $-$ (p*].

Proof: Since Con L $\neq 2^n$, there are prime intervals p and q such that $p^* \succ q^*$. Thus there exists a cycle $C = \{0, a, b, c, d \mid a \vee b = a \vee c = d, a \wedge b = a \wedge c = 0\}$ containing p and q. Clearly C contains N_5 as a sublattice and $\ell(C) \geq 3$. Since $\ell(L) \leq 4$, we can assume that $0 \in C$ and $p = [0, a]$. Thus p^* is exterior. By Lemma 2.7(ii), $\ell(L') \leq \ell(L) - 1$. If $\ell(L) = 3$, then max $\{\ell(a], \ell[b)\} = 2$ in Lemma 2.7(ii), hence $\ell(L') = 1$. Clearly $J(\text{Con } L') \cong J(\text{Con } L) - (p^*]$. ❑

Proposition 3.3 Let L be a lattice such that $\ell(L) = 3$. Then Con L $\cong 2^n$, $n \leq 3$ or J(Con L) is isomorphic to one of the posets as shown below (Figure 3.1). Further more,

(i) if L has a maximal chain of length 2 and $L \neq N_5$, then $J(\text{Con } L) \cong P_1$ and q_i are interior for all $q_i \in q_i^*$, $i = 1, \ldots, n$ if $n \geq 2$. If $n = 1$, then either q_1 is interior, or L is the lattice (or its dual) given by:

$\{0, 1, a, b, c_1, \ldots, c_n \mid n \geq 2,\ 0 \prec a \prec 1, 0 \prec b \prec c_1, \ldots, c_n \prec 1\}$.

(ii) if $J(\text{Con } L) \cong P_2$, then $L \cong N_5$ or L_2 (and its dual, see Figure 3.2).

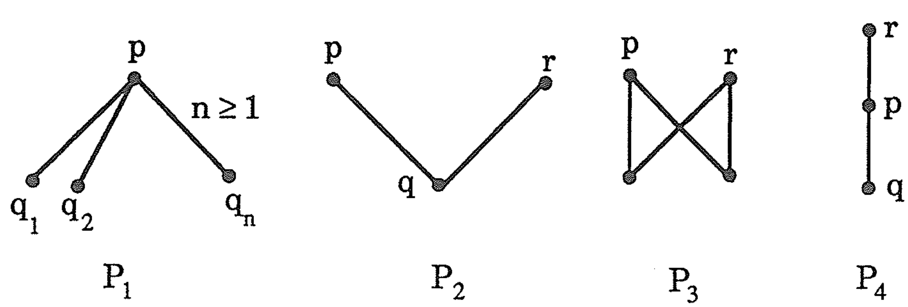


Figure 3.1

Proof: If Con L $\cong 2^n$, then $n \leq 3$ since $\ell(L) = 3$. Suppose that Con L $\neq 2^n$. Let $p \in p^*$ and $q_i \in q_i^*$ be such that $p^* \succ q_i^*$, $i = 1, \ldots, n$; q_i^* being minimal in

J(Con L). Let L' be the lattice as described in Lemma 3.2. Then $\ell(L') \leq 1$ and $|J(\text{Con } L')| \leq 1$.

If J(Con L') = Ø, then J(Con L) ≅ P_1. So we assume that J(Con L') = r*. Suppose that r* > p*. We will show that n = 1. Suppose that n ≥ 2 then by Lemma 2.7(ii), q_i* are interior for all i = 1, ..., n. Let $\Theta = \bigvee(\theta(q_i) \mid i = 1, \ldots, n)$ and L" = L/Θ. Then L" has at least two maximal chains of length 2. This implies that x* = ι for all exterior x of L" unless L" ≅ $\mathbf{2}^2$. But this is not the case since J(Con L") ≠ $\mathbb{1} \cup \mathbb{1}$. Thus p* is interior in L"; hence it is interior in L, which is not the case by Lemma 3.2. Thus n = 1 and J(Con L) ≅ P_4.

Suppose that r* and p* are not comparable. By applying Lemma 2.4, we have that r* > q_i* for all i. Now suppose that n ≥ 3, then q_i* are interior for all i and L"(as in the above paragraph) has at least 3 maximal chains of length 2. This implies that ι ∈ J(Con L") which is not the case. Thus n ≤ 2, and J(Con L) ≅ P_2 or P_3.

(i) Now we suppose that L has a maximal chain of length 2 and L is not N_5. We can assume that L has at least three atoms. In this case ι ∈ J(Con L), hence J(Con L) ≠ P_2 or P_3. Suppose that J(Con L) ≅ P_4, then L' = L/q* has at least two maximal chains, this implies p* is interior in L', hence it is interior in L. This contradicts Lemma 3.2. Thus J(Con L) ≅ P_1.

(ii) Let L ≠ N_5 and J(Con L) ≅ P_2. Let C = { 0, a, b, c, 1 } be a cycle of L containing prime intervals p = [0, a] and q = [b, c] with p* >– q*. We can assume L has no maximal chain of length 2 by (i), thus there exists d ∈ L such that a < d < 1. By Lemma 3.1(ii), L/q* cannot have chain of length 3, hence we may assume that [b, c]* = [0, e]* = [a, d]* for some e ∈ L. If

$L = \{0, a, b, c, d, e, 1\}$ then $L \cong L_2$ (see Figure 3.2). Otherwise let f be another atom or co-atom of L. If f is an atom, then $\iota \in J(\mathrm{Con}\, L)$, which is not the case. So we suppose that L has no new atom and f is a co-atom. Then we must have $f = a \vee b$ and $L \cong 2^3$, which is also not the case. Thus $L \cong L_2$.

Finally, we give examples of lattices whose congruences are as stated in the proposition. (Figure 3.2) ❑

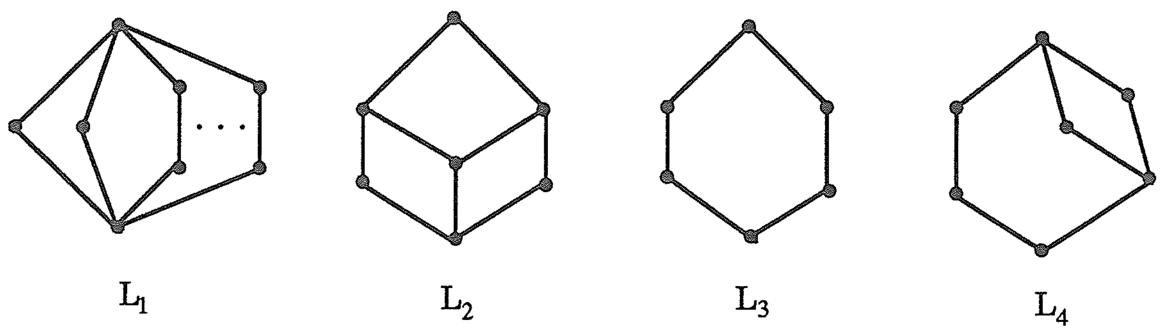


Figure 3.2

<u>Lemma 3.4</u> Let $\ell(L) = 4$ and $p^* \succ q_i^*$, $i = 1, \ldots, n$; $p^*, q_i^* \in J(\mathrm{Con}\, L)$. Let $L' = L/p^*$. Then

(i) if $\ell(L') = 3$, then L' has at least n maximal chains of length 2.

(ii) if $\ell(L') = 2$, then L' has at least n-1 maximal chains of length 2.

Proof: For each q_i^*, $i = 1, \ldots, n$, there is a cycle C_i as mentioned in Lemma 3.2 such that the congruence class of p^* containing the prime interval q_i has length 2 (it cannot be 3 as $\ell(L') \geq 2$). Denote such class by $\mathbf{q}_i$, it is easy to see that $\mathbf{q}_i \cap \mathbf{q}_j = \emptyset$ if $i \neq j$. If $\ell(L') = 3$, then none of the classes $\mathbf{q}_i$'s can be the 0 or 1 of L'. Thus L' has at least n maximal chains of length 2, i.e., $0 \prec \mathbf{q}_i \prec 1$ for $i = 1, \ldots, n$. If $\ell(L') = 2$, then at most two of $\mathbf{q}_i$'s can be the 0 or 1 of L'. If only one of the q_i's is the 0 or 1 of L', then the rest of the q_i's will give rise to n-1 maximal chains of length 2 in L'. If exactly two of the q_i's are respectively the 0 or 1 of L', then in this case, n = 2 and L' still has a maximal chain of length 2. ❑

Lemma 3.5 Let L be a lattice of length 4. Then any induced subposet of J(Con L) does not contain a cycle.

Proof: It suffices to consider the situation where the cycle of J(Con L) is $\{ a^*, b^*, c^*, d^* \mid a^* \prec (b^* \text{ and } c^*) \prec d^* \}$. By Lemma 3.4, $L_b = L/b^*$ is of length 3, and has a maximal chain of length 2. Since c* must be exterior in L_b, L_b contains a sublattice $\{ 0, 1, a, c, d_1, \ldots, d_n, \mid n \geq 2,\ 0 \prec a \prec 1, 0 \prec c \prec d_1, \ldots, d_n \prec 1 \}$ or its dual by proposition 3.3(i). But this implies that $L_c = L/c^*$ would have two maximal chains of length 2, (one as described in Lemma 3.4, and one arises from the fact that there is a congruence class of length 2 of c* containing only prime intervals in c*). This implies that b* is interior in L_c, hence in L. This contradicts Lemma 3.2. ❑

Proposition 3.6 Let L be a lattice such that $\ell(L) = 4$. Then Con L $\cong 2^n$, $n \leq 4$, or J(Con L) is isomorphic to one of the posets as shown in Figure 3.3.

Proof: Clearly, if Con L $\cong 2^n$, then $n \leq 4$ by Lemma 2.4. Suppose that Con L $\neq 2^n$. Let $L' = L/p^*$ where $p^*, q_i^* \in H_L/\sim$, $p^* \succ q_i^*$, and q_i^*, $i = 1, \ldots, n$ are minimal in J(Con L). We consider four different cases.

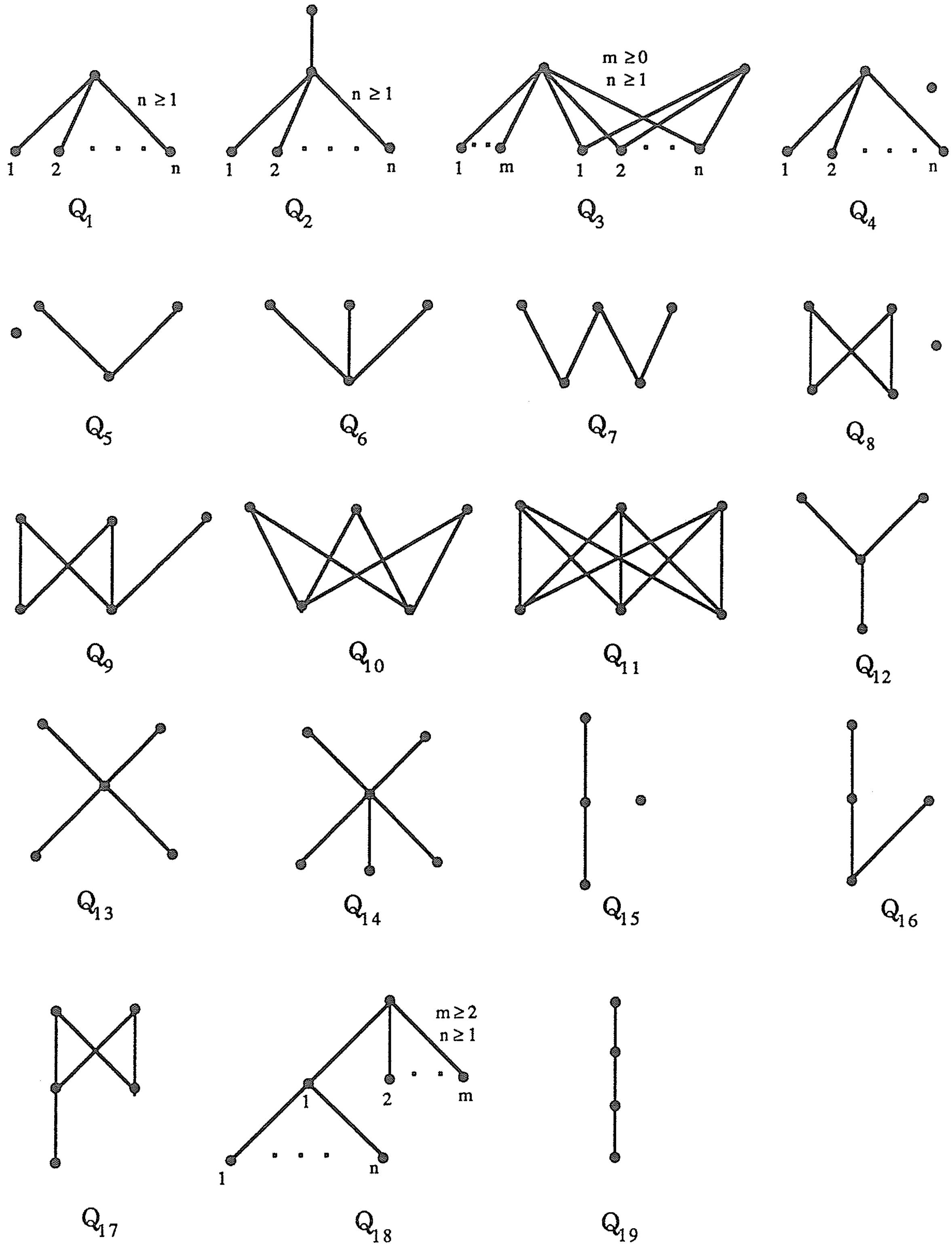

n ≥ 1
1 2 n
Q1
n ≥ 1
1 2 n
Q2
m ≥ 0
n ≥ 1
1 m 1 2 n
Q3
1 2 n
Q4
Q5
Q6
Q7
Q8
Q9
Q10
Q11
Q12
Q13
Q14
Q15
Q16
Q17
m ≥ 2
n ≥ 1
1 2 m
1 n
Q18
Q19

Figure 3.3

Case 1. $|J(\mathrm{Con}\ L')| \le 1$ or $\ell(L') \le 1$.

In this case, we have either $J(\mathrm{Con}\ L') = \emptyset$ or r^*. Hence $J(\mathrm{Con}\ L) \cong Q_1$, Q_2, Q_3 or Q_4.

Case 2. J(Con L) has three maximal elements.

Let p^*, q^*, r^* be the maximal elements of J(Con L) and let S be the poset $J(\mathrm{Con}\ L) - \{ p^*, q^*, r^* \}$. S is non-empty since $\mathrm{Con}\ L \neq \mathbf{2}^n$. Let $L'' = L/\theta(S)$ where $\theta(S) = \bigvee(y^* \mid y^* \in S)$. Then $\ell(L'') \ge 3$.

Suppose that $\ell(L'') = 4$, then L" is modular by Lemma 3.1(ii). Every maximal chain of L" has length 4. This implies $L'' = L$ and $S = \emptyset$ which is not the case by assumption. Hence $\ell(L'') = 3$ and L" is distributive by Lemma 3.1(i). If $|S| = 1$, then $J(\mathrm{Con}\ L) \cong Q_5$ or Q_6 by Lemma 2.4. Now suppose that $|S| \ge 2$. We first show that S is totally unordered. Assume that this is not the case, then there exist $u^*, v^* \in S$ such that $u^* \succ v^*$. Then L/u^* would have a maximal chain of length 2 by Lemma 3.4, which is a contradiction. Hence S is totally unordered. By applying Lemma 2.7 and proposition 3.3, one can see that $|S| \ge 2$ implies that for any $x^* \in S$, x is interior. Then $|S| \le 3$ since L" has width 3. Suppose that $|S| = 2$. If J(Con L) is disjoint, then one component is a singleton. By Lemma 2.7(iii) and proposition 3.3, $J(\mathrm{Con}\ L) \cong Q_8$. If J(Con L) is connected, then as a biparttie graph, it has 4, 5 or 6 edges. There are 4 of them, however, only three of them are possible and are given as Q_7, Q_9, and Q_{10}. If $|S| = 3$. Then J(Con L) must be connected. By using Lemma 3.4 and proposition 3.3, we have $J(\mathrm{Con}\ L/p^*) \cong \mathbf{1} \cup \mathbf{1}$, for any maximal p^* of J(Con L). Thus $J(\mathrm{Con}\ L) \cong Q_{11}$.

Case 3. $\ell(L') = 2$.

In this case, Con L' $\cong \mathbf{2}^1$ or $\mathbf{2}^2$. The former case is covered in case 1. For the latter case, we can assume that J(Con L') = { r*, s* } and either r* > p* or s* > p*. Suppose that both r* > p* and s* > p*. By Lemma 3.4, L' would have at least 3 maximal chains of length 2 if $n \geq 4$. This would implies that ι is a join irreducible of L', which is not the case. Thus $n \leq 3$. Therefore J(Con L) is isomorphic to one of the posets: Q_{12}, Q_{13} and Q_{14}. Now suppose r* > p* and s* and p* are not comparable. If s* is isolated, then J(Con L) $\cong Q_{15}$. Suppose that $s^* > q_i^*$ for some $1 \leq i \leq n$, and $n \geq 2$. Then by proposition 3.3(i), $s^* > q_i^*$ for all i. If $n \geq 2$, $L^* \cong L/s^*$ has length 3 and contains at least two maximal chains of length 2. This implies that p* is interior in L*, hence is interior in L. This is impossible by Lemma 3.2. Thus n = 1 and J(Con L) = Q_{16}.

Case 4. $\ell(L') = 3$.

The cases that Con L' $\cong \mathbf{2}^1$ or $\mathbf{2}^3$ are settled in case 1 and case 2 respectively. We show that Con L' $\neq \mathbf{2}^2$. Suppose that Con L' $\cong \mathbf{2}^2$, then L' is modular by Lemma 3.1. Every maximal chain of L' has length 3. By Lemma 3.4, L' has a maximal chain of length 2 which is a contradiction.

By proposition 3.3(i), J(Con L') $\cong P_1$ or P_2. For the case that J(Con L') $\cong P_2$ = { r*, s*, t* | r*, s* > t* }. We have L' $\cong N_5$ by proposition 3.3(i). This implies t* is minimal in J(Con L). By Lemma 2.4, proposition 3.3 and Lemma 3.4, we have r*, s* > p* and n = 1. Thus J(Con L) is isomorphic to

Q_{17}. Now suppose that J(Con L') ≅ P_1 = {r*, s_j* | r* > s_j*, j =1, ..., m}. We consider four different possibilities:

(i) r* and p* are not comparable and all the q_i*s and s_j*s are not comparable. In this case, we must have r* > q_i* for all i, for otherwise we would obtain a lattice L" which is the homomorphic image of L by collapsing all the congruences q_i which are coverved by both r* and p*, and J(Con L") ≅ **2** ∪ **2**. This is impossible by Lemma 2.4. Thus J(Con L) ≅ Q_3.

(ii) r* and p* are not comparable but s_1* > q_1*. By Lemma 3.4, we have n = 1. For otherwise, L' is a lattice having two maximal chains of length 2 and s* is interior in L' and L, which is not the case by Lemma 3.2. By applying Lemma 2.4, and Lemma 2.7, we conclude that m = 1. Hence J(Con L) ≅ Q_{16}.

(iii) r* >– p*. By Lemma 3.5, s_i* and q_j* are not comparable for all i, j; thus J(Con L) ≅ Q_{18}.

(iv) s_1* > p*. By Lemma 3.5, s_j* and q_i* are not comparable for j ≥ 2, and i ≥ 1. By Lemma 3.4 and proposition 3.3(i), one deduced that m = n = 1. Thus J(Con L) ≅ Q_{19}.

Finally, we give lattices K_i with J(Con K_i) ≅ Q_i. (Figure 3.4). For i = 4, 5 and 8, we can take L_1 X **2**, L_2 X **2** and L_3 X **2** as their respective lattices. ❑

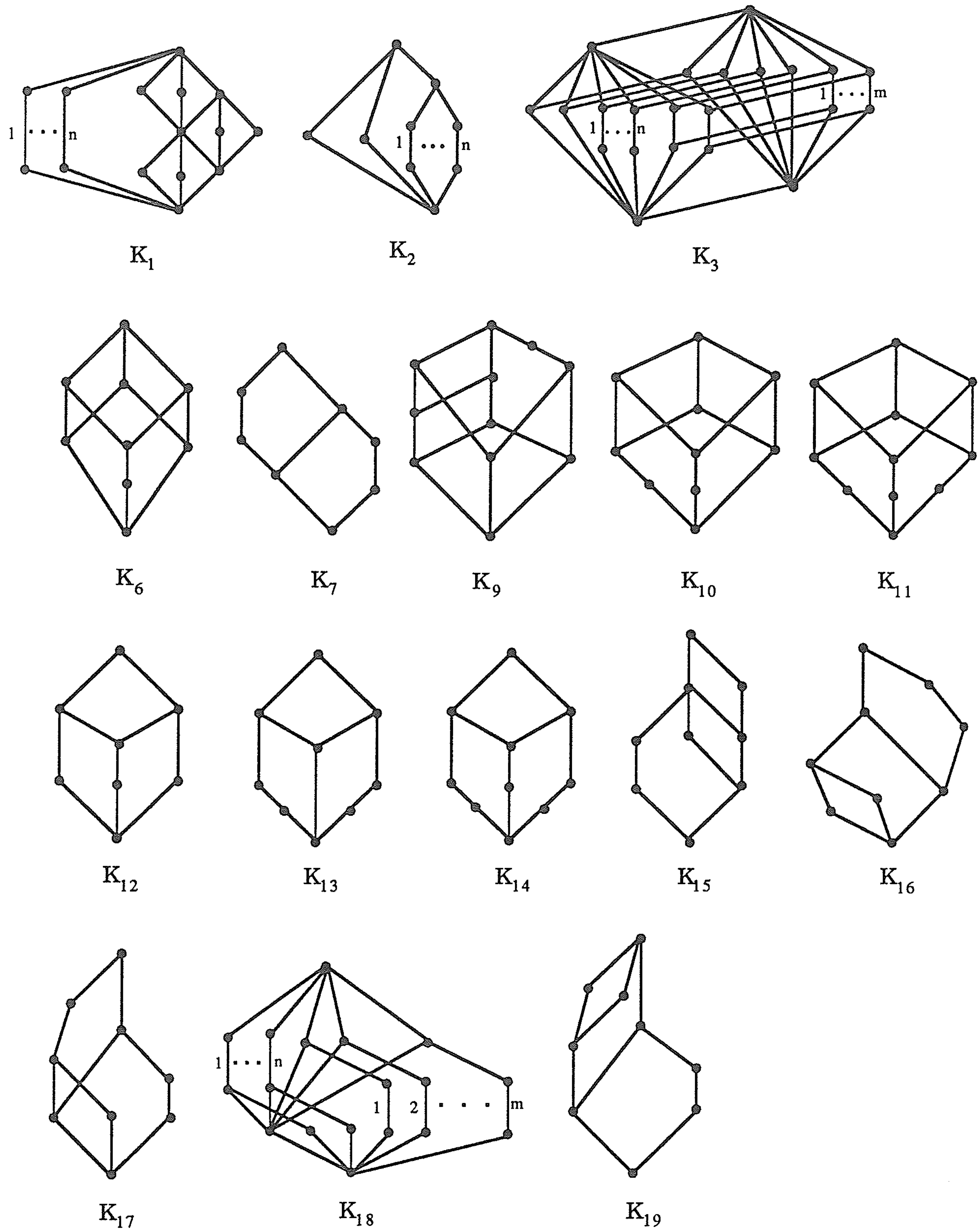

1 ... n
K_1
1 ... n
K_2
1 ... n
1 ... m
K_3
K_6
K_7
K_9
K_{10}
K_{11}
K_{12}
K_{13}
K_{14}
K_{15}
K_{16}
K_{17}
1 ... n
1 2 ... m
K_{18}
K_{19}

Figure 3.4

Chapter IV
Countable Semilattices of Compact Congruences

In this chapter, we show that every countable distributive semilattice with O is the compact congruence semilattice of some lattice. Our proof was based on the approach suggested in [Pu]. The same result was also proved by A. P. Huhn in [Hu]. In section 1, we give a brief description of the concept of representation as proposed in [Pu]. In section 2, we construct a representation for a countable distributive semilattice with O.

1. The Concept of Representation

Definition 4.1 Let $\mathcal{F}$, $\mathfrak{G}$ be two (covariant) functors from a category **A** to a category **B**. A natural equivalence of $\mathcal{F}$ in $\mathfrak{G}$ is a family $\ell : \{\ell_A \mid A \in \mathbf{A}\}$ of isomorphisms, such that given a morphism $\alpha : A_1 \to A_2$ in **A**, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(A_1) & \xrightarrow{\ \mathcal{F}(\alpha)\ } & \mathcal{F}(A_2) \\ \ell_{A_1} \big\downarrow \wr & & \wr \big\downarrow \ell_{A_2} \\ \mathfrak{G}(A_1) & \xrightarrow{\ \mathfrak{G}(\alpha)\ } & \mathfrak{G}(A_2) \end{array}$$

Definition 4.2 Let $\mathcal{F} : \mathbf{A} \to \mathbf{C}$, $\mathfrak{G} : \mathbf{B} \to \mathbf{C}$ be functors. A representation of $\mathcal{F}$ in $\mathfrak{G}$ is a pair $(\mathcal{H}, \ell)$, where $\mathcal{H} : \mathbf{A} \to \mathbf{B}$ is a functor and ℓ is a natural equivalence of $\mathcal{F}$ in $\mathfrak{G} \circ \mathcal{H}$.

If $(\mathcal{H}_1, \ell_1)$ is a representation of $\mathcal{F}$ in $\mathfrak{G}$ and $(\mathcal{H}_2, \ell_2)$ is a representation of $\mathfrak{G}$ in $\mathcal{V}$, then $(\mathcal{H}_2 \circ \mathcal{H}_1, \ell_2 \circ \ell_1)$ is a representation of $\mathcal{F}$ in $\mathcal{V}$.

A finite reflexive directed graph (X, R) is a finite set X with a reflexive relation R such that $a\,R\,b$ represents a directed edge from a to b. A subset U of X is called a segment, if whenever $y \in U$ and there is a finite sequence of directed edges from x to y, then $x \in U$. Let $U \subseteq X$. The segment generated by U is the set of all elements $y \in X$ such that there is a sequence of directed edges from y to x, where x is an element of U. A subset V of X is called a component if it is a maximal subset of X in which each of its elements generates the same segment. The set of all segments of X forms a distributive lattice under inclusion. We denote this by Seg X . Let C(X) be the set of the components of X. We can define a partial order $\leq$ on C(X) : $V_1 \leq V_2$ if and only if the segment generated by V_1 is contained in the segment generated by V_2. Each segment of X is generated as an element of the lattice Seg X by a subset of C(X). Indeed, $(J(\mathrm{Seg}(X)), \leq)$ is isomorphic to $(C(X), \leq)$.

Let (X_1, R_1) and (X_2, R_2) be finite reflexive directed graphs. Let $F : X_1 \to X_2$ be a partial mapping which satisfies the following conditions (IV-1):

(i) F is onto;

(ii) For all $a, b \in \mathrm{Dom}(F)$, $a\,R_1\,b$ implies that $F(a)\,R_2\,F(b)$;

(iii) If $F(a)\,R_2\,F(b)$, then there exists a $c \in \mathrm{Dom}(F)$, $a\,R_1\,c$ and $F(b) = F(c)$. **(IV-1)**

<u>**Definition 4.3**</u> We define the following categories:

(i) **Lat** is the category whose objects are lattices and morphisms are lattice embeddings which preserves congruences.

(ii) **SD** is the category whose objects are distributive 0-semilattices and morphisms are 0-embeddings.

(iii) **GrpD** is the category whose objects are reflexive directed graphs and morphisms are partial mappings as described above, under the usual composition (i. e., $\mathrm{Dom}(F_1 \circ F_2) = \mathrm{Dom}\ F_2 \cap F_2^{-1}(\mathrm{Dom}\ F_1)$).

The subscript "fin" will be used to denote the restriction of a category to its finite objects. The category $\mathbf{SD}_\wedge$ is the full sub-category of **SD** whose objects are distributive lattices (i.e., the meet of two compact elements is also compact). Let S be an object of **SD**. The full subcategory of **SD** whose object-set consists of all objects in **SD** which are also subsemilattices of S, is denoted by $\mathbf{SD} \rceil S$.

<u>**Definition 4.4**</u> We define the following functors:

(i) $Con^c : \mathbf{Lat} \to \mathbf{SD}$ is the covariant functor such that for a lattice L, Con^c L is its compact congruence semilattice (we abuse the notation Con^c L for Con^c L). For an embedding $\varphi : L_1 \to L_2$, $Con^c\ \varphi$ is the join homomorphism from $\mathrm{Con}^c L_1$ to $\mathrm{Con}^c L_2$, which maps each $\theta \in \mathrm{Con}^c L_1$ to the smallest congruence of L_2 containing the image of θ.

(ii) $Seg : \mathbf{GrpD} \to \mathbf{SD}$ is the contravariant functor such that for a directed graph X, Seg X = Seg X, the lattice of segments of X. For a morphism $F : X_1 \to X_2$, Seg F is a mapping from Seg X_2 to Seg X_1 which sends every segment U of X_2 to the smallest segment of X_1 containing $F^{-1}(U)$.

(iii) $Id : \mathbf{SD} \to \mathbf{SD}$ is the identity functor.

Lemma 4.5 Let C be any distributive 0-semilattice, then there is a directed family { $C_i \in \mathbf{SD}_{fin}$, $i \in I$ } such that C is the colimit of { C_i, $i \in I$ }.

Proof: Let $A \subseteq C$ be finite. We construct a distributive 0-subsemilattice of C containing A. Let <A> be the finite 0-sublattice generated by A in the ideal lattice of C. Let J be the set of the join irreducibles of <A>. If all the elements of J are in C, we are done. Otherwise, let $s \in J$ be a minimal non-compact element. There is an $s' \in C$ such that $s > s' > \vee (t \mid t < s, t \in J)$ and $(J - \{s\}) \cup \{s'\}$ generates a distributive 0-subsemilattice of C containing all the compact elements of <A>. We repeat this process to obtain the required 0-subsemilattice. Finally, the construction of the colimit is standard and is omitted. ❑

The following theorems and Lemma 4.5 provide the basic idea for one to obtain a lattice with a specified congruence lattice by the direct limit (colimit) construction.

Theorem 4.6 Let { $C_i \in \mathbf{SD}_{fin}$, $i \in I$ } be a directed family of distributive 0-semilattices having colimit C. Let { $L_i \in \mathbf{Lat}_{fin}$, $i \in I$ } be a directed family of lattices having colimit L. Suppose that the identity functor *Id*, restricted to { $C_i \in \mathbf{SD}_{fin}$, $i \in I$ }, has a representation in the functor Con^c, restricted to { $L_i \in \mathbf{Lat}_{fin}$, $i \in I$ }. Then $\mathrm{Con}^c L \cong C$.

Theorem 4.7 If $Id : \mathbf{SD}_{fin} \to \mathbf{SD}$ is representable in $Con^c : \mathbf{Lat}_{fin} \to \mathbf{SD}$, then $Id : \mathbf{SD} \to \mathbf{SD}$ is representable in $Con^c : \mathbf{Lat} \to \mathbf{SD}$.

Let X be a finite set, and let $\mathcal{P}(X)$ be the power set of X. A mapping $M : \mathcal{P}(X) \to \mathcal{P}(X)$ is called an m-operator if, for every A, B $\subseteq$ X, A $\subseteq$ B implies that $M(A) \subseteq M(B)$. Every m-operator determines a closure operator where A is closed iff $M(A) \subseteq A$.

Lemma 4.8 Let M_X, M_Y be m-operators on X, Y respectively. Let $G : X \to Y$ be a partial onto mapping such that for any $A \subseteq Y$, $G^{-1}(M_Y(A)) = M_X(G^{-1}(A))$. Then the mapping $A \to G^{-1}(A)$ is a 0-embedding of the lattice of closed subsets of Y into the lattice of closed subsets of X.

Proof: Let L(-) denotes the lattice of closed subsets of -. Let $A \in L(Y)$, then $M_X(G^{-1}(A)) = G^{-1}(M_Y(A)) \subseteq G^{-1}(A)$. Thus $G^{-1}(A) \in L(X)$. The mapping $L(Y) \to L(X)$ is one-one and preserving 0, since $G(G^{-1}(A)) = A$ as G is onto. It is clear that G^{-1} preserves meet as it is simply set intersection. As for the join, we have

$$\begin{aligned} G^{-1}(A \vee B) &= G^{-1}\left(\bigcap (C \in L(Y) \mid A \cup B \subseteq C) \right) \\ &= \bigcap (G^{-1}(C) \in L(X) \mid G^{-1}(A) \cup G^{-1}(B) \subseteq G^{-1}(C)) \\ &= G^{-1}(A) \vee G^{-1}(B). \end{aligned}$$ ❑

Let Z be a finite set and Q(x,y,z) be a ternary relation on Z. Suppose further that Q(a,b,c) implies that Q(a,c,b) for all $a \neq b \neq c \neq a$. Then Q determines an m- operator $M_Q(A) = \{ a \in Z \mid \exists\, b, c \in A, Q(a,b,c) \}$. Let $L_M(Z)$ denotes the corresponding closed-set lattice.

Lemma 4.9 $L_M(Z)$ is an atomistic lattice and there is a one-one correspondence, $\mathrm{Con}^c L \to \{ K \subseteq Z \mid Q(a,b,c) \text{ and } b \in K \text{ implies that } a \in K \}$. More precisely, for each $\Theta \in \mathrm{Con}^c L$, we have $\Theta \to K_\Theta = \{ a \in Z \mid \{a\}\ \Theta\ 0 \}$.

Proof: Clearly, $L_M(Z)$ is an atomistic lattice which has the set of atoms: $\{ \{a\} \mid a \in Z \}$. Thus every congruence Θ of $L_M(Z)$ is determined by the ideal generated by the set $K_\Theta = \{ \{a\} \mid \{a\} \Theta 0 \}$. It remains to show that K_Θ can indeed be characterized as stated in the lemma.

Let Θ be a congruence relation of $L_M(Z)$. Then Q(a,b,c) implies that $\{a\} = \{a\} \wedge (\{b\} \vee \{c\})$. Hence $\{b\} \Theta 0$ implies that $\{a\} \Theta (\{a\} \wedge \{c\})$, i.e. $\{a\} \Theta 0$ and $\{a\} \in K_\Theta$.

Conversely, let $K \subseteq Z$ be such that Q(a,b,c) and $b \in K$ implies that $a \in K$. We define a relation Θ on $L_M(Z)$ by: $A \Theta B$ if and only if $A \Delta B \subseteq K$ (where Δ is the symmetric difference operation of sets). Θ is clearly an equivalence relation which preserves meet. We thus need only show that Θ preserves join. i.e., for each $\Theta \in \mathrm{Con}^c L$, $(A \vee C) \Delta (B \vee C) \subseteq K$. Now suppose that $x \in (A \vee C) - (B \vee C)$. We have $A \vee C = \cup (S_n \mid n \geq 0)$, where $S_0 = A \cup C$ and, inductively $S_n = S_{n-1} \cup M(S_{n-1})$ for $n \geq 1$. Now $S_0 - (B \vee C)$ is clearly a subset of K, so we can assume that $x \notin K$ is chosen such that $x \in S_p - S_{p-1}$, and p (≥ 1) is of smallest possible value. Thus we have Q(x,y,z) for some $y, z \in S_{p-1}$. One of the y and z is not in $B \vee C$; for otherwise $x \in B \vee C$, a contradiction to the assumption on x. By the induction hypothesis, we have $y \in K$ or $z \in K$. This would imply that $x \in K$, which is a contradiction. Therefore $(A \vee C) - (B \vee C) \subseteq K$. Similarly, we have $(B \vee C) - (A \vee C) \subseteq K$. Hence Θ is indeed a congruence relation.

Let $(X, R) \in \mathbf{GprD_{fin}}$. Let $X^* = X \times 3$, where $3 = \{0,1,2\}$. Let Q_X be a ternary relation over X^* given by: $Q_X((a,i),(b,j),(c,k))$ iff ($i \neq j \neq k \neq i$),

(a = b or a = c) and (a R b and a R c). Then Q_X determines an m-operator M_X. Let $L(X^*)$ denotes the corresponding closed-set lattice of X^*.

__Lemma 4.10.__ Let $(X, R_X), (Y, R_Y) \in$ $\mathbf{GprD_{fin}}$ and let $F : (X, R_X) \to (Y, R_Y)$ be a morphism. Let $F^* : X^* \to Y^*$ be the map given by $F^*((a,i)) = (F(a),i)$. Then

(i) the mapping $A \to F^{*-1}(A)$ is an 0-embedding of $L(Y^*)$ into $L(X^*)$;

(ii) the mapping $i_X : Seg(X) \to Con^c(L(X^*))$, where $i_X(U)$ = the congruence of $L(X^*)$ generated by the segment U, is an isomorphism;

(iii) the family of isomorphisms $\{ i_X \mid (X, R_X) \in \mathbf{GprD_{fin}} \}$ is a natural equivalence of $Seg : \mathbf{GprD_{fin}} \to \mathbf{SD}$ in $Con^c : \mathbf{Lat_{fin}} \to \mathbf{SD}$.

Proof:

(i) We need to show that $F^{*-1}(M_Y(A)) = M_X(F^{*-1}(A))$ for any $A \subseteq Y^*$ by Lemma 4.8. Let $x = (a,i) \in M_X(F^{*-1}(A))$. Then there exist $(b,j), (c,k) \in F^{*-1}(A)$ such that $Q_X((a,i),(b,j),(c,k))$. Thus (a = b or a = c) and ($a\ R_X\ b$ and $a\ R_X\ c$). Hence (F(a) = F(b) or F(a) = F(c)), and ($F(a)\ R_X\ F(b)$ and $F(a)\ R_X\ F(c)$). Therefore $F^*(x) \in M_Y(A)$ and $x \in F^{*-1}(M_Y(A))$. Conversely, suppose that $x = (a,i) \in F^{*-1}(M_Y(A))$. Then there exist $(b',j), (c',k) \in A$ such that $Q_Y((F(a),i),(b',j),(c',k))$. Without loss of generality, we assume that F(a) = b'. Then there exist $d \in X$ such that $a\ R_X\ d$ and F(d) = c'. Thus, we have $Q_X((a,i),(a,j),(d,k))$ and $(a,i) \in M_X(F^{*-1}(A))$.

(ii) By Lemma 4.9 and the fact that for each $\Theta \in Con^c(L(X^*))$, $(a,i)\ \Theta\ 0$ iff $(a,j)\ \Theta\ 0$ for all j = 0,1,2. One see that the mapping $i_X : Seg(X) \to Con^c(L(X^*))$, is an isomorphism.

(iii) We need to show that $i_X \circ Seg(F) = Con^c(F) \circ i_Y$. Let $U \in Seg(Y)$, then $Seg(F)(U)$ is the smallest segment of X generated by $F^{-1}(U)$. Conversely, let $i_Y(U)$ be the congruence of $L(Y^*)$ corresponding to $U \in Seg(Y)$. Then $Con^c(F)(i_Y(U))$ is the congruence of $L(X^*)$ generated by $i_Y(U)$ under the embedding F^{*-1}. Now each atom (a,i), $a \in U$ is mapped to the join of all atoms of the form (b,i), $b \in F^{-1}(a)$. Thus $Con^c(F)(i_Y(U))$ is the congruence of $L(X^*)$ corresponding to the segment generated by $F^{-1}(U)$. Therefore, $i_X \circ Seg(F) = Con^c(F) \circ i_Y$. ❑

<u>**Theorem 4.11**</u> The functor $Seg : \mathbf{GrpD_{fin}} \to \mathbf{SD}$ is representable in $Con^c : \mathbf{Lat_{fin}} \to \mathbf{SD}$.

2. Representing Countable Semilattices as Compact Congruence of Lattices

The problem of representing a distributive 0-semilattice as the semilattice of the compact congruences of a lattice is now transformed to the problem of representation of the identity functor $Id : \mathbf{SD_{fin}} \to \mathbf{SD}$ in the functor Seg: $\mathbf{GrpD_{fin}} \to \mathbf{SD}$. However, this problem is still unsolved. It was shown in [Pu] that Id: $\mathbf{SD_{\wedge}} \to \mathbf{SD}$ is representable in $Con^c : \mathbf{Lat} \to \mathbf{SD}$. Hence the ideal lattice of a distributive lattice with 0 is the congruence lattice of some lattices. In this section, we use this approach to show that the ideal lattice of a countable distributive 0-semilattice is the congruence lattice of some lattices.

Let S be a countable distributive 0-semilattice, and let $s_1, s_2, ..., s_n, ...$ be an enumeration of the elements of $S - \{0\}$. Let S_1 be a finite distributive subsemilattice containing 0 and s_1, and inductively let S_n be a finite distributive subsemilattice containing S_{n-1} and $s_{f(n)}$ where $s_{f(n)}$ is the least element in the enumeration of $S - S_{n-1}$. Clearly, $S_1, S_2, ..., S_n, ...$ is an increasing chain whose colimit is S. As an application of Theorem 4.6, we construct a lattice whose semilattice of compact congruences is S by considering the representation of $\mathbf{SD} \rceil S$ through a chain of finite subsemilattices.

The Construction of the Graph for $S_{fin} \mid S$

Let S be a distributive semilattice. Let Σ_s^* be the set of all words generated by the alphabet set $S \cup \{ \Lambda \}$. Let $x = a_1a_2...a_k$, $a_1,...,a_k \in S$ be a word; we call $\imath(x) = a_1$ and $\ell(x) = a_k$ the first symbol and the last symbol respectively. The length of the word x is $|x| = k$. Let $x = a_1a_2...a_k$ and $y = b_1b_2...b_m$ be two words, the product $z = xy$ is the word $a_1a_2...a_kb_1b_2...b_m$. The symbol Λ is called the empty word and has the property that $x.\Lambda = \Lambda.x = x$ for all $x \in \Sigma_s^*$. We say that x is a sub-word of y if there is a $z \in \Sigma_s^* - \{ \Lambda \}$ such that $y = x.z$ (denoted by $x \subset y$). Let Σ_s be the set of all words $x = a_1a_2...a_k$ such that $a_1 < a_2 < ... < a_k$, $a_1 \in J(S)$, $a_2, a_3, ..., a_k \in S - J(S)$ where J(S) is the set of join-irreducibles of S. The set of all words in Σ_s with initial symbol a is called the a-tree of Σ_s. For a finite distributive semilattice S. We define the graph $H(S) = (X_s, R_s)$ as follows:

Let $X_s = \Sigma_s \cup \{ (x,y) \mid x \subset y \text{ in } \Sigma_s \text{ and } x, y \in \Sigma_s \}$, we say that $x R_s y$ iff one of the following conditions holds

(i) $x = y$.

(ii) $x, y \in \Sigma_s$, $x \subset y$.

(iii) $x, y \in \Sigma_s$, $\ell(x) < i(y)$.

(iv) $x \in \Sigma_s$, $y \in X_s - \Sigma_s$, $y = (z,x)$, $z \in \Sigma_s$.

(v) $x \in X_s - \Sigma_s$, $y \in \Sigma_s$, $x = (y,z)$, $z \in \Sigma_s$.

Under this construction, the components of H(S) are precisely the a-trees of Σ_s where $a \in J(S)$. Indeed, conditions (iv) and (v) above were inserted to guarantee this. Hence there is a natural isomorphism $e_s : S \to Seg(H(S))$. We take these isomorphisms to be the natural equivalence.

Let $j: C \to D$ be identical embedding of C into D. We define a partial mapping $F : H(D) \to H(C)$ as follows:

(a) if $x=a_1a_2...a_k \in \Sigma_D$, then F(x) is defined if and only if $a_k \in C$ and there is a largest integer $1 \leq i \leq k$ such that $a_i \in J(C)$, then F(x) is the word obtained from $a_ia_{i+1}...a_k$ by deleting all the symbols a_j, $i < j < k$ which are not in C.

(b) for $x = (y,z) \in X_D - \Sigma_D$, F(x) is defined if and only if F(y) and F(z) are defined and $F(y) \subset F(z)$, in this case $F(x) = (F(y),F(z))$.

<u>Claim 1.</u> F is onto.

If $y \in \Sigma_C$, we can write $y = b_1b_2...b_k$ where $b_1 \in J(C)$, and $b_2, ..., b_k \in C - J(C) \subseteq D - J(D)$. If $b_1 \in J(D)$, then $y \in \Sigma_D$ and $F(y) = y$; otherwise, let $a < b_1$ be such that $a \in J(D)$ and $y' = a.y \in \Sigma_D$, then we have $F(y') = y$.

Now suppose that $y = (x,z) \in X_C - \Sigma_C$ and $x \subset z$. By the above argument, there is a $z' \in \Sigma_D$ such that $F(z') = z$. We can choose a sub-word $x' \subset z'$ such that $F(x') = x$. Hence $(x',z') \in X_D - \Sigma_D$ and $F((x',z')) = (F(x),F(z)) = (x,z) = y$.

<u>Claim 2.</u> F preserves the relations.

Let $x, y \in X_D$ be such that both F(x), F(y) are defined and $x \; R_D \; y$. The case that x = y is obvious. We verify the following four cases.

(a) $x, y \in \Sigma_D$ and $x \subset y$.

Let $x = a_1 a_2 \ldots a_s$, $y = a_1 a_2 \ldots a_s a_{s+1} \ldots a_p$. If $F(x) \subset F(y)$, then $F(x) \; R_C \; F(y)$. Otherwise, by the definition of F, we have $\ell(F(x)) = a_s$ and $\imath(F(y)) > a_{s+1}$. Thus $F(x) \; R_C \; F(y)$.

(b) $x, y \in \Sigma_D$ and $\ell(x) < \imath(y)$.

In this case, it is clear that $\ell(F(x)) < \imath(F(y))$. Thus $F(x) \; R_C \; F(y)$.

(c) $x \in \Sigma_D$, $y \in X_D - \Sigma_D$ $y = (z,x)$, $z \subset x$.

Since $F(y) = (F(z),F(x))$ is defined, $F(z) \subset F(x)$. Thus $F(x) \; R_C \; F(y)$.

(d) $x \in X_D - \Sigma_D$, $y \in \Sigma_D$, $x = (y,z)$, $y \subset z$.

F(x) is defined implies that $F(x) = (F(y),F(z))$. Thus $F(x) \; R_C \; F(y)$.

<u>Claim 3.</u> F satisfies condition (iii) of (IV-1).

Let F(a) = x, F(b) = y and $x \; R_C \; y$. We consider four different cases.

(a) $x, y \in \Sigma_C$, $x \subset y$.

Let $x = b_1 b_2 \ldots b_k$ and $y = b_1 b_2 \ldots b_k b_{k+1} \ldots b_r$. Then $a \; R_D \; c$ where $c = a.b_{k+1} b_{k+2} \ldots b_r \in \Sigma_D$ and F (c) = F (b).

(b) $x, y \in \Sigma_C$, $\ell(x) < \imath(y)$.

Let $y = b_1 b_2 \ldots b_k$. If $b_1 \in J(D)$, then $y \in \Sigma_D$. If $b_1 \notin J(D)$, since $\ell(x) = \ell(F(a)) = \ell(a) < \iota(y) = b_1$, $a.y \in \Sigma_D$. In the former case we take $c = y$ and in the later case, we take $c = a.y$. Hence $a \, R_C \, c$ and $F(c) = y = F(b)$.

(c) $x \in \Sigma_C$, $y \in X_C - \Sigma_C$, $y = (z,x)$, $z \subset x$.

Since $F(a) = x$ contains z as a sub-word, we can always truncate a to obtain a sub-word d such that $F(d) = z$. Clearly $c = (d,a) \in \Sigma_D$, $a \, R_C \, c$ and $F(c) = (F(d),F(a)) = (z,x) = y = F(b)$.

(d) $x \in X_C - \Sigma_C$, $y \in \Sigma_C$, $x = (y,z)$, $y \subset z$.

In this case, we have $a \in X_D - \Sigma_D$. Let $a = (c,d)$, $c \subset d$. Then $F(a) = (F(c),F(d))$ implies $F(c) = y$ and $F(d) = z$. But then $a \, R_C c$ and $F(c) = y = F`(b)$.

Finally, we show that $e_D \circ j = Seg'(F) \circ e_C$. Let $x \in C$, then $j(x) = x \in D$ and $e_D \circ j(x)$ is the segment of H(D) which contains all the a-trees where $a \in J(D)$ and $a \leq x$. On the other hand, $e_C(x)$ is the segment of H(C) which contains all b-trees of H(C) where $b \in J(C)$, $b \leq x$. We have $x = a_1 \vee a_2 \vee \ldots \vee a_r = b_1 \vee b_2 \vee \ldots \vee b_s$ where $a_i \in J(D)$, $b_j \leq J(C)$. $a_i, b_j \leq x$. For each a_i, $i = 1, 2, \ldots, r$; we have $a_i \leq b_j$ for some j. Hence the smallest segment of H(D) containing F_n^{-1}(b_j trees) contains the a_i-trees of H(D). Thus the smallest segment containing $F_n^{-1}(\cup b_j$-trees) is exactly $\cup$ (a_i-trees), Hence $e_D \circ j = \text{Seg}(F) \circ e_C$.

Now, for the directed system $S_1 \to S_2 \to S_3 \to \ldots$, we have the inverse directed system $\ldots \to H(S_3) \to H(S_2) \to H(S_1)$. For $n > m+1$, we define the morphism $F_{n,m} : H(S_n) \to H(S_m)$ to be the composition $F_{m+1} \circ \ldots \circ F_n$, where $F_k : H(S_k) \to H(S_{k-1})$ is the morphism as described in the above construction.

Hence, we have proved:

Theorem 4.12 Every countable distributive semilattice with zero is the compact congruence semilattice of some lattice.

Chapter V
Complete Congruence Lattices

In this chapter, we answer the question raised in [Re, Wi]. For any complete lattice D, we construct a complete lattice L such that Com L is isomorphic to D. We first introduce some additional notation.

Let γ be an ordinal and let $\{ L_\alpha \mid \alpha < \gamma \}$ be a family of lattices. The sum $\Sigma (L_\alpha \mid \alpha < \gamma)$ is the lattice with underlying set $\cup (L_\alpha \mid \alpha < \gamma)$ and, besides the inherited order relations of each L_α, we have $x < y$ for all $x \in L_\alpha$, $y \in L_\beta$, $\alpha < \beta < \gamma$. Let L_1 and L_2 be lattices such the L_1 has a unit and L_2 has a zero, then $L_1 \oplus L_2$ is the lattice obtained from $L_1 + L_2$ by setting $I_{L_1} = O_{L_2}$. The dual of a lattice L is denoted by L^d. We shall be considering chains which can be obtained from ω and $\mathbf{n}$ by the operations +, $\oplus$, $(.)^d$. Thus it is appropriate for us to define, for a chain C, the support of C to be the set supp C = $\{ [x, y] \mid x \prec y \text{ in } C \}$. Let C_1 and C_2 be chains, we define supp $(C_1 \times C_2)$ to be the set $\{ [(x,y), (u,v)] \mid [x,u] \in \text{supp } C_1, [y,v] \in \text{supp } C_2 \}$. A valuation of a chain C by a set R is a mapping $\varphi : \text{supp } C \to R$. Let φ be a valuation of C, the induced valuation $\varphi \times \varphi : \text{supp } (C_1 \times C_2) \to R \times R$ is the mapping $\varphi \times \varphi ([(x,y), (u,v)]) = (\varphi[x,u], \varphi[y,v])$. The natural valuation of L^d obtained from φ is denoted by φ^d. Let C_1 and C_2 be two chains with valuations φ_1 and φ_2 respectively, then we simply use $\varphi_1 \cup \varphi_2$ to denote the valuation of the $C_1 \oplus C_2$ (or $C_2 \oplus C_1$) with $C_1 \cup C_2$ as underlying sets.

Let C be a chain and let φ be a valuation of C. We construct a lattice φ^*C as described below:

φ^*C has underlying set $(C\times C) \cup \{ u_\alpha \mid \alpha \in \text{supp}(C\times C) \cap (\varphi\times\varphi)^{-1}\Delta \}$ where Δ is the diagonal of $R\times R$ and, besides the inherited order relations of $C\times C$, we define $x \prec u_\alpha \prec y$ for each $\alpha = [x, y]$. (see Figure 5.1)

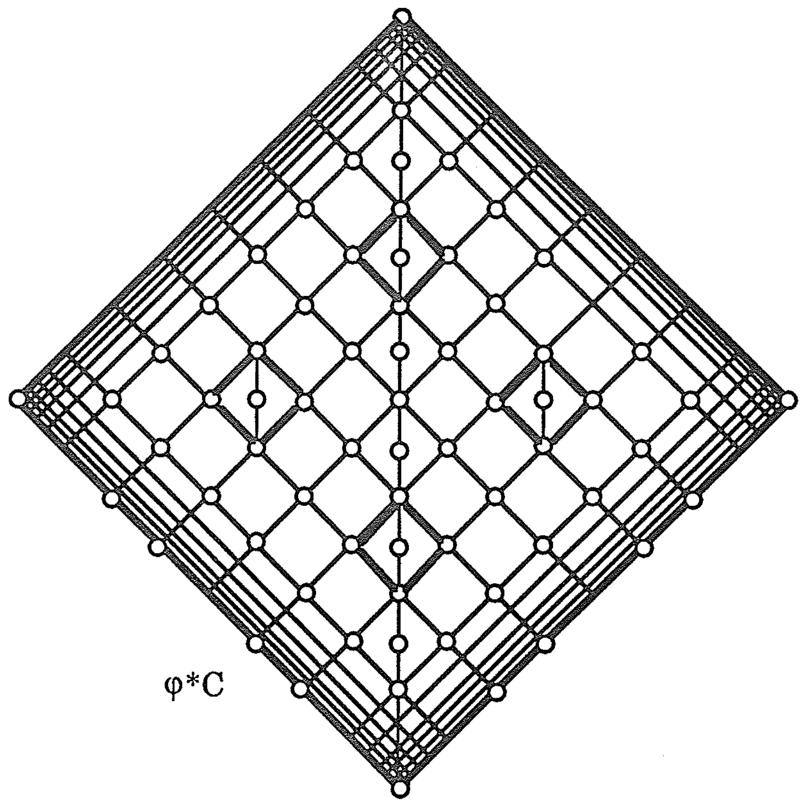


Figure 5.1

A final word about the notation. The elements of ω will be named by 0, 1, 2, ... in the usual order. The element $x \in L_\alpha$ in $\Sigma(L_\alpha \mid \alpha < \gamma)$ will be written as x_α and the corresponding element of $x \in L$ in L^d will be denoted as x^d. For ease of future reference, we shall reserve the letters x and y for the labelling of the elements of φ^*C mentioned above in the following manners: For each $(.,.) \in C\times C$, we label it by $x(.,.)$ and for each $u_\alpha \prec x(.,.)$, we label it as $y(.,.)$. An appropriate subscript will be added to x and y for different copies of lattices in our construction. Let $D = C^d \oplus C$ be a chain with valuation φ. Then we call the chain $\Delta(\varphi^*D) = \{ x(r,r) \mid r \in D \}$ and the chain $\Delta^+(\varphi^*D) = \{ x(r,r) \mid r \in C \}$ the diagonal and upper diagonal of φ^*D respectively.

Let L be a complete lattice. For a, b $\in$ L, let $\theta^*(a, b)$ be the principal complete congruence of L collapsing a and b ($\theta^*(a, b)$ is well defined as the intersection of arbitrary complete congruences is still a complete congruence). For $\Theta \in$ Com L, we have $\Theta = \underline{\vee} (\theta^*(a, b) \mid [a,b] \in I)$ where I ranges over all the closed interval [a,b] collapsed by Θ. We say that L is Θ-discrete if for each $\Theta \in$ Com L, the index set I can be restricted to the set of discrete intervals of L. Thus if C is a chain obtained from ω and $\mathbf{n}$ by the operations +, $\oplus$, $(.)^d$, then C is Θ-discrete. However, the real closed interval [0, 1] is not Θ-discrete.

The Construction of L

The construction of L is done in two parts. In part (I), we construct L for the case that K is finite. In part (II), we modify the construction of part (I) and construct a complete lattice L for arbitrary complete lattice K. For the infinite case, a similar construction was also given by G. Grätzer [Gr-2]. Let K be a complete lattice with zero $\varnothing$ and unit ι. Let $K^* = K - \{ \varnothing \}$.

<u>(I) K is finite</u>

Let the elements of K* be listed in a fixed sequence $a_1, a_2, \ldots, a_n = \iota$. Let $K^{(1)} = K^* - \{ \iota \}$ and $K^{(2)} = \{ \{a,b\} \mid a, b \in K^*$ and a, b are not comparable $\}$. We construct the following complete sublattices of L. For each a $\in K^{(1)}$, the sublattice L_a which reflects the order relation of K, and for each $\alpha \in K^{(2)}$, the sublattice L_α which reflects the join operation of K.

(i) sublattice L_0.

Let C_0 be the chain $\omega + 1$ and let the valuation φ : supp $C_0 \to K$ be given by $\varphi_0[2k-1, 2k] = a_k$ for $k = 1, 2, \ldots, n-1$, and $\varphi_0[k, k+1] = \iota$ otherwise. Let $D_0 = C_0{}^d \oplus C_0$ and let $\psi_0 = \varphi_0{}^d \cup \varphi_0$ be the natural valuation of D_0. Let $L_0 = (\psi_0{}^*D_0) \cup \{ z_0 \}$ be given additional order relation $O_{\psi_0{}^*D_0} -\!\!< z_0 -\!\!< I_{\psi_0{}^*D_0}$. Then L_0 is a complete lattice. The elements of $\psi_0{}^*D_0$ will be distinguished with a subscript 0.

(ii) For $a \in K^{(1)}$, sublattice L_a.

Let the subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_r}$ be a listing of $(a] - \{ \varnothing \}$. Let C_a be the chain $\omega + 1$ and let the valuation φ_a : supp $C_a \to K$ be given by:

$$\varphi_a[x, y] = \begin{cases} a & \text{if } [x, y] = [0, 1] \text{ or } [r+1, r+2], \\ a_{i_k} & \text{if } [x, y] = [k, k+1] \text{ or } [2k+r+1, 2k+r+2],\ k = 1, 2, \ldots, r, \\ \iota & \text{otherwise.} \end{cases}$$

Let $D_a = C_a{}^d \oplus C_a$ and let $\psi_a = \varphi_a{}^d \cup \varphi_a$ be the natural valuation of D_a. Let $L_a = (\psi_a{}^*D_a) \cup \{ w_a, z_a \}$ and let $O_{\psi_a{}^*D_a} -\!\!< z_a -\!\!< I_{\psi_a{}^*D_a}$, $y((r+1)^d, (r+1)^d) -\!\!< w_a -\!\!< y((r+2), (r+2))$. Then L_a is a complete lattice. The elements of $\psi_a{}^*D_a$ will be written with a subscript a.

(iii) For $\alpha \in K^{(2)}$, $\alpha = \{ a, b \}$, sublattice L_α.

Let the subsequence $a_{i_1}, a_{i_2}, a_{i_3}$ be a listing of $\{ a, b, a \vee b \}$. Let ω_1, ω_2 be two copies of ω and let C_α be the chain $\omega_1 + \omega_2 + 1$. Let the valuation φ_α be given as below:

$$\varphi_\alpha[x, y] = \begin{cases} a \vee b & \text{if } [x, y] = [0, 1], \\ a & \text{if } [x, y] = [(k)_1, (k+1)_1] \text{ and k is odd}, \\ b & \text{if } [x, y] = [(k)_1, (k+1)_1] \text{ and k is even}, \\ a_{i_k} & \text{if } [x, y] = [(2k-1)_2, (2k)_2] \text{ for } k = 1, 2, 3, \\ \iota & \text{otherwise}. \end{cases}$$

Let $D_\alpha = C_\alpha^{\ d} \oplus C_\alpha$ and let $\psi_\alpha = \varphi_\alpha^{\ d} \cup \varphi_\alpha$. Let $L_\alpha = (\psi_\alpha{}^*D_\alpha) \cup \{ w_\alpha, z_\alpha \}$ be given additional order relation $O_{\psi_\alpha{}^*D_\alpha} \prec z_\alpha \prec I_{\psi_\alpha{}^*D_\alpha}$ and $x(0_2^{\ d}, 0_2^{\ d}) \prec w_\alpha \prec x(0_2, 0_2)$. Then L_α is a complete lattice. The elements of $\psi_\alpha{}^*D_\alpha$ will be written with a subscript α.

A sketch of the valuation of the chains C_0, C_a, C_α and the lattices L_0, L_a and L_α are given in Figure 5.2 and Figure 5.3 respectively.

Let $L' = L_0 \cup \bigcup (L_a \mid a \in K^{(1)}) \cup \bigcup (L_\alpha \mid \alpha \in K^{(2)})$. We identified all the zeros of L_0, L_a, L_α and all the units of L_0, L_a, L_α. Furthermore, we introduce additional order relations so that the support of each of the L_0, L_a, and L_α which have the same value are projective to each other. We accomplish this by adding the order relations as descibed in (V-1) and (V-2):

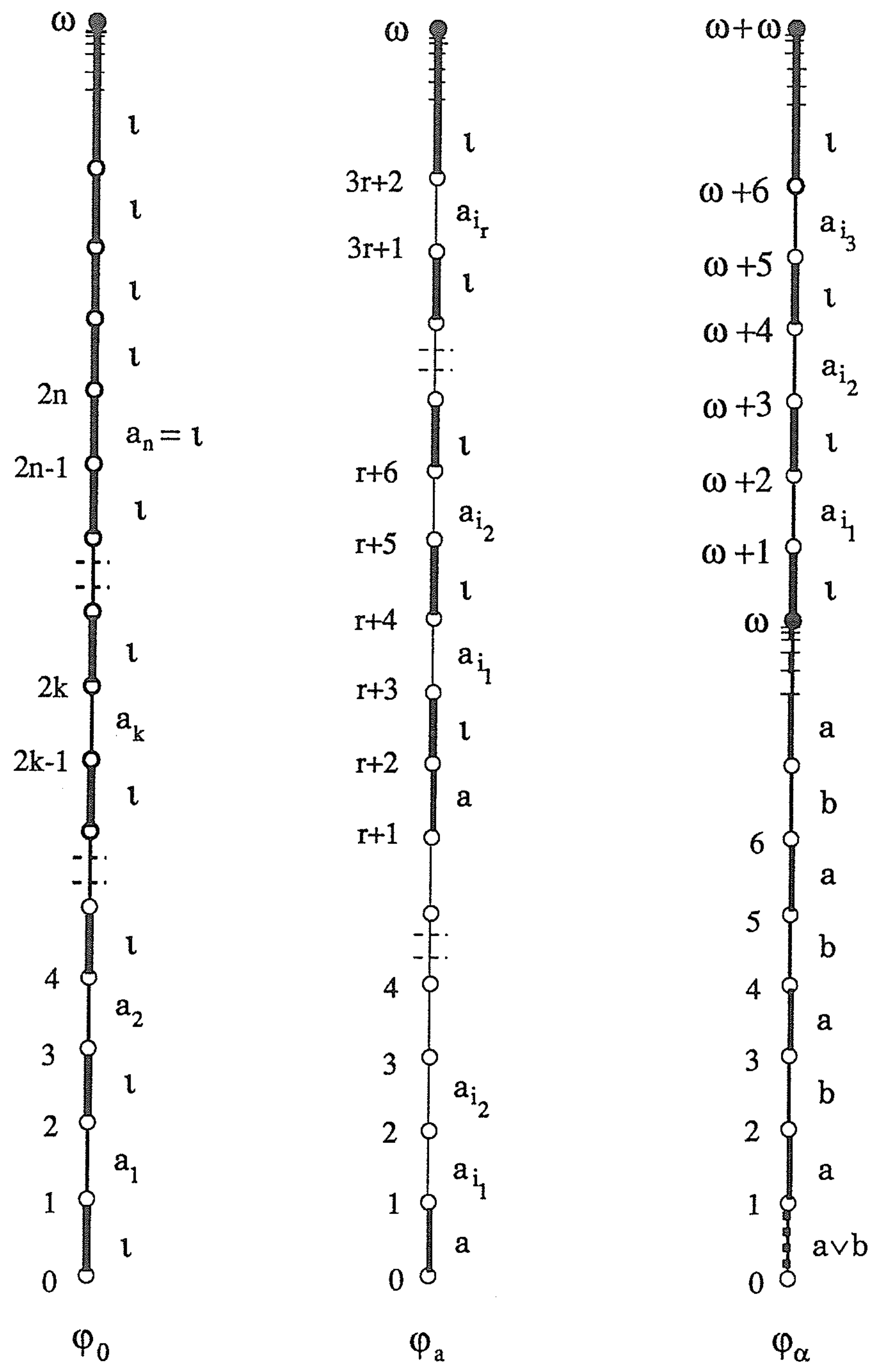

ω
ι
ι
ι
ι
2n
$a_n = ι$
2n-1
ι
ι
2k
a_k
2k-1
ι
ι
4
a_2
3
ι
2
a_1
1
ι
0
φ_0
ω
ι
3r+2
a_{i_r}
3r+1
ι
ι
r+6
a_{i_2}
r+5
ι
r+4
a_{i_1}
r+3
ι
r+2
a
r+1
4
3
a_{i_2}
2
a_{i_1}
1
a
0
φ_a
ω+ω
ι
ω +6
a_{i_3}
ω +5
ι
ω +4
a_{i_2}
ω +3
ι
ω +2
a_{i_1}
ω +1
ι
ω
a
b
6
a
5
b
4
a
3
b
2
a
1
a∨b
0
φ_α

Figure 5.2

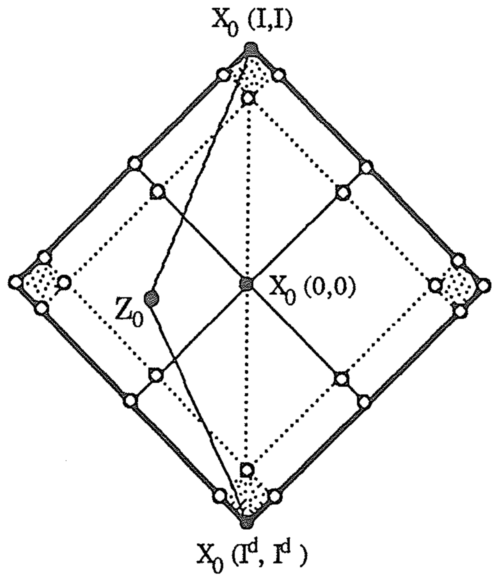


Figure 5.3(a)

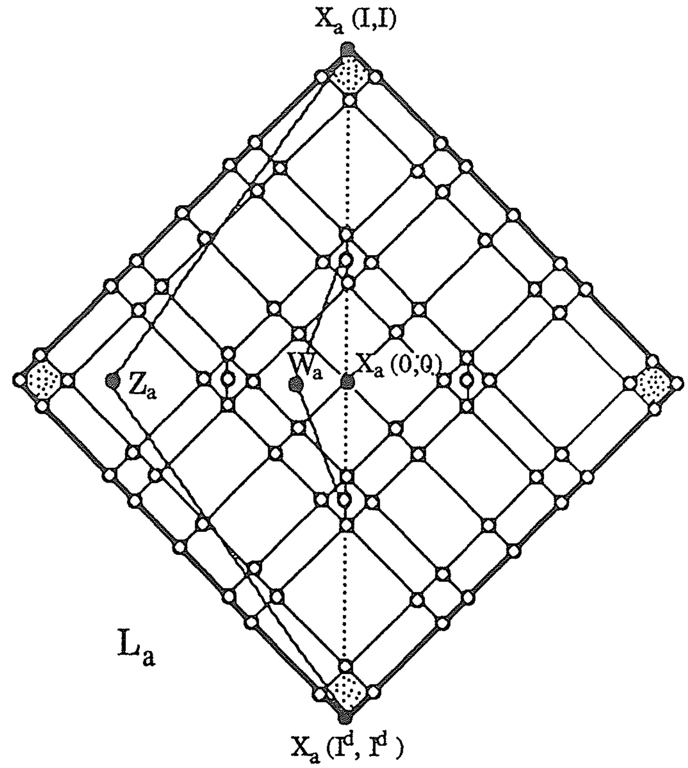


Figure 5.3(b)

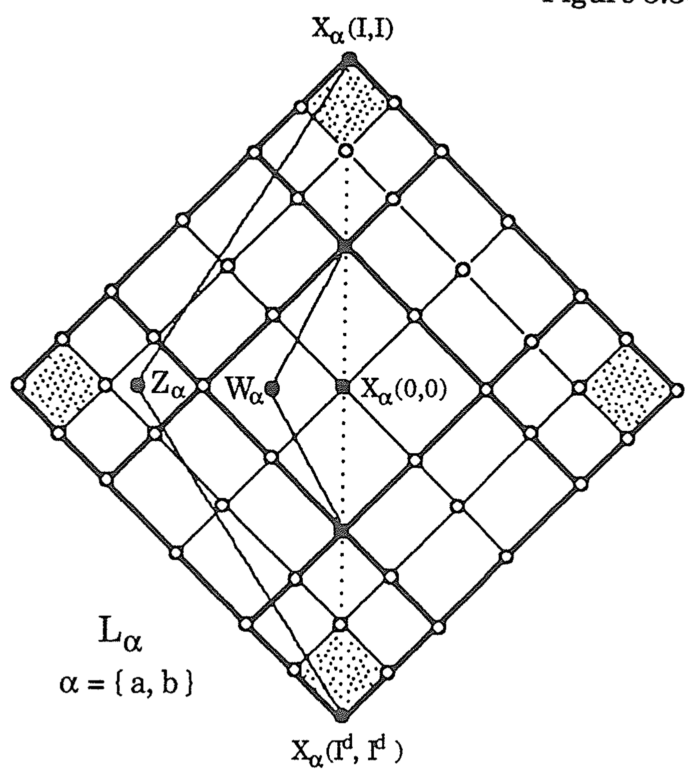


Figure 5.3(c)

(i) For each L_a, $a \in K^{(1)}$, $a_j = a_{i_k} \in (a] - \{\emptyset\}$, $k = 1, 2, ..., r$,

$$x_0(2j\text{-}1, 2j\text{-}1) \prec x_a(2k+r+1, 2k+r+1)$$
$$x_0(2j, 2j) \prec x_a(2k+r+2, 2k+r+2)$$
$$x_0(2n\text{-}1, 2n\text{-}1) \prec x_a(3r+3, 3r+3)$$
$$x_0(2n, 2n) \prec x_a(3r+4, 3r+4) \qquad \textbf{(V-1)}$$

(ii) For each L_α, $\alpha = \{a, b\} \in K^{(2)}$, $a_j = a_{i_k} \in \{a, b, a \vee b\}$, $k = 1, 2, 3$,

$$x_0(2j\text{-}1, 2j\text{-}1) \prec x_\alpha((2k\text{-}1)_2, (2k\text{-}1)_2)$$
$$x_0(2j, 2j) \prec x_\alpha((2k)_2, (2k)_2) \qquad \textbf{(V-2)}$$

Let L be the resulting poset. For a subset S of L, we write $S = S_0 \cup \bigcup (S_a \mid a \in K^{(1)}) \cup \bigcup (S_\alpha \mid \alpha \in K^{(2)})$ where $S_0 = S \cap L_0$, $S_a = S \cap L_a$ and $S_\alpha = S \cap L_\alpha$. By observing that the additional covering relations (V-1) and (V-2) are given along the upper diagonals of L_0, L_a and L_α, and are order preserving, i.e., $a < b$ and $a \prec c$, $b \prec d$ imply that $c < d$. We have the followings:

(i) For each S_i, $i \in K^{(1)} \cup K^{(2)}$, there is a largest element $p_0 \in \Delta(\psi_0{}^*D_0)$ such that $p_0 \leq \wedge_i S_i \in L_i$. Hence $\wedge S_i = \wedge_i S_i$ in L. Obviously $\wedge S_0 = \wedge_0 S_0$. Generally, if $x \in L$, we write $p_0(x)$ for the largest element of $\Delta(\psi_0{}^*D_0)$ such that $p_0(x) \leq x$. **(V-3)**

(ii) There is a least element $q_i \in \Delta^+(\psi_i{}^*D_i)$ such that $q_i \geq \vee_0 S_0 \in L_0$. Thus $\vee S_0 = \vee_0 S_0$ in L. Clearly $\vee S_i = \vee_i S_i$ for $i \in K^{(1)} \cup K^{(2)}$. Generally, if $x \in L$, we write $q_i(x)$ for the least element of $\Delta^+(\psi_i{}^*D_i)$ such that $q_i(x) \geq x$. **(V-4)**

Claim 1. L is a complete lattice.

Proof: In view of (V-3) and (V-4), we need only verify that any two elements of L has join and meet. Let r, s $\in$ L, up to symmetry, we have the following possibilities:

(i) Both r, s $\in L_0$ (or $\in L_i$, $i \in K^{(1)} \cup K^{(2)}$).

In this case, the join and meet of r and s are respectively the join and meet that have already exist in L_0 (or L_i).

(ii) $r \in L_i$, $s \in L_j$, $i, j \in K^{(1)} \cup K^{(2)}$, $i \neq j \neq 0$.

In this case, $r \vee s$ is always the unit of L and by (V-3), we have $r \wedge s = p_0(r) \wedge_0 p_0(s)$.

(iii) $r \in L_0$, $s \in L_i$.

By (V-3) and (V-4), we have $r \vee s = q_i(r) \vee_i s$ and $r \wedge s = r \wedge_0 p_0(s)$. ❑

Claim 2. Com L $\cong$ K.

Proof: It is not difficult to observe that each of the L_0 and L_i, $i \in K^{(1)} \cup K^{(2)}$ are Θ-discrete. Indeed, Com L_i, $i = 0$ or $i \in K^{(1)} \cup K^{(2)}$, is generated by $\Delta^+(\psi_i{*}D_i)$. Let $\Theta \in$ Com L, if there exist r and s such that $r \in L_i$, $s \in L_j$, $L_i \neq L_j$, and $r \equiv s$ (Θ), then $r \wedge s \equiv r \vee s$ (Θ). This implies Θ must collapse some interval [x, y] having value (ι, ι) and $\Theta = \iota$. Thus, if $\Theta \neq \iota$, every congruence classes of Θ must be a congruence classes of L_i when Θ is restricted to L_i. By using (V-1) and (V-2), we can conclude that every $\Theta \in$ Com L is generated by $\Delta^+(\psi_0{*}D_0)$. For each $a \in K^*$, it is not difficult to see that there is an $\Theta_i(a) \in$ Com L_i, $i = 0$ or $i \in K^{(1)} \cup K^{(2)}$, such that $[x, y] \in \Delta^+(\psi_i{*}D_i)$ and $x \equiv y$ $(\Theta_i(a))$, iff $\psi_i{\times}\psi_i [x, y] = (b, b)$, where $b \leq a$. Let $\Theta(a) = \cup\, (\Theta_i(a) \mid i = 0, i \in K^{(1)} \cup K^{(2)})$, then $\Theta(a) \in$ Com L. We have to verify the substitution property. Let x and y be the maximal and the minimal elements of a congruence class of Θ respectively. Let z be arbitrary element of L. Then we have the following cases:

(i) $x, y \in L_i$, $z \in L_j$, $i, j \in K^{(1)} \cup K^{(2)} \cup \{0\}$, $i = j$.

In this case, the substitution property is satisfied as Θ_i is a complete congruence relation of L_i.

(ii) $x, y \in L_i$, $z \in L_j$, $i, j \in K^{(1)} \cup K^{(2)} \cup \{0\}$, $0 = i \neq j$ or $i \neq j = 0$.

If $i = 0$, then $x \vee z = q_j(x) \vee z \equiv q_j(y) \vee z = y \vee z\ (\Theta_j)$, and $z \wedge x = p_0(z) \wedge x \equiv p_0(z) \wedge y = z \wedge y\ (\Theta_0)$. If $j = 0$, then $x \vee z = x \vee q_j(z) \equiv y \vee q_j(z) = y \vee z\ (\Theta_i)$, and $x \wedge z = z \wedge p_0(x) \equiv z \vee p_0(z) = z \vee y\ (\Theta_0)$.

(iii) $x, y \in L_i$, $z \in L_j$, $i \neq j \neq 0$.

In this case, $x \vee z = I \equiv I = y \vee z\ (\Theta)$, and $x \wedge z = p_0(x) \wedge p_0(z) \equiv p_0(y) \wedge p_0(z) = y \wedge z\ (\Theta_0)$.

For each $\Theta \in$ Com L. Let $\pi(\Theta) = \{ a \mid a \in K,\ \{x, y\} \subset \Delta^+(\psi_0{}^*D_0)$, and $x \equiv y\ (\Theta)$, $\psi_0 \times \psi_0\,[x, y] = (a, a) \}$. If $a, b \in \pi(\Theta)$, then $a \vee b \in \pi(\Theta)$ by $L_{\{a,b\}}$. If $b \leq a$ and $a \in \pi(\Theta)$, then $b \in \pi(\Theta)$ by L_a. Therefore, we have $\pi(\Theta(a)) = (a]$. Thus the mapping $\Theta \to \pi(\Theta)$ is an isomorphism of Com L to the principal ideal of K. Hence Com L $\cong$ K. ❑

<u>(II) K is infinite</u>

The construction for the infinite case is similar to the finite case. However, from the discussion in (I), we note that in order for the proof of

Claim 1 and Claim 2 to be valid, we need to accomplish the following two requirements:

(i) The properties stated in (V-3) and (V-4) must be preserved, i.e. the order relations given in (V-1) and (V-2) must preserved the upward and the downward continuity of join and meets. **(V-5)**

(ii) Given any infinite subset J of K, we must have $\Theta(\vee J) = \vee\,(\Theta(c) \mid c \in J)$ where the meaning of $\Theta(c)$ is explained in Claim 2. **(V-6)**

We assume the axiom of choice. For a set H, let γ_H denotes the least ordinal well ordering H, i.e. $H = \{ a_\alpha \mid \alpha < \gamma_H \}$. Let m_H denotes the cardinal of H. For an infinite cardinal $m \le m_K$, let $K^{(m)} = \{ J \mid J \subseteq K,\ m = m_J \}$. For an infinite set $J = \{ r_\alpha \mid \alpha < \gamma_J \} \in K^{(m)}$, we define $J^o = \{ s_\alpha \mid \alpha < \gamma_J,\ s_\alpha = \vee\,(r_\beta \mid \beta < \alpha) \}$. The elements of J^o form a chain (multi-chain) of K which is well-ordered by γ_J. Clearly $\vee(J) = \vee(J^o)$. The successor of the ordinal α is denoted by α^+.

In the following, we give various complete sublattices of L. They are similar to those given in (I) with some modifications: The sublattices L_0, L_a, L_α as described in (i), (ii) and (iii) serve the same purpose as their counterparts in the finite case. As for the infinite join of elements of K, we construct, in (iv), sublattices L_J for each $J \in K^{(m)}$, $m \le m_K$, m an infinite cardinal.

(i) the sublattice L_0.

Let $\gamma_{K(1)} = \kappa$ and let $K^{(1)} = \{ a_\alpha \mid \alpha < \kappa \}$. Let A, B, C be the chains $\mathbf{2}$, $\Sigma\,(\mathbf{2}_\alpha \mid \alpha < \kappa)$ and $\omega + \mathbf{1}$ respectively. Let $C_0 = A \oplus B + C$ (note: C_0 is a complete chain) and let the valuation φ_0 be given as: $\varphi_0\,[x, y] = a_\alpha$ if

$[x, y] = [0_\alpha, 1_\alpha]$, $\alpha < \kappa$, and $\varphi_0 [x, y] = \iota$ otherwise. Let $D_0 = C_0^d \oplus C_0$ with valuation $\psi_0 = \varphi_0^d \cup \varphi_0$. Then L_0 has underlying set $(\psi_0{*}D_0) \cup \{ z_0 \}$ with additional order relation $O_{\psi_0*D_0} \prec z_0 \prec I_{\psi_0*D_0}$.

(ii) the sublattices L_a for each $a \in K^{(1)}$.

Let $\gamma_{(a]-\{\varnothing\}} = \kappa$ and let $(a] - \{\varnothing\} = \{ a_\alpha \mid \alpha < \kappa \}$. Let A and B be the chains $\Sigma (\mathbb{1}_\alpha \mid \alpha < \kappa)$ and $\mathbb{2}$ respectively. Let $C_a = A + B \oplus C_0$ and let the valuation φ_a be given by:

$$\varphi_a[x, y] = \begin{cases} a_\alpha & \text{if } [x, y] = [0_\alpha, 0_{\alpha^+}] \ \ (0_{\alpha^+} = O_B \text{ if } \alpha^+ = \kappa), \\ a & \text{if } [x, y] = [O_B, I_B], \\ \varphi_0[x, y] & \text{if } [x, y] \in \operatorname{supp} C_0. \end{cases}$$

Let $D_a = C_a^d \oplus C_a$ and let the valuation $\psi_a = \varphi_a^d \cup \varphi_a$. Then L_a has underlying set $(\psi_a{*}D_a) \cup \{ w_a, z_a \}$ with additional order relations: $O_{\psi_a*D_a} \prec z_a \prec I_{\psi_a*D_a}$ and $y_a(O_B^d, O_B^d) \prec w_a \prec y_a(I_B, I_B)$.

(iii) the sublattices L_α for each $\{ a, b \} = \alpha \in K^{(2)}$.

Let C_α be the chain $\omega + C_0$ and let the valuation φ_α be:

$$\varphi_\alpha[x, y] = \begin{cases} a \vee b & \text{if } [x, y] = [0, 1], \\ a \text{ (or } b) & \text{if } [x, y] = [k, k+1], \ k \text{ is odd (or even)}, \\ \varphi_0[x, y] & \text{if } [x, y] \in \operatorname{supp} C_0. \end{cases}$$

Let $D_\alpha = C_\alpha{}^d \oplus C_\alpha$ and let the valuation $\psi_\alpha = \varphi_\alpha{}^d \cup \varphi_\alpha$. Then L_α has underlying set $(\psi_\alpha{}^*D_\alpha) \cup \{ w_\alpha, z_\alpha \}$ with additional order relations:
$O_{\psi_\alpha{}^*D_\alpha} \prec z_\alpha \prec I_{\psi_\alpha{}^*D_\alpha}$ and $x_\alpha(O_{C_0}{}^d, O_{C_0}{}^d) \prec w_\alpha \prec x_\alpha(O_{C_0}, O_{C_0})$.
(iv) the sublattices L_J for each $J \in K^{(m)}$, $m \leq m_K$, m an infinite cardinal.

Denote γ_J by κ and let $J^o = \{ s_\alpha \mid \alpha < \kappa \}$. Let A and B be the chains $\mathbf{2}$ and $\Sigma (\mathbb{1}_\alpha \mid \alpha < \kappa)$ respectively. Let $C_J = A \oplus B + C_0$ and let the valuation φ_J be given by:

$$\varphi_J[x, y] = \begin{cases} \bigvee (J) = \bigvee (J^o) & \text{if } [x, y] = [O_A, I_A], \\ s_\alpha & \text{if } [x, y] = [0_\alpha, 0_{\alpha^+}], \ (0_{\alpha^+} = O_{C_0} \text{ if } \alpha^+ = \kappa), \\ \varphi_0[x, y] & \text{if } [x, y] \in \text{supp } C_0. \end{cases}$$

Let $D_J = C_J{}^d \oplus C_J$ and let the valuation $\psi_J = \varphi_J{}^d \cup \varphi_J$. Then L_J has underlying set $(\psi_J{}^*D_J) \cup \{ w_J, z_J \}$ with additional order relations:
$O_{\psi_J{}^*D_J} \prec z_J \prec I_{\psi_J{}^*D_J}$ and $x_J(O_{C_0}{}^d, O_{C_0}{}^d) \prec w_J \prec x_J(O_{C_0}, O_{C_0})$.

Let $L' = L_0 \cup \bigcup (L_i \mid i \in K^{(1)} \cup K^{(2)}) \cup \bigcup (L_J \mid J \in K^{(m)}, m \leq m_K)$. Then L is obtained from L' by identifying all the zeros of L_0, L_i, L_J and all the units of L_0, L_i, L_J, with the following additional order relations:

For all $r \in C_0 - \{ O_{C_0}, I_{C_0} \}$, and $j \in K^{(1)} \cup K^{(2)} \cup \bigcup (K^{(m)} \mid m \leq m_K)$,

$$x_0(r, r) \prec x_j(r, r) \tag{V-7}$$

It is not difficult to see that (V-7) satisfies (V-5). Hence L is a complete lattice by a similar argument of Claim 1. As for (V-6), we use transfinite induction. Let β be an ordinal. Suppose that for all $J \subseteq K$ such that $\gamma_J < \beta$, we have $\Theta(\vee J) = \vee(\Theta(r_\alpha) \mid \alpha < \gamma_J)$. Let $J \subseteq K$ and $\gamma_J = \beta$. If $\beta = \alpha^+$, then $\vee(J) = s_\alpha \vee r_\alpha$, $\Theta(\vee J) = \Theta(s_\alpha \vee r_\alpha) = \Theta(s_\alpha) \vee \Theta(r_\alpha) = \vee(\Theta(r_\delta) \mid \delta < \alpha) \vee \Theta(r_\alpha) = \vee(\Theta(r_\delta) \mid \delta < \beta)$. If β is an limit ordinal, then for each $\alpha < \beta$, we have $\Theta(\vee J_\alpha) = \vee(\Theta(r_\delta) \mid \delta < \alpha)$, where J_α is the α-initial segment of J. Hence $\Theta(\vee J)$ collapses all interval [x, y] in $L_J \cap \Delta^+(\psi_J{}^*D_J)$ which has value $(\vee J_\alpha, \vee J_\alpha)$. But then $\Theta(\vee J)$, being a complete congruence relation, also collapses an interval in $L_J \cap \Delta^+(\psi_J{}^*D_J)$ having value $(\vee J, \vee J)$. Hence $\Theta(\vee J) = \vee(\Theta(\vee J_\alpha) \mid \alpha < \beta)) = \vee(\Theta(r_\alpha) \mid \alpha < \beta)$ and (V-6) is satisfied. Thus we have established an isomorphism between Com L and the set of principal ideals of K, i.e. Com L $\cong$ K. Thus we have proved:

Theorem 5.1 For every complete lattice D, there is a complete lattice L such that Com L $\cong$ D.

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