# Quantum State Transfer between Twins in Graphs

by

Hermie Monterde

A thesis submitted to the Faculty of Graduate Studies of
The University of Manitoba
in partial fulfillment of the requirements of the degree of

Master of Science

Department of Mathematics
University of Manitoba
Winnipeg

Copyright © 2021 by Hermie Monterde

## Abstract

A quantum spin network can be modelled by an undirected graph whose vertices and edges represent qubits and their interactions in the network, respectively. One major problem involving quantum spin networks is determining a time $\tau$ such that the state at one vertex is transported to another vertex at time $\tau$ with a particular level of probability, called the fidelity of quantum state transfer. Various useful types of quantum state transfer arise depending on the fidelity: periodicity, perfect state transfer, pretty good state transfer, and fractional revival.

In this thesis, we investigate quantum state transfer between twin vertices in weighted graphs with or without loops. We provide spectral and algebraic characterizations of twin vertices, and use these to identify which properties of twin vertices are essential for various types of quantum state transfer to occur between any two of them. A characterization of vertices in unweighted graphs that exhibit strong cospectrality with an additional vertex in the post-twinning graph is also given. Moreover, we determine necessary and sufficient conditions for periodicity, perfect state transfer, pretty good state transfer, and fractional revival to occur between twin vertices with respect to adjacency dynamics. Then, we apply our results to double cones on regular graphs, which are a special class of graphs with twin vertices. Finally, we explore quantum state transfer in some common families of graphs with respect to adjacency dynamics.

# Acknowledgements

First of all, I am grateful to Dr. Sarah Plosker and Dr. Steve Kirkland, for the opportunity to work with them, and for the unwavering support and guidance they provided me throughout the program. In this challenging era of CoVid-19, their generosity, optimism, and trust were vital in the completion of this work.

I would like to thank my supervisors, and acknowledge the Department of Mathematics, the Faculty of Science, and the Faculty of Graduate Studies at the University of Manitoba for the financial support.

I would like to thank the faculty members of the Department of Mathematics at the University of Manitoba, especially Dr. Julien Arino, Dr. Susan Cooper, Dr. Michelle Davidson, Dr. Karen Gunderson, and Dr. Derek Krepski, for being great mentors and teachers. Their compassion and excellence are inspirational.

A big thank-you to the staff of the Department of Mathematics at the University of Manitoba, especially Irene, John, Leah, Sara, and Clifford for all the help they have extended to me since day one. My life as a graduate student would not have been as easy without their great work and valuable support.

I am thankful for my family, especially my grandmother, Neria Oberio, for the love that transcends distance. I am thankful for my friends, in Canada (Avleen, Brock, Eugene, Ikaro, Kristaps, Lubana, Sooyeong), Philippines (Astrid, Carlo, Ciara, Janine, Jay, Jeff, Jethro, Karl, Ryan), and across the world (Arlou, Dem, Jj, Joanne, Jules, Kevin), for the good times. Their kindness, empathy, and humour kept me sane during the pandemic. I am also thankful to my homestay mom, Ninfa Bagan, for welcoming me to her home since I arrived in Winnipeg. I would also like to thank my comrades in Migrante Manitoba, especially Ali, Anthony, Diwa, Janice, kuya Levy, ate Liza, and ate Susan, for the kinship. Their moral, emotional, and social support meant a great deal to me as a Filipino migrant.

I would like to thank Dr. Andriy Prymak and Dr. Robert T.W. Martin for their helpful comments and suggestions.

Lastly, I am grateful to Landon, for the love and companionship.

To Landon and Gracie.

# Contents

# List of Figures

# List of Symbols

| | |
|---|---|
| $X$ | an undirected weighted graph with or without loops |
| $V(X)$ | the vertex set of $X$ |
| $E(X)$ | the edge set of $X$ |
| $N_X(u)$ | the set of neighbours of vertex $u$ in $X$ |
| $\overline{X}$ | the complement of graph $X$ |
| $X\backslash u$ | the resulting graph after deleting vertex $u$ from $X$ |
| $X_1 \cong X_2$ | $X_1$ and $X_2$ are isomorphic |
| $X_1 \cup X_2$ | the union of two graphs $X_1$ and $X_2$ |
| $X_1 \vee X_2$ | the join of two graphs $X_1$ and $X_2$ |
| $X_1 \square X_2$ | the cartesian product of two graphs $X_1$ and $X_2$ |
| $nX$ | the union of $n$ copies of $X$ |
| $X^{\square n}$ | the cartesian product of $n$ copies of $X$ |
| $X/\pi$ | the quotient of $X$ with respect to an equitable partition $\pi$ |
| $\widehat{X/\pi}$ | the symmetrized quotient of $X$ with respect to an equitable partition $\pi$ |
| $C_n$ | the cycle on $n$ vertices |
| $K_n$ | the complete graph on $n$ vertices |
| $K_n\backslash e$ | the complete graph on $n$ vertices minus an edge $e$ |
| $K_{m,n}$ | the complete bipartite graph on $n$ vertices |
| $O_n$ | the empty graph on $n$ vertices |
| $P_n$ | the cycle on $n$ vertices |
| $Q_n$ | the hypercube of dimension $n$, also known as the $n$-cube |
| $\overline{nK_2}$ | the cocktail party graph on $n$ vertices |
| $\mathbf{K}_n(\omega, \eta)$ | the complete graph on $n$ vertices with loops of weight $\omega$ and edges of weight $\eta$ |
| $\mathbf{K}_{1,n-1}(\omega, \eta)$ | the weighted star on $n$ vertices with loops of weight $\omega$ on vertices of degree one and edges of weight $\eta$ |
| $\mathbf{O}_n(\omega)$ | the empty graph on $n$ vertices with loops of weight $\omega$ |

| | |
|---|---|
| $\mathbf{K}_2(\omega,\eta) \vee X$ | the weighted double cone on $X$ with loops of weight $\omega$ and edge of weight $\eta$ between the apexes |
| $K_2 \vee X$ | the unweighted connected double cone on $X$ |
| $\overline{K_2} \vee X$ | the unweighted disconnected double cone on $X$ |
| $U(t)$ | the transition matrix of a graph $X$ with respect to $M(X)$ |
| $L(X)$ | the Laplacian matrix of a graph $X$ |
| $A(X)$ | the adjacency matrix of a graph $X$ |
| $M(X)$ | either $A(X)$ or $L(X)$ |
| $E_\lambda$ | orthogonal projection matrix onto the eigenspace associated to $\lambda$ |
| $\sigma(M(X))$ | spectrum of $M(X)$ |
| $\sigma_u(M(X))$ | eigenvalue support of a vertex $u$ in a graph $X$ with respect to $M(X)$ |
| $\sigma_u^{\pm}(M(X))$ | set of all eigenvalues $\lambda \in \sigma_u(M(X))$ such that $E_\lambda e_u = \pm E_\lambda e_v$ |
| $\phi(M,t)$ | the characteristic polynomial of a square matrix $M$ in the variable $t$ |
| $\rho$ | the minimum period of a vertex $u$ |
| $\\|\mathbf{x}\\|$ | the Euclidean norm of a vector $\mathbf{x}$ |
| $A \circ B$ | the Schur (Hadamard) product of matrices $A$ and $B$ |
| $A \otimes B$ | the tensor product of matrices $A$ and $B$ |
| $A^T$ | the transpose of a matrix $A$ |
| $A^*$ | the conjugate transpose of a matrix $A$ |
| $\mathbf{x}^T$ | the transpose of a vector $\mathbf{x}$ |
| $\mathbf{x}^*$ | the conjugate transpose of a vector $\mathbf{x}$ |
| $J_{m,n}$ | the all-ones $m \times n$ matrix |
| $J_n$ | the all-ones $n \times n$ matrix |
| $I_n$ | the identity $n \times n$ matrix |
| $\mathbf{1}_n$ | the all-ones vector in $\mathbb{C}^n$ |
| $\mathbf{0}_n$ | the all-zeros vector in $\mathbb{C}^n$ |
| $\mathbf{e}_1,\ldots,\mathbf{e}_n$ | the standard basis vectors for $\mathbb{C}^n$ |
| $\lvert a\rvert$ | the modulus of a complex number $a$ |
| $\mathbb{Z}[x]$ | the set of all polynomials with integer coefficients |
| $\nu_p(a)$ | the $p$-adic valuation of an integer $a$ |

# 1

# Introduction

The principles of quantum mechanics dictate that quantum states exhibit quantum superposition and entanglement. *Quantum superposition* is an intriguing ability of a quantum system to exist in multiple states at once until observed, while *quantum entanglement* allows quantum states to interact regardless of distance. Quantum computers exploit these two properties, and as a result, quantum computers solve certain problems exponentially faster than their classical counterparts. For this reason, quantum computing has attracted much attention in the last few decades.

Quantum states carry information, and in order to construct an operational quantum computer, two essential tasks must be accomplished: the accurate transmission of quantum states from one location in the quantum computer to another, and generating entanglements between quantum states. These two phenomena fall under the generational notion of *quantum state transfer*. In particular, *perfect state transfer* is achieved whenever a quantum state assigned at one location appears as the same quantum state (up to a phase factor) at a different location at a later time. However, if the final quantum state is not exactly the same as the initial quantum state, but can be made arbitrarily close, then we say that there is *pretty good state transfer*. Moreover, *fractional revival* happens when a quantum state at one location exists in multiple locations, including the initial location, at a later time. Identifying which quantum spin network topologies allow these types of quantum state transfer to occur has been an important area of research in quantum information theory. As it turns out, the combinatorial properties of the quantum spin network, as well as the spectral properties of a matrix associated to the quantum spin network, are the keys to understanding how quantum state transfer works.

In this thesis, we look at various types of quantum state transfer from a mathematical perspective, and use techniques from algebraic graph theory, combinatorial matrix theory, and spectral graph theory to derive our results.

## 1.1 Motivation

A *quantum state* is a complex unit vector associated to a *qubit* (*spin*). A network of $n$ qubits is called a *quantum spin network*, which can be mathematically modelled by an undirected weighted graph $X$ on $n$ vertices with or without loops whose vertex and edge sets are the qubits and their interactions, respectively. The weight of an edge can be physically interpreted as the *coupling strength* between the two interacting qubits, while the weight of a loop on a vertex can be viewed as a *potential* on a qubit, also known as *energy shift*, which can sometimes represent the strength of the magnetic field on the qubit. We then initialize our quantum spin network by assigning quantum states to the qubits in the network, and let the resulting *quantum spin system* evolve over time, i.e., let the initial quantum states travel across the network from qubit to qubit.

The *Hamiltonian* $\mathcal{H}$ of a quantum spin system is a time-independent Hermitian matrix that depends on the dynamics governing the evolution of the quantum spin system. Typically, there are two types of dynamics, one determined by the `XY` model (also called `XX`), and the other by the `XYZ` model (also called Heisenberg or `XXX`). For the `XY` model, the Hamiltonian is taken to be the adjacency matrix of $X$, while for the `XYZ` model, we take the (combinatorial) Laplacian matrix of $X$. For simplicity, we refer to the dynamics corresponding to the `XY` and `XYZ` models as *adjacency* and *Laplacian* dynamics, respectively.

Using the axioms of quantum mechanics, it is known that the state $\varphi(t)$ of the quantum spin system at any time $t \in \mathbb{R}$ is given by

$$\varphi(t) = U(t)\varphi(0), \tag{1.1.1}$$

where $\varphi(0)$ is the initial state of the system, and $U(t) = e^{i\mathcal{H}t}$ is called the transition matrix of quantum state transfer, which determines what is known as a *continuous-time quantum walk* on $X$. The square of the absolute value of the entry of $U(t)$ indexed by vertices $u$ and $v$ of $X$, called the *fidelity* of quantum state transfer from $u$ to $v$, is a number between 0 and 1, and signifies how close the quantum state in $v$ is from the initial quantum state in $u$ at time $t$. The concept of fidelity gives rise to various types of quantum state transfer. In particular, if the fidelity is 1, then we say that there is *perfect state transfer* between $u$ and $v$ at time $t$, while if the fidelity can be made arbitrarily close to 1 through appropriate choices of $t$, then we say that there is *pretty good state transfer* between $u$ and $v$ at time $t$. If the sum of the fidelities from $u$ to $u$, and $u$ to $v$ is 1, then we say that *fractional revival* occurs between $u$ and $v$ at time $t$.

In general, $\mathcal{H}$ has the property that the entry of $\mathcal{H}$ indexed by vertices $u$ and $v$ of $X$ is zero if and only if there is no edge between $u$ and $v$. In other words, regardless of edge weights, $\mathcal{H}$ respects the adjacencies of the vertices in $X$. For this reason, other authors have also investigated the case when the Hamiltonian is the signless Laplacian or the normalized Laplacian matrix of $X$. But for our purposes, we only consider the adjacency and Laplacian matrix of $X$.

In this thesis, we only focus on the single excitation case. That is, only one qubit in the network is assigned an initial quantum state. This additional condition reduces the size of $\mathcal{H}$ to $n \times n$, and yields the initial state

$$\varphi(0) = \gamma \mathbf{e}_u, \tag{1.1.2}$$

where $u$ is the vertex corresponding to the qubit holding the initial state. Equations (1.1.1) and (1.1.2) then allow us to define perfect state transfer, pretty good state transfer, and fractional revival in a matrix theoretic fashion in Chapter 3.2.

## 1.2 Brief literature review

The concept of a continuous-time quantum walk was first introduced by Farhi and Gutmann [30] in 1998, but it was not until 2003 that Bose proposed the use of paths to transmit quantum states [8]. Motivated by high fidelity quantum state transfer, Christandl et al. introduced perfect state transfer [23, 22] in 2005, and showed that unweighted paths only admit perfect state transfer for $n = 2, 3$ with respect to adjacency dynamics, and $n = 2$ with respect to Laplacian dynamics. This prompted researchers to search for new graphs admitting perfect state transfer. Some examples include families of cubelike graphs [21], integral circulant graphs [7], distance-regular graphs [26], Hadamard diagonalizable graphs [43], and even quotient graphs [5, 31], certain joins of graphs [3, 4], as well as non-complete extended $p$-sums (NEPS) of some graphs [53, 56]. Some authors also explored whether weighting the edges of the graph as well as adding loops to some vertices can induce perfect state transfer [3, 4, 13, 23, 46, 50]. However, due to its rarity, perfect state transfer was relaxed by several authors (Godsil [35], Vinet and Zhedanov [59]) to what is known as pretty

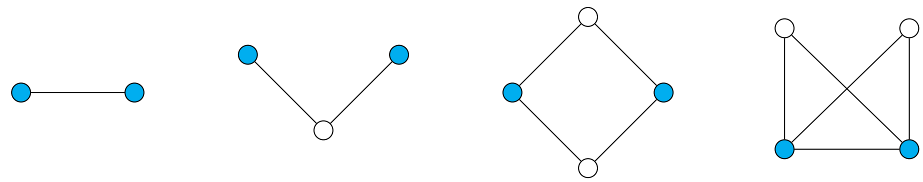

Figure 1.1: Small unweighted graphs that exhibit perfect state transfer between vertices marked blue

good state transfer. It turns out that an infinite family of unweighted paths exhibit pretty good state transfer as shown by Godsil et al. [37] and van Bommel [57] for adjacency dynamics, and Banchi et al. for Laplacian dynamics [6]. Pretty good state transfer was also investigated for cycles [55], a family of Cayley graphs [12], double stars [29], and even on weighted graphs with or without loops [28, 43, 47].

Unlike perfect state transfer, which has been extensively studied for more than 15 years, fractional revival, from a mathematical standpoint, is a relatively unexplored quantum phenomena. While earlier publications date back to 2007, it was not until 2019 when Chan et al. [16] explored the role of the underlying graph structure on the occurrence of fractional revival under adjacency dynamics. At the start of 2020, Kirkland and Zhang [50] characterized threshold graphs that admit fractional revival under Laplacian dynamics, and later than year, Chan et al. [18] characterized fractional revival in arbitrary graphs under Laplacian dynamics. In the same year, Chan et al. [15] developed a more general framework to study fractional revival, and extended the results in [18].

## 1.3 Summary and structure of this thesis

The complete graph on two vertices and the cycle on four vertices are well-known examples of small unweighted graphs that exhibit perfect state transfer between antipodal vertices with respect to adjacency and Laplacian dynamics. It is also known that the unweighted path on three vertices exhibits perfect state transfer between antipodal vertices with respect to adjacency dynamics, while the unweighted complete graph on four vertices minus an edge admits perfect state transfer between adjacent vertices with respect to Laplacian dynamics. Moreover, the cocktail party graph on $2n$ vertices, where $n$ is even, is an example of an infinite family of unweighted graphs that exhibit perfect state transfer between antipodal vertices. Upon examining the pairs of vertices in these graphs that admit perfect state transfer, one finds that they share the same neighbours, i.e., they are twins. As we shall see, a number of examples of quantum state transfer in the literature can be viewed through the notion of twins. Despite this fact, a detailed study of the role of twin vertices in quantum state transfer, to the best of our knowledge, has not been made yet. This motivates our study, and in this thesis, we provide a systematic approach to analyzing the properties of quantum state transfer between twin vertices in graphs. Our work represents both a new line of inquiry, as well as a unification of several seemingly unrelated results in the literature.

In Chapter 2, we present some graph, matrix, and number theoretic background for the reader, including the spectral decomposition, graph isomorphisms, graph partitions, and Kronecker's theorem for diophantine approximations, which are the main tools that we utilize in this thesis. This is followed by Chapter 3, which is an overview of various types of quantum state transfer (periodicity, perfect state transfer, pretty good state transfer, and fractional revival) and their properties. The discussion in this chapter includes a section on transition matrices and their properties, a section on strong cospectrality, which is a combinatorial condition required for two vertices to exhibit perfect state transfer or pretty good state transfer, and a section of examples. We then move on to Chapter 4, where we discuss the algebraic and spectral properties of twin vertices in positively weighted graphs with or without loops that are useful in quantum state transfer. In particular, we give a spectral characterization of twin vertices that are strongly cospectral with respect to the adjacency or Laplacian matrix. Moreover, in this chapter, we introduce the notion of twinning a vertex $u$ of an unweighted graph $X$, which is the process of adding a vertex $v$ such that $u$ and $v$ have the same neighbours in the post-twinning graph. We then completely characterize which vertices $u$ in a given unweighted graph $X$ are strongly cospectral with $v$ in the post-twinning graph with respect to the adjacency matrix. For the case of the Laplacian matrix, we give a partial characterization. In Chapter 5, we provide necessary and sufficient conditions for a pair of twin vertices in positively weighted graphs with or without loops to exhibit periodicity, perfect state transfer, pretty good state transfer, and fractional revival under adjacency dynamics. We also determine the minimum period of a periodic vertex in any connected and positively weighted graph $X$, and use this to determine the minimum time at which perfect state transfer occurs between two vertices in $X$. A similar result is also established if we add the assumption that the characteristic polynomial of the adjacency matrix of $X$ has integer coefficients. Next, in Chapter 6, we examine quantum state transfer in weighted double cones on regular graphs with respect to adjacency dynamics. We do this by applying the concept of equitable partitions to show that the apexes of a weighted double cone on a regular graph are always strongly cospectral, except for a few cases. For this reason, the family of weighted double cones on regular graphs provides promising candidates for perfect state transfer or pretty good state transfer. We then consider the case of unweighted connected and disconnected double cones on regular graphs, and identify which parameters induce periodicity, perfect state transfer, pretty good state transfer, and fractional revival between the apexes. For the case of weighted double cones on regular graphs, we provide a parametrization of the weights of the loops on the apexes, as well as the weight of the edge between them, such that the apexes exhibit periodicity and perfect state transfer. In particular, we show that the minimum period and minimum perfect state transfer time can be made arbitrarily small by a suitable choice of weights of either the loops or the edge connecting the apexes. In Chapter 7, we provide a survey of quantum state transfer for common families of unweighted graphs: complete graphs, complete bipartite graphs, complete graph minus an edge, cocktail party graphs, as well as paths and cycles. For the first four families of graphs mentioned, we apply our results in Chapter 6 to obtain a characterization of quantum state transfer between the apexes of these graphs when regarded as double cones over regular graphs. For the case of paths, we simply summarize knows results, while for cycles, we summarize known results and consider the weighted double cone on cycles. Moreover, we investigate whether adding loops or altering edge weights help induce perfect state transfer in these graphs. Finally, in Chapter 8, we list some problems for future study.

# 2

# Background

In this chapter, we provide an overview of the concepts relevant to the rest of the thesis. We start by introducing standard definitions and notation in graph and matrix theory.

## 2.1 Basic definitions and notation

In this thesis, we allow graphs with loops and weights, and unless otherwise stated, we assume all graphs to be undirected, simple and weighted. For more on the basics on graph theory, we refer the reader to [20, 33].

A *graph* $X$ consists of a non-empty vertex set $V(X)$ and an edge set $E(X)$, where an edge is an ordered pair of vertices in $V(X)$. If $(u, v) \in E(X)$, then we say that vertices $u$ and $v$ are *adjacent*, and the edge $(u, v)$ is *incident* to vertices $u$ and $v$. An edge of the form $(v, v)$ is called a *loop* on $v$, and in this case, we say that $v$ is adjacent to itself. We say that vertex $u$ is a neighbour of vertex $v$ if $u$ and $v$ are adjacent, and we denote the set of neighbours of $v$ in $X$ by $N_X(v)$. If $X$ has a loop on $v$, then $v \in N_X(v)$. We omit the subscript in $N_X(v)$ if $X$ is clear from the context.

A graph $X$ is *simple* if its edge set is not a multi-set and it does not contain loops. A simple graph with loops is called a *graph with loops*, and in this case, we only allow at most one loop at each vertex. If the edges $E(X)$ of a graph $X$ are unordered pairs of vertices, then we say that $X$ is an *undirected graph*. Otherwise, we say that $X$ is a *directed graph*, in which case the ordered pairs of vertices in $E(X)$ are called *arcs*, and $u$ is adjacent to $v$ whenever $(u, v) \in E(X)$. A *weighted graph* (resp., *weighted directed graph*) is a graph whose edges (resp., arcs) are assigned a real number, called the *weight* of the edge (resp., arcs). Moreover, if each edge (arc) has a positive weight, then we say that the weighted graph (resp., weighted directed graph) is *positively weighted.* In particular, if the weights are all one, then we say that the weighted graph (resp., weighted directed graph) is *unweighted.* That is, an unweighted graph (resp., unweighted directed graph) is a special case of a weighted graph (resp., weighted directed graph). A *negatively weighted* graph is defined in a similar manner.

The *degree* $\deg(v)$ of a vertex $v$ in a graph is the sum of the weights of the edges incident to $v$. Since a loop on $v$ contributes twice to its degree, one calculates $\deg(v)$ as the sum of the weights of all edges incident to $v$ that are not loops plus twice the weight of the loop on $v$.

A path of length $\ell$ from from vertex $u$ to vertex $v$ in a graph $X$ is a sequence of $\ell+1$ distinct vertices starting with $u$ and ending with $v$ such that any two consecutive vertices are adjacent. We say that a graph $X$ is *connected* if there is a path that joins every pair of vertices in $X$.

Let $X$ be a graph on $n \geq 2$ vertices. If $X$ has no edges, then we say that $X$ is the *empty graph* on $n$ vertices denoted by $O_n$. If any two vertices of $X$ are adjacent, then we call $X$ the *complete graph* on $n$ vertices denoted by $K_n$. We also denote the *complete graph on $n$ vertices minus an edge* by $K_n \backslash e$, where $e$ is the edge removed from $K_n$. We say that $X$ is *$k$-regular* if every vertex $v$ of $X$ has degree $k$. A *path* on $n \geq 3$ vertices, denoted by $P_n$, is a graph whose vertex set can be labelled as $V(P_n) = \{1, 2, \dots, n\}$ such that $E(P_n) = \{(j, j+1) : j = 1, \dots, n-1\}$. If we connect the two end vertices of $P_n$, then we obtain the *cycle* on $n \geq 3$ vertices, which we denote by $C_n$.

We say that a graph $X$ *bipartite* if its vertex set can partitioned into two subsets $V_1$ and $V_2$, called *partite sets*, such that each edge of $X$ has one end vertex in $V_1$, and the other end vertex in $V_2$. If $X$ is bipartite with partite sets $V_1$ and $V_2$ such that each vertex in each partite is adjacent to all vertices in the other partite, then we call $X$ the *complete bipartite graph* on $m = |V_1|$ and $n = |V_2|$ vertices, denoted $K_{m,n}$.

A graph $Y$ is a *subgraph* of a graph $X$ if $V(Y) \subseteq V(X)$ and $E(Y) \subseteq E(X)$. A subgraph $Y$ of $X$ is said to be *induced* if two vertices in $Y$ are adjacent if and only if they are adjacent in $X$. A connected subgraph of $X$ that is maximal is called a *component* of $X$. A *tree* is a connected graph that does not contain cycles as subgraphs. For a vertex $v$ in a graph $X$, we use $X \backslash u$ to denote the induced subgraph of $V(X) \backslash \{u\}$ which is called a *vertex-deleted* graph. The *complement* $\overline{X}$ of a graph $X$ is a graph with the same vertex set as $X$ such that any two vertices are adjacent in $\overline{X}$ if and only if they are not adjacent in $X$.

Let $X_1$ and $X_2$ be graphs with disjoint vertex sets. The *union* $X_1 \cup X_2$ is the graph such that $V(X_1 \cup X_2) = V(X_1) \cup V(X_2)$ and $E(X_1 \cup X_2) = E(X_1) \cup E(X_2)$. We denote the union of $n$ copies of $X_1$ by $nX_1$. The *cocktail party graph* on $2n$ vertices, denoted $\overline{nK_2}$, is the complement of the union of $n$ copies of $K_2$. The *join* $X_1 \vee X_2$ of $X_1$ and $X_2$ is the graph obtained from $X_1 \cup X_2$ by joining every vertex in $X_1$ to every vertex in $X_2$, and vice versa. Moreover, the *Cartesian product* $X_1 \square X_2$ of $X_1$ and $X_2$ is the graph such that $V(X_1 \square X_2) = V(X_1) \times V(X_2)$, where vertices $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent if and only if either $x_1 = x_2$ and $(y_1, y_2) \in E(Y)$ or $y_1 = y_2$ and $(x_1, x_2) \in E(X)$. Next, we consider basic notation from matrix theory. Let $A$ be an $m \times n$ matrix. We say that $A$ is a *complex* (*real*, *nonnegative*, respectively) matrix if $A$ is entry-wise complex (real, nonnegative, respectively). We denote the entry of $A$ in the $j$-th row and $\ell$-th column by $(A)_{j,\ell}$ for all $1 \leq j \leq m$ and $1 \leq \ell \leq n$. The entries of the form $(A)_{\ell,\ell}$ are called the diagonal entries of $A$. We denote the *transpose* of $A$ by $A^T$. If $A$ is a complex matrix, then we denoted by $\overline{A}$ the matrix derived from $A$ by taking the conjugates of all its entries. We then use $A^*$ to represent the conjugate transpose $(\overline{A})^T$. Note that if $A$ is a real matrix, then $A^* = A^T$. A nonnegative matrix whose sum of all entries in each row and column is one is called a *doubly stochastic matrix*. The *rank* of $A$ is denoted $\text{rank}(A)$. For subsets $\alpha \subset \{1, \ldots, m\}$ and $\beta \subset \{1, \ldots, n\}$, we use $A[\alpha, \beta]$ to denote the *submatrix* of $A$ whose rows and columns are indexed by $\alpha$ and $\beta$, respectively. If $B$ is an $p \times q$ matrix, then the *tensor product* of $A$ and $B$, denoted $A \otimes B$, is the $mp \times nq$ matrix which can be partitioned into $mn$ blocks with the same size as $B$, where the $(j, \ell)$ block is $a_{j,\ell}B$.

Now, let $A$ be an $n \times n$ matrix. If $\alpha \subset \{1, \ldots, n\}$, then we say that $A[\alpha, \alpha]$ is a *principal submatrix* of $A$, and it is known that $\text{rank}(A) \geq \text{rank}(A[\alpha, \alpha])$ for any subset $\alpha$ of $\{1, \ldots, n\}$. We say that $A$ is *Hermitian* if $A = A^*$, $A$ is *symmetric* if $A = A^T$, and *idempotent* if $A^2 = A$. If $A$ is real symmetric, then it follows that $A$ is Hermitian. We say that $A$ is *diagonal* if all nondiagonal entries of $A$ are zero, in which case we can write $A = \text{diag}(a_1, a_2, \ldots, a_n)$, where $a_j$ is the diagonal entry on $j$-th row and column of $A$. If $B$ is an $n \times n$ matrix, then the *Schur product* $A \circ B$ is the matrix whose entries are the entrywise product of $A$ and $B$. That is, $(A \circ B)_{j,\ell} = (A)_{j,\ell}(B)_{j,\ell}$.

A *permutation matrix* $P$ is a $(0, 1)$ square matrix in which 1 as an entry appears exactly once in each row and column of $P$. A permutation matrix $P$ is known to satisfy $P^TP = PP^T = I$. A complex square matrix $U$ is called *unitary* if $UU^* = U^*U = I$. If we add that $U$ is real, then $U$ is said to be an *real orthogonal matrix*.

Consequently, any permutation matrix is a real orthogonal matrix.

The *characteristic polynomial* of a square matrix $A$, denoted by $\phi(A,t)$, is the polynomial $\phi(A,t) = \det(tI - A)$. We call a root $\lambda$ of $\phi(A,t)$ as an *eigenvalue* of $A$. The algebraic multiplicity of an eigenvalue $\lambda$ of $A$ refers to the multiplicity $\lambda$ as a root of the characteristic polynomial of $A$. We say that $\lambda$ is a *simple* eigenvalue of $A$ if its algebraic multiplicity is one. We call the multi-set of eigenvalues of $A$ the *spectrum* of $A$, denoted $\sigma(A)$. For an eigenvalue $\lambda$ of $A$, any non-zero vector $\mathbf{w}$ such that $A\mathbf{w} = \lambda\mathbf{w}$ is called an *eigenvector* associated to $\lambda$, and we call the vector space of all eigenvectors corresponding $\lambda$ the eigenspace of $\lambda$.

We use $\mathbf{1}_n$ to denote the all ones column vector of length $n$, we use $I_n$ to denote the identity $n \times n$ matrix, we use $\mathbf{J}_{m,n}$ to denote the all ones $m \times n$ matrix, and we use $\mathbf{0}_n$ to denote the all zeros column vector of length $n$. If $m = n$, then we denote $\mathbf{J}_{m,n}$ by $\mathbf{J}_n$. We omit the subscripts of $\mathbf{1}_n$, $I_n$, $\mathbf{J}_{m,n}$, $\mathbf{0}_n$, and $\mathbf{J}_n$ if their sizes are clear from the context. We also adopt the convention of writing the standard basis vectors in an $n$-dimensional vector space as $\mathbf{e}_1, \dots, \mathbf{e}_n$.

Let $X$ be a directed graph on $n$ vertices labelled from 1 to $n$. The *adjacency matrix* $A(X)$ of $X$ is the $n \times n$ matrix given by

$$(A(X))_{j,\ell} = \begin{cases} \omega_{j,\ell}, & \text{if } j \text{ is adjacent to } \ell \\ 0, & \text{otherwise,} \end{cases}$$

where $\omega_{j,\ell}$ is the weight of the arc $(j,\ell)$. If $X$ is unweighted, then $\omega_{j,\ell} = 1$ whenever $j$ and $\ell$ are adjacent, while if $X$ is undirected, then $(A(X))_{j,\ell} = (A(X))_{\ell,j}$. The *degree matrix* $D(X)$ of $X$ is the the diagonal matrix whose $j$-th diagonal entry is the degree of vertex $j$. The *Laplacian matrix* of $X$ is defined as $L(X) = D(X) - A(X)$, where $A(X)$ and $D(X)$ are the adjacency and degree matrices of $X$, respectively. If $X$ is $k$-regular, then $D(X) = kI$, and so $L(X) = kI - A(X)$. If $X$ is undirected, then the adjacency and Laplacian matrices of $X$ are Hermitian, and if $X$ is positively weighted, then the adjacency matrix of $X$ is a nonnegative matrix. If the context is clear, then we simply write $A = A(X)$ and $L = L(X)$. We also define the characteristic vector $\mathbf{e}_S$ of a subset $S$ of the vertex set of $X$ as $\mathbf{e}_S = \sum_{u \in S} \mathbf{e}_v$, which has a one at the entries indexed by the elements of $S$, and zeroes elsewhere. If $S = \{u\}$ is a singleton, then $\mathbf{e}_u$ is called the characteristic vector of vertex $u$. Moreover, it is known that if $X_j$ is a weighted graph on $n_j$ vertices for $j = 1, 2$, then $M(X_1 \square X_2) = M(X_1) \oplus I_{n_1} + I_{n_2} \oplus M(X_2)$.

Lastly, we recall some algebraic and number theoretic concepts. An integer $\Delta$ is *square-free* if it is not divisible by a perfect square. For a prime $p$, the *p-adic* *valuation* of an integer $a \neq 0$, denoted $\nu_p(a)$, is the largest power of $p$ that divides $a$. It is known that every integer $a \neq 0$ can be uniquely written as $a = 2^{\nu_2(a)}\ell$, where $\ell$ is an odd number. We denote the set of all polynomials with integer coefficients as $\mathbb{Z}[x]$. An *algebraic number* $\lambda \in \mathbb{C}$ is a root of an irreducible polynomial $p(x) \in \mathbb{Z}[x]$. If $p(x)$ is monic, then we say that $\lambda$ is an *algebraic integer*, while if $p(x)$ has degree two, then we say that $\lambda$ is *quadratic*. If $\lambda$ is a root of a monic quadratic irreducible polynomial over $\mathbb{Z}$, then we say that $\lambda$ is a *quadratic integer*. Lastly, given $a, b, c \in \mathbb{R}$, we say that $a$ and $b$ are congruent modulo $c$, written as $a \equiv b \pmod{c}$, if and only if $a = b + 2kc$ for some integer $k$.

For more information on matrix theory, we refer the reader to [42], and for the basics of number theory, see [41].

## 2.2 Matrix theoretic results

In this section, we state facts from matrix theory that will prove useful in this work. We begin with the matrix exponential.

Let $M$ be an $n \times n$ complex matrix. The exponential $e^M$ of $M$ is defined as the power series $e^M = \sum_{j=0}^{\infty} \frac{1}{j!} M^j$, where $M^0 = I$.

**Proposition 2.2.1** ([40], Proposition 2.3)**.** *Let $M$ and $N$ be $n \times n$ complex matrices. The matrix exponential satifies the following properties.*

1. $e^0 = I$.
2. $e^{M^T} = \left(e^M\right)^T$.
3. $e^{M^*} = \left(e^M\right)^*$.
4. $e^{N^{-1}MN} = N^{-1}e^M N$, *whenever $N$ is nonsingular.*
5. $e^M e^N = e^{M+N}$, *whenever $MN = NM$.*

From these properties, it is clear that $M$ is symmetric (resp., Hermitian) if and only if $e^M$ is symmetric (resp., Hermitian). Moreover, if $M$ is Hermitian, then $e^{iM}\left(e^{iM}\right)^* = e^{iM}e^{-iM^*} = e^{iM}e^{-iM} = e^0 = I$ so that $e^{iM}$ is unitary.

A square matrix $A$ is said to be *normal* if $A^*A = AA^*$. A standard result in matrix theory states that a square matrix $A$ is normal if and only $A$ is unitarily diagonalizable, i.e., $A$ can be written as $A = U^*\mathcal{D}U$, where $U$ is a unitary matrix and $\mathcal{D}$ is a diagonal matrix of all eigenvalues of $A$. Since unitary and Hermitian matrices are normal, these matrices are unitarily diagonalizable.

Now, suppose $\lambda$ is an eigenvalue of $M$. As we know, an orthonormal basis for the eigenspace corresponding to $\lambda$ is not unique, but the orthogonal projection matrix onto the eigenspace corresponding to $\lambda$ is. That is, the orthogonal projection matrix onto the eigenspace $\lambda$ does not depend on the choice of an orthonormal basis for its eigenspace. Now, for any $t \in \mathbb{C}$, it is a fact that $\lambda$ is an eigenvalue of $M$ with eigenvector $w$ if and only if $\lambda + t$ is an eigenvalue of $M + tI$ with eigenvector $w$. Consequently, $M$ and $M + tI$ have the same set of orthogonal projection matrices.

Next, we deal with Hermitian matrices. It is well-known that every $n \times n$ Hermitian matrix has real spectrum with an orthonormal set of eigenvectors that forms a basis for $\mathbb{C}^n$. In particular, a real symmetric matrix has an orthonormal set of eigenvectors that forms a basis for $\mathbb{R}^n$. We call Hermitian matrix with nonnegative eigenvalues is called a positive semidefinite matrix. Since a Hermitian matrix is normal, every $n \times n$ Hermitian matrix $M$ can be unitarily diagonalized as $M = Q^*\mathcal{D}Q$, where $Q$ is a unitary matrix whose columns are the orthonormal eigenvectors for $M$ and $\mathcal{D} = \text{diag}(\lambda_1, \dots, \lambda_n)$, where each $\lambda_j$ is real. In particular, if $M$ is real symmetric, then $Q$ can be chosen to be real orthogonal. This fact has many important consequences, one of which is the following fact, known as the spectral decomposition for Hermitian matrices.

**Theorem 2.2.2** (Spectral Decomposition, see [33, 42])**.** *Let $M$ be an $n \times n$ complex Hermitian matrix. Then we can write $M = \sum_{j=1}^{r} \lambda_j E_j$, where $\lambda_1, \dots, \lambda_r$ are the distinct eigenvalues of $H$ and each $E_j$ is the orthogonal projection matrix onto the eigenspace associated to the eigenvalue $\lambda_j$ that satisfies the following conditions.*

1. *$E_j^T = E_j$, $E_j^2 = E_j$ and $E_j E_\ell = 0$ for $j \neq \ell$.*
2. *$\sum_{j=1}^{r} E_j = I_n$.*
3. *If $g$ is an analytic function which is defined at every eigenvalue of $M$, then $g(M) = \sum_{j=1}^{r} g(\lambda_j) E_j$.*

Throughout this paper, we call the matrices $E_j$ in Theorem 2.2.2 *spectral idempotents* corresponding to the eigenvalue $\lambda_j$. Note that the $E_j$'s are Hermitian, idempotent, and pairwise multiplicatively orthogonal matrices which sum to identity. Without using indices, we write $E_\lambda$ to refer to the spectral idempotent corresponding to $\lambda$. We can uniquely determine $E_\lambda$ given an orthonormal basis $\{\mathbf{w}_1, \dots, \mathbf{w}_\ell\}$ for the eigenspace corresponding to $\lambda$ by setting $E_\lambda = \sum_{j=1}^{\ell} \mathbf{w}_j \mathbf{w}_j^T$. Consequently, $M$ and $M + tI$ have the same set of spectral idempotents. Furthermore, observe that if $E_\ell$ is a spectral idempotent of $M$, then Theorem 2.2.2(1) yields

$$ME_\ell = \left(\sum_{j=1}^{r} \lambda_j E_j\right) E_\ell = \sum_{j=1}^{r} \lambda_j (E_j E_\ell) = \lambda_\ell E_\ell^2 = \lambda_\ell E_\ell$$

and similarly, $E_\ell M = \lambda_\ell E_\ell$. Consequently,

$$ME_\ell = E_\ell M = \lambda_\ell E_\ell \tag{2.2.1}$$

so that each spectral idempotent commutes with $M$. Also, notice from Theorem 2.2.2 statement (3) that $\sigma(g(M)) = \{g(\lambda) : \lambda \in \sigma(M)\}$, and $M$ and $g(M)$ both have the same set of spectral idempotents.

Let $X$ be a connected weighted graph on $n$ vertices with or without loops. We mention a few basic spectral properties of the adjacency matrix $A$ and Laplacian matrix $L$ of $X$. First, since $A$ and $L$ are Hermitian, they admit corresponding spectral decompositions. In particular, since $A$ is real symmetric, there exists an orthogonal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A$, and similarly, for $L$. Consequently, the spectral idempotents for $A$ and $L$ are all real symmetric. If we add that $X$ is connected and positively weighted, then $A$ is nonnegative while $L$ is not. The connectedness of $X$ allows us to invoke the famous Perron-Frobenius Theorem for nonnegative matrices which states that the largest eigenvalue $\lambda_{\max}$ of $A$ is simple and positive, and has an associated eigenvector that has all entries positive. Furthermore, if $X$ has no loops, then 0 is an eigenvalue of $L$ with associated eigenvector $\mathbf{1}$. Since the multiplicity of 0 as an eigenvalue of $L$ counts the number of connected components of $X$, it follows that 0 is a simple eigenvalue of $L$ if and only if $X$ is connected. Consequently, whether $M = A$ or $M = L$, the spectrum $\sigma(M)$ contains at least one eigenvalue with an eigenvector that has all entries positive, provided the underlying graph is connected. Moreover, if $X$ is positively weighted, then $L$ is positive semidefinite but $A$ is not. Furthermore, as we will see in Section 6.1, it is possible for $\phi(A,t) \in \mathbb{Z}[x]$, even if some entries of $A$ are not rational.

For a given subset $S$ of the vertex set of a graph $X$, we note that $M\mathbf{e}_S$ is the sum of all the columns of $M$ indexed by the elements of $S$. In particular, if $S = \{u\}$, then $M\mathbf{e}_u$ is simply the $u$-th column of $M$. Similarly, multiplying $\mathbf{e}_S^T$ and $\mathbf{e}_u^T$ to the right of $M$ yield corresponding results for rows. Thus, we have $(M)_{u,v} = \mathbf{e}_u^T M \mathbf{e}_v$. We also recall the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^n$ given by $\|\mathbf{z}\|^2 = \mathbf{z}^T\mathbf{z}$ for all $\mathbf{z} \in \mathbb{R}^n$. A nonzero vector $\mathbf{v}$ can be normalized to have norm 1 by dividing $\mathbf{v}$ by its norm.

## 2.3 Graph theoretic tools

We import more tools from graph theory that are fundamental in understanding quantum state transfer. We begin with graph isomorphisms.

### 2.3.1 Graph isomorphisms

An *isomorphism* between two graphs $X_1$ and $X_2$ is a bijection $f : V(X_1) \to V(X_2)$ with the property that $(u, v) \in E(X_1)$ if and only if $(f(u), f(v)) \in E(X_2)$, and the weights of $(u, v)$ and $(f(u), f(v))$ are equal. We say that two graphs $X_1$ and $X_2$ are *isomorphic*, denoted $X_1 \cong X_2$, if an isomorphism exists between them. One can think of isomorphic graphs as equal graphs, where one is obtained from the other by relabelling its vertices. This relabelling can be thought of as a permutation of the elements of $V(X)$. Thus, an isomorphism $X_1$ and $X_2$ can be represented as a permutation matrix $P$ that satisfies $M(X_2) = P^T M(X_1)P$, and we note that the same $P$ works whether $M = A$ or $M = L$. An *automorphism* $f$ of a graph is an isomorphism of a graph to itself. The least positive integer $n$ such that $f^n = I$ is called the *order* of $f$. If $f$ has order two, then we call $f$ an *involution*. The *orbit* of a vertex $u$ of $X$ under $f$, denoted $O_u$, is the set $O_u = \{f^j(u) : 1 \leq j \leq n\} \subseteq V(X)$.

For two vertices $u$ and $v$ of a graph $X$, we say that $u$ and $v$ are *cospectral* with respect to $M(X)$, where $M(X) = A(X)$ or $M(X) = L(X)$, if $M(X\backslash u)$ and $M(X\backslash v)$ have the same characteristic polynomial. Note that if $X\backslash u$ and $X\backslash v$ are isomorphic, then we can write $M(X\backslash u) = P^T M(X\backslash v)P$ for some permutation matrix $P$. This implies that $M(X\backslash u)$ and $M(X\backslash u)$ have the same characteristic polynomial, and so $u$ and $v$ are cospectral with respect to both $M(X) = A(X)$ and $M(X) = L(X)$. For more about automorphisms of a graph, see [33].

### 2.3.2 Graph partitions

We review another tool called equitable partitions that are important in the study of quantum state transfer under adjacency dynamics. We first recall the definition of a partition of a set.

**Definition 2.3.1.** Let $X$ be a set. A *partition* $\pi = (C_1, \ldots, C_r)$ of $X$ is a set of non-empty subsets $C_1, \ldots, C_r$ of $X$ called *cells* such that $X = \bigcup_{j=1}^{r} C_j$ and $C_j \cap C_\ell = \varnothing$ whenever $j \neq \ell$. In particular, if $X = \{x_1, x_2, \ldots\}$, then we define the *characteristic matrix* $\mathcal{P}$ of $\pi$ as the $|X| \times r$ matrix such that $(\mathcal{P})_{j,\ell} = 1$ if $x_j \in C_\ell$ and $(P)_{j,\ell} = 0$

otherwise. For $1 \leq \ell \leq r$, the $\ell$-th column of $\mathcal{P}$ is $\sum_{x_j \in C_\ell} \mathbf{e}_j$. If we normalize each column of $\mathcal{P}$, then we get the *normalized characteristic matrix* $\dot{\mathcal{P}}$ of $\pi$.

Note that the characteristic matrix $\dot{\mathcal{P}}$ of $\pi$ which satisfies $\dot{\mathcal{P}}\dot{\mathcal{P}}^T = I_k$. We now apply the previous definition to vertex sets of graphs. The following statement is an extension of the definition of equitable partition in [32, Chapter 5] to weighted graphs.

**Definition 2.3.2.** Let $X$ be a weighted graph with or without loops. We say that a partition $\pi = (C_1, \ldots, C_r)$ of $V(X)$ is *equitable* if for any $j, \ell \in \{1, \ldots, r\}$,

1. the number of neighbours in $C_\ell$ of each vertex in $C_j$ is equal, and
2. the weights of the edges between $C_\ell$ and $C_j$ are equal.

Alternatively, we say that $\pi = (C_1, \ldots, C_r)$ is an *equitable partition* of $X$.

Let $X$ be a weighted graph with or without loops. If $\pi = (C_1, \ldots, C_r)$ is an equitable partition of $X$, we denote by $c_{j,\ell}$ the number of neighbours in $C_\ell$ of any vertex in $C_j$. Note that $c_{j,\ell}$ and $c_{\ell,j}$ are not necessarily equal. However, since $X$ is undirected, it is clear that $\omega_{j,\ell} = \omega_{\ell,j}$. Now, observe that if $\pi$ is the partition of $V(X)$ whose cells are singletons containing each vertex of $X$, then $\pi$ is an equitable partition of $X$, in which case the characteristic matrix $\mathcal{P}$ of $\pi$ is simply $I_{|V(X)|}$. On the other hand, if $\pi$ is a partition of $V(X)$ containing only one cell $C_1$, then $X$ must be a regular graph, with regularity $k = c_{1,1}$.

An equitable partition of a graph gives rise to two graphs, a directed weighted graph, called a quotient graph, and an undirected weighted graph, called a symmetrized quotient graph. We again extend the definition of (symmetrized) quotient graph in [32, Chapter 5] to weighted graphs.

**Definition 2.3.3.** Let $X$ be a graph with or without loops, and $\pi = (C_1, \ldots, C_r)$ be an equitable partition of $X$.

1. The *quotient graph* $X/\pi$ of $X$ with respect to $\pi$ is a directed weighted graph whose vertex set are the cells of $\pi$, and with an arc of integer weight $c_{j,\ell}\omega_{j,\ell}$ from vertex $C_j$ to $C_\ell$ for each $c_{j,\ell} \neq 0$, where $\omega_{j,\ell}$ is the weight of every edge between $C_\ell$ and $C_j$.
2. The *symmetrized quotient graph* $\widehat{X/\pi}$ of $X$ with respect to $\pi$ is the undirected weighted graph whose vertex set are the cells of $\pi$, and edges $(C_j, C_\ell)$ of weight $\sqrt{c_{j\ell}c_{\ell j}}\omega_{j,\ell}$ whenever $c_{j,\ell}c_{\ell,j} > 0$, where $\omega_{j,\ell}$ is the weight of every edge between $C_\ell$ and $C_j$. We say that $\widehat{X/\pi}$ is a *contraction* of $X$, and $X$ is a *lift* of $\widehat{X/\pi}$.

We denote the adjacency matrices of $X/\pi$ and $\widehat{X/\pi}$ by $A(X/\pi)$ and $A(\widehat{X/\pi})$, respectively. Note that $(A(X/\pi))$ is not necessarily symmetric, but $A(\widehat{X/\pi})_{j,\ell}$ is Hermitian, and thus, $A(\widehat{X/\pi})_{j,\ell}$ admits a spectral decomposition. If $A(X/\pi)$ is symmetric, then $A(X/\pi) = A(\widehat{X/\pi})$, in which case $c_{j,\ell} = c_{\ell,j}$ for all $j$ and $\ell$.

To decide whether a partition of a graph is equitable, we can use its (normalized) characteristic matrix. In fact, a partition $\pi$ of a graph $X$ with characteristic matrix $\mathcal{P}$ is equitable if and only if either (i) $A(X)\mathcal{P} = \mathcal{P}B$ for some matrix $B$, in which case $B = A(X/\pi)$, or (ii) $A$ and $\dot{\mathcal{P}}\dot{\mathcal{P}}^T$ commute [33, Chapter 9].

## 2.4 Kronecker's Theorem

Lastly, we state a well-known fact in number theory called Kronecker's Theorem, a tool mainly used in Diophantine approximations. This fact will prove useful in establishing a specific type of state transfer called pretty good state transfer. There are many versions of this theorem, but we state a more general form.

**Theorem 2.4.1** (Kronecker's Theorem, see [52]). *Let $\lambda_1, \ldots, \lambda_r$ and $\zeta_1, \ldots, \zeta_r$ be arbitrary real numbers. For every $\epsilon > 0$, the system of inequalities*

$$|\lambda_j t - \zeta_j| < \epsilon \ (mod\ 2\pi) \tag{2.4.1}$$

*for $j = 1, \ldots, r$ admits a solution for $t$ if and only if for integers $\ell_1, \ldots, \ell_r$ such that*

$$\sum_{j=1}^{r} \ell_j \lambda_j = 0, \tag{2.4.2}$$

*then*

$$\sum_{j=1}^{r} \ell_j \zeta_j \equiv 0 \ (mod\ 2\pi). \tag{2.4.3}$$

Now, observe from Equation (2.4.2) that Theorem 2.4.1 is true whenever $\lambda_1, \ldots, \lambda_r$ are linearly indpenedent over the rationals. We state this observation as a corollary to Theorem 2.4.1.

**Corollary 2.4.2.** *Let $\lambda_1, \ldots, \lambda_r$ be linearly independent over $\mathbb{Q}$ and $\zeta_1, \ldots, \zeta_r$ be arbitrary real numbers. For every $\epsilon > 0$, the system of inequalities*

$$|\lambda_j t - \zeta_j| < \epsilon \ (mod\ 2\pi) \tag{2.4.4}$$

*for $j = 1, \ldots, r$ admits a solution for $t$.*

We remark that some of the earlier works [29, 37] regarding pretty good state transfer only made use of Corollary 2.4.2, while the more recent ones [6, 57] utilized the power of Theorem 2.4.1 to generate more general results.

# 3

# Quantum state transfer: an overview

We now formally define terminologies and review the basics of quantum state transfer. This chapter provides the reader a sufficient amount of discussion on the various types of quantum state transfer and their properties, as well as some examples.

Throughout this chapter, we assume all graphs are weighted with or without loops. We use $M(X)$ to denote the adjacency or Laplacian matrix of $X$. Unless otherwise specified, any statement pertaining to $M(X)$ applies to both the adjacency and Laplacian matrix of $X$. If the context is clear, we simply write $M = M(X)$, $A = A(X)$, and $L = L(X)$.

## 3.1 The transition matrix

We start by introducing the transition matrix of a graph and its properties.

**Definition 3.1.1.** Let $X$ be a weighted graph with or without loops.

1. The *transition matrix* with respect to $M(X)$ is the matrix $U(t) = e^{itM(X)}$.

2. The *fidelity of quantum state transfer* from vertex $u$ to vertex $v$ at time $t$ with respect to $M(X)$ is $\left|(U(t))_{u,v}\right|^2$.

Let $X$ be a weighted graph on $n$ vertices with or without loops, and $u, v \in V(X)$. Consider the transition matrix $U(t)$ with respect to $M(X)$. Since $M$ is Hermitian, it admits a spectral decomposition $M = \sum_{j=1}^{r} \lambda_j E_j$, where $\lambda_1, \ldots, \lambda_r$ are the distinct eigenvalues of $M(X)$. Since $e^x$ is an analytic function, Theorem 2.2.2(4) then yields

$$U(t) = \sum_{j=1}^{r} e^{it\lambda_j} E_j. \tag{3.1.1}$$

Because $M$ has real entries, there exists an orthonormal basis for $\mathbb{C}^n$ consisting of real eigenvectors of $M$, and so, each $E_j$ is real. Now, a unitary diagonalization of $M$ yields $M = Q^T \mathcal{D} Q$, where $Q$ is a real orthogonal matrix whose columns form an orthonormal set of eigenvectors for $M$ and $\mathcal{D} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix of eigenvalues of $M$. Making use of Proposition 2.2.1 and the fact that $e^{it\mathcal{D}} = \mathrm{diag}(e^{it\lambda_1}, \ldots, e^{it\lambda_n})$, we obtain the equation

$$\left|\mathbf{e}_u^T e^{itM} \mathbf{e}_v\right| = \left|\mathbf{e}_u^T e^{it\left(Q^T \mathcal{D} Q\right)} \mathbf{e}_v\right| = \left|\mathbf{e}_u^T Q^T e^{it\mathcal{D}} Q \mathbf{e}_v\right| = \left|(Q\mathbf{e}_u)^T e^{it\mathcal{D}} Q \mathbf{e}_v\right| = \left|\sum_{\ell=1}^{n} e^{it\lambda_\ell} (Q)_{u,\ell} (Q)_{\ell,v}\right|, \tag{3.1.2}$$

which allows us to compute the fidelities $\left|(U(t))_{u,v}\right|^2$ directly from $M$. Since $M$ is Hermitian, Theorem 3.1.1 and Proposition 2.2.1 imply that $U(t)$ is a complex symmetric unitary matrix that satisfies $U(0) = I_n$, $U(t)^* = U(-t)$, and

$$U(t_1 + t_2) = e^{i(t_1+t_2)M} = e^{it_1 M} e^{it_2 M} = U(t_1) U(t_2)$$

for any $t, t_1, t_2 \in \mathbb{R}$. Therefore, $(U(t))_{u,v} = (U(t))_{v,u}$, and so the fidelity of quantum state transfer from $u$ to $v$ is the same from $v$ to $u$. This allows us to talk about the fidelity of state transfer between two vertices without indicating the initial vertex of the transfer. Since $U(t)$ is unitary, each column of $U(t)$ has norm one, and thus

$$\sum_{j=1}^{n} \left|(U(t))_{u,j}\right|^2 = 1, \tag{3.1.3}$$

for any time $t$. From this, it follows that

$$0 \leq \left|(U(t))_{u,v}\right|^2 \leq 1 \tag{3.1.4}$$

for any pair of vertices $u$ and $v$ in $X$. Equations (3.1.2) and (3.1.4), and the fact that the exponential function is continuous imply that $\left|(U(t))_{u,v}\right|^2$ is a bounded continuous function of $t$. Moreover, Equations (3.1.3) and (3.1.4) allow us to interpret the fidelity $\left|(U(t))_{u,v}\right|^2$ as the probability that the initial state at vertex $u$ is transported to vertex $v$ at time $t$. Consequently, the transition matrix $U(t)$ determines a probability distribution on the vertex set of $X$, sometimes called the *continuous-time quantum walk* on $X$. For more information about quantum walks, see [48]. Note that the entries of $U(t)$ do not directly give the fidelities of quantum state transfer, and for this reason, we turn to the matrix $U(t) \circ U(-t)$, called the *mixing matrix* with respect to $M(X)$. Indeed, one checks that the entries of $U(t) \circ U(-t)$ are precisely the fidelities, which make $U(t) \circ U(-t)$ a symmetric doubly stochastic matrix for any $t \in \mathbb{R}$. As we will see in the next section, various types of quantum state transfer arise between $u$ and $v$ depending on the fidelity between them. We now turn our attention to transition matrices that are entry-wise equal in magnitude.

**Definition 3.1.2.** Let $U_1(t)$ and $U_2(t)$ be transition matrices with respect to $M_1(X)$ and $M_2(X)$, respectively. We say that $U_1(t)$ and $U_2(t)$ are *equivalent* if either $U_1(t) = \gamma U_2(t)$ or $U_1(t) = \gamma U_2(-t)$ for some unit complex number $\gamma$.

A more general definition of equivalent transition matrices is given in [2], but for our purposes, Definition 3.1.2 suffices. We call $\gamma$ in Definition 3.1.2 *phase factor*. While fidelities are measurable, it is known that phase factors are undetectable by quantum measurements. Hence, one may choose not to put too much attention to phase factors. Now, note that $(U(-t))_{u,v} = (U(t)^*)_{u,v} = \overline{(U(t))_{v,u}} = \overline{(U(t))_{u,v}}$. If $U_1(t) = \gamma U_2(-t)$ with $|\gamma| = 1$, then using the fact that $|\overline{z}| = |z|$ for all $z \in \mathbb{C}$, we get

$$\left|(U_1(t))_{u,v}\right| = \left|\gamma\,(U_2(-t))_{u,v}\right| = |\gamma| \cdot \left|\overline{(U_2(t))_{u,v}}\right| = \left|(U_2(t))_{u,v}\right|.$$

Similarly, if $U_1(t) = \gamma U_2(t)$ with $|\gamma| = 1$, then $\left|(U_1(t))_{u,v}\right| = \left|(U_2(t))_{u,v}\right|$. In both cases, we see that if $U_1(t)$ and $U_2(t)$ are equivalent, then the fidelities from $u$ to $v$ at time $t$ with respect $M_1$ and $M_2$ are equal, i.e., $U_1(t)$ and $U_2(t)$ determine the same probability distribution. This yields the following result.

**Proposition 3.1.3.** *If $U_1(t)$ and $U_2(t)$ are equivalent $n \times n$ transition matrices, then $\left|(U_1(t))_{u,v}\right| = \left|(U_2(t))_{u,v}\right|$ for all $u, v \in \{1, \dots, n\}$ and for all $t \in \mathbb{R}$.*

Now, let $X$ be a weighted graph with or without loops with transition matrix $U_1(t)$ with respect to $M(X)$. For $\alpha \in \mathbb{R}$, let $U_2(t)$ be the transition matrix with respect to $\alpha I \pm M(X)$. Since $\alpha I$ and $M(X)$ commute, Proposition 2.2.1 yields

$$U_2(t) = e^{it(\alpha I \pm M(X))} = e^{i\alpha t} e^{\pm itM(X)} = e^{i\alpha t} U_1(\pm t)$$

and hence $U_1(t)$ and $U_2(t)$ are equivalent. In particular, if $X$ is a $k$-regular graph with transition matrices $U_1(t)$ with respect to $L$ and $U_2(t)$ with respect to $A$, then $L = kI - A$, and thus $U_1(t) = e^{itk} U_2(-t)$. Since $|e^{itk}| = 1$, we conclude that $U_1(t)$ and $U_2(t)$ are equivalent.

**Proposition 3.1.4.** *If $X$ is a weighted $k$-regular graph with or without loops, then the transition matrices with respect to $A(X)$ and $L(X)$ are equivalent.*

As a consequence of Proposition 3.1.4, all results in quantum state transfer under adjacency dynamics are applicable to Laplacian dynamics, and vice versa, whenever the graph is regular.

Now, let $X$ be a weighted graph with or without loops, and $P$ be a permutation matrix representing an automorphism $f$ of $X$. Then $M(X) = P^T M(X) P$, and so

$$U(t) = e^{itM(X)} = e^{it(P^T M(X) P)} = P^T e^{itM(X)} P = P^T U(t) P, \qquad (3.1.5)$$

for any $t \in \mathbb{R}$. Let $u$, $v$ and $w$ are vertices of $X$. By Equation (3.1.5), we obtain $\mathbf{e}_u^T U(t) \mathbf{e}_v = (P\mathbf{e}_u)^T U(t) (P\mathbf{e}_v)$, and so $U(t)_{u,v} = U(t)_{f(u),f(v)}$. Moreover, if $f$ fixes $w$ but sends $u$ to $v$, then

$$(U(t))_{w,u} = \mathbf{e}_w^T U(t) \mathbf{e}_u = \mathbf{e}_w^T (P^T U(t) P) \mathbf{e}_u = (P\mathbf{e}_w)^T U(t) (P\mathbf{e}_u) = \mathbf{e}_w^T U(t) \mathbf{e}_v = (U(t))_{w,v}.$$

for any $t \in \mathbb{R}$. Thus, if $O_u$ is the orbit of $u$ under $f$, then $(U(t))_{w,u} = (U(t))_{w,v}$ for all $v \in O_u$. Since $U(t)$ is unitary, its $w$-th row gives us

$$1 = \sum_{j \in V(X)} |(U(t))_{w,j}|^2 = |O_u| \, |(U(t))_{w,u}|^2 + \sum_{j \notin O_u} |(U(t))_{w,j}|^2,$$

for any $t \in \mathbb{R}$. Thus, if $U(t))_{w,j} = 0$ for $j \notin O_u$, then $|(U(t))_{w,u}|^2 = \frac{1}{|O_u|}$. We summarize this in the following proposition.

**Proposition 3.1.5.** *Let $X$ be a weighted graph with or without loops, and with vertices $u$ and $v$. If $f$ is an automorphism of $X$, then $U(t)_{u,v} = U(t)_{f(u),f(v)}$ for any $t \in \mathbb{R}$. Moreover, if $f$ fixes $w$ and $O_u$ is the orbit of $u$ under $f$, then for any $t \in \mathbb{R}$, $(U(t))_{w,u} = (U(t))_{w,v}$ for all $v \in O_u$, and*

$$|(U(t))_{w,u}|^2 \leq \frac{1}{|O_u|},$$

*with equality if and only if $(U(t))_{w,j} = 0$ for all $j \notin O_u$.*

We also state a fact about transition matrices of cartesian products of graphs.

**Theorem 3.1.6** ([2, 35])**.** *Let $X_1$ and $X_2$ be weighted graphs with or without loops, and with transition matrices $U_{X_1}(t)$ and $U_{X_2}(t)$, respectively. If $U_{X_1 \square X_2}(t)$ is the* *transition matrix of* $X_1 \square X_2$*, then*

$$U_{X_1 \square X_2}(t) = U_{X_1}(t) \otimes U_{X_2}(t). \tag{3.1.6}$$

We close this section with a fact that establishes an entry-wise relationship between the transition matrices with respect to $A(X)$ and $A(\widehat{X/\pi})$ whenever $\pi$ is an equitable partition of $X$ satisfying a particular condition.

**Theorem 3.1.7** ([5], Theorem 2)**.** *Let* $X$ *be a weighted graph with or without loops, and* $\pi$ *be an equitable partition of* $X$ *with singleton cells* $\{u\}$ *and* $\{v\}$*. If* $U_X(t)$ *and* $U_{X/\pi}(t)$ *are the transition matrices with respect to* $A(X)$ *and* $A(\widehat{X/\pi})$ *respectively, then*

$$(U_X(t))_{u,v} = \left(U_{\widehat{X/\pi}}(t)\right)_{\{u\},\{v\}}, \tag{3.1.7}$$

*for any time* $t$.

## 3.2 Types of state transfer

This section is a survey of various types of quantum state transfer and the important properties that they exhibit. Since quantum state transfer can only happen between vertices in a connected graph, we assume that the graphs discussed herein are connected. The facts presented can then be applied to analyze quantum state transfer between vertices that belong to the same components of a disconnected graph. For more about quantum state transfer, we refer the reader to [25, 35].

### 3.2.1 Perfect State Transfer

We start with the most extensively studied type of quantum state transfer called perfect state transfer, which was first introduced by Christandl et al in 2005 [23, 22].

**Definition 3.2.1.** Let $X$ be a weighted graph with or without loops, and with vertices $u$ and $v$. We say that $X$ has *perfect state transfer* (PST) from $u$ to $v$ with respect to $M(X)$ if there exists a time $\tau$ and complex number $\gamma$ such that

$$U(\tau)\mathbf{e}_u = \gamma \mathbf{e}_v. \tag{3.2.1}$$

If $u = v$, then we say that vertex $u$ is *periodic* in $X$ with respect to $M(X)$, and we call $\tau$ the *period* of $u$ in $X$.

Note that $U(\tau)\mathbf{e}_u = \gamma\mathbf{e}_v$ if and only if $u$-th column of $U(\tau)$ is $\gamma\mathbf{e}_v$. Since $U(t)$ is unitary, Equation (3.1.3) implies that $\sum_{j=1}^{|V(X)|}\left|(U(t))_{u,j}\right|^2 = |\gamma|^2 = 1$. Consequently, $U(\tau)\mathbf{e}_u = \gamma\mathbf{e}_v$ if and only if $\mathbf{e}_v^T U(\tau)\mathbf{e}_u = \gamma$, and so

$$\left|(U(\tau))_{u,v}\right|^2 = \left|\mathbf{e}_v^T U(\tau)\mathbf{e}_u\right|^2 = |\gamma|^2 = 1 \tag{3.2.2}$$

This means that the existence of PST from $u$ to $v$ at time $\tau$ is equivalent to a unit fidelity of quantum state transfer from $u$ to $v$ at time $\tau$.

Now, let $X$ be a weighted graph with or without loops, and with vertices $u$ and $v$. Since $M$ is symmetric, $\overline{U(\tau)^*} = U(\tau)^T = U(\tau)$, and so

$$U(\tau)\mathbf{e}_u = \gamma\mathbf{e}_v \iff U(\tau)^*\mathbf{e}_v = \overline{\gamma}\mathbf{e}_u \iff \overline{U(\tau)^*\mathbf{e}_v} = \overline{\gamma\mathbf{e}_u} \iff U(\tau)\mathbf{e}_v = \gamma\mathbf{e}_u.$$

In other words, PST occurs from $u$ to $v$ if and only if PST occurs from $v$ to $u$, both of which happen at the same time with the same phase factor. We call this the *symmetry property* of PST, which allows us to view PST as a phenomenon between two vertices regardless of which vertex holds the initial state. Note that in terms of fidelities, this property follows directly the fact that $U(t)$ is symmetric.

**Proposition 3.2.2** (Symmetry property of PST)**.** *Let $X$ be a weighted graph with or without loops, and with vertices $u$ and $v$. Then perfect state transfer occurs from $u$ to $v$ if and only if perfect state transfer occurs from $v$ to $u$, at the same with the same phase factor.*

As a consequence of the fact that $U(t)$ is symmetric and unitary, we have that PST occurs between $u$ and $v$ if and only if the rows and columns of $U(\tau)$ indexed by $u$ and $v$ all have zero entries except the $(u, v)$ and $(v, u)$ entries which are both $\gamma$. If $P$ is a permutation matrix that represents a relabelling of the vertices such that $u$ and $v$ are labelled 1 and 2, then PST occurs between $u$ and $v$ with respect to $M$ if and only if there is a phase factor $\gamma$, and time $\tau$ such that

$$P^T U(\tau) P = \begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & N \end{bmatrix} \tag{3.2.3}$$

for some $(n-2)\times(n-2)$ unitary matrix $N$. Similarly, if vertex $u$ is periodic, then

$u = v$, and instead of Equation (3.2.3), we obtain

$$P^T U(\tau) P = \begin{bmatrix} \gamma & 0 \\ 0 & N' \end{bmatrix}, \tag{3.2.4}$$

for some $n-1 \times n-1$ unitary matrix $N'$. Now suppose PST occurs from $u$ to $v$ at time $\tau$ so that Equation (3.2.3) holds. Using the fact that $PP^T = I$, we have

$$P^T U(2\tau) P = P^T U(\tau) U(\tau) P = \left(P^T U(\tau) P\right)\left(P^T U(\tau) P\right) = \left(P^T U(\tau) P\right)^2$$

and therefore,

$$P^T U(2\tau) P = \begin{bmatrix} \gamma^2 & 0 & 0 \\ 0 & \gamma^2 & 0 \\ 0 & 0 & N^2 \end{bmatrix}. \tag{3.2.5}$$

From Equation (3.2.4), we can interpret Equation (3.2.5) as vertices $u$ and $v$ are periodic in $X$ with respect to $M$ at time $2\tau$ with phase factor $\gamma^2$. Alternatively, Proposition 2.2.1 and the symmetry property of PST yield $U(2\tau)\mathbf{e}_u = (U(t))^2\mathbf{e}_u = \gamma U(t)\mathbf{e}_v = \gamma^2\mathbf{e}_u$, and similarly, $U(2\tau)\mathbf{e}_v = \gamma^2\mathbf{e}_v$. We summarize this in the next proposition.

**Proposition 3.2.3.** *Let $X$ be a weighted graph with or without loops, and with vertices $u$ and $v$. If perfect state transfer occurs between $u$ and $v$ with respect to $M(X)$ at time $\tau$ and phase factor $\gamma$, then $u$ and $v$ are periodic with respect to $M(X)$ at time $2\tau$ and phase factor $\gamma^2$.*

Note that Proposition 3.2.3 implies that periodicity is a necessary condition for PST. However, the converse is not true. Indeed, if vertex $u$ is periodic and vertex $v$ is not, then PST cannot occur between $u$ and $v$ by Proposition 3.2.3. Meanwhile if two distinct vertices $u$ and $v$ are both periodic with respect to $M$ at time $\tau$ with corresponding phase factors $\gamma_u$ and $\gamma_v$, then Equation (3.2.5) yields

$$P^T U(\tau) P = \begin{bmatrix} \gamma_u & 0 & 0 \\ 0 & \gamma_v & 0 \\ 0 & 0 & N \end{bmatrix}. \tag{3.2.6}$$

for some $(n-2) \times (n-2)$ unitary matrix $N$. Letting $\tau = 2\delta$, we obtain $U(\tau) = U(\delta)^2$.

By Equation (3.2.6), we get that

$$P^T U(\delta) P = \begin{bmatrix} R & 0 \\ 0 & \sqrt{N} \end{bmatrix},$$

where $R^2 = \begin{bmatrix} \gamma_u & 0 \\ 0 & \gamma_v \end{bmatrix}$ and $\sqrt{N}$ is a square root of $N$. If $R = R'$, where

$$R' = \begin{bmatrix} \pm\sqrt{\gamma_u} & 0 \\ 0 & \pm\sqrt{\gamma_v} \end{bmatrix}$$

then $\sqrt{\gamma_u}$ and $\sqrt{\gamma_v}$ are unit complex numbers, and we see that $u$ and $v$ are not guaranteed to exhibit PST at time $\tau/2$ despite their periodicity at time $\tau$. We state this as a remark to Proposition 3.2.3.

**Remark 3.2.4.** The converse of Proposition 3.2.3 is not true. That is, a periodic vertex need not pair up with another vertex to exhibit perfect state transfer. As a concrete example, it can be checked that the middle vertex of unweighted $P_3$ exhibits periodicity, but not perfect state transfer.

Now, let us discuss the period $\tau$. For a given vertex $u$ of a graph $X$ that is periodic with respect to $M$, define $T = \{\tau : U(\tau)\mathbf{e}_u = \gamma_\tau \mathbf{e}_u \text{ for some } \gamma_\tau \in \mathbb{C}\}$ and let $\rho = \min_{\tau \in T_u} |\tau|$. It was shown by Godsil that $T$ is a cyclic subgroup of $\mathbb{R}$ with generator $\rho$, which is the least positive element of $T$ [35]. We call $\rho$ the *minimum period* of $u$ with respect to $M$. Godsil also showed that if vertex $u$ pairs up with vertex $v$ to exhibit PST, then the *minimum PST time* between $u$ and $v$ is $\tau = \rho/2$. Using Proposition 3.2.3, we get that the minimum period of $v$ is also $\rho$. These considerations yield the following proposition.

**Proposition 3.2.5** ([35], Lemma 3.1)**.** *Let $X$ be a weighted graph with or without loops, and with vertex $u$ that is periodic with respect to $M(X)$ with minimum period $\rho$. If perfect state transfer occurs between $u$ and $v$ with respect to $M(X)$ for some $v \neq u$, then the minimum time that it occurs is $\rho/2$, and the minimum period of $v$ is also $\rho$.*

Let $X$ be a weighted graph with or without loops, and with vertices $u$ and $v$ that exhibit PST with phase factor $\gamma$ at the minimum PST time $\rho/2$. By Proposition 3.2.3, we know that $u$ and $v$ are periodic with period $\rho$ and phase factor $\gamma^2$. If $\ell = 2m$ for some integer $m$, then $U\left(\frac{\rho}{2}\ell\right)\mathbf{e}_u = U(\rho)^m \mathbf{e}_u = \gamma^{2m}\mathbf{e}_u$ because $\rho$ is a period of $u$.

Similarly, $U(\frac{\rho}{2}\ell)\mathbf{e}_v = \gamma^{2m}\mathbf{e}_v$. In other words, $u$ and $v$ are periodic at $t = \frac{\rho}{2}\ell$ for every even integer $\ell$ with phase factor $\gamma^\ell$. Taking $\ell = 2$, we see that the minimum period of both $u$ and $v$ is $\rho$. On the other hand, if $\ell = 2m+1$ for some integer $m$, then we obtain $U\left(\frac{\rho}{2}\ell\right)\mathbf{e}_u = U\left(m\rho + \frac{\rho}{2}\right)\mathbf{e}_u = U(\rho)^m U\left(\frac{\rho}{2}\right)\mathbf{e}_u = U(\rho)^m\gamma\mathbf{e}_v = \gamma^{2m+1}\mathbf{e}_v$ because $\rho$ is the minimum period of $v$. Similarly, $U\left(\frac{\rho}{2}\ell\right)\mathbf{e}_v = \gamma^{2m+1}\mathbf{e}_u$. Consequently, PST occurs between $u$ and $v$ at $t = \frac{\rho}{2}\ell$ for every odd integer $\ell$ with phase factor $\gamma^\ell$.

**Proposition 3.2.6.** *Let $X$ be a weighted graph with or without loops, and with vertices $u$ and $v$ that exhibit perfect state transfer at the minimum time $\rho/2$ with phase factor $\gamma$ with respect to $M(X)$. Then the following statements hold with respect to $M(X)$.*

1. *If $\ell$ is even, then $u$ and $v$ are periodic at time $t = \frac{\rho}{2}\ell$ with phase factor $\gamma^\ell$. Moreover, the minimum period of both $u$ and $v$ is $\rho$.*

2. *If $\ell$ is odd, then $u$ and $v$ exhibit perfect state transfer at time $t = \frac{\rho}{2}\ell$ with phase factor $\gamma^\ell$.*

Now, suppose vertex $u$ has period $\tau$, not necessarily the minimum, and PST occurs between $u$ and $v$. We know from Proposition 3.2.5 that PST occurs between $u$ and $v$ at half the minimum period $\rho$ of $u$, so it only makes sense to check if PST also occurs between $u$ and $v$ at half of $\tau$. Since $\rho$ is a generator of $T$, we can write $\tau = \rho\ell$ for some integer $\ell$ so that $\frac{\tau}{2} = \frac{\rho}{2}\ell$. Applying Proposition 3.2.6, it follows that PST occurs between $u$ and $v$ at time $\frac{\tau}{2}$ if and only if $\ell$ is odd. This provides a necessary and sufficient condition for two vertices of the same period $\tau$ to exhibit PST at $\tau/2$, a result stronger than Remark 3.2.4.

**Corollary 3.2.7.** *Let $X$ be a weighted graph with or without loops, and with vertices $u$ and $v$ that are both periodic at time $\tau$ with respect to $M(X)$. If perfect state transfer occurs between $u$ and $v$, then it occurs at time $\frac{\tau}{2}$ if and only if $\tau = \rho\ell$ for some odd integer $\ell$, where $\rho$ is the minimum period of $u$.*

As a consequence of Corollary 3.2.7, the PST time between two vertices is always an odd multiple of the minimum PST time between them.

Finally, we prove the *monogamy property* of PST using the minimum period. This property was first shown by Kay [45] but we follow the proof of Coutinho [24]. Assume PST occurs from vertex $u$ to vertices $v$ and $w$. By Proposition 3.2.3, we have that vertex $u$ is periodic, with minimum period, say $\rho$. From Proposition 3.2.5, we know that the minimum PST time between $u$ and $v$, as well as between $v$ and $w$, is

$\rho/2$. Thus, $U(\rho/2)\mathbf{e}_u = \gamma_1\mathbf{e}_v$ and $U(\rho/2)\mathbf{e}_u = \gamma_2\mathbf{e}_w$, for some unit complex numbers $\gamma_1$ and $\gamma_2$. Consequently, $\gamma_1\mathbf{e}_v = \gamma_2\mathbf{e}_w$, which is only possible whenever $v = w$. That is, a vertex in a graph can only pair up to at most one vertex to exhibit PST.

**Proposition 3.2.8** (Monogamy property of PST, [45]). *Let $X$ be a weighted graph with or without loops, and with vertices $u, v$ and $w$. If perfect state transfer occurs from $u$ and $v$ and from $u$ to $w$, both with respect to $M(X)$, then $v = w$.*

Next, a straightforward application of Theorem 3.1.6 yields the following fact, which states that PST is preserved under Cartesian products.

**Theorem 3.2.9** ([2, 35]). *For $j = 1, 2$, if $X_j$ is a weighted graph with or without loops that exhibits perfect state transfer between vertices $u_j$ and $v_j$ at time $\tau$, then $X_1 \square X_2$ exhibits perfect state transfer between $(u_1, v_1)$ and $(u_2, v_2)$ at time $\tau$.*

By taking the cartesian products of multiple copies of a graph that exhibits PST, we obtain infinitely many examples of graphs that exhibit PST by virtue of Theorem 3.2.9. Before we move on to the notion of pretty good state transfer, we introduce a generalization of periodicity of a vertex.

**Definition 3.2.10.** Let $X$ be a weighted graph on $n$ vertices with or without loops, and with vertex $u$.

1. We say that $u$ is *almost periodic* with respect to $M(X)$ if there exists a sequence of times $\{\tau_\ell\}$ and a complex number $\gamma$ such that $\lim_{\ell\to\infty} ||U(\tau_\ell)\mathbf{e}_u - \gamma\mathbf{e}_u|| = 0$.
2. We say that $X$ is *almost periodic* with respect to $M(X)$ if there exists a sequence of times $\{\tau_\ell\}$ and a complex number $\gamma$ such that $\lim_{\ell\to\infty} ||U(\tau_\ell) - \gamma I_n|| = 0$.

Similar to the case of PST, we again note that the phase factor $\gamma$ in the above definitions is a unit complex number. Now, let $X$ be a graph on $n$ vertices with vertex $u$. Note that $u$ is almost periodic if and only if the fidelity $\left|(U(\tau_\ell))_{u,u}\right|^2$ is near $|\gamma|^2 = 1$ for a sufficiently large $\ell$. On the other hand, a graph $X$ is almost periodic if and only if $\lim_{\ell\to\infty} U(\tau_\ell) = \gamma I$, or equivalently, for every vertex $v$ of $X$, $U(\tau_\ell)\mathbf{e}_v$ is close to $\gamma\mathbf{e}_v$ for sufficiently large $\ell$. Thus, $X$ is almost periodic if and only if every vertex of $X$ are almost periodic. In [58], van Bommel showed that every graph is almost periodic with respect to $M(X)$, and consequently, every vertex of a graph is almost periodic with respect to $M(X)$.

For more information about PST, see the survey of Kendon and Tamon [49], Godsil [36] and Kay [44]. For more about periodicity, see the survey of Godsil [34].

### 3.2.2 Pretty Good State Transfer

In [37], Godsil showed that PST is a rare phenomenon in weighted graphs whose associated adjacency matrices have characteristic polynomials with integer coefficients. This prompted multiple authors (Godsil [37], Vinet and Zhedanov [59]) to introduce the notion of pretty good state transfer in 2012 as a natural relaxation of perfect state transfer.

**Definition 3.2.11** ([59])**.** Let $X$ be a weighted graph with or without loops, and with vertices $u$ and $v$. We say that *pretty good state transfer* (PGST) occurs from $u$ to $v$ with respect to $M(X)$ if there exists a sequence of times $\{\tau_\ell\}$ and complex number $\gamma$ such that

$$\lim_{\ell\to\infty} \|U(\tau_\ell)\mathbf{e}_u - \gamma\mathbf{e}_v\| = 0. \tag{3.2.7}$$

In addition, if perfect state transfer does not occur between $u$ and $v$, then we say that *proper* pretty good state transfer occurs from $u$ to $v$.

Note that if $u = v$, then the definition of PGST from $u$ to $v$ coincides with that of almost periodicity at $u$. But, as we know, every vertex of a graph is almost periodic. Thus, in the discussion of PGST, we need not consider the case when $u = v$.

Let us explore some properties of PGST. Let $X$ be a weighted graph with or without loops, and suppose PGST occurs from vertex $u$ to vertex $v$ of $X$. By Equation (3.2.7), $\lim_{\ell\to\infty} \left|(U(\tau_\ell))_{u,v} - \gamma\right| = 0$, and so

$$\lim_{\ell\to\infty} \left|(U(\tau_\ell))_{u,v}\right|^2 = \lim_{\ell\to\infty} \left|(U(\tau_\ell))_{v,u}\right|^2 = |\gamma|^2 = 1, \tag{3.2.8}$$

i.e., the fidelity between $u$ and $v$ gets arbitrarily close to one as $\ell$ tends to infinity.

**Remark 3.2.12.** By Equations (3.2.2) and (3.2.8), the phase factor $\gamma$ in Definition 3.2.1 and Definition 3.2.11 is a unit complex number. Like PST, Equation (3.2.8) also implies that PGST is symmetric, which again allows us to discuss PGST between two vertices, regardless of which vertex holds the initial state.

Now, if $\{\tau_j\}$ is a convergent subsequence of $\{\tau_\ell\}$ such that $\lim_{j\to\infty} \tau_j = \tau_0$, then using Equation (3.2.8) and the fact that $|U(t)_{u,v}|^2$ is a continuous function of $t$ yield $1 = \lim_{j\to\infty} \left|U(\tau_j)_{u,v}\right|^2 = \left|U(\lim_{j\to\infty} \tau_j)_{u,v}\right|^2 = |U(\tau_0)_{u,v}|^2$. Equivalently, PST occurs between $u$ and $v$ at time $t = \tau_0$. Consequently, if PST occurs between $u$ and $v$ at time $\tau_0$, then any sequence $\{\tau_j\}$ converging to $\tau_0$ satisfies Equation (3.2.7) so that PGST occurs between $u$ and $v$. Alternatively, if PST occurs between $u$ and $v$ with phase factor $\gamma$ and minimum PST time $\frac{\varrho}{2}$, then Proposition 3.2.6(2) implies that $\tau_\ell = \frac{\varrho}{2}(2\ell+1)$ satisfies $\left|(U(\tau_\ell))_{u,v}\right| = 1$ for every integer $\ell$. Thus, the sequence $\{\tau_\ell\}$ satisfies Equation (3.2.8), and hence PGST occurs between $u$ and $v$. In both cases, we see that the existence of PST implies that existence of PGST. However, the converse is not true. Using matrices, PGST occurs between $u$ and $v$ with respect to $M$ if and only if for every $\epsilon > 0$, there exists a time $\tau_\epsilon$ such that

$$P^T U(\tau_\epsilon) P = \begin{bmatrix} * & \gamma'_\epsilon & * \\ \gamma'_\epsilon & * & * \\ * & * & N_\epsilon \end{bmatrix} \tag{3.2.9}$$

for some $\gamma'_\epsilon \in \mathbb{C}$ with $|\gamma'_\epsilon| > 1-\epsilon$, and $(n-2)\times(n-2)$ matrix $N_\epsilon$. If $|\gamma'_\epsilon| < 1$ for every $\epsilon > 0$, then $P^T U(\tau_\epsilon) P$ cannot assume the block form in Equation (3.2.3), and so PST does not occur between $u$ and $v$. For this reason, one can think of PGST as a generalization of PST, and one need not check the existence of PGST between two vertices whenever PST occurs between them. But, unlike PST, proper PGST is not necessarily monogamous as illustrated by a weighted graph in [43, Example 2] that exhibits Laplacian PGST from one vertex to three distinct vertices. Next, we prove the following fact attributed to Dave Morris. We follow the proof of Godsil.

**Proposition 3.2.13** ([37], Lemma 13.1)**.** *Let $X$ be a weighted graph with or without loops, and with vertices $u$ and $v$. If pretty good state transfer occurs between $u$ and $v$ with respect to $M(X)$, then $E_\lambda \mathbf{e}_u = \pm E_\lambda \mathbf{e}_v$ for each $\lambda \in \sigma(M(X))$.*

*Proof.* Let $X$ be a weighted graph with or without loops, and with vertices $u$ and $v$. Assume $M = \sum_{\lambda\in\sigma(M)} \lambda E_\lambda$, and PGST occurs between $u$ and $v$ with respect to $M$. Using Equation (3.1.1), we can write $U(t) = \sum_{\lambda\in\sigma(M)} e^{it\lambda} E_\lambda$. For every $\lambda \in \sigma(M)$, define the sequence of complex numbers $\{e^{i\tau_\ell\lambda}\}$. Since each $\left|e^{i\tau_\ell\lambda}\right| = 1$, it follows that $\{e^{i\tau_\ell\lambda}\}$ is bounded, and hence contains a convergent subsequence $\{e^{i\tau_j\lambda}\}$. Set $\lim_{j\to\infty} e^{i\tau_j\lambda} = \zeta$. Evidently, $|\zeta| = 1$. Since $\lim\limits_{\ell\to\infty} U(\tau_\ell)\mathbf{e}_u = \gamma \mathbf{e}_v$, it also follows that $\lim\limits_{j\to\infty} U(\tau_j)\mathbf{e}_u = \gamma\mathbf{e}_v$. Making use of Equation (2.2.1), we obtain

$$\zeta E_\lambda \mathbf{e}_u = \left(\lim_{j\to\infty} e^{i\tau_j\lambda}\right) E_\lambda \mathbf{e}_u = \lim_{j\to\infty}\left(e^{i\tau_j\lambda} E_\lambda\right)\mathbf{e}_u = \lim_{j\to\infty}\left(E_\lambda U(\tau_j)\right) e_u = \gamma E_\lambda \mathbf{e}_v.$$

But since each spectral idempotent $E_\lambda$ is real, we get that $\zeta = \pm\gamma \neq 0$, and thus, $E_\lambda \mathbf{e}_u = \pm E_\lambda \mathbf{e}_v$ for every $\lambda \in \sigma(M)$. □

Proposition 3.2.13 motivates our next section on strong cospectrality. But before we move on to that topic, we consider another generalization of PST. For generalizations of pretty good state transfer, see [17, 19]

### 3.2.3 Fractional Revival

The last variant of quantum state transfer that we include here is a phenomenon where the probability of observing the initial state in vertex $u$ of a graph $X$ is entirely concentrated between two vertices $u$ and $v$ of $X$ at a particular time.

**Definition 3.2.14** ([16])**.** Let $X$ be a weighted graph with or without loops, and with distinct vertices $u$ and $v$. We say that $(\alpha, \beta)$-*fractional revival* (FR) occurs from $u$ to $v$ with respect to $M(X)$ if there exists a time $\tau$, and complex numbers $\alpha$ and $\beta$ such that

$$U(\tau)e_u = \alpha\mathbf{e}_u + \beta\mathbf{e}_v. \tag{3.2.10}$$

If $\alpha, \beta \neq 0$, then we say that the fractional revival from $u$ to $v$ is *proper*, while if $|\alpha| = |\beta|$, then we say that the fractional revival from $u$ to $v$ is *balanced*.

In physics terms, fractional revival is a unitary mapping of the initial state at vertex $u$ of $X$ to a superposition between vertices $u$ and $v$ in $X$ at a certain time $\tau$. If the values of $\alpha$ and $\beta$ do not matter, then we simply say fractional revival (FR).

**Remark 3.2.15.** Unlike PST and PGST, FR has two phase factors $\alpha$ and $\beta$, which satisfy $|\alpha|^2 + |\beta|^2 = 1$ because $U(t)$ is unitary.

By Remark 3.2.15, a balanced FR implies that $|\alpha|^2 = |\beta|^2 = \frac{1}{2}$, which corresponds to a maximum superposition of vertices $u$ and $v$. Moreover, if $\alpha = 0$ or $\beta = 0$, then the definition of FR is equivalent to that of PST or periodicity, respectively. Thus, FR generalizes PST and periodicity.

Let $X$ be a weighted graph with or without loops, and with vertices $u$ and $v$. To exclude periodicity from the discussion, we assume $\beta \neq 0$. If $(\alpha, \beta)$-FR occurs from $u$ to $v$, then Equation (3.2.10) yields $(U(\tau))_{v,u} = \mathbf{e}_v^T U(\tau)\mathbf{e}_u = \beta$ and $(U(\tau))_{u,u} = \mathbf{e}_u^T U(\tau)\mathbf{e}_u = \alpha$. Since $U(\tau)$ is symmetric, we get $(U(\tau))_{u,v} = \beta$. Now, since $U(\tau)$ is unitary, any two columns of $U(\tau)$ are orthogonal, and thus,

$$(U(\tau)\mathbf{e}_u)^*(U(\tau)\mathbf{e}_v) = \overline{(U(\tau))_{v,u}}\,(U(\tau))_{v,v} + \overline{(U(\tau))_{u,u}}\,(U(\tau))_{u,v} = \overline{\beta}\,(U(\tau))_{v,v} + \overline{\alpha}\beta = 0$$

which implies that $(U(\tau))_{v,v} = -\frac{\overline{\alpha}\beta}{\overline{\beta}}$, because $\beta \neq 0$. Thus, $U(t)\mathbf{e}_v = \left(-\frac{\overline{\alpha}\beta}{\overline{\beta}}\right)\mathbf{e}_v + \beta\mathbf{e}_u$, and since $\left|-\frac{\overline{\alpha}\beta}{\overline{\beta}}\right|^2 + |\beta|^2 = |\alpha|^2 + |\beta|^2 = 1$, it follows that $\left(-\frac{\overline{\alpha}\beta}{\overline{\beta}}, \beta\right)$-FR also occurs at time $\tau$ from $v$ to $u$. Conversely, if $\left(-\frac{\overline{\alpha}\beta}{\overline{\beta}}, \beta\right)$-FR occurs at time $\tau$ from $v$ to $u$, then one can show by reversing the steps that $(\alpha, \beta)$-FR occurs at time $\tau$ from $u$ to $v$. We call this the *weak symmetry property* of proper FR. Matrix theoretically, this means that $(\alpha, \beta)$-FR occurs at time $\tau$ from $u$ to $v$ if and only if there exists of a permutation matrix $P$ such that

$$P^T U(\tau) P = \begin{bmatrix} \alpha & \beta & 0 \\ \beta & -\frac{\overline{\alpha}\beta}{\overline{\beta}} & 0 \\ 0 & 0 & N \end{bmatrix} \tag{3.2.11}$$

for some $(n-2) \times (n-2)$ unitary matrix $N$. From Equation (3.2.11), the weak symmetry property of FR can be seen clearly. We summarize this discussion in the following proposition.

**Proposition 3.2.16** (Weak symmetry property of proper FR, [16])**.** *Let $X$ be a weighted graph with or without loops, and with distinct vertices $u$ and $v$. For $\beta \neq 0$, $(\alpha, \beta)$-fractional revival occurs at time $\tau$ from $u$ to $v$ with respect to $M(X)$ if and only if $\left(-\frac{\overline{\alpha}\beta}{\overline{\beta}}, \beta\right)$-fractional revival occurs at time $\tau$ from $v$ to $u$ with respect to $M(X)$.*

By Proposition 3.2.16, we need not specify the initial vertex of transfer when FR occurs between two vertices. Next, using Equation (3.1.1) and the fact that spectral idempotents $E_\lambda$ sum to identity, we can rewrite Equation (3.2.10) as

$$\sum_{\lambda \in \sigma(M)} e^{it\lambda} \left(E_\lambda \mathbf{e}_u\right) = \sum_{\lambda \in \sigma(M)} \alpha \left(E_\lambda \mathbf{e}_u\right) + \sum_{\lambda \in \sigma(M)} \beta \left(E_\lambda \mathbf{e}_v\right). \tag{3.2.12}$$

Since the $E_\lambda$'s are orthogonal projections, Equation (3.2.12) yields $(e^{it\lambda} - \alpha) E_\lambda \mathbf{e}_u = \beta E_\lambda \mathbf{e}_v$ for each $\lambda \in \sigma(M)$. If $\beta \neq 0$, then $|\alpha| < 1$, and so $e^{it\lambda} - \alpha \neq 0$. Consequently, we can write $E_\lambda \mathbf{e}_u = c E_\lambda \mathbf{e}_v$, where $c = \beta/(e^{it\lambda} - \alpha)$. This yields the following fact.

**Proposition 3.2.17** ([16], Proposition 4.2)**.** *Let $X$ be a weighted graph with or without loops, and with distinct vertices $u$ and $v$. If proper fractional revival occurs between $u$ and $v$ with respect to $M(X)$, then, for each $\lambda \in \sigma(M(X))$, $E_\lambda \mathbf{e}_u = c_\lambda E_\lambda \mathbf{e}_v$ for some complex number $c_\lambda \neq 0$.*

More specifically, in Laplacian dynamics, the following was shown using that the fact that $\mathbf{1}$ is an eigenvector for the Laplacian matrix.

**Proposition 3.2.18** ([18, 51])**.** *Let $X$ be a weighted graph with or without loops, and with distinct vertices $u$ and $v$. If proper Laplacian fractional revival occurs between $u$ to $v$, then $E_\lambda \mathbf{e}_u = \pm E_\lambda \mathbf{e}_v$ for each $\lambda \in \sigma(L(X))$.*

Now, let $X$ be a weighted graph with or without loops, and with vertices $u$ and $v$. If $f$ is an automorphism of $X$, then Proposition 3.1.5 implies that $U(t)_{u,v} = U(t)_{f(u),f(v)}$. In other words, PGST (resp., FR) occurs between $u$ and $v$ with respect to $M(X)$ if and only if PGST (resp., FR) occurs between $f(u)$ and $f(v)$ with respect to $M(X)$. Moreover, if $f$ fixes $u$ but does not fix $v$, then Proposition 3.1.5 implies that $|U(t)_{u,v}|^2 \leq \frac{1}{2}$, which implies that PGST cannot occur between $u$ and $v$. Moreover, since $U(t)_{u,v} = U(t)_{u,w}$ for all $w \in O_v$, no permutation matrix $P$ exists such that $P^T U(t) P$ assume the form in Equation (3.2.11). Thus, PGST and FR does not occur between $u$ and $v$. We summarize this in the following theorem.

**Theorem 3.2.19.** *Let $X$ be a weighted graph with or without loops.*

1. *If $f$ is an automorphism of $X$, then pretty good state transfer (resp., fractional revival) occurs between vertices $u$ and $v$ with respect to $M(X)$ if and only if pretty good state transfer (resp., fractional revival) occurs between $f(u)$ and $f(v)$ with respect to $M(X)$.*

2. *If there exists an automorphism of $X$ that fixes vertex $u$ but does not fix vertex $v$, then pretty good state transfer and fractional revival do not occur between $u$ and $v$ with respect to $M(X)$.*

For adjacency dynamics, the contrapositive of Theorem 3.2.19(2) for the case of PST and FR first appeared in [35, 16], respectively. Next, we state a corollary to Theorem 3.1.7.

**Corollary 3.2.20.** *Let $X$ be a weighted graph with or without loops, and $\pi$ be an equitable partition of $X$ with singleton cells $\{u\}$ and $\{v\}$. Then adjacency perfect state transfer (resp., adjacency pretty good state transfer and adjacency fractional revival) occurs between $u$ and $v$ in $X$ if and only if adjacency perfect state transfer occurs between $\{u\}$ and $\{v\}$ in $\widehat{X/\pi}$ (resp., adjacency pretty good state transfer and adjacency fractional revival). Moreover, the minimum perfect state transfer time (resp., fractional revival time) between $u$ and $v$ in $X$ is equal to the minimum perfect state transfer time (resp., fractional revival time) between $\{u\}$ and $\{v\}$ in $\widehat{X/\pi}$.*

Taking $u = v$ in Corollary 3.2.20, we get that $u$ is adjacency periodic in $X$ if and only if $\{u\}$ is adjacency periodic in $\widehat{X/\pi}$, and the minimum period of $u$ in $X$ is equal to the minimum period of $\{u\}$ in $\widehat{X/\pi}$.

It was shown that adjacency and Laplacian FR is a rare occurrence [16, 18] in weighted graphs whose associated adjacency and Laplacian matrices have characteristic polynomials with integer coefficients. Since PST is a special case of FR, we see

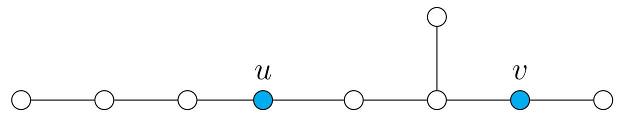


Figure 3.1: Schwenk's Tree with a pair of adjacency cospectral vertices $u$ and $v$

that Laplacian PST, like adjacency PST, is also rare. However, unlike PST, FR is not monogamous, as illustrated by an infinite family of regular graphs that exhibits FR from one vertex to two other vertices at different times [18, Theorem 12.2]. For more about FR, see [16, 18]. For generalizations of FR, see [15, 17, 19].

## 3.3 Strong Cospectrality

Motivated by Propositions 3.2.13 and 3.2.17, we formally introduce the notion of cospectral, parallel, and strongly cospectral vertices.

**Definition 3.3.1.** Let $X$ be a weighted graph with or without loops, and $M(X) = \sum_{j=1}^{r} \lambda_j E_j$ be the spectral decomposition of $M(X)$.

1. The *eigenvalue support* of vertex $u$ with respect to $M(X)$, denoted $\sigma_u(M(X))$, is the set of all eigenvalues $\lambda_j \in \sigma(M)$ such that $E_j \mathbf{e}_u \neq 0$.
2. We say that vertices $u$ and $v$ in $X$ are
   (a) *strongly cospectral* with respect to $M(X)$ if $E_j \mathbf{e}_u = \pm E_j \mathbf{e}_v$ for each $j$,
   (b) *cospectral* with respect to $M(X)$ if $(E_j)_{u,u} = (E_j)_{v,v}$ for each $j$, and
   (c) *parallel* with respect to $M(X)$ if $E_j \mathbf{e}_u$ and $E_j \mathbf{e}_v$ are parallel vectors for each $j$, i.e., there exists a constant $c$ such that $E_j \mathbf{e}_u = c E_j \mathbf{e}_v$.

If $M = A$, then we use the terms adjacency strongly cospectral, adjacency cospectral, and adjacency parallel interchangeably with strongly cospectral with respect to $A$, cospectral with respect to $A$, and parallel with respect to $A$, respectively. We use a similar language if $M = L$.

Now, let $X$ be a weighted graph with or without loops, and $M = \sum_{j=1}^{r} \lambda_j E_j$ be the spectral decomposition of $M$. If we consider the analytic function $g(t) = \frac{1}{\lambda - t}$, then Theorem 2.2.2 gives us

$$(\lambda I - M)^{-1} = g(\lambda I - M) = \sum_{j=1}^{r} g(\lambda - \lambda_j) E_j = \sum_{j=1}^{r} \frac{1}{\lambda - \lambda_j} E_j.$$

Now, Cramer's rule yields $((\lambda I - M)^{-1})_{u,u} = \frac{\phi(M(X\backslash u),t)}{\phi(M(X),t)}$. Therefore, vertices $u$ and $v$ of $X$ are cospectral with respect to $M$ if and only if

$$\begin{aligned}(E_j)_{u,u} = (E_j)_{v,v} \text{ for each } j &\iff \left((\lambda I - M)^{-1}\right)_{u,u} = \left((\lambda I - M)^{-1}\right)_{v,v} \\ &\iff \phi(M(X\backslash u), t) = \phi(M(X\backslash v), t).\end{aligned}$$

In other words, the definitions of cospectral vertices given in Definition 3.3.1 and that of Section 2.3.1 are equivalent. Recall that if $X\backslash u$ and $X\backslash v$ are isomorphic, then follows immediately that vertices $u$ and $v$ are cospectral. However, if $u$ and $v$ are cospectral, that is, $\phi(M(X\backslash u), t) = \phi(M(X\backslash v), t)$, then it does not follow that $X\backslash u$ and $X\backslash v$ are isomorphic. Take the graph $X$, for instance, as Schwenk's Tree in Figure 3.1. One checks that $u$ and $v$ are adjacency cospectral, but $X\backslash u \ncong X\backslash v$ [39].

Now that we have already defined parallel and strongly cospectral vertices, we restate Propositions 3.2.13, 3.2.17 and 3.2.18 into one lemma.

**Lemma 3.3.2.** *Let $X$ be a weighted graph with or without loops, and with vertices $u$ and $v$. The following statements hold.*

1. *If pretty good state transfer occurs between $u$ and $v$ with respect to $M(X)$, then $u$ and $v$ are strongly cospectral with respect to $M(X)$.*

2. *Assume proper fractional revival occurs between $u$ and $v$ with respect to $M(X)$. Then $u$ and $v$ are parallel with respect to $M(X)$, and $\sigma_u(M(X)) = \sigma_v(M(X))$. In particular, if $M = L$, then $u$ and $v$ are Laplacian strongly cospectral.*

Next, we state a consequence of cospectrality which we can use to rule out vertices in a graph which are not cospectral.

**Proposition 3.3.3.** *Let $X$ be a weighted graph with or without loops, and with vertices $u$ and $v$ that are cospectral with respect to $M(X)$. The following hold.*

1. $\sigma_u(M(X)) = \sigma_v(M(X))$

2. *For any analytic function $g$ defined on $\sigma(M(X))$, $g(M(X))_{u,u} = g(M(X))_{v,v}$.*

3. $(M(X)^{\ell})_{u,u} = (M(X)^{\ell})_{v,v}$ *for every integer $\ell \geq 0$.*

4. $(U(t))_{u,u} = (U(t))_{v,v}$ *for all $t \in \mathbb{R}$.*

5. *If $M(X) = L(X)$, then $deg(u) = deg(v)$.*

6. *If $M(X) = A(X)$ and $X$ is unweighted, then $deg(u) = deg(v)$.*

*Proof.* Let $X$ be a weighted graph with or without loops, and with cospectral vertices $u$ and $v$. Assume $M$ has spectral decomposition $M = \sum_{\lambda \in \sigma(M)} \lambda E_\lambda$. Since $u$ and $v$ are cospectral with respect to $M$, $(E_\lambda)_{u,u} = (E_\lambda)_{v,v}$ for all $\lambda \in \sigma(M)$. First, using the fact that each $E_j$ is symmetric and idempotent, we get

$$||E_j \mathbf{e}_u||^2 = (E_j \mathbf{e}_u)^T (E_j \mathbf{e}_u) = \mathbf{e}_u^T E_j^T E_j \mathbf{e}_u = \mathbf{e}_u^T E_j^2 \mathbf{e}_u = \mathbf{e}_u^T E_j \mathbf{e}_u = (E_j)_{u,u} \tag{3.3.1}$$

so that $E_j \mathbf{e}_u \neq 0$ if and only if $(E_j)_{u,u} \neq 0$. Thus, (1) is true. Next, suppose $g$ is an analytic function defined on each eigenvalue of $M$. By Theorem 2.2.2(3), $g(M) = \sum_{\lambda \sigma(A)} g(\lambda) E_\lambda$ so that $g(M)_{u,u} = g(M)_{v,v}$, and hence, (2) holds. Now, applying the functions $g(x) = x^\ell$ and $g(x) = e^{itx}$ to the spectral decomposition of $M$ yields $(M^\ell)_{u,u} = (M^\ell)_{v,v}$ for every nonnegative integer $\ell$ and $(U(t))_{u,u} = (U(t))_{v,v}$ for any $t \in \mathbb{R}$, respectively. This proves (3) and (4). Finally, if we take $M = L$, then we have that

$$\deg(u) = (L)_{u,u} = \sum_{\lambda \in \sigma(M)} \lambda \, (E_\lambda)_{u,u} = \sum_{\lambda \in \sigma(M)} \lambda \, (E_\lambda)_{v,v} = (L)_{v,v} = \deg(v),$$

which proves (5). Finally, let $M = A$ and $X$ be unweighted. We have

$$(A)^2_{u,u} = \sum_{\lambda \in \sigma(M)} \lambda^2 \, (E_\lambda)_{u,u} = \sum_{\lambda \in \sigma(M)} \lambda^2 \, (E_\lambda)_{v,v} = (A)^2_{v,v}.$$

Since $(A^k)_{u,v}$ is the number of paths of length $k$ from $u$ to $v$, we get that $(A^2)_{u,u}$ is twice the number of edges incident to vertex $u$, and thus, $\deg(u) = \deg(v)$. $\square$

For a characterization of adjacency cospectral vertices, see [[39], Theorem 3.1].

Now, let us explore the connection between strongly cospectral, cospectral and parallel vertices. Suppose $X$ is a weighted graph with or without loops, and with vertices $u$ and $v$. Let $M = \sum_{j=1}^r \lambda_j E_j$ be the spectral decomposition of $M$. Assume that $u$ and $v$ are strongly cospectral with respect to $M$. Since $E_j \mathbf{e}_u \neq 0$ if and only if $E_j \mathbf{e}_v \neq 0$ for each $j$, we obtain $\sigma_u(M) = \sigma_v(M)$. For $\lambda_j \in \sigma_u(M)$, let $\delta_j \in \{\pm 1\}$ such that $E_j \mathbf{e}_u = \delta_j E_j \mathbf{e}_v$, and define the sets

$$\sigma_u^+(M) = \{\lambda_j \in \sigma_u(M) : \delta_j = 1\} \text{ and } \sigma_u^-(M) = \{\lambda_j \in \sigma_u(M) : \delta_j = -1\}. \tag{3.3.2}$$

Then $\sigma_u^+(M) = \sigma_v^+(M)$, $\sigma_u^-(M) = \sigma_v^-(M)$, and $\sigma_u(M)$ is a disjoint union of $\sigma_u^+(M)$ and $\sigma_u^-(M)$. Moreover, it is clear that $u$ and $v$ are parallel with respect to $M$. Combining Equation (3.3.1) with the fact that $\|E_j \mathbf{e}_u\| = \|E_j \mathbf{e}_v\|$ yields $(E_j)_{u,u} =$

$(E_j)_{v,v}$ for each $j$, and so $u$ and $v$ are cospectral with respect to $M$. That is, strongly cospectral vertices are also cospectral and parallel. Conversely, let $u$ and $v$ be cospectral and parallel with respect to $M$. Then $E_j\mathbf{e}_u = cE_j\mathbf{e}_v$ for some complex number $c$. Since each $E_j$ is symmetric, we obtain

$$\begin{aligned}(E_j)_{u,u} &= \mathbf{e}_u^T E_j \mathbf{e}_u = c(\mathbf{e}_u^T E_j \mathbf{e}_v) = c(E_j)_{u,v} = c(E_j)_{v,u} \\ &= c(\mathbf{e}_v^T E_j \mathbf{e}_u) = c^2(\mathbf{e}_v^T E_j \mathbf{e}_v) = c^2(E_j)_{v,v}.\end{aligned}$$

And because $u$ and $v$ are cospectral, we get that $c = 1$, which, together with Equation (3.3.1), implies that $\|E_j\mathbf{e}_u\| = \|E_j\mathbf{e}_v\|$. Thus, $E_j\mathbf{e}_u = \pm E_j\mathbf{e}_v$, i.e., $u$ and $v$ are strongly cospectral with respect to $M$. These observations yield following proposition.

**Proposition 3.3.4** ([39], Lemma 4.1)**.** *Let $X$ be a weighted graph with or without loops, and with vertices $u$ and $v$. Then $u$ and $v$ are strongly cospectral with respect to $M(X)$ if and only if they are cospectral and parallel with respect to $M(X)$. Moreover, if $u$ and $v$ are strongly cospectral, then $\sigma_u(M(X)) = \sigma_u^+(M(X)) \cup \sigma_u^-(M)$, $\sigma_u^+(M(X)) = \sigma_v^+(M(X))$ and $\sigma_u^-(M(X)) = \sigma_v^-(M(X))$.*

A characterization of graphs whose vertices are all pairwise parallel is also known. We provide a proof for completeness.

**Proposition 3.3.5** ([39], Lemma 4.2)**.** *Let $X$ be a weighted graph with or without loops. Then all vertices of $X$ are pairwise parallel with respect to $M(X)$ if and only if all eigenvalues of $M(X)$ are simple.*

*Proof.* Let $X$ be a weighted graph with vertices $u$ and $v$. If every pair of vertices of $X$ are pairwise parallel with respect to $M(X)$, then $E_\lambda\mathbf{e}_j = cE_\lambda\mathbf{e}_u$ for every vertex $j$ of $X$. That is, each column of $E_\lambda$ is a scalar multiple of its $u$-th column so that $E_\lambda$ is a rank one matrix. Consequently, the eigenspace of each $\lambda \in \sigma(M(X))$ has dimension one, or equivalently, all eigenvalues of $M(X)$ are simple. Conversely, if all eigenvalues of $M(X)$ are simple, then each spectral idempotent corresponding to $\lambda \in \sigma(M(X))$ can be written as $E_\lambda = \frac{1}{\|\mathbf{v}_\lambda\|^2}\mathbf{v}_\lambda\mathbf{v}_\lambda^T$, where $\mathbf{v}_\lambda$ is the lone eigenvector corresponding to $\lambda$. Thus, it follows that $E_\lambda\mathbf{e}_u = cE_\lambda\mathbf{e}_v$, where $c = 0$ if $\mathbf{v}_\lambda^T\mathbf{e}_v = 0$, and $c = \frac{\mathbf{v}_\lambda^T\mathbf{e}_u}{\mathbf{v}_\lambda^T\mathbf{e}_v}$ otherwise. In other words, $u$ and $v$ are parallel. □

Combining Propositions 3.3.4 and 3.3.5 yields the following proposition.

**Proposition 3.3.6.** *Let $X$ be a weighted graph with or without loops, and with vertices $u$ and $v$. If each eigenvalue of $M(X)$ is simple, then $u$ and $v$ are cospectral with respect to $M(X)$ if and only if they are strongly cospectral with respect to $M(X)$.*

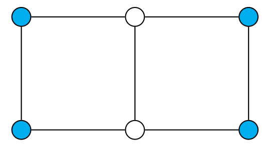

Figure 3.2: The graph $K_2 \square P_3$ with four pairwise adjacency strongly cospectral vertices marked blue

Now, by replacing $\sigma(M)$ by $\sigma_u(M)$ in the proof of the converse of Proposition 3.3.5, we get the following result.

**Corollary 3.3.7.** *Let $X$ be a weighted graph with or without loops. If $u$ and $v$ are vertices of $X$ such that all eigenvalues in $\sigma_u(M(X))$ and $\sigma_v(M(X))$ are simple, then $u$ and $v$ are parallel.*

By again using Proposition 3.3.4, we get a weaker version of Proposition 3.3.6.

**Corollary 3.3.8.** *Let $X$ be a weighted graph with or without loops, and with vertices $u$ and $v$. If $u$ and $v$ are vertices of $X$ such that $\sigma_u(M(X)) = \sigma_v(M(X))$ and all eigenvalues in $\sigma_u(M(X))$ are simple, then $u$ and $v$ are cospectral with respect to $M(X)$ if and only if they are strongly cospectral with respect to $M(X)$.*

Adjacency strong cospectrality between two vertices is also preserved under an equitable partition, provided they are singleton cells in $\pi$.

**Lemma 3.3.9** ([29], Lemma 3.2)**.** *Let $X$ be a weighted graph with or without loops, and $\pi$ be an equitable partition of $X$ with cells $\{u\}$ and $\{v\}$. Then $u$ and $v$ are strongly cospectral in $X$ if and only if $\{u\}$ and $\{v\}$ are strongly cospectral in $\widehat{X/\pi}$.*

Like PGST and FR, strong cospectrality is not necessarily monogamous. To see this, consider the graph $X = K_2 \square P_3$ in Figure 3.2. Let $S$ be the set of four degree two vertices of $X$. Since $X \backslash u \cong X \backslash v$ for any $u, v \in S$, each pair of vertices in $S$ are cospectral. Since $\sigma(A) = \{\pm 1, \pm 1 \pm \sqrt{2}\}$, Proposition 3.3.5 implies that any two vertices in $X$ are adjacency parallel, and so any pair of vertices in $S$ are adjacency strongly cospectral. Hence, we define a subset $S$ of $V(X)$ to be *strongly cospectral* with respect to $M$ if any two vertices in $S$ are strongly cospectral with respect to $M$.

Let $X$ be a weighted graph with or without loops, and with vertex $u$. Denote the set of all vertices that are strongly cospectral with $u$ by $S_u$. Then $u \in S_u$, and for any two vertices $v, w \in S_u$, $E_\lambda \mathbf{e}_u = \pm E_\lambda \mathbf{e}_v$ and $E_\lambda \mathbf{e}_u = \pm E_\lambda e_w$, and so $E_\lambda \mathbf{e}_v = \pm E_\lambda \mathbf{e}_w$ for all $\lambda \in \sigma_u(M)$. That is, $S_u$ is strongly cospectral with respect to $M$. Since strongly cospectral vertices are cospectral, Proposition 3.3.3(1) implies that $\sigma_u(M) = \sigma_v(M)$

for all $v \in S_u$. In [39, Lemma 7.3], Godsil showed that $|S_u|$ is bounded above by $|\sigma_u(M)|$. However, if $X \neq K_2$, then $|S_u|$ must be strictly less than $|V(X)|$, as the only graph that exhibits strong cospectrality between any pair of vertices is $K_2$ [39, Lemma 10.1]. Consequently, if $X \neq K_2$, then $|S_u| \leq \sigma_u(X) < |V(X)|$. Lastly, let $S'_u$ be the set of all vertices that exhibits PGST (resp., Laplacian FR) with $u$ with respect to $M$. Since strong cospectrality is a necessary condition for PGST (resp., Laplacian FR) to occur, we have $|S'_u| \leq |S_u| \leq \sigma_u(X) < |V(X)|$, whenever $X \neq K_2$. For more about strongly cospectral vertices, see [39].

## 3.4 Some examples

This section aims to illustrate the definitions and basic theorems that we presented in this chapter.

### 3.4.1 Hypercubes

We provide an infinite family of unweighted graphs that exhibit perfect state transfer. First, we consider the unweighted $K_2$ with vertices $v_1$ and $v_2$. The eigenvalues of $A = A(K_2)$ are $\pm 1$ with eigenvectors $(1, \pm 1)^T$. Thus $\sigma(A) = \sigma_j(A)$ for $j = 1, 2$, and the spectral idempotents are $E_1 = \frac{1}{2}\mathbf{J}_2$ and $E_{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Hence, $v_1$ and $v_2$ are adjacency strongly cospectral, with spectral decomposition of $A$ is $A = (1)E_1 + (-1)E_{-1}$. Since $g(x) = e^{itx}$ is analytic, we obtain

$$e^{itA} = e^{it}E_1 + e^{-it}E_{-1} = \frac{1}{2}\begin{bmatrix} e^{it} + e^{-it} & e^{it} - e^{-it} \\ e^{it} - e^{-it} & e^{it} + e^{-it} \end{bmatrix}.$$

Alternatively, we may use the power series expansions of $e^x$, $\cos x$ and $\sin x$ to derive an expression for $e^{itA}$. Noting that $A^{2k} = I$ and $A^{2k} = A$ for all integers $k$, we obtain

$$e^{itA} = \sum_{k=0}^{\infty} \frac{(itA)^k}{k!} = \sum_{k=0}^{\infty} \frac{(itA)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(itA)^{2k+1}}{(2k+1)!} = (\cos t)I + i(\sin t)A.$$

In both cases, we see that the transition matrix with respect to $A$ is given by

$$U(t) = \begin{bmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{bmatrix}.$$

Note that $|(U(t))_{1,2}|^2 = 1$ if and only if $t = \frac{k\pi}{2}$ for odd $k$. Thus, PST occurs between $v_1$ and $v_2$ with minimum PST time $\tau = \frac{\pi}{2}$ and phase factor $i$. Since

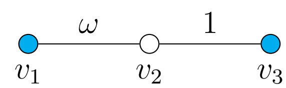


Figure 3.3: The graph $X(\omega)$

$U(2\tau) = U(\tau)^2 = I$, $v_1$ and $v_2$ are periodic at $2\tau$ with phase factor 1. Also, proper $(\cos t, i\sin t)$-FR occurs between $u$ and $v$ at time $t$ if and only if $t \neq \frac{\ell\pi}{2}$ for some integer $\ell$.

Now, for $n \geq 1$, consider the *hypercube* $Q_n$ (also called the $n$-cube). It is known that $Q_n \cong K_2^{\square n}$. Applying Theorem 3.2.9 to a cartesian product of $n$ copies of $K_2$, and using the fact that $K_2$ exhibits PST between its two antipodal vertices with minimum time $\tau = \frac{\pi}{2}$, we conclude that $Q_n$ exhibits PST between antipodal vertices with minimum time $\tau = \frac{\pi}{2}$. This shows that the hypercubes are a family of graphs that are excellent sources of PST. Lastly, since $Q_n$ is an $n$-regular graph, Proposition 3.1.4 implies that these results are also applicable for Laplacian dynamics.

**Proposition 3.4.1.** *For all $n \geq 1$, the hypercube $Q_n$ exhibits perfect state transfer between antipodal vertices with minimum time $\tau = \frac{\pi}{2}$.*

In Chapter 7, we provide a summary of quantum state transfer in some common families of graphs. For our next example, we look at two weighted versions of $P_3$.

### 3.4.2 Weighted $P_3$ with or without loops

Consider the graph $X = X(\omega)$ in Figure 3.3. Note that $X\backslash v_1 \cong K_2$, $X\backslash v_2 \cong O_2$ and $X\backslash v_3$ is the weighted $K_2$. Thus, if $\omega \neq \pm 1$, then no two vertices of $X$ are adjacency cospectral, while if $\omega \neq 1$, then no two vertices of $X$ are Laplacian cospectral. If $\omega = \pm 1$, then $v_1$ is adjacency cospectral with $v_3$, but not with $v_2$, while if $\omega = 1$, then $v_1$ is Laplacian cospectral with $v_3$, but not with $v_2$. Moreover,

$$A = \begin{bmatrix} 0 & \omega & 0 \\ \omega & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } L = \begin{bmatrix} \omega & -\omega & 0 \\ -\omega & 1+\omega & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

for any $\omega \in \mathbb{R}$. For the adjacency case, we have $\sigma(A) = \{0, \pm\sqrt{\omega^2+1}\}$ with eigenvectors $(1, 0, -\omega)^T$ and $(\omega, \pm\sqrt{\omega^2+1}, 1)^T$, which yields the spectral decomposition

$$A = (0)E_0 + \sqrt{\omega^2+1}E_{\sqrt{\omega^2+1}} + (-\sqrt{\omega^2+1})E_{-\sqrt{\omega^2+1}} \tag{3.4.1}$$

where

$$E_0 = \frac{1}{\omega^2+1}\begin{bmatrix} 1 & 0 & -\omega \\ 0 & 0 & 0 \\ -\omega & 0 & \omega^2 \end{bmatrix}$$

and

$$E_{\pm\sqrt{\omega^2+1}} = \frac{1}{2(\omega^2+1)}\begin{bmatrix} \omega^2 & \pm\omega\sqrt{\omega^2+1} & \omega \\ \pm\omega\sqrt{\omega^2+1} & \omega^2+1 & \pm\sqrt{\omega^2+1} \\ \omega & \pm\sqrt{\omega^2+1} & 1 \end{bmatrix}.$$

Thus, $v_1$ and $v_3$ are strongly cospectral if and only if $\omega = \pm 1$, in which case $\sigma_1(A) = \sigma_3(A) = \sigma(A)$. By Equation (3.4.1), we obtain

$$U(t) = E_0 + e^{it\sqrt{\omega^2+1}}E_{\sqrt{\omega^2+1}} + e^{-it\sqrt{\omega^2+1}}E_{-\sqrt{\omega^2+1}}. \tag{3.4.2}$$

Now, from Equation (3.4.2), it follows that

$$U(t)\mathbf{e}_1 = \begin{bmatrix} \frac{1}{\omega^2+1} + \frac{\omega^2 e^{it\sqrt{\omega^2+1}}}{2(\omega^2+1)} + \frac{\omega^2 e^{-it\sqrt{\omega^2+1}}}{2(\omega^2+1)} \\ \frac{\omega e^{it\sqrt{\omega^2+1}}\sqrt{\omega^2+1}}{2(\omega^2+1)} - \frac{\omega e^{-it\sqrt{\omega^2+1}}\sqrt{\omega^2+1}}{2(\omega^2+1)} \\ -\frac{\omega}{\omega^2+1} + \frac{\omega^2 e^{it\sqrt{\omega^2+1}}}{2(\omega^2+1)} + \frac{\omega^2 e^{-it\sqrt{\omega^2+1}}}{2(\omega^2+1)} \end{bmatrix}. \tag{3.4.3}$$

Let $\alpha = (U(t))_{1,1}$ and $\beta = (U(t))_{1,3}$. One checks that $(U(t))_{1,2} = 0$ if and only if $e^{it(2\sqrt{\omega^2+1})} = 1$, or equivalently, $t(2\sqrt{\omega^2+1}) \equiv 0 \pmod{2\pi}$. Combining this with Equation (3.4.3), we get $(\alpha,\beta)$-FR occurs from $v_1$ to $v_3$ at time $t$ if and only if $t(2\sqrt{\omega^2+1}) \equiv 0 \pmod{2\pi}$. Thus, the minimum $(\alpha,\beta)$-FR time is $\tau = \frac{\pi}{\sqrt{\omega^2+1}}$, in which case $\alpha = \frac{1-\omega^2}{\omega^2+1}$ and $\beta = \frac{-2\omega}{\omega^2+1}$. If $\omega \neq 0, \pm 1$, then proper $(\alpha,\beta)$-FR occurs from $v_1$ to $v_3$ at time $\tau$, which is balanced if and only if $\omega = -1 \pm \sqrt{2}$. However, if $\omega = \pm 1$, then PST occurs between $v_1$ and $v_3$ at time $\tau = \frac{\pi}{\sqrt{2}}$ with phase factor $-1$.

Next, consider $L$. As we know, no two vertices of $X$ are cospectral whenever $\omega \neq 1$, and hence no two vertices of $X$ are strongly cospectral. Thus, by Lemma 3.3.2, FR does not occur in $X$ whenever $\omega \neq 1$. Now, for $\omega = 1$, one checks that the eigenvalues of $L$ are $3, 1$ and $0$ with corresponding eigenvectors $(1,-2,1)^T$, $(-1,0,1)^T$ and $\mathbf{1}$. Using the spectral idempotents of $L$, we have that $v_1$ is strongly cospectral with $v_3$, but not with $v_2$. Now, one can compute $U(t)$ from the spectral decomposition of $L$, and verify that

$$U(t)\,\mathbf{e}_1 = \begin{bmatrix} \frac{1}{3} + \frac{1}{2}e^{it} + \frac{1}{6}e^{i3t} \\ \frac{1}{3} - \frac{1}{3}e^{i3t} \\ \frac{1}{3} - \frac{1}{2}e^{it} + \frac{1}{6}e^{i3t} \end{bmatrix}.$$

Since $(U(t))_{1,2} = 0$ if and only if $3t \equiv 0 \pmod{2\pi}$, we get $(\alpha, \beta)$-FR from $v_1$ to $v_3$ if and only if $3t \equiv 0 \pmod{2\pi}$. Thus, the minimum $(\alpha, \beta)$-FR time is $\tau = \frac{2\pi}{3}$, in which case $\alpha = \frac{1}{2}e^{i\pi/3}$ and $\beta = 1 - \frac{1}{2}e^{i\pi/3}$. Also, notice that $|\alpha| = |\beta|$ whenever $t = \tau$, and so we obtain a balanced FR. Lastly,

$$|(U(t)_{1,3})|^2 = \left(\frac{1}{3} - \frac{1}{2}\cos t + \frac{1}{6}\cos 3t\right)^2 + \left(\frac{1}{2}\sin t + \frac{1}{6}\sin 3t\right)^2 \leq \frac{1}{9}(5 + 2\sqrt{2}),$$

which implies that the fidelity of state transfer between $v_1$ and $v_3$ is at most $\approx 0.87$, and so no PGST occurs between $v_1$ and $v_3$. This yields the following proposition.

**Proposition 3.4.2.** *The following hold for the graph $X = X(\omega)$ in Figure 3.3.*

1. *Vertices $v_1$ and $v_3$ are adjacency strongly cospectral if and only if $\omega = \pm 1$.*
2. *Adjacency perfect state transfer occurs between $v_1$ and $v_3$ if and only if $\omega = \pm 1$, in which case the minimum perfect state transfer time is $\tau = \frac{\pi}{\sqrt{2}}$ with phase factor $\mp 1$.*
3. *Adjacency $(\alpha, \beta)$-fractional revival occurs from $v_1$ to vertex $v_3$ at time $t$ if and only if $t(2\sqrt{\omega^2 + 1}) \equiv 0 \pmod{2\pi}$. The minimum adjacency $(\alpha, \beta)$-fractional revival time between $v_1$ and $v_3$ is $\tau = \frac{\pi}{\sqrt{\omega^2+1}}$, in which case $\alpha = \frac{1-\omega^2}{\omega^2+1}$ and $\beta = \frac{-2\omega}{\omega^2+1}$. In particular, if $\omega \neq 0, \pm 1$, then proper adjacency $(\alpha, \beta)$-fractional revival occurs from $v_1$ to $v_3$ at time $\tau$, which is balanced if and only if $\omega = -1 \pm \sqrt{2}$.*
4. *Vertices $v_1$ and $v_3$ are Laplacian strongly cospectral if and only if $\omega = 1$.*
5. *Laplacian pretty good state transfer does not occur between $v_1$ and $v_3$.*
6. *Laplacian $(\alpha, \beta)$-fractional revival occurs from $v_1$ to $v_3$ at time $t$ if and only if $\omega = 1$ and $3t \equiv 0 \pmod{2\pi}$. The minimum Laplacian $(\alpha, \beta)$-fractional revival time between $v_1$ and $v_3$ is $\tau = \frac{2\pi}{3}$, in which case $\alpha = \frac{1}{2}e^{i\pi/3}$ and $\beta = 1 - \frac{1}{2}e^{i\pi/3}$, and yields a balanced Laplacian fractional revival.*

From these results, we see that altering the unit weight of the edge between $v_1$ and $v_2$ induces proper adjacency FR between $v_1$ and $v_3$. In particular, replacing the unit weight of the edge between $v_1$ and $v_2$ by a negative unit weight preserves the occurrence of adjacency PST between $v_1$ and $v_3$ in the unweighted version of $P_3$. We also note that Proposition 3.4.2 statements (2) and (3) appears in [16, Example 4.3].

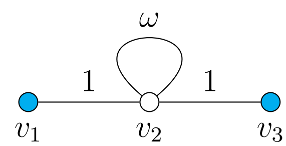


Figure 3.4: The graph $Y(\omega)$

Now, consider the graph $Y = Y(\omega)$ in Figure 3.4. Then

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & \omega & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & \omega+2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

for all $\omega \in \mathbb{R}$. For the adjacency case, $\sigma(A) = \{0, \lambda^{\pm}\}$, where $\lambda^{\pm} = \frac{1}{2}\left(\omega \pm \sqrt{\omega^2+8}\right)$, with eigenvectors $\mathbf{e}_1 - \mathbf{e}_3$ and $(1, \lambda^{\pm}, 1)$. This yields the spectral decomposition

$$A = (0)E_0 + \lambda^+ E_{\lambda^+} + \lambda^- E_{\lambda^-}, \tag{3.4.4}$$

where

$$E_0 = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \text{ and } E_{\lambda^{\pm}} = \frac{1}{(\lambda^{\pm})^2+2}\begin{bmatrix} 1 & \lambda^{\pm} & 1 \\ \lambda^{\pm} & (\lambda^{\pm})^2 & \lambda^{\pm} \\ 1 & \lambda^{\pm} & 1 \end{bmatrix}.$$

From the spectral idempotents, it follows that $v_1$ and $v_3$ are adjacency strongly cospectral for all $\omega$. Using Equation (3.4.4), we get

$$U(t) = E_0 + e^{it\lambda^+} E_{\lambda^+} + e^{-it\lambda^-} E_{\lambda^-}. \tag{3.4.5}$$

From Equation (3.4.5), it follows that

$$U(t)e_1 = \begin{bmatrix} \frac{1}{2} + \frac{e^{it\lambda^+}}{(\lambda^+)^2+2} + \frac{e^{it\lambda^-}}{(\lambda^-)^2+2} \\ \frac{\lambda^+ e^{it\lambda^+}}{(\lambda^+)^2+2} + \frac{\lambda^- e^{it\lambda^-}}{(\lambda^-)^2+2} \\ -\frac{1}{2} + \frac{e^{it\lambda^+}}{(\lambda^+)^2+2} + \frac{e^{it\lambda^-}}{(\lambda^-)^2+2} \end{bmatrix}. \tag{3.4.6}$$

Now, let $\alpha = (U(t))_{1,1}$ and $\beta = (U(t))_{1,3}$. One checks that $\frac{\lambda^+}{\lambda^-} \cdot \frac{(\lambda^-)^2+2}{(\lambda^+)^2+2} = -1$ for all $\omega \in \mathbb{R}$, we obtain $(U(t))_{1,2} = 0$ if and only if $e^{it(\lambda^+ - \lambda^-)} = 1$, i.e., $t(\sqrt{\omega^2+8}) \equiv 0 \pmod{2\pi}$. Combining this with Equation (3.4.6), we conclude that $(\alpha, \beta)$-FR from

$v_1$ to $v_3$ at time $t$ if and only if

$$t(\sqrt{\omega^2+8}) \equiv 0 \pmod{2\pi}. \tag{3.4.7}$$

Thus, the minimum $(\alpha,\beta)$-FR time is $\tau = \frac{2\pi}{\sqrt{\omega^2+8}}$. Since $(U(t))_{1,2}=0$ and $\frac{\lambda^+-\lambda^-}{\lambda^+\left((\lambda^-)^2+2\right)} = \frac{1}{2}$ for all $\omega\in\mathbb{R}$, we can write

$$\begin{aligned}(U(t))_{1,3} &= -\frac{1}{2}+\frac{e^{it\lambda^+}}{(\lambda^+)^2+2}+\frac{e^{it\lambda^-}}{(\lambda^-)^2+2} = -\frac{1}{2}-\frac{\lambda^- e^{it\lambda^-}}{\lambda^+\left((\lambda^-)^2+2\right)}+\frac{e^{it\lambda^-}}{(\lambda^-)^2+2}\\ &= -\frac{1}{2}+\frac{e^{it\lambda^-}\left(\lambda^+-\lambda^-\right)}{\lambda^+\left((\lambda^-)^2+2\right)} = -\frac{1}{2}+\frac{1}{2}e^{it\lambda^-},\end{aligned}$$

and so $\left|(U(t))_{1,3}\right|^2 = \frac{1}{2}(1-\cos t\lambda^-)$. Now, for integers $m$ and $r$ such that either $-m > \frac{r}{2} > 0$ or $m > -\frac{r}{2} > 0$, define

$$\theta_{m,r} = 2(r+m)\sqrt{\frac{-2}{r(r+2m)}}. \tag{3.4.8}$$

Now, proper $(\alpha,\beta)$-FR occurs between $v_1$ and $v_3$ if and only if $\cos t\lambda^- \neq \pm 1$. Equivalently, $t \neq \frac{2r\pi}{\omega-\sqrt{\omega^2+8}}$ for any integer $r$. Combining this with Equation (3.4.7), we get that $\frac{2m\pi}{\sqrt{\omega^2+8}} \neq \frac{2r\pi}{\omega-\sqrt{\omega^2+8}}$ for any integer $m$. That is, $\omega\neq\theta_{m,r}$ for all integers $m$ and $r$. Moreover, $v_1$ is adjacency periodic if and only if $\cos t\lambda^- = 1$, in which case $\omega=\theta_{m,r}$ for some even $r$. Similarly, PST occurs between $v_1$ and $v_3$ if and only if $\cos t\lambda^- = -1$, in which case $\omega=\theta_{m,r}$ for some odd $r$. Lastly, balanced $(\alpha,\beta)$-FR occurs between $v_1$ and $v_3$ if and only if $\cos t\lambda^- = 0$. That is, $t = \frac{r\pi}{\omega-\sqrt{\omega^2+8}}$. Combining this with Equation (3.4.7), we obtain $\omega=\zeta_{m,r}$, where

$$\zeta_{m,r} = 2(r+2m)\sqrt{-\frac{2}{r(r+4m)}} \tag{3.4.9}$$

for some integers $m$ and $r$ such that either $-m > \frac{r}{4} > 0$ or $m > -\frac{r}{4} > 0$. We remark that the time at which all of these occur is at $t = \frac{2m\pi}{\sqrt{\omega^2+8}}$. Next, let us investigate when PGST occurs between $u$ and $v$. Let $\{\epsilon_j\}$ be a sequence of positive numbers converging to 0. Using the definition, PGST occurs between $v_1$ and $v_3$ if and only if

$$|U(t_j)\mathbf{e}_1-\gamma\mathbf{e}_3| < \epsilon_j, \tag{3.4.10}$$

for some unit $\gamma = e^{i\zeta}\in\mathbb{C}$ and a sequence of times $\{t_j\}$. Note that we can write

$U(t)\mathbf{e}_1 = E_0\mathbf{e}_1 + e^{it\lambda^+}E_{\lambda^+}\mathbf{e}_1 + e^{it\lambda^-}E_{\lambda^-}\mathbf{e}_1$, and $\gamma\mathbf{e}_3 = \gamma E_0\mathbf{e}_3 + \gamma E_{\lambda^+}\mathbf{e}_3 + \gamma E_{\lambda^-}\mathbf{e}_3$ because spectral idempotents sum to identity. Since $E_0\mathbf{e}_1 = -E_0\mathbf{e}_3$ and $E_{\lambda^\pm}\mathbf{e}_1 = E_{\lambda^\pm}\mathbf{e}_3$, Equation 3.4.10 then yields

$$|t_j - (\zeta + \pi)| < \epsilon_j \pmod{2\pi} \text{ and } |t_j\lambda^\pm - \zeta| < \epsilon_j \pmod{2\pi}. \tag{3.4.11}$$

We now invoke Theorem 2.4.1. Let $\ell_1, \ell_2, \ell_3$ be integers such that

$$(0)\ell_1 + \lambda^+\ell_2 + \lambda^-\ell_3 = 0. \tag{3.4.12}$$

We want to prove that $\ell_1(\zeta + \pi) + \ell_2\zeta + \ell_3\zeta \equiv 0 \pmod{2\pi}$. If $\ell_1$ is even, then we simply take $\zeta = 2\pi$. However, if $\ell_1$ is odd, then the $\ell_j$'s must satisfy $\ell_1 + \ell_2 + \ell_3 \neq 0$ to ensure a solution for $\zeta$ exists. Now, for every pair of integers $a$ and $b$ such that $ab > 0$, define

$$\delta_{a,b} = \begin{cases} -\sqrt{\frac{2}{ab}}(a-b) < 0, & \text{if } a > b > 0 \\ -\sqrt{\frac{2}{ab}}(a-b) > 0, & \text{if } b > a > 0 \\ \sqrt{\frac{2}{ab}}(a-b) > 0, & \text{if } 0 > a > b \\ \sqrt{\frac{2}{ab}}(a-b) < 0, & \text{if } 0 > b > a. \end{cases} \tag{3.4.13}$$

Using the form of $\lambda^\pm$, we can rewrite Equation (3.4.12) as

$$(\ell_2 + \ell_3)\omega + (\ell_2 - \ell_3)\sqrt{\omega^2 + 8} = 0. \tag{3.4.14}$$

Note that $\ell_2 = 0$ if and only if $\ell_3 = 0$. Thus, if $\ell_2 = 0$, then $\ell_1 + \ell_2 + \ell_3 = \ell_1 \neq 0$. Moreover, if $\ell_2 \neq 0$, then Equation (3.4.14) is true if and only if $\omega = \delta_{\ell_2,\ell_3}$. Thus, if $\omega$ is not of the form $\delta_{a,b}$, then Equation (3.4.14) fails to hold, as well as Equation (3.4.12). Applying Theorem 2.4.1, we get that the inequalities in Equation (3.4.11) admit a solution $t_j$ for each $j$, and we get PGST between $v_1$ and $v_3$. On the other hand, if $\omega = \delta_{a,b}$ for some integers $a$ and $b$ such that $ab > 0$, then Equation (3.4.11) holds with $\ell_2 = ac$ and $\ell_3 = bc$ for any nonzero integer $c$. Assume $\ell_1$ is odd. If $a$ and $b$ have the same parity, then $\ell_1 + \ell_2 + \ell_3 = \ell_1 + c(a+b) \neq 0$ for any $c$ because $a+b$ is even, while if $a$ and $b$ have the opposite parities, then by taking $\ell_1 = -c(a+b)$, where $c$ is odd, we get that $\ell_1 + \ell_2 + \ell_3 = 0$. Thus, for the case that $\omega = \delta_{a,b}$ for some integers $a$ and $b$ such that $ab > 0$, Theorem 2.4.1 yields a solution $t_j$ for the inequalities in Equation (3.4.11) if and only if $a$ and $b$ have the same parity, in which case we get PGST between $v_1$ and $v_3$.

For the Laplacian case, $\sigma(L) = \{1, \lambda^\pm\}$, where $\lambda^\pm = \frac{1}{2}\left(\omega + 3 \pm \sqrt{\omega^2 + 2\omega + 9}\right)$

with eigenvectors $\mathbf{e}_1 - \mathbf{e}_3$ and $(1, z^{\pm}, 1)$, where $z^{\pm} = \lambda^{\pm} - (\omega + 2)$. Thus,

$$A = E_1 + \lambda^+ E_{\lambda^+} + \lambda^- E_{\lambda^-},$$

where

$$E_1 = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \text{ and } E_{\lambda^{\pm}} = \frac{1}{\left(z^{\pm}\right)^2 + 2}\begin{bmatrix} 1 & z^{\pm} & 1 \\ z^{\pm} & (z^{\pm})^2 & z^{\pm} \\ 1 & z_{\pm} & 1 \end{bmatrix}$$

so that $v_1$ and $v_3$ are strongly cospectral for all $\omega \in \mathbb{R}$. Moreover,

$$U(t) = e^{it}E_1 + e^{it\lambda^+}E_{\lambda^+} + e^{-it\lambda^-}E_{\lambda^-}.$$

Similar to the adjacency case, $(\alpha, \beta)$-FR occurs from $v_1$ to $v_3$ at time $t$ if and only if $\left(U(t)\right)_{1,2} = 0$, in which case

$$t(\sqrt{\omega^2 + 2w + 9}) \equiv 0 \pmod{2\pi}. \tag{3.4.15}$$

Thus, the minimum $(\alpha, \beta)$-FR time is $\tau = \frac{2\pi}{\sqrt{\omega^2+2w+9}}$. Since $\left(U(t)\right)_{1,2} = 0$ and $\frac{z^+ - z^-}{z^+\left((z^-)^2+2\right)} = -\frac{1}{2}$ for all $\omega \in \mathbb{R}$, we can write $\left(U(t)\right)_{1,3} = -\frac{1}{2}\left(e^{it} + e^{it\lambda^-}\right)$, and so

$$\left|\left(U(t)\right)_{1,3}\right|^2 = \frac{1}{2}(1 + \cos t\cos(t\lambda^-) + \sin t \sin(t\lambda^-)) = \frac{1}{2}(1 + \cos(t(\lambda^- - 1))).$$

Thus, proper $(\alpha, \beta)$-FR occurs between $v_1$ and $v_3$ if and only if $\cos(t(\lambda^- - 1)) \neq \pm 1$. Equivalently, $t \neq \frac{2r\pi}{\omega+1-\sqrt{\omega^2+2\omega+9}}$ for any integer $r$. Combining this with Equation (3.4.15), we get that $\frac{2m\pi}{\sqrt{\omega^2+8}} \neq \frac{2r\pi}{\omega-\sqrt{\omega^2+8}}$ for any integer $m$. That is, $\omega + 1 \neq \theta_{m,r}$ for all integers $m$ and $r$. Moreover, $v_1$ is periodic if and only if $\cos(t(\lambda^- - 1)) = -1$, in which case $\omega + 1 = \theta_{m,r}$ for some integer $m$ and odd $r$. Similarly, PST occurs between $v_1$ and $v_3$ if and only if $\cos(t(\lambda^- - 1)) = 1$, in which case $\omega + 1 = \theta_{m,r}$ for some integer $m$ and even $r$. Lastly, balanced $(\alpha, \beta)$-FR occurs between $v_1$ and $v_3$ if and only if $\cos(t(\lambda^- - 1) = 0$. That is, $t = \frac{r\pi}{\omega+1-\sqrt{\omega^2+2\omega+9}}$. Combining this with Equation (3.4.15), we obtain $\omega + 1 = \zeta_{m,r}$. We remark that the time at which all of these occur is at $t = \frac{2m\pi}{\sqrt{\omega^2+2\omega+9}}$. Finally, to show PGST, we use a similar argument to the case under adjacency dynamics. Let $\ell_1, \ell_2, \ell_3$ be integers such that

$$\ell_1 + \lambda^+\ell_2 + \lambda^-\ell_3 = 0, \tag{3.4.16}$$

and $\ell_1$ is odd. Using the form of $\lambda^{\pm}$, we can rewrite Equation (3.4.16) as

$$2\ell_1 + (\ell_2 + \ell_3)(\omega + 3) + (\ell_2 - \ell_3)\sqrt{(\omega + 1)^2 + 8} = 0. \tag{3.4.17}$$

If $\ell_2$ and $\ell_3$ are zero, then $\ell_1 + \ell_2 + \ell_3 = \ell_1 \neq 0$. Meanwhile, if one of $\ell_2$ and $\ell_3$ is zero, then Equation (3.4.17) yields

$$\omega + 1 = \frac{b^2 - 2ab - a^2}{b^2 + ab} \tag{3.4.18}$$

for some integer $b$ and odd $a$. Lastly, let $\ell_2$ and $\ell_3$ be nonzero. If $\ell_2 + \ell_3$ is even, then we are done. Otherwise, $\ell_2 \pm \ell_3$ is odd, and hence, nonzero. Thus, Equation (3.4.17) can be written as $a(\omega + 1)^2 + b(\omega + 1) + c = 0$, where $a, b, c \in \mathbb{Z}$ and $a \neq 0$. That is, $\omega + 1$ is a quadratic algebraic number. Thus, if we choose $\omega + 1 \neq 1$ to be any real number that is not of the form given in Equation (3.4.18), and is not a quadratic algebraic number, then Equation (3.4.17) fails to hold. By Theorem 2.4.1, we get PGST between $v_1$ and $v_3$. We summarize these results in the following proposition.

**Proposition 3.4.3.** *Consider $\theta_{m,r}$ given in Equation (3.4.8), and $\zeta_{m,r}$ given in Equation (3.4.9). The following statements hold for the graph $Y = Y(\omega)$ in Figure 3.4.*

1. *Vertices $v_1$ and $v_3$ are adjacency and Laplacian strongly cospectral for all $\omega \in \mathbb{R}$.*

2. *Adjacency $(\alpha, \beta)$-fractional revival occurs from $v_1$ to $v_3$ at time $t$ if and only if $t(\sqrt{\omega^2 + 8}) \equiv 0 \pmod{2\pi}$. The minimum adjacency $(\alpha, \beta)$-fractional revival time is $\tau = \frac{2\pi}{\sqrt{\omega^2+8}}$. Moreover, the following statements hold.*

   *(a) Proper adjacency $(\alpha, \beta)$-fractional revival occurs between $v_1$ and $v_3$ if and only if $\omega$ is not of the form $\theta_{m,r}$. Moreover, $v_1$ is adjacency periodic if and only if $\omega = \theta_{m,r}$ for even $r$, while adjacency perfect state transfer occurs between $v_1$ and $v_3$ if and only if $\omega = \theta_{m,r}$ for odd $r$.*

   *(b) Balanced adjacency $(\alpha, \beta)$-fractional revival occurs between $v_1$ and $v_3$ if and only if $\omega = \zeta_{m,r}$.*

   *In addition, (a) or (b) occurs at time $t = \frac{2m\pi}{\sqrt{\omega^2+8}}$.*

3. *Adjacency pretty good state transfer occurs between $v_1$ and $v_3$ if and only if either $\omega$ is not of the form $\delta_{a,b}$, or $\omega = \delta_{a,b}$ such that $a$ and $b$ have the same parity, where $\delta_{a,b}$ is given in Equation (3.4.13).*

4. *Laplacian $(\alpha,\beta)$-fractional revival occurs from $v_1$ to $v_3$ at time $t$ if and only if $t(\sqrt{\omega^2+2\omega+9}) \equiv 0 \pmod{2\pi}$. The minimum Laplacian $(\alpha,\beta)$-fractional revival time is $\tau = \frac{2\pi}{\sqrt{\omega^2+2\omega+9}}$. Moreover, the following hold.*

   (a) *Proper Laplacian $(\alpha,\beta)$-fractional revival occurs between $v_1$ and $v_3$ if and only if $\omega+1$ is not of the form $\theta_{m,r}$. Moreover, $v_1$ is Laplacian periodic if and only if $\omega+1=\theta_{m,r}$ for odd $r$, while Laplacian perfect state transfer occurs between $v_1$ and $v_3$ if and only if $\omega+1=\theta_{m,r}$ for even $r$.*

   (b) *Balanced Laplacian $(\alpha,\beta)$-fractional occurs between $v_1$ and $v_3$ if and only if $\omega+1=\zeta_{m,r}$.*

   *In addition, (a) or (b) occurs at time $t = \frac{2m\pi}{(\omega+1)-\sqrt{\omega^2+2\omega+9}}$.*

5. *If we choose $\omega+1 \neq 1$ to be any real number that is not of the form given in Equation (3.4.18), and is not a quadratic algebraic number, then Laplacian pretty good state transfer occurs between $v_1$ and $v_3$.*

Note that the graph $Y(\omega)$ in Proposition 3.4.3 appears in [2, Fact 6]. Unlike the graph $X(\omega)$ in Proposition 3.4.2, Proposition 3.4.3 statements (2) and (4) offer numerous choices for $\omega$, all of which are quadratic algebraic numbers of specific form, such that adding a loop of weight $\omega \neq 0$ on $v_2$ yields adjacency and Laplacian PST, as well as balanced FR, between $v_1$ and $v_3$. Moreover, from Proposition 3.4.3 statements (3) and (5), we observe that adjacency and Laplacian PGST is induced between $v_1$ and $v_3$ for almost all weights $\omega$, except for a few exceptions

It is also interesting to compare the the shortest adjacency $(\alpha,\beta)$-FR time between $X(\omega)$ and $Y(\omega)$. For $Y(\omega)$, we see from Proposition 3.4.3(2) that the minimum adjacency $(\alpha,\beta)$-FR time is $\tau_Y = \frac{2\pi}{\sqrt{\omega^2+8}}$, while for $X(\omega)$, we have $\tau_X = \frac{\pi}{\sqrt{\omega^2+1}}$ by Proposition 3.4.2(3). Thus, $\tau_X \geq \tau_Y$ if and only if $|\omega| \leq \pm\frac{2\sqrt{3}}{3}$, with equality whenever $\omega = \pm\frac{2\sqrt{3}}{3}$. That is, as long as the weights are small, the graph $Y(\omega)$ yields a shorter adjacency $(\alpha,\beta)$-FR time than $X(\omega)$, while if the weights are larger, then $X(\omega)$ yields a shorter adjacency $(\alpha,\beta)$-FR time than $Y(\omega)$. Meanwhile, Proposition 3.4.2(3) tells us that Laplacian $(\alpha,\beta)$-FR is achieved in $X(\omega)$ only when $\omega = 1$, while various weights $\omega$ work so that Laplacian $(\alpha,\beta)$-FR is achieved in $Y(\omega)$ by Proposition 3.4.3(4). From this example, it is evident that by adding loops or varying the edge weights of a given graph, one may be able to induce a specific type of state transfer under a given dynamics, and/or improve the time at which it occurs.

# 4

# Properties of twin vertices

We now examine some properties of twin vertices that are useful in quantum state transfer. We start with the definition of twins in weighted graphs. Throughout this chapter, we assume all graphs to be weighted with or without loops.

**Definition 4.0.1.** Let $X$ be a weighted graph with or without loops. We say that two distinct vertices $u$ and $v$ of $X$ are *twins* if the following conditions hold:

1. $N_X(u)\backslash\{v\} = N_X(v)\backslash\{u\}$,
2. the edges $(u, w)$ and $(v, w)$ have equal weights for each $w \in N_X(u)\backslash\{v\}$, and
3. if there are loops on $u$ and $v$, then they must have equal weights.

In addition, if $u$ and $v$ are adjacent, then we say that $u$ and $v$ are *true twins*. Otherwise, we say that $u$ and $v$ are *false twins*.

Note that if either $u$ and $v$ are false twins or if $u$ and $v$ are true twins with loops, then $N_X(u) = N_X(v)$. However, this is not the case if either $u$ and $v$ are true twins without loops or if $u$ and $v$ are false twins with loops. Now, if $X$ is a simple unweighted graph, then $u$ and $v$ are twins if and only if $N_X(u)\backslash\{v\} = N_X(v)\backslash\{u\}$. From this, we see that our definition generalizes the definition of twin vertices from simple, unweighted graphs to weighted graphs with or without loops. In literature, false twins are also called duplicates and clones without an edge, while true twins are also called co-duplicates or clones with an edge, see [1, 10].

Let $X$ be a weighted graph with or without loops, and $\omega, \eta \in \mathbb{R}$. We say that a subset $U = U(\omega, \eta)$ of $V(X)$ is a *set of twins* in $X$ if any two vertices in $U$ are pairwise twins, where each vertex in $U$ has a loop of weight $\omega$, and the loops are absent if $\omega = 0$, and every pair of vertices in $U$ are connected by an edge with weight

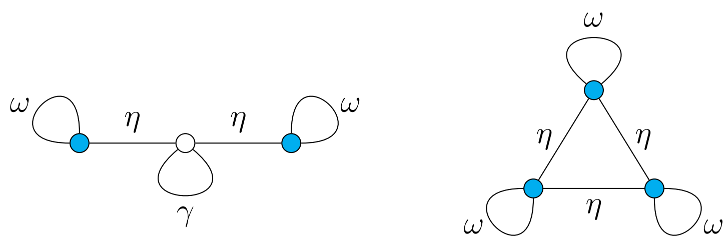


Figure 4.1: Weighted graphs with twin vertices marked blue: weighted $P_3$ with loops (left), and the weighted complete graph on three vertices with loops $\mathbf{K}_3(\omega, \eta)$ (right)

$\eta \neq 0$, and no pair of vertices in $U$ are adjacent whenever $\eta = 0$. Let $u, v, w \in U$ such that $u$ and $v$ are false twins, and $v$ and $w$ are true twins. Since $v$ and $w$ are true twins, we get that $w \in N(v)$. But using the fact that $u$ and $v$ are false twins, we also have that $u \notin N(v)$ and $w \in N(u)$, and hence, $u \in N(w)$. Thus, $u \notin N(v)$ and $u \in N(w)$, which contradicts the fact that $v$ and $w$ are true twins. For this reason, when we say that $U = U(\omega, \eta)$ is a set of twins in $X$, then either every pair of distinct vertices in $U$ are true twins, in which case $\eta \neq 0$, or false twins, in which case $\eta = 0$. We present an example.

**Example 4.0.2.** In the unweighted complete graph $K_n$, any pair of vertices are true twins. Thus, $U = V(K_n)$ is a set of true twins in $K_n$. Meanwhile, in the unweighted complete graph minus an edge $K_n \backslash e$, the two non-adjacent vertices form a set of false twins, while the rest of the $n-2$ vertices form a set of true twins.

For $\omega, \eta \in \mathbb{R}$, let $\mathbf{K}_n(\omega, \eta)$ denote the *weighted complete graph* on $n$ vertices with or without loops, where every loop on each vertex has weight $\omega$, and every edge between two distinct vertices has weight $\eta$; $\mathbf{K}_{1,n-1}(\omega, \eta)$ denote the *weighted star* on $n$ vertices with or without loops, where every loop on each vertex of degree one has weight $\omega$, and every edge has weight $\eta$; and $\mathbf{O}_n(\omega)$ denote the *weighted empty graph with loops* on $n$ vertices with or without loops, where every loop on each vertex has weight $\omega$. If $\omega = 0$, then the loops on the vertices of these graphs are absent.

Let $X$ be a weighted graph on $n$ vertices, and $U = U(\omega, 0)$ be a set of false twins in $X$ with $|U| = m$. Then no pair of distinct vertices in $U$ are adjacent, and so the induced subgraph of $U$ in $X$ is isomorphic to $\mathbf{O}_m(\omega)$. If we add that $X$ is connected, then $U \neq V(X)$, otherwise $X$ is a set of $n$ isolated vertices, and therefore, not connected. Thus, $|U| \leq n-1$. In particular, if $|U| = n-1$, then each vertex in $U$ is connected to the single vertex in $V(X) \backslash U$ so that $X \cong \mathbf{K}_{1,n-1}(\omega, \eta)$. Moreover, if $V(X)\backslash U$ is also a set of false twins in $X$, then $X \cong \mathbf{O}_m(\omega) \vee \mathbf{O}_{n-m}(\omega')$, which can be viewed as a weighted complete bipartite graph with or without loops.

Now, let $X$ be a connected weighted graph on $n$ vertices, and $U = U(\omega, \eta)$ be a set of true twins in $X$ with $|U| = m$ and $\eta \neq 0$. Then every pair of vertices in $U$ are adjacent so that $u \in N_X(v)$ for all $u, v \in U$ with $u \neq v$. Equivalently, the induced subgraph of $U$ in $X$ is isomorphic to $\mathbf{K}_m(\omega, \eta)$. In particular, $U = V(X)$ if and only if $X \cong \mathbf{K}_n(\omega, \eta)$. Next, assume $U \neq V(X)$. If $|U| = n - 1$, then the lone vertex $u \in V(X)\backslash U$ is adjacent to one of the vertices in $U$, otherwise $X$ is disconnected. Since $U$ is a set of true twins, it follows that each vertex in $U$ is adjacent to $u$, and hence $u \in U$, which is a contradiction. Thus, $|U| \leq n - 2$. If $V(X)\backslash U$ is another set of true twins in $X$, then at least one vertex in $U$ is connected to at least one vertex in $V(X)\backslash U$, otherwise $X$ is disconnected. But since $U$ and $V(X)\backslash U$ are sets of true twins, every vertex in $U$ is connected to every vertex in $V(X)\backslash U$. Thus, $X \cong \mathbf{K}_m(\omega, \eta) \vee \mathbf{K}_{n-m}(\omega', \eta')$, where either $\omega \neq \omega'$ or $\eta \neq \eta'$, i.e, $X$ is a weighted complete graph on $n$ vertices. Since $\omega \neq \omega'$ or $\eta \neq \eta'$, we have $\mathbf{K}_m(\omega, \eta) \vee \mathbf{K}_{n-m}(\omega', \eta') \ncong \mathbf{K}_n(\gamma, \zeta)$. However, if $U \neq V(X)$ and $V(X)\backslash U$ is a set of false twins in $X$, then $X \cong \mathbf{K}_m(\omega, \eta) \vee \mathbf{O}_{n-m}(\omega')$. In particular, if $|U| = n-2$, then $X \cong \mathbf{K}_{n-2}(\omega, \eta) \vee \mathbf{O}_2(\omega')$, which can be viewed as a weighted complete graph minus an edge with or without loops. These observations yield the following proposition.

**Proposition 4.0.3.** *Let $X$ be a weighted graph $n$ vertices, and $U = U(\omega, \eta)$ be a set of twins in $X$. The following statements hold.*

1. *If $U$ is a set of false twins in $X$, then $\eta = 0$ and the induced subgraph of $U$ in $X$ is isomorphic to $\boldsymbol{O}_{|U|}(\omega)$. If we add that $X$ is connected, then $|U| \leq n - 1$. In particular, if $|U| = n - 1$, then $X \cong \boldsymbol{K}_{1,n-1}(\omega, \eta)$, while if $V(X)\backslash U$ is also a set of false twins in $X$, then $X \cong \boldsymbol{O}_{|U|}(\omega) \vee \boldsymbol{O}_{n-|U|}(\omega')$.*

2. *Let $U$ be a set of true twins in $X$. Then $\eta \neq 0$ and the induced subgraph of $U$ in $X$ is isomorphic to $\boldsymbol{K}_{|U|}(\omega, \eta)$. If $X$ is connected, then the following hold.*

   *(a) $U = V(X)$ if and only if $X \cong \boldsymbol{K}_n(\omega, \eta)$.*

   *(b) If $U \neq V(X)$, then $|U| \leq n - 2$. If we add that $V(X)\backslash U$ is a set of true twins, then $X \cong \boldsymbol{K}_{|U|}(\omega, \eta) \vee \boldsymbol{K}_{n-|U|}(\omega', \eta')$, where either $\omega \neq \omega'$ or $\eta \neq \eta'$, while if $V(X)\backslash U$ is a set of false twins, then $X \cong \boldsymbol{K}_{|U|}(\omega, \eta) \vee \boldsymbol{O}_{n-|U|}(\omega')$.*

Our main goal in this chapter is to give necessary and sufficient conditions for a pair of twin vertices in a graph to be strongly cospectral. To do this, we begin by examining the size of the eigenvalue support of a vertex in a connected graph. Throughout this chapter, we use $M(X)$ to denote the adjacency and Laplacian matrices of a graph $X$, unless otherwise specified. If the context is clear, then we simply write $M = M(X)$, $A = A(X)$ and $L = L(X)$.

## 4.1 Size of eigenvalue supports

Let $X$ be a connected weighted graph, and $u$ be a vertex of $X$. If $|\sigma_u(A)| = 1$, then there exists $\theta$ such that $E_\theta \mathbf{e}_u \neq 0$ and $E_\lambda \mathbf{e}_u = 0$ for all $\lambda \in \sigma(A)\backslash\{\theta\}$. Since spectral idempotents sum to identity, we obtain

$$\mathbf{e}_u = \sum_{\lambda \in \sigma(A)} E_\lambda \mathbf{e}_u = E_\theta \mathbf{e}_u.$$

But using the spectral decomposition of $A$, we have that

$$A\mathbf{e}_u = \sum_{\lambda \in \sigma(A)} \lambda(E_\lambda \mathbf{e}_u) = \theta E_\theta \mathbf{e}_u = \theta \mathbf{e}_u.$$

That is, the $u$th column of $A$ has all entries zero except possibly at the $u$th row. Equivalently, $X$ is disconnected, which is a contradiction. Therefore, $|\sigma_u(A)| \geq 2$. If we add that $X$ is positively weighted, then by the Perron Frobenius Theorem, the largest eigenvalue $\lambda_{\max}$ of the adjacency matrix $A$ of $X$ is simple, and there is a corresponding eigenvector $\mathbf{w}$ that has all entries positive. Hence, $\lambda_{\max} \in \sigma_u^+(A)$.

Next, we consider the Laplacian matrix. Assume $X$ has no loops. Since $X$ is connected, we know that 0 is a simple eigenvalue of $L$ with eigenvector $\mathbf{1}$, and hence $0 \in \sigma_u(L)$. Similarly, if 0 is the only eigenvalue in $\sigma_u(L)$, then $E_\lambda \mathbf{e}_u = 0$ for all nonzero eigenvalues $\lambda$ of $L$. Since spectral idempotents sum to identity, we get

$$\mathbf{e}_u = \sum_{\lambda \in \sigma(L)} E_\lambda \mathbf{e}_u = E_0 \mathbf{e}_u = \mathbf{1},$$

which is a contradiction. Therefore, $|\sigma_u(L)| \geq 2$. In particular, if $v$ is another vertex of $X$ such that $u$ and $v$ are Laplacian strongly cospectral, then $0 \in \sigma_u^+(L)$. From the above considerations, we have the following result.

**Proposition 4.1.1.** *Let $X$ be a connected weighted graph with or without loops. If $u \in V(X)$, then $|\sigma_u(M(X))| \geq 2$. In particular, if $X$ is positively weighted, then $\lambda_{\max} \in \sigma_u(A(X))$, while if $X$ has no loops, then $0 \in \sigma_u(L(X))$. Moreover, if*

$v \in V(X)$ *strongly cospectral with* $u$*, then* $\lambda_{\max} \in \sigma_u^+(A(X))$ *provided* $X$ *is positively weighted, while* $0 \in \sigma_u^+(L(X))$ *provided* $X$ *has no loops.*

## 4.2 Spectral properties of graphs with twins

Next, we look at the implications of having twins in a graph to its associated adjacency and Laplacian matrices. Let $X$ be a graph and $U = U(\omega, \eta)$ be a set of twins in $X$. Assume $u, v \in U$. Since $N(u)\backslash\{v\} = N(v)\backslash\{u\}$, the columns of $A$ indexed by $u$ and $v$ differ by a multiple of $\mathbf{e}_u - \mathbf{e}_v$. In particular, one checks that

$$A(\mathbf{e}_u - \mathbf{e}_v) = (\omega - \eta)(\mathbf{e}_u - \mathbf{e}_v) \tag{4.2.1}$$

That is, $\mathbf{e}_u - \mathbf{e}_v$ is an eigenvector for $A$ corresponding to $\theta$, where $\theta = \omega$ if $u$ and $v$ are false twins, and $\theta = \omega - \eta$ if $u$ and $v$ are true twins. Conversely, if $\mathbf{e}_u - \mathbf{e}_v$ is an eigenvector for $A$ corresponding to some eigenvalue $\theta$, then we have that $A(\mathbf{e}_u - \mathbf{e}_v) = \theta(\mathbf{e}_u - \mathbf{e}_v)$. Rearranging this equation gives us

$$(A - \theta I)\mathbf{e}_u = (A - \theta I)\mathbf{e}_v. \tag{4.2.2}$$

Comparing the $j$th entries in Equation (4.2.2), we deduce that $(A)_{j,u} = (A)_{j,v}$ for $j \neq u, v$. Equation (4.2.2) also gives us $(A)_{u,u} - \theta = (A)_{u,v}$ and $(A)_{v,v} - \theta = (A)_{v,u}$, which implies that $(A)_{u,u} = (A)_{v,v}$. Since $A$ is symmetric, $(A)_{u,v} = (A)_{v,u} = \omega - \theta$. Consequently, $u$ and $v$ are twins. Moreover, if we let $(A)_{u,u} = \omega$ and $(A)_{u,v} = \eta$, then we obtain $\theta = \omega - \eta$.

For case of $L$, if $u, v \in U(\omega, \eta)$, then $L(\mathbf{e}_u - \mathbf{e}_v) = \deg(u)(\mathbf{e}_u - \mathbf{e}_v) - A(\mathbf{e}_u - \mathbf{e}_v)$. Combining this with Equation (4.2.1), we get

$$L(\mathbf{e}_u - \mathbf{e}_v) = (\deg(u) - \omega + \eta)(\mathbf{e}_u - \mathbf{e}_v) \tag{4.2.3}$$

Hence, $\mathbf{e}_u - \mathbf{e}_v$ is an eigenvector for $L$ corresponding to the eigenvalue $\theta$, where $\theta = \deg(u) - \omega$ if $u$ and $v$ are false twins, while $\theta = \deg(u) - \omega + \eta$ if $u$ and $v$ are true twins. Conversely, if $\mathbf{e}_u - \mathbf{e}_v$ is an eigenvector for $L$ corresponding to the eigenvalue $\theta$, then $L(\mathbf{e}_u - \mathbf{e}_v) = \theta(\mathbf{e}_u - \mathbf{e}_v)$, which we can rewrite as

$$(L - \theta I)\mathbf{e}_u = (L - \theta I)\mathbf{e}_v. \tag{4.2.4}$$

Comparing the $j$th entries of Equation (4.2.4) yields $(L)_{j,u} = (L)_{j,v}$ for $j \neq u, v$,

$(L)_{u,u} - \theta = (L)_{u,v}$ and $(L)_{v,v} - \theta = (L)_{v,u}$. The former equation implies that $(A)_{j,u} = (A)_{j,v}$ for $j \neq u, v$, while the two latter ones yield $(L)_{u,u} = (L)_{v,v}$, or equivalently, $\deg(u) - (A)_{u,u} = \deg(v) - (A)_{v,v}$. Since $\deg(u) = \sum_{j \neq u,v} (A)_{j,u} + 2(A)_{u,u}$, we have that $(A)_{u,u} = (A)_{v,v}$. Thus, $u$ and $v$ are twins, and solving for $\theta$ gives us $\theta = \deg(u) - \omega + \eta$, where $\omega$ is the weight of the loops on $u$ and $v$, and $\eta$ is the weight of the edge between them. These observations yield the following proposition.

**Proposition 4.2.1.** *Let $X$ be a weighted graph with or without loops, and $U(\omega, \eta)$ be a set of twins in $X$. Then $u, v \in U(\omega, \eta)$ if and only if $\mathbf{e}_u - \mathbf{e}_v$ is an eigenvector for $M(X)$ corresponding to the eigenvalue $\theta$, where*

$$\theta = \begin{cases} \omega - \eta, & \text{if } M(X) = A(X) \\ deg(u) - \omega + \eta, & \text{if } M(X) = L(X). \end{cases} \tag{4.2.5}$$

In Proposition 4.2.1, if $X$ is simple and unweighted, then $\omega = 0$ and $\eta = 1$. Thus, if $u$ and $v$ are twins in $X$, then we can write $\theta$ in Equation 4.2.5 as

$$\omega - \eta = \begin{cases} -1, & \text{if } u \text{ and } v \text{ are true twins} \\ 0, & \text{if } u \text{ and } v \text{ are false twins} \end{cases} \tag{4.2.6}$$

for the adjacency case, while

$$\deg(u) - \omega + \eta = \begin{cases} \deg(u) + 1, & \text{if } u \text{ and } v \text{ are true twins} \\ \deg(u), & \text{if } u \text{ and } v \text{ are false twins} \end{cases} \tag{4.2.7}$$

for the Laplacian case.

## 4.3 Algebraic properties of graphs with twins

Now, we prove a basic algebraic fact about graphs with twin vertices. Assume $X$ is a graph with twin vertices $u$ and $v$. Define the function $f$ on $V(X)$ given by $f(u) = v$, $f(v) = u$, and $f(a) = a$ for all $a \in V(X) \backslash \{u, v\}$. Since $N(u) \backslash \{v\} = N(v) \backslash \{u\}$, the edges $(u, w)$ and $(v, w)$ have equal weights for each $w \in N_X(u) \backslash \{v\}$, and the loops, if there are any, on $u$ and $v$ have equal weights, it follows that $f$ is an automorphism of $X$. In particular, since $f^2$ is the identity function on $V(X)$, we conclude that $f$ is an involution that switches $u$ and $v$ and fixes all other vertices. Conversely, suppose there exists an involution $f$ of $X$ that switches $u$ and $v$ and fixes all other vertices.

Let $N(u)\backslash v = \{u_1, \ldots, u_j\}$. Since $f$ fixes all vertices other than $u$ and $v$, we get

$$N(v)\backslash\{u\} = N(f(u))\backslash f(v) = \{f(u_1), \ldots, f(u_j)\} = \{u_1, \ldots, u_j\} = N(u)\backslash\{v\}.$$

Moreover, since $f$ preserves the weights of adjacent vertices, it follows that $u$ and $v$ are twins. We also note that $f|_{V(X\backslash u)}$ is an isomorphism of the vertex deleted graphs $X\backslash u$ and $X\backslash v$. Consequently, $u$ and $v$ are cospectral with respect to $M$. We state these facts in the following proposition and corollary.

**Proposition 4.3.1.** *Let $X$ be a weighted graph with or without loops. Then vertices $u$ and $v$ are twins in $X$ if and only if there exists an involution on $X$ that switches $u$ and $v$ and fixes all other vertices.*

**Corollary 4.3.2.** *Let $X$ be a weighted graph with or without loops. If vertices $u$ and $v$ are twins in $X$, then $u$ and $v$ are cospectral with respect to $M(X)$.*

Proposition 4.3.1 has an interesting consequence on the entries of the transition matrix indexed by twin vertices.

**Theorem 4.3.3.** *Let $X$ be a weighted graph with or without loops. Then vertices $u$ and $v$ are twins in $X$ if and only if $(U(t))_{u,u} = (U(t))_{v,v}$, $(U(t))_{u,v} = (U(t))_{v,u}$, and $(U(t))_{w,u} = (U(t))_{w,v}$ for all $w \in V(X)\backslash\{u, v\}$ and for any $t \in \mathbb{R}$. Moreover, $(U(t))_{u,u} \neq (U(t))_{v,u}$ for any $t \in \mathbb{R}$.*

*Proof.* Let $X$ be a weighted graph with or without loops. To prove necessity, let $u$ and $v$ be twin vertices in $X$. By Corollary 4.3.1, there exists an automorphism $f$ of $X$ that switches $u$ and $v$, and fixes all other vertices. Thus for any $t \in \mathbb{R}$, Proposition 3.1.5 implies that $(U(t))_{w,u} = (U(t))_{w,v}$ for all $w \in V(X)\backslash\{u, v\}$. Now, by Corollary 4.3.2, $u$ and $v$ are cospectral, and thus, $(U(t))_{u,u} = (U(t))_{v,v}$ for any $t \in \mathbb{R}$. Lastly, because $U(t)$ is symmetric, $(U(t))_{u,v} = (U(t))_{v,u}$ for any $t \in \mathbb{R}$. To prove sufficiency, let $t \in \mathbb{R}$, and suppose $a = (U(t))_{u,u} = (U(t))_{v,v}$, $b = (U(t))_{u,v} = (U(t))_{v,u}$, and $(U(t))_{w,u} = (U(t))_{w,v}$ for all $w \in V(X)\backslash\{u, v\}$. Note that $a - b \neq 0$, otherwise the columns of $U(t)$ indexed by $u$ and $v$ are equal, which is a contradiction because $U(t)$ is nonsingular. Now, a simple computation reveals that $U(t)(\mathbf{e}_u - \mathbf{e}_v) = (a-b)(\mathbf{e}_u - \mathbf{e}_v)$ so that $\mathbf{e}_u - \mathbf{e}_v$ is an eigenvector for $U(t)$. Since $U(t)$ and $M$ have the same set of eigenvectors, it follows that $\mathbf{e}_u - \mathbf{e}_v$ is an eigenvector for $M$. By Proposition 4.2.1, we get that $u$ and $v$ are twins in $X$. The latter statement is true because $a - b \neq 0$. □

If $u$ and $v$ are twins, then Theorem 4.3.3 implies that $U(t)\mathbf{e}_u$ and $U(t)\mathbf{e}_v$ have equal entries except for those indexed by $u$ and $v$. A statement similar to Theorem

4.3.3 appears in [[24],Theorem 8.1.3], although we point out that since $(U(t))_{u,u} \neq (U(t))_{v,u}$, it cannot happen that $U(t)\mathbf{e}_u = U(t)\mathbf{e}_v$. Otherwise, $U(t)(\mathbf{e}_u - \mathbf{e}_v) = 0$, i.e., 0 is an eigenvalue of $U(t)$, which is a contradiction because $U(t)$ is unitary.

Now, in the proof of Theorem 4.3.3, $a - b = (U(t))_{u,u} - (U(t))_{u,v}$ is an eigenvalue of $U(t)$ for any $t \in \mathbb{R}$ with associated eigenvector $\mathbf{e}_u - \mathbf{e}_v$ whenever $u$ and $v$ are twins. Using triangle inequality and the fact that $U(t)$ is unitary, we obtain

$$1 = |(U(t))_{u,u} - (U(t))_{u,v}| \leq |(U(t))_{u,u}| + |(U(t))_{u,v}|.$$

so that $|(U(t))_{u,v}| = 1$ if and only if $(U(t))_{u,u} = 0$ and $|(U(t))_{u,u}| = 1$ if and only if $(U(t))_{u,v} = 0$. However, if $U$ is a set of twins such that $u, v \in U$ and $|U| \geq 3$, then Theorem 4.3.3 implies that $|(U(t))_{u,v}| = |(U(t))_{u,w}| = 1$ whenever $w \in U \backslash \{u, v\}$, which is a contradiction because $U$ is unitary. Consequently, if $U$ is a set of twins such that $|U| \geq 3$, then $(U(t))_{u,u} \neq 0$ for all $t \in \mathbb{R}$ and for all $u \in U$. We summarize these in the following corollary.

**Corollary 4.3.4.** *Let $X$ be a weighted graph with or without loops, and let $U$ be a set of twins in $X$ such that $|U| \geq 2$. If $u$ and $v$ are distinct vertices in $U$, then $|(U(t))_{u,u}| + |(U(t))_{u,v}| \geq 1$ for all $t \in \mathbb{R}$. Moreover, the following hold.*

1. *If $|U| = 2$, then perfect state transfer occurs between $u$ and $v$ at time $t$ if and only if $(U(t))_{u,u} = 0$, and pretty good state transfer occurs between $u$ and $v$ if and only if there exists a sequence of times $\{\tau_j\}$ such that $\lim\limits_{j\to\infty}(U(\tau_j))_{u,u} = 0$.*
2. *If $|U| \geq 3$, then $(U(t))_{u,u} \neq 0$ for all $t \in \mathbb{R}$ and for all $u \in U$.*
3. *Vertex $u$ is periodic with period $t$ if and only if $(U(t))_{u,v} = 0$.*

Let $X$ be a weighted graph on $m \geq 2$ vertices with or without loops, and $U$ be a set of twins in $X$ such that $|U| \geq 2$. If $X = \mathbf{K}_m(\omega, \eta)$, then Proposition 4.0.3 yields $U = V(X)$. Using Theorem 4.3.3, for each $t \in \mathbb{R}$, we can write $U(t) = (a-b)I_m + b\mathbf{J}_m$ for some complex numbers $a$ and $b$ such that $a \neq b$. If $m = 2$, then $|a|^2 + |b|^2 = 1$. While if $m \geq 3$, then $a \neq 0$ by Corollary 4.3.4(2), and combining this with the fact that $U$ is unitary yields $|b|^2 < \frac{1}{m-1}$ for all $t \in \mathbb{R}$. Now, suppose $X \neq \mathbf{K}_m(\omega, \eta)$, and let $P$ be a permutation matrix such that the first $|U|$ columns of $P^T U(t) P$ are indexed by $U$. Then Proposition 4.0.3 yields $m > |U|$, and Theorem 4.3.3 allows us to write $P^T U(t) P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ for every $t \in \mathbb{R}$, where $A = (a-b)I_{|U|} + b\mathbf{J}_{|U|}$ and $C = B^T = c\mathbf{J}_{m-|U|,|U|}$, for some complex numbers $a$, $b$ and $c$ such that $a \neq b$.

By Corollary 4.3.4, $a \neq 0$, and because $U(t)$ is unitary, we have $|b|^2 < \frac{1}{|U|-1}$ for all $t \in \mathbb{R}$. Similarly, $|c|^2 < \max\left\{\frac{1}{m-|U|}, \frac{1}{|U|}\right\}$ for all $t \in \mathbb{R}$. In particular, if $n \geq 2|U|$, then $|c|^2 < \frac{1}{|U|}$. Otherwise, $|c|^2 < \frac{1}{m-|U|}$.

**Corollary 4.3.5.** *Let $X$ be a weighted graph on $m \geq 2$ vertices with or without loops, and $U$ be a set of twins in $X$ such that $|U| \geq 2$.*

1. *If $X = \boldsymbol{K}_m(\omega, \eta)$, then for all $t \in \mathbb{R}$, $U(t) = (a-b)I_m + b\boldsymbol{J}_m$ for some complex numbers $a \neq 0$ and $b$ such that $a \neq b$. In addition, if $m = 2$, then $|a|^2 + |b|^2 = 1$, while if $m \geq 3$, then $|b|^2 < \frac{1}{m-1}$.*

2. *If $X \neq \boldsymbol{K}_m(\omega, \eta)$, then $m > |U|$ and for all $t \in \mathbb{R}$,*

$$U(t) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{4.3.1}$$

*where $A = (a-b)I_{|U|} + b\boldsymbol{J}_{|U|}$ is indexed by $U$ and $C = B^T = c\boldsymbol{J}_{m-|U|,|U|}$ for some complex numbers $a \neq 0$, $b$ and $c$ such that $a \neq b$. In addition,*

$$|b|^2 < \frac{1}{|U|-1} \text{ and } |c|^2 < \max\left\{\frac{1}{m-|U|}, \frac{1}{|U|}\right\}. \tag{4.3.2}$$

*In particular, if $n \geq 2|U|$, then $|c|^2 < \frac{1}{|U|}$. Otherwise, $|c|^2 < \frac{1}{m-|U|}$.*

Using Corollaries 4.3.2 and 4.3.5, if $U$ is a set of twins such that $|U| \geq 3$ and $\delta := \frac{|U|-2}{|U|-1}$, then any two distinct vertices $u, v \in U$ are cospectral and satisfies $|b|^2 = |U(t)_{u,v}|^2 < 1 - \delta$ for all $t$. This answers a question posed by Godsil in [39], which implies that the converse of [[39], Theorem 13.3] does not hold. In particular, Corollary 4.3.5(1) gives a bound for the fidelity $|b|^2$ between any pair of distinct vertices in $\mathbf{K}_m(\omega, \eta)$. By explicitly computing the eigenvalues and spectral idempotents of $\mathbf{K}_m(\omega, \eta)$, we will see that this bound can be improved. For $m \geq 2$, the adjacency matrix of $\mathbf{K}_m(\omega, \eta)$ is given by $A = (\omega - \eta)I_m + \eta\mathbf{J}_m$. It can easily be shown that the eigenvalues of $A$ are $\theta := \omega + (m-1)\eta$ and $\gamma := \omega - \eta$, with associated eigenvectors $\mathbf{1}_m$ and $\mathbf{e}_1 - \mathbf{e}_j$ for $j = 2, \ldots, m$. Thus, $E_\theta = \frac{1}{m}\mathbf{J}_m$ and orthogonalizing the set $\{\mathbf{e}_1 - \mathbf{e}_j : j = 2, \ldots, m\}$ yields $E_\gamma = I_m - \frac{1}{n}\mathbf{J}_m$. Thus, the spectral decomposition of $A$ is given by

$$A = \theta E_\theta + \gamma E_\gamma,$$

and taking the exponential of $itA$ gives us the transition matrix

$$U(t) = e^{it\theta}E_\theta + e^{it\gamma}E_\gamma = \frac{1}{m}e^{it\theta}\mathbf{J}_m + e^{it\gamma}\left(I_m - \frac{1}{m}\mathbf{J}_m\right) = e^{it\gamma}I_n + \frac{1}{m}\left(e^{it\theta} - e^{it\gamma}\right)\mathbf{J}_m. \tag{4.3.3}$$

Our computation reveals that indeed, as Corollary 4.3.5(1) states, we can write $U(t) = (a-b)I_m + b\mathbf{J}_m$ for some $a, b \in \mathbb{C}$, where $a \neq 0$ and $a \neq b$. In this case, we have that $b = \frac{1}{m}\left(e^{it\theta} - e^{it\gamma}\right)$, and consequently,

$$\begin{aligned} |b|^2 &= \frac{1}{m^2}[(\cos(\theta t) - \cos(\gamma t))^2 + (\sin(\theta t) - \sin(\gamma t))^2] \\ &= \frac{2}{m^2}[1 - (\cos(\theta t)\cos(\gamma t) + \sin(\theta t)\sin(\gamma t))] \\ &= \frac{2}{m^2}[1 - \cos((\theta - \gamma)t)] = \frac{2}{m^2}(1 - \cos(m\eta t)). \end{aligned}$$

Thus, we see that $0 \leq |b|^2 \leq \frac{4}{m^2}$. Now, $|b|^2$ is maximum if and only if $t = \frac{j\pi}{m\eta}$ for some odd $j$, while it is minimum if and only if $t = \frac{j\pi}{m\eta}$ for some even $j$. Meanwhile, since $U(t)$ is unitary, we have that $|a|^2 = 1 - (m-1)|b|^2$ whose minimum and maximum values $1 - \frac{4(m-1)}{m^2}$ and 1 are attained whenever $|b|^2$ is maximum and minimum, respectively. We summarise these in the following theorem.

**Theorem 4.3.6.** *Let $m \geq 2$. The transition matrix $U(t)$ of $\boldsymbol{K}_m(\omega, \eta)$ satisfies Equation (4.3.3). In particular, for any two distinct vertices $u$ and $v$ of $\boldsymbol{K}_m(\omega, \eta)$, the following statements hold.*

1. $|(U(t))_{u,v}|^2 = \frac{2}{m^2}(1 - \cos(m\eta t))$ *and* $|(U(t))_{u,u}|^2 = 1 - (m-1)|(U(t))_{u,v}|^2$.
2. $|(U(t))_{u,v}|^2$ *has maximum and minimum values* $\frac{4}{m^2}$ *and* $0$*, which are attained at* $t = \frac{j\pi}{m\eta}$ *for odd* $j$ *and even* $j$*, respectively.*
3. $|(U(t))_{u,u}|^2$ *has minimum and maximum values* $1 - \frac{4(m-1)}{m^2}$ *and* $1$*, which are attained whenever* $|b|^2$ *is maximum and minimum, respectively.*

The fidelities of quantum state transfer for the unweighted $K_m$ was first given by Bose et. al. in [[9], Theorem 4]. Theorem 4.3.6 generalizes their result to the weighted complete graph $\mathbf{K}_m(\omega, \eta)$. Using Theorem 4.3.6, we observe that $\mathbf{K}_m(\omega, \eta)$, like the unweighted $K_m$, admits PST if and only if $m = 2$. Moreover, $\mathbf{K}_m(\omega, \eta)$ does not admit PGST and FR, but exhibits periodicity at every vertex with minimum period $\rho = \frac{\pi}{m\eta}$. We also note that a larger value of $\eta$ yields a shorter minimum period.

However, it is interesting to see that $\omega$ does not affect the fidelities $|(U(t))_{u,v}|^2$ and $|(U(t))_{u,u}|^2$. We will revisit the weighted complete graph $\mathbf{K}_m(\omega,\eta)$ in Chapter 6.

Now, let $X$ be a weighted graph on $n \geq 3$ vertices with or without loops, and $U$ be a set of twins in $X$ such that $|U| \geq 3$. Assume $u$, $v$ and $w$ are distinct vertices in $U$. By Proposition 4.3.1, there exists an automorphism $f$ of $X$ that switches $u, v \in U$ and fixes every vertex in $V(X)\backslash\{u,v\}$. By virtue of Theorem 3.2.19, any vertex in $V(X)\backslash\{u,v\}$ cannot be involved in PGST and FR with $u$ and $v$. Using the same argument, any vertex in $V(X)\backslash\{u,w\}$ cannot be involved in PGST and FR with $u$ and $w$. Since $u, v$ and $w$ are arbitrary, we conclude that any vertex in $U$ cannot be involved in PGST and FR with any vertex in $X$. This yields our next result.

**Corollary 4.3.7.** *Let $X$ be a weighted graph with or without loops, and $U$ be a set of twins in $X$ such that $|U| \geq 3$. Then any vertex in $U$ cannot be involved in pretty good state transfer and fractional revival in $X$.*

## 4.4 Strongly cospectral twin vertices

Let $X$ be a connected weighted graph on $n \geq 3$ vertices, and $u$ and $v$ be vertices in $X$ that are strongly cospectral with respect to $M$. As we know, $\sigma_u(M) = \sigma_v(M)$. In particular, $\sigma_u^+(M) = \sigma_v^+(M)$ and $\sigma_u^-(M) = \sigma_v^-(M)$. Consequently,

$$E_{\lambda_j}\mathbf{e}_u = E_{\lambda_j}\mathbf{e}_v \text{ and } E_{\lambda_\ell}\mathbf{e}_u = -E_{\lambda_\ell}\mathbf{e}_v, \tag{4.4.1}$$

for all $\lambda_j \in \sigma_u^+(M)$ and for all $\lambda_\ell \in \sigma_u^-(M)$. Define

$$\mathbf{w}^+ := \sum_{\lambda_j \in \sigma_u^+(M)} E_{\lambda_j}\mathbf{e}_u \text{ and } \mathbf{w}^- := \sum_{\lambda_\ell \in \sigma_u^-(M)} E_{\lambda_\ell}\mathbf{e}_u. \tag{4.4.2}$$

From Equation (4.4.1), it follows that

$$\mathbf{w}^+ = \sum_{\lambda_j \in \sigma_u^+(M)} E_{\lambda_j}\mathbf{e}_v \text{ and } \mathbf{w}^- = -\sum_{\lambda_\ell \in \sigma_u^-(M)} E_{\lambda_\ell}\mathbf{e}_v. \tag{4.4.3}$$

Since spectral idempotents sum to identity, Equations (4.4.2) and 4.4.3 yield

$$\mathbf{e}_u = \mathbf{w}^+ + \mathbf{w}^- \text{ and } \mathbf{e}_v = \mathbf{w}^+ - \mathbf{w}^-.$$

Solving for $\mathbf{w}^+$ and $\mathbf{w}^-$ in terms of $\mathbf{e}_u$ and $\mathbf{e}_v$, we get

$$\mathbf{w}^+ = \frac{1}{2}(\mathbf{e}_u + \mathbf{e}_v) \text{ and } \mathbf{w}^- = \frac{1}{2}(\mathbf{e}_u - \mathbf{e}_v), \tag{4.4.4}$$

from which it follows that $\sigma_u^+(M)$ and $\sigma_u^-(M)$ contain at least one element.

Now, let us look at the size of the sets $\sigma_u^+(M)$, $\sigma_u^-(M)$ and $\sigma_u(M)$. We first consider the case when $M = A$. If $X$ is positively weighted, then we know from Proposition 4.1.1 that $\lambda_{max} \in \sigma_u^+(A)$. If $\lambda_{max}$ is the only element in $\sigma_u^+(A)$, then $\mathbf{w}^+ = (\mathbf{v}\mathbf{v}^T)\mathbf{e}_u = (\mathbf{v}^T\mathbf{e}_u)\mathbf{v}$. Since $\mathbf{w}^T\mathbf{e}_u$ is a positive scalar, all entries of $\mathbf{w}^+$ must also positive, a contradiction to the first equation in (4.4.4). Hence, $|\sigma_u^+(A)| \geq 2$, and so $|\sigma_u(A)| \geq 3$. If $X$ is negatively weighted, then $A(X) = -(-(A(X)))$, where $-A(X)$ is the adjacency matrix of $X$ with all its edge weights positive. Since $A(X)$ and $-A(X)$ have the same set of eigenvectors, the results for positively weighted graphs also apply to negatively weighted graphs. Next, assume $\sigma_u^+(A)$ has one element $\lambda$ so that $X$ is neither positively nor negatively weighted. That is, $X$ has positive and negative edge weights. From Equation (4.4.2), we obtain $\mathbf{w}^+ = E_\lambda \mathbf{e}_u$, and using the fact that spectral idempotents are pairwise orthogonal, we get

$$A\mathbf{w}^+ = \left(\sum_{\mu\in\sigma(A)} \mu E_\mu\right)(E_\lambda \mathbf{e}_u) = \lambda E_\lambda^2 \mathbf{e}_u = \lambda E_\lambda \mathbf{e}_u = \lambda \mathbf{w}^+. \tag{4.4.5}$$

In other words, $A(\mathbf{e}_u + \mathbf{e}_v) = \lambda(\mathbf{e}_u + \mathbf{e}_v)$. This yields $(A)_{j,u} = -(A)_{j,v}$ for all $j \in V(X)\backslash\{u, v\}$ and $(A)_{u,u} = (A)_{v,v}$ so that $u$ and $v$ are not twins. By Proposition 4.2.1, $\mathbf{w}^-$ is not an eigenvector for $A$, which implies that $|\sigma_u^-(A)| \geq 2$. Thus, $|\sigma_u(A)| = |\sigma_u^+(A)| + |\sigma_u^-(A)| \geq 3$. In both cases, we see that $|\sigma_u(A)| \geq 3$.

For the Laplacian case, assuming that $X$ has no loops, then regardless of the signs of the edge weights on the graph, Proposition 4.1.1 implies that $0 \in \sigma_u^+(L)$ with associated eigenvector $\mathbf{1}$. If 0 is the only element in $\sigma_u^+(L)$, then $\mathbf{w}^+ = \mathbf{J}\mathbf{e}_u = \mathbf{1}$, which contradicts the second equation in (4.4.4). Thus, $|\sigma_u^+(L)| \geq 2$, and so $|\sigma_u(L)| \geq 3$.

Lastly, if $|\sigma_u^-(M)| = 1$, then a calculation similar to Equation (4.4.5) reveals that $\mathbf{w}^-$ is an eigenvector for $M$. By Proposition 4.2.1, $u$ and $v$ are twins in $X$. We state these facts in the following proposition.

**Proposition 4.4.1.** *Let $X$ be a connected weighted graph on $n \geq 3$ vertices. If $u$ and $v$ are strongly cospectral vertices of $X$ with respect to $M(X)$, then $|\sigma_u(M(X))| \geq 3$. In particular, the following statements hold.*

1. *If $X$ is either positively or negatively weighted, then $|\sigma_u^+(A(X))| \geq 2$, and $|\sigma_u^-(A(X))| \geq 1$*

2. *If $|\sigma_u^+(A(X))| = 1$, then $X$ has positive and negative edge weights, $\mathbf{e}_u + \mathbf{e}_v$ is an eigenvector for $A(X)$, $u$ and $v$ are not twins, and $|\sigma_u^-(A(X))| \geq 2$.*

*3. If $X$ has no loops, then $|\sigma_u^+(L(X))| \geq 2$, and $|\sigma_u^-(L(X))| \geq 1$.*

*In addition, if $|\sigma_u^-(M(X))| = 1$, then $u$ and $v$ are twins.*

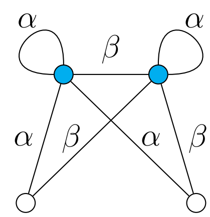


Figure 4.2: The weighted graph $X(\alpha, \beta)$

We remark that the result in Proposition 4.4.1 for the case of Laplacian dynamics was first observed by Coutinho et al. [[27], Lemma 3.1] in 2014, and then by Chan et al. [[18], Corollary 6.3] in 2020.

From the sketch of the proof of Theorem 4.3.6, we know that $\mathbf{K}_m(\omega, \eta)$ only has two eigenvalues, and so $|\sigma(\mathbf{K}_m(\omega, \eta))| = 2$. Thus, by Proposition 4.4.1, any two vertices in $\mathbf{K}_m(\omega, \eta)$ are not strongly cospectral. Since every pair of vertices in $\mathbf{K}_m(\omega, \eta)$ are twins, and hence cospectral, we conclude that any two vertices in $\mathbf{K}_m(\omega, \eta)$ are not parallel. By Lemma 3.3.2, FR and PGST do not occur in $\mathbf{K}_m(\omega, \eta)$.

Now, to illustrate Proposition 4.4.1, we give the following example.

**Example 4.4.2.** Consider the weighted graph $X(1, -1)$ in Figure 4.2 where the vertices marked blue are labelled $u$ and $v$, while the other two are labelled $a$ and $b$. If we index the first two rows of $A$ by $u$ and $v$, then we obtain

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of $A$ are 0 (multiplicity two) and $1 \pm \sqrt{5}$, with corresponding eigenvectors $\mathbf{e}_u + \mathbf{e}_v$, $\mathbf{e}_a - \mathbf{e}_b$ and $\left(\frac{1}{2}\left(1 \pm \sqrt{5}\right), \frac{1}{2}\left(-1 \mp \sqrt{5}\right), 1, 1\right)^T$. Note that $u$ and $v$ are true twins. By computing the spectral idempotents of $A$, one checks that

$$E_0\mathbf{e}_u = E_0\mathbf{e}_v \text{ and } E_{1\pm\sqrt{5}}\mathbf{e}_u = -E_{1\pm\sqrt{5}}\mathbf{e}_v,$$

while

$$E_0\mathbf{e}_a = -E_0\mathbf{e}_b \text{ and } E_{1\pm\sqrt{5}}\mathbf{e}_u = E_{1\pm\sqrt{5}}\mathbf{e}_v.$$

Consequently, $u$ and $v$ are adjacency strongly cospectral, as well as $a$ and $b$. Moreover, observe that $\sigma_u^+(A) = \sigma_a^-(A) = \{0\}$, i.e., $|\sigma_u^+(A)| = |\sigma_a^-(A)| = 1$, and $|\sigma_u^-(A)| = |\sigma_a^+(A)| = 2$. Indeed, by Proposition 4.4.1, $X$ has positive and negative edge weights, $|\sigma_u(A)| = |\sigma_a(A)| = 3$, $\mathbf{e}_u + \mathbf{e}_v$ is an eigenvector for $A$, and $a$ and $b$ are true twins.

Our main goal in this section is to characterize twin vertices that are strongly cospectral. To do this, we first state a consequence of consequence of Proposition 4.4.8 and Corollary 4.3.2.

**Theorem 4.4.3.** *Let $X$ be a weighted graph with or without loops, and assume vertices $u$ and $v$ are twins in $X$. Then $u$ and $v$ are strongly cospectral with respect to $M(X)$ if and only if they are parallel with respect to $M(X)$.*

From Proposition 4.4.8, we know that strong cospectrality is equivalent to cospectrality and parallelism. However, by imposing additional conditions on the vertices, we are able to determine sufficient conditions such that strong cospectrality is equivalent to either cospectrality or parallelism. For vertices with equal eigenvalue supports containing simple eigenvalues, strong cospectrality and cospectrality are equivalent by Proposition 3.3.8, while for twin vertices, strong cospectrality and parallelism are equivalent by Theorem 4.4.3. Next, we prove the following result.

**Theorem 4.4.4.** *Let $X$ be a connected weighted graph with or without loops. Assume $U(\omega, \eta)$ is a set of twins in $X$, and consider $\theta$ in Equation (4.2.5).*

1. *Let $u \in U$. If $\Omega$ is an orthogonal set of eigenvectors for $\theta$ such that $\mathbf{e}_u - \mathbf{e}_v \in \Omega$ for each $v \in U \backslash \{u\}$, then $E_\theta \mathbf{e}_u = cE_\theta \mathbf{e}_v$ if and only if $c = -1$. Moreover, if $E_\theta \mathbf{e}_u = -E_\theta \mathbf{e}_v$, then $|U| = 2$ and either $|\Omega| = 1$ or $\boldsymbol{w}^T \mathbf{e}_u = \boldsymbol{w}^T \mathbf{e}_v = 0$ for all $\boldsymbol{w} \in \Omega \backslash \{\mathbf{e}_u - \mathbf{e}_v\}$.*

2. *If $u \in U$ and $v \in V(X) \backslash U$, then $E_\theta \mathbf{e}_u \neq cE_\theta \mathbf{e}_v$ for any $c \in \mathbb{R}$.*

3. *For all $\mu \in \sigma_u(M(X))$ with $\mu \neq \theta$, $E_\mu \mathbf{e}_u = E_\mu \mathbf{e}_v$ for all $u, v \in U(\omega, \eta)$.*

*Proof.* Let $X$ be a connected positively weighted graph with or without loops, and $U = U(\omega, \eta)$ be a set of twins in $X$. Assume $m = |U| \geq 3$, and without loss of generality, suppose the first $|U|$ columns of $M = M(X)$ are indexed by the elements of $U$. By Proposition 4.1.1, $\{\mathbf{e}_1 - \mathbf{e}_j : j = 2, \ldots, m\}$ is an orthogonal set of eigenvectors of $M$ corresponding to the eigenvalue $\theta$ defined in Equation (4.2.5). Orthogonalizing this set yields an orthogonal subset

$$W = \{\mathbf{e}_1 + \ldots + \mathbf{e}_{j-1} - (j-1)\mathbf{e}_j : j = 2, \ldots, m\}$$

of eigenvectors for $M$ corresponding to $\theta$. Let $\Omega'$ be an orthogonal set of eigenvectors for $M$, and $\Omega$ be an orthogonal set of eigenvectors for $M$ corresponding to $\theta$ such that $W \subseteq \Omega$. For each $\mathbf{w} = (x_1, \ldots, x_n) \in \Omega' \backslash W$, $\mathbf{w} \cdot (\mathbf{e}_1 + \ldots + \mathbf{e}_{j-1} - (j-1)\mathbf{e}_j) = 0$ for each $j = 2, \ldots, m$, and so we obtain.

$$\mathbf{w} = (x, \ldots, x, x_{m+1}, \ldots, x_n), \tag{4.4.6}$$

for some $x, x_{m+1}, \ldots, x_n \in \mathbb{R}$. Suppose $W \neq \Omega$ and let $\mathbf{w} \in W \backslash \Omega$. If $x \neq 0$ and $x_j = 0$ for all $j = m+1, \ldots, n$, then $[\mathbf{1}_{|U|} \quad \mathbf{0}_{n-|U|}]^T$ is an eigenvector for $M$. For the case that $M = A$, we can write $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, where $A_1 = A(K_m(\omega, \eta))$ and $A_2 = A_3^T = [\delta_{1,m+1}\mathbf{1}_m \ \cdots \ \delta_{1,n}\mathbf{1}_m]$, where $j = m+1, \ldots, n$, and $\delta_{1,j} = 0$ if $j \notin N_X(1)$ and $\delta_{1,j} > 0$ otherwise. Observe that

$$A \begin{bmatrix} \mathbf{1}_m \\ \mathbf{0}_{n-m} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} \mathbf{1}_m \\ \mathbf{0}_{n-m} \end{bmatrix} = \begin{bmatrix} A_1\mathbf{1}_m \\ A_3\mathbf{1}_m \end{bmatrix} = \begin{bmatrix} (\omega + (n-1)\eta)\mathbf{1}_m \\ |U|(\delta_{1,m+1}, \ldots, \delta_{1,n})^T \end{bmatrix}.$$

Since $A \begin{bmatrix} \mathbf{1}_m \\ \mathbf{0}_{n-m} \end{bmatrix} = \theta \begin{bmatrix} \mathbf{1}_m \\ \mathbf{0}_{n-m} \end{bmatrix}$, it follows that $\delta_{1,j} = 0$ for each $j = m+1, \ldots, n$. Equivalently, $X$ is disconnected, which is a contradiction. Thus, $x = 0$ or $x_j \neq 0$ for at least one $j \in \{m+1, \ldots, n\}$, which implies that if $\mathbf{w} \in \Omega \backslash W$, then either $\mathbf{w} = (0, \ldots, 0, x_{m+1}, \ldots, x_n)$ or $\mathbf{w} = (1, \ldots, 1, x_{m+1}, \ldots, x_n)$. A similar calculation for the Laplacian case also yields the same result. Consequently, we may write $E_\theta = E_W + E_{\Omega \backslash W}$, where

$$E_W = \sum_{\mathbf{w} \in W} \frac{1}{||\mathbf{w}||^2} \mathbf{w}\mathbf{w}^T = \left( I_m - \frac{1}{m}\mathbf{J}_m \right) \oplus \mathbf{0}_{n-m} \tag{4.4.7}$$

and

$$E_{\Omega \backslash W} = \sum_{\mathbf{w} \in \Omega \backslash W} \frac{1}{||\mathbf{w}||^2} \mathbf{w}\mathbf{w}^T = \sum_{\mathbf{z} \in Z} \frac{1}{||\mathbf{z}||^2} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{z}\mathbf{z}^T \end{bmatrix} + \sum_{\mathbf{z} \in Z'} \frac{1}{||\mathbf{z}||^2} \begin{bmatrix} \mathbf{J}_m & \mathbf{1}_m\mathbf{z}^T \\ (\mathbf{z})(\mathbf{1}_m^T) & \mathbf{z}\mathbf{z}^T \end{bmatrix}, \tag{4.4.8}$$

where $Z = \{\mathbf{z} : (\mathbf{0}_m, \mathbf{z}) \in \Omega \backslash W\}$, $Z' = \{\mathbf{z} : (\mathbf{1}_m, \mathbf{z}) \in \Omega \backslash W\}$ and $E_{\Omega \backslash W}$ is absent of $W = \Omega$. Note that $Z$ or $Z'$ can be empty, and in case they are nonempty, then they are linearly independent sets. Morover, if $z \in Z$ or $z \in Z'$, then we have that $\mathbf{z} = (x_{m+1}, \ldots, x_n) \neq \mathbf{0}$.

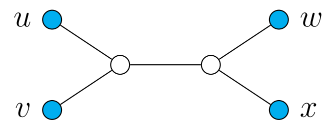


Figure 4.3: A graph with four pairwise cospectral vertices marked blue

Let $u \in U$. Then one can check using Equations (4.4.7) and (4.4.8) that

$$E_\theta \mathbf{e}_u = E_W \mathbf{e}_u + E_{\Omega\backslash W}\mathbf{e}_u = \begin{bmatrix} \mathbf{y} \\ \mathbf{0}_{n-m} \end{bmatrix} + \sum_{\mathbf{z}\in Z'} \frac{1}{||\mathbf{z}||^2} \begin{bmatrix} \mathbf{1}_m \\ \mathbf{z} \end{bmatrix} \tag{4.4.9}$$

where $y = \left(-\frac{1}{m}, \ldots, -\frac{1}{m}, 1-\frac{1}{m}, -\frac{1}{m}, \ldots, -\frac{1}{m}\right)^T$ and the entry of $y$ equal to $1-\frac{1}{m}$ is indexed by $u$. Assume $v \in U\backslash\{u\}$. Comparing the $u$th and $v$th entries of $E_\theta \mathbf{e}_u$ and $E_\theta \mathbf{e}_v$ using Equation (4.4.9), we get that $E_\theta \mathbf{e}_u = cE_\theta \mathbf{e}_v$ for some $c \neq 0$ if and only if $c + (c-1)\left(-\frac{1}{m} + \sum_{\mathbf{z}\in Z'} \frac{1}{||\mathbf{z}||^2}\right) = 0$ and $-1 + (c-1)\left(-\frac{1}{m} + \sum_{\mathbf{z}\in Z'} \frac{1}{||\mathbf{z}||^2}\right) = 0$. Equivalently, $c = -1$. This yields $-\frac{1}{m} = -\frac{m-1}{m}$, i.e., $m = 2$ so that $|W| = 1$. Moreover, comparing the last $n-(m+1)$ entries of $E_\theta \mathbf{e}_u$ and $-E_\theta \mathbf{e}_v$ gives us $\sum_{\mathbf{z}\in Z'} \frac{1}{||\mathbf{z}||^2}\mathbf{z} = 0$. Since $Z'$ is a linearly independent set, we get that $z = 0$, which is impossible. Thus, we conclude that $Z' = \varnothing$. If $Z \neq \varnothing$, then $\mathbf{w}^T\mathbf{e}_u = \mathbf{w}^T\mathbf{e}_v = 0$ while if $Z = \varnothing$, then $|\Omega| = |W| = 1$. This proves (1).

Next, we show (2). Assume $v \in V(X)\backslash U$. Then we obtain

$$E_\theta \mathbf{e}_v = E_W \mathbf{e}_v + E_{\Omega\backslash W}\mathbf{e}_v = \sum_{\mathbf{z}\in Z} \frac{x_v}{||\mathbf{z}||^2} \begin{bmatrix} \mathbf{0}_m \\ \mathbf{z} \end{bmatrix} + \sum_{\mathbf{z}\in Z'} \frac{x_v}{||\mathbf{z}||^2} \begin{bmatrix} \mathbf{1}_m \\ \mathbf{z} \end{bmatrix}, \tag{4.4.10}$$

where $x_v$ is the $v$th entry of $\mathbf{w}$. If $E_\theta \mathbf{e}_u = cE_\theta \mathbf{e}_v$ for any $c \in \mathbb{R}$, then comparing the entries of $E_\theta \mathbf{e}_u$ and $cE_\theta \mathbf{e}_v$ indexed by $U$ using Equations (4.4.9) and (4.4.10) yields $-\frac{1}{m} = \frac{m-1}{m}$, which is a contradiction. This proves (2).

Finally, from Proposition 4.1.1, we know that $|\sigma_u(M)| \geq 2$, and so there exists an eigenvalue $\mu \in \sigma_u(M)$ distinct from $\theta$. Let $\mathbf{w}_1, \ldots, \mathbf{w}_s \in \Omega'$ be eigenvectors corresponding to $\mu$. From Equation (4.4.6), we have $\mathbf{w}_j^T\mathbf{e}_u = \mathbf{w}_j^T\mathbf{e}_v$ for all $u, v \in U$, and consequently,

$$E_\mu \mathbf{e}_u = \sum_{j=1}^{s}(\mathbf{w}_j\mathbf{w}_j^T)\mathbf{e}_u = \sum_{j=1}^{s}(\mathbf{w}_j^T\mathbf{e}_u)\mathbf{w}_j = \sum_{j=1}^{s}(\mathbf{w}_j^T\mathbf{e}_v)\mathbf{w}_j = \sum_{j=1}^{s}(\mathbf{w}_j\mathbf{w}_j^T)\mathbf{e}_v = E_\mu \mathbf{e}_v.$$

Thus, (3) is true. □

By Theorem 4.4.4(2), any vertex $u \in U$ is not parallel to any vertex $v \in V(X)\backslash U$.

However, it is possible for a vertex in a graph with a twin to be cospectral to a vertex that is not its twin. Take for instance the graph in Figure 4.3. Observe that $u$ and $v$ are false twins in $X$, and $w$ is not twins with $u$ and $v$. Since $X\backslash u \cong X\backslash v \cong X\backslash w$, it follows that $\{u, v, w\}$ are pairwise cospectral.

We also remark that by Theorem 4.4.4(1), if $|U| \geq 3$, then any two vertices in $U$ are not parallel. Thus, combining Theorem 4.4.4 statements (1) and (2) yields the following corollary.

**Corollary 4.4.5.** *Let $X$ be a connected weighted graph with or without loops, and $U = (\omega, \eta)$ be a set of twins in $X$. If $|U| \geq 3$, then each $u \in U$ is not parallel, and hence not strongly cospectral, with any vertex $v \in V(X)\backslash\{u\}$.*

From Lemma 3.3.2, we know that strong cospectrality is a necessary condition for twin vertices to exhibit PGST and FR. Thus, if $|U| \geq 3$, then Corollary 4.4.5 implies that any vertex in $U$ cannot be involved in PGST and FR, a result that coincides with Corollary 4.3.7. Moreover, the contrapositive of Corollary 4.4.5 tells us that a set $U$ of twins in $X$ that contains a pair of distinct parallel, and hence strongly cospectral, vertices must have size two.

In [39], Godsil posed the question: is there an (unweighted) tree that contains three vertices, any two of which are strongly cospectral? We do not know if such a tree exists. However, if $X$ is an unweighted tree with twin vertices $u$ and $v$, then $\deg(u) = \deg(v) = 1$, i.e., $u$ and $v$ are leaves sharing the same neighbour. Otherwise, $X$ contains a four cycle containing $u$ and $v$, which is a contradiction. Thus, if $X$ contains three vertices that are pairwise parallel, then no two of them are leaves sharing the same neighbour.

Let us take a look at the following examples.

**Example 4.4.6.** Consider the weighted complete graph $\mathbf{K}_n(\omega, \eta)$ on $n \geq 3$ vertices. Then any two vertices of $\mathbf{K}_n(\omega, \eta)$ are true twins. By Corollary 4.4.5, any two vertices of $\mathbf{K}_n(\omega, \eta)$ are not parallel, and hence, are strongly cospectral with respect to $M(\mathbf{K}_n(\omega, \eta))$.

**Example 4.4.7.** Let $n_1, n_2 \geq 1$, and consider the unweighted complete bipartite graph $K_{n_1,n_2}$ with partite sets $V_j$ such that $|V_j| = n_j$ for $j = 1, 2$. Note that any two vertices in $V_j$ are false twins. Thus, if $|V_j| \geq 3$, then by Corollary 4.4.5, any two vertices in $V_j$ are not parallel, and hence, not strongly cospectral with respect to $M(K_{n_1,n_2})$. Consequently, the only complete bipartite graphs with a pair of strongly cospectral vertices on one vertex partition are $K_{2,n}$ and $K_{n,2}$ for $n \geq 1$. In particular, the cycle $K_{2,2} = C_4$ is the only complete bipartite graph whose vertices on each partite set are strongly cospectral with respect to $M(C_4)$.

Next, combining Theorem 4.4.4 and Theorem 4.4.3, we acquire a spectral characterization of strongly cospectral twin vertices.

**Corollary 4.4.8.** *Let $X$ be a connected weighted graph with or without loops. Assume $U(\omega, \eta) = \{u, v\}$ is a set of twins in $X$, and consider $\theta$ in Equation (4.2.5). If $\Omega$ is an orthogonal set of eigenvectors for $\theta$ such that $\mathbf{e}_u - \mathbf{e}_v \in \Omega$, then $u$ and $v$ are strongly cospectral with respect to $M(X)$ if and only if $|\Omega| = 1$ or $\boldsymbol{w}^T\mathbf{e}_u = \boldsymbol{w}^T\mathbf{e}_v = 0$ for all $\boldsymbol{w} \in \Omega\backslash\{\mathbf{e}_u - \mathbf{e}_v\}$. Moreover, if $u$ and $v$ are strongly cospectral with respect to $M(X)$, then $\sigma_u^-(M(X)) = \{\theta\}$, $\sigma_u^+(M(X)) = \sigma_u(M(X))\backslash\{\theta\}$, and $u$ and $v$ cannot be strongly cospectral to any $w \in V(X)\backslash\{u, v\}$.*

If the eigenvalue $\theta$ in Corollary 4.4.8 is simple, then we get the following result.

**Corollary 4.4.9.** *Let $X$ be a connected weighted graph with or without loops. Assume $U(\omega, \eta) = \{u, v\}$ is a set of twins in $X$, and consider $\theta$ in Equation (4.2.5). If $\theta$ is a simple eigenvalue of $M(X)$, then $u$ and $v$ are strongly cospectral with respect to $M(X)$, and $E_\theta = \frac{1}{2}(\mathbf{e}_u - \mathbf{e}_v)(\mathbf{e}_u - \mathbf{e}_v)^T$.*

To illustrate the previous corollary, we give an unweighted example.

**Example 4.4.10.** For $\omega \in \mathbb{R}$, let $Y = Y(\omega)$ be the weighted $P_3$ in Figure 3.4 with end vertices $u$ and $v$. Since $u$ and $v$ are false twins in $X$, Proposition 4.2.1 yields 0 and $\deg(u) = 1$ as simple eigenvalues of $A$ and $L$ respectively, both with associated eigenvector $\mathbf{e}_u - \mathbf{e}_v$. By Corollary 4.4.9, we conclude that $u$ and $v$ are adjacency and Laplacian strongly cospectral for all $\omega \in \mathbb{R}$, a result that is consistent with Proposition 3.4.3(1).

The next example shows that the converse of Corollary 4.4.9 does not hold.

**Example 4.4.11.** Let $X$ be the unweighted $C_4$ and assume that $u_j$ and $v_j$ are a pair of nonadjacent vertices in $X$ for $j = 1, 2$. Since $u_j$ and $v_j$ are false twins in $X$, Proposition 4.2.1 yields 0 as an eigenvalue of $A$ with eigenvector $\mathbf{e}_{u_j} - \mathbf{e}_{v_j}$ for $j = 1, 2$. Indeed, one checks that $A$ has eigenvalues 0 (with multiplicity 2) with corresponding eigenvectors $\mathbf{e}_{u_j} - \mathbf{e}_{v_j}$ and $\pm 2$ with corresponding eigenvectors $\mathbf{1}$ and $\mathbf{e}_{u_1} - \mathbf{e}_{u_2} + \mathbf{e}_{v_1} - \mathbf{e}_{u_1}$. Clearly, 0 is not a simple eigenvalue of $A$. However, since $(\mathbf{e}_{u_j} - \mathbf{e}_{v_j})^T\mathbf{e}_{u_k} = (\mathbf{e}_{u_j} - \mathbf{e}_{v_j})^T\mathbf{e}_{v_k} = 0$ for $j \neq k$, Corollary 4.4.8 yields adjacency strong cospectrality between $u_j$ and $v_j$ with $\sigma_{u_j}^+(A) = \{\pm 2\}$ and $\sigma_{u_j}^-(A) = \{0\}$, for $j = 1, 2$. Indeed, one computes from the eigenvectors of $A$ that $E_{\pm 2}\mathbf{e}_{u_j} = E_{\pm 2}\mathbf{e}_{v_j}$ and $E_0\mathbf{e}_{u_j} = -E_0\mathbf{e}_{v_j}$ for $j = 1, 2$.

Another consequence of Corollary 4.4.8 is a characterization of strongly cospectral vertices which are also twins.

**Corollary 4.4.12.** *Let $X$ be a connected weighted graph with or without loops. Assume vertices $u$ and $v$ are strongly cospectral vertices with respect to $M(X)$. Then $u$ and $v$ are twins if and only if $|\sigma_u^-(M(X))| = 1$.*

*Proof.* The necessity follows from Proposition 4.4.1, while the sufficiency follows from Proposition 4.4.4. □

Using Proposition 4.4.1 and Corollary 4.4.12, we give a lower bound for the sizes of the eigenvalue supports of strongly cospectral vertices in positively or negatively weighted graphs which are twins, and those which are not.

**Corollary 4.4.13.** *Let $X$ be a connected positively or negatively weighted graph with or without loops. If $u$ and $v$ are twin vertices in $X$ that are strongly cospectral, then $|\sigma_u(M(X))| \geq 3$. Otherwise, $|\sigma_u(M(X))| \geq 4$.*

In other words, the only strongly cospectral vertices in a connected positively or negatively weighted graph that can have an eigenvalue supports of size three are twins. However, we remark that the last statement in Corollary 4.4.13 is not necessarily true for the case of connected graphs with positive and negative edge weights, as illustrated by the graph $X(1, -1)$ in Example 4.2 where $u$ and $v$ are not twins but $|\sigma_u(A(X))| = 3$.

## 4.5 Twinning vertices in a graph

In this section, we characterize the vertices $u$ of a graph $G$ such that $u$ and $v$, for some vertex $v$ not in $V(G)$, are strongly cospectral with respect to $A(X)$, where $X$ is the resulting graph after twinning vertex $u$ of $G$. To do this, we first make the notion of twinning a vertex of a graph more precise.

Let $G$ be a weighted graph with or without loops, and $\omega, \eta \in \mathbb{R}$. For $u \in V(G)$ and $v \notin V(G)$, define $X = X(\omega, \eta)$ as the graph such that $V(X) = V(G) \cup \{v\}$ and $E(X) = E(G) \cup \{(v, w) : w \in N(u)\backslash\{u\}\} \cup F_\eta \cup F_\omega$ such that

- the edges $(u, w)$ and $(v, w)$ have equal weights for all $w \in N_G(u)\backslash\{u\}$;
- either $F_\eta = \varnothing$ (i.e., $u$ and $v$ are false twins in $X$) or $F_\eta = \{(u, v)\}$ (i.e. $u$ and $v$ are true twins in $X$) and $\eta > 0$ as the weight of $(u, v)$; and

- if $G$ has a loop on $u$ with weight $\omega$, then $F_\omega = \{(v,v)\}$, in which case $v$ also has a loop in $X(\omega,\eta)$ with weight $\omega$. Otherwise, $F_\omega = \varnothing$.

We call $X(\omega,\eta)$ the resulting graph after *twinning* vertex $u$ in $G$ by $v \notin V(G)$. In particular, if $F_\eta = \varnothing$, then $X(\omega,0)$ is the resulting graph after false twinning $u \in V(G)$ by $v \notin V(G)$, while if $F_\eta = \{(u,v)\}$, then $X(\omega,\eta)$ is the resulting graph after true twinning $u \in V(G)$ by $v \notin V(G)$ and $\eta$ is the weight of $(u,v)$. We also remark that if $X(\omega,\eta)$ is simple and unweighted, then $\omega = 0$ and $\eta \in \{0,1\}$. In literature, twinning is also called vertex duplication or vertex cloning, see [1].

Now, by Proposition 4.2.1, we know that $\mathbf{e}_u - \mathbf{e}_v$ is an eigenvector of $M$ corresponding to $\theta$ defined in Equation (4.2.5). Thus, 0 is an eigenvalue of $A(X)-(\omega-\eta)I$ and $L(X) - (\deg(u) - \omega + \eta)I$. This yields the following proposition.

**Proposition 4.5.1.** *Assume $G$ is a weighted graph with or without loops and let $X = X(\omega,\eta)$ for some $\omega,\eta \in \mathbb{R}$ be the resulting graph after twinning $u \in V(G)$ by $v \notin V(G)$. Then 0 is an eigenvalue of $A(X)-(\omega-\eta)I$ and $L(X)-(deg(u)-\omega+\eta)I$.*

Assume $G$ be a weighted graph on $n$ vertices with or without loops. Suppose $X = X(\omega,\eta)$ is the resulting graph after twinning $u \in V(G)$ by $v \notin V(G)$. Our main goal in this section is to characterize the vertices $u$ of $G$ such that $u$ and $v$ are strongly cospectral with respect to $M$. Without loss of generality, suppose the last two columns of $M$ are indexed by $u$ and $v$, with $v$ as the last. Define the vector $\mathbf{y}$ of length $n-1$ such that $(\mathbf{y})_j = \omega_j$ if $j \in N_G(u)$, where $\omega_j$ is the weight of the edge $(j,u)$, and $(\mathbf{y})_j = 0$ otherwise. Consider the case $M(X) = A(X)$ and let $\mu = \omega - \eta$. Since $u$ and $v$ are twins in $X$, we can write

$$A(X) - \mu I = \left[\begin{array}{c|c|c} A(G\backslash u) - \mu I & \mathbf{y} & \mathbf{y} \\ \hline \mathbf{y}^T & \eta & \eta \\ \hline \mathbf{y}^T & \eta & \eta \end{array}\right] \tag{4.5.1}$$

where $A(G) - \mu I = \left[\begin{array}{c|c} A(G\backslash u) - \mu I & \mathbf{y} \\ \hline \mathbf{y}^T & \eta \end{array}\right]$. By Proposition 4.5.1, we know that 0 is an eigenvalue of $A(X) - \mu I$. Using this fact, we proceed with two cases, the first of which is when 0 is a simple eigenvalue of $A(X) - \mu I$.

**Lemma 4.5.2.** *Let $G$ be a weighted graph with or without loops, and $X = X(\omega,\eta)$ for some $\omega,\eta \in \mathbb{R}$ be the resulting graph after twinning $u \in V(G)$ by $v \notin V(G)$. Then 0 is a simple eigenvalue of $A(X) - (\omega-\eta)I$ if and only if $A(G) - (\omega-\eta)I$ is nonsingular.*

*Proof.* Suppose $G$ is a weighted graph on $n$ vertices with or without loops. Let $X = X(\omega, \eta)$ for some $\omega, \eta \in \mathbb{R}$ be the resulting graph after twinning $u \in V(G)$ by $v \notin V(G)$, and $\mu := \omega - \eta$. Assume 0 is a simple eigenvalue of $A(X) - \mu I$ so that its rank is $n$. By Proposition 4.2.1, $\mathbf{e}_u - \mathbf{e}_v$ is the only eigenvector of $A(X) - \mu I$ associated to 0. Since $A(G) - \mu I$ is an $n \times n$ principal submatrix of $A(X) - \mu I$, Cauchy's Interlacing Theorem implies that either $A(G) - \mu I$ is nonsingular, or $A(G) - \mu I$ has an eigenvector $\mathbf{w}$ associated to 0 such that $\begin{bmatrix} \mathbf{y} \\ \eta \end{bmatrix}^T \mathbf{w} = 0$. If the latter holds, then Equation (4.5.1) yields $(A(X) - \mu I) \begin{bmatrix} \mathbf{w} \\ 0 \end{bmatrix} = 0$, i.e., 0 as an eigenvalue of $(A(X) - \mu I)$ has at least two eigenvectors, a contradiction. Thus, $A(G) - \mu I$ is nonsingular. Conversely, let $A(G) - \mu I$ be nonsingular. Since $A(G) - \mu I$ is a full rank $n \times n$ principal submatrix of $A(X) - \mu I$, $\text{rank}(A(X) - \mu I) \geq n$. If $\text{rank}(A(X) - \mu I) = n + 1$, then $A(X) - \mu I$ is nonsingular, a contradiction to Proposition 4.5.1. Thus, $\text{rank}(A(X) - \mu I) = n$, i.e., 0 is a simple eigenvalue of $A(X) - \mu I$. □

For the case that the multiplicity of 0 as an eigenvalue of $A(X) - (\omega - \eta)I$ is at least two, we have the following result.

**Lemma 4.5.3.** *Let $G$ be a weighted graph with or without loops, and $X = X(\omega, \eta)$ for some $\omega, \eta \in \mathbb{R}$ be the resulting graph after twinning $u \in V(G)$ by $v \notin V(G)$. Without loss of generality, assume the last two columns of $A(X)$ are indexed by $u$ and $v$, with $v$ as the last. Define the vector $\boldsymbol{y}$ of length $|V(G)| - 1$ such that $(\boldsymbol{y})_j = \omega_j$ if $j \in N_G(u)$, where $\omega_j$ is the weight of the edge $(j, u)$, and $(\boldsymbol{y})_j = 0$ otherwise. The following statements hold.*

1. *Let $\Omega$ be an orthogonal set of eigenvectors for the eigenvalue 0 of $A(X) - (\omega - \eta)I$ such that $\mathbf{e}_u - \mathbf{e}_v \in \Omega$ and $|\Omega| \geq 2$. If $u$ and $v$ are adjacency strongly cospectral in $X$, then*
$$W = \{\boldsymbol{z} : [\boldsymbol{z}\ 0\ 0]^T \in \Omega \backslash \{\mathbf{e}_u - \mathbf{e}_v\}\} \tag{4.5.2}$$
*is a maximal orthogonal subset of eigenvectors for the eigenvalue 0 of $A(G \backslash u) - (\omega - \eta)I$ such that $W \subseteq \{\boldsymbol{y}\}^\perp$ and $|W| \geq 1$.*

2. *Let $W$ be a maximal orthogonal subset of eigenvectors for the eigenvalue 0 of $A(G \backslash u) - (\omega - \eta)I$ such that $W \subseteq \{\boldsymbol{y}\}^\perp$ and $|W| \geq 1$. Then $u$ and $v$ are adjacency strongly cospectral in $X$ if and only if*
$$\Omega = \{[\boldsymbol{z}\ 0\ 0]^T : \boldsymbol{z} \in W\} \cup \{\mathbf{e}_u - \mathbf{e}_v\} \tag{4.5.3}$$

*is an orthogonal set of eigenvectors for the eigenvalue* $0$ *of* $A(X) - (\omega - \eta)I$ *such that* $\mathbf{e}_u - \mathbf{e}_v \in \Omega$ *and* $|\Omega| \geq 2$.

*In both cases,* $A(G) - (\omega - \eta)I$ *is singular.*

*Proof.* Let $G$ be a weighted graph with or without loops and $X = X(\omega, \eta)$ for some $\omega, \eta \in \mathbb{R}$ be the resulting graph after twinning $u \in V(G)$ by $v \notin V(G)$. Suppose $\mu = \omega - \eta$. We first prove (1). Let $\Omega$ be an orthogonal set of eigenvectors for the eigenvalue 0 of $A(X) - \mu I$ such that $\mathbf{e}_u - \mathbf{e}_v \in \Omega$ and $|\Omega| \geq 2$. Assume $u$ and $v$ are adjacency strongly cospectral in $X$. Then by Corollary 4.4.8, we have $\mathbf{w}^T\mathbf{e}_u = \mathbf{w}^T\mathbf{e}_v = 0$ for all $\mathbf{w} \in \Omega \backslash \{\mathbf{e}_u - \mathbf{e}_v\}$. Consequently, we can write each $\mathbf{w} \in \Omega \backslash \{\mathbf{e}_u - \mathbf{e}_v\}$ as $\mathbf{w} = [\mathbf{z}\ 0\ 0]^T$ for some vector $\mathbf{z}$ of length $|V(G)| - 1$. Using Equation (4.5.1), we get

$$(A(X) - \mu I)\mathbf{w} = \left[\begin{array}{c|c|c} A(G\backslash u) - \mu I & \mathbf{y} & \mathbf{y} \\ \hline \mathbf{y}^T & \eta & \eta \\ \hline \mathbf{y}^T & \eta & \eta \end{array}\right] \begin{bmatrix} \mathbf{z} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (A(G\backslash u) - \mu I)\mathbf{z} \\ \mathbf{y}^T\mathbf{z} \\ \mathbf{y}^T\mathbf{z} \end{bmatrix} = \mathbf{0}, \tag{4.5.4}$$

which yields $(A(G\backslash u) - \mu I)\mathbf{z} = \mathbf{0}$ and $\mathbf{y}^T\mathbf{z} = 0$ for each $\mathbf{w} \in \Omega \backslash \{\mathbf{e}_u - \mathbf{e}_v\}$. Equivalently, 0 is an eigenvalue of $A(G\backslash u) - \mu I$ and the set $W' = \{\mathbf{z} : [\mathbf{z}\ 0\ 0]^T \in \Omega \backslash \{\mathbf{e}_u - \mathbf{e}_v\}\}$ is an orthogonal subset of eigenvectors for the eigenvalue 0 of $A(G\backslash u) - \mu I$ such that $W' \subseteq \{\mathbf{y}\}^\perp$. Let $W$ be an orthogonal subset of eigenvectors for the eigenvalue 0 of $A(G\backslash u) - \mu I$ such that $W' \subseteq W \subseteq \{\mathbf{y}\}^\perp$. Assume $\mathbf{x} \in W \backslash W'$ so that $A(G\backslash u) - \mu I)\mathbf{x} = 0$ and $\mathbf{y}^T\mathbf{x} = 0$. Using the same computation in Equation (4.5.4), we have that $[\mathbf{x}\ 0\ 0]^T$ is an eigenvector for the eigenvalue 0 of $A(X) - \mu I$. Thus, $\mathbf{x} \in W'$, which is a contradiction. Therefore, $W = W'$, and $|W| \geq 1$.

Now, let us prove (2). Suppose $W$ is a maximal orthogonal subset of eigenvectors for the eigenvalue 0 of $A(G\backslash u) - \mu I$ such that $W \subseteq \{\mathbf{y}\}^\perp$ and $|W| \geq 1$. Define the set $\Omega' = \{[\mathbf{z}\ 0\ 0]^T : \mathbf{z} \in W\}$. A computation similar to Equation (4.5.4) shows that $\Omega'$ is an orthogonal subset of eigenvectors for the eigenvalue 0 of $A(X) - \mu I$. Since $u$ and $v$ are twins in $X$, Proposition 4.2.1 yields $\mathbf{e}_u - \mathbf{e}_v$ as an eigenvector for 0, and consequently, $\Omega' \cup \{\mathbf{e}_u - \mathbf{e}_v\}$ is an orthogonal subset of eigenvectors for the eigenvalue 0 of $A(X) - \mu I$. Now, let $\Omega$ be a full orthogonal set of eigenvectors for the eigenvalue 0 of $A(X) - \mu I$ such that $\Omega' \cup \{\mathbf{e}_u - \mathbf{e}_v\} \subseteq \Omega$. If $\mathbf{w} \in \Omega \backslash \Omega'$ and $\mathbf{w} \neq \mathbf{e}_u - \mathbf{e}_v$, then $\mathbf{w}^T\mathbf{e}_u \neq 0$ or $\mathbf{w}^T\mathbf{e}_v \neq 0$. Otherwise, $\mathbf{w} = [\mathbf{z}\ 0\ 0]$, which is a contradiction. Applying Corollary 4.4.8, we get that $u$ and $v$ are not adjacency strong cospectral in $X$. Therefore, $u$ and $v$ are adjacency strongly cospectral in $X$ if and only if $\Omega = \Omega' \cup \{\mathbf{e}_u - \mathbf{e}_v\}$.


Figure 4.4: Post-twinning graphs for $P_4$ with twin vertices marked blue

Finally, we add that in both cases, for each $\mathbf{z} \in W$, we have that $[\mathbf{z}\ 0]^T$ is an eigenvector for the eigenvalue 0 of $A(G) - \mu I$, so that $A(G) - \mu I$ is singular. □

Combining Lemmas 4.5.2 and 4.5.3(2) yields the following fact which is useful in identifying which vertices of a graph will induce adjacency strong cospectrality in the post-twinning graph.

**Theorem 4.5.4.** *Let $G$ be a weighted graph with or without loops and $X = X(\omega, \eta)$ be the resulting graph after twinning $u \in V(G)$ by $v \notin V(G)$. Then $u$ and $v$ are adjacency strongly cospectral in $X$ if and only if one of the following conditions hold:*

1. *$A(G) - (\omega - \eta)I$ is nonsingular; or*

2. *$A(G) - (\omega - \eta)I$ is singular, and given a maximal orthogonal subset $W$ of eigenvectors for the eigenvalue 0 of $A(G\backslash u) - (\omega - \eta)I$ such that $W \subseteq \{\boldsymbol{y}\}^\perp$, where is defined in Lemma 4.5.3, and $|W| \geq 1$, we obtain an orthogonal set $\Omega = \{[\boldsymbol{z}\ 0\ 0]^T : \boldsymbol{z} \in W\} \cup \{\mathbf{e}_u - \mathbf{e}_v\}$ of eigenvectors for the eigenvalue 0 of $A(X) - (\omega - \eta)I$.*

*In particular, if condition (1) is true, then twinning any $u \in V(G)$ by $v \notin V(G)$ results in adjacency strong cospectrality between $u$ and $v$ in $X$.*

*Proof.* Let $G$ be a weighted graph with or without loops and $X = X(\omega, \eta)$ be the resulting graph after twinning $u \in V(G)$ by $v \notin V(G)$. By Lemma 4.5.2, statement (1) is equivalent to 0 being a simple eigenvalue of $A(X) - (\omega - \eta)I$, and a direct application of Corollary 4.4.9 yields adjacency strong cospectrality between $u$ and $v$ in $X$. Next, a direct application of Corollary 4.4.8 yields statement (2). □

**Remark 4.5.5.** We note that in Theorem 4.5.4(2), $A(G\backslash u) - (\omega - \eta)I$ is singular. Thus, if $A(G) - (\omega - \eta)I$ is singular but $A(G\backslash u) - (\omega - \eta)I$ is nonsingular, then $u$ and $v$ are not adjacency strongly cospectral in $X(\omega, \eta)$.

To illustrate Theorem 4.5.4, we give two examples, one where the resulting graph is weighted, and another where the resulting graph is unweighted.

**Example 4.5.6.** Let $G = P_4$, and $X = X(0, \eta)$ for $\eta \geq 0$ be the resulting graph after twinning $u \in V(G)$ by $v \notin V(G)$, see Figure 4.4. Since $\sigma(A(G)) = \left\{\frac{1}{2}(\pm 1 \pm \sqrt{5})\right\}$, we have $A(G)+\eta I$ is nonsingular if and only if $\eta \neq \frac{1}{2}(\pm 1 \pm \sqrt{5})$. Thus, if $\eta \neq \frac{1}{2}(\pm 1 \pm \sqrt{5})$, then Theorem 4.5.4(1) yields adjacency strong cospectrality of $u$ and $v$ in $X$, for any $u \in V(G)$. Moreover, if $\eta \in \sigma(A(G))$, then $A(G) + \eta I$ is singular. If $u \in \{1, 4\}$, then $G \backslash u = P_3$ so that $A(G \backslash u) + \eta I$ is nonsingular. By Theorem 4.5.4(2), we conclude that $u$ and $v$ are not adjacency strongly cospectral in $X$. Meanwhile, if $u \in \{2, 3\}$, then $G \backslash u = K_2 \cup \{a\}$, where $a \in \{1, 4\}$ so that $A(G \backslash u) + \eta I$ is nonsingular. By Theorem 4.5.4(2), $u$ and $v$ are also not adjacency strongly cospectral in $X$.

**Example 4.5.7.** Let $G = P_5$ with $V(G) = \{1, 2, 3, 4, 5\}$, and $X = X(0, \eta)$ be the resulting graph after twinning $u \in V(G)$ by $v \notin V(G)$, where $\eta \in \{0, 1\}$. Moreover, $\sigma(A(G)) = \{0, \pm 1, \pm\sqrt{3}\}$, and so $A(G) + \eta I$ is singular.

- If $u = 1$, then $G \backslash u = P_4$, and so $A(G \backslash u) + \eta I$ is nonsingular by Example 4.5.6. Applying Theorem 4.5.4(2), we conclude that $u$ and $v$ are not adjacency strongly cospectral in $X$.

- If $u = 2$, then $G \backslash u = P_3 \cup K_1$. From Example 4.4.10, the eigenvalues of $A(G \backslash u)$ are 0 (multiplicity 2), and $\pm\sqrt{2}$. Consequently, $A(G \backslash u)+\eta I$ is singular if $\eta = 0$, and nonsingular if $\eta = 1$. If $\eta = 1$, then it follows from Theorem 4.5.4(2) that $u$ and $v$ are not adjacency strongly cospectral in $X$. Now, suppose $\theta = 0$. Assume that the last row of $A(G)$ is indexed by $u$ so that the eigenvectors for $A(G \backslash u)$ corresponding to 0 are $(0, -1, 0, 1)^T$ and $(1, 0, 0, 0)^T$. This yields an orthogonal subset $\{(1, -1, 0, 1)^T, (0, -1, 0, 1)^T\}$ of eigenvectors for $A(G \backslash u)$ corresponding to 0, from which we obtain a maximal orthogonal subset $W = \{(1, -1, 0, 1)^T\}$ of eigenvectors for $A(G \backslash u)$ corresponding to 0 with $W \subseteq \{\mathbf{y}\}^\perp$. Finally, since $\Omega = \{[\mathbf{z}\ 0\ 0]^T : \mathbf{z} \in W\} \cup \{\mathbf{e}_u - \mathbf{e}_v\} = \{(1, -1, 0, 1, 0, 0)^T, \mathbf{e}_u - \mathbf{e}_v\}$ is an orthogonal set of eigenvectors for the eigenvalue 0 of $A(X)$, invoking Theorem 4.5.4(2) yields adjacency strong cospectrality between $u$ and $v$ in $X$.

Next, consider the case $M(X) = L(X)$ and let $\mu = \deg(u) - \omega + \eta$. Since $u$ and $v$ are twins in $X$, we can write

$$L(X) - \mu I = \left[\begin{array}{c|c} L(G) + D_v - \mu I & -\mathbf{y} \\ \hline -\mathbf{y}^T & 0 \end{array}\right], \tag{4.5.5}$$

where $D_v$ is the $n \times n$ diagonal matrix such that $(D_v)_{j,j} = \omega_j$ if $j \in N_G(v)$, where $\omega_j$ is the weight of the edge $(j, v)$, and $(D_v)_{j,j} = 0$ otherwise. We know from Proposition

4.5.1 that 0 is an eigenvalue of $L(X) - \mu I$. By using the same argument, we get an analog of Lemma 4.5.2 for the Laplacian case when 0 is a simple eigenvalue of $L(X) - \mu I$.

**Lemma 4.5.8.** *Let $G$ be a weighted graph with or without loops, and $X = X(\omega, \eta)$ be the resulting graph after twinning $u \in V(G)$ by $v \notin V(G)$. Then 0 is a simple eigenvalue of $L(X) - (deg(u) - \omega + \eta)I$ if and only if $L(G) + D_v - (deg(u) - \omega + \eta)I$ is nonsingular, where $D_v$ is defined in Equation (4.5.5).*

The previous lemma, together with Corollary 4.4.9, yields the following result.

**Lemma 4.5.9.** *Let $G$ be a weighted graph with or without loops, and $X = X(\omega, \eta)$ be the resulting graph after twinning $u \in V(G)$ by $v \notin V(G)$. If $L(G) + D_v - (deg(u) - \omega + \eta)I$ is nonsingular, then $u$ and $v$ are Laplacian strongly cospectral in $X$, where $D_v$ is defined in Equation (4.5.5).*

# 5

# State transfer between twins in graphs

This chapter explores quantum state transfer with respect to the adjacency matrix $A(X)$ of a graph $X$. If the context is clear, then we simply write $A$. Moreover, we assume all graphs in this chapter are connected with at least two vertices. If $X$ is not connected, then we may apply our results to the components of $X$.

## 5.1 Periodicity

We begin this section with a characterization of periodic vertices in a graph.

**Theorem 5.1.1** (Ratio Condition,[34]). *Let $X$ be a weighted graph with or without loops. Then the following statements are equivalent.*

1. *Vertex $u$ of $X$ is periodic.*
2. *For all $\lambda_p, \lambda_q, \lambda_r, \lambda_s \in \sigma_u(A(X))$ with $\lambda_r \neq \lambda_s$, we have*

$$\frac{\lambda_p - \lambda_q}{\lambda_r - \lambda_s} \in \mathbb{Q}. \tag{5.1.1}$$

*In addition, if $\phi(A(X), t) \in \mathbb{Z}[x]$, then $u$ is periodic if and only if either all eigenvalues in $\sigma_u(A(X))$ are integers, or there is a square-free integer $\Delta$ such that all eigenvalues in $\sigma_u(A(X))$ are quadratic integers in $\mathbb{Q}(\sqrt{\Delta})$, and the difference of any two eigenvalues in $\sigma_u(A(X))$ is an integer multiple of $\sqrt{\Delta}$.*

Let $X$ be a weighted graph with or without loops. Assume $\phi(A(X), t) \in \mathbb{Z}[x]$ and $u \in V$. By Theorem 5.1.1, the eigenvalues in $\sigma_u(A)$ assume a particular form.

If two of these eigenvalues $\lambda_1$ and $\lambda_2$ are integers, then Equation 5.1.1 yields

$$\frac{\lambda - \lambda_1}{\lambda_1 - \lambda_2} \in \mathbb{Q},$$

for all $\lambda \in \sigma_u(A)$, which implies that $\sigma_u(A)$ consists of all integers. However, if at most one of the eigenvalues in $\sigma_u(A)$ is an integer, then either all eigenvalues in $\sigma_u(A)$ are integer multiples of $\sqrt{\Delta}$ or all eigenvalues in $\sigma_u(A)$ are of the form $\frac{1}{2}\left(a + b_j\sqrt{\Delta}\right)$ where $a$ and each $b_j$ are integers and $\Delta$ is a square-free integer. On the other hand, if $\phi(A(X), t) \notin \mathbb{Z}[x]$, then $X$ can still exhibit periodicity at a vertex, and the eigenvalues in the support of that vertex need not be quadratic integers.

Now, let $X$ be a connected weighted graph with or without loops. From the definition, vertex $u$ is periodic in $X$ if and only if there is a time $t$ such that

$$U(t)\mathbf{e}_u = \gamma \mathbf{e}_u \tag{5.1.2}$$

for some unit $\gamma \in \mathbb{C}$. Using the spectral decomposition of $U(t)$ and the fact that the spectral idempotents sum to identity, we can write Equation 5.1.2 as

$$\sum_{\lambda \in \sigma_u(A)} e^{it\lambda} E_\lambda \mathbf{e}_u = \sum_{\lambda \in \sigma_u(A)} \gamma E_\lambda \mathbf{e}_u. \tag{5.1.3}$$

Multiplying Equation 5.1.3 by a spectral idempotent $E_\lambda$ yields

$$e^{it\lambda} = \gamma \tag{5.1.4}$$

for each $\lambda \in \sigma_u(A)$. In other words, $e^{it\lambda}$ is a phase factor of periodicity for every $\lambda \in \sigma_u(A)$. Now, from Proposition 4.1.1, we know that the size of $\sigma_u(A)$ is at least two. Fix $\lambda_0 \in \sigma_u(A)$. From Equation 5.1.4, we get that $e^{it\lambda} = e^{it\lambda_0}$, or equivalently,

$$e^{it(\lambda_0 - \lambda)} = 1 \tag{5.1.5}$$

for each $\lambda \in \sigma_u(A)\backslash\{\lambda_0\}$. Consequently,

$$t(\lambda_0 - \lambda) = m_\lambda \pi \tag{5.1.6}$$

for some even integer $m_\lambda$. This shows that a period $t$ of a periodic vertex $u$ only depends on the eigenvalue support of $u$. Now, suppose $\lambda_1$ and $\lambda_2$ are distinct eigenvalues in $\sigma_u(A)$ with $\lambda_1 > \lambda_2$. If $|\sigma_u(A)| = 2$, then $\rho = \frac{2\pi}{\lambda_1 - \lambda_2}$ satisfies Equation 5.1.6 so that $u$ is periodic with period $\sigma_u(A)\backslash\{\lambda_1, \lambda_2\}$. If there exists another period $\rho'$ of

$u$ such that $0 < \rho' \leq \rho$, then taking $\lambda_0 = \lambda_1$ in Equation 5.1.6 yields $\rho'(\lambda_1 - \lambda_2) = m\pi$ for some even integer $m$. Consequently, $\frac{m\pi}{\lambda_1 - \lambda_2} \leq \frac{2\pi}{\lambda_1 - \lambda_2}$, or equivalently, $m \leq 2$. Since $m \geq 0$ and $m = 0$ yields $\rho = 0$, it follows that $m = 2$, and hence $\rho' = \rho$. In other words, if $u$ is a vertex with $|\sigma_u(A)| = 2$, then $u$ is periodic with minimum period $\rho = \frac{2\pi}{\lambda_1 - \lambda_2}$. Now, consider the case that $|\sigma_u(A)| \geq 3$, and assume that vertex $u$ is periodic. Suppose $\lambda_1 = \lambda_{\max}$ is the largest eigenvalue of $A$ and $\lambda_2$ is the smallest eigenvalue in $\sigma_u(A)$. Then by Theorem 5.1.1, every $\lambda \in \sigma_u(A) \backslash \{\lambda_1, \lambda_2\}$ satisfies

$$\frac{\lambda_1 - \lambda}{\lambda_1 - \lambda_2} = \frac{p_\lambda}{q_\lambda} \tag{5.1.7}$$

for some integers $p_\lambda$ and $q_\lambda$ with $\gcd(p_\lambda, q_\lambda) = 1$. If we let

$$\rho = \frac{2\pi q}{\lambda_1 - \lambda_2}, \tag{5.1.8}$$

where $q = \operatorname{lcm}\{q_\lambda : \lambda \in \sigma_u(A) \backslash \{\lambda_1, \lambda_2\}\}$, then $\rho(\lambda_1 - \lambda_2) = 2\pi q$, and applying Equation 5.1.6 with $\lambda_0 = \lambda_1$ gives us

$$\rho(\lambda_1 - \lambda) = \frac{2\pi q}{\lambda_1 - \lambda_2} \cdot \frac{p_\lambda(\lambda_1 - \lambda_2)}{q_\lambda} = m_\lambda \pi, \tag{5.1.9}$$

where $m_\lambda = 2p_\lambda \left(\frac{q}{q_\lambda}\right)$ is an even integer for each $\lambda \in \sigma_u(A) \backslash \{\lambda_1, \lambda_2\}$. In other words, $\rho$ is a period of $u$. Now, suppose there is another period $\rho'$ of $u$ such that $0 < \rho' \leq \rho$. Taking $\lambda_0 = \lambda_1$ and $\lambda = \lambda_2$ in Equation 5.1.6 implies that $\rho' = \frac{m\pi}{\lambda_1 - \lambda_2}$, where $m$ is an even integer. Thus, $\rho' = \frac{m\pi}{\lambda_1 - \lambda_2} \leq \frac{2\pi q}{\lambda_1 - \lambda_2} = \rho$, from which we get

$$m \leq 2q \tag{5.1.10}$$

Now, taking $\lambda_0 = \lambda_1$ in Equation 5.1.6 and making use of Equation 5.1.7 yields

$$\rho'(\lambda_1 - \lambda) = \frac{m\pi}{\lambda_1 - \lambda_2} \cdot \frac{p_\lambda(\lambda_1 - \lambda_2)}{q_\lambda} = 2\pi \left(\frac{m}{2}\right) \left(\frac{p_\lambda}{q_\lambda}\right). \tag{5.1.11}$$

Note that $\left(\frac{m}{2}\right)\left(\frac{p_\lambda}{q_\lambda}\right)$ is an integer because $\rho'$ is a period of $u$. Since $m$ is even, $\frac{m}{2}$ is an integer, and since $\gcd(p_\lambda, q_\lambda) = 1$, it follows that $q_\lambda$ divides $\frac{m}{2}$ for each $\lambda \in \sigma_u(A) \backslash \{\lambda_1, \lambda_2\}$. Consequently, $q \leq \frac{m}{2}$, and combining this with Equation 5.1.10 yields $q = \frac{m}{2}$, from which it is clear that $\rho' = \rho$. That is, $\rho$ in Equation 5.1.8 is the minimum period of $u$. Lastly, if $|\sigma_u(A)| = 3$ with $\sigma_u(A) = \{\lambda_1, \lambda_2, \lambda_3\}$, then $q = q_{\lambda_3}$, and it follows from Equation 5.1.8 that the minimum period of $u$ is given by

$$\rho = \frac{2\pi q}{\lambda_1 - \lambda_2} = \frac{2\pi q}{\frac{q(\lambda_1-\lambda_3)}{p}} = \frac{2\pi p}{\lambda_1 - \lambda_3}, \tag{5.1.12}$$

where the second equality holds because of Equation 5.1.7. We summarize these results in the following theorem.

**Theorem 5.1.2.** *Let $X$ be a connected weighted graph with or without loops. Assume $u$ and $v$ are vertices of $X$, and $\lambda_1$ and $\lambda_2$ be the largest and smallest eigenvalues in $\sigma_u(A(X))$, respectively. The following statements hold.*

1. *If $|\sigma_u(A(X))| = 2$, then $u$ is periodic with minimum period $\rho = \frac{2\pi}{\lambda_1-\lambda_2}$. In addition, if perfect state transfer occurs between $u$ and $v$, then it occurs with minimum time $\tau = \frac{\pi}{\lambda_1-\lambda_2}$.*
2. *If $|\sigma_u(A)| \geq 3$ and $u$ is periodic, then the minimum period of $u$ is $\rho = \frac{2\pi q}{\lambda_1-\lambda_2}$, where $q = lcm\{q_\lambda : \lambda \in \sigma_u(A)\backslash\{\lambda_1, \lambda_2\}\}$, and $p_\lambda$ and $q_\lambda$ are integers such that $gcd(p_\lambda, q_\lambda) = 1$ and $\frac{\lambda_1-\lambda}{\lambda_1-\lambda_2} = \frac{p_\lambda}{q_\lambda}$ for each $\lambda \in \sigma_u(A)\backslash\{\lambda_1, \lambda_2\}$. In addition, if perfect state transfer occurs between $u$ and $v$, then it occurs with minimum time $\tau = \frac{\pi q}{\lambda_1-\lambda_2}$. In particular, if $\sigma_u(A(X)) = \{\lambda_1, \lambda_2, \lambda_3\}$, then the minimum period of $u$ is $\rho = \frac{2\pi p}{\lambda_1-\lambda_3}$, where $p = p_{\lambda_3}$, and if perfect state transfer occurs between $u$ and $v$, then the minimum time it occurs is $\tau = \frac{\pi p}{\lambda_1-\lambda_3}$.*

We remark that Theorem 5.1.2 and Propositon 3.2.5 imply that the minimum period of a vertex as well as the minimum PST time between two vertices solely depend on the eigenvalue support.

In [35], the Godsil showed that the minimum period of a periodic vertex is at least $\frac{2\pi}{\lambda_1-\lambda_2}$. However, we observe in Theorem 5.1.2(2) that since $q > p \geq 1$ whenever $|\sigma_u(A)| \geq 3$, we get that $\rho = \frac{2\pi q}{p(\lambda_1-\lambda_2)} > \frac{2\pi}{\lambda_1-\lambda_2}$. That is, the minimum period whenever $|\sigma_u(A)| \geq 3$ always exceeds $\frac{2\pi}{\lambda_1-\lambda_2}$. Consequently, the bound given in [35] is tight if and only if $|\sigma_u(A)| = 2$. Indeed, if $\sigma_u(A)$ is large enough, then it is also likely that the set $\{q_\lambda : \lambda \in \sigma_u(A)\backslash\{\lambda_1, \lambda_2\}\}$ are large, which likely yields a large value for $q$, and in turn, a large value for $\rho$. In other words, the minimum period at a vertex is likely long if the size of its eigenvalue support is large.

With the assumptions in Theorem 5.1.2, we further suppose that $\phi(A, t) \in \mathbb{Z}[x]$. By Theorem 5.1.1, we can write each $\lambda \in \sigma_u(A)\backslash\{\lambda_1\}$ as $\lambda_1 - \lambda = b_\lambda\sqrt{\Delta}$ for some integer $b_\lambda > 0$. Now, let

$$g = \gcd\{b_\lambda : \lambda \in \sigma_u(A)\backslash\{\lambda_1\}\} \tag{5.1.13}$$

and $h_\lambda$ be the integer such that $b_\lambda = gh_\lambda$. Define $\rho = 2\pi/g\sqrt{\Delta}$. Since

$$\rho(\lambda_1 - \lambda) = \left(2\pi/g\sqrt{\Delta}\right)\left(b_\lambda\sqrt{\Delta}\right) = 2\pi h_\lambda,$$

Equation 5.1.6 tells us that $\rho$ is a period of $u$. Now, we claim that $\rho$ is the minimum period of $u$. To see this, suppose the minimum period of $u$ is $\rho'$, so that $0 < \rho' \leq \rho$. From Equation 5.1.6, each $\lambda \in \sigma_u(A)\backslash\{\lambda_1\}$ satisfies $\rho' b_\lambda\sqrt{\Delta} = m_\lambda\pi$ for some even integer $m_\lambda$. Since $\rho'$ generates all periods, it follows that $\rho$ is an integer multiple of $\rho'$. That is, there exists an integer $s$ such that

$$\rho = s\rho' \iff \frac{2\pi}{g\sqrt{\Delta}} = \frac{sm_\lambda\pi}{b_\lambda\sqrt{\Delta}} \iff b_\lambda = s\left(\frac{m_\lambda}{2}\right)g$$

for each $\lambda \in \sigma_u(A)\backslash\{\lambda_1\}$. Note that each $\frac{m_\lambda}{2}$ is an integer because each $m_\lambda$ is even. Since $s$ divides every $b_\lambda$ and $g = \gcd\{b_\lambda : \lambda \in \sigma_u(A)\backslash\{\lambda_1\}\}$, we conclude that $s = 1$, and so $\rho = \rho'$. This proves that $\rho$ is the minimum period of $u$. Lastly, we show that $g$ in Equation 5.1.13 does not depend on the choice of the eigenvalue excluded in $\sigma_u(A)$. To see this, suppose $\sigma_u(A) = \{\lambda_1, \ldots, \lambda_n\}$, and for some $\ell \in \{1, \ldots, n\}$, define $g' = \gcd\{c_j : \lambda_j \in \sigma_u(A)\backslash\{\lambda_\ell\}\}$, where $c_j$ is an integer such that $\lambda_\ell - \lambda_j = c_j\sqrt{\Delta}$. Without loss of generality, we may assume $\Delta = 1$. Then we can write $g = \gcd\{\lambda_1 - \lambda_j : j \neq 1\}$ and $g' = \gcd\{\lambda_\ell - \lambda_j : j \neq \ell\}$. Using the fact that $\gcd(a, b) = \gcd(a, b - a) = \gcd(-a, b)$, we obtain

$$\begin{aligned} g &= \gcd\left(\{\lambda_1 - \lambda_\ell\} \cup \{\lambda_1 - \lambda_j : j \neq 1, \ell\}\right) \\ &= \gcd\left(\{\lambda_1 - \lambda_\ell\} \cup \{(\lambda_1 - \lambda_j) - (\lambda_1 - \lambda_\ell) : j \neq 1, \ell\}\right) \\ &= \gcd\left(\{\lambda_1 - \lambda_\ell\} \cup \{(\lambda_\ell - \lambda_j) : j \neq 1, \ell\}\right) \\ &= \gcd\left(\{\lambda_\ell - \lambda_1\} \cup \{(\lambda_\ell - \lambda_j) : j \neq 1, \ell\}\right) = g'. \end{aligned}$$

This discussion yields the following result which also appears in [[25], Corollary 7.6.2].

**Corollary 5.1.3.** *Let $X$ be a connected weighted graph with or without loops, and $u$ be a periodic vertex in $X$. If $\phi(A(X), t) \in \mathbb{Z}[x]$, then for any $\lambda_0 \in \sigma_u(A)$, the minimum period $\rho$ of $u$ is $\rho = 2\pi/g\sqrt{\Delta}$, where $g = gcd\{b_\lambda : \lambda \in \sigma_u(A)\backslash\{\lambda_0\}\}$ and $b_\lambda$ is an integer such that $\lambda_0 - \lambda = b_\lambda\sqrt{\Delta}$ for all $\lambda \in \sigma_u(A)\backslash\{\lambda_0\}$. In addition, if perfect state transfer occurs between $u$ and $v$, then the minimum time it occurs is $\tau = \frac{\pi}{g\sqrt{\Delta}}$.*

From Theorem 5.1.2, we remark that the minimum period of a periodic vertex may exceed $2\pi$. Take for instance the weighted $K_2$, where the weight of the single edge is $0 < \omega < 0.5$. Note that $\phi(A, t) = x^2 - \omega^2 \notin \mathbb{Z}[x]$, and the eigenvalues of

$A(K_2)$ are $\pm\omega$. Hence, the minimum period of either vertex is $\rho = \frac{2\pi}{\omega-(-\omega)} > 2\pi$. In this case, if a periodic vertex is involved in PST, then the minimum PST time exceeds $\pi$. However, from Proposition 5.1.3, we observe that if $\phi(A(X), t) \in \mathbb{Z}[x]$, then we are guaranteed that the minimum period of $u$ is at most $2\pi$. Consequently, if a periodic vertex is involved in PST, then the minimum PST time is at most $\pi$.

Next, we say that a subset $U$ of $V(X)$ is *periodic* if every vertex in $U$ is periodic. By Theorem 5.1.1, we remark that if two vertices $u$ and $v$ in a graph $X$ have the same eigenvalue support, then $u$ is periodic if and only if $v$ is periodic. Consequently, if $u$ and $v$ are periodic with the same eigenvalue support, then $u$ and $v$ have the same set of periods, and hence must have the same minimum periods. We state this observation as follows.

**Proposition 5.1.4.** *Let $X$ be a connected weighted graph with or without loops, and $U$ be a subset of $V(X)$. If all vertices in $U$ have the same eigenvalue support, then $U$ is periodic if and only if one vertex in $U$ is periodic. In addition, if $U$ is periodic, then every vertex in $U$ has the same minimum period.*

If $U \subseteq V(X)$ whose all elements are pairwise cospectral, then Corollary 3.3.3(1) implies that all vertices in $U$ have the same eigenvalue support, and so Proposition 5.1.4 applies to $U$. We now characterize vertices with twins that are periodic whenever $\phi(A(X), t) \in \mathbb{Z}[x]$.

**Theorem 5.1.5.** *Let $X$ be a connected weighted graph with or without loops. Suppose $\phi(A(X), t) \in \mathbb{Z}[x]$, and $U = U(\omega, \eta)$ is a set of twins in $X$ such that $|U| \geq 2$. Then $U$ is periodic if and only if the eigenvalues in the support of a vertex in $U$ are all the form $\omega - \eta + b_j\sqrt{\Delta}$, where $b_j$ is an integer, and either $\Delta = 1$ for each $j$ or $\Delta$ is a square-free integer for each $j$. In addition, if $U$ is periodic, then every vertex in $U$ has the minimum period $\rho = 2\pi/g\sqrt{\Delta}$, where $g = gcd(b_1, \ldots, b_r)$.*

*Proof.* Let $X$ be a graph such that $\phi(A(X), t) \in \mathbb{Z}[x]$, and $U = U(\omega, \eta)$ be a set of twins in $X$ such that $|U| \geq 2$. Suppose $u \in U$, and let $v \in U$ with $v \neq u$. Then, from Proposition 4.2.1, we know that $\omega - \eta$ is an eigenvalue of $A$ with eigenvector $\mathbf{e}_u - \mathbf{e}_v$. Consequently, $\omega - \eta \in \sigma_u(A)$. Since $\phi(A(X), t) \in \mathbb{Z}[x]$, Theorem 5.1.1 implies that the eigenvalues in the support of vertex $u$ are all of the form $\omega - \eta + b_j\sqrt{\Delta}$, where $b_j$ is an integer, and either $\Delta = 1$ for each $j$ or $\Delta$ is a square-free integer for each $j$. Finally, since $U$ is a set of twins in $X$ then Corollary 4.3.2 implies that the vertices in $U$ are pairwise cospectral, and Proposition 5.1.4 implies that they have the same minimum period, which, by Proposition 5.1.3, is equal to $\rho = 2\pi/g\sqrt{\Delta}$, where $g = \gcd(b_1, \ldots, b_r)$. □

## 5.2 Perfect state transfer

Let $X$ be a connected weighted graph with or without loops containing twin vertices $u$ and $v$. We want to know when can PST occur between $u$ and $v$ in $X$. From Lemma 3.3.2(1), $u$ and $v$ need to be strongly cospectral so that $\sigma_u(A) = \sigma_v(A)$. Assume $U = U(\omega, \eta)$ in $X$ is a set of twins in $X$ such that $u, v \in U$. Then Corollary 4.4.9 implies that $U = \{u, v\}$, and Proposition 4.2.1 yields $\theta = \omega - \eta$ as an eigenvalue of $A(X)$. Now, let $\theta, \lambda_1, \ldots, \lambda_r$ be the eigenvalues in $\sigma_u(A)$ with corresponding spectral idempotents $E_\theta, E_1, \ldots, E_r$. By definition, there is PST between $u$ and $v$ in $X$ if and only if

$$U(t)\mathbf{e}_u = \gamma \mathbf{e}_v \tag{5.2.1}$$

for some unit $\gamma \in \mathbb{C}$. Using the spectral decomposition of $U(t)$ and the fact that the spectral idempotents sum to identity, we can write Equation 5.2.1 as

$$\sum_{\lambda \in \sigma_u(A)} e^{it\lambda} E_\lambda \mathbf{e}_u = \sum_{\lambda \in \sigma_u(A)} \gamma E_\lambda \mathbf{e}_v. \tag{5.2.2}$$

By Corollary 4.4.8, we know that $\sigma_u^- = \{\theta\}$ and $\sigma_u^+ = \sigma_u(A) \backslash \{\theta\}$. Consequently, $E_\theta e_u = -E_\theta e_v$ and $E_j e_u = E_j e_v$, and hence, Equation 5.2.2 is true if and only if $\gamma = e^{it\lambda_j} = -e^{it\theta}$ for each $j$. Equivalently,

$$e^{it(\lambda_j - \theta)} = -1, \tag{5.2.3}$$

and so each $\lambda_j$ satisfies

$$t(\lambda_j - \theta) = m_j \pi$$

for some odd integer $m_j$. From Proposition 3.2.3, we know that $u$ and $v$ are periodic. Moreover, since $u$ and $v$ are twin vertices that are strongly cospectral, Corollary 4.4.13 yields $|\sigma_u(A)| \geq 3$. Thus, invoking Theorem 5.1.2(2) and Proposition 3.2.5, we get that the minimum PST time between $u$ and $v$ is given by

$$\rho = \frac{\pi q}{\Lambda_1 - \Lambda_2}, \tag{5.2.4}$$

where $\Lambda_1$ and $\Lambda_2$ are the largest and smallest eigenvalues in $\sigma_u(A)$ respectively, and $q$ is an integer given in Theorem 5.1.2(2). In particular, if $\sigma_u(A) = \{\theta, \lambda_1, \lambda_2\}$, then the minimum period of $u$ is $\rho = \frac{\pi p}{\Lambda_1 - \Lambda}$, where $\Lambda = \max\{\lambda : \lambda \in \sigma \backslash \{\Lambda_1\}\}$ and $p$ is an integer given in Theorem 5.1.2(2).

**Theorem 5.2.1.** *Let $X$ be a connected weighted graph with or without loops, and $U(\omega, \eta) = \{u, v\}$ be a set of twins in $X$. Assume $\sigma_u(A) = \{\omega - \eta, \lambda_1, \ldots, \lambda_r\}$, and $\Lambda_1$ and $\Lambda_2$ be the largest and smallest eigenvalues in $\sigma_u(A)$. Then perfect state transfer occurs between $u$ and $v$ if and only if there exists a time $\tau$ such that for each $j = 1, \ldots, r$,*

$$\tau(\lambda_j - (\omega - \eta)) = m_j \pi \tag{5.2.5}$$

*for some odd $m_j$. In addition, if perfect state transfer occurs between $u$ and $v$, then the minimum time it occurs is $\tau = \frac{\pi q}{\Lambda_1 - \Lambda_2}$, where $q$ is an integer given in Theorem 5.1.2(2). In particular, if $\sigma_u(A) = \{\omega - \eta, \lambda_1, \lambda_2\}$, then $\tau = \frac{\pi p}{\Lambda_1 - \Lambda}$, where $\Lambda = \max\{\lambda : \lambda \in \sigma \backslash \{\Lambda_1\}\}$ and $p$ is an integer given in Theorem 5.1.2(2).*

With the assumption in Theorem 5.2.1, we further suppose that $\phi(A(X), t) \in \mathbb{Z}[x]$. Then by Theorem 5.1.5, we can write each $\lambda_j = \theta + b_j\sqrt{\Delta}$, where $b_j$ is an integer, and either $\Delta = 1$ for each $j$ or $\Delta$ is a square-free integer for each $j$. We proceed with two cases.

- For each $j$, let $\nu_2(b_j) = q \geq 0$ so that we can write each $\lambda_j - \theta = 2^q \ell_j \sqrt{\Delta}$ for some odd $\ell_j$. That is, the largest powers of two that divide all the $b_j$'s are equal. Consider $g = \gcd(\ell_1, \ldots, \ell_r)$, and for each $j$, let $\ell_j = g h_j$. Since each $\ell_j$ is odd, $g$ is odd and each $h_j$ is odd. Therefore, at time $\tau = \pi/2^q g\sqrt{\Delta}$, we get

$$e^{i\tau(\lambda_j - \theta)} = e^{i\left(\pi/2^q g\sqrt{\Delta}\right)(2^q \ell_j \sqrt{\Delta})} = e^{i\pi h_j} = -1.$$

  Thus, Equation 5.2.3 holds, i.e., PST occurs between $u$ and $v$ at $\tau = \pi/2^q g\sqrt{\Delta}$. We claim that $\tau$ is the minimum PST time between $u$ and $v$. To see this, suppose the minimum PST time is $\tau'$ with $0 < \tau' \leq \tau$. From Equation 5.2.5, $\tau'$ satisfies $\tau'(\lambda_j - \theta) = m_j \pi$. But from Corollary 3.2.7, we know that $\tau$ is an odd multiple of $\tau'$. That is, there exists an odd integer $s$ such that

$$\tau = s\tau' \iff \frac{\pi}{2^q g\sqrt{\Delta}} = \frac{s m_j \pi}{2^q \ell_j \sqrt{\Delta}} \iff \ell_j = s m_j g$$

  for each $j$. Since $s$ divides every $\ell_j$ and $g = \gcd(\ell_1, \ldots, \ell_r)$, we conclude that $s = 1$ so that $\ell_j = m_j g$. Equivalently, $m_j = h_j$, and we get that $\tau = \tau'$.

- For some $j$ and $r$, let $q_1 = \nu_2(b_j)$ and $q_2 = \nu_2(b_r)$, and assume $q_1 \neq q_2$. That is, the largest powers of two that divide two of the $b_j$'s are not equal. Without loss of generality, suppose $q_1 > q_2$. Note that we can write $\lambda_j - \theta = 2^{q_1} \ell_j \sqrt{\Delta}$ and $\lambda_r - \theta = 2^{q_2} \ell_r \sqrt{\Delta}$ for some odd $\ell_j$ and $\ell_r$. From Equation 5.2.5, we get

$t = \frac{m_j \pi}{2^{q_1}\ell_j\sqrt{\Delta}} = \frac{m_r \pi}{2^{q_2}\ell_r\sqrt{\Delta}}$, and thus,

$$m_j\ell_r = m_r 2^{q_1-q_2}\ell_j. \tag{5.2.6}$$

Since $q_1 - q_2 > 0$, the right hand side of Equation 5.2.6 is even, while the left hand side is odd, a contradiction. Thus, we do not get PST between $u$ and $v$.

We summarize these results in the following theorem.

**Theorem 5.2.2.** *Let $X$ be a connected weighted graph with or without loops. Suppose $\phi(A(X), t) \in \mathbb{Z}[x]$, and $U(\omega, \eta) = \{u, v\}$ be a set of twins in $X$ such that $\sigma_u(A(X)) = \{\omega - \eta, \lambda_1, \ldots, \lambda_r\}$. Then perfect state transfer occurs between $u$ and $v$ if and only if the following conditions hold.*

1. *Vertices $u$ and $v$ are strongly cospectral, in which case $\sigma_u^-(A(X)) = \{\omega - \eta\}$ and $\sigma_u^+(A(X)) = \{\lambda_1, \ldots, \lambda_r\}$.*
2. *For each $j$, $\lambda_j = \omega - \eta + b_j\sqrt{\Delta}$, where $b_j$ is an integer, and either $\Delta = 1$ for each $j$ or $\Delta$ is a square-free integer for each $j$.*
3. *For each $j$, $\nu_2(b_j) = q$, where $q$ is a nonnegative integer.*

*In addition, if perfect state transfer occurs between $u$ and $v$ at time $t$, then the minimum time it occurs is $\tau = \frac{\pi}{2^q g'\sqrt{\Delta}}$, where $g' = gcd(\ell_1, \ldots, \ell_r)$ and each $\ell_j = \frac{b_j}{2^q}$.*

In Theorem 5.2.2, conditions (1) and (2) reflect the fact strong cospectrality and periodicity are necessary conditions for PST between two vertices. To check strong cospectrality between twin vertices, one may use Corollary 4.4.8. Lastly, observe that $g = \gcd(b_1, \ldots, b_r) = \gcd(2^q\ell_1, \ldots, 2^q\ell_r) = 2^q g'$, so that the minimum PST time in Theorem 5.2.2 is indeed half of the minimum period indicated in Theorem 5.1.5.

## 5.3 Pretty good state transfer

Let $X$ be a connected weighted graph with or without loops containing twin vertices $u$ and $v$. Like PST, if PGST occurs between $u$ and $v$ and $U(\omega, \eta)$ is a set of twins such that $u, v \in U$, then it follows from Corollary 4.4.9 that $U(\omega, \eta) = \{u, v\}$. Assume $\sigma_u(A) = \{\lambda_1, \ldots, \lambda_r\}$, and define $\delta_j := (1 - \zeta_j)/2$, where $\zeta_j$ satisfies $E_j\mathbf{e}_u = \zeta_j E_j \mathbf{e}_v$. Since $u$ and $v$ are strongly cospectral, we have

$$\sigma_u^+(A) = \{\lambda_j \in \sigma_u(A) : \delta_j = 0\} \text{ and } \sigma_u^-(A) = \{\lambda_j \in \sigma_u(A) : \delta_j = 1\}. \tag{5.3.1}$$

Our goal is to characterize twin vertices in a graph that exhibit PGST. To do this, we state the following fact which was proved using Theorem 2.4.1.

**Theorem 5.3.1** ([6], Theorem 2)**.** *Let $X$ be a connected weighted graph with or without loops. Then pretty good state transfer occurs between vertices $u$ and $v$ of $X$ if and only if both conditions below are satisfied.*

1. *Vertices $u$ and $v$ are strongly cospectral.*

2. *If there is a set of integers $\{\ell_j\}$ such that*

$$\sum_{\lambda_j \in \sigma_u(A(X))} \ell_j \lambda_j = 0 \text{ and } \sum_{\lambda_j \in \sigma_u(A(X))} \ell_j \delta_j \text{ is odd} \tag{5.3.2}$$

   *then*

$$\sum_{\lambda_j \in \sigma_u(A(X))} \ell_j \neq 0. \tag{5.3.3}$$

The proof of Theorem 5.3.1 utilizes Theorem 2.4.1, and works whether or not the characteristic polynomial has integer coefficients. Thus, we can apply Theorem 5.3.1 to weighted graphs with or without loops. The following theorem characterizes twin vertices in weighted graphs with or without loops that exhibit PGST.

**Theorem 5.3.2.** *Let $X$ be a connected weighted graph with or without loops, and $U(\omega,\eta) = \{u, v\}$ be a set of twins in $X$ with $\sigma_u(A) = \{\omega-\eta, \lambda_1, \dots, \lambda_n\}$. Then pretty good state transfer occurs between $u$ and $v$ if and only if the following conditions hold.*

1. *Vertices $u$ and $v$ are strongly cospectral, in which case $\sigma_u^-(A(X)) = \{\omega - \eta\}$ and $\sigma_u^+(A(X)) = \{\lambda_1, \dots, \lambda_r\}$.*

2. *Let $\{\ell_1, \dots, \ell_r\}$ be a set of integers.*

   (a) *If $\omega - \eta \neq 0$, then*

$$\frac{1}{\omega-\eta}\sum_{j=1}^{r} \ell_j \lambda_j \text{ is odd} \tag{5.3.4}$$

   *implies that*

$$\sum_{j=1}^{r} \ell_j \left(1 - \frac{\lambda_j}{\omega-\eta}\right) \neq 0. \tag{5.3.5}$$

   (b) *If $\omega - \eta = 0$, then*

$$\sum_{j=1}^{r} \ell_j \lambda_j = 0 \tag{5.3.6}$$

*implies that*

$$\sum_{j=1}^{r} \ell_j \textit{ is even.} \tag{5.3.7}$$

*Proof.* Let $X$ be a connected weighted graph with or without loops, and $U(\omega, \eta) = \{u, v\}$ be a set of twins in $X$ with $\sigma_u(A) = \{\lambda_1, \ldots, \lambda_r, \lambda_{r+1}\}$, where $\lambda_{r+1} = \omega - \eta$. We show that condition (2) is equivalent to Theorem 5.3.1(2). From Corollary 4.4.8, $\sigma_u^+(A) = \{\lambda_1, \ldots, \lambda_r\}$ and $\sigma_u^-(A) = \{\lambda_{r+1}\}$. Now, assume there are integers $\ell_1, \ldots, \ell_{r+1}$ such that

$$\sum_{j=1}^{r+1} \ell_j \lambda_j = 0 \text{ and } \sum_{j=1}^{r+1} \ell_j \delta_j \text{ is odd} \implies \sum_{j=1}^{r+1} \ell_j \neq 0 \tag{5.3.8}$$

From Equation 5.3.1, we have that $\delta_j = 0$ for all $j = 1, \ldots, r$ and $\delta_{r+1} = 1$, and hence, the condition $\sum_{j=1}^{r+1} \ell_j \delta_j$ is odd in Equation 5.3.8 necessitates that $\ell_{r+1}$ is odd. Moreover, we can rewrite the condition $\sum_{j=1}^{r+1} \ell_j \lambda_j = 0$ in Equation 5.3.8 to $\sum_{j=1}^{r} \ell_j \lambda_j = -\ell_{r+1}(\omega - \eta)$. Consequently, the implication in 5.3.8 is equivalent to

$$\sum_{j=1}^{r} \ell_j \lambda_j = -\ell_{r+1}(\omega - \eta) \text{ and } \ell_{r+1} \text{ is odd} \implies \sum_{j=1}^{r+1} \ell_j \neq 0 \tag{5.3.9}$$

Now, if $\omega - \eta \neq 0$, then $\ell_{r+1} = -\frac{1}{\omega - \eta} \sum_{j=1}^{r} \ell_j \lambda_j$ is odd. Thus,

$$\sum_{j=1}^{r+1} \ell_j = \sum_{j=1}^{r} \ell_j + \ell_{r+1} = \sum_{j=1}^{r} \ell_j - \frac{1}{\omega - \eta} \sum_{j=1}^{r} \ell_j \lambda_j = \sum_{j=1}^{r} \ell_j \left(1 - \frac{\lambda_j}{\omega - \eta}\right) \neq 0. \tag{5.3.10}$$

That is, the implication in 5.3.9 is equivalent to

$$\frac{1}{\omega - \eta} \sum_{j=1}^{r} \ell_j \lambda_j \text{ is odd} \implies \sum_{j=1}^{r} \ell_j \left(1 - \frac{\lambda_j}{\omega - \eta}\right) \neq 0. \tag{5.3.11}$$

On the other hand, if $\omega - \eta = 0$, then $\sum_{j=1}^{r} \ell_j = 0$. Since $\sum_{j=1}^{r} \ell_j \neq -\ell_{r+1}$ and $\ell_{r+1}$ is an arbitrary odd integer, we conclude that $\sum_{j=1}^{r} \ell_j$ is even. Consequently, the implication in 5.3.9 can be written as

$$\sum_{j=1}^{r} \ell_j \lambda_j = 0 \implies \sum_{j=1}^{r} \ell_j \text{ is even.} \tag{5.3.12}$$

This proves that condition (2) is equivalent to Theorem 5.3.1(2). □

Next, we prove the following useful fact. We follow the proof of Pal.

**Theorem 5.3.3** ([54]). *Let $X$ be a connected weighted graph with or without loops. If $u$ and $v$ are vertices in $X$ such that either $u$ or $v$ is periodic, then the existence of pretty good state transfer and perfect state transfer between $u$ and $v$ are equivalent.*

*Proof.* Let $X$ be a connected weighted graph with or without loops. Assume $u$ and $v$ are vertices in $X$, and without loss of generality, suppose $u$ is periodic at time $\rho$ so that $U(\rho)\mathbf{e}_u = \gamma\mathbf{e}_u$ for some unit $\gamma \in \mathbb{C}$. Since PST is a special case of PGST, we only need to show that the converse holds. First, observe that

$$|U(t+\rho)_{u,v}|^2 = |\mathbf{e}_v^T U(t+\rho)\mathbf{e}_u|^2 = |\mathbf{e}_v^T U(t)U(\rho)\mathbf{e}_u|^2 = |\gamma\mathbf{e}_v^T U(t)\mathbf{e}_u|^2 = |U(t)_{u,v}|^2,$$

which implies that $|U(t)_{u,v}|^2$ is a periodic function of $t$ with period $\rho$. Since $|U(t)_{u,v}|^2$ is a continuous function of $t$, it follows that the image of $|U(t)_{u,v}|^2$ is a closed and bounded subset of $[0,1]$. Now, suppose PGST occurs between $u$ and $v$. That is, there exists a sequence $\{\tau_j\}$ such that $\lim_{j\to\infty} |U(\tau_j)_{u,v}|^2 = 1$. That is, 1 is an accumulation point of the image of $|U(t)_{u,v}|^2$. Since the image of $|U(t)_{u,v}|^2$ is closed, it contains 1. Using the fact that $|U(t)_{u,v}|^2$ is continuous, the Extreme Value Theorem guarantees the existence of $\tau_0 \in [0,\rho]$ such that $|U(\tau_0)_{u,v}|^2 = 1$. Equivalently, PST occurs between $u$ and $v$ at time $\tau_0$. □

Applying Theorem 5.3.4 yields the following corollary.

**Corollary 5.3.4.** *Let $X$ be a connected weighted graph with or without loops. If $u$ and $v$ are vertices in $X$ such that either $u$ or $v$ is periodic, then proper pretty good state transfer does not occur between $u$ and $v$.*

## 5.4 Fractional revival

Lastly, we explore fractional revival between twins in graphs. Let $X$ be a connected weighted graph with or without loops, and $U$ be a set of twins in $X$. If $u, v \in U$ and proper FR occurs between $u$ and $v$, then we know from Lemma 3.3.2 that $u$ and $v$ are parallel. However, since $u$ and $v$ are twins, we have that $u$ and $v$ are also cospectral by Proposition 4.3.2. Consequently, twin vertices that exhibit fractional revival are strongly cospectral, and by Corollary 4.4.5, we have that $|U| = 2$. Consequently, FR between twin vertices is monogamous. Now, let us characterize twin vertices that exhibit FR. We need the following fact which was stated in the context of unweighted graphs, but also holds for weighted graphs.

**Lemma 5.4.1** ([16], Proposition 5.1)**.** *Let $X$ be a connected weighted graph with or without loops, and suppose vertices $u$ and $v$ of $X$ admit $(\alpha,\beta)$-fractional revival. Then $u$ and $v$ are strongly cospectral if and only if there exists $\gamma,\zeta\in\mathbb{R}$ such that $\alpha=e^{i\zeta}\cos\gamma$ and $\beta=ie^{i\zeta}\sin\gamma$.*

Using Lemma 5.4.1, we provide a characterization of twins in a graph that exhibit $(\alpha,\beta)$-FR. Assume $X$ be a connected weighted graph with or without loops, and $U=U(\omega,\eta)=\{u,v\}$ be a set of twins in $X$ that are strongly cospectral. Let $\theta=\omega-\eta$, and suppose $\theta,\lambda_1,\ldots,\lambda_r$ are the distinct eigenvalues in $\sigma_u(A)$ with corresponding spectral idempotents $E_\theta,E_1,\ldots,E_r$, where $\theta$ is defined in Equation 4.2.5. Using the definition, $(\alpha,\beta)$-FR occurs between $u$ and $v$ in $X$ if and only if there exists a time $\tau$ and some $\alpha,\beta\in\mathbb{C}$ satisfying $|\alpha|^2+|\beta|^2=1$ such that

$$U(\tau)\mathbf{e}_u=\alpha\mathbf{e}_u+\beta\mathbf{e}_v \tag{5.4.1}$$

Note that the spectral decomposition of $U(t)$ is given by

$$U(t)\mathbf{e}_u=\sum_{\lambda\in\sigma_u(A)}e^{it\lambda}E_\lambda \tag{5.4.2}$$

Moreover, from Corollary 4.4.8, we have $\sigma_u^+(A)=\{\lambda_1,\ldots,\lambda_r\}$ and $\sigma_u^-(A)=\{\theta\}$. Using this, and the fact that the spectral idempotents sum to identity, we get

$$\alpha\mathbf{e}_u+\beta\mathbf{e}_v=\sum_{\lambda\in\sigma_u(A)}(\alpha E_\theta\mathbf{e}_u+\beta E_\theta\mathbf{e}_v)=(\alpha-\beta)E_\theta\mathbf{e}_u+(\alpha+\beta)\sum_{j=1}^{r}E_j\mathbf{e}_u. \tag{5.4.3}$$

Applying Equations 5.4.2 and 5.4.3 to Equation 5.4.1 yields

$$\alpha-\beta=e^{i\tau\theta}\text{ and }\alpha+\beta=e^{-i\tau\lambda_j} \tag{5.4.4}$$

for each $j$. Invoking Lemma 5.4.1, there exists $\gamma,\zeta\in\mathbb{R}$ such that $\alpha=e^{i\zeta}\cos\gamma$ and $\beta=ie^{i\zeta}\sin\gamma$. Since $\alpha-\beta=e^{i\zeta}(\cos\gamma-i\sin\gamma)=e^{i\zeta}e^{-i\gamma}=e^{i(\zeta-\gamma)}$, and similarly, $\alpha+\beta=e^{i(\zeta+\gamma)}$, we can rewrite Equation 5.4.4 as

$$e^{i\tau\theta}=e^{i(\zeta-\gamma)}\text{ and }e^{i\tau\lambda_j}=e^{i(\zeta+\gamma)}, \tag{5.4.5}$$

and so $\tau\theta\equiv\zeta-\gamma\pmod{2\pi}$. The two equations in 5.4.5 yields $e^{i\tau(\lambda_j-\theta)}=e^{i2\gamma}$. Thus,

$$\tau(\lambda_j-\theta)\equiv 2\gamma\pmod{2\pi}. \tag{5.4.6}$$

In particular, proper $(\alpha, \beta)$-FR occurs between $u$ and $v$ if and only if $|\cos\gamma| \neq 0$ and $|\sin\gamma| \neq 0$, which happens if and only if $\gamma \neq \frac{\pi}{2}\ell$ for any integer $\ell$. Meanwhile, balanced $(\alpha, \beta)$-FR occurs between $u$ and $v$ if and only if $|\cos\gamma| = |\sin\gamma|$, i.e., $\gamma = \frac{\pi}{4} + \frac{j\pi}{2} = \frac{(2j+1)\pi}{4}$ for some integer $j$. This yields the following fact.

**Theorem 5.4.2.** *Let $X$ be a connected weighted graph with or without loops, and $U(\omega, \eta) = \{u, v\}$ be a set of twins in $X$ with $\sigma_u(A) = \{\omega - \eta, \lambda_1, \ldots, \lambda_r\}$. Then $(\alpha, \beta)$-fractional revival occurs from $u$ to $v$ at time $\tau$ if and only if the following conditions hold.*

1. *Vertices $u$ and $v$ are strongly cospectral.*
2. *There exists $\zeta, \gamma \in \mathbb{R}$ such that $\alpha = e^{i\zeta}\cos\gamma$ and $\beta = ie^{i\zeta}\sin\gamma$.*
3. *For each $j$, $\tau(\lambda_j - \omega + \eta) \equiv 2\gamma \pmod{2\pi}$ and $\tau(\omega - \eta) \equiv \zeta - \gamma \pmod{2\pi}$.*

*In particular, proper $(\alpha, \beta)$-fractional revival occurs between $u$ and $v$ if and only if $\gamma \neq \frac{\pi}{2}\ell$ for any integer $\ell$, while balanced $(\alpha, \beta)$-fractional revival occurs between $u$ and $v$ if and only if $\gamma = \frac{\pi}{4}\ell$ for some odd integer $\ell$.*

In Theorem 5.4.2, we remark that if $\omega - \eta = 0$, then condition 3 is equivalent to $\tau\lambda_j \equiv 2\gamma \pmod{2\pi}$ and $\zeta \equiv \gamma \pmod{2\pi}$, which yields $\alpha = e^{i\zeta}\cos\zeta$ and $\beta = ie^{i\zeta}\cos\zeta$. However, calculating the minimum $(\alpha, \beta)$-FR time for this case is not an easy task, as $\tau$ in the second equation of condition 3 is annihilated by the fact that $\omega - \eta = 0$. But if $\omega - \eta \neq 0$, then determining the minimum $(\alpha, \beta)$-FR time will be straightforward from condition 3. In particular, if $\omega - \eta > 0$ and $\zeta - \gamma > 0$, then the minimum $(\alpha, \beta)$-FR time is given by $\tau = \frac{\zeta - \gamma}{\omega - \eta}$, while if $\omega - \eta > 0$ and $\zeta - \gamma < 0$, then the minimum $(\alpha, \beta)$-FR time is $\tau = \frac{\zeta - \gamma + 2\ell\pi}{\omega - \eta}$, where $\ell$ is the least integer such that $\zeta - \gamma + 2\ell\pi > 0$.

# 6

# Double cones on regular graphs

This chapter was inpired by the work of multiple authors in [3, 4, 31] about double cones on regular graphs. We provide a systematic approach in discussing periodicity, PST, PGST and FR in weighted double cones on regular graphs. We begin with equitable partitions.

## 6.1 An equitable partition

Let $H$ be a graph on $n$ vertices. We call $\mathbf{K}_2(\omega,\eta)\vee H$ a *weighted double cone* on $H$ with *apexes* $u$ and $v$, where either $\eta\neq 0$ or $\omega\neq 0$. In particular, if $\eta\neq 0$, then $\mathbf{K}_2(\omega,\eta)\vee H$ is a *connected double cone* on $H$, in which case $u$ and $v$ are true twins in $\mathbf{K}_2(\omega,\eta)\vee H$. Otherwise, $\mathbf{K}_2(\omega,0)\vee H$ is a *disconnected double cone* on $H$, in which case $\omega\neq 0$, and $u$ and $v$ are false twins in $\mathbf{K}_2(\omega,0)\vee H$. Note that $\mathbf{K}_2(\omega,\eta)$ is a connected graph for any $\omega,\eta\in\mathbb{R}$.

Let $H$ be a weighted $k$-regular graph on $n$ vertices, and $X(\gamma):=\mathbf{K}_2(\omega,\eta)\vee H$ be the double cone on $H$ such that the loops on $u$ and $v$ have weights $\omega$, the edge $(u,v)$ has weight $\eta$, and the edges $(u,w)$ and $(v,w)$ have weights equal to $\gamma$ for all $w\in V(H)$. We determine the conditions that yield strong cospectrality between the apexes of $X(\gamma)$. Define a partition $\pi=\{C_1,C_2,C_3\}$ in $X$ such that $C_1=\{u\}$, $C_2=\{v\}$ and $C_3=V(H)$. Then one checks that $\pi$ is equitable and

$$A(\widehat{X(\gamma)/\pi})=\begin{bmatrix}\omega & \eta & \sqrt{n}\gamma\\ \eta & w & \sqrt{n}\gamma\\ \sqrt{n}\gamma & \sqrt{n}\gamma & k\end{bmatrix},\tag{6.1.1}$$

If $\eta=0$, then we may view $\widehat{X/\pi}$ as a weighted path on three vertices with end vertices

$C_1$ and $C_2$ such that each edge has weight $\sqrt{n}\gamma$, and the end and middle vertices have loop of weights $\omega$ and $k$, respectively. Note that, in Example 3.4.2, we have already discussed the case when a weighted path on three vertices is isomorphic to the graph $Y(\omega')$ in Figure 3.4, in which case $\omega = \eta = 0$ and $\omega' = \sqrt{n}\gamma$. Meanwhile, if $\eta \neq 0$, then $\widehat{X/\pi}$ may be viewed as a weighted complete graph on three vertices such that $C_3$ has a loop of weight $k$, $C_1$ and $C_2$ have loops of weight $\omega$, the edge $(C_1, C_2)$ has weight $\eta$, and the remaining two edges have weight $\sqrt{n}\gamma$. In other words, a weighted $P_3$ is a contraction of a weighted disconnected double cone while a weighted $K_3$ is a contraction of a weighted connected double cone.

Note that if $H = \mathbf{K}_n(\omega, \eta)$ and $\eta = \gamma$, then $X(\gamma) = \mathbf{K}_2(\omega, \eta) \vee H \cong \mathbf{K}_{n+2}(\omega, \eta)$. Since every pair of vertices in $X(\gamma)$ are twins, it follows from Corollaries 4.3.2 and 4.4.5 that $u$ and $v$ are not parallel. Now, suppose $H \ncong \mathbf{K}_n(\omega, \eta)$ or $\eta \neq \gamma$, so that $X(\gamma) \ncong \mathbf{K}_{n+2}(\omega, \eta)$. Using Equation 6.1.1, the eigenvalues of $\widehat{X(\gamma)/\pi}$ are

$$\lambda^{\pm} = \frac{1}{2}\left(k + \omega + \eta \pm \sqrt{(k - \omega - \eta)^2 + 8n\gamma^2}\right) \text{ and } \omega - \eta, \tag{6.1.2}$$

with corresponding eigenvectors

$$\left(\frac{n\gamma^2 - \eta(\lambda^{\pm} - \omega - \eta)}{\sqrt{n}\gamma(\lambda^{\mp} - \omega + \eta)}, \frac{n\gamma^2 - \eta(\lambda^{\pm} - \omega - \eta)}{\sqrt{n}\gamma(\lambda^{\mp} - \omega + \eta)}, 1\right) \text{ and } e_1 - e_2. \tag{6.1.3}$$

Thus, $C_1$ and $C_2$ are strongly cospectral in $\widehat{X(\gamma)/\pi}$, and $\lambda^{\pm} \in \sigma_{C_1}^{+}(A(\widehat{X(\gamma)/\pi}))$ and $\omega - \eta \in \sigma_{C_2}^{-}(A(\widehat{X(\gamma)/\pi}))$. We summarize these facts in the following lemma.

**Lemma 6.1.1.** *Let $H$ be a weighted $k$-regular graph, and $X(\gamma) = \boldsymbol{K}_2(\omega, \eta) \vee H$ be a double cone on $H$ with apexes $u$ and $v$. The partition $\pi = \{C_1, C_2, C_3\}$ of $X$ such that $C_1 = \{u\}$, $C_2 = \{v\}$ and $C_3 = V(H)$ is equitable. Moreover, the following hold.*

1. *If $H = \boldsymbol{K}_n(\omega, \eta)$ and $\eta = \gamma$, then $C_1$ and $C_2$ are not parallel in $\widehat{X(\gamma)/\pi}$.*
2. *If $H \neq \boldsymbol{K}_n(\omega, \eta)$ or $\eta \neq \gamma$, then $C_1$ and $C_2$ are strongly cospectral in $\widehat{X(\gamma)/\pi}$ with $\sigma_{C_1}^{+}(A(\widehat{X(\gamma)/\pi}) = \{\lambda^{\pm}\}$ and $\sigma_{C_1}^{-}(A(\widehat{X(\gamma)/\pi}) = \{\omega - \eta\}$.*

Moreover, since $C_1$ and $C_2$ are singleton cells that contain $u$ and $v$ respectively, applying Lemma 3.3.9 to Lemma 6.1.1 yields the following characterization of strongly cospectrality apexes in weighted double cones.

**Corollary 6.1.2.** *Let $H$ be a weighted $k$-regular graph, and $X(\gamma) = \boldsymbol{K}_2(\omega, \eta) \vee H$ be a double cone on $H$ with apexes $u$ and $v$. Then $u$ and $v$ are strongly cospectral in $X(\gamma)$ if and only if either $H \ncong \boldsymbol{K}_n(\omega, \eta)$ or $\eta \neq \gamma$.*

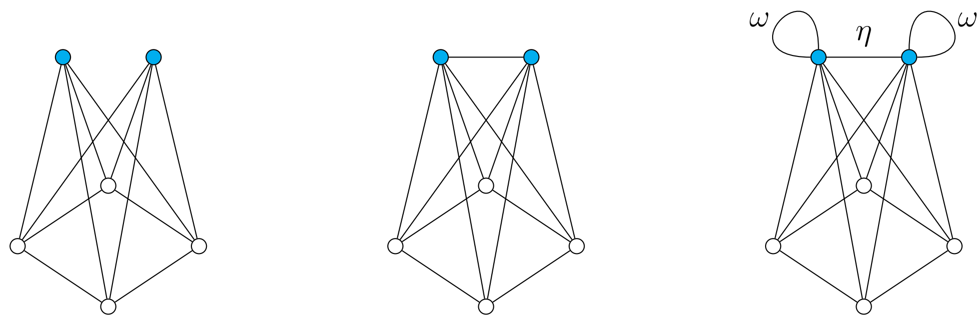


Figure 6.1: Double cones on $C_4$ with apexes marked blue: $\overline{K_2} \vee C_4$ (left), $K_2 \vee C_4$ (center), and $\mathbf{K}_2(\omega, \eta) \vee C_4$ (right)

By Corollary 6.1.2, the family of weighted double cones on regular graphs are promising candidates for PST, PGST and FR. As we will see later on, unweighted complete graphs are the only unweighted double cones on regular graphs that do not exhibit strong cospectrality. Next, we characterize quantum state transfer in weighted complete graphs.

**Theorem 6.1.3** (Quantum state transfer in weighted Complete Graphs)**.** *Let $m \geq 3$, and suppose $X = \boldsymbol{K}_m(\omega, \eta)$ with vertices $u$ and $v$. The following hold.*

1. *Vertices $u$ and $v$ are not parallel.*
2. *Pretty good state transfer and fractional revival do not occur between $u$ and $v$.*
3. *Vertices $u$ and $v$ are periodic with minimum period $\rho = \frac{2\pi}{m\eta}$.*

*Proof.* For $m \geq 3$, let $X = \mathbf{K}_m(\omega, \eta)$ with vertices $u$ and $v$. Note that (1) and (2) follow from Lemmas 6.1.1, 3.3.9, and 3.3.2. Now, one checks that the eigenvalues of $X$ are $\omega - \eta$ and $\omega + (m-1)\eta$ (multiplicity $m-1$). Applying Proposition 4.1.1 and Theorem 5.1.2(1) yield (3). □

**Remark 6.1.4.** Let $Y$ be a weighted complete graph on $m \geq 3$ vertices with vertex partition $Y_1, \ldots, Y_n$. If all the $Y_j$'s are sets of twins such that $|Y_j| \geq 3$ for each $j$, then by virtue of Corollary 4.4.5, any pair of vertices in $Y$ are not strongly cospectral, and hence, does not exhibit PGST and FR. As a consequence, if $U$ is a subset of the vertex set of $\mathbf{K}_m(\omega, \eta)$ with $|U| \geq 3$, then either altering the weight of the loops on all vertices in $U$ from $\omega$ to $\omega'$, or altering the weight of the edge from $\eta$ to $\eta'$ between all pairs of vertices in $U$, does not induce PGST nor FR between any pair of vertices in the resulting graph.

It is also interesting to compare Theorem 6.1.3 with Theorem 4.3.6. From Theorem 4.3.6(2), it is evident that PGST does not occur between any two vertices of $\mathbf{K}_m(\omega, \eta)$, as the maximum fidelity between them is $\frac{4}{m^2}$. Moreover, since the fidelity of state transfer between any two distinct vertices at any time $t$ are equal, there is no time $t$ such that the state at a vertex will be concentrated at only two vertices, i.e., FR does not occur between any two vertices of $\mathbf{K}_m(\omega, \eta)$.

Next, we deal with unweighted double cones. Suppose $H$ is an unweighted $k$-regular graph and $\gamma = 1$. Then $k \in \{0, 1 \ldots, n-1\}$, and because a $k$-regular graph exists on $n$ vertices if and only if $nk$ is even, we have that at least one of $n$ and $k$ is even. We call $X := X(1) = \mathbf{K}_2(0, \eta) \vee H$ the unweighted double cone on $H$, where $\eta \in \{0, 1\}$. For brevity, we write $X = K_2 \vee H$ whenever $\eta = 1$, in which case we call $X$ an *unweighted connected double cone* on $H$, while we write $X = \overline{K_2} \vee H$ whenever $\eta = 0$, in which case we call $X$ an *unweighted disconnected double cone* on $H$. If $X = K_2 \vee H$, then $\omega - \eta = -1$, and the eigenvalues of $A(\widehat{X/\pi})$ are

$$\lambda^{\pm} = \frac{1}{2}\left(k + 1 \pm \sqrt{(k-1)^2 + 8n}\right) \text{ and } -1, \tag{6.1.4}$$

while if $X = \overline{K_2} \vee H$, then $\omega - \eta = 0$, and the eigenvalues of $A(\widehat{X/\pi})$ are

$$\lambda^{\pm} = \frac{1}{2}\left(k \pm \sqrt{k^2 + 8n}\right) \text{ and } 0. \tag{6.1.5}$$

These observations yield a corollary of Lemma 6.1.1 for unweighted double cones.

**Corollary 6.1.5.** *Let $H$ be an unweighted $k$-regular graph on $n$ vertices.*

1. *If $X = K_2 \vee H$, then $C_1$ and $C_2$ are strongly cospectral in $\widehat{X/\pi}$ if and only if $k \neq n-1$, in which case $\sigma^+_{C_1}(A(\widehat{X/\pi}) = \{\lambda^{\pm}\}$ and $\sigma^-_{C_1}(A(\widehat{X/\pi}) = \{-1\}$.*

2. *If $X = \overline{K_2} \vee H$, then $C_1$ and $C_2$ are strongly cospectral in $\widehat{X/\pi}$ for every $k \in \{0, 1, \ldots, n-1\}$, in which case $\sigma^+_{C_1}(A(\widehat{X/\pi})) = \{\lambda^{\pm}\}$ and $\sigma^-_{C_1}(A(\widehat{X/\pi})) = \{0\}$.*

Note that from Equations (6.1.4) and (6.1.5), it follows that $\phi(A(\widehat{X/\pi})) \in \mathbb{Z}[x]$ whenever $\omega = 0$ and $\eta \in \{0, 1\}$, even if $n$ is not a perfect square. That is, it is possible for the characteristic polynomial of the adjacency matrix of a connected weighted undirected graph to have integer coefficients even if some entries of the adjacency matrix are not rational.

We now apply Lemma 3.3.9 to Corollary 6.1.5 to get the following result.

**Corollary 6.1.6.** *Let $H$ be an unweighted $k$-regular graph on $n$ vertices.*

1. *The apexes of $K_2 \vee H$ are strongly cospectral if and only if $k \neq n-1$.*
2. *For every $k \in \{0, 1, \ldots, n-1\}$, the apexes of $\overline{K_2} \vee H$ are strongly cospectral.*

In other words, the only unweighted double cone on a regular graph that does not yield strong cospectrality between its apexes is the connected double cone on a complete graph. Indeed, a connected double cone on $K_n$ results in $K_{n+2}$, and we know from Example 4.4.6 that any two vertices of the complete graph are not strongly cospectral.

In the next two sections, we characterize the parameters (number of vertices $n$ and regularity $k$) that yield PST, PGST and FR between the apexes of unweighted disconnected and connected double cones on regular graphs. To do this, we import some facts from elementary number theory. An integer $T$ is called a *triangular number* if there exists an integer $z$ such that $T = \frac{z(z+1)}{2}$. It is known that an integer $T$ is a triangular number if and only if $8T + 1$ is a perfect square. Also, we recall that squares of even numbers are divisible by 4, and hence, even numbers of the form $4\ell + 2$ are not perfect squares. Moreover, perfect squares of odd numbers are of the form $4\ell + 1$ so that odd numbers of the form $4\ell + 3$ are not perfect squares.

Now, suppose $n$ is a positive integer, and $k \in \{0, 1, \ldots, n-1\}$. We examine when $k^2 + 8n$ is a perfect square by looking at the following cases.

- If $k = 4s + 1$, then $n$ is even so that $nk$ is even, and

  $$k^2 + 8n = 16s^2 + 8s + 1 + 8n = 8(2s^2 + s + n) + 1$$

  which is a perfect square if and only if $2s^2 + s + n$ is a triangular number.

- If $k = 4s+3$, then a calculation to that in the previous case implies that $k^2+8n$ is a perfect square if and only if $2s^2 + 3s + n + 1$ is a triangular number.

- If $k = 4s + 2$, then $k^2 + 8n = 4(4s^2 + 4s + 2n + 1)$. If $n = 2m + 1$, then $k^2 + 8n = 4[4(s^2 + s + m) + 3]$, where $4(s^2 + s + m) + 3$ is not a perfect square. If $n = 2m$, then

  $$k^2 + 8n = 4[4(s^2 + s + m) + 1]$$

  which is a perfect square if and only if $s^2 + s + m$ is twice a triangular number.

- If $k = 4s$, then $k^2 + 8n = 16s^2 + 8n$. If $n = 2m$, then

  $$k^2 + 8n = 16(s^2 + m)$$

which is a perfect square if and only if $s^2+m$ is a perfect square. If $n = 2m+1$, then $k^2 + 8n = 16s^2 + 16m + 8 = 4[4(s^2 + m) + 2]$, where $4(s^2 + m) + 2$ is not a perfect square.

We summarize these in the following lemma.

**Lemma 6.1.7.** *$\sqrt{k^2 + 8n}$ is an integer if and only if one of the conditions hold.*

1. *$k = 4s+3$ and $T = 2s^2+3s+n+1$ is a triangular number, with $k^2+8n = 8T+1$.*
2. *$k = 4s + 2$, $n = 2m$ and $2T = s^2 + s + m$, where $T$ a triangular number, with $k^2 + 8n = 4(8T + 1)$.*
3. *$k = 4s + 1$ and $T = 2s^2 + s + n$ is a triangular number, with $k^2 + 8n = 8T + 1$.*
4. *$k = 4s$, $n = 2m$ and $s^2 + m$ is a perfect square, with $k^2 + 8n = 16(s^2 + m)$.*

## 6.2 Disconnected double cones

With the help of Corollary 6.1.5 and Lemma 6.1.7, we now characterize quantum state transfer between the apexes of unweighted disconnected double cones on unweighted $k$-regular graphs.

**Theorem 6.2.1.** *Let $H$ be an unweighted $k$-regular graph on $n$ vertices. The following hold for the apexes $u$ and $v$ of $\overline{K_2} \vee H$.*

1. *Vertices $u$ and $v$ are periodic if and only if $k = 0$ or $k^2+8n$ is a perfect square, in which case the minimum period is given by*
$$\rho = \begin{cases} \frac{2\pi}{\sqrt{2n}}, & \text{if } k = 0 \\ \frac{2\pi}{g}, & \text{if } k^2 + 8n \text{ is a perfect square,} \end{cases}$$
*where $g = gcd\left(\frac{1}{2}\left(k + \sqrt{k^2 + 8n}\right), \frac{1}{2}\left(k - \sqrt{k^2 + 8n}\right)\right)$.*
2. *If $k = 0$, then perfect state transfer occurs between $u$ and $v$, with minimum time $\tau = \frac{\pi}{\sqrt{2n}}$.*
3. *Let $k > 0$ and $k^2 + 8n$ is a perfect square. Then perfect state transfer occurs between $u$ and $v$ if and only if $k = 4s$, $n = 2m$, $s^2 + m$ is a perfect square, where $m = 2^a b$ and $s = 2^p q$ for some integers $a$ and $p$ with $0 \leq a \leq 2p$ and odd integers $b$ and $q$, in which case the minimum time is $\tau = \frac{\pi}{2^{\frac{a}{2}+1} g}$, where $g = gcd\left(2^{p-\frac{a}{2}} q + \sqrt{2^{2p-a} q^2 + b}, 2^{p-\frac{a}{2}} q - \sqrt{2^{2p-a} q^2 + b}\right)$.*

4. *If $k=0$ or $k^2+8n$ is a perfect square, then proper pretty good state transfer does not occur between $u$ and $v$.*

5. *If $k>0$ and $k^2+8n$ is not a perfect square, then pretty good state transfer occurs between $u$ and $v$.*

6. *$(\alpha,\beta)$-fractional revival occurs between $u$ and $v$ at time $\tau$ if and only if $\alpha = e^{i\gamma}\cos\gamma$ and $\beta = ie^{i\gamma}\sin\gamma$, where*

$$\gamma = \frac{r\left(k-\sqrt{k^2+8n}\right) - q\left(k+\sqrt{k^2+8n}\right)}{2\sqrt{k^2+8n}}\pi$$

*for some integers $r$ and $q$ such that $r>q$, in which case $\tau = \frac{2(r-q)\pi}{\sqrt{k^2+8n}}$. In particular, the minimum $(\alpha,\beta)$-fractional revival time is $\tau = \frac{2\pi}{\sqrt{k^2+8n}}$ which occurs if and only if $r-q=1$, in which case we may take $\gamma = \frac{(k-\sqrt{k^2+8n})\pi}{2\sqrt{k^2+8n}}$. If $q=0$, then proper $(\alpha,\beta)$-fractional revival occurs between $u$ and $v$ if and only if $\frac{rk}{\sqrt{k^2+8n}}$ is not an integer, while balanced $(\alpha,\beta)$-fractional revival occurs between $u$ and $v$ if and only if*

$$2rk = (2r+\ell)\sqrt{k^2+8n} \tag{6.2.1}$$

*for some odd integer $\ell$, in which case $k=4s$ and $n=2m$ such that $s^2+m$ is a perfect square, and $s$ and $m$ have the same parity. In particular, if $s$ and $m$ are odd, then taking $r=\frac{1}{2}\sqrt{s^2+m}$ and $\ell = s-\sqrt{s^2+m}$ yields balanced $(\alpha,\beta)$-FR between $u$ and $v$ at $\tau=\frac{\pi}{4}$.*

*Proof.* Let $H$ be an unweighted $k$-regular graph on $n$ vertices, and $X = \overline{K_2}\vee H$ with apexes $u$ and $v$. By Lemma 6.1.5(2), we know that $\pi = \{C_1, C_2, C_3\}$ is an equitable partition of $X$, where $C_1=\{u\}$, $C_2=\{v\}$ and $C_3=V(H)$. Since $X$ is connected, $\widehat{X/\pi}$ is also a connected weighted graph. Moreover, one checks that $\phi(A(\widehat{X/\pi}),t)\in\mathbb{Z}[x]$, and Equation 6.1.5 yields the eigenvalues $\lambda^{\pm}$ and 0 for $A(\widehat{X/\pi})$, where $\lambda^{\pm} = \frac{1}{2}\left(k\pm\sqrt{k^2+8n}\right)$. Furthermore, the cells $C_1$ and $C_2$ are strongly cospectral in $\widehat{X/\pi}$ with $\sigma_{C_1}(A(\widehat{X/\pi})) = \sigma(A(\widehat{X/\pi}))$.

Let us prove (1). Since $C_1$ and $C_2$ are strongly cospectral in $\widehat{X/\pi}$, they have the same eigenvalue support. Hence, from Proposition 5.1.4, we have that $C_1$ is periodic if and only if $C_2$ is periodic. Now, using the fact that $0\in\sigma_{C_1}(A(\widehat{X/\pi}))$, the ratio condition tells us that $C_1$ is periodic in $\widehat{X/\pi}$ if and only if $\lambda^{\pm}-0=\lambda^{\pm}$ is an integer multiple of $\sqrt{\Delta}$, where $\Delta=1$ or $\Delta$ is a square-free integer. Equivalently, $k=0$ or

$k^2+8n$ is a perfect square. Moreover, Theorem 5.1.5 yields the minimum period $\rho$ for both $C_1$ and $C_2$ as

$$\rho = \begin{cases} \frac{2\pi}{\sqrt{2n}}, & \text{if } k=0 \\ \frac{2\pi}{g}, & \text{if } k^2+8n \text{ is a perfect square,} \end{cases}$$

where $g = \gcd\left(\frac{1}{2}\left(k+\sqrt{k^2+8n}\right), \frac{1}{2}\left(k-\sqrt{k^2+8n}\right)\right)$. Thus, (1) holds. To prove (2), assume $k=0$ so that $\sigma_{C_1}(A(\widehat{X/\pi})) = \{0, \pm\sqrt{2n}\}$. Note that we can write $\pm\sqrt{2n} = \pm b\sqrt{\Delta}$ for some integers $b$ and $\Delta$, where $\Delta = 1$ or $\Delta$ is square-free. Since $\nu_2(b) = \nu_2(-b)$, we conclude from Theorem 5.2.2 that PST occurs between $C_1$ and $C_2$ in $\widehat{X/\pi}$. Moreover, since the minimum period of $C_1$ is $\rho = \frac{2\pi}{\sqrt{2n}}$, the minimum PST time between $C_1$ and $C_2$ is given by $\tau = \frac{\pi}{\sqrt{2n}}$.

Now, let us prove (3). Suppose $k>0$ and $k^2+8n$ is a perfect square so that Lemma 6.1.7 holds and $\lambda^{\pm}$ are integers. We proceed with cases.

- Let $k$ be odd. Then $k^2+8n$ is odd, and so $\sqrt{k^2+8n}$ is also odd. If we write $k = 2m+1$ and $\sqrt{k^2+8n} = 2\ell+1$, then

  $$\lambda^{\pm} = \frac{1}{2}(k \pm \sqrt{k^2+8n}) = \frac{1}{2}(2m+1 \pm (2\ell+1))$$

  which yields $\lambda^+ = m+\ell+1$ and $\lambda^- = m-\ell$. Since $m \pm \ell$ have the same parity, it follows that $\lambda^{\pm}$ have opposite parities so that $\nu_2(\lambda^+) \neq \nu_2(\lambda^-)$. By Theorem 5.2.2, there is no PST between $C_1$ and $C_2$ in $\widehat{X/\pi}$.

- Let $k = 4s+2$, $n = 2m$ and $2T = s^2+s+m$ be twice a triangular number. Then $k^2+8n = 4(8T+1)$ is a perfect square so that

  $$\lambda^{\pm} = \frac{1}{2}(k \pm 2\sqrt{8T+1}) = 2s+1 \pm \sqrt{8T+1}$$

  If we write $\sqrt{8T+1} = 2r+1$, then $\lambda^+ = 2(s+r+1)$ and $\lambda^- = 2(s-r)$. Note that $s \pm r$ have the same parity, and so $s+r+1$ and $s-r$ have opposite parities. This implies that $\nu_2(\lambda^+) \neq \nu_2(\lambda^-)$, and hence Theorem 5.2.2 yields no PST between $C_1$ and $C_2$ in $\widehat{X/\pi}$.

- Let $k = 4s$, $n = 2m$ and $s^2+m$ be a perfect square so that $k^2+8n = 16(s^2+m)$ is a perfect square. Then

  $$\lambda^{\pm} = \frac{1}{2}(k \pm 4\sqrt{s^2+m}) = 2(s \pm \sqrt{s^2+m}). \qquad (6.2.2)$$

– Suppose $m$ is even and $s$ is odd. Then $\sqrt{s^2+m}$ is odd. If we write $s = 2y+1$ and $\sqrt{s^2+m} = 2z+1$, then $\lambda^{\pm} = 2(2y+1 \pm (2z+1))$. That is, $\lambda^+ = 4(y+z+1)$ and $\lambda^+ = 4(y-z)$. Since $y+z+1$ and $y-z$ have opposite parities, we get that $\nu_2(\lambda^+) \neq \nu_2(\lambda^-)$. By Theorem 5.2.2, we get no PST between $C_1$ and $C_2$ in $\widehat{X/\pi}$.

– Suppose $m$ is odd. If $s$ is even, then $\sqrt{s^2+m}$ is odd, and so $s \pm \sqrt{s^2+m}$ is also odd. If $s$ is odd, then $\sqrt{s^2+m}$ is even, and so $s \pm \sqrt{s^2+m}$ is again odd. Thus, $s \pm \sqrt{s^2+m}$ is always odd whenever $m$ odd so that $\nu_2(\lambda^{\pm}) = 1$. By Theorem 5.2.2, PST occurs between $C_1$ and $C_2$ in $\widehat{X/\pi}$ with minimum time $\tau = \frac{\pi}{2g}$, where $g = \gcd\left(s + \sqrt{s^2+m}, s - \sqrt{s^2+m}\right)$.

– Suppose $m$ and $s$ are even. Then we can write $m = 2^a b$ and $s = 2^p q$ for some odd numbers $b, q$ and integers $a, p \geq 1$ and we obtain

$$s^2 + m = 2^{2p}q^2 + 2^a b \tag{6.2.3}$$

If $a > 2p$, then Equation 6.2.3 gives us $s^2 + m = 2^{2p}(q^2 + 2^{a-2p}b)$. Thus, we can write Equation 6.2.2 as

$$\lambda^{\pm} = 2\left(2^p q \pm 2^p \sqrt{q^2 + 2^{2a-p}b}\right) = 2^{p+1}\left(q \pm \sqrt{q^2 + 2^{2a-p}b}\right)$$

where $\sqrt{q^2 + 2^{2a-p}b}$ is odd because $q$ is odd. Now, suppose we write $\sqrt{q^2 + 2^{2a-p}b} = 2z+1$ and $q = 2y+1$ for some integers $y$ and $z$. Then we obtain $\lambda^{\pm} = 2^{p+1}\left[2y+1 \pm (2z+1)\right]$, and we get that $\lambda^+ = 2^{p+2}(y+z+1)$ and $\lambda^- = 2^{p+2}(y-z)$. Since $y \pm z$ have same parities, it follows that $y+z+1$ and $y-z$ have opposite parities, and thus $\nu_2(\lambda^+) \neq \nu_2(\lambda^-)$. By Theorem 5.2.2, we get no PST between $C_1$ and $C_2$ in $\widehat{X/\pi}$. Next, $a < 2p$, then $s^2 + m = 2^a(2^{2p-a}q^2 + b)$, where $2^{2p-a}q^2 + b$ is odd, since $b$ is odd. Since $s^2+m$ is a perfect square, $\sqrt{s^2+m} = 2^{\frac{a}{2}}\sqrt{2^{2p-a}q^2+b}$ and $a$ must be even. Thus, it follows from Equation 6.2.2 that

$$\begin{aligned}\lambda^{\pm} = 2(s \pm \sqrt{s^2+m}) &= 2\left(2^p q \pm 2^{\frac{a}{2}}\sqrt{2^{2p-a}q^2+b}\right)\\ &= 2^{\frac{a}{2}+1}\left(2^{p-\frac{a}{2}}q \pm \sqrt{2^{2p-a}q^2+b}\right).\end{aligned}$$

Since $b$ is odd, $2^{2p-a}q^2 + b$ is odd, and so $2^{p-\frac{a}{2}}q \pm \sqrt{2^{2p-a}q^2+b}$ is also odd. Consequently, $\nu_2(\lambda^{\pm}) = \frac{a}{2} + 1$, and we conclude using Theorem 5.2.2 that there PST between $C_1$ and $C_2$ in $\widehat{X/\pi}$ with minimum time $\tau = \frac{\pi}{2^{\frac{a}{2}+1}g}$, where $g = \gcd\left(2^{p-\frac{a}{2}}q + \sqrt{2^{2p-a}q^2+b}, 2^{p-\frac{a}{2}}q - \sqrt{2^{2p-a}q^2+b}\right)$. Finally, if $a = 2p$, then we can write Equation 6.2.3 as $s^2 + m = 2^{2p}(q^2+b)$. Using Equation 6.2.2, we obtain

$$\lambda^{\pm} = 2^{p+1}\left(q \pm \sqrt{q^2+b}\right).$$

Since $b$ and $q$ are odd, $\sqrt{q^2+b}$ is even, and hence $q \pm \sqrt{q^2+b}$ is odd. Thus, $\nu_2(\lambda^{\pm}) = p$. By Theorem 5.2.2, we obtain PST between $C_1$ and $C_2$ in $\widehat{X/\pi}$ with minimum time $\tau = \frac{\pi}{2^{p+1}g}$, where $g = \gcd\left(q+\sqrt{q^2+b}, q-\sqrt{q^2+b}\right)$. We also remark that the result for the case when $m$ is odd concides with that of $a < 2p$ if $s$ is even and $a = 2p$ if $s$ is odd.

Note that (4) is an immediate consequence of Corollary 5.3.4.

Let us show (5). Suppose $k > 0$ and $\Delta^2 = k^2 + 8n$ is not a perfect square. If $\ell_1, \ell_2$ are integers such that $\ell_1\lambda^+ + \ell_2\lambda^- = 0$, then we obtain

$$\ell_1\lambda^+ + \ell_2\lambda^- = \frac{1}{2}\ell_1(k+\Delta) + \frac{1}{2}\ell_2(k-\Delta) = \frac{1}{2}(\ell_1+\ell_2)k + \frac{1}{2}(\ell_1-\ell_2)\Delta = 0. \quad (6.2.4)$$

But since $\Delta$ is irrational, Equation 6.2.4 is possible if and only if $\ell_1 = \ell_2$. Thus, $k\ell_1 = 0$, and so $\ell_1 = \ell_2 = 0$. That is, $\lambda^+$ and $\lambda^-$ are linearly independent over $\mathbb{Q}$, and hence, by Corollary 2.4.2, there is PGST between $C_1$ and $C_2$ in $\widehat{X/\pi}$.

Lastly, taking $\theta = \omega - \eta = 0$ in Theorem 5.4.2, we get $\alpha, \beta$)-FR between $u$ and $v$ at time $t$ if and only if $\alpha = e^{i\gamma}\cos\gamma$, $\beta = ie^{i\gamma}\sin\gamma$, and

$$t\lambda^+ = 2\gamma + 2r\pi \text{ and } t\lambda^- = 2\gamma + 2q\pi$$

for some integers $r$ and $q$. Equivalently,

$$t = \frac{2(\gamma+r\pi)}{\lambda^+} = \frac{2(\gamma+q\pi)}{\lambda^-} \iff \gamma = \frac{(r\lambda^- - q\lambda^+)\pi}{\lambda^+ - \lambda^-}. \quad (6.2.5)$$

Since $\lambda^+ - \lambda^- = \sqrt{k^2+8n} > 0$, solving for $t$ gives us

$$t = \frac{2(r-q)\pi}{\sqrt{k^2+8n}}. \quad (6.2.6)$$

In order to get a positive value for $t$, we restrict $r - q$ to be positive. In particular, the minimum $(\alpha,\beta)$-FR time $\tau$ occurs whenever $r = 1$ and $q = 0$, in which case $\tau = \frac{2\pi}{\sqrt{k^2+8n}}$ and $\gamma = \frac{(k-\sqrt{k^2+8n})\pi}{2\sqrt{k^2+8n}}$. Taking $q = 0$ in Equation 6.2.5, and then applying Theorem 5.4.2 yields $\gamma = \frac{r(k-\sqrt{k^2+8n})\pi}{2\sqrt{k^2+8n}}$. We obtain a proper $(\alpha,\beta)$-FR between $C_1$

and $C_2$ in $\widehat{X/\pi}$ if and only if $\frac{r(k-\sqrt{k^2+8n})}{2\sqrt{k^2+8n}} \neq \frac{\ell}{2}$ for any integer $\ell$, or equivalently, $\frac{rk}{\sqrt{k^2+8n}}$ is not an integer. Moreover, we obtain a balanced $(\alpha, \beta)$-FR between $C_1$ and $C_2$ in $\widehat{X/\pi}$ if and only if

$$\frac{r(k-\sqrt{k^2+8n})}{2\sqrt{k^2+8n}} = \frac{\ell}{4}$$

for some odd integer $\ell$. Equivalently,

$$2rk = (2r+\ell)\sqrt{k^2+8n} \tag{6.2.7}$$

so that $k^2+8n$ is an even perfect square because $\ell$ is odd. In particular, since $k$ and $n$ must be even so that $k^2+8n$ is even, we must have $\nu_2(\sqrt{k^2+8n}) \geq 2$. Applying Lemma 6.1.7 yields two cases.

- If $k = 4s+2$, $n = 2m$ and $2T = s^2+s+m$, where $T$ is a triangular number, then $\sqrt{k^2+8n} = 2\sqrt{8T+1}$. Since $8T+1$ is odd, we have $\nu_2(\sqrt{k^2+8n}) = 1$, which is a contradiction, and so balanced $(\alpha, \beta)$-FR does not occur between $C_1$ and $C_2$ in $\widehat{X/\pi}$.

- If $k = 4s$, $n = 2m$ and $s^2+m$ is a perfect square, then $\sqrt{k^2+8n} = 4\sqrt{s^2+m}$. Using this, we can rewrite Equation 6.2.7 as $2rs = (2r+\ell)\sqrt{s^2+m}$. Thus, $\sqrt{s^2+m}$ is even, so that $s$ and $m$ have the same parity. Further, assume $s$ and $m$ are odd so that $\sqrt{s^2+m}$ is even and $s-\sqrt{s^2+m}$ is odd. Taking $r = \frac{1}{2}\sqrt{s^2+m}$ and $\ell = s-\sqrt{s^2+m}$, then Equation 6.2.7 holds, and hence, there is balanced $(\alpha, \beta)$-FR between $C_1$ and $C_2$ in $\widehat{X/\pi}$, which, by Equation 6.2.6, occurs at time $\tau = \frac{2r\pi}{\sqrt{k^2+8n}} = \frac{\pi}{4}$.

Finally, since $C_1 = \{u\}$ and $C_2 = \{v\}$ are singleton cells in $\pi$, applying Theorem 3.1.7 yields the desired result. □

Some results in Theorem 6.2.1(6) were first observed by Chan et al. in [16, Example 6.3]. Moreover, in Theorem 6.2.1(6), balanced $(\alpha, \beta)$-FR between $u$ and $v$ can be achieved by taking different values for $q$ in Equation 6.2.5. Next, we state a fact from [4, Corollary 13], which was derived from a theorem about unweighted joins of regular graphs.

**Corollary 6.2.2.** *For any unweighted $k$-regular graph $H$ on $n$ vertices, the apexes of $\overline{K_2} \vee H$ exhibit perfect state transfer if $\Delta = \sqrt{k^2+8n}$ is an integer and $k, \Delta \equiv 0$ (mod 4) and $\nu_2(k) \neq \nu_2(\Delta)$.*

We remark that Corollary 6.2.2 does not cover the case $k = 0$, in which case we have perfect state transfer between the apexes of $\overline{K_2} \vee H$ by Theorem 6.2.1, regardless of whether $\Delta = \sqrt{2n}$ is an integer or not. Now, assume $\Delta = \sqrt{k^2 + 8n}$ is an integer and $k, \Delta \equiv 0 \pmod 4$ and $\nu_2(k) \neq \nu_2(\Delta)$. We show that PST occurs between the apexes of $\overline{K_2} \vee H$ using Theorem 6.2.1. Since $k, \Delta \equiv 0 \pmod 4$, we get that $k = 4s$ and $\Delta = 4\ell$ for some integers $s$ and $\ell$. Then $\Delta^2 = k^2 + 8n = 16s^2 + 8n = 16\ell^2$, which implies that $n = 2m$ and $\ell = \sqrt{s^2 + m}$. Let $s = 2^p q$ and $\ell = 2^z r$ for some odd integers $q$ and $r$. Then $\Delta = 4\ell = 2^{z+2} r$, and since $k = 4s = 2^{p+2} q$ and $\nu_2(k) \neq \nu_2(\Delta)$, we know that $p \neq z$. Now,

$$m = \ell^2 - s^2 = 2^{2z} r^2 - 2^{2p} q^2 = \begin{cases} 2^{2p} \left(2^{2(z-p)} r^2 - q^2\right), & \text{if } z > p \\ 2^{2z} \left(r^2 - 2^{2(p-z)} q^2\right), & \text{if } z < p. \end{cases}$$

Thus, if $z > p$, then $\nu_2(m) = 2p$, while if $z < p$, then $\nu_2(m) = 2z < 2p$. In both cases, Theorem 6.2.13(b) yields PST between the apexes of $\overline{K_2} \vee H$. Moreover, if $\nu_2(k) = \nu_2(\Delta)$, then $p = z$, and hence $m = \ell^2 - s^2 = 2^{2z} r^2 - 2^{2p} q^2 = 2^{2p}(r^2 - q^2)$. Since $r^2 - q^2$ is even, we get that $\nu_2(m) \geq 2p + 1$, and so we get no PST between the apexes of $\overline{K_2} \vee H$ by Theorem 6.2.13(b). In other words, the converse of Corollary 6.2.2 is true provided $k \neq 0$. This yields the following characterization of PST between the apexes of $\overline{K_2} \vee H$, which is a more succinct version of Theorem 6.2.1(2,3).

**Corollary 6.2.3.** *For any unweighted $k$-regular graph $H$ on $n$ vertices, the apexes of $\overline{K_2} \vee H$ exhibit perfect state transfer if and only if either (i) $k = 0$, or (ii) $\Delta = \sqrt{k^2 + 8n}$ is an integer and $k, \Delta \equiv 0$ (mod 4) and $\nu_2(k) \neq \nu_2(\Delta)$.*

## 6.3 Connected double cones

Next, let us look at the apexes of connected double cones. Unlike the unweighted disconnected double cone, we deal with the case $k = n - 1$ separately. The next proposition follows directly from Theorem 6.1.3.

**Proposition 6.3.1.** *Let $H$ be an unweighted $(n-1)$-regular graph on $n$ vertices. The following hold for the apexes $u$ and $v$ of $K_2 \vee H$.*

1. *Vertices $u$ and $v$ are not strongly cospectral.*

2. *Perfect state transfer, pretty good state transfer, and fractional revival does not occur between $u$ and $v$.*

3. *Vertices u and v are periodic with minimum period* $\rho = \frac{2\pi}{n+2}$.

Next, we deal with the more general case where $k \neq n-1$.

**Theorem 6.3.2.** *Let $H$ be an unweighted $k$-regular graph on $n$ vertices, where $k \neq n-1$. Then the following hold for the apexes $u$ and $v$ of $K_2 \vee H$.*

1. *Vertices $u$ and $v$ are periodic if and only if $(k-1)^2+8n$ is a perfect square, with minimum period $\rho = \frac{2\pi}{g}$, where $g = gcd\left(\frac{1}{2}\left(\kappa+2+\sqrt{\kappa^2+8n}\right), \frac{1}{2}\left(\kappa+2-\sqrt{\kappa^2+8n}\right)\right)$.*

2. *Perfect state transfer occurs between $u$ and $v$ if and only if $k-1=4s$, $n=2m$, $s^2+m$ is a perfect square, where $m = 2^a b - 1$ and $s = 2^p q - 1$ for some integers $a$ and $p$ with $a, p \geq 0$ and odd integers $b$ and $q$ such that one of the following conditions hold.*

   (a) *$a < p+1$, where $a \geq 0$ is even, with minimum time $\tau = \frac{\pi}{2^{\frac{a}{2}+1} g}$, where $g = gcd\left(2^{p-\frac{a}{2}}q + \sqrt{2^{2p-a}q^2 - 2^{p-a+1}q + b}, 2^{p-\frac{a}{2}}q - \sqrt{2^{2p-a}q^2 - 2^{p-a+1}q + b}\right)$.*

   (b) *$a = p+1$, where $p \geq 3$ is odd, and $y \neq \frac{p-1}{2}$, where $y$ is an integer such that $2^y z = \sqrt{2^{p-1}q^2 - q + b}$ for some odd integer $z$, with minimum time*

      i. *$\tau = \frac{\pi}{2^{p+1} g}$, where $g = gcd\left(q + 2^{y-\frac{p-1}{2}}z, q - 2^{y-\frac{p-1}{2}}z\right)$, if $y > \frac{p-1}{2}$; or*

      ii. *$\tau = \frac{\pi}{2^{\frac{2y+p+1}{2}} g}$, where $g = gcd\left(2^{y-\frac{p-1}{2}}q + z, 2^{y-\frac{p-1}{2}}q - z\right)$, if $y < \frac{p-1}{2}$.*

   (c) *$a > p+1$, where $p \geq 1$ is odd, with minimum time $\tau = \frac{\pi}{2^{\frac{p+3}{2}} g}$, where $g = gcd\left(2^{\frac{p-1}{2}}q + \sqrt{2^{p-1}q^2 - q + 2^{a-p-1}b}, 2^{\frac{p-1}{2}}q - \sqrt{2^{p-1}q^2 - q + 2^{a-p-1}b}\right)$.*

3. *If $(k-1)^2+8n$ is a perfect square, then proper pretty good state transfer does not occur between $u$ and $v$.*

4. *If $(k-1)^2+8n$ is not a perfect square, then pretty good state transfer occurs between $u$ and $v$.*

5. *$(\alpha, \beta)$-fractional revival occurs between $u$ and $v$ at time $\tau$ if and only if $\alpha = e^{i\zeta}\cos\gamma$ and $\beta = ie^{i\zeta}\sin\gamma$, where*

$$\gamma = \frac{r\left(k+4-\sqrt{(k-1)^2+8n}\right) - q\left(k+4+\sqrt{(k-1)^2+8n}\right)}{2\sqrt{(k-1)^2+8n}}\pi$$

   *and*

$$\zeta = \frac{r\left(k-\sqrt{(k-1)^2+8n}\right) - q\left(k+\sqrt{(k-1)^2+8n} + p\sqrt{(k-1)^2+8n}\right)}{2\sqrt{(k-1)^2+8n}}\pi$$

*for some integers p, r, and q such that $r > q$, in which case $\tau = \frac{2(r-q)\pi}{\sqrt{(k-1)^2+8n}}$. In particular, the minimum $(\alpha, \beta)$-fractional revival time is*

$$\tau = \begin{cases} \frac{2\pi}{\sqrt{(k-1)^2+8n}}, & \text{if } \gamma - \zeta \neq 0 \\ 2\pi, & \text{if } \gamma - \zeta = 0, \end{cases}$$

*which is achieved if and only if $r - q = 1$. If $\zeta = 0$ and $q = 0$, then proper $(\alpha, \beta)$-fractional revival occurs between u and v if and only if $\frac{r-p\sqrt{(k-1)^2+8n}}{\sqrt{(k-1)^2+8n}}$ is not an integer, while balanced $(\alpha, \beta)$-fractional revival occurs between u and v if and only if*

$$8r = (8p + \ell)\sqrt{(k-1)^2 + 8n} \tag{6.3.1}$$

*for some odd integer $\ell$, in which case $k - 1 = 4s$ and $n = 2m$ such that $s^2 + m$ is a perfect square, and s and m have the same parity. In particular, if $p, \ell$ and r are chosen such that $r = \frac{1}{2}(8p + \ell)\sqrt{s^2 + m}$, then we get balanced $(\alpha, \beta)$-FR between u and v at $\tau = (8p + \ell)\pi$.*

*Proof.* Let $H$ be an unweighted $k$-regular graph on $n$ vertices, where $k \neq n - 1$, and $X = K_2 \vee H$ with apexes $u$ and $v$. Define $\kappa = k - 1$ so that $\kappa \in \{-1, 0, 1, \ldots, n-2\}$. Using the equitable partition $\pi$ for $X$ in Lemma 6.1.5(1), we obtain a connected weighted graph $\widehat{X/\pi}$ with $\phi(A(\widehat{X/\pi}), t) \in \mathbb{Z}[x]$. From Equation 6.1.4, we know that the eigenvalues of $A(\widehat{X/\pi})$ are $\lambda^{\pm}$ and $-1$, where $\lambda^{\pm} = \frac{1}{2}\left(\kappa + 2 \pm \sqrt{\kappa^2 + 8n}\right)$. Moreover, $C_1 = \{u\}$ and $C_2 = \{v\}$ of $\pi$ are strongly cospectral in $\widehat{X/\pi}$ with $\sigma_{C_1}(A(\widehat{X/\pi})) = \sigma(A(\widehat{X/\pi}))$.

We prove (1). In order for $C_1$ and $C_2$ to be periodic, the ratio condition requires both $\lambda^{\pm}$ to be either integers or of the form $-1 + a_j\sqrt{\Delta}$ for some square-free integer $\Delta$. If $\kappa^2 + 8n$ is not a perfect square, then $\lambda^{\pm}$ cannot assume the form $-1 + a_j\sqrt{\Delta}$ as $\frac{1}{2}(\kappa + 2) \neq -1$ for every $\kappa$. Thus, $C_1$ and $C_2$ are periodic in $\widehat{X/\pi}$ if and only if $\kappa^2 + 8n$ is a perfect square, in which case $\lambda^{\pm}$ are both integers. Theorem 5.1.5 then yields the minimum period $\rho$ for both $C_1$ and $C_2$ as $\rho = \frac{2\pi}{g}$, where $g = \gcd\left(\frac{1}{2}\left(\kappa + 2 + \sqrt{\kappa^2 + 8n}\right), \frac{1}{2}\left(\kappa + 2 - \sqrt{\kappa^2 + 8n}\right)\right)$. Thus, (1) holds.

Next, we show (2). For PST to occur between $C_1$ and $C_2$ in $\widehat{X/\pi}$, both cells must be periodic, which implies that $\kappa^2 + 8n$ is a perfect square. We proceed with cases by virtue of Lemma 6.1.7.

- Let $\kappa$ be odd. Then $\kappa^2 + 8n$ is odd, and hence $\sqrt{\kappa^2 + 8n}$ is also odd. If we write $\kappa = 2r + 1$ and $\kappa^2 + 8n = 2s + 1$, then

$$\lambda^{\pm} = \tfrac{1}{2}(\kappa + 2 \pm \sqrt{\kappa^2 + 8n}) = \tfrac{1}{2}((2r+3) \pm (2s+1))$$

which gives us $\lambda^+ = r+s+2$ and $\lambda^- = r-s+1$. Since $r \pm s$ have the same parity, it follows that $\lambda^{\pm}$ have opposite parities. Thus, $\nu_2(\lambda^+ + 1) \neq \nu_2(\lambda^- + 1)$, and by Theorem 5.2.2, there is no PST between $C_1$ and $C_2$ in $\widehat{X/\pi}$.

- Let $\kappa = 4s+2$, $n = 2m$ and $2T = s^2 + s + m$, where $T$ a triangular number. Then $\kappa^2 + 8n = 4(8T+1)$ is a perfect square so that

$$\lambda^{\pm} = \tfrac{1}{2}((4s+2) + 2 \pm 2\sqrt{8T+1}) = 2s + 2 \pm \sqrt{8T+1}.$$

If we write $\sqrt{8T+1} = 2r+1$, then $\lambda^+ = 2(s+r)+3$ and $\lambda^- = 2(s-r)-1$. That is, $\lambda^+ + 1 = 2(s+r+1)$ and $\lambda^- + 1 = 2(s-r)$. Since $s \pm r$ have the same parity, $s+r+1$ and $s-r$ have opposite parities, and so $\lambda^+$ and $\lambda^-$ also have opposite parities. Consequently, $\nu_2(\lambda^+ + 1) \neq \nu_2(\lambda^- + 1)$, and by Theorem 5.2.2, there is no PST between $C_1$ and $C_2$ in $\widehat{X/\pi}$.

- Let $\kappa = 4s$, $n = 2m$ and $s^2 + m$ be a perfect square so that $\kappa^2 + 8n = 16(s^2+m)$ is a perfect square. Then $\lambda^{\pm} = \tfrac{1}{2}(4s + 2 \pm 4\sqrt{s^2+m}) = 2s + 1 \pm 2\sqrt{s^2+m})$. Consequently,

$$\lambda^{\pm} + 1 = 2(s + 1 \pm \sqrt{s^2+m}) \tag{6.3.2}$$

  - Suppose $m$ is odd and $s$ is even so that $\sqrt{s^2+m}$ is odd. If we write $s = 2z$ and $\sqrt{s^2+m} = 2y+1$, then $\lambda^{\pm} + 1 = 2(2z + 1 \pm (2y+1))$, which gives us $\lambda^+ + 1 = 4(y+z+1)$ and $\lambda^+ + 1 = 4(y-z)$. Since $y \pm z$ have the same parity, $y+z+1$ and $y-z$ have opposite parities, which implies $\nu_2(\lambda^+ + 1) \neq \nu_2(\lambda^+ - 1)$, there is no PST between $C_1$ and $C_2$ in $\widehat{X/\pi}$ by Theorem 5.2.2.
  - Suppose $m$ is even. If $s$ is odd, then $\sqrt{s^2+m}$ is odd, and so $s+1 \pm \sqrt{s^2+m}$ is odd, while if $s$ is even, then $\sqrt{s^2+m}$ is even, and so $s+1 \pm \sqrt{s^2+m}$ is again odd. Thus, if $m$ is even, then $\nu_2(\lambda^{\pm} + 1) = 1$, and hence PST occurs between $C_1$ and $C_2$ in $\widehat{X/\pi}$ by Theorem 5.2.2, with minimum PST time $\tau = \frac{\pi}{2g}$, where $g = \gcd\left(s + 1 + \sqrt{s^2+m}, s + 1 - \sqrt{s^2+m}\right)$.
  - Suppose $m$ and $s$ are odd. Then we can write $m + 1 = 2^a b$ and $s + 1 = 2^p q$ for some integers $a$ and $p$ with $a, p \geq 1$ and odd integers $b$ and $q$, which yields

$$s^2 + m = 2^{2p} q^2 - 2^{p+1} q + 2^a b$$

If $a < p+1$, then we can write $s^2 + m = 2^a(2^{2p-a}q^2 - 2^{p-a+1}q + b)$, where $2^{2p-a}q^2 - 2^{p-a+1}q + b$ is odd because $b$ is odd. Since $s^2 + m$ is a perfect square, we get that $a \geq 2$ is even, and Equation 6.3.2 yields

$$\lambda^{\pm} + 1 = 2^{\frac{a}{2}+1}\left(2^{p-\frac{a}{2}}q \pm \sqrt{2^{2p-a}q^2 - 2^{p-a+1}q + b}\right)$$

Since $p - \frac{a}{2} > 0$, $2^{p-\frac{a}{2}}q$ is even, and so $2^{p-\frac{a}{2}}q \pm \sqrt{2^{2p-a}q^2 - 2^{p-a+1}q + b}$ is odd. Thus, $\nu_2(\lambda^{\pm} + 1) = \frac{a}{2} + 1$, and consequently, we get PST between $C_1$ and $C_2$ in $\widehat{X/\pi}$ by Theorem 5.2.2, with minimum time $t = \frac{\pi}{2^{\frac{a}{2}+1}g}$, where $g = \gcd\left(2^{p-\frac{a}{2}}q + \sqrt{2^{2p-a}q^2 - 2^{p-a+1}q + b}, 2^{p-\frac{a}{2}}q - \sqrt{2^{2p-a}q^2 - 2^{p-a+1}q + b}\right)$. We also note that the case when $m$ is even, in which case $a = 0$, coincides with that of $a < p + 1$.

If $a = p + 1$, then Equation 6.3.2 gives us

$$\lambda^{\pm} + 1 = 2\left(2^p q \pm 2^{\frac{p+1}{2}}\sqrt{2^{p-1}q^2 - q + b}\right),$$

and so $p$ is odd. In particular, if $p = 1$, then $\lambda^{\pm}+1 = 4\left(q \pm \sqrt{q^2 - q + b}\right)$, where $\sqrt{q^2 - q + b}$ is odd because $b$ and $q$ are odd. Let $q = 2y + 1$ and $\sqrt{q^2 - q + b} = 2z + 1$. Then $\lambda^{\pm} + 1 = 4\left(2y + 1 \pm (2z + 1)\right)$, and this yields $\lambda^{+} + 1 = 8(y + z + 1)$ and $\lambda^{-} - 1 = 8(y - z)$. Since $y \pm z$ have the same parity, it follows that $y+z+1$ and $y-z$ have opposite parities. This means that $\nu_2(\lambda^{+}+1) \neq \nu_2(\lambda^{-}-1)$, and so there is no PST between $C_1$ and $C_2$ in $\widehat{X/\pi}$ by Theorem 5.2.2. On the other hand, if $p \geq 3$, then $2p > p + 1$, and so

$$\lambda^{\pm} + 1 = 2^{\frac{p+3}{2}}\left(2^{\frac{p-1}{2}}q \pm \sqrt{2^{p-1}q^2 - q + b}\right)$$

Since $b$ and $q$ are odd, we get that $\sqrt{2^{p-1}q^2 - q + b}$ is even. Let us write $\sqrt{2^{p-1}q^2 - q + b} = 2^y z$ for some odd $z$. Then $\lambda^{\pm}+1 = 2^{\frac{p+3}{2}}\left(2^{\frac{p-1}{2}}q \pm 2^y z\right)$. We have three subcases, namely, $y > \frac{p-1}{2}$, $y < \frac{p-1}{2}$ and $y = \frac{p-1}{2}$. First, suppose $y > \frac{p-1}{2}$. Then $\lambda^{\pm} + 1 = 2^{p+1}\left(q \pm 2^{y-\frac{p-1}{2}}z\right)$, where $q \pm 2^{y-\frac{p-1}{2}}z$ is odd. This gives us $\nu_2(\lambda^{\pm} + 1) = p + 1$, and applying Theorem 5.2.2 yields PST between $C_1$ and $C_2$ in $\widehat{X/\pi}$ with minimum PST time $\tau = \frac{\pi}{2^{p+1}g}$, where $g = \gcd\left(q + 2^{y-\frac{p-1}{2}}z, q - 2^{y-\frac{p-1}{2}}z\right)$. For the case that $y < \frac{p-1}{2}$, we have that $\lambda^{\pm} + 1 = 2^{\frac{p+3}{2}} \cdot 2^y\left(2^{\frac{p-1}{2}y}q \pm z\right)$, where $2^{y-\frac{p-1}{2}}q \pm z$ is odd. Consequently, $\nu_2(\lambda^{\pm} + 1) = 2^{\frac{2y+p+1}{2}}$, and we get PST between $C_1$ and $C_2$

in $\widehat{X/\pi}$ by Theorem 5.2.2, with minimum PST time $\tau = \frac{\pi}{2^{\frac{2y+p+1}{2}}g}$, where $g = \gcd\left(2^{y-\frac{p-1}{2}}q + z, 2^{y-\frac{p-1}{2}}q - z\right)$. Lastly, suppose $y = \frac{p-1}{2}$. Then we get $\lambda^{\pm} + 1 = 2^{\frac{p+3}{2}} \cdot 2^{\frac{p-1}{2}}(q \pm z)$. Since $q$ and $z$ are odd, it follows that $q \pm z$ are even. However, because $\nu_2(q+z) \neq \nu_2(q-z)$, it follows that $\nu_2(\lambda^+ + 1) \neq \nu_2(\lambda^- + 1)$, and hence, there no PST between $C_1$ and $C_2$ in $\widehat{X/\pi}$ by Theorem 5.2.2.

Finally, if $a > p+1$, then $s^2 + m = 2^{p+1}(2^{p-1}q^2 - q + 2^{a-p-1}b)$ so that

$$\lambda^{\pm} + 1 = 2\left(2^p q \pm 2^{\frac{p+1}{2}}\sqrt{2^{p-1}q^2 - q + 2^{a-p-1}b}\right)$$

and $p$ is odd. In particular, if $p = 1$, then $\lambda^{\pm}+1 = 4\left(q \pm \sqrt{q^2 - q + 2^{a-2}b}\right)$. Since $a > 2$ and $q$ is odd, we get that $q \pm \sqrt{q^2 - q + 2^{a-2}b}$ is odd. Thus, $\nu_2(\lambda^{\pm} + 1) = 2$, and therefore there is PST between $C_1$ and $C_2$ in $\widehat{X/\pi}$ by Theorem 5.2.2, with minimum PST time $\tau = \frac{\pi}{4g}$, where $g = \gcd\left(q + \sqrt{q^2 - q + 2^{a-2}b}, q - \sqrt{q^2 - q + 2^{a-2}b}\right)$. On the other hand, if $p \geq 3$, then $2p > p+1$, and we get

$$\lambda^{\pm} + 1 = 2^{\frac{p+3}{2}}\left(2^{\frac{p-1}{2}}q \pm \sqrt{2^{p-1}q^2 - q + 2^{a-p-1}b}\right)$$

Since $q$ is odd, it follows that $2^{\frac{p-1}{2}}q \pm \sqrt{2^{p-1}q^2 - q + 2^{a-p-1}b}$ is also odd. Therefore, $\nu_2(\lambda^{\pm}+1) = \frac{p+3}{2}$, and therefore there is PST between $C_1$ and $C_2$ in $\widehat{X/\pi}$ by Theorem 5.2.2, with minimum PST time $\tau = \frac{\pi}{2^{\frac{p+3}{2}}g}$, where $g = \gcd\left(2^{\frac{p-1}{2}}q + \sqrt{2^{p-1}q^2 - q + 2^{a-p-1}b}, 2^{\frac{p-1}{2}}q - \sqrt{2^{p-1}q^2 - q + 2^{a-p-1}b}\right)$.

Note that (4) is an immediate consequence of Corollary 5.3.4.

Let us show (5). Assume $\Delta^2 = \kappa^2 + 8n$ is not a perfect square. If $\ell_1, \ell_2$ are integers such that $\ell_1\lambda^+ + \ell_2\lambda^- = 0$, then we obtain

$$\ell_1\lambda^+ + \ell_2\lambda^- = \frac{1}{2}\ell_1(\kappa+2+\Delta) + \frac{1}{2}\ell_2(\kappa+2-\Delta) = \frac{1}{2}(\ell_1+\ell_2)(\kappa+2) + \frac{1}{2}(\ell_1-\ell_2)\Delta = 0. \tag{6.3.3}$$

But because $\Delta$ is irrational, Equation 6.3.3 is possible if and only if $\ell_1 = \ell_2$. Thus, $k\ell_1 = 0$, and so $\ell_1 = \ell_2 = 0$. That is, $\lambda^+$ and $\lambda^-$ are linearly independent over $\mathbb{Q}$, and hence, by Corollary 2.4.2, there is PGST between $C_1$ and $C_2$ in $\widehat{X/\pi}$.

Lastly, taking $\theta = \omega - \eta = -1$ in Theorem 5.4.2, we get $\alpha, \beta)$-FR between $u$ and

$v$ at time $t$ if and only if $\alpha = e^{i\zeta}\cos\gamma$, $\beta = ie^{i\zeta}\sin\gamma$,

$$t = \gamma - \zeta + 2p\pi > 0, \tag{6.3.4}$$

and

$$t(\lambda^+ + 1) = 2\gamma + 2r\pi \text{ and } t(\lambda^- + 1) = 2\gamma + 2q\pi$$

for some integers $p$, $r$, and $q$. Equivalently,

$$t = \frac{2(\gamma + r\pi)}{\lambda^+ + 1} = \frac{2(\gamma + q\pi)}{\lambda^- + 1} \iff \gamma = \frac{[r(\lambda^- + 1) - q(\lambda^+ + 1)]\,\pi}{\lambda^+ - \lambda^-}. \tag{6.3.5}$$

Since $\lambda^+ - \lambda^- = \sqrt{\kappa^2 + 8n} > 0$, solving for $t$ gives us

$$t = \frac{2(r - q)\pi}{\sqrt{\kappa^2 + 8n}}, \tag{6.3.6}$$

where we restrict $r - q > 0$ so that $t > 0$. Combining Equations 6.3.6 and 6.3.4, we obtain

$$\gamma - \zeta = \frac{\left(r - q - p\sqrt{\kappa^2 + 8n}\right) 2\pi}{\sqrt{\kappa^2 + 8n}}. \tag{6.3.7}$$

In particular, if $\gamma - \zeta > 0$, then we may take $p = 0$ so that $\gamma - \zeta = \frac{2(r-q)\pi}{\sqrt{\kappa^2+8n}}$. However, if $\gamma - \zeta = 0$, then $p > 0$ by Equation 6.3.4 so that $(\alpha, \beta)$-FR takes place at $t = 2p\pi$. And if $\gamma - \zeta < 0$, then by Equation 6.3.7, $p$ must satisfy $p\sqrt{\kappa^2 + 8n} + q - r > 0$, or equivalently, $p > \frac{r-q}{\sqrt{\kappa^2+8n}}$. Moreover, Equations 6.3.7 and 6.3.5 althogether yields

$$\zeta = \frac{\left(r(\lambda^- - 1) - q(\lambda^+ - 1) + 2p\sqrt{\kappa^2 + 8n}\right) \pi}{\sqrt{\kappa^2 + 8n}} \tag{6.3.8}$$

Now, the minimum $(\alpha, \beta)$-FR time $\tau$ is given by

$$\tau = \begin{cases} \frac{2\pi}{\sqrt{\kappa^2+8n}}, & \text{if } \gamma - \zeta \neq 0 \\ 2\pi, & \text{if } \gamma - \zeta = 0 \end{cases}$$

which is achieved if and only if $r - q = 1$. In particular, if $\zeta = 0$ and $q = 0$, then Equation 6.3.7 yields

$$\gamma = \frac{\left(r - p\sqrt{\kappa^2 + 8n}\right) 2\pi}{\sqrt{\kappa^2 + 8n}}.$$

Consequently, we obtain proper $(\alpha, \beta)$-FR between $C_1$ and $C_2$ in $\widehat{X/\pi}$ if and only if $\frac{2\left(r - p\sqrt{\kappa^2+8n}\right)}{\sqrt{\kappa^2+8n}} \neq \frac{\ell}{2}$ for any integer $\ell$, or equivalently, $\frac{r-p\sqrt{\kappa^2+8n}}{\sqrt{\kappa^2+8n}}$ is not an integer.

Moreover, we get balanced $(\alpha, \beta)$-FR between $C_1$ and $C_2$ in $\widehat{X/\pi}$ if and only if

$$\frac{2\left(r - p\sqrt{\kappa^2 + 8n}\right)}{\sqrt{\kappa^2 + 8n}} = \frac{\ell}{4}$$

for some odd integer $\ell$. Equivalently,

$$8r = (8p + \ell)\sqrt{\kappa^2 + 8n} \tag{6.3.9}$$

so that $\kappa^2 + 8n$ is an even perfect square because $\ell$ is odd. In particular, we have that $\nu_2(\sqrt{\kappa^2 + 8n}) \geq 3$. Applying Lemma 6.1.7 yields two cases.

- If $\kappa = 4s+2$, $n = 2m$ and $2T = s^2+s+m$, where $T$ is a triangular number, then $\sqrt{\kappa^2 + 8n} = 2\sqrt{8T + 1}$, and so $\nu_2(\sqrt{\kappa^2 + 8n}) = 1$, which is a contradiction. Thus, balanced $(\alpha, \beta)$-FR does not occur between $C_1$ and $C_2$ in $\widehat{X/\pi}$.
- If $\kappa = 4s$, $n = 2m$ and $s^2 + m$ is a perfect square, then $\sqrt{\kappa^2 + 8n} = 4\sqrt{s^2 + m}$. Using this, we can rewrite Equation 6.3.9 as $2r = (8p + \ell)\sqrt{s^2 + m}$. Thus, $\sqrt{s^2 + m}$ is even. Hence, if $p, \ell$ and $r$ with $r = \frac{1}{2}(8p + \ell)\sqrt{s^2 + m}$, Equation 6.3.9 holds, and hence, there is balanced $(\alpha, \beta)$-FR between $C_1$ and $C_2$ in $\widehat{X/\pi}$, which, by Equation 6.3.6, occurs at time $\tau = \frac{2r\pi}{\sqrt{\kappa^2+8n}} = (8p + \ell)\pi$.

Finally, since $C_1 = \{u\}$ and $C_2 = \{v\}$ are singleton cells in $\pi$, applying Theorem 3.1.7 yields the desired result. □

**Remark 6.3.3.** In Theorem 6.3.2(2bi), $g = \gcd(q + 2^{y-\frac{p-1}{2}}z, q - 2^{y-\frac{p-1}{2}}z)$ can be further simplified. If $d = \gcd(q, z)$, then $d|g$. However, since $q$ and $z$ are odd, and $y - \frac{p-1}{2} > 0$, it follows that $g$ is odd. Now, because $g|2q$ and $g|2^{y-\frac{p-1}{2}+1}z$, we find that $g|q$ and $g|z$, and so $g|d$. Consequently, $d = g$. The $g$'s in some statements of Theorem 6.3.2, as well as Theorem 6.2.1, can also be simplified in a similar manner.

We note in Theorem 6.2.1 that balanced $(\alpha, \beta)$-FR between $C_1$ and $C_2$ may also be achieved by taking different combinations of values for $\zeta$ and $q$ in Equation 6.3.7. Next, we state [4, Corollary 15].

**Corollary 6.3.4.** *For any unweighted $k$-regular graph $H$ on $n$ vertices, the apexes of $K_2 \vee H$ exhibit perfect state transfer if $\Delta = \sqrt{(k-1)^2 + 8n}$ is an integer and $k - 1, \Delta \equiv 0$ (mod 8).*

As in Corollary 6.3.4, assume $\Delta^2 = (k-1)^2 + 8n$ is an integer and $k - 1, \Delta \equiv 0$ (mod 8). Then we can write $k-1 = 4s$ for some even integer $s$ and $\Delta = 8r$ for integer

$r$. If $s = 2q$, then $64r^2 = \Delta^2 = (k-1)^2 + 8n = 64p^2 + 8n \iff 8(r^2 - q^2) = n$. That is, $n = 2m$ for some even $m$. Consequently, $\nu_2(s+1) = \nu_2(m+1) = 0$. Applying Theorem 6.3.2(2a) yields PST between the apexes of $K_2 \vee H$. Now, suppose $k-1 = 4s$ for odd $s$ so that $k - 1 \not\equiv 0 \pmod 8$. If we add that $n = 4r$ for any integer $r$, then $\Delta = \sqrt{16s^2 + 32r} = 4\sqrt{s^2 + 2r} \not\equiv 0 \pmod 8$ because $s$ is odd. Since $\nu_2(s+1) > 0$ and $\nu_2(m+1) = 0$, we have that $\nu_2(m+1) < \nu_2(s+1)$, and hence Theorem 6.3.2(2a) yields PST between the apexes of $K_2 \vee H$. This shows that Corollary 6.3.4 is a special case of Theorem 6.3.2(2).

## 6.4 The weighted case

Let $H$ be a positively weighted $k$-regular graph on $n$ vertices. For $\gamma > 0$ and $\omega, \eta \geq 0$, consider again the graph $X(\gamma) = \mathbf{K}_2(\omega, \eta) \vee H$ with apexes $u$ and $v$, which is also positively weighted. In this section, similar to the proof of [3, Corollary 2], we provide a parametrization of the weights $\omega$ and $\eta$ such that $u$ and $v$ exhibit periodicity and perfect state transfer in $X(\gamma)$ whenever $n$ and $\gamma$ are fixed. Now, since $u$ and $v$ are twins in $X$, Corollary 4.3.2 yields cospectrality between $u$ and $v$ in $X$, and by Proposition 5.1.4, $u$ is periodic if and only if $v$ is periodic, in which case both vertices have the same minimum period. Applying the equitable partition $\pi$ in Lemma 6.1.1 gives us a positively weighted graph $\widehat{X(\gamma)/\pi}$, where $\{u\}$ and $\{v\}$ are singleton cells in $\pi$. Corollary 3.2.20 then implies that $u$ is periodic in $X$ if and only if $\{u\}$ is periodic in $\widehat{X(\gamma)/\pi}$, and the minimum period of $u$ in $X$ is equal to the minimum period of $\{u\}$ in $\widehat{X(\gamma)/\pi}$. Thus, in order to assess periodicity between $u$ and $v$ in $X$, it suffices to examine the periodicity of the cells $\{u\}$ and $\{v\}$ in $\widehat{X(\gamma)/\pi}$.

Recall from Equation 6.1.2 that the eigenvalues of $\widehat{X(\gamma)/\pi}$ are

$$\lambda^{\pm} = \frac{1}{2}\left(k + \omega + \eta \pm \sqrt{(k - \omega - \eta)^2 + 8n\gamma^2}\right) \text{ and } \omega - \eta$$

Evidently, $\widehat{X(\gamma)/\pi}$ is positively weighted so that $\lambda^+$ is the largest eigenvalue of $\widehat{X(\gamma)/\pi}$. By Theorem 5.1.1 and Corollary 3.2.20, $u$ and $v$ are periodic in $X(\gamma)$ if and only if

$$\lambda^+ - \lambda^- = \sqrt{(k - \omega - \eta)^2 + 8n\gamma^2} = \ell\Delta \tag{6.4.1}$$

and

$$\lambda^+ - (\omega - \eta) = \frac{1}{2}\left(k - \omega + 3\eta + \sqrt{(k - \omega - \eta)^2 + 8n\gamma^2}\right) = m\Delta, \tag{6.4.2}$$

for some $0 < \Delta \in \mathbb{R}$ and positive integers $\ell$ and $m$ such that $\ell^2\Delta^2 - 8n\gamma^2 \geq 0$. By Theorem 5.1.2(2), the minimum period $\rho$ of both $u$ and $v$ is given by $\rho = \frac{2\pi}{g\Delta}$, where $g = \gcd(\ell, m)$. Now, define $\zeta = \sqrt{\ell^2\Delta^2 - 8n\gamma^2}$. Then Equation 6.4.1 yields

$$k - \omega - \eta = \mp\zeta. \tag{6.4.3}$$

Let $\eta^\pm = \frac{1}{4}\left[(2m-\ell)\Delta \pm \zeta\right]$. Making use of Equations 6.4.2 and 6.4.3, we obtain

$$\eta = \begin{cases} \eta^+, & \text{if } (2m-\ell)\Delta \geq -\zeta \\ \eta^-, & \text{if } (2m-\ell)\Delta \geq \zeta, \end{cases} \tag{6.4.4}$$

where the conditions in Equation 6.4.4 ensure that $\eta > 0$. Moreover, Equation 6.4.2 also gives us $\omega = k + 3\eta - (2m-\ell)\Delta$. Now, let $\omega^\pm = k - \frac{1}{4}(2m-\ell)\Delta \pm \frac{3}{4}\zeta$. Making use of Equation 6.4.4 yields

$$\omega = \begin{cases} \omega^+, & \text{if } 4k + 3\zeta \geq (2m-\ell)\Delta \text{ and } \eta = \eta^+ \\ \omega^-, & \text{if } 4k - 3\zeta \geq (2m-\ell)\Delta \text{ and } \eta = \eta^-, \end{cases} \tag{6.4.5}$$

where the conditions in Equation 6.4.5 ensure that $\omega > 0$. Since $n$ and $\gamma$ are fixed, Equations 6.4.5 and 6.4.4 yield the desired parametrizations of $\omega$ and $\eta$ that induce periodicity between $u$ and $v$ in $X(\gamma)$. In particular, since $(2m-\ell)\Delta \geq \zeta > -\zeta$ and $4k+3\zeta > 4k-3\zeta \geq (2m-\ell)\Delta$, if we take $\ell$, $m$ and $\Delta$ such that $4k-3\zeta \geq (2m-\ell)\Delta \geq \zeta$, then we may choose $\omega$ and $\eta$ such that either $\omega = \omega^+$ and $\eta = \eta^+$, or $\omega = \omega^-$ and $\eta = \eta^-$. Now, in order to guarantee PST between $u$ and $v$, we further assume that either $H \ncong \mathbf{K}_n(\omega, \eta)$ or $\eta \neq \gamma$ so that by Lemma 6.1.2, $\{u\}$ and $\{v\}$ are strongly cospectral in $\widehat{X(\gamma)/\pi}$, with $\lambda^\pm \in \sigma^+_{\{u\}}(A(\widehat{X(\gamma)/\pi}))$ and $\omega - \eta \in \sigma^-_{\{u\}}(A(\widehat{X(\gamma)/\pi}))$. Applying Theorem 2.2.2 to Equation 5.2.1, the occurence of PST between $u$ and $v$ is equivalent to existence of a time $\tau$ such that $e^{-i\lambda^+\tau} = e^{-i\lambda^-\tau} = -e^{-i(\omega-\eta)\tau}$. Equivalently,

$$(\lambda^+ - \lambda^-)t = 2\alpha\pi \text{ and } (\lambda^+ - \omega + \eta)t = \beta\pi, \tag{6.4.6}$$

where $\alpha$ is an integer and $\beta$ is odd. Using Equations 6.4.1 and 6.4.2, we can rewrite Equation 6.4.6 as

$$\ell\Delta t = 2\alpha\pi \text{ and } m\Delta t = \beta\pi. \tag{6.4.7}$$

That is, $t = \frac{\beta\pi}{m\Delta} = \frac{2\alpha\pi}{\ell\Delta}$ so that $\beta\ell = 2\alpha m$. Since $\beta$ is odd, this is equivalent to the condition that $\nu_2(\ell) > \nu_2(m)$. Indeed, if $\ell$ and $m$ are integers such that $\nu_2(\ell) > \nu_2(m)$, then we get PST between $u$ and $v$ with minimum time $\tau = \frac{\pi}{g\Delta}$, where $g = \gcd(\ell, m)$.

These observations yield the following theorem.

**Theorem 6.4.1.** *Let $H$ be a positively weighted $k$-regular graph on $n$ vertices, and $X(\gamma) = \boldsymbol{K}_2(\omega,\eta) \vee H$ be positively weighted with apexes $u$ and $v$ for a fixed $\gamma > 0$. Then $u$ and $v$ are periodic in $X(\gamma)$ if and only if*

$$\eta = \begin{cases} \eta^+, & \text{if } (2m-\ell)\Delta \geq -\zeta \\ \eta^-, & \text{if } (2m-\ell)\Delta \geq \zeta, \end{cases}$$

*and*

$$\omega = \begin{cases} \omega^+, & \text{if } 4k+3\zeta \geq (2m-\ell)\Delta \text{ and } \eta = \eta^+ \\ \omega^-, & \text{if } 4k-3\zeta \geq (2m-\ell)\Delta \text{ and } \eta = \eta^-, \end{cases}$$

*where $\eta^{\pm} = \frac{1}{4}\left[(2m-\ell)\Delta \pm \zeta\right]$ and $\omega^{\pm} = k - \frac{1}{4}(2m-\ell)\Delta \pm \frac{3}{4}\zeta$ for some positive real $\Delta$, and positive integers $\ell$ and $m$ such that $\ell^2\Delta^2 - 8n\gamma^2 \geq 0$, $\zeta = \sqrt{\ell^2\Delta^2 - 8n\gamma^2}$ and $4k + 3\zeta \geq (2m-\ell)\Delta \geq -\zeta$. These parameters yield the minimum period $\rho = \frac{2\pi}{g\Delta}$, where $g = \gcd(\ell, m)$. In particular, if $\ell$, $m$ and $\Delta$ are chosen such that $4k - 3\zeta \geq (2m-\ell)\Delta \geq \zeta$, then we may take $\omega$ and $\eta$ such that either $\omega = \omega^+$ and $\eta = \eta^+$, or $\omega = \omega^-$ and $\eta = \eta^-$. In addition, if either $H \neq \boldsymbol{K}_n(\omega,\eta)$ or $\eta \neq \gamma$, then perfect state state transfer occurs between $u$ and $v$ in $X(\gamma)$ if and only if $\nu_2(\ell) > \nu_2(m)$, with minimum time $\tau = \frac{\rho}{2}$.*

**Remarks.**

1. If we want $u$ and $v$ to be periodic in $X(\gamma)$ with period $\tau$, then it suffices to choose $\Delta = \frac{2\pi}{\tau}$, and positive integers $\ell$ and $m$ such that $\ell^2\Delta^2 - 8n\gamma^2 \geq 0$. Moreover, if we want PST between $u$ and $v$, then we add that $\nu_2(\ell) > \nu_2(m)$, in which case PST occurs at $\frac{\tau}{2}$.

2. The weights $\omega$ and $\eta$ are functions of $\ell$, $m$, $\Delta$, $n$ and $\gamma$, whereas the minimum period and minimum PST time are functions of $\ell$, $m$ and $\Delta$ only.

3. The weights $\omega$ and $\eta$ can be taken to be either be algebraic or transcendental over $\mathbb{Q}$. In particular, assuming $\gamma$ is algebraic, $\omega$ and $\eta$ are algebraic whenever $\Delta$ is algebraic, while $\omega$ and $\eta$ are transcendental whenever $\Delta$ is transcendental.

4. A shorter minimum PST time is achieved by choosing $\ell$ and $m$ such that $g = \gcd(\ell, m)$ is larger, or by taking larger values of $\Delta$. In particular, if $\Delta$ is algebraic (resp., transcendental) over $\mathbb{Q}$, then the periods and PST times between $u$ and $v$ are algebraic (resp., transcendental) multiples of $\pi$.

5. If $\ell = 2m$, then we obtain $\eta^{\pm} = \frac{1}{4}\zeta$ and $w^{\pm} = k \pm \frac{3}{4}\zeta$. In particular, if $\zeta = 0$, then $\eta = 0$ and $\omega = k$, while if $\zeta = \frac{4}{3}k$, then $\eta = \frac{1}{3}k$ and $\omega = 0$, and if $\zeta = 4$, then $\eta = 1$ and $\omega = k + 3$.

**Example 6.4.2.** Let $n \geq 3$, and consider $X(1) = \mathbf{K}_2(\omega, \eta) \vee C_n$ with apexes $u$ and $v$. As we know, $C_n$ is a 2-regular graph. Now, assume $\ell = 2$ and $m = 1$. Applying Theorem 6.4.1, we get the following values for the weights $\eta$ and $\omega$ given our choice of $\Delta$.

- If $\Delta = \sqrt{2n}$, then $\eta = 0$ and $\omega = 2$.
- If $\Delta = \sqrt{2\left(n + \frac{8}{9}\right)}$, then $\eta = \frac{2}{3}$ and $\omega = 0$.
- If $\Delta = \pi\sqrt{2n}$. Then $\eta = \frac{1}{4}\sqrt{(\pi^2 - 1)8n}$ and $\omega = 2 + \frac{3}{4}\sqrt{(\pi^2 - 1)8n}$.

These weights induce PST between $u$ and $v$ in $X(1)$ with minimum PST time $\tau = \frac{\pi}{\Delta}$. Notice that in (1) and (2), the weights are algebraic, and the PST time is an algebraic multiple of $\pi$. However, in (3), the weights are transcendental, and the PST time is a transcendental multiple of $\pi$.

Now, observe that if we take $\Delta > \sqrt{2n}$, $\ell = 2$ and $m = 1$, then Theorem 6.4.1 yields the parameters $\eta = \frac{1}{2}\sqrt{\Delta^2 - 2n}$ and $\omega = k + \frac{3}{4}\sqrt{\Delta^2 - 2n}$ that induce perfect state transfer between the apexes of $X(\gamma) = \mathbf{K}_2(\omega, \eta) \vee H$ with minimum time $\tau = \frac{\pi}{\Delta}$. Consequently, we can make the PST time shorter by taking larger values of $\Delta$. We state this observation in the following corollary.

**Corollary 6.4.3.** *Let $H$ be a positively weighted $k$-regular graph on $n$ vertices, and $\gamma > 0$ be fixed. For every $\epsilon > 0$, there exists positive weights $\omega$ and $\eta$ such that induce perfect state transfer occurs between the apexes of $X(\gamma) = \boldsymbol{K}_2(\omega, \eta) \vee H$ with minimum time $\tau < \epsilon$.*

We remark that the downside of taking larger values of $\Delta$ is that they also yield larger values for the weights $\omega$ and $\eta$, which might not be realizable in practice.

Now, when the underlying regular graph is unweighted, Theorem 6.4.1 guarantees the existence of weights $\omega$ and $\eta$ such that adding loops of weight $\omega$ on the apexes, as well as altering the unit weight of the edge between them to $\eta$ results in periodicity and PST between the apexes in the resulting weighted graph. We state this as a corollary.

**Corollary 6.4.4.** *Let $H$ be an unweighted $k$-regular graph on $n$ vertices. Then there exist positive weights $\omega$ and $\eta$ such that the apexes of $X(1) = \boldsymbol{K}_2(\omega, \eta) \vee H$*

*exhibit periodicity and perfect state transfer. In particular, if $\omega$ is fixed, then $\eta$ exists provided $k-\omega \geq -\zeta$, while if $\eta$ is fixed, then $\omega$ exists provided $k-\eta \geq -\zeta$, where $\zeta$ is a parameter given in Theorem 6.4.1.*

Note that Corollary 6.4.4 is the same result as [3, Corollary 2]. The main difference is that the proof of Corollary 2 utilized a parametrization for $\omega$ and $\eta$ that only allows algebraic numbers that are quadratic, whereas the parametrization of $\omega$ and $\eta$ that we provided in Theorem 6.4.1 includes all possible algebraic and transcendental values. An interesting result was established by Ge et al. in [31, Corollary 8], which states that in the presence of weights, any disconnected double cone $\overline{K_2} \vee G$, where $G$ is not necessarily regular, admits PST between its apexes. It is unknown whether the same is true for the case of the connected double cone $K_2 \vee G$, where $G$ is not necessarily regular. We leave this as an open question. Next, taking either $\eta = 0$ or $\omega = 0$ in Corollary 6.4.4, and applying Corollary 6.4.3 yields the following fact.

**Corollary 6.4.5.** *Let $H$ be an unweighted $k$-regular graph on $n$ vertices.*

1. *For every $\epsilon > 0$, there exists a positive weight $\omega$ such that adding loops on the apexes $u$ and $v$ of $\overline{K_2} \vee H$ induces perfect state transfer between $u$ and $v$ in the resulting weighted graph with minimum time $\tau < \epsilon$.*

2. *For every $\epsilon > 0$, there exists a positive weight $\eta$ such that altering the unit weight of the edge between the apexes $u$ and $v$ of $K_2 \vee H$ to $\eta$ induces perfect state transfer between $u$ and $v$ in the resulting weighted graph with minimum time $\tau < \epsilon$.*

In particular, by taking $\ell = 2$, $m = 1$ and $\Delta = \sqrt{2n}$, one checks using Remark 5 that $\eta = 0$ and $\omega = k$. This means that we can add loops of weight $\omega = k$ on the apexes $u$ and $v$ of $\overline{K_2} \vee H$ induces PST between $u$ and $v$ in the resulting weighted graph, with minimum PST time $\frac{\pi}{\sqrt{2n}}$. Similarly, by taking $\ell = 2$, $m = 1$ and $\Delta = \frac{1}{3}\sqrt{4k^2 + 18n}$ yields $\eta = \frac{1}{3}k$ and $\omega = 0$. In other words, altering the unit edge between the apexes of $K_2 \vee H$ to $\eta = \frac{1}{3}k$ induces perfect state transfer between them in the resulting weighted graph, with minimum PST time $\frac{3\pi}{\sqrt{4k^2+18n}}$.

# 7

# State transfer in common families of graphs

In addition to Hypercubes in Section 3.4.1, we look into quantum state transfer in some well-known families of unweighted graphs. We begin with complete graphs.

## 7.1 Complete graphs

**Theorem 7.1.1** (Quantum state transfer in Complete Graphs)**.** *For $n \geq 3$, the following statements hold for $K_n$.*

1. *Every vertex in $K_n$ is periodic, with minimum period $\rho = \frac{2\pi}{n}$.*

2. *Any pair of vertices in $K_n$ does not admit perfect state transfer, pretty good state transfer, and fractional revival.*

3. *For every $\epsilon > 0$, there exists a positive weight $\omega$ such that adding loops of weight $\omega$ on vertices $u$ and $v$ of $K_n$ induces perfect state transfer between $u$ and $v$ in the resulting weighted graph with minimum PST time $\tau < \epsilon$. In particular, the weight $\omega = n$ yields perfect state transfer between $u$ and $v$ in the resulting graph with loops with minimum time $\tau = \frac{\pi}{\sqrt{2n}}$.*

4. *For every $\epsilon > 0$, there exists a positive weight $\eta$ such that altering the unit weight of the edge between vertices $u$ and $v$ of $K_n$ to $\eta$ induces perfect state transfer occurs between $u$ and $v$ in the resulting weighted graph with minimum time $\tau < \epsilon$. In particular, for $n \geq 3$, the weight $\eta = \frac{1}{3}(n-3)$ yields perfect state transfer between $u$ and $v$ in the resulting weighted graph with minimum time $\tau = \frac{3\pi}{\sqrt{4(n-3)^2+18n}}$.*

5. *Adding loops of weight $\omega$ to at least three vertices of $K_n$ or altering the unit weight of the edges between three or more vertices of $K_n$ to $\eta$ does not yield pretty good state transfer and fractional revival in the resulting graph.*

*Proof.* Let $n \geq 2$ and $u, v \in V(K_n)$. Then we can write $K_n \cong \mathbf{K}_n(0,1) \cong K_2 \vee K_{n-2}$ with apexes $u$ and $v$. Thus, (1) and (2) follows immediately from Theorem 6.1.3. Next, fixing $\eta = 1$ in Corollary 6.4.4, and combining this with Corollary 6.4.3 proves (3). In particular, if we choose $\ell = 2$, $m = 1$ and $\Delta = 2\sqrt{2n+4}$, then $\eta = 1$ and $\omega = n$, which, by Theorem 6.4.1, induce PST between $u$ and $v$ in the resulting weighted graph with minimum time $\tau = \frac{\pi}{\sqrt{2n}}$. Now, applying Corollary 6.4.5(2) to $K_n \cong K_2 \vee K_{n-2}$ proves (4). In particular, for $n \geq 3$, if we choose $\Delta = \frac{2}{3}\sqrt{4(n-3)^2 + 18n}$, $\ell = 2$ and $m = 1$, then $\eta = \frac{1}{3}(n-3)$ and $\omega = 0$ induces PST between $u$ and $v$ in the resulting weighted graph with minimum time $\tau = \frac{3\pi}{\sqrt{4(n-3)^2+18n}}$. The last statement is a straightforward consequence of Remark 6.1.4. □

We remark that since $K_n$ is $(n-1)$-regular, Proposition 3.1.4 implies that Theorem 7.1.1 applies to both adjacency and Laplacian dynamics. The absence of PST in unweighted complete graphs as stated in Theorem 7.1.1(2) was first proven by Bose et al. [9, Theorem 4], while Casaccino et al. showed in [13, Theorem 1] that adding a loop of weight $\omega = n$ to any two vertices of $K_n$ induces PST between them in the resulting weighted graph at time $\tau = \frac{\pi}{\sqrt{2n}}$. We extend this result in Theorem 7.1.1(3) by offering a plethora of weights that can work to induce PST between any two vertices of $K_n$ in the resulting weighted graph with a shorter PST time. These weights are obtained by fixing $\eta = 1$ in Corollary 6.4.4, and then using Theorem 6.4.1 to generate all possible weights $\omega$. Next, we look at complete bipartite graphs.

## 7.2 Complete bipartite graphs

Let $m, n \geq 1$ and $A = A(K_{m,n})$. The eigenvalues of $A$ are $\pm\sqrt{mn}$ and 0 (multiplicity $m+n-2$), with corresponding eigenvectors $\left[\frac{\pm\sqrt{mn}}{m}\mathbf{1}_m^T, \mathbf{1}_n^T\right]^T$, $\mathbf{e}_1 - \mathbf{e}_j$ for $j = 2, \ldots, m$ and $e_{m+1} - e_k$ for $k = m+2, \ldots, m+n$. Thus, all three eigenvalues of $A$ belong $\sigma_u(A)$ for every vertex $u$ of $K_{m,n}$. Applying Theorems 5.1.1 and 5.1.2, we get that each vertex of $K_{m,n}$ is periodic with minimum period $\rho = \frac{2\pi}{\sqrt{mn}}$.

Now, we investigate whether $K_{m,n}$ admits PST, PGST and FR. Let $u$ and $v$ be vertices in $K_{m,n}$ that belong to different partite sets, and either $m \neq n$ or $m = n \geq 2$. Without loss of generality, suppose the first $m$ columns of $A$ are indexed by the partite set that contains $u$ such that the first and $(m+1)$st columns of $A$ are indexed by $u$ and $v$. Since the vectors $\frac{1}{\sqrt{(\ell-1)+(\ell-1)^2}}\left(\sum_{j=1}^{\ell-1}\mathbf{e}_j-(\ell-1)\mathbf{e}_\ell\right)$ for $\ell=2,\ldots,m$ together with $\frac{1}{\sqrt{(\ell-1)+(\ell-1)^2}}\left(\sum_{j=m+1}^{m+\ell-1}\mathbf{e}_j-(m+\ell-1)\mathbf{e}_{m+\ell}\right)$ for $\ell=2,\ldots,n$ form an orthonormal basis for the eigenspace associated to the eigenvalue 0 of $A$, one checks that $E_0=(I_m-\frac{1}{m}J_m)\oplus(I_n-\frac{1}{n}J_n)$, and so $E_0\mathbf{e}_u$ and $E_0\mathbf{e}_v$ are not parallel. Moreover, by Example 4.4.7, any two distinct vertices in $K_{m,n}$ that belong to the same partite set with size at least three are not parallel, and hence, cannot be involved in PGST and FR. These observations allow us to narrow it down to two cases.

- Let $m=n=1$. Then $K_{1,1}\cong K_2$ which, as we know from an example in Section 3.4.1, admits perfect state transfer between $u$ and $v$ with minimum perfect state transfer time $\tau=\frac{\pi}{2}$, as well as $(\cos t, i\sin t)$-fractional revival between $u$ and $v$ at any time $t$, which is proper if and only if $t\neq\frac{\ell\pi}{2}$ for some integer $\ell$.

- Let $m=2$ and $n\geq 1$, and $u$ and $v$ be the vertices in the partite set of size two. Then $K_{2,n}\cong K_{n,2}\cong\overline{K_2}\vee O_n$, where the apexes $u$ and $v$ of $\overline{K_2}\vee O_n$. Applying Theorem 6.2.1(2), we get PST between $u$ and $v$ with minimum PST time $\tau=\frac{\pi}{\sqrt{2n}}$. Moreover, Theorem 6.2.1(6) implies that $(\alpha,\beta)$-FR occurs between $u$ and $v$ if and only if $\alpha=e^{i\gamma}\cos\gamma$ and $\beta=e^{i\gamma}\sin\gamma$, where $\gamma=\frac{\ell\pi}{2}$ for some integer $\ell$, which only yields either periodicity or PST. Thus, no proper fractional revival occurs between $u$ and $v$.

We summarize these in the following corollary.

**Theorem 7.2.1** (Quantum state transfer in Complete Bipartite Graphs)**.** *Let $m,n\geq 1$, and $u$ and $v$ be distinct vertices of $K_{m,n}$. Then the following statements hold.*

1. *Every vertex of $K_{m,n}$ is periodic with minimum period $\rho=\frac{2\pi}{\sqrt{mn}}$.*
2. *If either (i) $u$ and $v$ belong to different partite sets, and either $m\neq n$ or $m=n\geq 2$, or (ii) $u$ and $v$ belong to the same partite set whose size is at least three, then $u$ and $v$ are not parallel.*
3. (a) *If $m=n=1$, and $u$ and $v$ belong to different partite sets, then $K_{m,n}=K_2$ which admits perfect state transfer between $u$ and $v$ with minimum time $\tau=\frac{\pi}{2}$, as well as $(\cos t, i\sin t)$-fractional revival between $u$ and $v$ at any time $t$, which is proper if and only if $t\neq\frac{\ell\pi}{2}$ for some integer $\ell$.*

   (b) *If $m=2$ and $n\geq 1$, and $u$ and $v$ belong to the same partite set, then perfect state transfer occurs between $u$ and $v$ with minimum time $\tau=\frac{\pi}{\sqrt{2n}}$, and no proper fractional revival occurs between $u$ and $v$.*

By Theorem 7.2.1(2), we conclude that a pair of vertices that either belong to different partite sets which are not both singletons, or belong to the same partite set whose size is at least three does not admit PST, PGST and FR. Now, even though we did not include weighted complete bipartite graphs in the discussion, it is interesting to note that for $n \geq 3$, PST cannot be induced between any two adjacent vertices of $K_{n,n}$ by adding loops nor by altering the unit weight of the edge between them [3, Corollary 4]. For the case that $m \neq n$, adding loops nor changing the edge weight between any two adjacent vertices of $K_{m,n}$ does not induce cospectrality, and hence, does not yield strong cospectrality. Consequently, for all $m, n \geq 3$, we conclude that by adding loops nor by altering the unit weight of the edge between them, PST, as well as PGST whenever $m \neq n$, cannot be induced between any two adjacent vertices of $K_{m,n}$ in the resulting graph.

## 7.3 Complete graph minus an edge

Next, we characterize state transfer for complete graphs minus an edge.

**Theorem 7.3.1** (Quantum state transfer in Complete Graphs minus an edge)**.** *Let $n \geq 4$, and $u$ and $v$ be vertices of $K_n \backslash e$.*

1. *If $u$ and $v$ are not adjacent, then the following statements hold.*
   (a) *Vertices $u$ and $v$ are strongly cospectral, but are not cospectral with other vertices in $K_n \backslash e$.*
   (b) *Vertices $u$ and $v$ are not periodic.*
   (c) *Pretty good state transfer occurs between $u$ and $v$.*
   (d) *$(\alpha, \beta)$-fractional revival occurs between $u$ and $v$ at time $\tau$ if and only if $\alpha = e^{i\gamma} \cos \gamma$ and $\beta = ie^{i\gamma} \sin \gamma$, where*

$$\gamma = \frac{r\left(n - 3 - \sqrt{n^2 + 2n - 1}\right) - q\left(n - 3 + \sqrt{n^2 + 2n - 1}\right)}{2\sqrt{n^2 + 2n - 1}}\pi$$

*for some integers $r$ and $q$ such that $r > q$, in which case $\tau = \frac{2(r-q)\pi}{\sqrt{n^2+2n-1}}$. In particular, the minimum $(\alpha, \beta)$-fractional revival time is $\tau = \frac{2\pi}{\sqrt{n^2+2n-1}}$ which occurs if and only if $r - q = 1$. If $q = 0$, then proper $(\alpha, \beta)$-fractional revival occurs between $u$ and $v$, while balanced $(\alpha, \beta)$-fractional revival does not occur between $u$ and $v$.*

(e) *For every $\epsilon > 0$, there exists a positive weight $\omega$ such that adding loops of weight $\omega$ on $u$ and $v$ induces perfect state transfer between them in the resulting weighted graph with minimum time $\tau < \epsilon$. In particular, the weight $\omega = n-3$ yields perfect state transfer between $u$ and $v$ in the resulting weighted graph with minimum time $\tau = \frac{\pi}{\sqrt{2n}}$.*

2. *If $u$ and $v$ are adjacent, then the following statements hold.*

   (a) *If $n = 4$, then the following statements hold.*

   i. *Vertices $u$ and $v$ are not periodic.*

   ii. *Pretty good state transfer occurs between $u$ and $v$.*

   iii. *$(\alpha, \beta)$-fractional revival occurs between $u$ and $v$ at time $\tau$ if and only if $\alpha = e^{i\zeta}\cos\gamma$ and $\beta = ie^{i\zeta}\sin\gamma$, where*

$$\gamma = \frac{r\left(4-\sqrt{17}\right) - q\left(4+\sqrt{17}\right)}{2\sqrt{17}}\pi \quad \text{and} \quad \zeta = \frac{-r\sqrt{17} - q\sqrt{17} + p\sqrt{17}}{2\sqrt{17}}\pi$$

*for some integers $p$, $r$ and $q$ such that $r > q$, in which case $\tau = \frac{2(r-q)\pi}{\sqrt{17}}$. In particular, the minimum $(\alpha, \beta)$-fractional revival time is*

$$\tau = \begin{cases} \frac{2\pi}{\sqrt{17}}, & \text{if } \gamma - \zeta \neq 0 \\ 2\pi, & \text{if } \gamma - \zeta = 0, \end{cases}$$

*which is achieved if and only if $r - q = 1$. If $\zeta = 0$ and $q = 0$, then proper $(\alpha, \beta)$-fractional revival occurs between $u$ and $v$, while balanced $(\alpha, \beta)$-fractional revival does not occur between $u$ and $v$.*

   (b) *If $n \geq 5$, then $u$ and $v$ are not periodic, and are not parallel.*

*Proof.* Let $n \geq 4$, and consider $K_n \backslash e$, where $e = (u, v)$. Note that we can write $K_n \backslash e \cong \overline{K_2} \vee K_{n-2}$ with apexes $u$ and $v$, and $K_{n-2}$ is $(n-3)$-regular. By Lemma 6.1.6, $u$ and $v$ are strongly cospectral. But since $X\backslash u \cong X\backslash v \cong K_{n-1}$, and $X\backslash w \cong K_{n-1}\backslash e$ for every vertex $w$ of $X$ with $w \neq u, v$, we have that $u$ and $v$ are not cospectral with other vertices. Thus, (a) is true. Now, if $n$ is odd, then $k$ is even, and so by Lemma 6.1.7, $\sqrt{k^2 + 8(n-2)}$ is not a perfect square. If $n$ is even, then $k$ is odd and we can write $n = 2m$ for some integer $m$. This yields

$$\begin{aligned} k^2 + 8(n-2) &= (n-3)^2 + 8(n-2) = n^2 + 2n - 1 \\ &= 4m^2 + 12m + 1 = 8\left(\frac{m(m+2)}{2}\right) + 1 \end{aligned}$$

which, by Lemma 6.1.7, is a perfect square if and only if $\frac{m(m+2)}{2}$ is a triangular number. But since every triangular number is of the form $\frac{z(z+1)}{2}$, we get that $k^2 + 8(n-2)$ is not a perfect square. In both cases, $\sqrt{k^2+8(n-2)}$ is not a perfect square. Hence, (b) is true. Next, applying Theorem 6.2.1 statements (5) and (6) proves (c) and (d), respectively. Lastly, invoking Corollary 6.4.5(1) to $K_n \backslash e \cong \overline{K_2} \vee K_{n-2}$ proves (e). In particular, for $n \geq 3$, if we choose $\ell\Delta = 2\sqrt{2n}$ and $\ell = 2m$, then one checks that $\omega = n-3$, which, by Theorem 6.4.1 induces PST between $u$ and $v$ in the resulting weighted graph with minimum time $\tau = \frac{\pi}{\sqrt{2n}}$.

Next, assume $u$ and $v$ are distinct vertices of $X$ that are not incident with $e$. If $n = 4$, then $u$ and $v$ are true twins, and we can write $K_4 \backslash e \cong K_2 \vee O_2$, where $O_2$ is a 0-regular graph. Since $(k-1)^2 + 8(n-2) = 17$ is not a perfect square, Theorem 6.3.2(1), implies that $u$ and $v$ are not periodic, and hence, do not exhibit PST. However, we obtain PGST by 6.3.2(4), and invoking Theorem 6.3.2(5) yields the result for FR. Now, suppose $n \geq 5$. Observe that all vertices of $X$, except the two joining $e$, are pairwise true twins. Since there are $n \geq 3$ of them, we conclude from Corollary 4.4.5 that $u$ and $v$ are not strongly cospectral. In particular, since $u$ and $v$ are cospectral because they are twins, it follows that $u$ and $v$ are not parallel. Now, the eigenvalues of $A(K_n \backslash e)$ are $-1, 0$ and $\lambda^{\pm}$, where $\lambda^{\pm} = \frac{1}{2}(n-3 \pm \sqrt{n^2+2n-7})$. The eigenvector corresponding to 0 is $e_j - e_\ell$, where $e = (j, \ell)$. Since $u \neq j, \ell$, we get that $0 \notin \sigma_u(A(K_n \backslash e))$, and one checks that $\sigma_u(A(K_n \backslash e)) = \{-1, \lambda^{\pm}\}$, from which it is clear that Theorem 5.1.1 does not hold. Consequently, $u$ is not periodic. Lastly, since $u$ and $v$ are cospectral, they have the same eigenvalue support. Applying Proposition 5.1.4, we get that $v$ is also not periodic. □

By Theorem 7.3.1(2b), any pair of adjacent vertices in $K_n \backslash e$ does not exhibit PST, PGST and FR for all $n \geq 5$. Similar to the case of complete graphs, Casaccino et al. in [13, Theorem 2] showed that adding a loop of weight $\omega = n-3$ to the two non-adjacent vertices of $K_n \backslash e$ induces PST between them in the resulting weighted graph at time $\tau = \frac{\pi}{\sqrt{2n}}$. We extend again this result in Theorem 7.3.1(1e) to include a multitude of weights that can work to induce PST between any two vertices of $K_n \backslash e$ in the resulting weighted graph. Moreover, based on numerical solutions, the authors in [13] conjectured that in the absence of loops, PST can be achieved between the two non-adjacent vertices of $K_n \backslash e$ for every $n \geq 4$. This turns out to be false, and what the authors observed was in fact PGST, and not PST. Indeed, the paper was published around 2009, and PGST was not formally introduced until 2012. Next, we move on to cocktail party graphs.

## 7.4 Cocktail party graphs

**Theorem 7.4.1** (Quantum state transfer in Cocktail Party Graphs). *Let $m \geq 2$, and $u$ and $v$ be distinct vertices of $\overline{mK_2}$. Then the following statements hold.*

1. *Every vertex of $\overline{mK_2}$ is periodic with minimum period $\tau = \pi$.*

2. *If $u$ and $v$ are non-adjacent, then the following statements hold.*

   (a) *Vertices $u$ and $v$ are strongly cospectral.*

   (b) *Perfect state transfer occurs between $u$ and $v$ if and only if $m$ is even with minimum time $\tau = \frac{\pi}{2g}$, where*

$$g = gcd\left(\frac{m-2}{2} + \sqrt{\left(\frac{m-2}{2}\right)^2 + (m-1)}, \frac{m-2}{2} - \sqrt{\left(\frac{m-2}{2}\right)^2 + (m-1)}\right)$$

   (c) *If $m$ is odd, then proper pretty good state transfer does not occur between $u$ and $v$.*

   (d) *$(\alpha, \beta)$-FR occurs between $u$ and $v$ at time $\tau$ if and only if $\alpha = e^{i\gamma}\cos\gamma$ and $\beta = ie^{i\gamma}\sin\gamma$, where*

$$\gamma = -\frac{r + q(m-1)}{m}\pi$$

   *for some integers $r$ and $q$ such that $r > q$, in which case $\tau = \frac{(r-q)\pi}{m}$. In particular, the minimum $(\alpha, \beta)$-fractional revival time is $\tau = \frac{\pi}{m}$ which occurs if and only if $\gamma = -\frac{(r-1)m+1}{m}\pi$. If $q = 0$, then proper $(\alpha, \beta)$-fractional revival occurs between $u$ and $v$ if and only if $\frac{r}{m}$ is not an integer, while balanced $(\alpha, \beta)$-fractional revival occurs between $u$ and $v$ if and only if $2r = m\ell$ for some odd integer $\ell$, in which case $m$ is even.*

   (e) *For every $\epsilon$, there exists a positive weight $\omega$ such that adding loops of weight $\omega$ on $u$ and $v$ induces perfect state transfer between $u$ and $v$ in the resulting weighted graph with minimum time $\tau < \epsilon$. In particular, the weight $\omega = 2m - 4$ yields perfect state transfer between $u$ and $v$ in the resulting weighted graph with minimum time $\tau = \frac{\pi}{\sqrt{2n}}$.*

3. *If $u$ and $v$ are adjacent, then $u$ and $v$ are not parallel.*

*Proof.* Suppose $m \geq 2$, and consider $\overline{mK_2}$ with nonadjacent vertices $u$ and $v$. As we know, $\overline{mK_2}$ is a $(2m-2)$-regular graph on $2m$ vertices, and we can write $\overline{mK_2} \cong$

$\overline{K_2} \vee \overline{(m-1)K_2}$ with apexes $u$ and $v$. That is, $\overline{mK_2}$ is a double cone on a $k$-regular graph on $n = 2(m-1)$ vertices, where $k = 2m-4$. Thus, $k^2+8n = 4m^2$ is a perfect square for all $m \geq 2$, and $g = \gcd\left(\frac{1}{2}(k+\sqrt{k^2+8n}), \frac{1}{2}(k-\sqrt{k^2+8n})\right) = \gcd(2(m+1), -2) = 2$. By Theorem 6.2.1(1), $u$ and $v$ are periodic with minimum period $\rho = \frac{2\pi}{g} = \pi$. This proves (1).

Next, let us show (2). Note that (a) follows from Lemma 6.1.6(1). If $m$ is odd, then $k = 2(m-2)$, where $m-2$ is odd, and so by Theorem 6.2.1(2), PST does not occur between $u$ and $v$. Moreover, since each vertex of $\overline{mK_2}$ is periodic and $k^2+8n$ is a perfect square, it follows from Theorem 6.2.1(4) that proper PGST does not occur between $u$ and $v$. Thus, (c) is true. On the other hand, if $m$ is even, then $m-1$ is odd and $k = 4\left(\frac{m-2}{2}\right)$. Consequently, $0 = \nu_2(m-1) \leq \nu_2\left(\frac{m-2}{2}\right)$, and by Theorem 6.2.1(3), PST occurs between $u$ and $v$ with minimum PST time $\tau = \frac{\pi}{2g}$, where $g = \gcd\left(\frac{m-2}{2} + \sqrt{\left(\frac{m-2}{2}\right)^2 + (m-1)}, \frac{m-2}{2} - \sqrt{\left(\frac{m-2}{2}\right)^2 + (m-1)}\right)$. This proves (3). A direct application of Theorems 5.4.2 and 6.2.1(6) proves (d). Lastly, invoking Corollary 6.4.5(2) to $\overline{mK_2} = \overline{K_2} \vee \overline{(m-1)K_2}$ proves (e). In particular, if we choose $\ell = 2$, $m' = 1$ and $\Delta = 4\sqrt{m-1}$, then one checks that $\omega = 2m-4$, which, by Theorem 6.4.1 induces PST between $u$ and $v$ in the resulting weighted graph with minimum time $\tau = \frac{\pi}{2\sqrt{m-1}}$.

Finally, since every pair of nonadjacent vertices $u$ and $v$ in $\overline{mK_2}$ are false twins, and are adjacent but not twins with any $w \in V(\overline{mK_2})\backslash\{u, v\}$, Theorem 4.4.4(2) implies that $u$ and $v$ are not parallel with any $w \in V(\overline{mK_2})\backslash\{u, v\}$. Consequently, any pair of adjacent vertices of $\overline{mK_2}$ are not parallel, and hence, not strongly cospectral. Thus, PST, PGST, and fractional revival do not occur between any pair of adjacent vertices of $\overline{mK_2}$. This proves (3). □

By Theorem 7.4.1(3), any pair of adjacent vertices of $\overline{nK_2}$ does not exhibit PST, PGST and FR.

## 7.5 Paths and cycles

For completeness, we also include a summary of quantum state transfer in paths and cycles. We begin with paths.

**Theorem 7.5.1** (Quantum state transfer in Paths)**.** *Let $n \geq 2$, and $u$ and $v$ be vertices of $P_n$. Then the following statements hold.*

1. *([57], Lemma 2) Vertices $u$ and $v$ are strongly cospectral if and only if $u+v = n+1$.*

2. *[22, 35] Perfect state transfer occurs between* $u$ *and* $v$ *if and only if* $u+v=n+1$ *and* $n=2,3$.

3. *([57], Theorem 4) Pretty good state transfer occurs between* $u$ *and* $v$ *if and only if* $u+v=n+1$ *and either*

   (a) $n=2^t-1$*, where* $t$ *is a positive integer;*

   (b) $n=p-1$*, where* $p$ *is an odd prime or*

   (c) $n=2^tp-1$*, where* $t$ *is a positive integer,* $p$ *is an odd prime, and* $u$ *is an integer multiple of* $2^{t-1}$.

4. *([16], Theorem 8.1) Fractional revival occurs between* $u$ *and* $v$ *if and only if* $u+v=n+1$ *and* $n=2,3,4$.

In 2004, it was shown by Christandl et al. in [23, Section VIII] that for all $n\geq 2$, if we set the weight of the edge $(j,j+1)$ to $\sqrt{j(n-j)}$ for each $j\in\{1,\ldots,n-1\}$, then PST can be achieved between end vertices in the resulting weighted path. Moreover, in 2016, Kempton et al. established in [46, Theorem 1.3] that adding loops to the vertices of $P_n$ does not induce PST between end points of the resulting weighted path. However, in 2019, Kempton et al. proved in [47, Theorem 1.3] that there exists a weight $\omega$ such that adding loops of weight $\omega$ to the end vertices of $P_n$ induces PGST between them in resulting weighted path. Lastly, we remark that before a complete characterization of PGST in paths was achieved by van Bommel in [57], a characterization of paths exhibiting PGST between end vertices was first given by Godsil et al. [37]. Next, we gather known results about cycles.

**Theorem 7.5.2** (Quantum state transfer in Cycles)**.** *Let* $n\geq 3$*, and* $u$ *and* $v$ *be vertices of* $C_n$*. Then the following statements hold.*

1. *([55], Lemma 5) Vertices* $u$ *and* $v$ *are strongly cospectral if and only if* $n$ *is even and* $|u-v|=\frac{n}{2}$.

2. *([7], Theorem 8) Perfect state transfer occurs between* $u$ *and* $v$ *if and only if* $n=4$ *and* $|u-v|=2$.

3. *([55], Theorem 13) Pretty good state transfer occurs between* $u$ *and* $v$ *in* $C_n$*, as well as its complement, if and only if* $n=2^k$ *for any integer* $k\geq 2$ *and* $|u-v|=\frac{n}{2}$.

4. *([16], Theorem 7.3) Fractional revival occurs between* $u$ *and* $v$ *if and only if* $n=4,6$ *and* $|u-v|=\frac{n}{2}$.

Now, unlike paths, cycles are regular graphs. Thus, we can take the double cone on $C_n$ and check which types of quantum state transfer occur on the apexes. The following result is a straightforward application of Theorems 6.2.1 and 6.3.2, as well as Corollaries 6.4.3 and 6.4.5, and Example 6.4.2.

**Theorem 7.5.3** (Quantum state transfer in Double Cones on Cycles)**.** *Let $n \geq 3$.*

1. *The following statements hold for the apexes $u$ and $v$ of $\overline{K_2} \vee C_n$.*

   (a) *Vertices $u$ and $v$ are periodic if and only if $n = 4T$ for some triangular number $T$, with minimum period $\frac{2\pi}{g}$, where $g = gcd\left(1 + \sqrt{8T+1}, 1 - \sqrt{8T+1}\right)$.*

   (b) *If $n = 4T$ for some triangular number $T$, then perfect state transfer and proper pretty good state transfer do not occur between $u$ and $v$.*

   (c) *If $n \neq 4T$ for all triangular numbers $T$, then pretty good state transfer occurs between $u$ and $v$.*

   (d) *$(\alpha, \beta)$-fractional revival occurs between $u$ and $v$ at time $\tau$ if and only if $\alpha = e^{i\gamma}\cos\gamma$ and $\beta = ie^{i\gamma}\sin\gamma$, where*

   $$\gamma = \frac{r\left(1 - \sqrt{2n+1}\right) - q\left(1 + \sqrt{2n+1}\right)}{2\sqrt{2n+1}}\pi$$

   *for some integers $r$ and $q$ such that $r > q$, in which case $\tau = \frac{(r-q)\pi}{\sqrt{2n+1}}$. In particular, the minimum $(\alpha, \beta)$-fractional revival time is $\tau = \frac{\pi}{\sqrt{2n+1}}$ which occurs if and only if $r - q = 1$, in which case we may take $\gamma = \frac{(1-\sqrt{2n+1})\pi}{2\sqrt{2n+1}}$. If $q = 0$, then proper $(\alpha, \beta)$-fractional revival occurs between $u$ and $v$ if and only if $\frac{r}{\sqrt{2n+1}}$ is not an integer, while balanced $(\alpha, \beta)$-fractional revival does not occur between $u$ and $v$.*

2. *The following statements hold for the apexes $u$ and $v$ of $K_2 \vee C_n$.*

   (a) *Vertices $u$ and $v$ are periodic if and only if $n$ is a triangular number, with minimum period $\rho = \frac{2\pi}{g}$, where $g = gcd\left(\frac{1}{2}\left(3 + \sqrt{8n+1}\right), \frac{1}{2}\left(3 - \sqrt{8n+1}\right)\right)$.*

   (b) *If $n$ is a triangular number, then perfect state transfer and pretty good state transfer do not occur between $u$ and $v$.*

   (c) *If $n$ is not a triangular number, then pretty good state transfer occurs between $u$ and $v$.*

(d) $(\alpha, \beta)$-fractional revival occurs between $u$ and $v$ at time $\tau$ if and only if $\alpha = e^{i\zeta}\cos\gamma$ and $\beta = ie^{i\zeta}\sin\gamma$, where

$$\gamma = \frac{r\left(6-\sqrt{8n+1}\right) - q\left(6+\sqrt{8n+1}\right)}{2\sqrt{8n+1}}\pi$$

and

$$\zeta = \frac{r\left(2-\sqrt{8n+1}\right) - q\left(2+\sqrt{8n+1}\right) + p\sqrt{8n+1}}{2\sqrt{8n+1}}\pi$$

for some integers $p$, $r$, and $q$ such that $r > q$, in which case $\tau = \frac{2(r-q)\pi}{\sqrt{8n+1}}$. In particular, the minimum $(\alpha, \beta)$-fractional revival time is

$$\tau = \begin{cases} \frac{2\pi}{\sqrt{8n+1}}, & \text{if } \gamma - \zeta \neq 0 \\ 2\pi, & \text{if } \gamma - \zeta = 0, \end{cases}$$

which is achieved if and only if $r - q = 1$. If $\zeta = 0$ and $q = 0$, then proper $(\alpha, \beta)$-fractional revival occurs between $u$ and $v$ if and only if $\frac{r-p\sqrt{8n+1}}{\sqrt{8n+1}}$ is not an integer, while balanced $(\alpha, \beta)$-fractional revival does not occur between $u$ and $v$.

3. For every $\epsilon > 0$, there exist positive weights $\omega$, $\omega'$, $\eta$ and $\eta'$ such that the apexes of $K_2(\omega, \eta) \vee C_n$, $K_2(\omega', 0) \vee C_n$, and $K_2(0, \eta') \vee C_n$ have perfect state transfer with minimum time $\tau < \epsilon$. In particular, the apexes of $\boldsymbol{K}_2(\omega, \eta) \vee C_n$, where $\omega = 2 + \frac{3}{4}\sqrt{(\pi^2-1)8n}$ and $\eta = \frac{1}{4\sqrt{(\pi^2-1)8n}}$, as well as $\boldsymbol{K}_2(2, 0) \vee C_n$, and $\boldsymbol{K}_2\left(0, \frac{2}{3}\right) \vee C_n$ have perfect state transfer with minimum time $\frac{\pi}{\sqrt{2n}}$, $\frac{3\pi}{\sqrt{2(9n+8)}}$ and $\frac{1}{\sqrt{2n}}$, respectively.

# 8

# Future work

Here, we provide a compilation of problems motivated by the previous chapters in this thesis.

## 8.1 Properties of twin vertices

In Chapter 4, we introduced the concept of twinning a vertex of a graph, and provided a characterization of vertices $u$ in a given unweighted graph $X$ such that $u$ and an additional vertex $v$ are adjacency strongly cospectral in the post-twinning graph. We are now working to achieve a complete characterization for the case of Laplacian matrix.

**Problem 1.** Characterize vertices $u$ in an arbitrary unweighted graph $X$ such that $u$ and $v$ are Laplacian strongly cospectral with $v$ in the post-twinning graph.

## 8.2 State transfer between twins in graphs

Next, in Chapter 5, we provided necessary and sufficient conditions for periodicity, perfect state transfer, pretty good state transfer, and fractional revival to occur between twin vertices in a graph with respect to adjacency dynamics. We have started working on the Laplacian case. Nonetheless, we still pose the problem here.

**Problem 2.** Determine necessary and sufficient conditions for periodicity, perfect state transfer, pretty good state transfer, and fractional revival to occur between twin vertices in a graph with respect to Laplacian dynamics.

Recall that in Chapter 3, we gave an overview of periodicity, perfect state transfer, pretty good state transfer, and fractional revival. While a great deal of work in quantum state transfer research is related to these four types, our discussion does not encompass all types known in literature. Uniform mixing [14, 38], $K$-fractional revival [15], pretty good fractional revival [17, 19], and group state transfer [11] are examples of other types that we have not covered. Thus, we pose the following problem.

**Problem 3.** Provide necessary and sufficient conditions for other types of quantum state transfer, including but not limited to uniform mixing, $K$-fractional revival, pretty good fractional revival, and group state transfer, to occur in a graph with twin vertices.

## 8.3 Double cones on regular graphs

In Chapter 6, we explored quantum state transfer between the apexes of positively weighted double cone graphs under adjacency dynamics. We have started working to extend our results to Laplacian dynamics.

**Problem 4.** Extend results in Chapter 6 about quantum state transfer in double cones on regular graphs to Laplacian dynamics. That is, characterize Laplacian perfect state transfer, Laplacian periodicity, and Laplacian pretty good state transfer, and Laplacian fractional revival between the apexes of a double cone on a regular graph.

Our computations show that we can still induce periodicity and perfect state transfer even if the weight of either the loops or the edge between the apexes are negative. Thus, we pose the following problem.

**Problem 5.** Identify which negative weights $\omega$ and $\eta$ induce periodicity and perfect state transfer between the apexes of $X(\gamma) = K_2(\omega, \eta) \vee H$, where $\gamma > 0$ is fixed and $H$ is a weighted $k$-regular graph.

Moreover, we have seen that there are weights $\omega$ and $\eta$ that induce perfect state transfer between the apexes of $X(\gamma) = K_2(\omega, \eta) \vee H$ at an arbitrarily small time. However, these weights become very large as the minimum perfect state transfer time becomes shorter, and hence may not be feasible in practice.

**Problem 6.** Determine optimality properties of the weights $\omega$ and $\eta$ that induce perfect state transfer between the apexes of $X(\gamma) = K_2(\omega, \eta) \vee H$, where $H$ is a $k$-regular graph. Can $\omega$ and $\eta$ be negative?

And again, we plan to extend our results in double cone graphs to other types of quantum state transfer.

**Problem 7.** Investigate which other types of quantum state transfer occur between the apexes of $X(\gamma) = K_2(\omega, \eta) \vee H$, where $\gamma > 0$ is fixed and $H$ is a $k$-regular graph.

## 8.4 State transfer in common families of graphs

Lastly, in Chapter 7, we provided a survey of quantum state transfer for common families of unweighted graphs under adjacency dynamics. We are working to extend this to include Laplacian dynamics.

**Problem 8.** Extend results in Chapter 7 about quantum state transfer in common families of graphs to Laplacian dynamics. That is, characterize Laplacian perfect state transfer, Laplacian periodicity, and Laplacian pretty good state transfer, and Laplacian fractional revival in common families of graphs.

We also consider extensions of our results in Chapter 7 to other types quantum of state transfer.

**Problem 9.** Characterize other types of quantum state transfer in common families of unweighted graphs, including but not limited to uniform mixing, $K$-fractional revival, pretty good fractional revival, and group state transfer.

For complete graphs minus an edge and cocktail party graphs, we examined whether adding loops or altering edge weights between nonadjacent vertices helps induce perfect state transfer between them. It would be interesting to see what happens if we apply this instead to a pair of adjacent vertices.

**Problem 10.** Determine weights $\omega$ and $\eta$ such that adding loops of weight $\omega$, or altering the unit weight of the edge between any two adjacent vertices of $K_n \backslash e$ and $\overline{nK_2}$ to $\eta$ induce perfect state transfer.

Finally, since we considered double cones on cycles, it is natural to extend our results to double cones on paths.

**Problem 11.** Characterize weights $\omega$ and $\eta$ that induce perfect state transfer between the apexes of $X(\gamma) = K_2(\omega, \eta) \vee P_n$, for a fixed $\gamma > 0$. For the unweighted case, provide a complete characterization of quantum state transfer between the apexes of connected and disconnected double cones on paths.

# Index

# References

[1] John Ahn, Christine Alar, Beth Bjorkman, Steve Butler, Joshua Carlson, Audrey Goodnight, Haley Knox, Casandra Monroe, and Michael C. Wigal. Ordered multiplicity inverse eigenvalue problem for graphs on six vertices. *The Electronic Journal of Linear Algebra*, 37, 2021.

[2] Rachael Alvir, Sophia Dever, Benjamin Lovitz, James Myer, Christino Tamon, Yan Xu, and Hanmeng Zhan. Perfect state transfer in Laplacian quantum walk. *Journal of Algebraic Combinatorics*, 43(4):801–826, 2016.

[3] Ricardo J. Angeles-Canul, Rachael M. Norton, Michael C. Opperman, Christopher C. Paribello, Matthew C. Russell, and Christino Tamon. On quantum perfect state transfer in weighted join graphs. *International Journal of Quantum Information*, 7(8):1429–1445, 2009.

[4] Ricardo J. Angeles-Canul, Rachael M. Norton, Michael C. Opperman, Christopher C. Paribello, Matthew C. Russell, and Christino Tamon. Perfect state transfer, integral circulants, and join of graphs. *Quantum Information and Computation*, 10(3-4):0325–0342, 2010.

[5] Rachel Bachman, Eric Fredette, Jessica Fuller, Michael Landry, Michael Opperman, Christino Tamon, and Andrew Tollefson. Perfect state transfer on quotient graphs. *Quantum Information and Computation*, 12(3-4):293–313, 2012.

[6] Leonardo Banchi, Gabriel Coutinho, Chris Godsil, and Simone Severini. Pretty good state transfer in qubit chains-The Heisenberg Hamiltonian. *Journal of Mathematical Physics*, 58(3):32202, 2017.

[7] Milan Bašić, Marko D. Petković, and Dragan Stevanović. Perfect state transfer in integral circulant graphs. *Applied Mathematics Letters*, 22(7):1117–1121, 2009.

[8] Sougato Bose. Quantum communication through an unmodulated spin chain. *Physical Review Letters*, 91(20):207901, 2003.

[9] Sougato Bose, Andrea Casaccino, Stefano Mancini, and Simone Severini. Communication in XYZ all-to-all quantum networks with a missing link. *International Journal of Quantum Information*, 7(4):713–723, 2009.

[10] Johann A. Briffa and Irene Sciriha. On the displacement of eigenvalues when removing a twin vertex. *Discussiones Mathematicae - Graph Theory*, 40(2):435–450, 2020.

[11] Luke C. Brown, William J. Martin, and Duncan Wright. Continuous time quantum walks on graphs: group state transfer. arXiv:2103.08837, 2021.

[12] Xiwang Cao, Dandan Wang, and Keqin Feng. Pretty good state transfer on Cayley graphs over dihedral groups. *Discrete Mathematics*, 343(1):111636, 2020.

[13] Andrea Casaccino, Seth Lloyd, Stefano Mancini, and Simone Severini. Quantum state transfer through a qubit network with energy shifts and fluctuations. *International Journal of Quantum Information*, 7(8):1417–1427, 2009.

[14] Ada Chan. Complex Hadamard matrices, instantaneous uniform mixing and cubes. *Algebraic Combinatorics*, 3(3):757–774, 2020.

[15] Ada Chan, Gabriel Coutinho, Whitney Drazen, Or Eisenberg, Chris Godsil, Gabor Lippner, Mark Kempton, Christino Tamon, and Hanmeng Zhan. Fundamentals of fractional revival in graphs. arXiv:2004.01129, 2020.

[16] Ada Chan, Gabriel Coutinho, Christino Tamon, Luc Vinet, and Hanmeng Zhan. Quantum fractional revival on graphs. *Discrete Applied Mathematics*, 269:86–98, 2019.

[17] Ada Chan, Whitney Drazen, Or Eisenberg, Mark Kempton, and Gabor Lippner. Approximate quantum fractional revival in paths and cycles. arXiv:2005.00492, 2020.

[18] Ada Chan, Bobae Johnson, Mengzhen Liu, Malena Schmidt, Zhanghan Yin, and Hanmeng Zhan. Laplacian Fractional Revival on Graphs. arXiv:2010.10413, 2020.

[19] Ada Chan, Bobae Johnson, Mengzhen Liu, Malena Schmidt, Zhanghan Yin, and Hanmeng Zhan. Laplacian pretty good fractional revival. arXiv:2010.10465, 2020.

[20] G Chartrand, L Lesniak, and P Zhang. *Graphs & Digraphs*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, sixth edition, 2016.

[21] Wang Chi Cheung and Chris Godsil. Perfect state transfer in cubelike graphs. *Linear Algebra and Its Applications*, 435(10):2468–2474, 2011.

[22] Matthias Christandl, Nilanjana Datta, Tony C. Dorlas, Artur Ekert, Alastair Kay, and Andrew J. Landahl. Perfect transfer of arbitrary states in quantum spin networks. *Physical Review A - Atomic, Molecular, and Optical Physics*, 71(3), 2005.

[23] Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew Landahl. Perfect state transfer in quantum spin networks. *Physical Review Letters*, 92(18):187902, 2004.

[24] Gabriel Coutinho. Quantum state transfer in graphs. Ph.D. Thesis, University of Waterloo, 2014.

[25] Gabriel Coutinho and Chris Godsil. Graph spectra and continuous quantum walks. Unpublished, 2021.

[26] Gabriel Coutinho, Chris Godsil, Krystal Guo, and Frederic Vanhove. Perfect state transfer on distance-regular graphs and association schemes. *Linear Algebra and Its Applications*, 478:108–130, 2015.

[27] Gabriel Coutinho and Henry Liu. No Laplacian perfect state transfer in trees. *SIAM Journal on Discrete Mathematics*, 29(4):2179–2188, 2015.

[28] Or Eisenberg, Mark Kempton, and Gabor Lippner. Pretty good quantum state transfer in asymmetric graphs via potential. *Discrete Mathematics*, 342(10):2821–2833, 2019.

[29] Xiaoxia Fan and Chris Godsil. Pretty good state transfer on double stars. *Linear Algebra and Its Applications*, 438(5):2346–2358, 2013.

[30] Edward Farhi and Sam Gutmann. Quantum computation and decision trees. *Physical Review A - Atomic, Molecular, and Optical Physics*, 58(2):915–928, 1998.

[31] Yang Ge, Benjamin Greenberg, Oscar Perez, and Christino Tamon. Perfect state transfer, graph products and equitable partitions. *International Journal of Quantum Information*, 9(3):823–842, 2011.

[32] C. Godsil. *Algebraic Combinatorics*. Chapman Hall/CRC Mathematics Series. Taylor & Francis, 1993.

[33] C. Godsil and G. Royle. *Algebraic Graph Theory*, volume 207 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[34] Chris Godsil. Periodic graphs. *Electronic Journal of Combinatorics*, 18(1):1–15, 2011.

[35] Chris Godsil. State transfer on graphs. *Discrete Mathematics*, 312(1):129–147, 2012.

[36] Chris Godsil. When can perfect state transfer occur? *Electronic Journal of Linear Algebra*, 23:877–890, 2012.

[37] Chris Godsil, Stephen Kirkland, Simone Severini, and Jamie Smith. Number-theoretic nature of communication in quantum spin systems. *Physical Review Letters*, 109(5):3–6, 2012.

[38] Chris Godsil, Natalie Mullin, and Aidan Roy. Uniform mixing and association schemes. *Electronic Journal of Combinatorics*, 24(3), 2017.

[39] Chris Godsil and Jamie Smith. Strongly cospectral vertices. arXiv:1709.07975, 2017.

[40] B. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Graduate Texts in Mathematics. Springer International Publishing, 2015.

[41] G. H. Hardy, E. M. Wright, D. R. Heath-Brown, R. Heath-Brown, J. Silverman, and A. Wiles. *An Introduction to the Theory of Numbers*. Oxford mathematics. OUP Oxford, 2008.

[42] R Horn and C Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, second edition, 2013.

[43] Nathaniel Johnston, Steve Kirkland, Sarah Plosker, Rebecca Storey, and Xiaohong Zhang. Perfect quantum state transfer using Hadamard diagonalizable graphs. *Linear Algebra and Its Applications*, 531:375–398, 2017.

[44] Alastair Kay. Perfect, efficient, state transfer and its application as a constructive tool. *International Journal of Quantum Information*, 8(4):641–676, 2010.

[45] Alastair Kay. Basics of perfect communication through quantum networks. *Physical Review A - Atomic, Molecular, and Optical Physics*, 84(2), 2011.

[46] Mark Kempton, Gabor Lippner, and Shing Tung Yau. Perfect state transfer on graphs with a potential. *Quantum Information and Computation*, 17(3-4):0303–0327, 2017.

[47] Mark Kempton, Gabor Lippner, and Shing-Tung Yau. Pretty good state transfer in graphs with an involution. arXiv:1702.07000, 2017.

[48] Vivien M. Kendon. Quantum walks on general graphs. *International Journal of Quantum Information*, 4(5):791–805, 2006.

[49] Vivien M. Kendon and Christino Tamon. Perfect state transfer in quantum walks on graphs. *Journal of Computational and Theoretical Nanoscience*, 8(3):422–433, 2011.

[50] Steve Kirkland, Darian McLaren, Rajesh Pereira, Sarah Plosker, and Xiaohong Zhang. Perfect quantum state transfer in weighted paths with potentials (loops) using orthogonal polynomials. *Linear and Multilinear Algebra*, 67(5):1043–1061, 2019.

[51] Steve Kirkland and Xiaohong Zhang. Fractional Revival of Threshold Graphs under Laplacian Dynamics. *Discussiones Mathematicae - Graph Theory*, 40(2):585–600, 2020.

[52] B.M. Levitan, V.V.Z. Boris Moiseevich Levitan, V.V. Jikov, V.V. Zhikov, L.V. Longdon, and L.W. Longdon. *Almost Periodic Functions and Differential Equations*. Cambridge University Press, 1982.

[53] Yipeng Li, Xiaogang Liu, Shenggui Zhang, and Sanming Zhou. Perfect state transfer in NEPS of complete graphs. *Discrete Applied Mathematics*, 289:98–114, 2021.

[54] Hiranmoy Pal. Quantum state transfer on a class of circulant graphs. *Linear and Multilinear Algebra*, 2019.

[55] Hiranmoy Pal and Bikash Bhattacharjya. Pretty good state transfer on circulant graphs. *Electronic Journal of Combinatorics*, 24(2), 2017.

[56] Hiranmoy Pal and Bikash Bhattacharjya. Pretty good state transfer on some NEPS. *Discrete Mathematics*, 340(4):746–752, 2017.

[57] Christopher Martin van Bommel. A complete characterization of pretty good state transfer on paths. *Quantum Information and Computation*, 19(7-8):601–608, 2019.

[58] Christopher Martin van Bommel. Quantum walks and pretty good state transfer on paths. Ph.D. Thesis, University of Waterloo, 2019.

[59] Luc Vinet and Alexei Zhedanov. Almost perfect state transfer in quantum spin chains. *Physical Review A - Atomic, Molecular, and Optical Physics*, 86(5), 2012.