# ON NEWTON'S PROBLEM in the CALCULUS of VARIATIONS

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## in the
# Calculus of Variations

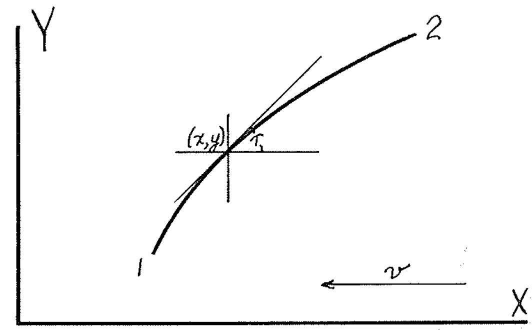

Fig. 1.

The Problem. *0

The problem which is to be solved by the use of the calculus of variations may be expressed as follows: To find the form of curve joining two given points, 1 and 2, in the xy plane, such that the surface of revolution generated by the arc 12 on rotating about the x-axis, shall encounter a minimum resistance to translatory motion along the negative direction of the x-axis, when the law of resistance is as follows: the normal pressure on the surface is proportional to the square of the normal component of velocity.

Suppose the point (x,y) in Fig. 1 to be a point on the curve, $\tau$ the angle of slope of the curve at the point, v the velocity, and k a constant of proportionality, and s the length of the curve from 1 to (x,y). The element of resistance is then found to be

(1) $$2\pi k^2 v^2 y \sin^3 \tau \, ds$$

Now, if we agree to represent the arc in the form

(2) $$x = \varphi(t) \ ; \quad y = \psi(t) \ ; \quad t_1 \leqq t \leqq t_2$$

*0 See notes at end.

then the element of resistance becomes

$$2\pi k^2 v^2 \frac{y\, y'^3}{x'^2 + y'^2}\, dt$$

and the total resistance on the curve between the points 1 and 2 becomes

$$(3) \qquad 2\pi k^2 v^2 \int_{t_1}^{t_2} \frac{y\, y'^3}{x'^2 + y'^2}\, dt$$

For physical reasons this integral can only represent the resistance on an arc on which the conditions $x' \geqq 0$ ; $y' \geqq 0$ are true. This is clear from figure 2. If $x' < 0$ such a region as A would be formed in which the medium would be carried along and the ordinary law of resistance would not hold. If $y' < 0$ such a region as B would be formed in which the surface would be protected from the medium by the surface in front of it. The latter would also make the resistance integral negative along a portion of the arc, which is also impossible. We must also have y greater than zero except possibly at 1, as an arc y = 0 would have no meaning.

The problem then becomes one of minimizing the above integral, for which purpose we shall use the general theory of the problem of the calculus of variations in parametric form, due to Weierstrass. The problem as we shall discuss it analytically may now be stated thus:

Among all "ordinary curves"*1 joining a point 1 to a point 2 in the xy plane, and on which the following

*1 See notes at end.

restrictions hold,

(4) $y > 0$ for $t_1 < t \leq t_2$

$x' \geqq 0$, $y' \geqq 0$ for $t_1 \leq t \leq t_2$

to determine that which will give a minimum value to

(5)
$$J = \int_{t_1}^{t_2} \frac{y\, y'^3}{x'^2 + y'^2}\, dt$$

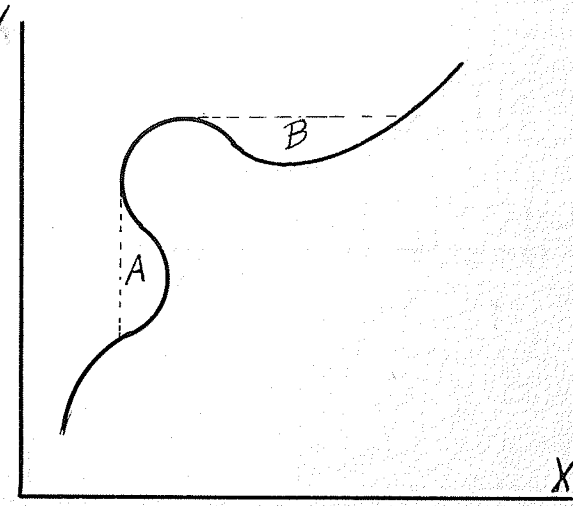


Fig. 2.

A curve fulfilling the conditions (4) will be called "admissible". The integrand function will hereafter be denoted by F, or F(x,y,x',y'), where

(6)
$$F(x,y,x'y') = \frac{y\, y'^3}{x'^2 + y'^2}$$

The Weierstrass Theory is evidently applicable to the integral since F is of class C''' at least *2 and also satisfies the homogeneity condition,*3 i.e.,

(7)
$$F(x,y,kx',ky') = kF(x,y,x',y')$$

<u>The Derivatives of F</u>

The derivatives of F which will be used and the related function $F_1$*4 are here set down for later convenience.

(8)
$$F_{x'} = \frac{-2yy'^3x'}{(x'^2+y'^2)^2}; \qquad F_{y'} = \frac{yy'^2(3x'^2 + y'^2)}{(x'^2 + y'^2)^2}$$

(9)
$$F_1 = \frac{2yy'(3x'^2 - y'^2)}{(x'^2 + y'^2)^3}$$

Solution of Euler's Differential Equation.*[5]

To solve Euler's equation we may use the form

$$F_x - \frac{d}{dt}(F_{x'}) = 0$$

We have $F_x = 0$ and so, using equation (8), we may write

(11) $$-F_{x'}/2 = \frac{yy'^3x'}{(x'^2 + y'^2)^2} = a$$

We must have $a \geq 0$ on account of (6).

If now we put

(12) $$x'/y' = q$$

and substitute in (11) we obtain

(13) $$y = \frac{a(1 + q^2)^2}{q}$$

Then putting $x'$ in terms of $q$ and integrating

(14) $$x - b = a\left(\frac{3q^4}{4} + q^2 - \log q\right)$$

These two equations define a solution of Euler's differential equation which we shall call a Newton Curve.

It is also to be noted that apart from the extraneous solution $y' = 0$, there are when $a = 0$ the two solutions

(15) $$x = \alpha \qquad \text{and} \qquad y = \beta$$

The minimizing curve must then be sought among arcs composed of segments of these three curves.

The Newton Curve.

The Newton curve is plotted on page 5 for $a = 1$, $b = 0$, in terms of $(x, y, p)$ where $p = 1/q$ is the slope of the curve. The form given is easily verified from the equations: