

# Positive singular solutions of a certain Elliptic PDE

by

Negar Mohammadnejad

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## Abstract

In this thesis, we study the existence of positive singular solutions of a system of partial differential equations on a bounded domain. We first consider the solution of the following problem:

$$\begin{cases} -\Delta w = |\nabla w|^p & \text{in } B_1 \setminus \{0\}, \\ w = 0 & \text{on } \partial B_1. \end{cases} \quad (1)$$

Then we use its well-known solution to study the positive singular solutions of its perturbations on  $B_1$  which is a unit ball centered at the origin in  $\mathbb{R}^N$  and where we assume  $N \geq 3$  and  $\frac{N}{N-1} < p < 2$

$$\begin{cases} -\Delta u = (1 + \kappa_1(x))|\nabla v|^p & \text{in } B_1 \setminus \{0\}, \\ -\Delta v = (1 + \kappa_2(x))|\nabla u|^p & \text{in } B_1 \setminus \{0\}, \\ u = v = 0 & \text{on } \partial B_1. \end{cases} \quad (2)$$

In this equation,  $\kappa_1$  and  $\kappa_2$  are both non-negative, continuous functions such that  $\kappa_1(0) = \kappa_2(0) = 0$ . We want to show the existence of positive singular solutions of equation (2) on the domain.

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# 1

## Introduction

Many studies have been done on the existence of positive singular solutions of different partial differential equations. An example is the Lane-Emden equation

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $p > 1$  and  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  ( $N \geq 3$ ) with smooth boundary. The existence versus the non-existence of this problem has been studied on different bounded domains and also in cases where  $p$  belongs to other intervals; see [13, 9, 8, 10, 7]. A closer example to our equation (2) can be the equation (1). The case  $0 < p < 1$  was studied in [2]. In [12, 14], the authors study boundary blow up versions of (1) where they remove the minus sign in front of the Laplacian. In [3, 4, 5, 6, 11], studies on equations similar to (1) was done. In this work, we would like to prove the existence of positive singular solutions of the equation

$$\begin{cases} -\Delta u = (1 + \kappa_1(x))|\nabla v|^p & \text{in } B_1 \setminus \{0\}, \\ -\Delta v = (1 + \kappa_2(x))|\nabla u|^p & \text{in } B_1 \setminus \{0\}, \\ u = v = 0 & \text{on } \partial B_1, \end{cases} \quad (1.2)$$

where  $\frac{N}{N-1} < p < 2$ , and also  $\kappa_1 \geq 0$ , and  $\kappa_2 \geq 0$  are continuous with  $\kappa_i(0) = 0$ . Note that  $B_1$  is the unit ball centered at the origin in  $\mathbb{R}^N$  ( $N \geq 3$ ). Here we do not want any smallness conditions on  $\kappa_i$  except it is zero at the origin. Our main result is the following.

**Theorem 1.1.** Suppose  $N \geq 3$ ,  $\frac{N}{N-1} < p < 2$  and  $\kappa_1, \kappa_2$  are non-negative, continuous functions with  $\kappa_i(0) = 0$ . There exists an infinite number of positive singular solutions  $u_t, v_t$  of (1.2) which blow up at the origin. Moreover,  $u_t$  and  $v_t$  converge uniformly to zero away from the origin as  $t$  approaches infinity.

**The parameters.** The parameters  $p, \xi, \beta$ , and  $\sigma$  for the remainder of this work will be as follows unless otherwise stated.

First we consider the following singular solution of a scalar problem.

**Example 1.2.** Let  $N \geq 3$ ,  $\frac{N}{N-1} < p < 2$ ,  $\xi := (p-1)(N-1)$  (note this implies  $\xi > 1$ ),  $\beta := \frac{p-1}{\xi-1} > 0$ . Then for all  $t \geq 0$  the radial function

$$w_t(r) = \int_r^1 \frac{dy}{(ty^\xi + \beta y)^{\frac{1}{p-1}}} \quad (1.3)$$

is the singular solution of

$$\begin{cases} -\Delta w = |\nabla w|^p & \text{in } B_1 \setminus \{0\}, \\ w = 0 & \text{on } \partial B_1. \end{cases} \quad (1.4)$$

Our approach for finding a solution  $(u, v)$  of (1.2) will be to look for solutions of the form  $u = w_t + \phi$  and  $v = w_t + \psi$ . Note that  $\phi, \psi$  are unknown functions, but they are equal to zero on the boundary of  $B_1$  and  $w_t$  is the solution of (1.4). By considering our solutions  $u$  and  $v$  in this new form, we should also note that  $\phi$  and  $\psi$  need to

satisfy

$$\begin{cases} -\Delta\phi - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\psi = \kappa_1(x)|\nabla w_t + \nabla\psi|^p + I(\psi) & \text{in } B_1 \setminus \{0\}, \\ -\Delta\psi - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\phi = \kappa_2(x)|\nabla w_t + \nabla\phi|^p + I(\phi) & \text{in } B_1 \setminus \{0\}, \\ \phi = \psi = 0 & \text{on } \partial B_1, \end{cases} \quad (1.5)$$

where

$$I(\zeta) := |\nabla w_t + \nabla\zeta|^p - |\nabla w_t|^p - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\zeta.$$

To find a solution of (1.5), we will apply a fixed point argument. A key step will be to understand the linear operator on the left hand side of (1.5), namely the solvability of

$$\begin{cases} -\Delta\phi - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\psi = f & \text{in } B_1 \setminus \{0\}, \\ -\Delta\psi - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\phi = g & \text{in } B_1 \setminus \{0\}, \\ \phi = \psi = 0 & \text{on } \partial B_1, \end{cases} \quad (1.6)$$

for  $(\phi, \psi)$ , given  $f$  and  $g$ .

We define the nonlinear mapping  $T_t(\phi, \psi) = (\hat{\phi}, \hat{\psi})$  via

$$\begin{cases} -\Delta\hat{\phi} - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\hat{\psi} = \kappa_1(x)|\nabla w_t + \nabla\psi|^p + I(\psi) & \text{in } B_1 \setminus \{0\}, \\ -\Delta\hat{\psi} - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\hat{\phi} = \kappa_2(x)|\nabla w_t + \nabla\phi|^p + I(\phi) & \text{in } B_1 \setminus \{0\}, \\ \phi = \psi = 0 & \text{on } \partial B_1. \end{cases} \quad (1.7)$$

We will show that  $T_t$  is a contraction on a suitable complete metric space. Then, we use this to show the existence of the positive singular solutions of (1.2).

# 2

## Mathematical Background

### 2.1 Basic definitions

**Theorem 2.1.** *Local regularity of elliptic equations.* Let  $\Omega$  denote a bounded domain in  $\mathbb{R}^N$  and  $\Omega_0 \subset\subset \Omega$ . Then there is some  $C > 0$  such that for all  $u$  smooth one has

$$\|u\|_{C^{2,\alpha}(\Omega_0)} \leq C\|\Delta u\|_{C^{0,\alpha}(\bar{\Omega})} + C\|u\|_{C^{0,\alpha}(\bar{\Omega})}.$$

**Definition 2.2.** *Compact Set.* A subset  $K$  of a metric space is called compact if every sequence  $x_n$  in  $K$  has a subsequence which converges to a limit in  $K$ .

**Theorem 2.3.** *Diagonalisation Argument.* Let  $\Omega \subset \mathbb{R}^N$  be an open set. If  $f_n$  is a sequence in  $C^0(\Omega)$  which is bounded and continuous, then there exists a subsequence  $f_{n_k}$  and a function  $f \in \Omega$  such that for any compact subset  $K \subset \Omega$  the restrictions  $f_{n_k}|_K$  converge uniformly to  $f|_K$ . We say  $f_{n_k}$  converges uniformly to  $f$  on compact sets.

**Theorem 2.4.** Let  $(X, A, \mu)$  be a measure space and  $\{f_n\}_n$  a sequence of real-valued measurable bounded functions. Suppose  $f_n$  converges to  $f$  uniformly on  $X$



and suppose that  $\mu(X) < +\infty$ . Then

$$\int_X f_n d\mu \rightarrow \int_X f d\mu$$

as  $n$  goes to infinity.

**Theorem 2.5.** *Banach's Fixed Point Theorem.* Let  $(X, d)$  be a complete metric space and let  $T : X \rightarrow X$  be a contraction on  $X$  meaning there exists a positive constant  $K < 1$  such that

$$d(T(x), T(y)) \leq Kd(x, y) \quad \text{for all } x, y \in X.$$

Then  $T$  has a unique fixed point  $x \in X$  such that  $T(x) = x$ .

# 3

## Main approach

Our approach to find a solution  $(u, v)$  of (1.2) will be to look for solutions of the form  $u = w_t + \phi$  and  $v = w_t + \psi$  where  $\phi, \psi$  are unknown. We mentioned that

$$w_t(r) = \int_r^1 \frac{dy}{(ty^\xi + \beta y)^{\frac{1}{p-1}}}, \quad (3.1)$$

is a singular solution of

$$\begin{cases} -\Delta w = |\nabla w|^p & \text{in } B_1 \setminus \{0\}, \\ w = 0 & \text{on } \partial B_1, \end{cases} \quad (3.2)$$

where  $N \geq 3$ ,  $\frac{N}{N-1} < p < 2$ ,  $\xi = (p-1)(N-1)$ ,  $\beta = \frac{p-1}{\xi-1} > 0$ . To show this, note that for a radial function  $w(r)$  where  $|x| = r$  we have  $\nabla w(x) = w'(x) \frac{x}{r}$  and  $\Delta w = w'' + \frac{N-1}{r} w'$ . This means that

$$|\nabla w(x)| = |w'(x)| \frac{|x|}{r} = |w'(r)|.$$

Thus, assuming the solution is decreasing, we can rewrite 3.2 as:

$$-w'' - \frac{N-1}{r} w' = (-w'(r))^p$$

where  $w(1) = 0$ . Set  $v(r) := -w'(r)$ . By using the integrating factor, we can rewrite the equation as follows:

$$\begin{aligned} v'(r) + \frac{N-1}{r}v(r) &= (v(r))^p \\ \iff r^{N-1}v'(r) + (N-1)rv(r) &= r^{N-1}(v(r))^p \\ \iff \frac{d}{dr} \left( r^{N-1}v(r) \right) &= r^{N-1}(v)^p. \end{aligned}$$

After setting  $u(r) = r^{N-1}v(r)$ , we get

$$\frac{du}{dr} = u^p r^{(N-1)(1-p)}.$$

The method of the separation of variables yields

$$\begin{aligned} u(r)^{-p} du &= r^{(N-1)(1-p)} dr \\ \iff \frac{u(r)^{1-p}}{1-p} &= \frac{r^{(N-1)(1-p)+1}}{(N-1)(1-p)+1} + C \\ \iff 1 &= \frac{(p-1)}{Np-p-N} u(r)^{p-1} r^{(1-p)(N-1)+1} + C(p-1)u(r)^{p-1}. \end{aligned}$$

Since we have  $u^{p-1} = v^{(p-1)}r^{(N-1)(p-1)}$ , we get:

$$\begin{aligned} 1 &= \frac{(p-1)r}{Np-p-N} v^{p-1} + C(p-1)v^{p-1}r^{(N-1)(p-1)} \\ \iff 1 &= v(r)^{p-1} \left( \frac{(p-1)r}{Np-p-N} + Cr^{(N-1)(p-1)} \right). \end{aligned}$$

Then we have:

$$v_t(y)^{p-1} = \frac{1}{ty^\xi + \beta y}$$

where  $\beta = \frac{p-1}{Np-p-N}$  and  $\xi = (N-1)(p-1)$ . It follows that

$$v_t(y) = \frac{1}{(ty^\xi + \beta y)^{\frac{1}{p-1}}}$$

and so the solution is

$$w_t(r) = \int_r^1 \frac{dy}{(ty^\xi + \beta y)^{\frac{1}{p-1}}}.$$

If we set  $u = w_t + \phi$  and  $v = w_t + \psi$ , an immediate computation shows that  $\phi$  and  $\psi$  need to satisfy

$$\begin{cases} -\Delta\phi - p|\nabla w|^{p-2}\nabla w \cdot \nabla\psi = \kappa_1(x)|\nabla w + \nabla\psi|^p + I(\psi) & \text{in } B_1 \setminus \{0\}, \\ -\Delta\psi - p|\nabla w|^{p-2}\nabla w \cdot \nabla\phi = \kappa_2(x)|\nabla w + \nabla\phi|^p + I(\phi) & \text{in } B_1 \setminus \{0\}, \\ \phi = \psi = 0 & \text{on } \partial B_1, \end{cases} \quad (3.3)$$

where

$$I(\zeta) = |\nabla w + \nabla\zeta|^p - |\nabla w|^p - p|\nabla w|^{p-2}\nabla w \cdot \nabla\zeta.$$

Our goal in this next section will be to understand the solvability of

$$\begin{cases} -\Delta\phi - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\psi = f & \text{in } B_1 \setminus \{0\}, \\ -\Delta\psi - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\phi = g & \text{in } B_1 \setminus \{0\}, \\ \phi = \psi = 0 & \text{on } \partial B_1, \end{cases} \quad (3.4)$$

for  $(\phi, \psi)$  given  $f$  and  $g$ . There are two natural approaches to solve this problem.

**Approach 1.** Fix  $(f, g)$  and given  $\phi$  we solve the second equation for  $\psi$ , namely

$$-\Delta\psi = p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\phi + g \quad \text{in } B_1 \setminus \{0\},$$

with  $\psi = 0$  on  $\partial B_1$ . We now input this  $\psi$  into the first equation and solve for  $\phi$ , which we now call  $\hat{\phi}$ , i.e.

$$-\Delta\hat{\phi} = p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\psi + f \quad \text{in } B_1 \setminus \{0\},$$

with  $\hat{\phi} = 0$  on  $\partial B_1$ . This defines the operator  $T_t(\phi) = \hat{\phi}$ . Note that if we can find a fixed point of  $T_t$  (say  $T_t(\phi) = \phi$ ) then  $(\phi, \psi)$  satisfies the linear problem.

We will instead use another approach to decouple the system.

**Approach 2.** Fix  $(f, g)$ , and set  $F := f + g$  and  $G := f - g$ . Suppose that  $\zeta_i$  is a solution of the scalar problems

$$\begin{cases} -\Delta\zeta_1 - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\zeta_1 = F & \text{in } B_1 \setminus \{0\}, \\ \zeta_1 = 0 & \text{on } \partial B_1, \end{cases} \quad (3.5)$$

$$\begin{cases} -\Delta\zeta_2 + p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\zeta_2 = G & \text{in } B_1 \setminus \{0\}, \\ \zeta_2 = 0 & \text{on } \partial B_1. \end{cases} \quad (3.6)$$

The problem in (3.5) has been studied in the previous work [1], but (3.6) needs examination. A linear algebra argument shows that if  $\zeta_i$  satisfies the above, then for  $(\phi, \psi)$  to be a solution of (3.4), we need  $\phi, \psi$  to satisfy  $\phi + \psi = \zeta_1$ , and  $\phi - \psi = \zeta_2$ . From this, we see that

$$\phi = \frac{\zeta_1 + \zeta_2}{2} \quad \text{and} \quad \psi = \frac{\zeta_1 - \zeta_2}{2}. \quad (3.7)$$

So to understand the solvability of (3.4), it is enough to understand the solvability of the two scalar problems given by (3.5) and (3.6). By using the solution of (3.2), one can write out the left hand side of (3.5) explicitly as

$$-\Delta\zeta_1 + \frac{p}{(t|x|^{\xi-1} + \beta)} \frac{x \cdot \nabla\zeta_1}{|x|^2}.$$

Similarly, the right hand side of 3.6 is given by:

$$-\Delta\zeta_2 - \frac{p}{(t|x|^{\xi-1} + \beta)} \frac{x \cdot \nabla\zeta_2}{|x|^2}.$$

This motivates the definition of the linear operators

$$L_t^\pm(\zeta) = -\Delta\zeta \pm \frac{p}{(t|x|^{\xi-1} + \beta)} \frac{x \cdot \nabla\zeta}{|x|^2}. \quad (3.8)$$

So  $L_t^+$  is the linear operator associated with the left hand side of (3.5) and  $L_t^-$  is the linear operator associated with the left-hand side of (3.6). Later we will need the following asymptotic result:

- $\lim_{r \rightarrow 0} r^{\sigma+1} w_t'(r) = -C_\beta$  where  $C_\beta = \frac{1}{\beta^{\frac{1}{p-1}}}$  (3.9)
- for all  $t \geq 0$ , there exists a constant  $C$  such that  $\lim_{r \searrow 0} r^\sigma w_t(r) = C$ .

# 4

## The linear theory

### 4.1 Analysis of the Linear operators $L_t^\pm$

Define the following norms

$$\|\phi\|_X := \sup_{0 < |x| \leq 1} \{|x|^\sigma |\phi| + |x|^{\sigma+1} |\nabla \phi|\}$$

$$\|f\|_Y := \sup_{0 < |x| \leq 1} \{|x|^{\sigma+2} |f(x)|\}$$

where  $\sigma = \frac{2-p}{p-1}$ . Let  $X$  denote the set of functions  $\phi$  such that  $\|\phi\|_X < \infty$  and vanish on the boundary of  $B_1$  and we let  $Y$  denote the set of functions  $f$  such that  $\|f\|_Y < \infty$ . The goal is to show that our nonlinear mapping  $T_t(\phi, \psi)$  is a contraction by applying Banach's Fixed Point Theorem on the complete metric space

$$\mathcal{F}_R := \{(\phi, \psi) \in X \times X : \|\phi\|_X, \|\psi\|_X \leq R\}.$$

On this space, we have

$$\|(\phi, \psi)\|_{X \times X} := \|\phi\|_X + \|\psi\|_X.$$

We can now state our main proposition in this chapter.

**Proposition 4.1.** Let  $N \geq 3$ , such that  $\frac{N}{N-1} < p < 2$ ,  $\xi = (p-1)(N-1) > 1$ ,  $\beta = \frac{p-1}{\xi-1} > 0$  and  $\sigma = \frac{2-p}{p-1}$ . There is a positive constant  $C$  such that for all  $f, g \in Y$ , there are functions  $\phi$  and  $\psi$  in  $X$  which satisfy the equation

$$\begin{cases} -\Delta\phi - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\psi = f & \text{in } B_1 \setminus \{0\}, \\ -\Delta\psi - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\phi = g & \text{in } B_1 \setminus \{0\}, \\ \phi = \psi = 0 & \text{on } \partial B_1, \end{cases} \quad (4.1)$$

and the estimate

$$\|\phi\|_X + \|\psi\|_X \leq C\|f\|_Y + C\|g\|_Y.$$

With the approach we have taken in this section, we use the change of notation from (3.7) and restate our proposition as follows.

**Proposition 4.2.** Let  $N \geq 3$ , such that  $\frac{N}{N-1} < p < 2$ ,  $\xi = (p-1)(N-1) > 1$ ,  $\beta = \frac{p-1}{\xi-1} > 0$  and  $\sigma = \frac{2-p}{p-1}$ . There is some positive constant  $C$  such that for all  $f \in Y$  and non-negative  $t$ , there is some  $\phi \in X$  which satisfies

$$\begin{cases} L_t^\pm(\phi) = f & \text{in } B_1 \setminus \{0\}, \\ \phi = 0 & \text{on } \partial B_1. \end{cases} \quad (4.2)$$

Moreover, one has the estimate  $\|\phi\|_X \leq C\|f\|_Y$ .

**Spherical harmonics.** Note  $\Delta_\theta = \Delta_{S^{N-1}}$  is the Laplace-Beltrami operator on  $S^{N-1}$  with eigenpairs  $(\psi_k, \lambda_k)$  satisfying

$$-\Delta_\theta \psi_k(\theta) = \lambda_k \psi_k(\theta) \quad \text{for } \theta \in S^{N-1}$$

and  $\psi_0 = 1$ ,  $\lambda_0 = 0$ ,  $\lambda_1 = N-1$ ,  $\lambda_2 = 2N$ . We normalize  $\psi_k$  in  $L^2(S^{N-1})$  such that  $\|\psi_k\|_{L^2(S^{N-1})} = 1$ .



Using this, given  $f \in Y$ , and  $\phi \in X$ , we can decompose  $\phi$  and  $f$  into various modes by writing

$$f(x) = \sum_{k=0}^{\infty} b_k(r)\psi_k(\theta) \quad \text{and} \quad \phi(x) = \sum_{k=0}^{\infty} a_k(r)\psi_k(\theta).$$

Note that  $a_k(1) = 0$  after considering the boundary condition of  $\phi$ . A computation shows that  $\phi$  satisfies (4.2) provided  $a_k$  satisfies

$$-a_k''(r) - \frac{(N-1)a_k'(r)}{r} + \frac{\lambda_k a_k(r)}{r^2} \pm \frac{p a_k'(r)}{\beta r + t r^\xi} = b_k(r) \quad \text{for } 0 < r < 1 \quad (4.3)$$

with  $a_k(1) = 0$ .

Let  $X_1$  and  $Y_1$  be the closed subspaces of  $X$  and  $Y$  defined by

$$X_1 := \left\{ \phi \in X : \phi = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta) \right\} \quad \text{and} \quad Y_1 := \left\{ f \in Y : f = \sum_{k=1}^{\infty} b_k(r)\psi_k(\theta) \right\}.$$

Note that  $X_1$  and  $Y_1$  are just the same representations as  $X$  and  $Y$  except that the summations start at  $k = 1$  and not  $k = 0$ .

To show that they are closed subspaces of  $X$  and  $Y$ , we need to show that if any sequence  $\phi_m \in X_1$  is such that  $\phi_m$  converges to some  $\phi$  in  $X$ , then  $\phi$  is in  $X_1$ . We need a similar result for  $Y$  and  $Y_1$ . We first note that if we have  $\phi(x) \in X_1$  then we can write  $\phi(r, \theta) = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta)$  and by taking the integral of both sides with respect to  $\theta$ , we get

$$\int_{S^{N-1}} \phi(r, \theta) d\theta = \sum_{k=1}^{\infty} a_k(r) \int_{S^{N-1}} \psi_k(\theta) d\theta = 0.$$

This is due to the orthonormality of  $\psi_k$ 's and  $\psi_0 = 1$ . Thus, we can conclude that for all  $\phi$  in  $X$  we have

$$\phi \in X_1 \iff \int_{S^{N-1}} \phi(r, \theta) d\theta = 0 \quad \text{for all } 0 < r \leq 1. \quad (4.4)$$

Since by the assumption we have  $\phi_m(r, \theta) \in X_1$ , it follows that  $\int_{S^{N-1}} \phi_m(r, \theta) = 0$  for all  $0 < r \leq 1$ . Fix  $0 < r \leq 1$  and set  $\zeta_m(\theta) = \phi_m(r, \theta)$  and  $\zeta(\theta) = \phi(r, \theta)$ . We claim that  $\zeta_m$  converges uniformly to  $\zeta$  in  $X$  on  $S^{N-1}$ . Assuming the claim is true, we can fix  $0 < r \leq 1$  and by the uniform convergence of  $\zeta_m$  to  $\zeta$  on  $S^{N-1}$  we get

$$0 = \int_{S^{N-1}} \phi_m(r, \theta) \longrightarrow \int_{S^{N-1}} \phi(r, \theta).$$

Thus we have  $\int_{S^{N-1}} \phi(r, \theta) = 0$  and by (4.4) we can see that  $\phi \in X_1$ .

Let us prove the claim. If we fix  $\epsilon > 0$ , we can write

$$\begin{aligned} \|\phi_m - \phi\|_X &\geq \sup_{0 < |x| \leq 1} |x|^\sigma |\phi_m(x, \theta) - \phi(x, \theta)| \geq \sup_{\epsilon < |x| \leq 1} |x|^\sigma |\phi_m(x, \theta) - \phi(x, \theta)| \\ &\geq \epsilon^\sigma |\phi_m(x, \theta) - \phi(x, \theta)|. \end{aligned}$$

Note that  $r = |x|$  and  $\epsilon$  are fixed and we know that since  $\phi_m$  converges to  $\phi$  in  $X$  we have  $\|\phi_m - \phi\|_X \rightarrow 0$ . So we get the uniform convergence of  $\phi_m$  to  $\phi$  away from the origin. We can use a similar proof for  $Y_1$ . Thus  $X_1, Y_1$  are closed subspaces of  $X$  and  $Y$  respectively.

#### 4.1.1 Kernel of $L_0^\pm$

Using the definition of our linear operator (3.8) in the case of  $t = 0$ , we can write it explicitly as

$$L_0^\kappa(\phi) = -\Delta\phi + \frac{\kappa x \cdot \nabla\phi(x)}{\beta|x|^2}$$

where  $\kappa$  could be either equal to  $+p$  or  $-p$ . For simplicity, we set  $L_0^{+p} := L_0^+$  and  $L_0^{-p} := L_0^-$ .

**Lemma 4.3.** Let  $\frac{N}{N-1} < p < 2$ ,  $\xi = (p-1)(N-1) > 1$ ,  $\beta = \frac{p-1}{\xi-1} > 0$ ,  $\sigma = \frac{2-p}{p-1}$  and suppose  $\phi \in X_1$  is such that  $L_0^\kappa(\phi) = 0$  in  $B_R \setminus \{0\}$  with  $\phi = 0$  on  $\partial B_R$  in the case of  $R$  finite where we have  $\kappa = p$  or  $\kappa = -p$ , then  $\phi = 0$ .

*Proof.* We saw that we can decompose  $\phi(x)$  into  $\phi(x) = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta)$ , and since  $L_0^\kappa = 0$  in  $B_R \setminus \{0\}$ , then  $a_k$  should satisfy

$$-a_k''(r) - \frac{(N-1)a_k'(r)}{r} + \frac{\lambda_k a_k(r)}{r^2} + \frac{\kappa a_k'(r)}{\beta r} = 0 \quad \text{for } 0 < r < R$$

where  $a_k(R) = 0$  in the case of  $R < \infty$ .

Also we have

$$\sup_{0 < r < R} \{r^\sigma |a_k(r)| + r^{\sigma+1} |a_k'(r)|\} < \infty. \quad (4.5)$$

*Proof of the statement.* To show that (4.5) is true, note that given  $\phi \in X$ , we have

$$\phi(x) = \sum_{k=0}^{\infty} a_k(r)\psi_k(\theta)$$

with  $a_k(R) = 0$ . By using the property of orthonormality of  $\psi_k(\theta)$ 's, we have

$$a_k(r) = \int_{S^{N-1}} \phi(r, \theta) \psi_k(\theta) d\theta.$$

Thus, by noting that  $|\phi(r, \theta)|r^\sigma \leq \|\phi\|_X$ , we obtain

$$|a_k(r)| \leq \sup_{S^{N-1}} |\psi_k(\theta)| \int_{|\theta|=1} \frac{\|\phi\|_X}{r^\sigma} d\theta.$$

Since  $\phi$  is in  $X$ , there exists a positive  $C_k$  such that  $\|\phi\|_X \leq C$ . Thus we get

$|a_k(r)| \leq \frac{C_k}{r^\sigma}$  and we deduce

$$r^\sigma |a_k(r)| \leq C_k. \quad (4.6)$$

We know the gradient expressed in n-dimensional spherical coordinates is given by:

$$\nabla \phi = \phi_r \cdot \hat{r} + \frac{1}{r} \frac{\partial \phi}{\partial \theta_1} \cdot \hat{\theta}_1 + \frac{1}{\sin \theta_1} \cdot \frac{\partial \phi}{\partial \theta_2} \hat{\theta}_2 + \dots + \frac{1}{r \sin \theta_1 \dots \sin \theta_{n-2}} \cdot \frac{\partial \phi}{\partial \psi} \hat{\psi}.$$

Since  $\hat{r}$  is orthogonal to  $\hat{\theta}_i$  and  $\hat{\psi}$ , we obtain  $\phi_r = \nabla \phi(x) \cdot \hat{r}$  which gives us  $|\phi_r| \leq |\nabla \phi|$ .

Again for  $\phi \in X$  we can write  $\phi_r(r, \theta) = \sum_{k=0}^{\infty} a'_k(r) \psi_k(\theta)$  where  $a_k(1) = 0$ . We use the orthonormality of  $\psi_k(\theta)$ 's, and we obtain

$$a'_k(r) = \int_{S^{N-1}} \phi_r(r, \theta) \psi_k(\theta) d\theta.$$

Thus, noting that we have  $|\nabla \phi(r, \theta)| r^{\sigma+1} \leq \|\phi\|_X$ , we can write

$$|a'_k(r)| \leq \sup_{|\theta|=1} |\psi_k(\theta)| \int_{S^{N-1}} |\phi_r(r, \theta)| d\theta \leq \hat{C}_k \int_{|\theta|=1} |\nabla \phi(r\theta)| d\theta \leq \hat{C}_k \int_{|\theta|=1} \frac{\|\phi\|_X}{r^{\sigma+1}} \leq \frac{\tilde{C}_k}{r^{\sigma+1}}.$$

From this, we deduce

$$r^{\sigma+1} |a'_k(r)| < \tilde{C}. \quad (4.7)$$

Thus (4.6) and (4.7) give us the following result

$$\sup_{0 < r < R} \{r^\sigma |a_k(r)| + r^{\sigma+1} |a'_k(r)|\} < \infty.$$

□

Our equation

$$-a''_k(r) - \frac{(N-1)a'_k(r)}{r} + \frac{\lambda_k a_k(r)}{r^2} + \frac{\kappa a'_k(r)}{\beta r} = 0 \quad \text{for } 0 < r < R \quad (4.8)$$

is an Euler ODE and we can find its solutions. So we write

$$r^2 a''_k(r) + r a'_k(r) \left( (N-1) - \frac{\kappa}{\beta} \right) - \lambda_k a_k(r) = 0$$

and its solution is

$$a_k(r) = C_k r^{\gamma_k^+} + D_k r^{\gamma_k^-}$$

where  $\gamma^\pm$  are given by

$$\gamma_k^\pm(\kappa) = \frac{-(N-2-\frac{\kappa}{\beta})}{2} \pm \frac{\sqrt{(N-2-\frac{\kappa}{\beta})^2 + 4\lambda_k}}{2}.$$

• First we set

$$\kappa = +p.$$

This means that we are going to look at the kernel of  $L_0^+ = L_0^p$ . Using (4.8) our ODE is

$$-a_k''(r) - \frac{(N-1)a_k'(r)}{r} + \frac{\lambda_k a_k(r)}{r^2} + \frac{pa_k'(r)}{\beta r} = 0 \quad \text{for } 0 < r < R \quad (4.9)$$

with the solution

$$a_k(r) = C_k r^{\gamma_k^+} + D_k r^{\gamma_k^-}$$

for some  $C_k, D_k \in \mathbb{R}$  where  $\gamma_k^\pm$  are given by

$$\gamma_k^\pm(p) = \frac{-(N-2-\frac{p}{\beta})}{2} \pm \frac{\sqrt{(N-2-\frac{p}{\beta})^2 + 4\lambda_k}}{2}.$$

We claim  $\gamma_k^- + \sigma \leq -1 < 0$ . We also have that  $\gamma_k^+$  is nonzero and  $\gamma_k^+ + \sigma > 0$  for  $k \geq 1$ .

*Proof of the claim.* To show this, we define a function  $f(\gamma)$  as  $f(\gamma) = \gamma^2 + (N-2 - \frac{p}{\beta})\gamma - \lambda_1$ . We first note that  $-\sigma - 1 = \frac{-1}{p-1} < 0$ , so we get

$$f(-\sigma - 1) = \frac{1 + (1-p)(N-2-\frac{p}{\beta}) - (N-1)(p-1)^2}{(p-1)^2}$$

and by a computation, we have

$$f(-\sigma - 1) = \frac{\xi(p-1) - \xi(p-1)}{(p-1)^2} = 0.$$

It shows that  $\gamma_1^- = -\sigma - 1$ . Thus, we have  $\gamma_k^- + \sigma \leq \gamma_1^- + \sigma = -1$  for all  $k \geq 1$  by

monotonicity in  $k$ . We note that by definition, we have

$$\gamma_1^+(p) = \frac{-(N-2-\frac{p}{\beta})}{2} + \frac{\sqrt{(N-2-\frac{p}{\beta})^2 + 4(N-1)}}{2} > 0.$$

So again by the monotonicity in  $k$  we can see that  $\gamma_k^+ + \sigma > \gamma_1^+ + \sigma > 0$ .

□

We now show that  $\phi = 0$ . First we consider the case where  $0 < R < \infty$ . By considering our boundary condition, the equation should satisfy  $a_k(R) = C_k R^{\gamma_k^+} + D_k R^{\gamma_k^-} = 0$ . So there exists some non-zero  $\eta_k$  such that  $D_k = \eta_k(R)C_k$ . We can rewrite the equation as

$$a_k(r) = C_k(r^{\gamma_k^+} + \eta_k(R)r^{\gamma_k^-}).$$

For  $k \geq 1$ , we saw that  $\gamma_k^- + \sigma \leq -1 < 0$ . Thus, when  $r$  goes to zero,  $r^{\gamma_k^- + \sigma}$  would go to infinity. But since  $a_k$  should satisfy (4.5), we must have  $C_k = 0$ .

We now consider the case where  $R = \infty$ . We have that  $a_k(r) = C_k r^{\gamma_k^+} + D_k r^{\gamma_k^-}$  and  $\gamma_k^- + \sigma < 0$ ,  $\gamma_k^+ + \sigma > 0$  are both nonzero and distinct. So when  $r$  goes to zero and infinity, in order to have  $a_k$  satisfy (4.5), we must have  $C_k = D_k = 0$ . This shows that for all  $k \geq 1$  we have  $a_k = 0$ , thus

$$\phi = 0. \tag{4.10}$$

• We now set

$$\kappa = -p.$$

We will be looking at the kernel of  $L_0^- = L_0^{-p}$ . Again we can use equation (4.8).

Thus our ODE is

$$-a_k''(r) - \frac{(N-1)a_k'(r)}{r} + \frac{\lambda_k a_k(r)}{r^2} - \frac{p a_k'(r)}{\beta r} = 0 \quad \text{for } 0 < r < R \tag{4.11}$$

and its solution is  $a_k(r) = C_k r^{\gamma_k^+} + D_k r^{\gamma_k^-}$  for some  $C_k, D_k \in \mathbb{R}$  where  $\gamma^\pm$  are given by

$$\gamma_k^\pm(-p) = \frac{-(N-2+\frac{p}{\beta})}{2} \pm \frac{\sqrt{(N-2+\frac{p}{\beta})^2 + 4\lambda_k}}{2}. \quad (4.12)$$

Similarly we claim that  $\gamma_k^- + \sigma \leq -1$  for  $k \geq 1$ . To show this, define a function  $f(\gamma) = \gamma^2 + (N-2+\frac{p}{\beta})\gamma - \lambda_1$ . Noting that  $-\sigma - 1 = \frac{-1}{p-1}$ , and by considering the values of  $\beta$  and  $\xi$ , we get

$$\begin{aligned} f(-\sigma - 1) &= \left(\frac{-1}{p-1}\right)^2 + (N-2+\frac{p}{\beta})\left(\frac{-1}{p-1}\right) - (N-1) \\ &= \frac{-\xi - \xi(p-1) + 2p}{(p-1)^2} = \frac{2p(1-\xi)}{(p-1)^2}. \end{aligned} \quad (4.13)$$

Since  $\xi > 1$ , we have  $f(-\sigma - 1) < 0$ . Note that  $f(\gamma)$  is a quadratic function with  $\gamma_1^\pm$  as its roots. We showed that  $f(-\sigma - 1) < 0$ . Thus, we can conclude  $\gamma_1^- \leq -\sigma - 1 \leq \gamma_1^+$  and this means that  $\gamma_k^- \leq -\sigma - 1$  for all  $k \geq 1$  by the monotonicity in  $k$ . We now show that  $\gamma_k^+ + \sigma$  is nonzero and positive. Note that we have

$$\gamma_1^+(-p) = \frac{-(N-2+\frac{p}{\beta})}{2} + \frac{\sqrt{(N-2+\frac{p}{\beta})^2 + 4(N-1)}}{2} > 0.$$

Thus again by the monotonicity in  $k$ , we see that  $\gamma_k^+ + \sigma$  is positive for all  $k \geq 1$ . We first consider the case where  $0 < R < \infty$ . We use a similar approach to the case  $\kappa = p$ . By considering the boundary condition, to have  $a_k(r)$  satisfy (4.5), we must have  $a_k(r) = 0$ . For the case  $R = \infty$ , we can again use a similar approach as before and we note that  $\gamma_k^- + \sigma$  is negative, and  $\gamma_k^+ + \sigma$  is positive and they are distinct. By sending  $r$  to zero and infinity, we deduce that in order to have  $a_k$  satisfy (4.5), we must have  $a_k = 0$ . This shows that for all  $k \geq 1$

$$\phi = 0. \quad (4.14)$$

Our two results from (4.10) and (4.14) complete the lemma, and so we showed that for  $\phi \in X_1$  if  $L_0^\kappa(\phi) = 0$  in  $B_R \setminus \{0\}$  with  $\phi = 0$  on  $\partial B_R$  in the case of  $R$  finite where  $\kappa = p$  or  $\kappa = -p$ , then  $\phi = 0$ .

□

### 4.1.2 Kernel of $L_t^\pm$

Recall that we have our linear operator  $L_t^\kappa$  defined as :

$$L_t^\kappa(\phi) = -\Delta\phi \pm \frac{p}{(t|x|^{\xi-1} + \beta)} \frac{x \cdot \nabla\phi}{|x|^2} \quad (4.15)$$

where  $\kappa$  is either equal to  $p$  or  $-p$ .

**Lemma 4.4.** Let  $0 < R \leq \infty$ ,  $0 \leq t \leq \infty$ . When  $\kappa = p$ , suppose  $\phi \in X_1$  is such that  $L_t^p(\phi) = 0$  in  $B_R \setminus \{0\}$  with  $\phi = 0$  on  $\partial B_R$  where  $R$  is finite then  $\phi = 0$ . When  $\kappa = -p$ , suppose  $\phi \in X$  is such that  $L_t^{-p}(\phi) = 0$  in  $B_R \setminus \{0\}$  with  $\phi = 0$  on the boundary of  $B_R$  when  $R$  is finite, then  $\phi = 0$  and in case of  $R = \infty$ , then  $\phi$  is constant.

*Proof.* For the proof we will switch notations, and hence by (4.15) we can write  $L_t^+ = L_t^p$  and  $L_t^- = L_t^{-p}$ . Suppose  $R, t, \phi$  are as in the hypothesis.

- Set

$$\kappa = +p.$$

So we are looking at the kernel of  $L_t^+$ . Further, since  $L_\infty = -\Delta$  and this result is well known for the Laplacian, we suppose  $0 < t < \infty$ . We write  $\phi(x) = \sum_{k=1}^\infty a_k(r)\psi_k(\theta)$ , and since  $\phi \in X_1$ , there is no  $k = 0$  mode. Then  $a_k$  satisfies

$$-a_k''(r) - \frac{(N-1)a_k'(r)}{r} + \frac{\lambda_k a_k(r)}{r^2} + \frac{pa_k'(r)}{\beta r + tr^\xi} = 0 \quad \text{for } 0 < r < R \quad (4.16)$$



where  $a_k(R) = 0$  in the case of  $R < \infty$ . Also note that

$$\sup_{0 < r < R} \{r^\sigma |a_k(r)| + r^{\sigma+1} |a'_k(r)|\} < \infty. \quad (4.17)$$

Fix  $k \geq 1$  and we set  $\omega(\tau) := r^\sigma a_k(r)$  where  $\tau = \ln(r)$ . A computation shows that  $\omega = \omega(\tau)$  satisfies

$$\omega_{\tau\tau} + g(\tau)\omega_\tau + C_k\omega = 0 \quad \text{for } \tau \in (-\infty, \ln(R)) \quad (4.18)$$

where

$$g(\tau) = N - 2 - 2\sigma - \frac{p}{\beta + te^{(\xi-1)\tau}}$$

$$C_k(\tau) = -\lambda_k + \frac{p\sigma}{\beta + te^{(\xi-1)\tau}} - \sigma(N - 2 - \sigma).$$

We claim we have the improved decay estimate. Thus  $r^\sigma |a_k(r)| \rightarrow 0$  as  $r$  goes to zero and in the case of  $R$  being infinite, we have  $r^\sigma |a_k(r)| \rightarrow 0$  as  $r \rightarrow \infty$ .

For the moment, we assume we have the claim. Then note that we should have  $\omega \rightarrow 0$  as  $\tau$  goes to  $-\infty$ , and similarly in the case of  $R = \infty$ , we have  $\omega \rightarrow 0$  as  $\tau$  approaches infinity. By multiplying by  $-1$  if needed, we can assume that if  $\omega \neq 0$ , (and since  $\omega(-\infty) = \omega(\ln(R)) = 0$ ) there exists some  $\tau_0 \in (-\infty, \ln(R))$  such that  $\omega(\tau_0) = \max \omega > 0$ . It follows that we should have  $\omega_{\tau\tau}(\tau_0) \leq 0$  and  $\omega_\tau(\tau_0) = 0$ . Hence from the equation, we deduce that  $g(\tau_0)\omega_\tau(\tau_0) = 0$  and  $\omega_{\tau\tau}(\tau_0) + C_k\omega(\tau_0) = 0$ .

We get

$$-\omega_{\tau\tau} = C_k\omega(\tau_0) \geq 0. \quad (4.19)$$

This means that we should have  $C_k(\tau_0) \geq 0$ . Using the monotonicity of  $C_k$  in  $\tau$  and  $k$ , we see that for all  $\tau \in (-\infty, \ln(R))$  we have

$$C_k(\tau) \leq C_k(-\infty) \leq C_1(-\infty) = -(N - 1) + \frac{p\sigma}{\beta} - \sigma(N - 2 - \sigma).$$

By considering the bound on  $p$ , we can show that  $-(N-1) + \frac{p\sigma}{\beta} - \sigma(N-2-\sigma)$  is negative. To see this, note that we have

$$\begin{aligned} -(N-1) + \frac{p\sigma}{\beta} - \sigma(N-2-\sigma) &= -N + 1 + \frac{2\sigma}{\beta} - \sigma N + 2\sigma + \sigma^2 \\ &= \frac{(p-1)(pN - p + p^2 - Np^2 - 1)}{(p-1)^2} = \frac{(p-1)(N-1)(p-p^2) + (1-p)}{(p-1)^2}. \end{aligned}$$

Thus,  $(N-1) + \frac{p\sigma}{\beta} - \sigma(N-2-\sigma) < 0$  which means that  $C_k(\tau) < 0$  for all  $\tau \in (-\infty, \ln(R))$ . This is a contradiction and we must have  $\omega = 0$ . Hence  $a_k = 0$  for all  $k \geq 1$ .

We now prove the improved decay estimate.

*Proof of the improved decay estimate.* Fix  $k \geq 1$  and set  $a(r) = a_k(r)$ . First we assume  $R$  is finite. So we have

$$-\Delta a(r) + \frac{\lambda_k a(r)}{r^2} + \frac{pa'(r)}{\beta r + tr^\xi} = 0 \quad \text{in } 0 < r < R,$$

with  $a(R) = 0$ . We suppose the claim is false. Then there exists some  $r_m$  approaching zero such that  $r_m^\sigma |a(r_m)| \geq \epsilon_0 > 0$ . Define the rescaled function  $a^m(r) := r_m^\sigma a(r_m r)$  and note that we also have  $|a_m(1)| \geq \epsilon_0$  and  $r^\sigma |a^m(r)| \leq C$ . A computation shows that

$$r_m^{\sigma+2} \left[ -\Delta a(r_m r) + \frac{\lambda_k a(r_m r)}{(r_m r)^2} + \frac{pa'(r_m r)}{\beta r_m r + t(r_m r)^\xi} \right] = 0 \quad \text{in } 0 < r_m r < R,$$

so we get

$$-\Delta a^m(r) + \frac{\lambda_k a^m(r)}{r^2} + \frac{p(a^m)'(r)}{\beta r + tr^\xi r_m^{\xi-1}} = 0 \quad \text{in } 0 < r < \frac{R}{r_m}.$$

By passing to the limit, we can find some  $a^\infty$  bounded away from zero such that

$r^\sigma |a^\infty(r)| + r^{\sigma+1} |(a^\infty)'(r)| \leq C$ . Thus we have

$$-\Delta a^\infty(r) + \frac{\lambda_k a^\infty(r)}{r^2} + \frac{p(a^\infty)'(r)}{\beta r} = 0 \quad \text{in } 0 < r < \infty.$$

Let  $\phi = a^\infty \psi_k(\theta)$ . So  $\phi$  is nonzero too and it is in the kernel of  $L_0^+$ . This is a contradiction with Lemma 4.3 which stated the kernel of  $L_0^+$  is trivial.

In the case of  $R = \infty$ , we assume there exists some  $r_m$  approaching infinity as  $m$  goes to infinity. Again we can pass to the limit in

$$-\Delta a^m(r) + \frac{\lambda_k a^m(r)}{r^2} + \frac{p(a^m)'(r)}{\beta r + tr^\xi r_m^{\xi-1}} = 0 \quad \text{in } 0 < r < \infty$$

and so we get

$$-(a^\infty)''(r) - \frac{(N-1)(a^\infty)'}{r} + \frac{\lambda_k a^\infty(r)}{r^2} = 0.$$

This is an Euler ODE and the solution is  $a_k(r) = C_k r^{\gamma_k^+} + D_k r^{\gamma_k^-}$  where

$$\gamma_k^\pm = \frac{-(N-2) \pm \sqrt{(N-2)^2 + 4\lambda_k}}{2}.$$

We claim  $\gamma_k^- + \sigma < 0 < \gamma_k^+ + \sigma$ . By assuming we have the claim, we can send  $r$  to zero and infinity. Thus we can deduce that to have  $a_k$  satisfy (4.17), we must have  $C_k = D_k = 0$ . As a result,  $a_k$  should be zero for all  $k \geq 1$ .

We now prove the claim. Recall that  $\sigma = \frac{2-p}{p-1} > 0$  and also note that

$$\gamma_1^+ = \frac{-(N-2) + \sqrt{(N-2)^2 + 4(N-1)}}{2} = \frac{-N+2}{2} + \frac{N}{2} = 1. \quad (4.20)$$

Thus  $\gamma_1^+ + \sigma$  is positive and we know by monotonicity in  $k$ , we have that  $\gamma_1^+ < \gamma_k^+$  for all  $k \geq 1$ . So we deduce that  $\gamma_k^+ + \sigma > 0$ .

We now show that  $\gamma_1^- + \sigma < 0$ . A simple computation shows that  $\gamma_1^- = -N + 1$  so

$$\gamma_1^- + \sigma = -N + 1 + \sigma = -N + 1 + \frac{2-p}{p-1} = \frac{-Np + N + 1}{p-1} = \frac{N(-p+1) + 1}{p-1}.$$

Considering the bounds on  $p$ , we obtain

$$\gamma_1^- + \sigma < \frac{N(\frac{-N}{N-1} + 1) + 1}{p-1} = \frac{(\frac{-N}{N-1}) + 1}{p-1} = \frac{-N + N - 1}{(N-1)(p-1)} = \frac{-1}{(N-1)(p-1)} < 0. \quad (4.21)$$

So  $\sigma + \gamma_1^-$  is negative and by the monotonicity in  $k$  we have  $\gamma_k^- < \gamma_1^-$  for all  $k \geq 1$ , so we find that  $\gamma_k^- + \sigma < 0$ . This completes the proof of the improved decay estimate.  $\square$

• We now set

$$\kappa = -p.$$

In this part, we are looking at the kernel of  $L_t^-$ . First we set  $k \geq 1$ . Thus  $\phi$  is in  $X_1$ . Suppose  $0 < t < \infty$ . We write  $\phi$  as  $\phi(x) = \sum_{k=1}^{\infty} a_k(r) \psi_k(\theta)$ , and then  $a_k$  should satisfy

$$-a_k''(r) - \frac{(N-1)a_k'(r)}{r} + \frac{\lambda_k a_k(r)}{r^2} - \frac{p a_k'(r)}{\beta r + t r^\xi} = 0 \quad \text{for } 0 < r < R$$

where  $a_k(R) = 0$  when  $R < \infty$ . Also, similar to the last case we should have

$$\sup_{0 < r < R} \{r^\sigma |a_k(r)| + r^{\sigma+1} |a_k'(r)|\} < \infty. \quad (4.22)$$

Hence, we use a similar approach to the case  $\kappa = +p$ . We fix  $k \geq 1$  and we set  $\omega(\tau) = r^\sigma a_k(r)$  where  $\tau = \ln(r)$ . By a computation, we find that  $\omega = \omega(\tau)$  satisfies

$$\omega_{\tau\tau} + g(\tau)\omega_\tau + C_k\omega = 0 \quad \tau \in (\infty, \ln(R))$$

where

$$g(\tau) = N - 2 - 2\sigma + \frac{p}{\beta + te^{(\xi-1)\tau}}$$

$$C_k(\tau) = -\lambda_k - \frac{p\sigma}{\beta + te^{(\xi-1)\tau}} - \sigma(N - 2 - \sigma).$$

We claim we have the improved decay estimate meaning  $r^\sigma |a_k(r)| \rightarrow 0$  as  $r$  goes to zero and when  $R$  is infinite we have  $r^\sigma |a_k(r)| \rightarrow 0$  as  $r \rightarrow \infty$ . For the moment, we assume we have the claim. Then note that the same as the last case, this gives us  $\omega \rightarrow 0$  as  $\tau \rightarrow -\infty$  and in the case of  $R = \infty$  we have the same result when  $\tau \rightarrow \infty$ . By multiplying by  $-1$ , if needed, we can assume that if  $\omega \neq 0$  (and since  $\omega(-\infty) = \omega(\ln(R)) = 0$ ), there exists some  $\tau_0 \in (-\infty, \ln(R))$  such that  $\omega(\tau_0) = \max \omega > 0$ . Then we get  $\omega_{\tau\tau}(\tau_0) \leq 0$  and  $\omega_\tau(\tau_0) = 0$  and hence from the equation, we get  $g(\tau_0)\omega_\tau(\tau_0) = 0$  and

$$\omega_{\tau\tau}(\tau_0) + C_k\omega(\tau_0) = 0 \implies -\omega_{\tau\tau}(\tau_0) = C_k\omega(\tau_0) \geq 0.$$

From this, we see that we must have

$$C_k(\tau_0) = -\lambda_k - \frac{p\sigma}{\beta + te^{(\xi-1)\tau_0}} - \sigma(N - 2 - \sigma) \geq 0.$$

We know  $\lambda_k$  is positive for all  $k \geq 1$  and it is obvious that  $\frac{p\sigma}{\beta + te^{(\xi-1)\tau}} > 0$  for all  $\tau \in (-\infty, \ln(R))$ . Also note that

$$N - 2 - \sigma = N - 2 - \frac{2-p}{p-1} = N - 2 - \frac{2}{p-1} + \frac{p}{p-1} = N - 1 - \frac{1}{p-1}.$$

By considering the restrictions on  $p$ , we find that  $N - 1 - \frac{1}{p-1} > 0$  for  $N \geq 3$ . Thus we can deduce that

$$-\lambda_k < 0, \quad -\frac{p\sigma}{\beta + te^{(\xi-1)\tau}} < 0, \quad \text{and} \quad -\sigma(N - 2 - \sigma) < 0.$$

So  $C_k(\tau)$  is positive for all  $\tau \in (-\infty, \ln(R))$  which means  $C_k(\tau_0) < 0$ . Hence, we have a contradiction and thus  $\omega = 0$ . This gives us  $a_k = 0$  for all  $k \geq 1$ .

We now prove the claimed decay estimate.

*Proof of the claimed decay estimate.* We again use a similar proof to the case  $\kappa = +p$ .

We fix  $k \geq 1$  and set  $a(r) = a_k(r)$ , so we have

$$-\Delta a(r) + \frac{\lambda_k a(r)}{r^2} - \frac{pa'(r)}{\beta r + tr^\xi} = 0 \quad \text{in } 0 < r < R,$$

with  $a(R) = 0$ . Suppose the claim is false. Then there is some  $r_m$  that goes to zero such that  $r_m^\sigma |a(r_m)| \geq \epsilon_0 > 0$ . Define the rescaled function  $a^m(r) := r_m^\sigma a(r_m r)$  and note that  $|a_m(1)| \geq \epsilon_0$  and  $r^\sigma |a^m(r)| \leq C$ . A computation shows that

$$r_m^{\sigma+2} \left[ \Delta a(r_m r) + \frac{\lambda_k a(r_m r)}{(r_m r)^2} - \frac{pa'(r_m r)}{\beta r_m r + t(r_m r)^\xi} \right] = 0 \quad \text{in } 0 < r_m r < R,$$

so we get

$$-\Delta a^m(r) + \frac{\lambda_k a^m(r)}{r^2} - \frac{p(a^m)'(r)}{\beta r + tr^\xi r_m^{\xi-1}} = 0 \quad \text{in } 0 < r < \frac{R}{r_m}. \quad (4.23)$$

By passing to the limit, we find  $a^\infty$  such that it is bounded away from zero with  $r^\sigma |a^\infty(r)| + r^{\sigma+1} |(a^\infty)'(r)| \leq C$ . Thus, we have

$$-\Delta a^\infty(r) + \frac{\lambda_k a^\infty(r)}{r^2} + \frac{p(a^\infty)'(r)}{\beta r} = 0 \quad \text{in } 0 < r < \infty.$$

Let  $\phi(x) = a^\infty(r)\psi_k(\theta)$ . So  $\phi$  is nonzero too and it is in the kernel of  $L_0^-$ . This is a contradiction with Lemma 4.3 which stated the kernel of  $L_0^-$  is trivial.

In the case of  $R = \infty$ , we assume there exists some  $r_m$  approaching infinity as  $m$  goes to infinity. Again we pass to the limit in

$$-\Delta a^m(r) + \frac{\lambda_k a^m(r)}{r^2} - \frac{p(a^m)'(r)}{\beta r + tr^\xi r_m^{\xi-1}} = 0 \quad \text{in } 0 < r < \infty. \quad (4.24)$$

and we get

$$-(a^\infty)''(r) - \frac{(N-1)(a^\infty)'}{r} + \frac{\lambda_k a^\infty(r)}{r^2} = 0 \quad \text{in } 0 < r < \infty.$$

This is again an Euler ODE and the characteristic equation would be  $\gamma^2 + (N-2)\gamma - \lambda_k = 0$  such that

$$\gamma_k^\pm = \frac{-(N-2) \pm \sqrt{(N-2)^2 + 4\lambda_k}}{2}.$$

Thus, the solution is  $a_k(r) = C_k r^{\gamma_k^+} + D_k r^{\gamma_k^-}$ . In (4.20) and (4.21), we showed that  $\gamma_k^- + \sigma < 0 < \gamma_k^+ + \sigma$ . Thus we can deduce that when we send  $r$  to zero and infinity, we must have  $C_k = D_k = 0$  in order to have  $a_k$  in the required space. This result give us that  $a_k = 0$  for all  $k \geq 1$ .

• We now set  $k = 0$ . We have  $L_t^- = 0$  thus we can write

$$-a_k''(r) - \frac{(N-1)a_k'(r)}{r} - \frac{pa_k'(r)}{\beta r + tr^\xi} = 0 \quad \text{for } 0 < r < R \quad (4.25)$$

such that  $\sup_{0 < r < 1} \{r^\sigma |a_t(r)| + r^{\sigma+1} |a_t'(r)|\} \leq C$ . We know the integrating factor associated with our ODE (4.25) is  $\mu_t(r) = e^{P(r)}$  where

$$\begin{aligned} P(r) &= \int_1^r \left( \frac{N-1}{\tau} + \frac{p}{\beta\tau + t\tau^\xi} \right) d\tau = \left( (N-1) \ln \tau + \frac{p}{\beta(1-\xi)} \ln(\beta\tau^{\xi-1} + t) \right) \Big|_1^r \\ &= (N-1) \ln r + \frac{p}{\beta(1-\xi)} \left( \ln r^{1-\xi} + \ln \left( \frac{tr^{\xi-1} + \beta}{t + \beta} \right) \right). \end{aligned}$$

So we can write the integrating factor as

$$\begin{aligned} \mu_t(r) &= e^{(N-1) \ln r + \frac{p}{\beta(1-\xi)} \left( \ln r^{1-\xi} + \ln \left( \frac{tr^{\xi-1} + \beta}{t + \beta} \right) \right)} \\ &= r^{N-1 + \frac{p}{\beta}} \left( \frac{tr^{\xi-1} + \beta}{t + \beta} \right)^{\left(-\frac{p}{\beta(1-\xi)}\right)} = r^{N-1 + \frac{p}{\beta}} \left( \frac{tr^{\xi-1} + \beta}{t + \beta} \right)^{\left(\frac{p}{1-p}\right)}. \end{aligned}$$

Thus from (4.25), using the integrating factor, we get:

$$\frac{d}{dr}(\mu_t(r)a'_t(r)) = 0.$$

This means that we should have:

$$\mu_t a'(r) = C$$

where  $C$  is a constant. Thus we have

$$r^{\sigma+1}|a'_t(r)| = \frac{Cr^{\sigma+1}}{\mu_t(r)} = Cr^{\sigma+1-N+1-\frac{p}{\beta}} \left( \frac{tr^{\xi-1} + \beta}{t + \beta} \right)^{\frac{p}{p-1}}.$$

Note that  $\sigma + 2 - N - \frac{p}{\beta} = \frac{(\xi-1)(-1-p)}{p-1} < 0$ . Since  $\phi$  is in  $X$ , we require  $a'_t$  to satisfy the required bounds meaning  $r^{\sigma+1}|a'_t(r)|$  should be bounded. So we can deduce that  $C$  should be zero. So we have

$$\mu_t a'_t(r) = 0.$$

By considering the boundary condition  $a_t(R) = 0$ , we obtain  $a_t(r) = 0$  for  $r \in (0, R)$  and  $a_t(r)$  is constant on the whole space  $\mathbb{R}^N$ . We now consider the case where  $t$  approaches infinity. Thus we have the ODE

$$-a''_k(r) - \frac{(N-1)a'_k}{r} = 0 \tag{4.26}$$

where  $a(1) = 0$ . We know the integrating factor associated with our ODE (4.26) is  $\mu_t(r) = e^{P(r)}$  where

$$P(r) = \int_1^r \frac{N-1}{r} = (N-1) \ln r.$$



So we can write the integrating factor as  $\mu_t(r) = r^{(N-1)}$ . Thus, we have

$$\frac{d}{d\tau} (\mu_t(\tau)a'_t(\tau)) = 0.$$

Now by solving this equation, we obtain

$$a(r) = C \int_r^1 \frac{1}{s^{N-1}} ds = \frac{C}{2-N} s^{2-N} \Big|_r^1 = C \left( \frac{1}{2-N} - \frac{r^{2-N}}{2-N} \right).$$

We know  $a(r)$  needs to satisfy  $r^\sigma |a_t(r)| \leq C_1$  for some positive constant  $C_1$ . So we should have

$$r^\sigma \left| C \left( \frac{1}{2-N} - \frac{r^{2-N}}{2-N} \right) \right| = \left| C \left( \frac{r^\sigma}{2-N} - \frac{r^{\sigma+2-N}}{2-N} \right) \right| \leq C_1.$$

Noting that  $\sigma = \frac{2-p}{p-1} > 0$  and  $\frac{N}{N-1} < p < 2$ , by a computation we can see that  $\sigma + 2 - N$  is negative. So for  $a_t(r)$  to satisfy the required bound when  $r$  approaches zero, we should have  $C = 0$ . This shows that  $a_t(r) = 0$  when  $t$  goes to infinity and so we should have  $\phi = 0$ .

Thus the lemma is complete and we proved if  $\kappa = p$  and  $\phi \in X_1$  is such that  $L_t^p(\phi) = 0$  in  $B_R \setminus \{0\}$  with  $\phi = 0$  on  $\partial B_R$  in the case of  $R$  finite then  $\phi = 0$  and when  $\kappa = -p$  and  $\phi \in X$  is such that  $L_t^{-p}(\phi) = 0$  in  $B_R \setminus \{0\}$  with  $\phi = 0$  on  $\partial B_R$  in the case of  $R$  finite then  $\phi = 0$  and in case of  $k = 0$  and  $R$  infinite  $\phi$  is a constant.  $\square$

### 4.1.3 Linear theory of $L_t^\pm$ on $X_1$ ; A priori estimate

**Theorem 4.5.** There is some positive constant  $C$  such that for all positive  $t_m$ , and functions  $\phi_m \in X_1$  and  $f^m \in Y_1$  we have

$$\begin{cases} L_{t_m}^\pm(\phi_m) = f^m & \text{in } B_1 \setminus \{0\}, \\ \phi_m = 0 & \text{on } \partial B_1. \end{cases} \quad (4.27)$$

One has the estimate  $\|\phi_m\|_X \leq C\|f^m\|_Y$ .

*Proof.* If we assume the result is false, then by passing to a subsequence (without renaming) there is some  $C_m > 0$ ,  $t_m \geq 0$ ,  $f^m \in Y_1$  and  $\phi_m \in X_1$  with

$$\begin{cases} -\Delta\phi_m + \frac{\kappa}{(t_m|x|^{\xi-1} + \beta)} \frac{x \cdot \nabla\phi_m}{|x|^2} = f^m & \text{in } B_1 \setminus \{0\}, \\ \phi_m = 0, & \text{on } \partial B_1, \end{cases} \quad (4.28)$$

and

$$\|\phi_m\|_X > C_m\|f^m\|_Y.$$

By normalizing, we get  $\|\phi_m\|_X = 1$  and  $\|f^m\|_Y \rightarrow 0$ .

We claim

$$\sup_{0 < |x| \leq 1} \{|x|^{\sigma+1} |\nabla\phi_m|\} \rightarrow 0. \quad (4.29)$$

We will show that proving this claim results in:  $\sup_{0 < |x| \leq 1} \{|x|^\sigma |\phi_m|\} \rightarrow 0$ .

*proof of the claim.* Suppose there is some  $0 < |x_m| < 1$  and  $\epsilon_0 > 0$  such that

$$\epsilon_0 \leq |x_m|^{\sigma+1} |\nabla\phi_m(x_m)| \leq 1.$$

There are two cases that should be considered. Either  $|x_m|$  could be bounded away from zero or it could be approaching zero.

- We first set  $\kappa = p$ . So there exists some  $C > 0$ , and there are some  $\phi_m \in X_1$ , and  $f^m \in Y_1$  such that they satisfy

$$\begin{cases} -\Delta\phi_m + \frac{p}{(t_m|x|^{\xi-1} + \beta)} \frac{x \cdot \nabla\phi_m}{|x|^2} = f^m & \text{in } B_1 \setminus \{0\}, \\ \phi_m = 0, & \text{on } \partial B_1, \end{cases} \quad (4.30)$$

and we have

$$\|\phi_m\|_X > C\|f^m\|_Y$$

where  $\|\phi_m\|_X = 1$ ,  $\|f^m\|_Y \rightarrow 0$ . Also there are some  $0 < |x_m| < 1$  and  $\epsilon_0 > 0$  such

that

$$\epsilon_0 \leq |x_m|^{\sigma+1} |\nabla \phi_m(x_m)| \leq 1. \quad (4.31)$$

Now we prove the first case for  $L_{t_m}^+$ .

**Case 1:** Assume  $|x_m|$  is bounded away from zero. Define  $A_k$  and  $\tilde{A}_k$  for  $k \geq 2$  as

$$A_k = \left\{ x \in B_1 : \frac{1}{k} < |x| < 1 \right\} \quad \text{and} \quad \tilde{A}_k = \left\{ x \in B_1 : \frac{1}{2k} < |x| < 1 \right\}.$$

Note that  $A_k \subset \tilde{A}_k$ . From the equation (4.30), we have

$$-\Delta \phi_m = f^m - \frac{p}{(t_m |x|^{\xi-1} + \beta)} \frac{x \cdot \nabla \phi_m}{|x|^2} \quad \text{in } \tilde{A}_k.$$

Set  $g^m := f^m - \frac{p}{(t_m |x|^{\xi-1} + \beta)} \frac{x \cdot \nabla \phi_m}{|x|^2}$ . We can see that

$$\begin{aligned} \left| \frac{p}{(t_m |x|^{\xi-1} + \beta)} \frac{x \cdot \nabla \phi_m}{|x|^2} \right| &\leq \frac{p}{(t_m |x|^{\xi-1} + \beta)} \left| \frac{x \cdot \nabla \phi_m}{|x|^2} \right| \\ &\leq \frac{p}{(t_m |x|^{\xi-1} + \beta)} \frac{|x|^{\sigma+1} |\nabla \phi_m|}{|x|^{\sigma+2}} \\ &\leq \frac{p}{\beta} \frac{\|\phi_m\|_X}{|x|^{\sigma+2}} = \frac{p}{\beta} \frac{1}{|x|^{\sigma+2}} \leq \frac{p}{\beta} (2k)^{2+\sigma} := \gamma_k. \end{aligned} \quad (4.32)$$

Also we find that,

$$\sup_{x \in \tilde{A}_k} |f^m| = \sup_{x \in \tilde{A}_k} \frac{|x|^{\sigma+2} |f^m|}{|x|^{\sigma+2}} \leq \left( \sup_{x \in \tilde{A}_k} \frac{\|f^m\|_Y}{|x|^{\sigma+2}} \right) \leq (2k)^{2+\sigma} \|f^m\|_Y = \gamma_k \|f^m\|_Y \quad (4.33)$$

which by (4.32) and (4.33) we can see that  $g_m$  is bounded in  $\tilde{A}_k$ . So there exists a positive constant  $C$  such that

$$\|g^m(x)\|_{L^\infty(\tilde{A}_k)} \leq C$$

meaning

$$\|\Delta\phi_m\|_{L^\infty(\bar{A}_k)} \leq C.$$

By elliptic regularity 2.1, we can say that for some  $0 < \lambda < 1$ , there exists  $C_1 > 0$  such that

$$\|\phi_m\|_{C^{1,\lambda}(\bar{A}_k)} \leq C_1.$$

Thus  $\{\phi_m\}_m$  is a bounded sequence in  $C^{1,\lambda}(\bar{A}_k)$  for all  $k \geq 2$ . By the standard compactness argument and a diagonal argument there exists some subsequence  $\{\phi_{m_i}\}_i \subset \{\phi_m\}_m$  and  $\phi \in C^{1,\frac{\lambda}{2}}(\bar{A}_k)$  such that

$$\phi_{m_i} \rightarrow \phi \quad C^{1,\frac{\lambda}{2}}(\bar{A}_k).$$

So we can write

$$\phi_{m_i} \rightarrow \phi \quad C_{loc}^{1,\frac{\lambda}{2}}(\bar{B}_1 \setminus \{0\}).$$

Suppose  $t_m$  converges to some  $t \in [0, \infty]$  and by passing the limit in (4.30). We see that when  $t \in [0, \infty)$ ,  $\phi$  solves

$$-\Delta\phi + \frac{p}{(t|x|^{\xi-1} + \beta)} \frac{x \cdot \nabla\phi}{|x|^2} = 0 \quad \text{in } B_1 \setminus \{0\}$$

with  $\phi = 0$  on  $\partial B_1$ . When  $t \rightarrow \infty$ , we get

$$-\Delta\phi = 0 \quad \text{in } B_1 \setminus \{0\}$$

with  $\phi = 0$  on  $\partial B_1$ . Using the completeness of  $\mathbb{R}^N$ , we can pass to a subsequence and so  $x_m$  converges to some  $x_0$  such that  $|x_0|$  is bounded away from zero. We can now pass the limit in (4.31) to see that  $|x_0|^{\sigma+1} |\nabla\phi(x_0)| \geq \epsilon_0$  and this means  $\phi \neq 0$ . We need to show that  $\phi$  is in  $X_1$ . So we need to show that  $\phi$  belongs to  $X$  and it has no  $k = 0$  mode. We showed that for every fixed  $0 < |x| < 1$ , we have

$|x|^{\sigma+1}|\nabla\phi_{m_i}(x)| \leq 1$ . So by passing to the limit, we get  $|x|^{\sigma+1}|\nabla\phi| \leq 1$ . Thus by integration, we can show that we have  $|x|^\sigma|\phi(x)| \leq 1$ , so we get

$$\sup_{0 < |x| \leq 1} \left\{ |x|^\sigma |\phi(x)| + |x|^{\sigma+1} |\nabla\phi(x)| \right\} < \infty.$$

This means that  $\phi$  is in  $X$ . First note that since  $\phi_m$  belong to  $X_1$  for all  $0 < r \leq 1$ , we can write

$$\int_{|\theta|=1} \phi_m(r\theta) d\theta = \sum_{k=1}^{\infty} a_k(r) \int_{|\theta|=1} \psi_{k,m}(\theta) d\theta = 0. \quad (4.34)$$

This shows that  $\phi_m$  has zero average over all the sphere for radius  $0 < r \leq 1$ . Now we use the convergence we obtained above, and we can write for all  $0 < r \leq 1$  we have that  $\theta \mapsto \phi_m(r\theta)$  converges uniformly on  $S^{N-1}$  to  $\theta \mapsto \phi(r\theta)$ . Thus for all  $0 < r \leq 1$  we have

$$0 = \int_{|\theta|=1} \phi_m(r\theta) d\theta \rightarrow \int_{|\theta|=1} \phi(r\theta) d\theta.$$

So  $\phi$  also has zero average over all the sphere for radius  $0 < r \leq 1$  and hence  $\phi$  is in  $X_1$ . This means that  $\phi \in X_1$  is nonzero. Thus for  $t \in [0, \infty)$  we have a contradiction with the results from the previous lemmas. We now show that in case of  $t \rightarrow \infty$  we also get a contradiction. We saw that when  $t = \infty$ , we have

$$\begin{cases} \Delta\phi = 0 & \text{in } B_1 \setminus \{0\}, \\ \phi = 0 & \text{on } \partial B_1. \end{cases} \quad (4.35)$$

Then we write  $\phi = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta)$ , and solving the equation gives us

$$a_k = C_k r^{\gamma_k^+} + D_k r^{\gamma_k^-}$$

where

$$\gamma_k^\pm = \frac{-(N-2)}{2} \pm \frac{\sqrt{(N-2)^2 + 4\lambda_k}}{2}.$$

In Lemma 4.4, we showed that  $\gamma_k^- + \sigma < 0 < \gamma_k^+ + \sigma$  for all  $k \geq 1$ . Hence, to have  $\phi$  in the required space, we must have  $C_k = D_k = 0$  and so  $\phi = 0$ . It is a contradiction with  $\phi$  being nonzero.

**Case 2:** In this case, we assume there is some  $\{x_m\}$  such that  $|x_m| \rightarrow 0$  and  $|x_m|^{\sigma+1} |\nabla \phi_m(x_m)| \geq \epsilon_0 > 0$ . Set  $s_m = |x_m|$ , so we have  $s_m \rightarrow 0$ . Define  $z_m := s_m^{-1} x_m$  a sequence and note that by definition  $|z_m| = 1$ . Thus,

$$|z_m|^{\sigma+1} s_m^{\sigma+1} |\nabla \phi_m(s_m z_m)| = s_m^{\sigma+1} |\nabla \phi_m(s_m z_m)| \geq \epsilon_0 > 0. \quad (4.36)$$

Define  $\zeta_m(z) := s_m^\sigma \phi_m(s_m z)$  for  $0 < |s_m z| < 1$ . By (4.36)

$$|\nabla \zeta_m(z_m)| = s_m^{\sigma+1} |\nabla \phi_m(s_m z_m)| \geq \epsilon_0 \quad (4.37)$$

and also we have the bounds

$$|z|^\sigma |\zeta_m(z)| \leq 1, \quad \text{and} \quad |z|^{\sigma+1} |\nabla \zeta_m(z)| \leq 1. \quad (4.38)$$

By (4.30), we can write

$$-\Delta \phi_m(s_m z) + \frac{p}{\beta + t_m |s_m z|^{\xi-1}} \frac{s_m z \cdot \nabla \phi_m(s_m z)}{|s_m z|^2} = f^m(s_m z).$$

Using our definition, we can obtain  $\Delta \zeta_m(z) = s_m^{\sigma+2} \Delta \zeta_m(s_m z)$ , and  $\nabla \zeta_m(z) = s_m^{\sigma+1} \nabla \zeta_m(s_m z)$ .

Thus a computation shows that

$$-\Delta \zeta_m(z) + \frac{p}{\beta + t_m s_m^{\xi-1} |z|^{\xi-1}} \frac{z \cdot \nabla \zeta_m(z)}{|z|^2} = s_m^{\sigma+2} f^m(s_m z). \quad (4.39)$$

Note that by setting  $g^m(z) := s_m^{\sigma+2} f^m(s_m z)$ , we showed that

$$L_{t_m s_m^{\xi-1}}^+(\zeta_m(z)) = g^m(z) \quad \text{in} \quad B_{\frac{1}{s_m}} = E_m = \left\{ z : 0 < |z| < \frac{1}{s_m} \right\}$$

with  $\zeta_m = 0$  on  $\partial E_m$ . Thus

$$-\Delta \zeta_m(z) = g^m(z) - \frac{p}{\beta + t_m s_m^{\xi-1} |z|^{\xi-1}} \frac{z \cdot \nabla \zeta_m(z)}{|z|^2} \quad \text{in } E_m.$$

We define  $A_k$  and  $\tilde{A}_k$  for  $k \geq 2$  as

$$A_k = \left\{ z \in B_1 : \frac{1}{k} < |z| < k \right\} \quad \text{and} \quad \tilde{A}_k = \left\{ z \in B_1 : \frac{1}{2k} < |z| < 2k \right\}.$$

We have that  $A_k \subset \tilde{A}_k$  and also  $\tilde{A}_k \subset E_m$  for  $m$  big enough. Thus we obtained

$$-\Delta \zeta_m(z) = g^m(z) - \frac{p}{\beta + t_m s_m^{\xi-1} |z|^{\xi-1}} \frac{z \cdot \nabla \zeta_m(z)}{|z|^2} \quad \text{in } \tilde{A}_k.$$

First note that

$$\sup_{z \in \tilde{A}_k} |g^m| = \sup_{z \in \tilde{A}_k} s_m^{\sigma+2} |f^m(s_m z)| = \sup_{z \in \tilde{A}_k} \frac{|s_m|^{\sigma+2} |z|^{\sigma+2} |g^m|}{|z|^{\sigma+2}} \leq \left( \sup_{z \in \tilde{A}_k} \frac{\|f^m\|_Y}{|z|^{\sigma+2}} \right) \leq (2k)^{2+\sigma} \|f^m\|_Y. \quad (4.40)$$

Also we have

$$\begin{aligned} \left| \frac{p}{(t_m s_m^{\xi-1} |z|^{\xi-1} + \beta)} \frac{z \cdot \nabla \zeta_m}{|z|^2} \right| &\leq \frac{p}{(t_m |x|^{\xi-1} + \beta)} \left| \frac{z \cdot \nabla \zeta_m}{|x|^2} \right| \\ &\leq \frac{p}{(t_m s_m^{\xi-1} |z|^{\xi-1} + \beta)} \frac{|z|^{\sigma+1} |\nabla \zeta_m|}{|z|^{\sigma+2}} \\ &\leq \frac{p}{(t_m s_m^{\xi-1} |z|^{\xi-1} + \beta)} \frac{|z|^{\sigma+1} |s_m|^{\sigma+1} |\nabla \phi_m(s_m z)|}{|z|^{\sigma+2}} \\ &\leq \frac{p \|\phi_m\|_X}{\beta |z|^{\sigma+2}} = \frac{p}{\beta} \frac{1}{|z|^{\sigma+2}} \leq \frac{p}{\beta} (2k)^{2+\sigma} := \gamma_k. \end{aligned} \quad (4.41)$$

Set  $G = g^m(z) - \frac{p}{\beta + t_m s_m^{\xi-1} |z|^{\xi-1}} \frac{z \cdot \nabla \zeta_m(z)}{|z|^2}$ . By (4.40) and (4.41), we can show that  $G$  is bounded in  $\tilde{A}_k$ . Thus, there exists a  $C > 0$  such that

$$\|G\|_{L^\infty(\tilde{A}_k)} \leq C$$

and since  $-\Delta\zeta_m = G$ , we have

$$\|\Delta\zeta_m\|_{L^\infty(\bar{A}_k)} \leq C.$$

By the elliptic regularity, we can say that for  $0 < \lambda < 1$  there exists a positive constant  $C_1$  such that

$$\|\zeta_m\|_{C^{1,\lambda}(\bar{A}_k)} \leq C_1.$$

Thus  $\{\zeta_m\}_m$  is a bounded sequence in  $C^{1,\lambda}(\bar{A}_k)$  for all  $k \geq 2$ . By standard compactness argument and a diagonal argument there exist some subsequence  $\{\zeta_{m_i}\}_i \subset \{\zeta_m\}_m$  and  $\zeta \in C^{1,\frac{\lambda}{2}}(\bar{A}_k)$  such that:

$$\zeta_{m_i} \rightarrow \zeta \quad C^{1,\frac{\lambda}{2}}(\bar{A}_k)$$

so we get

$$\zeta_{m_i} \rightarrow \zeta \quad C_{loc}^{1,\frac{\lambda}{2}}(\mathbb{R}^N \setminus \{0\}).$$

Suppose that  $t_m s_m^{\xi-1}$  converges to some  $t \in [0, \infty]$ . By passing the limit in (4.39) we see that when  $t < \infty$ ,  $\zeta$  solves

$$-\Delta\zeta + \frac{p}{(t|z|^{\xi-1} + \beta)} \frac{z \cdot \nabla\zeta}{|z|^2} = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}$$

and in the case of  $t = \infty$ , we get

$$\Delta\zeta = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{4.42}$$

Using the completeness of  $\mathbb{R}^N$ , we can pass to a subsequence such that  $z_m \rightarrow z_0$  with  $z_0$  bounded away from zero and  $|z_0| = 1$ . We can now pass the limit in (4.37) to see that  $|\nabla\zeta(z_0)| \geq \epsilon_0$  and this means that  $\zeta$  is nonzero. We now need to show that  $\zeta$  is in  $X_1$ . So we need to show that  $\zeta$  belongs to  $X$  and it has no  $k = 0$  mode. We showed



that for every fixed  $0 < |z| < 1$ , we have  $|z|^\sigma |\zeta_m(z)| \leq 1$ , and  $|z|^{\sigma+1} |\nabla \zeta_m(z)| \leq 1$ , so by passing the limit in these two expressions, we get  $|z|^{\sigma+1} |\nabla \zeta(x)| \leq 1$  and  $|z|^\sigma |\zeta| \leq 1$  in  $\mathbb{R}^N \setminus \{0\}$ . Thus,  $\zeta$  is in  $X$ . We know that  $\zeta_m$  has no  $k = 0$  mode so  $\zeta_m \in X_1$ . With a similar approach to case 1, we can show that since  $\zeta_m$  is in  $X_1$ , it has zero average over the sphere with radius  $0 < r \leq 1$ . With the convergence that we have obtained, we find that  $\zeta$  has also zero average over the sphere and so  $\zeta$  is also in  $X_1$ . This shows that  $\zeta \in X_1$  is nonzero and it satisfies  $L_t(\zeta) = 0$  in  $\mathbb{R}^N \setminus \{0\}$  which is a contradiction with the kernel results we obtained before.

When  $t = \infty$ , we have

$$\Delta \zeta = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \quad (4.43)$$

We can write  $\zeta = \sum_{k=1}^{\infty} a_k(r) \psi_k(\theta)$ . Solving the equation gives us

$$a_k = C_k(r^{\gamma_k^+}) + D_k(r^{\gamma_k^-})$$

where  $\gamma_k^\pm = \frac{-(N-2)}{2} \pm \frac{\sqrt{(N-2)^2 + 4\lambda_k}}{2}$ . In Lemma 4.4, we showed that  $\gamma_k^+ + \sigma$  is positive and  $\gamma_k^- + \sigma$  is negative and they are both nonzero and distinct. Hence, to have  $a_k$  in the required space, we should have  $C_k = D_k = 0$ . So  $\zeta = 0$  and it is a contradiction with  $\zeta$  being nonzero.

The results from case 1 and case 2 complete the proof of the claim we made in (4.29) which means we have

$$\sup_{0 < |x| \leq 1} |x|^{\sigma+1} |\phi_m(x)| \rightarrow 0.$$

□

As we mentioned before, this result gives us

$$\sup_{0 < |x| \leq 1} \{|x|^\sigma |\phi_m|\} \rightarrow 0. \quad (4.44)$$

To show this, fix  $0 < |x| < 1$  and let  $\hat{x}$  be a point on the boundary of  $B_1 \setminus \{0\}$ . Also,

let  $t_1 = \frac{1}{|x|}$  such that  $g(t) = \phi_m(tx)$ . Then,  $g'(t) = \nabla\phi_m \cdot x$  and this gives us

$$|g(t_1) - g(1)| \leq \int_1^{t_1} |g'(t)| dt.$$

Thus, we can write

$$|\phi_m(\hat{x}) - \phi_m(x)| = |\phi_m(x)| \leq \int_1^{t_1} |\nabla\phi_m(tx)||x| dt. \quad (4.45)$$

We showed that

$$|\nabla\phi_m(z)||z|^{\sigma+1} \leq \epsilon \quad \text{for } 0 < |z| \leq 1$$

and we can write

$$|\nabla\phi_m(tx)|t^{\sigma+1}|x|^{\sigma+1} \leq \epsilon. \quad (4.46)$$

By (4.45) and (4.46), we find that

$$|\phi_m(x)| \leq \int_1^{t_1} \frac{\epsilon|x|}{t^{\sigma+1}|x|^{\sigma+1}} dt = \frac{\epsilon}{|x|^\sigma} \int_1^{t_1} \frac{1}{t^{\sigma+1}} dt = \frac{\epsilon}{|x|^\sigma} \left[ \frac{t^{-\sigma}}{-\sigma} \right]_1^{t_1} = \frac{\epsilon}{|x|^\sigma \sigma} \left[ \frac{-1}{t_1^\sigma} + 1 \right]$$

so we have

$$|x|^\sigma |\phi_m(x)| \leq \frac{\epsilon}{\sigma} \left[ 1 - \frac{1}{t_1^\sigma} \right] \leq \frac{\epsilon}{\sigma}. \quad (4.47)$$

When  $\epsilon$  goes to zero, we get that  $|\phi_m(tx)|t^{\sigma+1}|x|^{\sigma+1}$  approaches zero. By (4.47) we also find that  $|x|^\sigma |\phi_m(x)|$  goes to zero. Thus we have

$$\sup_{0 < |x| \leq 1} \{ |x|^{\sigma+1} |\nabla\phi_m(x)| + |x|^\sigma |\phi_m(x)| \} \rightarrow 0.$$

This is a contradiction with  $\|\phi\|_X = 1$ .

□

- Set  $\kappa = -p$  such that  $\{\phi_m\}_m \subset X_1$ ,  $f^m \in Y_1$  satisfy

$$\begin{cases} -\Delta\phi_m - \frac{p}{(t_m|x|^{\xi-1} + \beta)} \frac{x \cdot \nabla\phi_m}{|x|^2} = f^m & \text{in } B_1 \setminus \{0\}, \\ \phi_m = 0, & \text{on } \partial B_1, \end{cases} \quad (4.48)$$

and we have the estimate

$$\|\phi_m\|_X > C\|f^m\|_Y$$

such that

$$\epsilon_0 \leq |x_m|^{\sigma+1} |\nabla\phi_m(x_m)| \leq 1. \quad (4.49)$$

Now we prove the first case for  $L_{t_m}^-$ .

**Case 1:** Assume  $|x_m|$  is bounded away from zero. Define  $A_k$  and  $\tilde{A}_k$  for  $k \geq 2$  the same as before

$$A_k = \left\{ x \in B_1 : \frac{1}{k} < |x| < 1 \right\} \quad \text{and} \quad \tilde{A}_k = \left\{ x \in B_1 : \frac{1}{2k} < |x| < 1 \right\}.$$

We note that  $A_k \subset \tilde{A}_k$ . From (4.39), we have

$$-\Delta\phi_m = f^m + \frac{p}{(t_m|x|^{\xi-1} + \beta)} \frac{x \cdot \nabla\phi_m}{|x|^2} \quad \text{in } \tilde{A}_k. \quad (4.50)$$

Similarly we set  $g^m := f^m + \frac{p}{(t_m|x|^{\xi-1} + \beta)} \frac{x \cdot \nabla\phi_m}{|x|^2}$ , and we have

$$\left| \frac{p}{(t_m|x|^{\xi-1} + \beta)} \frac{x \cdot \nabla\phi_m}{|x|^2} \right| \leq \frac{p|x|^{\sigma+1} |\nabla\phi_m|}{(t_m|x|^{\xi-1} + \beta)|x|^{\sigma+2}} \leq \frac{p\|\phi_m\|_X}{\beta|x|^{\sigma+2}} = \frac{p}{\beta} \frac{1}{|x|^{\sigma+2}} \leq \frac{p}{\beta} (2k)^{2+\sigma} := \gamma_k.$$

Similar to (4.33), we can show that  $\sup_{x \in \tilde{A}_k} |f^m| \leq \gamma_k \|f^m\|_Y$ . This means that  $g_m$  is bounded. So there exists a positive constant  $C$  such that

$$\|g^m(x)\|_{L^\infty(\tilde{A}_k)} \leq C$$

meaning

$$\|\Delta\phi_m\|_{L^\infty(\bar{A}_k)} \leq C.$$

By elliptic regularity, we can say that for  $0 < \lambda < 1$  there exists a positive constant  $C_1$  such that

$$\|\phi_m\|_{C^{1,\lambda}(\bar{A}_k)} \leq C_1.$$

Thus  $\{\phi_m\}_m$  is a bounded sequence in  $C^{1,\lambda}(\bar{A}_k)$  for all  $k \geq 2$ . By standard compactness argument and a diagonal argument there exists a subsequence  $\{\phi_{m_i}\}_i \subset \{\phi_m\}_m$  and  $\phi \in C^{1,\frac{\lambda}{2}}(\bar{A}_k)$  such that

$$\phi_{m_i} \rightarrow \phi \quad C^{1,\frac{\lambda}{2}}(\bar{A}_k)$$

so we can write

$$\phi_{m_i} \rightarrow \phi \quad C_{loc}^{1,\frac{\lambda}{2}}(\bar{B}_1 \setminus \{0\}).$$

Suppose that  $t_m$  converges to some  $t \in [0, \infty]$ . By passing the limit in (4.39), we see that  $\phi$  solves

$$-\Delta\phi - \frac{p}{(t|x|^{\xi-1} + \beta)} \frac{x \cdot \nabla\phi}{|x|^2} = 0 \quad \text{in } B_1 \setminus \{0\}$$

with  $\phi = 0$  on  $\partial B_1$  when  $t \in [0, \infty)$ . Also, when  $t \rightarrow \infty$  we get

$$-\Delta\phi = 0 \quad \text{in } B_1 \setminus \{0\}$$

with  $\phi = 0$  on  $\partial B_1$ . Using the completeness of  $\mathbb{R}^N$ , we can pass to a subsequence  $x_m$  converging to  $x_0$  such that  $|x_0|$  is bounded away from zero. Now we can pass the limit in (4.49) to see that  $|x_0|^{\sigma+1} |\nabla\phi(x_0)| \geq \epsilon_0$  and this means  $\phi$  is not zero. At this step, we need to show that  $\phi \in X_1$ . So we need to show that  $\phi$  is in  $X$  and it has no  $k = 0$  mode. We again follow the same approach as the case  $\kappa = +p$ . We showed that for every fixed  $0 < |x| < 1$ , we have  $|x|^{\sigma+1} |\nabla\phi_{m_i}(x)| \leq 1$ . So by passing

to the limit, we get  $|x|^{\sigma+1}|\nabla\phi| \leq 1$ . Thus by integration, similar to (4.47) we can show that we have  $|x|^\sigma|\phi(x)| \leq 1$  thus

$$\sup_{0 < |x| \leq 1} \left\{ |x|^\sigma |\phi(x)| + |x|^{\sigma+1} |\nabla\phi(x)| \right\} < \infty.$$

This shows that  $\phi$  is in  $X$ . First note that since  $\phi_m$  belong to  $X_1$  for all  $0 < r \leq 1$ , we can write

$$\int_{|\theta|=1} \phi_m(r\theta) d\theta = \sum_{k=1}^{\infty} a_k(r) \int_{|\theta|=1} \psi_{k,m}(\theta) d\theta = 0$$

This means that  $\phi_m$  has zero average over all the sphere for radius  $0 < r \leq 1$ . Now we use the convergence we obtained above and we can write for all  $0 < r \leq 1$  we have that  $\theta \mapsto \phi_m(r\theta)$  converges uniformly on  $S^{N-1}$  to  $\theta \mapsto \phi(r\theta)$ . So for all  $0 < r \leq 1$  we have

$$0 = \int_{|\theta|=1} \phi_m(r\theta) d\theta \rightarrow \int_{|\theta|=1} \phi(r\theta) d\theta.$$

So  $\phi$  has also zero average over all the sphere for radius  $0 < r \leq 1$  and hence  $\phi \in X_1$ . We can conclude that  $\phi \in X_1$  is nonzero. Thus for  $t \in [0, \infty)$  we have a contradiction with the results from the previous lemmas.

We now show in case of  $t \rightarrow \infty$ , we will also get a contradiction. We saw that when  $t = \infty$ , we have

$$\begin{cases} \Delta\phi = 0 & \text{in } B_1 \setminus \{0\}, \\ \phi = 0 & \text{on } \partial B_1. \end{cases} \quad (4.51)$$

Then we write  $\phi = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta)$ , and solving the equation gives us

$$a_k = C_k(r^{\gamma_k^+} + D_k r^{\gamma_k^-})$$

where  $\gamma_k^\pm = \frac{-(N-2)}{2} \pm \frac{\sqrt{(N-2)^2 + 4\lambda_k}}{2}$ . In Lemma 4.4, we showed that  $\gamma_k^- + \sigma < 0 < \gamma_k^+ + \sigma$  for all  $k \geq 1$ . Hence, to have  $\phi$  in the required space, we must have  $C_k = D_k = 0$  and so  $\phi = 0$ . It is a contradiction with  $\phi$  being nonzero.

**Case 2:** In this case, we assume there exists some  $\{x_m\}$  such that  $|x_m| \rightarrow 0$  and  $|x_m|^{\sigma+1}|\nabla\phi_m(x_m)| \geq \epsilon_0 > 0$ . Set  $s_m = |x_m|$ , so we have  $s_m \rightarrow 0$ . Define  $z_m := s_m^{-1}x_m$  a sequence and note that by definition  $|z_m| = 1$ . Thus,

$$s_m^{\sigma+1}|\nabla\phi_m(s_m z_m)| \geq \epsilon_0 > 0. \quad (4.52)$$

Define  $\zeta_m(z) := s_m^\sigma\phi_m(s_m z)$  for  $0 < |s_m z| < 1$ . By (4.52),  $|\nabla\zeta_m(z_m)| = s_m^{\sigma+1}|\nabla\phi_m(s_m z_m)|$  thus

$$|\nabla\zeta_m(z_m)| = s_m^{\sigma+1}|\nabla\phi_m(s_m z_m)| \geq \epsilon_0 \quad (4.53)$$

and also we have the bounds

$$|z|^\sigma|\zeta_m(z)| \leq 1 \quad \text{and} \quad |z|^{\sigma+1}|\nabla\zeta_m(z)| \leq 1. \quad (4.54)$$

By (4.28), we can write:

$$-\Delta\phi_m(s_m z) - \frac{p}{\beta + t_m|s_m z|^{\xi-1}} \frac{s_m z \cdot \nabla\phi_m(s_m z)}{|s_m z|^2} = f^m(s_m z). \quad (4.55)$$

Using our definition, we can obtain  $\Delta\zeta_m(z) = s_m^{\sigma+2}\Delta\zeta_m(s_m z)$ , and  $\nabla\zeta_m(z) = s_m^{\sigma+1}\nabla\zeta_m(s_m z)$ .

Thus a computation shows that

$$-\Delta\zeta_m(z) - \frac{p}{\beta + t_m s_m^{\xi-1}|z|^{\xi-1}} \frac{z \cdot \nabla\zeta_m(z)}{|z|^2} = s_m^{\sigma+2} f^m(s_m z). \quad (4.56)$$

Note that by setting  $g^m(z) := s_m^{\sigma+2} f^m(s_m z)$ , we showed that

$$L_{t_m s_m^{\xi-1}}^-(\zeta_m(z)) = g^m(z) \quad \text{in} \quad B_{\frac{1}{s_m}} = E_m = \left\{ z : 0 < |z| < \frac{1}{s_m} \right\}$$

with  $\zeta_m = 0$  on  $\partial E_m$ . Thus,

$$-\Delta \zeta_m(z) = g^m(z) + \frac{p}{\beta + t_m s_m^{\xi-1} |z|^{\xi-1}} \frac{z \cdot \nabla \zeta_m(z)}{|z|^2} \quad \text{in } E_m.$$

The same as before define  $A_k$  and  $\tilde{A}_k$  for  $k \geq 2$ ,

$$A_k = \left\{ z \in B_1 : \frac{1}{k} < |z| < k \right\} \quad \text{and} \quad \tilde{A}_k = \left\{ z \in B_1 : \frac{1}{2k} < |z| < 2k \right\}.$$

We have that  $A_k \subset \tilde{A}_k$  and also  $\tilde{A}_k \subset E_m$  for  $m$  big enough. Thus we obtained

$$-\Delta \zeta_m(z) = g^m(z) + \frac{p}{\beta + t_m s_m^{\xi-1} |z|^{\xi-1}} \frac{z \cdot \nabla \zeta_m(z)}{|z|^2} \quad \text{in } \tilde{A}_k.$$

First note that similar to (4.40), and (4.41) we can show that  $\sup_{z \in \tilde{A}_k} |g^m| \leq (2k)^{2+\sigma} \|f^m\|_Y$ ,

and we have

$$\left| \frac{-p}{(t_m s_m^{\xi-1} |z|^{\xi-1} + \beta)} \frac{z \cdot \nabla \zeta_m}{|z|^2} \right| \leq \frac{p}{(t_m |x|^{\xi-1} + \beta)} \left| \frac{z \cdot \nabla \zeta_m}{|x|^2} \right| \leq \frac{p}{\beta} (2k)^{2+\sigma} := \gamma_k.$$

Set  $G = g^m(z) - \frac{p}{\beta + t_m s_m^{\xi-1} |z|^{\xi-1}} \frac{z \cdot \nabla \zeta_m(z)}{|z|^2}$ . Hence  $G$  is bounded in  $\tilde{A}_k$ . Thus there exists a constant  $C > 0$  such that

$$\|G\|_{L^\infty(\tilde{A}_k)} \leq C$$

and since  $-\Delta \zeta_m = G$ , we have

$$\|\Delta \zeta_m\|_{L^\infty(\tilde{A}_k)} \leq C.$$

By elliptic regularity, we can say that for  $0 < \lambda < 1$ , there exists a positive constant  $C_1$  such that

$$\|\zeta_m\|_{C^{1,\lambda}(\tilde{A}_k)} \leq C_1.$$

Thus  $\{\zeta_m\}_m$  is a bounded sequence in  $C^{1,\lambda}(\tilde{A}_k)$  for all  $k \geq 2$ . By standard com-

compactness argument and a diagonal argument, there exist some subsequence  $\{\zeta_{m_i}\}_i \subset \{\zeta_m\}_m$  and  $\zeta$  such that

$$\zeta_{m_i} \rightarrow \zeta \quad C^{1, \frac{\lambda}{2}}(\bar{A}_k)$$

so we have

$$\zeta_{m_i} \rightarrow \zeta \quad C_{loc}^{1, \frac{\lambda}{2}}(\mathbb{R}^N \setminus \{0\}).$$

Suppose that  $t_m s_m^{\xi-1}$  converges to some  $t \in [0, \infty]$ . By passing the limit in (4.22), we see that when  $t < \infty$ ,  $\zeta$  solves

$$-\Delta\zeta - \frac{p}{(t|z|^{\xi-1} + \beta)} \frac{z \cdot \nabla\zeta}{|z|^2} = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}$$

and in the case of  $t = \infty$  we get

$$\Delta\zeta = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{4.57}$$

Using the completeness of  $\mathbb{R}^N$ , we can pass to a subsequence such that  $z_m$  converges to some  $z_0$  such that  $z_0$  is bounded away from zero and  $|z_0| = 1$ . We can now pass the limit in (4.53) to see that  $|\nabla\zeta(z_0)| \geq \epsilon_0$  and this means that  $\zeta$  is nonzero. We now need to show that  $\zeta$  is in  $X_1$ . So we need to show that  $\zeta$  belongs to  $X$  and it has no  $k = 0$  mode. We showed that for every fixed  $0 < |z| < 1$ , we have  $|z|^\sigma |\zeta_m(z)| \leq 1$ , and  $|z|^{\sigma+1} |\nabla\zeta_m(z)| \leq 1$ , so by passing the limit in these two expressions, we get  $|z|^{\sigma+1} |\nabla\zeta(x)| \leq 1$  and  $|z|^\sigma |\zeta| \leq 1$  in  $\mathbb{R}^N \setminus \{0\}$ . Thus,  $\zeta$  is in  $X$ . We know that  $\zeta_m$  has no  $k = 0$  mode so  $\zeta_m \in X_1$  and with a similar approach as in case 1 we can show that since  $\zeta_m$  is in  $X_1$ , it has zero average over the sphere with radius  $0 < r \leq 1$ . With the convergence that we have obtained, we find that  $\zeta$  has also zero average over the sphere and so  $\zeta$  is also in  $X_1$ . This means that  $\zeta \in X_1$  is nonzero and it satisfies  $L_t(\zeta) = 0$  in  $\mathbb{R}^N \setminus \{0\}$  which is a contradiction with the kernel results we



obtained before. We saw that when  $t = \infty$ , we have

$$\Delta\zeta = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \quad (4.58)$$

Then, we write  $\zeta = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta)$ , and solving the equation gives us

$$a_k = C_k r^{\gamma_k^+} + D_k r^{\gamma_k^-}$$

where  $\gamma_k^{\pm} = \frac{-(N-2)}{2} \pm \sqrt{\frac{(N-2)^2 + 4\lambda_k}{2}}$ . In Lemma 4.4 we showed that  $\gamma_k^- + \sigma < 0 < \gamma_k^+ + \sigma$  for all  $k \geq 1$ . Hence, to have  $\zeta$  in the required space, we must have  $C_k = D_k = 0$  and so  $\zeta = 0$ . It is a contradiction with  $\zeta$  being nonzero.

Similarly, the results from case 1 and case 2 complete the proof of the claim we made in (4.29) which means that we have

$$\sup_{0 < |x| \leq 1} |x|^{\sigma+1} |\phi_m(x)| \rightarrow 0.$$

With a similar proof as (4.47) in the case of  $\kappa = p$ , we can obtain  $\sup_{0 < |x| \leq 1} \{|x|^{\sigma} |\phi_m^{\pm}|\} \rightarrow 0$ . Thus we have

$$\sup_{0 < |x| \leq 1} \{|x|^{\sigma+1} |\nabla \phi_m(x)| + |x|^{\sigma} |\phi_m(x)|\} \rightarrow 0$$

which is a contradiction with  $\|\phi\|_X = 1$ . This completes the proof of Theorem 4.5.

We showed that there is a constant  $C > 0$  such that for all  $t_m \geq 0$ ,  $\{\phi_m^{\pm}\}_m \subset X_1$  and  $f^m \in Y_1$  that they satisfy

$$\begin{cases} L_{t_m}^{\pm}(\phi_m) = f^m & \text{in } B_1 \setminus \{0\}, \\ \phi_m^{\pm} = 0 & \text{on } \partial B_1, \end{cases} \quad (4.59)$$

one has the estimate  $\|\phi_m\|_X \leq C \|f^m\|_Y$ . □

For the case  $\kappa = +p$  one can use a continuation argument along with Theorem 4.5 and the studies that has been done on  $L_0$  in [2] and [1] to show the following. At this part, we skip the details on the continuation argument in the case of  $\kappa = -p$  and they will be studied later in Lemma 4.8 and Lemma 4.9 .

**Corollary 4.6.** There exists a positive constant  $C$  such that for all  $f \in Y_1$  there is some  $\phi \in X_1$  such that  $L_t^+(\phi) = f$  in  $B_1 \setminus \{0\}$  with  $\phi = 0$  on  $\partial B_1$  and  $\|\phi\|_X \leq C\|f\|_Y$ .

To get the desired result on the full space  $X$ , we need to recombine it with the result for the  $k = 0$  mode.

**Lemma 4.7.** ( $k = 0$  mode for  $L_t^+$ ) We are considering the case where  $k = 0$  in (4.3) and for all positive  $t$ ,  $a_t(r)$  (with dependence on  $t$ ) solves the equation. We are also assuming

$$\sup_{0 < r \leq 1} r^{\sigma+2}|b(r)| \leq 1 \quad (4.60)$$

thus we have

$$-a_t''(r) - \frac{(N-1)a_t'(r)}{r} + \frac{pa_t'(r)}{\beta r + tr^\xi} = b(r) \quad 0 < r < 1 \quad (4.61)$$

with  $a_t(1) = 0$ . Note that  $a_t(r)$  also satisfies

$$\sup_{0 < |x| \leq 1} \left\{ r^\sigma |a_t(r)| + r^{\sigma+1} |a_t'(r)| \right\} \leq C. \quad (4.62)$$

*Proof.* We know the integrating factor associated with our ODE (4.61) is  $\mu_t(r) = e^{P(r)}$  where

$$\begin{aligned} P(r) &= \int_1^r \left( \frac{N-1}{\tau} - \frac{p}{\beta\tau + t\tau^\xi} \right) d\tau = \left( (N-1) \ln \tau - \frac{p}{\beta(1-\xi)} \ln(\beta\tau^{\xi-1} + t) \right) \Big|_1^r \\ &= (N-1) \ln r - \frac{p}{\beta(1-\xi)} \left( \ln r^{1-\xi} + \ln \left( \frac{tr^{\xi-1} + \beta}{t + \beta} \right) \right). \end{aligned}$$

So we can find the integrating factor

$$\begin{aligned}\mu_t(r) &= e^{(N-1)\ln r - \frac{p}{\beta(1-\xi)} \left( \ln r^{1-\xi} + \ln \left( \frac{tr^{\xi-1} + \beta}{t+\beta} \right) \right)} = e^{\ln r^{(N-1)}} \cdot e^{\ln r^{(1-\xi)\left(-\frac{p}{\beta(1-\xi)}\right)}} \cdot e^{\ln \left( \frac{tr^{\xi-1} + \beta}{t+\beta} \right)^{\left(-\frac{p}{\beta(1-\xi)}\right)}} \\ &= r^{N-1-\frac{p}{\beta}} \left( \frac{tr^{\xi-1} + \beta}{t+\beta} \right)^{\left(-\frac{p}{\beta(1-\xi)}\right)} = r^{N-1-\frac{p}{\beta}} \left( \frac{tr^{\xi-1} + \beta}{t+\beta} \right)^{\left(\frac{p}{p-1}\right)}.\end{aligned}$$

Thus from (4.61), using the integrating factor, we get:

$$-\frac{d}{d\tau}(\mu_t(\tau)a'_t(\tau)) = \mu_t(\tau)b(\tau).$$

By noting that  $\mu_t(1) = 1$ , we can integrate both sides and we obtain

$$-\int_s^1 \frac{d}{d\tau}(\mu_t(\tau)a'_t(\tau))d\tau = \int_s^1 \mu_t(\tau)b(\tau)d\tau \implies -(\mu_t(1)a'_t(1) - \mu_t(s)a'_t(s)) = \int_s^1 \mu_t(\tau)b(\tau)d\tau.$$

We can now deduce that

$$a'_t(s) = \frac{1}{\mu_t(s)} \left( a'_t(1) + \int_s^1 \mu_t(\tau)b(\tau)d\tau \right).$$

By integrating again with respect to  $s$  from  $r$  to 1, and considering  $a_t(1) = 0$ , we get

$$a_t(r) = - \int_r^1 \frac{1}{\mu_t(s)} \left( a'_t(1) + \int_s^1 \mu_t(\tau)b(\tau)d\tau \right) ds.$$

Set  $a'_t(1) = - \int_{R_t}^1 \mu_t(\tau)b(\tau)d\tau$ , where  $R_t^{\xi-1}t = 1$  then we have

$$\begin{aligned}a'_t(r) &= -(-1) \frac{1}{\mu_t(s)} \left[ - \int_{R_t}^1 \mu_t(\tau)b(\tau)d\tau + \int_r^1 \mu_t(\tau)b(\tau)d\tau \right] \\ &= \frac{1}{\mu_t(s)} \int_r^{R_t} \mu_t(\tau)b(\tau)d\tau = - \frac{1}{\mu_t(r)} \int_{R_t}^r \mu_t(\tau)b(\tau)d\tau.\end{aligned}$$

So we can write  $a_t$  as

$$a_t(r) = \int_r^1 \left( \frac{1}{\mu_t(s)} \int_{R_t}^s \mu_t(\tau) b(\tau) d\tau \right) ds, \quad 0 < r \leq 1.$$

We now need to check that  $a_t$  satisfies the bounds independent of  $t$  for large  $t$ .

We should consider two cases:

(i)  $0 < r < R_t$ ,

(ii)  $R_t < r \leq 1$ .

For  $r < R_t$ , we have

$$\begin{aligned} r^{\sigma+1} |a'_t(r)| &\leq r^{\sigma+1+1-N+\frac{p}{\beta}} \left( \frac{tr^{\xi-1} + \beta}{t + \beta} \right)^{-\left(\frac{p}{p-1}\right)} \int_r^{R_t} \mu_t(\tau) |b(\tau)| d\tau \\ &\leq r^{\sigma+2-N+\frac{p}{\beta}} \left( \frac{tr^{\xi-1} + \beta}{t + \beta} \right)^{-\left(\frac{p}{p-1}\right)} \int_r^{R_t} \frac{\tau^{\sigma+2}}{\tau^{\sigma+2}} \mu_t(\tau) |b(\tau)| d\tau. \end{aligned}$$

Note that  $-\sigma - 3 + N - \frac{p}{\beta} = -\xi$ , so by (4.60), we can write

$$\begin{aligned} r^{\sigma+1} |a'_t(r)| &\leq r^{\xi-1} \left( \frac{t + \beta}{tr^{\xi-1} + \beta} \right)^{\left(\frac{p}{p-1}\right)} \int_r^{R_t} \tau^{N-1-\frac{p}{\beta}} \left( \frac{t\tau^{\xi-1} + \beta}{t + \beta} \right)^{\left(\frac{p}{p-1}\right)} \frac{|b(\tau)| \tau^{\sigma+2}}{\tau^{\sigma+2}} d\tau \\ &\leq r^{\xi-1} \int_r^{R_t} \left( \frac{t\tau^{\xi-1} + \beta}{tr^{\xi-1} + \beta} \right)^{\left(\frac{p}{p-1}\right)} \frac{1}{\tau^\xi} d\tau \leq r^{\xi-1} \left( \frac{tR_t^{\xi-1} + \beta}{tr^{\xi-1} + \beta} \right)^{\left(\frac{p}{p-1}\right)} \int_r^{R_t} \frac{1}{\tau^\xi} d\tau \\ &\leq \left( \frac{1 + \beta}{tr^{\xi-1} + \beta} \right)^{\left(\frac{p}{p-1}\right)} \left( \frac{1 - \left(\frac{R_t}{r}\right)^{1-\xi}}{\xi - 1} \right) \leq \left( \frac{\beta + 1}{\beta} \right)^{\frac{p}{p-1}} \frac{1}{\xi - 1}. \end{aligned}$$

Thus we proved

$$r^{\sigma+1} |a'_t(r)| \leq \left( \frac{\beta + 1}{\beta} \right)^{\frac{p}{p-1}} \frac{1}{\xi - 1} \quad \text{for } 0 < r < R. \quad (4.63)$$

For  $r > R_t$ , we there exists a constant  $c_q > 0$  such that we have  $(a + b)^q \leq c_q(a^q + b^q)$  for  $q > 1$  and by using this inequality and noting that  $\xi - \frac{p(\xi-1)}{p-1} < 0$ , and  $\xi > 1$ , we

can write

$$\begin{aligned}
r^{\sigma+1}|a'_t(r)| &\leq \frac{r^{\xi-1}}{(\beta + tr^{\xi-1})^{\frac{p}{p-1}}} \int_{R_t}^r \frac{(\beta + t\tau^{\xi-1})^{\frac{p}{p-1}}}{\tau^\xi} d\tau \\
&\leq \frac{r^{\xi-1}}{(\beta + tr^{\xi-1})^{\frac{p}{p-1}}} \int_{R_t}^r \left( \frac{C\beta^{\frac{p}{p-1}}}{\tau^\xi} + \frac{Ct^{\frac{p}{p-1}}}{\tau^{\xi - \frac{p(\xi-1)}{p-1}}} \right) d\tau \\
&\leq \frac{Cr^{\xi-1}}{(\beta + tr^{\xi-1})^{\frac{p}{p-1}}} \left( \left( \frac{\beta^{\frac{p}{p-1}} \tau^{1-\xi}}{1-\xi} \right) \Big|_{R_t}^r + \left( \frac{t^{\frac{p}{p-1}} \tau^{\frac{(\xi-1)}{p-1}}}{\frac{(\xi-1)}{p-1}} \right) \Big|_{R_t}^r \right) \\
&\leq \frac{Cr^{\xi-1}}{(\beta + tr^{\xi-1})^{\frac{p}{p-1}}} \left( \left( \frac{\beta^{\frac{p}{p-1}}}{1-\xi} (r^{1-\xi} - R_t^{1-\xi}) \right) + \left( (p-1) \frac{t^{\frac{p}{p-1}}}{\xi-1} (r^{\frac{(\xi-1)}{p-1}} - R_t^{\frac{(\xi-1)}{p-1}}) \right) \right) \\
&\leq \frac{C_1 \beta^{\frac{p}{p-1}} r^{\xi-1} R_t^{1-\xi}}{(\beta + tr^{\xi-1})^{\frac{p}{p-1}}} + \frac{C_1 (tr^{\xi-1})^{\frac{p}{p-1}}}{(\beta + tr^{\xi-1})^{\frac{p}{p-1}}}
\end{aligned}$$

where  $C_1$  is a constant independent of  $t$ . Recall we have  $tR_t^{\xi-1} = 1$ , thus

$$r^{\sigma+1}|a'_t(r)| \leq \frac{C_1 \beta^{\frac{p}{p-1}} tr^{\xi-1}}{(\beta + tr^{\xi-1})^{\frac{p}{p-1}}} + \frac{C_1 (tr^{\xi-1})^{\frac{p}{p-1}}}{(\beta + tr^{\xi-1})^{\frac{p}{p-1}}} \leq \frac{C_1 \beta^{\frac{p}{p-1}}}{(tr^{\xi-1})^{\frac{p}{p-1}}} + C_1.$$

Since for  $r \geq R_t$  we have  $tr^{\sigma-1} \geq tR_t^{\sigma-1} = 1$ , we can deduce that

$$r^{\sigma+1}|a'_t(r)| \leq C_1 \beta^{\frac{p}{p-1}} + C_1 = C_1(1 + \beta^{\frac{p}{p-1}}) \quad \text{for } r \geq R_t. \quad (4.64)$$

Combining (4.63) and (4.64) gives us

$$\sup_{0 < r \leq 1} r^{\sigma+1}|a'_t(r)| \leq \max \left\{ C_1(1 + \beta^{\frac{p}{p-1}}), \left( \frac{\beta + 1}{\beta} \right)^{\frac{p}{p-1}} \frac{1}{\xi - 1} \right\}.$$

This shows that  $a_t(r)$  satisfies the equation and is bounded independent of  $t$ . □

We will delay the proof of the mapping properties of  $L_t^-$  for now and we turn to completing the proof of the main linear result assuming we have the  $L_t^-$  mapping properties.

**Completion of the proof of Proposition 4.1.** Here by assuming we have the

result from Lemma 4.9, we can combine it with Theorem 4.5 and Lemma 4.7 to complete the proof of Proposition 4.1. Let  $f \in Y$  and  $\phi \in X$  satisfy (4.2) and we write  $f(x) = f_0(r) + f_1(x)$ , and  $\phi(x) = \phi_0(r) + \phi_1(x)$ , where we have split off the  $k = 0$  mode from  $\phi_1 \in X_1$ , and  $f_1 \in Y_1$ . Then by (4.5), we can write

$$\|\phi\|_X \leq \|\phi_0\|_X + \|\phi_1\|_X \leq C\|f_0\|_Y + C\|f_1\|_Y.$$

Hence, if we can show there is some constant  $D > 0$  (independent of  $f$ ) such that  $\|f_0\|_Y + \|f_1\|_Y \leq D\|f_0 + f_1\|_Y$  then we would be done.

Given  $f \in Y$ , we have

$$f(x) = \sum_{k=0}^{\infty} b_k(r)\psi_k(\theta) = b_0(r)\psi_0(\theta) + \sum_{k=1}^{\infty} b_k(r)\psi_k(\theta) = b_0(r) + \sum_{k=1}^{\infty} b_k(r)\psi_k(\theta) = f_0 + f_1$$

where  $\psi_0(\theta) = 1$  and  $b_k(r) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} f(r, \theta)\psi_k(\theta)d\theta$ .

Noting that

$$\int_{S^{N-1}=1} |f(r, \theta)|d\theta \leq \frac{\|f\|_Y}{r^{\sigma+2}} |S^{N-1}|,$$

we can obtain

$$\|f_0\|_Y = \sup_{0 < |x| \leq 1} r^{\sigma+2}|b_0(r)| = \sup_{0 < |x| \leq 1} r^{\sigma+2} \left| \frac{1}{|S^{N-1}|} \int_{S^{N-1}} f(r\theta)d\theta \right| \leq \sup_{0 < |x| \leq 1} r^{\sigma+2} \frac{\|f\|_Y}{r^{\sigma+2}} = \|f\|_Y. \quad (4.65)$$

We know  $f_1 = f - f_0$  so we can write

$$\|f_1\|_Y = \|f - f_0\|_Y \leq \|f\|_Y + \|f_0\|_Y \leq 2\|f\|_Y. \quad (4.66)$$

By (4.65) and (4.66) we have

$$\|f_0\|_Y + \|f_1\|_Y \leq D\|f\|_Y = D\|f_0 + f_1\|_Y$$

thus

$$\|\phi\|_X \leq D\|f_0 + f_1\|_Y = D\|f\|_Y \quad (4.67)$$

where  $D$  is a positive constant independent of  $f$ . Now we go can back to our previous notation to show that the main result we want in Proposition 4.1. We proved that if we fix  $(f, g)$  and set  $F = f + g$  and  $G = f - g$  and consider  $\zeta_i$  to be a solution of the scalar problems

$$\begin{cases} -\Delta\zeta_1 - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\zeta_1 = F, & \text{in } B_1 \setminus \{0\} \\ \zeta_1 = 0 & \text{on } \partial B_1, \end{cases} \quad (4.68)$$

$$\begin{cases} -\Delta\zeta_2 + p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\zeta_2 = G & \text{in } B_1 \setminus \{0\}, \\ \zeta_2 = 0 & \text{on } \partial B_1, \end{cases} \quad (4.69)$$

we can define the operators

$$L_t^\pm(\zeta) = -\Delta\zeta \pm \frac{p}{(t|x|^{\xi-1} + \beta)} \frac{x \cdot \nabla\zeta}{|x|^2}.$$

By (4.67) for some positive constants  $C_1$  and  $C_2$  we have the estimates

$$\|\zeta_1\|_X \leq C_1\|F\|_Y \quad \text{and} \quad \|\zeta_2\|_X \leq C_2\|G\|_Y.$$

We know  $\phi$ , and  $\psi$  satisfy  $\phi + \psi = \zeta_1$  and  $\phi - \psi = \zeta_2$  From this we saw that

$$\phi = \frac{\zeta_1 + \zeta_2}{2}, \quad \psi = \frac{\zeta_1 - \zeta_2}{2}. \quad (4.70)$$

So we get

$$\|\phi\|_X = C_1\|\zeta_1 + \zeta_2\|_X \leq C_1(\|\zeta_1\| + \|\zeta_2\|) \leq C(\|F\|_Y + \|G\|_Y),$$

and

$$\|\psi\|_X = C_2 \|\zeta_1 - \zeta_2\|_X \leq C_2 (\|\zeta_1\| + \|\zeta_2\|) \leq C (\|F\|_Y + \|G\|_Y).$$

Since  $F = f + g$ , and  $G = f - g$  we have

$$\|F\|_Y \leq \|f\|_Y + \|g\|_Y \quad \text{and} \quad \|G\|_Y \leq \|f\|_Y + \|g\|_Y.$$

So there is some positive constant  $C$  such that for all  $t \geq 0$  and for all  $f$ , and  $g$  in  $Y$ , there are some  $\phi$  and  $\psi$  in  $X$  which satisfy

$$\begin{cases} -\Delta\phi - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\psi = f & \text{in } B_1 \setminus \{0\}, \\ -\Delta\psi - p|\nabla w_t|^{p-2}\nabla w_t \cdot \nabla\phi = g & \text{in } B_1 \setminus \{0\}, \\ \phi = \psi = 0 & \text{on } \partial B_1, \end{cases} \quad (4.71)$$

and we have the estimate

$$\|\phi\|_X + \|\psi\|_X \leq C\|f\|_Y + C\|g\|_Y.$$

This completes the main result of our linear theory, and Proposition 4.1. □

We now turn to the case of  $\kappa = -p$ . Consider (4.3) in the case of  $\kappa = -p$  and  $t = 0$  given by

$$-a_k''(r) - \frac{(N-1)a_k'}{r} + \frac{\lambda_k a_k(r)}{r^2} - \frac{p a_k'(r)}{\beta r} = b_k(r) \quad \text{for } 0 < r < 1 \quad (4.72)$$

with  $a_k(1) = 0$ .

**Lemma 4.8.** For all  $k \geq 0$  there is some  $C_k > 0$  such that for all functions  $b(r)$  with  $\sup_{0 < r < 1} r^{\sigma+2}|b(r)| \leq 1$  there is some function  $a(r)$  which satisfies (4.72) and



$$\sup_{0 < r < 1} r^\sigma |a(r)| \leq C_k.$$

*Proof.* Here we assume  $k \geq 1$  and  $t = 0$ . In Lemma 4.9 we will prove the result for the case where  $k$  is zero and  $t$  is positive. Our result will also hold true for  $t = 0$ . By assuming  $k \geq 1$ , we have an Euler type equation. We know that the fundamental set of solutions of homogeneous version of (4.72) play a crucial role. The solution of the homogeneous equation is given by  $a_k(r) = C_1 r^{\gamma_k^+} + C_2 r^{\gamma_k^-}$  where

$$\gamma_k^\pm = \frac{-(N - 2 + \frac{p}{\beta})}{2} \pm \frac{\sqrt{(N - 2 + \frac{p}{\beta})^2 + 4\lambda_k}}{2}.$$

We can now use variation of parameters to write out the particular solution of (4.72) as

$$a_{k,p}(r) = u_1(r)r^{\gamma_k^+} + u_2(r)r^{\gamma_k^-}. \quad (4.73)$$

We know that

$$u_1'(r)r^{\gamma_k^+} + u_2'(r)r^{\gamma_k^-} = 0. \quad (4.74)$$

Thus we need to solve for  $u_1$  and  $u_2$ . So we write

$$\begin{aligned} a_{k,p}(r) &= u_1(r)r^{\gamma_k^+} + u_2(r)r^{\gamma_k^-} \\ \iff a'_{k,p}(r) &= u_1'(r)r^{\gamma_k^+} + u_1(r)\gamma_k^+ r^{\gamma_k^+ - 1} + u_2'(r)r^{\gamma_k^-} + u_2(r)\gamma_k^- r^{\gamma_k^- - 1}. \end{aligned} \quad (4.75)$$

By (4.74), we get

$$\begin{aligned} a'_{k,p}(r) &= u_1\gamma_k^+ r^{\gamma_k^+} + u_2\gamma_k^- r^{\gamma_k^-} \\ \iff a''_{k,p}(r) &= u_1'\gamma_k^+ r^{\gamma_k^+ - 1} + u_1(\gamma_k^+)(\gamma_k^+ - 1)r^{\gamma_k^+ - 2} + u_2'\gamma_k^- r^{\gamma_k^- - 1} + u_2\gamma_k^-(\gamma_k^- - 1)r^{\gamma_k^- - 2}. \end{aligned} \quad (4.76)$$

We now plug in these values in (4.72) and we get

$$\begin{aligned}
& u_1' \gamma_k^+ r^{\gamma_k^+ - 1} + u_1(\gamma_k^+)(\gamma_k^+ - 1)r^{\gamma_k^+ - 2} + u_2' \gamma_k^- r^{\gamma_k^- - 1} + u_2 \gamma_k^- (\gamma_k^- - 1)r^{\gamma_k^- - 2} \quad (4.77) \\
& + \left[ N - 1 - \frac{p}{\beta} \right] \frac{1}{r} \left( u_1' r^{\gamma_k^+ - 1} + u_1(\gamma_k^+) \right) - \frac{\lambda_k}{r^2} \left[ u_1 r^{\gamma_k^+} + u_2 r^{\gamma_k^-} \right] = -b_k(r) \\
\iff & u_1' r^{\gamma_k^+ - 1} + u_2' \gamma_k^- r^{\gamma_k^- - 1} + u_1 r^{\gamma_k^+ - 2} \left[ \gamma_k^{+2} - \gamma_k^+ + \gamma_k^+ \left[ N - 1 - \frac{p}{\beta} \right] - \lambda_k \right] \\
& + u_2 r^{\gamma_k^- - 2} \left[ \gamma_k^{-2} - \gamma_k^- + \gamma_k^- \left[ N - 1 - \frac{p}{\beta} \right] - \lambda_k \right] = -b_k(r).
\end{aligned}$$

A computation show that

$$\left[ \gamma_k^{+2} - \gamma_k^+ + \gamma_k^+ \left[ N - 1 - \frac{p}{\beta} \right] - \lambda_k \right] = \left[ \gamma_k^{-2} - \gamma_k^- + \gamma_k^- \left[ N - 1 - \frac{p}{\beta} \right] - \lambda_k \right] = 0. \quad (4.78)$$

Thus we have

$$u_1' \gamma_k^+ r^{\gamma_k^+ - 1} + u_2' \gamma_k^- r^{\gamma_k^- - 1} = -b_k(r) \quad \text{and} \quad u_1' r^{\gamma_k^+} + u_2' r^{\gamma_k^-} = 0. \quad (4.79)$$

From this two equations, we can solve for  $u_1$  and  $u_2$  and we get:

$$u_1(r) = \int_{T_1}^r \frac{b_k(\tau) d\tau}{\tau^{\gamma_k^+ - 1} (\gamma_k^- - \gamma_k^+)} \quad \text{and} \quad u_2(r) = \int_{T_2}^r \frac{-b_k(\tau) d\tau}{\tau^{\gamma_k^- - 1} (\gamma_k^- - \gamma_k^+)} \quad (4.80)$$

where  $C_1$ ,  $C_2$  and  $T_1$  and  $T_2$  are to be picked later. Thus the solution of the equation (4.72) can be written as:

$$\begin{aligned}
a_k(r) &= C_1 r^{\gamma_k^+} + C_2 r^{\gamma_k^-} + r^{\gamma_k^+} u_1 + r^{\gamma_k^-} u_2 \quad (4.81) \\
&= C_1 r^{\gamma_k^+} + C_2 r^{\gamma_k^-} + r^{\gamma_k^+} \int_{T_1}^r \frac{b_k(\tau) d\tau}{\tau^{\gamma_k^+ - 1} (\gamma_k^- - \gamma_k^+)} + r^{\gamma_k^-} \int_{T_2}^r \frac{-b_k(\tau) d\tau}{\tau^{\gamma_k^- - 1} (\gamma_k^- - \gamma_k^+)}.
\end{aligned}$$

By the boundary condition, we need to have  $a_k(1) = 0$ . Thus we pick  $T_1 = 1$ ,  $C_2 = 0$ ,

$T_2 = 0$  and  $C_1$  such that we have

$$C_1 + \int_0^1 \frac{-b_k(\tau)d\tau}{\tau^{\gamma_k^- - 1}(\gamma_k^- - \gamma_k^+)} = 0 \implies C_1 = \int_0^1 \frac{b_k(\tau)d\tau}{\tau^{\gamma_k^- - 1}(\gamma_k^- - \gamma_k^+)}. \quad (4.82)$$

We first show that  $C_1$  is well defined. Note that in (4.13) we showed that  $\gamma_k^- + \sigma + 1 \leq 0$  and

$$\frac{|b_k(\tau)|}{\tau^{\gamma_k^- - 1}} \leq \frac{1}{\tau^{\gamma_k^- + \sigma + 1}}. \quad (4.83)$$

Thus we have

$$\begin{aligned} |C_1| &\leq \left| \int_0^1 \frac{b_k(\tau)d\tau}{\tau^{\gamma_k^- - 1}(\gamma_k^- - \gamma_k^+)} \right| \leq \frac{1}{|\gamma_k^- - \gamma_k^+|} \int_0^1 \frac{|b_k(\tau)|d\tau}{\tau^{\gamma_k^- - 1}} \\ &\leq \frac{1}{|\gamma_k^- - \gamma_k^+|} \int_0^1 \frac{d\tau}{\tau^{\sigma + \gamma_k^- + 1}} \leq \frac{1}{|\gamma_k^- - \gamma_k^+|} \frac{1}{|\sigma + \gamma_k^-|} (1 - 0) \leq C \end{aligned} \quad (4.84)$$

where  $C$  is a positive constant. This shows that  $C_1$  is bounded. So we found the equation of  $a_k(r)$  as following

$$a_k(r) = r^{\gamma_k^+} \int_0^1 \frac{b_k(\tau)d\tau}{\tau^{\gamma_k^- - 1}(\gamma_k^- - \gamma_k^+)} + r^{\gamma_k^+} \int_1^r \frac{b_k(\tau)d\tau}{\tau^{\gamma_k^+ - 1}(\gamma_k^- - \gamma_k^+)} + r^{\gamma_k^-} \int_0^r \frac{-b_k(\tau)d\tau}{\tau^{\gamma_k^- - 1}(\gamma_k^- - \gamma_k^+)}. \quad (4.85)$$

With this choice of parameters,  $a_k(r)$  satisfies the equation (4.72) and the boundary condition. We now need to show that it satisfies the estimate as well. We have

$$r^\sigma |a_k(r)| \leq \left| \frac{r^\sigma r^{\gamma_k^+}}{\gamma_k^- - \gamma_k^+} \int_0^1 \frac{b_k(\tau)d\tau}{\tau^{\gamma_k^- - 1}} + \frac{r^\sigma r^{\gamma_k^+}}{\gamma_k^- - \gamma_k^+} \int_r^1 \frac{b_k(\tau)d\tau}{\tau^{\gamma_k^+ - 1}} + \frac{r^\sigma r^{\gamma_k^-}}{\gamma_k^- - \gamma_k^+} \int_0^r \frac{b_k(\tau)d\tau}{\tau^{\gamma_k^- - 1}} \right|.$$

Noting that we have the estimate (4.83) and  $\sigma + \gamma_k^+ > 0$ , we see that there exists a positive constant  $C$  such that

$$\begin{aligned} \left| \frac{r^\sigma r^{\gamma_k^+}}{\gamma_k^- - \gamma_k^+} \int_r^1 b_k(\tau) \frac{d\tau}{\tau^{\gamma_k^+ - 1}} \right| &\leq r^{\gamma_k^+ + \sigma} \int_r^1 |b_k(\tau)| \frac{d\tau}{\tau^{\gamma_k^+ - 1}} \\ &\leq r^{\gamma_k^+ + \sigma} \int_r^1 \frac{d\tau}{\tau^{\gamma_k^+ + \sigma + 1}} \leq \frac{r^{\gamma_k^+ + \sigma}}{|\sigma + \gamma_k^+|} \left( 1 - \frac{1}{r^{\sigma + \gamma_k^+}} \right) \leq Cr^{\gamma_k^+ + \sigma} \leq C. \end{aligned} \quad (4.86)$$

We now need to examine the other two terms. Similarly, we should note that we have  $\sup_{0 < r < 1} r^{\sigma+2}|b_k(r)| \leq 1$ , and  $\sigma + \gamma_k^+ > 0$ . Thus we get

$$\begin{aligned}
& \left| \frac{r^{\gamma_k^+ + \sigma}}{\gamma_k^- - \gamma_k^+} \int_0^1 \frac{b_k(\tau) d\tau}{\tau^{\gamma_k^- - 1}} + \frac{r^\sigma r^{\gamma_k^-}}{\gamma_k^- - \gamma_k^+} \int_0^r b_k(\tau) \frac{d\tau}{\tau^{\gamma_k^- - 1}} \right| \\
& \leq \frac{1}{|\gamma_k^- - \gamma_k^+|} \left( r^{\gamma_k^+ + \sigma} \int_0^1 \frac{|b_k(\tau)| d\tau}{\tau^{\gamma_k^- - 1}} + r^{\gamma_k^- + \sigma} \int_0^r \frac{|b_k(\tau)| d\tau}{\tau^{\gamma_k^- - 1}} \right) \\
& \leq \frac{1}{|\gamma_k^- - \gamma_k^+|} \left( r^{\gamma_k^+ + \sigma} \int_0^1 \frac{d\tau}{\tau^{\sigma + \gamma_k^- + 1}} + r^{\gamma_k^- + \sigma} \int_0^r \frac{d\tau}{\tau^{\sigma + \gamma_k^- + 1}} \right) \\
& \leq \frac{1}{|\gamma_k^- - \gamma_k^+|} \left( \frac{r^{\gamma_k^+ + \sigma}}{|\sigma + \gamma_k^-|} + \frac{r^{\gamma_k^- + \sigma}}{|\sigma + \gamma_k^-| r^{\gamma_k^- + \sigma}} \right) \leq r^{\gamma_k^+ + \sigma} + 1 \leq C \quad (4.87)
\end{aligned}$$

The bounds we get from (4.86) and (4.87) give us the desired estimate on  $a_k(r)$ . □

**Proof of Proposition 4.2.** Recall we are trying to show there is some  $C > 0$  such that for all  $f \in Y$  and  $t \geq 0$  there is some  $\phi \in X$  which satisfies

$$\begin{cases} L_t^\pm(\phi) = f & \text{in } B_1 \setminus \{0\}, \\ \phi = 0 & \text{on } \partial B_1. \end{cases} \quad (4.88)$$

Moreover one has  $\|\phi\|_X \leq C\|f\|_Y$ .

For the case  $\kappa = +p$ , the result has been proved in [1]. We are going to show the result is also true for the case  $\kappa = -p$ . First we prove the result on space  $X_1$  and to get the desired result on the full space  $X$ , we will recombine it with the result for the  $k = 0$  mode in Lemma 4.9. We fix some  $0 < T < \infty$ , and define the set of  $A$  to be all  $t \in [0, T]$  such that there exists a  $C_t > 0$  that for all  $f \in Y_1$  there exists a  $\phi \in X_1$  satisfying (4.88) and the estimate

$$\|\phi\|_X \leq C_T \|f\|_Y. \quad (4.89)$$

In Lemma 4.8, we showed that  $0 \in A$ , thus the set  $A$  is non-empty. We are going to show that  $A$  is closed and open.

First to show that is open, we let  $t_0$  be in the set  $A$ , and we are going to show that for some small  $\epsilon$  we have that  $t = t_0 + \epsilon$  is also in the set  $A$ . This means that we need to show that there exists a  $C_t > 0$  such that if we let  $f \in Y_1$  then there exists  $\phi \in X_1$  such that

$$\begin{cases} L_t^-(\phi) = f, & \text{in } B_1 \setminus \{0\}, \\ \phi = 0, & \text{on } \partial B_1, \end{cases} \quad (4.90)$$

and they satisfy the estimate. Since  $t_0$  is in the set  $A$ , thus there exists a  $C_{t_0}$  such that for all  $f \in Y_1$  there exists a  $\phi_0$  such that they satisfy

$$\begin{cases} L_{t_0}^-(\phi_0) = f, & \text{in } B_1 \setminus \{0\}, \\ \phi_0 = 0, & \text{on } \partial B_1, \end{cases} \quad (4.91)$$

and the estimate

$$\|\phi_0\|_X \leq C_{t_0} \|f\|_Y. \quad (4.92)$$

We look for a solution of the form  $\phi = \phi_0 + \psi$  where  $\psi$  is unknown. We let  $L_t^-(\phi) = -\Delta\phi + a_t\nabla\phi$  where  $a_t = \frac{-px}{(t|x|^{\epsilon-1} + \beta)|x|^2}$ . Thus we want

$$\begin{aligned} & -\Delta(\phi_0 + \psi) + a_t(\nabla\phi_0 + \nabla\psi) = f \\ \iff & -\Delta\phi_0 + a_{t_0}\nabla\phi_0 - \Delta\psi + [a_{t_0+\epsilon} - a_{t_0}]\nabla\phi_0 + a_{t_0+\epsilon}\nabla\psi = f. \end{aligned}$$

By (4.91), we find that

$$\begin{aligned} & f - \Delta\psi + [a_{t_0+\epsilon} - a_{t_0}]\nabla\phi_0 + a_{t_0+\epsilon}\nabla\psi = f \\ \iff & -\Delta\psi + a_{t_0}\nabla\psi + [a_{t_0+\epsilon} - a_{t_0}]\nabla\phi_0 + [a_{t_0+\epsilon} - a_{t_0}]\nabla\psi = 0 \\ \iff & L_{t_0}^-(\psi) = [a_{t_0} - a_{t_0+\epsilon}]\nabla\phi_0 + [a_{t_0} - a_{t_0+\epsilon}]\nabla\psi. \end{aligned} \quad (4.93)$$

Thus, we need to find  $\psi$  such that it satisfies (4.93). Now let  $f \in Y_1$  be nonzero and set  $F := \frac{f}{\|f\|}$  so  $\|F\|_Y = 1$ . By noting that  $\phi_0$  satisfies (4.92) and  $\|\psi\|_X \leq 1$ , we want to show that  $\phi$  is a solution of  $L_t^-(\phi) = f$ , so we are going to apply a fixed point argument. Define the mapping  $T_\epsilon(\psi) = \hat{\psi}$  such that

$$L_{t_0}^-(\hat{\psi}) = [a_{t_0} - a_{t_0+\epsilon}] \nabla \phi_0 + [a_{t_0} - a_{t_0+\epsilon}] \nabla \psi := f_1. \quad (4.94)$$

We are going to do a fixed point argument on  $T_\epsilon : B_1 \rightarrow B_1$  where  $B_1 = \{\psi \in X; \|\psi\|_X \leq 1\}$ .

We need to show that for some  $\epsilon$  there exists a small  $\epsilon_0 > 0$  such that for all  $|\epsilon| < \epsilon_0$

(I)  $T_\epsilon(B_1) \subset B_1$ ,

(II) there exists some  $\gamma \in (0, 1)$  such that for all  $\psi_1, \psi_2 \in B_1$  we have  $\|T_\epsilon(\psi_2) - T_\epsilon(\psi_1)\|_X \leq \gamma \|\psi_2 - \psi_1\|_X$ .

(I) *Into.* We have  $L_{t_0}^-(\hat{\psi}) = f_1$ . Let  $f_1 := K + I$  where  $K := [a_{t_0} - a_{t_0+\epsilon}] \nabla \phi_0$  and  $I := [a_{t_0} - a_{t_0+\epsilon}] \nabla \psi$ . Thus we have

$$\|f_1\| \leq \|K\|_Y + \|I\|_Y. \quad (4.95)$$

We can find

$$\begin{aligned} \|K\|_Y &\leq \sup_{0 < |x| < 1} |x|^{\sigma+2} [a_{t_0} - a_{t_0+\epsilon}] \nabla \phi_0 \\ &\leq \sup_{0 < |x| < 1} |x|^{\sigma+2} \left| \frac{px}{((t_0 + \epsilon)|x|^{\xi-1} + \beta)} - \frac{px}{((t_0)|x|^{\xi-1} + \beta)} \right| \frac{|\nabla \phi_0|}{|x|^2} \\ &\leq \sup_{0 < |x| < 1} |x|^{\sigma+1} |\nabla \phi_0| \left| \frac{p}{((t_0 + \epsilon)|x|^{\xi-1} + \beta)} - \frac{p}{((t_0)|x|^{\xi-1} + \beta)} \right|. \end{aligned} \quad (4.96)$$

Note that we have

$$\begin{aligned} \left| \frac{p}{(t_0 + \epsilon)|x|^{\xi-1} + \beta} - \frac{p}{t_0|x|^{\xi-1} + \beta} \right| &= \left| \frac{p(t_0|x|^{\xi-1} + \beta) - p((t_0 + \epsilon)|x|^{\xi-1} + \beta)}{[(t_0 + \epsilon)|x|^{\xi-1} + \beta][t_0|x|^{\xi-1} + \beta]} \right| \\ &= \left| \frac{p \epsilon |x|^{\xi-1}}{[(t_0 + \epsilon)|x|^{\xi-1} + \beta][t_0|x|^{\xi-1} + \beta]} \right|. \end{aligned} \quad (4.97)$$

Since  $\phi_0 \in X$ , we know that  $\sup_{0 < |x| < 1} |x|^{\sigma+1} |\nabla \phi_0| \leq 1$ , so we get

$$\|K\|_Y \leq \sup_{0 < |x| < 1} |x|^{\sigma+1} |\nabla \phi_0| \left| \frac{p \epsilon |x|^{\xi-1}}{[(t_0 + \epsilon)|x|^{\xi-1} + \beta][(t_0)|x|^{\xi-1} + \beta]} \right| \leq \epsilon \frac{p}{\beta^2} \quad (4.98)$$

and thus as  $\epsilon$  goes to zero,  $\|K\|_Y$  goes to zero. For  $\|I\|_Y$ , similarly we have

$$\begin{aligned} \|I\|_Y &\leq \sup_{0 < |x| < 1} |x|^{\sigma+2} [a_{t_0} - a_{t_0+\epsilon}] \nabla \psi & (4.99) \\ &\leq \sup_{0 < |x| < 1} |x|^{\sigma+2} \left| \frac{px}{((t_0 + \epsilon)|x|^{\xi-1} + \beta)} - \frac{px}{((t_0)|x|^{\xi-1} + \beta)} \right| \frac{|\nabla \psi|}{|x|^2} \\ &\leq \sup_{0 < |x| < 1} |x|^{\sigma+1} |\nabla \psi| \left| \frac{p \epsilon |x|^{\xi-1}}{[(t_0 + \epsilon)|x|^{\xi-1} + \beta][(t_0)|x|^{\xi-1} + \beta]} \right|. \end{aligned}$$

Since  $\psi \in X$ , we know that  $\sup_{0 < |x| < 1} |x|^{\sigma+1} |\nabla \psi| \leq 1$  so we get

$$\|I\|_Y \leq \epsilon \frac{p}{\beta^2} \quad (4.100)$$

and thus as  $\epsilon$  goes to zero  $\|I\|_Y$  goes to zero. By (4.95), we can deduce that for  $\epsilon$  small we have  $T_\epsilon(B_1) \subset B_1$  and so  $T_\epsilon$  is into.

(II) *Contraction.* We need to show that for some  $\gamma \in (0, 1)$ , we have

$$\|T_\epsilon(\psi_2) - T_\epsilon(\psi_1)\|_X \leq \gamma \|\psi_2 - \psi_1\|_X. \quad (4.101)$$

We set the right hand side of (4.94) to be  $f_1$  and we can write it as  $f_1 := K(\phi_0) + I(\psi)$  where  $K := [a_{t_0} - a_{t_0+\epsilon}] \nabla \phi_0$  and  $I := [a_{t_0} - a_{t_0+\epsilon}] \nabla \psi$ . Thus we can write  $\|T_\epsilon(\psi_2) - T_\epsilon(\psi_1)\|_X$  as

$$\begin{aligned} \|K(\phi_0) + I(\psi_2) - K(\phi_0) - I(\psi_1)\|_Y &= \|I(\psi_2) - I(\psi_1)\|_Y \leq \sup_{0 < |x| < 1} |x|^{\sigma+2} |\nabla \psi_2 - \nabla \psi_1| [a_{t_0} - a_{t_0+\epsilon}] \\ &\leq \sup_{0 < |x| < 1} |x|^{\sigma+1} |\nabla(\psi_2 - \psi_1)| \left| \frac{p \epsilon |x|^{\xi-1}}{[(t_0 + \epsilon)|x|^{\xi-1} + \beta][(t_0)|x|^{\xi-1} + \beta]} \right|. \end{aligned}$$

Similar to (4.99), we can find that

$$\|I(\psi_2) - I(\psi_1)\|_Y \leq \|\psi_2 - \psi_1\|_X \left| \frac{p \epsilon |x|^{\xi-1}}{[(t_0 + \epsilon)|x|^{\xi-1} + \beta][t_0|x|^{\xi-1} + \beta]} \right| \leq \epsilon \frac{p}{\beta^2} \|\psi_2 - \psi_1\|_X \quad (4.102)$$

For  $\epsilon$  small, we get that there exists some  $\gamma \in (0, 1)$  such that

$$\|I(\psi_2) - I(\psi_1)\|_Y \leq \gamma \|\psi_2 - \psi_1\|_X \quad (4.103)$$

which gives us

$$\|T_\epsilon(\psi_2) - T_\epsilon(\psi_1)\|_X \leq \gamma \|\psi_2 - \psi_1\|_X. \quad (4.104)$$

Thus  $T_\epsilon$  is also a contraction. So we can apply Banach's Fixed point Theorem and thus there exists  $\psi \in X_1$  such that it is the fixed point of  $T_\epsilon$ . So we showed that there exists a constant  $C_t$  such that for  $F \in Y_1$  where  $\|F\| = 1$  there exists a  $\phi$  in  $X_1$  such that  $L_t(\phi) = F$ . We show that there exists a  $C_{t_1} > 0$  such that for all  $f \in Y_1$  there exists  $\phi \in X_1$  such that they satisfy  $L_t(\phi) = f$  and the estimate. Thus, using the linearity of  $L_t$ , we can write

$$L_t(\phi) = F = \frac{f}{\|f\|_Y} \Rightarrow (\|f\|_Y L_t^-(\phi)) = L_t^-(\|f\|_Y \phi) = f$$

Now we set  $\phi := \|f\|_Y \phi$ , so we have

$$L_t^-(\phi) = f \quad \text{and} \quad \|\phi\|_X \leq \| \|f\|_Y \phi \|_X \leq \|\phi\|_X \|f\|_Y \leq C_{t_1} \|f\|_Y. \quad (4.105)$$

Thus for all  $f \in Y$  there exists some  $\phi \in X_1$  such that  $L_t^-(\phi) = f$  and  $\|\phi\|_X \leq C_{t_1} \|f\|$ . This means that  $t = t_0 + \epsilon$  is in the set  $A$  and thus  $A$  is open.

We now show that  $A$  is also closed. Let  $t_m$  be in  $A$  such that  $t_m$  converges to  $t \in [0, T]$ . The goal is to show that  $t$  is also in  $A$ . So we need to show that if we have  $f \in Y_1$  we can find  $\phi \in X_1$  such that  $L_t(\phi) = f$  and they satisfy the estimate.



Since  $t_m \in A$  thus for all  $f \in Y_1$  there exists  $\phi_m \in X_1$  such that  $L_{t_m}(\phi_m) = f$  and  $\|\phi_m\|_X \leq C_{t_m} \|f\|_Y$ . From  $L_{t_m}(\phi_m) = f$ , we get

$$-\Delta\phi_m + a_t \nabla\phi_m = f \quad (4.106)$$

where  $a_t = \frac{-px}{(t|x|^{\xi-1} + \beta)|x|^2}$ . Thus we have

$$-\Delta\phi_m = -a_t \nabla\phi_m + f := g_m. \quad (4.107)$$

We first assume  $C_{t_m}$  is bounded. Similar to before for  $k \geq 2$  we define the two sets

$$A_k = \left\{ x \in B_1 : \frac{1}{k} < |x| < 1 \right\} \quad \text{and} \quad \tilde{A}_k = \left\{ x \in B_1 : \frac{1}{2k} < |x| < 1 \right\}.$$

such that  $A_k \subset \tilde{A}_k$ . With a similar approach as (4.50) we can show that  $g_m$  is bounded in  $\tilde{A}_k$  and we have  $\|\Delta\phi_m\|_{L^\infty(\tilde{A}_k)} \leq C$ . By elliptic regularity we get that  $\|\phi_m\|_{C^{1,\lambda}(\bar{A}_k)} \leq C_1$ . Thus, we can use the compactness argument and the diagonal argument to deduce that there exists a subsequence  $\{\phi_{m_i}\}_i \subset \{\phi_m\}_m$  and  $\phi \in C^{1,\frac{\lambda}{2}}(\bar{A}_k)$  such that  $\phi_{m_i} \rightarrow \phi$  in  $C_{loc}^{1,\frac{\lambda}{2}}(\bar{B}_1 \setminus \{0\})$  and thus  $\phi$  satisfies

$$-\Delta\phi - \frac{p}{(t|x|^{\xi-1} + \beta)} \frac{x \cdot \nabla\phi}{|x|^2} = f \quad \text{in} \quad B_1 \setminus \{0\}. \quad (4.108)$$

Note that for fixed  $0 < |x| < 1$  we have

$$\|\phi\|_X = \sup_{0 < |x| \leq 1} \left\{ |x|^\sigma |\phi| + |x|^{\sigma+1} |\nabla\phi| \right\} \leq \lim_{m \rightarrow \infty} \sup_{0 < |x| \leq 1} \left\{ |x|^\sigma |\phi_m| + |x|^{\sigma+1} |\nabla\phi_m| \right\} \leq \lim_{m \rightarrow \infty} \|\phi_m\|_X. \quad (4.109)$$

Thus, since  $t_m \in A$ , we get

$$\|\phi\|_X \leq \lim_{m \rightarrow \infty} \|\phi_m\|_X \leq C_{t_m} \|f\|_Y. \quad (4.110)$$

This give us that  $\phi$  satisfies  $L_t^-(\phi) = f$  and the estimate thus  $t$  is in  $A$ .

We now assume  $C_{t_m}$  is unbounded. Since by the assumption  $C_m$  is the smallest possible constant that we have the estimate for, we can say that we can find some  $\phi_m \in X_1$  and  $f \in Y_1$  such that  $L_{t_m}^-(\phi_m) = f$  and  $\|\phi_m\|_X \leq C_{t_m}\|f\|_Y$ . But for  $(C_{t_m} - 1)$ , we do not have the estimate and thus we get  $\|\phi_m\|_X \geq (C_{t_m} - 1)\|f\|_Y$ . We first normalize and we get  $\|\phi_m\|_X = 1$  and  $\|f\|_Y \rightarrow 0$ . We also have

$$-\Delta\phi_m = -a_t\nabla\phi_m + f := g_m \quad (4.111)$$

and with the same approach as above there exists a subsequence  $\{\phi_{m_i}\}_i \subset \{\phi_m\}_m$  and  $\phi \in C^{1, \frac{\lambda}{2}}(\bar{A}_k)$  such that

$$-\Delta\phi - \frac{p}{(t|x|^{\xi-1} + \beta)} \frac{x \cdot \nabla\phi}{|x|^2} = 0 \quad \text{in } B_1 \setminus \{0\}. \quad (4.112)$$

With a similar argument as in Theorem 4.5, we find that  $\phi \in X_1$  is nonzero and it in the kernel of  $L_t^-$  which a contradiction with our kernel results. Thus we can deduce that  $C_{t_m}$  should be bounded. We have shown that  $A$  is non-empty, open and closed meaning that for all  $t \in (0, \infty)$  there exists  $C_t$  such that for all  $f \in Y_1$  there exists  $\phi \in X_1$  such that

$$\begin{cases} L_t^-(\phi) = f, & \text{in } B_1 \setminus \{0\}, \\ \phi = 0, & \text{on } \partial B_1, \end{cases} \quad (4.113)$$

and

$$\|\phi\|_X \leq C_t\|f\|_Y. \quad (4.114)$$

So for all  $0 < T < \infty$  there exists a  $C^T \leq \infty$  such that  $|C_t| \leq C^T$  for all  $0 \leq t \leq T$ . We should note that  $C_t$  can not approach infinity since if we assume that  $C_t \rightarrow \infty$  as  $t \rightarrow \infty$ , we get the same equation as (4.51) and we can apply the same argument to show that we get a contradiction. This shows that  $C_t$  should be bounded and

thus the proof is complete. □

Now to get the desired result on the full space  $X$ , we need to recombine it with the result for the  $k = 0$  mode.

**Lemma 4.9.** ( $k = 0$  mode for  $L_t^-$ ) There is some positive constant  $C$  such that for all positive  $t$  and all  $b(r)$  defined on  $r \in (0, 1)$  with infinite  $\sup_{0 < r < 1} r^{\sigma+2}|b(r)|$ , there exists some  $a_t$  which solves

$$-a_t''(r) - \frac{(N-1)a_t'(r)}{r} - \frac{pa_t'(r)}{\beta r + t r^\xi} = b(r) \quad 0 < r \leq 1 \quad (4.115)$$

with  $a_t(1) = 0$ . Also, there exists  $C > 0$  (independent of  $t$ ) such that  $a_t(r)$  satisfies

$$\sup_{0 < r < 1} r^{\sigma+1}|a_t'(r)| \leq C \sup_{0 < r < 1} r^{\sigma+2}|b(r)|. \quad (4.116)$$

We are also assuming

$$\sup_{0 < r \leq 1} r^{\sigma+2}|b(r)| \leq 1. \quad (4.117)$$

*Proof.* The same as before, we know the integrating factor associated with our ODE (4.115) is given by  $\mu_t(r) = e^{P(r)}$  where

$$P(r) = \int_1^r \left( \frac{N-1}{\tau} + \frac{p}{\beta\tau + t\tau^\xi} \right) d\tau = (N-1) \ln r + \frac{p}{\beta(1-\xi)} \left( \ln \left( \frac{\beta r^{1-\xi} + t}{t + \beta} \right) \right).$$

By noting that  $\beta(1-\xi) = 1-p$ , the integrating factor is

$$\mu_t(r) = e^{(N-1) \ln r + \frac{p}{\beta(1-\xi)} \left( \ln \left( \frac{\beta r^{1-\xi} + t}{t + \beta} \right) \right)} = e^{\ln r^{(N-1)}} \cdot e^{\ln \left( \frac{\beta r^{1-\xi} + t}{t + \beta} \right)^{\left( \frac{p}{\beta(1-\xi)} \right)}} = r^{N-1 + \frac{p}{\beta}} \left( \frac{tr^{1-\xi} + \beta}{t + \beta} \right)^{\left( \frac{p}{1-p} \right)}.$$

Thus from (4.115), using the integrating factor, we get

$$-\frac{d}{d\tau}(\mu_t(\tau)a'_t(\tau)) = \mu_t(\tau)b(\tau).$$

By considering  $\mu_t(1) = 1$ , we can integrate both sides and obtain

$$-\int_s^1 \frac{d}{d\tau}(\mu_t(\tau)a'_t(\tau))d\tau = \int_s^1 \mu_t(\tau)b(\tau)d\tau \implies -(\mu_t(1)a'_t(1) - \mu_t(s)a'_t(s)) = \int_s^1 \mu_t(\tau)b(\tau)d\tau.$$

Thus, we have

$$a'_t(s) = \frac{1}{\mu_t(s)} \left( a'_t(1) + \int_s^1 \mu_t(\tau)b(\tau)d\tau \right). \quad (4.118)$$

To obtain  $a_t(r)$ , we integrate (4.118) with respect to  $s$  from  $r$  to 1 and we consider  $a_t(1) = 0$ , so we deduce

$$a_t(r) = -\int_r^1 \frac{1}{\mu_t(s)} \left( a'_t(1) + \int_s^1 \mu_t(\tau)b(\tau)d\tau \right) ds.$$

We set  $a'_t(1) = -\int_{R_t}^1 \mu_t(\tau)b(\tau)d\tau$  where  $R_t^{\xi-1}t = 1$  then we have

$$a'_t(r) = -\frac{1}{\mu_t(r)} \int_{R_t}^r \mu_t(\tau)b(\tau)d\tau.$$

So we can write  $a_t$  as

$$a_t(r) = \int_r^1 \left( \frac{1}{\mu_t(s)} \int_{R_t}^s \mu_t(\tau)b(\tau)d\tau \right) ds \quad \text{for } 0 < r \leq 1.$$

Thus by considering (4.117), we can find that

$$\begin{aligned}
a_t(r) &\leq \int_r^1 \left( \frac{1}{\mu_t(s)} \int_{R_t}^s \frac{\mu_t(\tau)}{\tau^{\sigma+2}} \tau^{\sigma+2} |b(\tau)| d\tau \right) ds \\
&\leq \int_r^1 s^{1-N-\frac{p}{\beta}} \left( \frac{ts^{\xi-1} + \beta}{t + \beta} \right)^{-\left(\frac{p}{1-p}\right)} \int_{R_t}^s \left( \frac{t\tau^{\xi-1} + \beta}{t + \beta} \right)^{\left(\frac{p}{1-p}\right)} \tau^{-\xi+2\frac{p}{\beta}} \|b(r)\|_Y \\
&\leq C_t \|b(r)\|_Y.
\end{aligned} \tag{4.119}$$

So we have shown that  $a_t$  satisfies the equation with an estimate, but  $C$  can possibly depend on  $t$ . Now assume the result is false and we suppose that there exists some positive  $t_m$  such that  $C_{t_m} > M$ . Also,  $a_{t_m}$ , and  $b_m$  satisfy (4.115) and

$$\|a_{t_m}\|_X > M \|b_m\|_Y.$$

By normalizing, we can assume that

$$\|b_m\|_Y = \sup_{0 < r < 1} \{r^{\sigma+2} |b_m(r)|\} \rightarrow 0 \quad \text{and} \quad \|a_{t_m}\|_X = \sup_{0 < r < 1} \{r^\sigma |a_{t_m}(r)| + r^{\sigma+1} |a'_{t_m}(r)|\} = 1. \tag{4.120}$$

We claim

$$\sup_{0 < r < 1} \{r^{\sigma+1} |a'_{t_m}|\} \rightarrow 0. \tag{4.121}$$

Suppose there are some  $0 < r_m < 1$  and  $\epsilon > 0$  such that

$$r_m^{\sigma+1} |a_{t_m}(r_m)| \geq \epsilon. \tag{4.122}$$

We need to consider two cases:

(I)  $r_m$  is bounded away from zero,

(II)  $r_m \searrow 0$ .

In either case, recall that  $a_m$  satisfies

$$-a_{t_m}''(r) - \frac{(N-1)a_{t_m}'(r)}{r} - \frac{pa_{t_m}'(r)}{\beta r + t_m r^\xi} = b_m(r) \quad \text{for } 0 < r < 1 \quad (4.123)$$

with  $a_{t_m}(1) = 0$ .

**Case (I).** Note that for all small positive  $\varepsilon$ , we have  $\sup_{\varepsilon < r < 1} |b_m(r)| \rightarrow 0$ . Fix  $\varepsilon_0 > 0$  small. We have that  $|\frac{pa_{t_m}'(r)}{\beta r + t_m r^\xi}|$  is bounded by some positive  $C$  on  $\varepsilon_0 < r < 1$ . Using the regularity theorem of elliptic PDE, we can say for all  $\varepsilon_0 > 0$ , there exist some  $0 < \lambda < 1$  and a positive constant  $C_1$  such that

$$\|a_{t_m}\|_{C^{1,\lambda}(2\varepsilon_0 < r < 1)} < C_1.$$

So by the compactness argument and the diagonal argument, there exists some subsequence (without renaming)  $a_{t_m}$  such that  $a_{t_m} \rightarrow a$  in  $C_{loc}^{1,\frac{\lambda}{2}}(0, 1]$ . Now suppose  $t_m$  converges to some  $t \in [0, \infty]$  and so we can pass to the limit in (4.123) to arrive at

$$-a''(r) - \frac{(N-1)a'(r)}{r} - \frac{pa'(r)}{\beta r + tr^\xi} = 0 \quad 0 < r < 1 \quad (4.124)$$

with  $a(1) = 0$  when  $t$  is finite. Also, in the case where  $t = \infty$  the equation is

$$-a''(r) - \frac{(N-1)a'(r)}{r} = 0 \quad 0 < r < 1$$

with  $a(1) = 0$ . Note that we can pass to a subsequence of  $r_m$  such that  $r_m \rightarrow r_0 \in (0, 1]$  and using the convergence we have, we get  $r_0^{\sigma+1}|a'(r_0)| > \varepsilon$ . The kernel results we obtained for  $\kappa = -p$  in Lemma 4.4 shows that this kernel is trivial and hence we have a contradiction.

**Case (II).** We first define

$$z_m(r) := r_m^\sigma \{a_{t_m}(r_m r) - a_{t_m}(r_m)\} \quad \text{for } 0 < r < \frac{1}{r_m}$$

where  $z_m(1) = 0$ . We have

$$-a_{t_m}''(rr_m) - \frac{(N-1)a_{t_m}'(rr_m)}{rr_m} - \frac{pa_{t_m}'(rr_m)}{\beta rr_m + t_m(rr_m)^\xi} = b_m(rr_m) \quad 0 < rr_m < 1 \quad (4.125)$$

so we get

$$-r_m^{\sigma+2}a_{t_m}''(rr_m) - \frac{(N-1)r_m^{\sigma+1}a_{t_m}'(rr_m)}{r} - \frac{pr_m^{\sigma+1}a_{t_m}'(rr_m)}{\beta r + t_m(r)^\xi(r_m)^\xi} = r_m^{\sigma+2}b_m(rr_m).$$

Thus

$$-z_m''(r) - \frac{(N-1)z_m'(r)}{r} - \frac{pz_m'(r)}{\beta r + r^\xi t_m(r_m)^\xi} = r_m^{\sigma+2}b_m(rr_m) =: g_m(r) \quad 0 < r < \frac{1}{r_m} \quad (4.126)$$

where  $z_m(1) = 0$ . Also note that we have

$$\sup_{0 < r < \infty} r^{\sigma+2}|g_m(r)| = \sup_{0 < rr_m < 1} (rr_m)^{\sigma+2}|b_m(rr_m)| \rightarrow 0.$$

Considering the boundary condition and (4.122), we can write

$$z_m(1) = 0, \quad |z_m'(1)| = r_m^{\sigma+1}|a_{t_m}'(r_m)| \geq \epsilon, \quad \text{and} \quad r^{\sigma+1}|z_m'(r)| = (rr_m)^{\sigma+1}|a_{t_m}'(r_m r)| \leq 1. \quad (4.127)$$

Since  $\left| \frac{pz_m'(r)}{\beta r + r^\xi t_m(r_m)^\xi} \right|$  is bounded, we can apply a similar argument as before and so we have  $\{z_m\}_m$  is a bounded sequence is  $C^{1,\lambda}$  away from the origin and infinity, thus there exists a subsequence (without renaming)  $\{z_m\}_m$  such that  $z_m$  converges to some  $z$  in  $C^{1,\frac{\lambda}{2}}$ . Now we can pass to the limit in (4.126). We need to consider three cases depending on the limiting behaviour of  $t_m(r_m)^\xi$ :

(i)  $t_m(r_m)^{\xi-1} \rightarrow 0$ ,

(ii)  $t_m(r_m)^{\xi-1} \rightarrow t \in (0, \infty)$ ,

(iii)  $t_m(r_m)^{\xi-1} \rightarrow \infty$ .

Case (i): Assuming that  $t_m(r_m)^{\xi-1}$  goes to zero, by passing the limit in equation (4.126), we get

$$-z''(r) - \frac{(N-1)z'(r)}{r} - \frac{pz'(r)}{\beta r} = 0 \quad 0 < r < \infty$$

where  $z(1) = 0$ . Since away from the origin and infinity we have the  $C^{1,\lambda}$  convergence, we can pass the limit in (4.127). Thus we have

$$|z'(1)| \geq \epsilon, \tag{4.128}$$

and

$$r^{\sigma+1}|z'(r)| \leq 1.$$

Since  $z(r)$  is in the required space, we can use the kernel results we obtained in the case of  $k = 0$ , and  $t = 0$ . So we should have  $z(r) = 0$  on  $0 < r < \infty$ . But this is a contradiction with (4.128).

Case (ii): In this case, we can use a similar approach as case (i) and by passing the limit in equation (4.126), we get

$$-z''(r) - \frac{(N-1)z'(r)}{r} - \frac{pz'(r)}{\beta r + t} = 0 \quad 0 < r < \infty$$

and so using the convergence we have obtained, we can pass to the limit in (4.127).

Thus we have

$$|z'(1)| \geq \epsilon, \tag{4.129}$$



and

$$r^{\sigma+1}|z'(r)| \leq 1.$$

We again use the Lemma 4.4 in the case  $t \neq 0$  where similarly gives us the result of  $z$  being zero. Thus have the same contradiction as case (i).

Case (iii): When  $t_m(r_m)^{\xi-1}$  approaches infinity, we get

$$-z''(r) - \frac{(N-1)z'(r)}{r} = 0 \quad 0 < r < \infty$$

where  $z(1) = 0$ . We can use the result from Lemma 4.4 in the case of  $t$  approaching infinity which stated we should have  $z(r) = 0$ . Thus again we have a contradiction with (4.129).

The contradictions from these 3 cases prove that we have the estimate (4.119) where  $C$  is independent of  $t$ . This can complete the result and thus we have shown that  $a_t$  satisfies the equation and we have the estimate independent of  $t$ .

□

# 5

## The non-linear theory

The goal in this chapter is to show that our nonlinear mapping  $T_t(\phi, \psi)$  is into and a contraction. Thus by applying Banach's Fixed Point Theorem we can obtain its fixed point and complete the proof of Theorem 1.1. Before that, we need to show the estimates and the asymptotic results that we will need for the main proof.

### 5.1 Estimates

**Lemma 5.1.** 1. Suppose  $1 < p \leq 2$ . Then there exists some positive  $\hat{C}_{1,1}$  such that for  $x, y \in \mathbb{R}^N$

$$\left| |x + y|^p - |x|^p - p|x|^{p-2}x \cdot y \right| \leq \hat{C}_{1,1}|y|^p. \quad (5.1)$$

2. For  $x, y, z \in \mathbb{R}^N$ , and  $1 < p \leq 2$ , there exists some positive  $\hat{C}_{1,2}$  such that

$$\left| |x + y|^p - p|x|^{p-2}x \cdot y - |x + z|^p + p|x|^{p-2}x \cdot z \right| \leq \hat{C}_{1,2} \left( |y|^{p-1} + |z|^{p-1} \right) |y - z|. \quad (5.2)$$

3. For  $x, y, z \in \mathbb{R}^N$ , and  $1 < p \leq 2$ , there exists some positive  $\hat{C}_{1,3}$  such that

$$||x + y|^p - |x + z|^p| \leq \hat{C}_{1,3} (|x|^{p-1} + |y|^{p-1} + |z|^{p-1}) |y - z|. \quad (5.3)$$

*Proof.* 1. We start with the proof of the first inequality for  $x, y, z \in \mathbb{R}^N$ , and  $1 < p \leq 2$ . First we prove it for the case where  $|y|$  is small. Fix  $p$  and let  $\hat{x} = \frac{x}{|x|}$  and  $\hat{y} = \frac{y}{|y|}$  then we get  $|\hat{x}| = 1$  and  $|\hat{y}| \leq \epsilon$  where  $\epsilon$  is small.

Let  $x = \hat{x}$  and  $y = \hat{y}$ , thus

$$|x + y|^p = (|x + y|^2)^{\frac{p}{2}} = [|x|^2 + 2x \cdot y + |y|^2]^{\frac{p}{2}} = [1 + 2x \cdot y + |y|^2]^{\frac{p}{2}}.$$

Let  $t = 2x \cdot y + |y|^2$  then by using the binomial theorem we get

$$|x + y|^p = [1 + t]^{\frac{p}{2}} = 1 + \frac{p}{2}t + \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} t^k.$$

We now add  $(-1 - p x \cdot y)$  to the both sides of the equality, so for some positive constants  $C_1, C_2, \dots$ , we have

$$\begin{aligned} \left| |x + y|^p - 1 - p x \cdot y \right| &= \left| -1 - p x \cdot y + 1 + p x \cdot y + \frac{p}{2}|y|^2 + \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} t^k \right| \\ &\leq \left| \frac{p}{2}|y|^2 + \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} t^k \right| \leq \frac{p}{2}|y|^2 + \left| (C_1 y^2 + C_2 y^3 + \dots) \right|. \end{aligned}$$

Thus for small  $\epsilon$  such that  $|y| \leq \epsilon$  and fixed  $1 < p \leq 2$ , we get

$$\begin{aligned} \left| |x + y|^p - 1 - p x \cdot y \right| &\leq \frac{p}{2}|y|^2 + (C_1|y|^2 + C_2|y|^3 + \dots) \\ &\leq \frac{p}{2}|y|^p + \tilde{C}_1|y|^p + \tilde{C}_2|y|^p + \tilde{C}_3|y|^p + \dots \leq \tilde{C}|y|^p. \end{aligned}$$

So we can deduce that

$$\left| |x + y|^p - 1 - p x \cdot y \right| \leq \tilde{C}|y|^p. \quad (5.4)$$

We now let  $|x| = 1$ , and  $|y| > \epsilon$ . First note that we know there exists a constant  $C_p > 0$  such that  $|x + y|^p \leq C_p(|x|^p + |y|^p)$ .

We want to show that there exists a positive constant  $\hat{C}$  such that for  $|y| \geq \epsilon$ , and  $1 < p \leq 2$

$$\left| |x + y|^p - 1 - p x \cdot y \right| \leq \hat{C}|y|^p.$$

For  $|x| = 1$  and  $|y| > \epsilon$ , we have

$$\frac{\left| |x + y|^p - 1 - p x \cdot y \right|}{|y|^p} \leq \frac{|x + y|^p + 1 + p|y|}{|y|^p} \leq \frac{C_p + C_p|y|^p + 1 + p|y|}{|y|^p}$$

and so when  $|y|$  approaches infinity we get

$$\sup_{|x|=1, |y|>\epsilon} \frac{\left| |x + y|^p - 1 - p x \cdot y \right|}{|y|^p} = C < \infty,$$

which shows that for  $|y| \geq \epsilon$ , when  $1 < p \leq 2$  there exists some  $\hat{C} > 0$  such that

$$\left| |x + y|^p - 1 - p x \cdot y \right| \leq \hat{C}|y|^p. \quad (5.5)$$

Thus by (5.4) and (5.5), for  $|\hat{x}| = 1$  and  $\hat{y} \in \mathbb{R}^N$ , let  $C_{1,1} = \max\{\tilde{C}, \hat{C}\}$  so we have

$$\left| |\hat{x} + \hat{y}|^p - 1 - p \hat{x} \cdot \hat{y} \right| \leq C_{1,1}|\hat{y}|^p. \quad (5.6)$$

Note that we had  $\hat{x} = \frac{x}{|x|}$  and  $\hat{y} = \frac{y}{|y|}$ . Then we rewrite (5.6) as

$$\begin{aligned} & \left| \left| \frac{x}{|x|} + \frac{y}{|y|} \right|^p - 1 - p \frac{x}{|x|} \cdot \frac{y}{|y|} \right| \leq C_{1,1} \left| \frac{y}{|y|} \right|^p \\ \implies & \frac{1}{|x|^p} \left| |x+y|^p - |x|^p - p|x|^{p-2} x \cdot y \right| \leq \frac{1}{|x|^p} C_{1,1} |y|^p. \end{aligned}$$

Thus for  $x$  and  $y \in \mathbb{R}^N$  we have:

$$\left| |x+y|^p - |x|^p - p|x|^{p-2} x \cdot y \right| \leq C_{1,1} |y|^p.$$

2. We want to prove that for all  $x, y, z \in \mathbb{R}^N$ , and  $1 < p \leq 2$ , there exists a positive constant  $\hat{C}_{1,2} > 0$  such that

$$\left| |x+y|^p - p|x|^{p-2} x \cdot y - |x+z|^p + p|x|^{p-2} x \cdot z \right| \leq \hat{C}_{1,2} (|y|^{p-1} + |z|^{p-1}) |y-z|.$$

Similarly, we first prove it for the case where  $|y|$  and  $|z|$  are small. Fix  $p$  and let  $\hat{x} = \frac{x}{|x|}$ ,  $\hat{y} = \frac{y}{|y|}$ , and  $\hat{z} = \frac{z}{|z|}$  thus we get  $|\hat{x}| = 1$ ,  $|\hat{y}| \leq \epsilon_1$ , and  $|\hat{z}| \leq \epsilon_2$  where  $\epsilon_1$  and  $\epsilon_2$  are small. Let  $x = \hat{x}$ ,  $y = \hat{y}$ , and  $z = \hat{z}$ . We have

$$\begin{aligned} |x+y|^p &= (|x+y|^2)^{\frac{p}{2}} = (|x|^2 + 2x \cdot y + |y|^2)^{\frac{p}{2}} = (1 + 2x \cdot y + |y|^2)^{\frac{p}{2}} \\ \iff |x+z|^p &= (|x+z|^2)^{\frac{p}{2}} = (|x|^2 + 2x \cdot z + |z|^2)^{\frac{p}{2}} = (1 + 2x \cdot z + |z|^2)^{\frac{p}{2}}. \end{aligned}$$

Set  $\{t = 2x \cdot y + |y|^2\}$  and  $\{\tau = 2x \cdot z + |z|^2\}$  then by the binomial theorem, we get

$$|x+y|^p = (1+t)^{\frac{p}{2}} = 1 + \frac{p}{2}t + \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} t^k, \quad \text{and} \quad |x+z|^p = (1+\tau)^{\frac{p}{2}} = 1 + \frac{p}{2}\tau + \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} \tau^k.$$

Thus for a fixed  $k$ , we can write

$$\begin{aligned}
|x+y|^p - |x+z|^p &= 1 + \frac{p}{2}t - 1 - \frac{p}{2}\tau + \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} t^k - \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} \tau^k \\
&= p x \cdot y - p x \cdot z + \frac{p}{2} |y|^2 - \frac{p}{2} |z|^2 + \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} t^k - \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} \tau^k \\
&= p x \cdot y - p x \cdot z + \frac{p}{2} |y|^2 - \frac{p}{2} |z|^2 + \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} (t^k - \tau^k) \\
&= p x \cdot y - p x \cdot z + \frac{p}{2} |y|^2 - \frac{p}{2} |z|^2 \\
&\quad + \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} (t - \tau)(t^{k-1} + t^{k-2}\tau + t^{k-3}\tau^2 + \dots + t\tau^{k-2} + \tau^{k-1}).
\end{aligned}$$

So by taking  $\{p x \cdot y - p x \cdot z\}$  to the other side, we get

$$\begin{aligned}
|x+y|^p - |x+z|^p - p x \cdot y + p x \cdot z &= \frac{p}{2} (|y|^2 - |z|^2) + \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} (t - \tau) (t^{k-1} + t^{k-2}\tau + t^{k-3}\tau^2 + \dots + t\tau^{k-2} + \tau^{k-1}) \\
&= \frac{p}{2} (y - z) \cdot (y + z) + (t - \tau) \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} (t^{k-1} + t^{k-2}\tau + t^{k-3}\tau^2 + \dots + t\tau^{k-2} + \tau^{k-1}) \\
&= \frac{p}{2} (y - z) \cdot (y + z) + (2 x \cdot y - 2 x \cdot z + |y|^2 - |z|^2) \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} (t^{k-1} + t^{k-2}\tau + \dots + t\tau^{k-2} + \tau^{k-1}) \\
&= \frac{p}{2} (y - z) \cdot (y + z) + (2 x \cdot (y - z) + (y - z)(y + z)) \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} (t^{k-1} + t^{k-2}\tau + \dots + t\tau^{k-2} + \tau^{k-1}) \\
&= (y - z) \cdot \left[ \frac{p}{2} (y + z) + (2 x + (y + z)) \cdot \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} (t^{k-1} + t^{k-2}\tau + \dots + t\tau^{k-2} + \tau^{k-1}) \right].
\end{aligned}$$

Since we have  $|t| \leq 2|y| + |y|^2$ , and  $|\tau| \leq 2|z| + |z|^2$  and also  $|y| < \epsilon_1$  and  $|z| < \epsilon_2$ , we can deduce that there exists a positive constant  $C$  such that

$$||x+y|^p - |x+z|^p - p x \cdot y + p x \cdot z| \leq C|y-z|.$$

We now prove the inequality for all  $x, y$ , and  $z \in \mathbb{R}^N$ . Let  $f(x) = |x|^p$  and we take

$|x| = 1$  and at least one of  $y, z$  are bounded away from zero. Consider

$$g(t) = f(x + y + t(z - y)) - f(x) - t\nabla f(x) \cdot (z - y)$$

thus

$$g(1) = f(x + y + (z - y)) - f(x) - \nabla f(x) \cdot (z - y) = f(x + z) - f(x) - \nabla f(x) \cdot (z - y),$$

$$g(0) = f(x + y) - f(x).$$

Also note that

$$g'(t) = (z - y) \cdot \nabla f(x + y + t(z - y)) - \nabla f(x) \cdot (z - y).$$

We can write  $|g(1) - g(0)| \leq \int_0^1 |g'(t)| dt$  which gives us

$$\begin{aligned} |g(1) - g(0)| &= |f(x + z) - f(x) - \nabla f(x) \cdot (z - y) - f(x + y) + f(x)| & (5.7) \\ &= |f(x + z) - f(x + y) - \nabla f(x)(z - y)| \\ &= |f(x + z) - f(x + y) - p|x|^{p-1}(z - y)| \\ &= |f(x + z) - f(x + y) - pz + py| \end{aligned}$$

and also note that we have

$$\begin{aligned} \int_0^1 |g'(t)| dt &= \int_0^1 |(z - y)p \left| |x + y + t(z - y)|^{p-1} - |x|^{p-1}(z - y) \right| & (5.8) \\ &= \int_0^1 \left| p(z - y) \left| |x + y + t(z - y)|^{p-1} - 1 \right| \right| \\ &\leq |z - y| \left( |x|^{p-1} + |y|^{p-1} + |z|^{p-1} - 1 \right). \end{aligned}$$

By (5.7) and (5.8), we can deduce that

$$|f(x+z) - f(x+y) - pz + py| \leq |z-y|(|x|^{p-1} + |y|^{p-1} + |z|^{p-1} - 1).$$

Thus, there exists a positive constant  $C$  such that for  $|\hat{x}| = 1$ , and  $\hat{y}, \hat{z} \in \mathbb{R}^N$  we have

$$||\hat{x} + \hat{y}|^p - p\hat{y} - |\hat{x} + \hat{z}|^p + p\hat{z}| \leq C (|\hat{y}|^{p-1} + |\hat{z}|^{p-1}) |\hat{y} - \hat{z}|.$$

Note that we had  $\hat{x} = \frac{x}{|x|}$ ,  $\hat{y} = \frac{y}{|x|}$ , and  $\hat{z} = \frac{z}{|x|}$ . Then we rewrite this as

$$\begin{aligned} & \left| |\hat{x} + \hat{y}|^p - p\hat{y} - |\hat{x} + \hat{z}|^p + p\hat{z} \right| \leq C (|\hat{y}|^{p-1} + |\hat{z}|^{p-1}) |\hat{y} - \hat{z}| \\ \iff & \left| \left| \frac{x}{|x|} + \frac{y}{|x|} \right|^p - p \frac{y}{|x|} - \left| \frac{x}{|x|} + \frac{z}{|x|} \right|^p + p \frac{z}{|x|} \right| \leq C \left( \left| \frac{y}{|x|} \right|^{p-1} + \left| \frac{z}{|x|} \right|^{p-1} \right) \left| \frac{y}{|x|} - \frac{z}{|x|} \right| \\ \iff & \frac{1}{|x|^p} \left| |x+y|^p - p|x|^{p-1}y - |x+z|^p + p|x|^{p-1}z \right| \leq \frac{1}{|x|^p} C (|y|^{p-1} + |z|^{p-1}) |y-z|. \end{aligned}$$

So we get

$$\left| |x+y|^p - p|x|^{p-2}x \cdot y - |x+z|^p + p|x|^{p-2}x \cdot z \right| \leq C (|y|^{p-1} + |z|^{p-1}) |y-z|.$$

3. We want to prove for all  $x, y$ , and  $z \in \mathbb{R}^N$ , and  $1 < p \leq 2$ , there exists a positive constant  $\hat{C}_{1,3}$  such that

$$||x+y|^p - |x+z|^p| \leq \hat{C}_{1,2} (|x|^{p-1} + |y|^{p-1} + |z|^{p-1}) |y-z|.$$

Similarly at first we prove the estimate for  $|y|$  and  $|z|$  small. Fix  $p$  and let  $\hat{x} = \frac{x}{|x|}$ ,  $\hat{y} = \frac{y}{|x|}$ ,  $\hat{z} = \frac{z}{|x|}$ . Then we get  $|\hat{x}| = 1$  and also  $|\hat{y}| \leq \epsilon_1$ , and  $|\hat{z}| \leq \epsilon_2$  where  $\epsilon_1$  and  $\epsilon_2$



are small. Set  $x = \hat{x}$ ,  $y = \hat{y}$ , and  $z = \hat{z}$ . We have

$$\begin{aligned} |x + y|^p &= |(x + y)^2|^{\frac{p}{2}} = \left| |x|^2 + 2x \cdot y + |y|^2 \right|^{\frac{p}{2}} = \left| 1 + 2x \cdot y + |y|^2 \right|^{\frac{p}{2}} \\ |x + z|^p &= |(x + z)^2|^{\frac{p}{2}} = \left| |x|^2 + 2x \cdot z + |z|^2 \right|^{\frac{p}{2}} = \left| 1 + 2x \cdot z + |z|^2 \right|^{\frac{p}{2}}. \end{aligned}$$

Set  $\{t = 2x \cdot y + |y|^2\}$  and  $\{\tau = 2x \cdot z + |z|^2\}$  then by the binomial theorem we get

$$|x + y|^p = (1 + t)^{\frac{p}{2}} = 1 + \frac{p}{2}t + \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} t^k, \quad \text{and} \quad |x + z|^p = |1 + \tau|^{\frac{p}{2}} = 1 + \frac{p}{2}\tau + \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} \tau^k.$$

Thus, for a fixed  $k$ , we can get

$$\begin{aligned} |x + y|^p - |x + z|^p &= 1 + \frac{p}{2}t - 1 - \frac{p}{2}\tau + \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} t^k - \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} \tau^k. \\ &= p x \cdot y - p x \cdot z + \frac{p}{2} |y|^2 - \frac{p}{2} |z|^2 + \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} t^k - \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} \tau^k \\ &= p x \cdot y - p x \cdot z + \frac{p}{2} |y|^2 - \frac{p}{2} |z|^2 + \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} (t^k - \tau^k) \\ &= p x \cdot y - p x \cdot z + \frac{p}{2} (y^2 - z^2) \\ &\quad + \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} (t - \tau)(t^{k-1} + t^{k-2}\tau + t^{k-3}\tau^2 + \dots + t\tau^{k-2} + \tau^{k-1}). \end{aligned}$$

So we have

$$\begin{aligned} |x + y|^p - |x + z|^p &= p x \cdot y - p x \cdot z + \frac{p}{2} (y - z)(y + z) + (t - \tau) \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} (t^{k-1} + t^{k-3}\tau^2 + \dots + t\tau^{k-2} + \tau^{k-1}) \\ &= px(y - z) + \frac{p}{2} (y - z)(y + z) + (2x \cdot y - 2x \cdot z + |y|^2 - |z|^2) \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} (t^{k-1} + \dots + \tau^{k-1}) \\ &= px(y - z) + \frac{p}{2} (y - z)(y + z) + (2x \cdot (y - z) + (y - z)(y + z)) \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} (t^{k-1} + \dots + \tau^{k-1}) \\ &= (y - z) \left[ px + \frac{p}{2} (y + z) + (2x + (y + z)) \sum_{k=2}^{\infty} \binom{\frac{p}{2}}{k} (t^{k-1} + t^{k-2}\tau + \dots + t\tau^{k-2} + \tau^{k-1}) \right]. \end{aligned}$$

Since we have  $|t| \leq 2|y| + |y|^2$ , and  $|\tau| \leq 2|z| + |z|^2$  and also by noting that  $|y| < \epsilon_1$  and  $|z| < \epsilon_2$ , we can say there exists a positive constant  $C$  such that

$$||x + y|^p - |x + z|^p| \leq C|y - z|.$$

Let us prove the inequality for all  $x, y$ , and  $z \in \mathbb{R}^N$ . Let  $f(x) = |x|^p$  and we take  $|x| = 1$  and at least one of  $y$  or  $z$  is bounded away from zero. Consider the function

$$g(t) = f(x + y + t(z - y))$$

thus we have

$$\begin{aligned} g(1) &= f(x + y + (z - y)) = f(x + z), \\ g(0) &= f(x + y). \end{aligned}$$

We find that

$$g'(t) = (z - y) \nabla f(x + y + t(z - y)).$$

We can write  $|g(1) - g(0)| \leq \int_0^1 |g'(t)| dt$  so we have

$$|g(1) - g(0)| = |f(x + z) - f(x + y) + f(x)| \tag{5.9}$$

and

$$\begin{aligned} \int_0^1 |g'(t)| dt &= \int_0^1 |(z - y)p |x + y + t(z - y)|^{p-1}| dt \\ &= \int_0^1 |p(z - y)|x + y + t(z - y)|^{p-1}| dt \\ &\leq |z - y|(|x|^{p-1} + |y|^{p-1} + |z|^{p-1}). \end{aligned} \tag{5.10}$$

By (5.9) and (5.10), we can deduce that

$$|f(x+z) - f(x+y) + f(x)| \leq |z-y|(|x|^{p-1} + |y|^{p-1} + |z|^{p-1}).$$

Thus, there exists a constant  $C > 0$  such that for  $|\hat{x}| = 1$  and  $\hat{y}, \hat{z} \in \mathbb{R}^N$ , we have

$$||\hat{x} + \hat{y}|^p - |\hat{x} + \hat{z}|^p| \leq C \left(1 + |\hat{y}|^{p-1} + |\hat{z}|^{p-1}\right) |\hat{y} - \hat{z}|.$$

Note that we had  $\hat{x} = \frac{x}{|x|}$ ,  $\hat{y} = \frac{y}{|x|}$ , and  $\hat{z} = \frac{z}{|x|}$ . Then we rewrite this as

$$\begin{aligned} & ||\hat{x} + \hat{y}|^p - |\hat{x} + \hat{z}|^p| \leq C \left(1 + |\hat{y}|^{p-1} + |\hat{z}|^{p-1}\right) |\hat{y} - \hat{z}| \\ \iff & \left| \left| \frac{x}{|x|} + \frac{y}{|x|} \right|^p - \left| \frac{x}{|x|} + \frac{z}{|x|} \right|^p \right| \leq C \left(1 + \left| \frac{y}{|x|} \right|^{p-1} + \left| \frac{z}{|x|} \right|^{p-1}\right) \left| \frac{y}{|x|} - \frac{z}{|x|} \right| \\ \iff & \frac{1}{|x|^p} ||x+y|^p - |x+z|^p| \leq \frac{1}{|x|^p} C \left(|x|^{p-1} + |y|^{p-1} + |z|^{p-1}\right) |y-z|. \end{aligned}$$

Thus we have:

$$||x+y|^p - |x+z|^p| \leq C \left(|x|^{p-1} + |y|^{p-1} + |z|^{p-1}\right) |y-z|.$$

□

## 5.2 Asymptotics

To prove Theorem 1.1, we will also need some asymptotics of  $w_t$ . Note that we had

$$w_t(r) = \int_r^1 \frac{dy}{(ty^\xi + \beta y)^{\frac{1}{p-1}}}, \quad \text{and} \quad w'_t(r) = 0 - \frac{1}{(tr^\xi + \beta r)^{\frac{1}{p-1}}} = \frac{-1}{(tr^\xi + \beta r)^{\frac{1}{p-1}}}.$$

This means that for  $0 < r < 1$ , we have

$$|w'_t(r)| \leq \min\left\{\frac{C_\beta}{r^{\frac{1}{p-1}}}, \frac{1}{t^{\frac{1}{p-1}}r^{N-1}}\right\}$$

where  $C_\beta = \frac{1}{\beta^{\frac{1}{p-1}}}$ . This gives us

$$r^{\sigma+1}|w'_t(r)| \leq \min\left\{C_\beta, \frac{1}{t^{\frac{1}{p-1}}r^{N-2-\sigma}}\right\}.$$

We should also note that for any positive  $t$ , we have

$$\lim_{r \rightarrow 0} r^{\sigma+1}w'_t(r) = -C_\beta. \quad (5.11)$$

To show this, note that

$$\lim_{r \rightarrow 0} r^{\sigma+1}w'_t(r) = \lim_{r \rightarrow 0} \frac{-r^{\sigma+1}}{(tr^\xi + \beta r)^{\frac{1}{p-1}}} = \lim_{r \rightarrow 0} \frac{-r^{\sigma+1}}{r^{\frac{1}{p-1}}(tr^{\xi-1} + \beta)^{\frac{1}{p-1}}}$$

and since  $\sigma + 1 = \frac{1}{p-1}$ , we have

$$\lim_{r \rightarrow 0} r^{\sigma+1}w'_t(r) = \lim_{r \rightarrow 0} \frac{-r^{\sigma+1}}{r^{\sigma+1}(tr^{\xi-1} + \beta)^{\frac{1}{p-1}}} = \lim_{r \rightarrow 0} \frac{-1}{(tr^{\xi-1} + \beta)^{\frac{1}{p-1}}} = \frac{-1}{\beta^{\frac{1}{p-1}}} = -C_\beta.$$

Also, away from zero, we can see that  $w_t(r)$  and  $w'_t(r)$  converge uniformly to zero.

### 5.3 The Fixed Point Argument

We defined the nonlinear mapping  $T_t(\phi, \psi) = (\hat{\phi}, \hat{\psi})$  via

$$\begin{cases} -\Delta \hat{\phi} - p|\nabla w|^{p-2}\nabla w \cdot \nabla \hat{\psi} = \kappa_1(x)|\nabla w + \nabla \psi|^p + I(\psi) & \text{in } B_1 \setminus \{0\}, \\ -\Delta \hat{\psi} - p|\nabla w|^{p-2}\nabla w \cdot \nabla \hat{\phi} = \kappa_2(x)|\nabla w + \nabla \phi|^p + I(\phi) & \text{in } B_1 \setminus \{0\}, \\ \phi = \psi = 0 & \text{on } \partial B_1, \end{cases} \quad (5.12)$$

where

$$I(\zeta) = |\nabla w + \nabla \zeta|^p - |\nabla w|^p - p|\nabla w|^{p-2} \nabla w \cdot \nabla \zeta.$$

So for  $R > 0$ , we define the space  $\mathcal{F}_R$  as

$$\mathcal{F}_R := \{(\phi, \psi) \in X \times X : \|\phi\|_X, \|\psi\|_X \leq R\}.$$

We will show that  $T_t$  is a contraction on  $\mathcal{F}_R$  for suitable  $R$  and  $t$ . Also, note that on this space we have

$$\|(\phi, \psi)\|_{X \times X} := \|\phi\|_X + \|\psi\|_X. \quad (5.13)$$

### 5.3.1 Into

**Lemma 5.2.** To show that our mapping  $T_t$  is into, we need to show that for all  $(\phi, \psi) \in \mathcal{F}_R$ , we have  $T_t(\phi, \psi) \in \mathcal{F}_R$ . We know that by (5.13), we have

$$\|T_t(\phi, \psi)\|_{X \times X} = \|(\hat{\phi}, \hat{\psi})\|_{X \times X} = \|\hat{\phi}\|_X + \|\hat{\psi}\|_X.$$

First we set the right hand side of (5.12) to be  $H_1(\psi)$  and  $H_2(\phi)$  as  $H_1(\psi) = J_t(\psi) + Q_t(\psi)$  and  $H_2(\phi) = I_t(\phi) + K_t(\phi)$  where

$$J_t(\psi) := |\nabla w_t + \nabla \psi|^p - |\nabla w_t|^p - p|\nabla w_t|^{p-2} \nabla w_t \nabla \psi, \quad \text{and} \quad Q_t(\psi) := \kappa_1 |\nabla w_t + \nabla \psi|^p,$$

$$I_t(\phi) := |\nabla w_t + \nabla \phi|^p - |\nabla w_t|^p - p|\nabla w_t|^{p-2} \nabla w_t \nabla \phi, \quad \text{and} \quad K_t(\phi) := \kappa_2 |\nabla w_t + \nabla \phi|^p.$$

To get the result we need, we should prove two statements for both  $H_1(\psi)$  and  $H_2(\phi)$ .

First we state them for  $H_1(\phi)$ .

1. There is some positive constant  $C$  such that for  $R \in (0, 1)$ ,  $0 < \delta < 1$ ,  $t > 1$ ,

and  $\psi \in B_R \subset X$  one has

$$\|\kappa_1|\nabla w_t + \nabla\psi|^p\|_Y \leq C \left( R^p + \sup_{|z|<\delta} |k_1(z)| + \frac{1}{t^{\frac{p}{p-1}} \delta^{(N-1)p-\sigma-2}} \right).$$

2. There is some positive constant  $C$  such that for all  $t > 1$ ,  $0 < R < 1$  and  $\psi \in B_R$  one has

$$\| |\nabla w_t + \nabla\psi|^p - |\nabla w_t|^p - p|\nabla w_t|^{p-2}\nabla w_t \nabla\psi \|_Y \leq CR^p.$$

*Proof.* Fix  $R$ , and  $\psi$  as in the hypothesis and  $C$  would be a constant independent of these parameters. We have the estimates

$$(|x|^{\sigma+1}|\nabla\phi(x)|)^p \leq R^p \quad \text{and} \quad (|x|^{\sigma+1}|\nabla w_t(x)|)^p \leq C. \quad (5.14)$$

First note that we have:

$$\begin{aligned} \|Q_t(\psi)\|_Y &= \sup_{0<|x|\leq 1} |x|^{\sigma+2}|\kappa_1|(|\nabla w_t + \nabla\psi|^p) \leq C \sup_{0<|x|\leq 1} |x|^{\sigma+2}|\kappa_1|(|\nabla w_t|^p + |\nabla\psi|^p) \\ &\leq C \sup_{0<|x|\leq 1} |x|^{\sigma+2}|x|^{-(\sigma+1)p}|\kappa_1| \left( (|x|^{\sigma+1}|\nabla w_t|)^p + (|x|^{\sigma+1}|\nabla\psi|)^p \right). \end{aligned}$$

We know that  $\sigma + 2 - (\sigma + 1)p = \frac{2-p}{p-1}(1-p) + 2 - p = 0$ , thus we get

$$\|Q_t(\psi)\|_Y \leq C \sup_{0<|x|\leq 1} |\kappa_1|(|x|^{\sigma+1}|\nabla w_t|)^p + \sup_{0<|x|\leq 1} |\kappa_1|(|x|^{\sigma+1}|\nabla\psi|)^p.$$

We saw that  $w_t(r) = \int_r^1 \frac{dy}{(ty^\xi + \beta y)^{\frac{1}{p-1}}}$ , so

$$\|Q_t(\psi)\|_Y \leq \sup_{0<|x|\leq 1} |\kappa_1| \left( |x|^{\sigma+1}(t|x|^\xi + \beta|x|)^{\frac{-1}{p-1}} \right)^p + \sup_{0<|x|\leq 1} |\kappa_1|(|x|^{\sigma+1}|\nabla\psi|)^p$$

and since  $\sigma + 1 = \frac{1}{p-1}$ , we get

$$\|Q_t(\psi)\|_Y \leq \sup_{0 < |x| \leq 1} |\kappa_1| (t|x|^{\xi-1} + \beta)^{\frac{-p}{p-1}} + \sup_{0 < |x| \leq 1} |\kappa_1| (|x|^{\sigma+1} |\nabla\psi|)^p.$$

Fix  $0 < \delta \ll 1$ . We have  $\kappa_1(x)$ , and  $\kappa_2(x)$  are positive continuous functions such that

$\kappa_1(0) = \kappa_2(0) = 0$ . We write the  $\sup_{0 < |x| \leq 1} |\kappa_1(x)| (t|x|^{\xi-1} + \beta)^{\frac{-p}{p-1}} + \sup_{0 < |x| \leq 1} |\kappa_1(x)| (|x|^{\sigma+1} |\nabla\psi|)^p$  as a supremum over  $B_\delta$  and  $\delta < |x| \leq 1$  and this gives us

$$\begin{aligned} \|Q_t(\psi)\|_Y &\leq \sup_{B_\delta} |\kappa_1(x)| (t|x|^{\xi-1} + \beta)^{\frac{-p}{p-1}} + \sup_{\delta < |x| \leq 1} |\kappa_1(x)| (t|x|^{\xi-1} + \beta)^{\frac{-p}{p-1}} \\ &\quad + \sup_{B_\delta} |\kappa_1(x)| (|x|^{\sigma+1} |\nabla\psi|)^p + \sup_{\delta < |x| \leq 1} |\kappa_1(x)| (|x|^{\sigma+1} |\nabla\psi|)^p \\ &\leq C \sup_{B_\delta} |\kappa_1(x)| + CR^p + \sup_{\delta < |x| \leq 1} \frac{1}{t^{\frac{p}{p-1}} |x|^{(\xi-1)\frac{p}{p-1}}} \\ &\leq CR^p + C \sup_{B_\delta} |\kappa_1(x)| + \frac{C}{t^{\frac{p}{p-1}} |\delta|^{(\xi-1)\frac{p}{p-1}}}. \end{aligned}$$

Thus we have

$$\|Q_t(\psi)\|_Y \leq C \left( R^p + \sup_{B_\delta} |\kappa_2(x)| + \frac{1}{t^{\frac{p}{p-1}} |\delta|^{(\xi-1)\frac{p}{p-1}}} \right). \quad (5.15)$$

For  $J_t(\psi)$ , we apply the estimate (5.1) where we set  $x = \nabla w_t$  and  $y = \nabla\psi$ . So we get

$$\begin{aligned} \|J_t(\psi)\|_Y &\leq \sup_{0 < |x| \leq 1} |x|^{\sigma+2} |\nabla w_t + \nabla\psi|^p - |\nabla w_t|^p - p |\nabla w_t|^{p-2} \nabla w_t \nabla\psi \\ &\leq \sup_{0 < |x| \leq 1} |x|^{\sigma+2} |\nabla\psi|^p \leq \sup_{0 < |x| \leq 1} (|x|^{\sigma+1} |\nabla\psi|)^p \leq CR^p. \end{aligned} \quad (5.16)$$

By (5.15) and (5.16) we can deduce

$$\begin{aligned} \|H_1(\psi)\|_Y &= \sup_{0 < |x| \leq 1} |x|^{\sigma+2} |J_t(\psi) + Q_t(\psi)| \leq \sup_{0 < |x| \leq 1} |x|^{\sigma+2} |J_t(\psi)| + \sup_{0 < |x| \leq 1} |x|^{\sigma+2} |Q_t(\psi)| \\ &\leq C \left( R^p + \sup_{B_\delta} |\kappa_1|(x) + \frac{1}{t^{\frac{p}{p-1}} |\delta|^{(\xi-1)\frac{p}{p-1}}} \right). \end{aligned}$$

□

We now prove similar statements for  $H_2(\phi)$ .

1. There is some positive constant  $C$  such that for  $R \in (0, 1)$ ,  $0 < \delta < 1$ ,  $t > 1$ , and  $\phi \in B_R$  one has

$$\|\kappa_2 |\nabla w_t + \nabla \phi|^p\|_Y \leq C \left( R^p + \sup_{|z| < \delta} |\kappa_2(z)| + \frac{1}{t^{\frac{p}{p-1}} \delta^{(N-1)p-\sigma-2}} \right).$$

2. There is some positive constant  $C$  such that for all  $t > 1$ ,  $0 < R < 1$  and  $\phi \in B_R$  one has

$$\| |\nabla w_t + \nabla \phi|^p - |\nabla w_t|^p - p |\nabla w_t|^{p-2} \nabla w_t \nabla \phi \|_Y \leq CR^p.$$

*Proof.* Fix  $R$ , and  $\phi$  as in the hypothesis and  $C$  would be a constant independent of these parameters and we again use the estimates  $(|x|^{\sigma+1} |\nabla \psi(x)|)^p \leq R^p$ , and  $(|x|^{\sigma+1} |\nabla w_t(r)|)^p \leq C$ . First note that we have:

$$\begin{aligned} \|k_t(\phi)\|_Y &= \sup_{0 < |x| \leq 1} |x|^{\sigma+2} |\kappa_2| (|\nabla w_t + \nabla \phi|^p) \\ &\leq C \sup_{0 < |x| \leq 1} |x|^{\sigma+2} |\kappa_2| (|\nabla w_t|^p + |\nabla \phi|^p) \\ &\leq C \sup_{0 < |x| \leq 1} |x|^{\sigma+2} |x|^{-(\sigma+1)p} |\kappa_2| \left( (|x|^{\sigma+1} |\nabla w_t|)^p + (|x|^{\sigma+1} |\nabla \phi|)^p \right). \end{aligned}$$



We saw that  $\sigma + 2 - (\sigma + 1)p = 0$ , thus we get:

$$\|k_t(\phi)\|_Y \leq C \sup_{0 < |x| \leq 1} |\kappa_2|(|x|^{\sigma+1} |\nabla w_t|)^p + \sup_{0 < |x| \leq 1} |\kappa_2|(|x|^{\sigma+1} |\nabla \phi|)^p.$$

By using the equation of  $w_t$ , we have

$$\|k_t(\phi)\|_Y \leq \sup_{0 < |x| \leq 1} |\kappa_2| \left( |x|^{\sigma+1} (t|x|^\xi + \beta|x|)^{\frac{-1}{p-1}} \right)^p + \sup_{0 < |x| \leq 1} |\kappa_2|(|x|^{\sigma+1} |\nabla \phi|)^p$$

and since  $\sigma + 1 = \frac{1}{p-1}$ , we get

$$\|k_t(\phi)\|_Y \leq \sup_{0 < |x| \leq 1} |\kappa_2|(t|x|^{\xi-1} + \beta)^{\frac{-p}{p-1}} + \sup_{0 < |x| \leq 1} |\kappa_2|(|x|^{\sigma+1} |\nabla \phi|)^p.$$

Similarly, fix  $0 < \delta \ll 1$ . We know that  $\kappa_1(x), \kappa_2(x) > 0$  are continuous and  $\kappa_1(0) = \kappa_2(0) = 0$ . We write the  $\sup_{0 < |x| \leq 1} |\kappa_2(x)|(t|x|^{\xi-1} + \beta)^{\frac{-p}{p-1}} + \sup_{0 < |x| \leq 1} |\kappa_2(x)|(|x|^{\sigma+1} |\nabla \phi|)^p$  as a supremum over  $B_\delta$  and  $\delta < |x| \leq 1$ . Noting that  $(\xi - 1)\frac{p}{p-1} > 0$ , we get

$$\begin{aligned} \|k_t(\phi)\|_Y &\leq \sup_{B_\delta} |\kappa_2(x)|(t|x|^{\xi-1} + \beta)^{\frac{-p}{p-1}} + \sup_{\delta < |x| \leq 1} |\kappa_2(x)|(t|x|^{\xi-1} + \beta)^{\frac{-p}{p-1}} \\ &\quad + \sup_{B_\delta} |\kappa_2(x)|(|x|^{\sigma+1} |\nabla \phi|)^p + \sup_{\delta < |x| \leq 1} |\kappa_2(x)|(|x|^{\sigma+1} |\nabla \phi|)^p \\ &\leq C \sup_{B_\delta} |\kappa_2(x)| + CR^p + \sup_{\delta < |x| \leq 1} \frac{1}{t^{\frac{p}{p-1}} |x|^{(\xi-1)\frac{p}{p-1}}} \\ &\leq CR^p + C \sup_{B_\delta} |\kappa_2(x)| + \frac{C}{t^{\frac{p}{p-1}} |\delta|^{(\xi-1)\frac{p}{p-1}}}. \end{aligned}$$

Thus we have

$$\|k_t(\phi)\|_Y \leq C \left( R^p + \sup_{B_\delta} |\kappa_2(x)| + \frac{1}{t^{\frac{p}{p-1}} |\delta|^{(\xi-1)\frac{p}{p-1}}} \right). \quad (5.17)$$

For  $I_t(\phi)$ , we can apply (5.1), where we set  $x = \nabla w_t$  and  $y = \nabla \phi$ . So we get:

$$\begin{aligned} \|I_t(\phi)\|_Y &\leq \sup_{0 < |x| \leq 1} |x|^{\sigma+2} |\nabla w_t + \nabla \phi|^p - |\nabla w_t|^p - p |\nabla w_t|^{p-2} \nabla w_t \nabla \phi \quad (5.18) \\ &\leq \sup_{0 < |x| \leq 1} |x|^{\sigma+2} |\nabla \phi|^p \leq \sup_{0 < |x| \leq 1} (|x|^{\sigma+1} |\nabla \phi|)^p \leq CR^p. \end{aligned}$$

By (5.17) and (5.18) we obtain

$$\begin{aligned} \|H_2(\phi)\|_Y &= \sup_{0 < |x| \leq 1} |x|^{\sigma+2} |I_t(\phi) + K_t(\phi)| \leq \sup_{0 < |x| \leq 1} |x|^{\sigma+2} |I_t(\phi)| + \sup_{0 < |x| \leq 1} |x|^{\sigma+2} |K_t(\phi)| \\ &\leq C \left( R^p + \sup_{B_\delta} |\kappa_2|(x) + \frac{1}{t^{\frac{p}{p-1}} |\delta|^{(\xi-1)\frac{p}{p-1}}} \right). \end{aligned}$$

□

### 5.3.2 Contraction

**Lemma 5.3.** To show our mapping is a contraction, we are going to show that if we have  $0 < R < 1$ ,  $0 < \delta < 1$ ,  $t > 1$ , and  $\phi_i, \psi_i \in B_R$  and we set  $T_t(\phi_i, \psi_i) = (\hat{\phi}_i, \hat{\psi}_i)$ , there exists some  $C > 0$  such that

$$\|T_t(\phi_2, \psi_2) - T_t(\phi_1, \psi_1)\|_{X \times X} \leq C \left\{ R^{p-1} + \sup_{B_\delta} \{|\kappa_1| + |\kappa_2|\} + \frac{1}{t \delta^{\xi-1}} \right\} \|(\phi_2, \psi_2) - (\phi_1, \psi_1)\|_{X \times X}.$$

*Proof.* First note that we have:

$$\begin{aligned} \|T_t(\phi_2, \psi_2) - T_t(\phi_1, \psi_1)\|_{X \times X} &= \|(\hat{\phi}_2, \hat{\psi}_2) - (\hat{\phi}_1, \hat{\psi}_1)\|_{X \times X} \\ &= \|[(\hat{\phi}_2 - \hat{\phi}_1), (\hat{\psi}_2 - \hat{\psi}_1)]\|_{X \times X} \\ &= \|\hat{\phi}_1 - \hat{\phi}_2\|_X + \|\hat{\psi}_1 - \hat{\psi}_2\|_X. \end{aligned}$$

We set the right hand sides of (5.12) to be  $H_1(\psi)$  and  $H_2(\phi)$  where

$$H_1(\psi) := |\nabla w_t + \nabla \psi|^p - |\nabla w_t|^p - p|\nabla w_t|^{p-2} \nabla w_t \nabla \psi + \kappa_1 |\nabla w_t + \nabla \psi|^p,$$

$$H_2(\phi) := |\nabla w_t + \nabla \phi|^p - |\nabla w_t|^p - p|\nabla w_t|^{p-2} \nabla w_t \nabla \phi + \kappa_2 |\nabla w_t + \nabla \phi|^p.$$

For  $H_1(\psi)$  let us define  $I_1$  and  $K_1$  such that

$$H_1(\psi_2) - H_1(\psi_1) = I_1 + k_1$$

where

$$I_1(\psi_1, \psi_2) := |\nabla w_t + \nabla \psi_2|^p - p|\nabla w_t|^{p-2} \nabla w_t \nabla \psi_2 - |\nabla w_t + \nabla \psi_1|^p + p|\nabla w_t|^{p-2} \nabla w_t \nabla \psi_1 \quad (5.19)$$

and

$$k_1(\psi_1, \psi_2) := \kappa_1(x)(|\nabla w_t + \nabla \psi_2|^p - |\nabla w_t + \nabla \psi_1|^p). \quad (5.20)$$

Similarly, we write  $H_2(\phi)$  as

$$H_2(\phi_2) - H_2(\phi_1) = J_2 + Q_2$$

where

$$I_2(\phi_1, \phi_2) := |\nabla w_t + \nabla \phi_2|^p - p|\nabla w_t|^{p-2} \nabla w_t \nabla \phi_2 - |\nabla w_t + \nabla \phi_1|^p + p|\nabla w_t|^{p-2} \nabla w_t \nabla \phi_1$$

$$Q_2(\phi_1, \phi_2) := \kappa_2(x)(|\nabla w_t + \nabla \phi_2|^p - |\nabla w_t + \nabla \phi_1|^p).$$

We prove two claims for both  $H_1$  and  $H_2$  to show that  $T_t$  is a contraction. First we state and prove our claims for  $H_1(\psi)$ .

**Claim 1:** There exists a positive constant  $\hat{C}$  such that for  $0 < R < 1$ ,  $t \geq 0$  and

$\phi_i, \psi_i \in B_R$  we have

$$\sup_{0 < |x| \leq 1} |x|^{\sigma+2} |k_1(\psi_1, \psi_2)| \leq \hat{C} \left( \sup_{B_\delta} \kappa_1(x) + \frac{1}{t\delta^{\xi-1}} + 2R^{p-1} \right) \|(\phi_2, \psi_2) - (\phi_1, \psi_1)\|_{X \times X}.$$

**Claim 2.** There exists a positive constant  $\tilde{C}$  such that for  $0 < R < 1$ ,  $t > 1$ , and  $\phi_i, \psi_i \in B_R$  we have:

$$\sup_{0 < |x| \leq 1} |x|^{\sigma+2} |I_1(\psi_1, \psi_2)| \leq \tilde{C} 2R^{p-1} \|(\phi_2, \psi_2) - (\phi_1, \psi_1)\|_{X \times X}.$$

*Proof of claim 1.* We use (3.1) and (5.20) and apply (5.3) where we take  $x = \nabla w_t$ ,  $y = \nabla \psi_2$  and  $z = \nabla \psi_1$ . Also we note that  $(p-1)(\sigma+1) = 1$ , thus we have:

$$\begin{aligned} \sup_{0 < |x| \leq 1} |x|^{\sigma+2} |k_1(\psi_1, \psi_2)| &\leq \sup_{0 < |x| \leq 1} |x|^{\sigma+2} \kappa_1(x) \left( (|\nabla w_t + \nabla \psi_2|^p - |\nabla w_t + \nabla \psi_1|^p) \right) \\ &\leq C_1 \sup_{0 < |x| \leq 1} |x|^{\sigma+2} \kappa_1(x) \left( |\nabla w_t|^{p-1} + |\nabla \psi_1|^{p-1} + |\nabla \psi_2|^{p-1} \right) |\nabla \psi_1 - \nabla \psi_2| \\ &\leq C \sup_{0 < |x| \leq 1} |x|^{\sigma+2} \kappa_1(x) \left( \frac{1}{(t|x|^\xi + \beta|x|)} + |\nabla \psi_1|^{p-1} + |\nabla \psi_2|^{p-1} \right) |\nabla \psi_1 - \nabla \psi_2| \\ &\leq C_1 \sup_{0 < |x| \leq 1} |x|^{\sigma+2} \kappa_1(x) \left( (t|x|^\xi + \beta|x|)^{-1} \right) |\nabla \psi_1 - \nabla \psi_2| \\ &\quad + C_1 \sup_{0 < |x| \leq 1} |x|^{\sigma+2} \kappa_1(x) \left( |\nabla \psi_1|^{p-1} + |\nabla \psi_2|^{p-1} \right) |\nabla \psi_1 - \nabla \psi_2| \\ &\leq C_1 \sup_{0 < |x| \leq 1} |x|^{\sigma+1} |\nabla \psi_1 - \nabla \psi_2| \left( \frac{\kappa_1(x)|x|}{t|x|^\xi + \beta|x|} \right. \\ &\quad \left. + \kappa_1(x)|x|^{(1-(p-1)(\sigma+1))} \left( (|x|^{\sigma+1} |\nabla \psi_2|)^{p-1} + (|x|^{\sigma+1} |\nabla \psi_1|)^{p-1} \right) \right) \\ &\leq C_1 \sup_{0 < |x| \leq 1} |x|^{\sigma+1} |\nabla(\psi_1 - \psi_2)| \left( \frac{\kappa_1(x)|x|}{t|x|^\xi + \beta|x|} + \kappa_1(x) \left( (R)^{p-1} + (R)^{p-1} \right) \right) \\ &\leq C_1 \sup_{0 < |x| \leq 1} \left( \frac{\kappa_1(x)}{t|x|^{\xi-1} + \beta} + 2C_1 R^{p-1} \kappa_1(x) \right) \|\psi_2 - \psi_1\|_X. \end{aligned}$$

Let  $0 < \delta < 1$ , so we find that:

$$\begin{aligned}
\sup_{0 < |x| \leq 1} \frac{\kappa_1(x)}{t|x|^{\xi-1} + \beta} &\leq \sup_{0 < |x| \leq \delta} \frac{\kappa_1(x)}{t|x|^{\xi-1} + \beta} + \sup_{\delta < |x| \leq 1} \frac{\kappa_1(x)}{t|x|^{\xi-1} + \beta} \\
&\leq \sup_{0 < |x| \leq \delta} \frac{\kappa_1(x)}{\beta} + \sup_{\delta < |x| \leq 1} \frac{\kappa_1(x)}{t\delta^{\xi-1}} \\
&\leq C \sup_{B_\delta} \kappa_1(x) + \frac{C}{t\delta^{\xi-1}}.
\end{aligned} \tag{5.21}$$

Thus, there exists a positive constant  $\hat{C}$  such that

$$\sup_{0 < |x| \leq 1} |x|^{\sigma+2} |k_1(\psi_1, \psi_2)| \leq \hat{C} \left( \sup_{B_\delta} \kappa_1(x) + 2R^{p-1} + \frac{1}{t\delta^{\xi-1}} \right) \|\psi_2 - \psi_1\|_X.$$

We should note that

$$\|\psi_2 - \psi_1\|_X \leq \|(\phi_2, \psi_2) - (\phi_1, \psi_1)\|_{X \times X} = \|\psi_2 - \psi_1\|_X + \|\phi_2 - \phi_1\|_X \tag{5.22}$$

which means

$$\sup_{0 < |x| \leq 1} |x|^{\sigma+2} |k_1(\psi_1, \psi_2)| \leq \hat{C} \left( \sup_{B_\delta} \kappa_1(x) + 2R^{p-1} + \frac{1}{t\delta^{\xi-1}} \right) \|(\phi_2, \psi_2) - (\phi_1, \psi_1)\|_{X \times X}. \tag{5.23}$$

□

*Proof of claim 2.* We apply (5.3) where we set  $\nabla w_t = x$ ,  $\nabla \psi_2 = y$ ,  $\nabla \psi_1 = z$ , thus

we have

$$\begin{aligned}
& \sup_{0 < |x| \leq 1} |x|^{\sigma+2} |I_1(\psi_1, \psi_2)| \\
&= \sup_{0 < |x| \leq 1} |x|^{\sigma+2} \left| |\nabla w_t + \nabla \psi_2|^p - p |\nabla w_t|^{p-2} \nabla w_t \cdot \nabla \psi_2 - |\nabla w_t + \nabla \psi_1|^p + p |\nabla w_t|^{p-2} \nabla w_t \cdot \nabla \psi_1 \right| \\
&\leq C \sup_{0 < |x| \leq 1} |x|^{\sigma+2} \left( (|\nabla \psi_2|^{p-1} + |\nabla \psi_1|^{p-1}) |\nabla \psi_2 - \nabla \psi_1| \right) \\
&\leq C \sup_{0 < |x| \leq 1} |x| (|\nabla \psi_2|^{p-1} + |\nabla \psi_1|^{p-1}) |x|^{\sigma+1} |\nabla \psi_2 - \nabla \psi_1| \\
&\leq C \sup_{0 < |x| \leq 1} |x|^{1-(p-1)(\sigma+1)} \left( (|x|^{(\sigma+1)} |\nabla \psi_2|^{p-1}) + (|x|^{(\sigma-1)} |\nabla \psi_1|^{p-1}) \right) |x|^{\sigma+1} |\nabla(\psi_2 - \psi_1)| \\
&\leq C \sup_{0 < |x| \leq 1} \left( (|x|^{(\sigma+1)} |\nabla \psi_2|^{p-1}) + (|x|^{(\sigma-1)} |\nabla \psi_1|^{p-1}) \right) \|\psi_2 - \psi_1\|_X \\
&\leq \tilde{C} 2R^{p-1} \|\psi_2 - \psi_1\|_X.
\end{aligned}$$

Thus by (5.22), we can write

$$\sup_{0 < |x| \leq 1} |x|^{\sigma+2} |I_1(\psi_1, \psi_2)| \leq \tilde{C} 2R^{p-1} \|(\phi_2, \psi_2) - (\phi_1, \psi_1)\|_{X \times X}. \quad (5.24)$$

By (5.23) and (5.24), we can deduce that

$$\|\hat{\psi}_2 - \hat{\psi}_1\|_X \leq \hat{C} \left( \sup_{B_\delta} \kappa_1(x) + \frac{1}{t\delta^{\xi-1}} + 2R^{p-1} \right) \|(\phi_2, \psi_2) - (\phi_1, \psi_1)\|_{X \times X}. \quad (5.25)$$

□

We now state similar claims for  $H_2(\phi)$ .

**Claim 3:** There exists a positive constant  $\hat{C}$  such that for  $0 < R < 1$ ,  $t \geq 0$  and  $\phi_i, \psi_i \in B_R$

$$\sup_{0 < |x| \leq 1} |x|^{\sigma+2} |Q_2(\phi_1, \phi_2)| \leq \hat{C} \left( \sup_{B_\delta} \kappa_2(x) + 2R^{p-1} + \frac{1}{t\delta^{\xi-1}} \right) \|(\phi_2, \psi_2) - (\phi_1, \psi_1)\|_{X \times X}.$$

**Claim 4:** There exists  $\tilde{C} > 0$  such that for  $0 < R < 1$ ,  $t > 1$ , and  $\phi_i, \psi_i \in B_R$  we

have:

$$\sup_{0 < |x| \leq 1} |x|^{\sigma+2} |J_2(\phi_2, \phi_2)| \leq \tilde{C} 2R^{p-1} \|(\phi_2, \psi_2) - (\phi_1, \psi_1)\|_{X \times X}.$$

*Proof of claim 3.* With a similar proof to (5.23), we can apply (5.3) where we set  $x = \nabla w_t$ ,  $y = \nabla \psi_2$  and  $z = \nabla \psi_1$  and we have:

$$\begin{aligned} \sup_{0 < |x| \leq 1} |x|^{\sigma+2} |Q_1(\phi_1, \phi_2)| &\leq \sup_{0 < |x| \leq 1} |x|^{\sigma+2} \kappa_2(x) \left( (|\nabla w_t + \nabla \phi_2|^p - |\nabla w_t + \nabla \phi_1|^p) \right) \\ &\leq C_1 \sup_{0 < |x| \leq 1} |x|^{\sigma+2} \kappa_2(x) \left( |\nabla w_t|^{p-1} + |\nabla \phi_1|^{p-1} + |\nabla \phi_2|^{p-1} \right) |\nabla \phi_1 - \nabla \phi_2| \\ &\leq C \sup_{0 < |x| \leq 1} |x|^{\sigma+2} \kappa_2(x) \left( \frac{1}{(t|x|^\xi + \beta|x|)} + |\nabla \phi_1|^{p-1} + |\nabla \phi_2|^{p-1} \right) |\nabla \phi_1 - \nabla \phi_2| \\ &\leq C_1 \sup_{0 < |x| \leq 1} |x|^{\sigma+1} |\nabla(\phi_1 - \phi_2)| \left( \frac{\kappa_2(x)|x|}{t|x|^\xi + \beta|x|} + \kappa_2(x) \left( (R)^{p-1} + (R)^{p-1} \right) \right) \\ &\leq C_1 \sup_{0 < |x| \leq 1} \left( \frac{\kappa_2(x)}{t|x|^{\xi-1} + \beta} + 2C_1 R^{p-1} \kappa_2(x) \right) \|\phi_2 - \phi_1\|_X. \end{aligned}$$

Similar to (5.21), by letting  $0 < \delta < 1$ , we have:

$$\sup_{0 < |x| \leq 1} \frac{\kappa_2(x)}{t|x|^{\xi-1} + \beta} \leq C \sup_{B_\delta} \kappa_2(x) + \frac{C}{t\delta^{\xi-1}}.$$

Thus, by (5.22) there exists a positive constant  $\hat{C}$  such that

$$\sup_{0 < |x| \leq 1} |x|^{\sigma+2} |Q_1(\phi_1, \phi_2)| \leq \hat{C} \left( \sup_{B_\delta} \kappa_2(x) + 2R^{p-1} + \frac{1}{t\delta^{\xi-1}} \right) \|(\phi_2, \psi_2) - (\phi_1, \psi_1)\|_{X \times X}. \quad (5.26)$$

*Proof of claim 4* Similar to the proof of (5.24), we can apply (5.2) where we take

$\nabla w_t = x$ ,  $\nabla \psi_2 = y$ ,  $\nabla \psi_1 = z$ , and thus we have

$$\begin{aligned}
& \sup_{0 < |x| \leq 1} |x|^{\sigma+2} |J_1(\phi_1, \psi_2)| \\
&= \sup_{0 < |x| \leq 1} |x|^{\sigma+2} \left| |\nabla w_t + \nabla \psi_2|^p - p |\nabla w_t|^{p-2} \nabla w_t \cdot \nabla \phi_2 - |\nabla w_t + \nabla \phi_1|^p + p |\nabla w_t|^{p-2} \nabla w_t \cdot \nabla \phi_1 \right| \\
&\leq C \sup_{0 < |x| \leq 1} |x|^{\sigma+2} \left( (|\nabla \phi_2|^{p-1} + |\nabla \phi_1|^{p-1}) |\nabla \phi_2 - \nabla \phi_1| \right) \\
&\leq C \sup_{0 < |x| \leq 1} \left( (|x|^{\sigma+1} |\nabla \phi_2|^{p-1}) + (|x|^{\sigma-1} |\nabla \phi_1|^{p-1}) \right) \|\phi_2 - \phi_1\|_X \\
&\leq \tilde{C} 2R^{p-1} \|\phi_2 - \phi_1\|_X.
\end{aligned}$$

So by (5.22), we can write

$$\sup_{0 < |x| \leq 1} |x|^{\sigma+2} |J_1(\phi_1, \phi_2)| \leq \tilde{C} 2R^{p-1} \|(\phi_2, \psi_2) - (\phi_1, \psi_1)\|_{X \times X}. \quad (5.27)$$

By (5.26) and (5.27), we can deduce that

$$\|\hat{\phi}_2 - \hat{\phi}_1\|_X \leq \hat{C} \left( \sup_{B_\delta} \kappa_2(x) + \frac{1}{t\delta^{\xi-1}} + 2R^{p-1} \right) \|(\phi_2, \psi_2) - (\phi_1, \psi_1)\|_{X \times X}. \quad (5.28)$$

So by (5.25) and (5.28), we have

$$\begin{aligned}
& \|T_t(\phi_2, \psi_2) - T_t(\phi_1, \psi_1)\|_{X \times X} = \|(\hat{\phi}_2, \hat{\psi}_2) - (\hat{\phi}_1, \hat{\psi}_1)\|_{X \times X} \\
&= \|[(\hat{\phi}_2 - \hat{\phi}_1), (\hat{\psi}_2 - \hat{\psi}_1)]\|_{X \times X} = \|\hat{\phi}_1 - \hat{\phi}_2\|_X + \|\hat{\psi}_1 - \hat{\psi}_2\|_X \\
&\leq \hat{C} \left( \sup_{B_\delta} \kappa_1 + 2R^{p-1} + \frac{1}{t\delta^{\xi-1}} \right) \|(\phi_2, \psi_2) - (\phi_1, \psi_1)\|_{X \times X} \\
&+ \tilde{C} \left( \sup_{B_\delta} \kappa_2 + 2R^{p-1} + \frac{1}{t\delta^{\xi-1}} \right) \|(\phi_2, \psi_2) - (\phi_1, \psi_1)\|_{X \times X} \\
&\leq C \left( \sup_{B_\delta} \{\kappa_1 + \kappa_2\} + 2R^{p-1} + \frac{1}{t\delta^{\xi-1}} \right) \|(\phi_2, \psi_2) - (\phi_1, \psi_1)\|_{X \times X}.
\end{aligned}$$

□



## 5.4 Proof of the main theorem

**Proof of Theorem (1.1)** We can now complete the proof of our main theorem. Recall that we want to find some  $\phi$  and  $\psi$  which satisfy (3.4) such that  $u(x) = w_t(x) + \phi(x)$ , and  $v(x) = w_t(x) + \psi(x)$  satisfy equation (1.2). We will show that the mapping  $T_t$  is a contraction on  $\mathcal{F}_R$  for suitable  $0 < R < 1$ , and large  $t$ .

*Into.* Let  $0 < R < 1$ ,  $0 < \delta < 1$ ,  $t > 1$ , and  $\phi, \psi \in B_R$ . Set  $T_t(\phi, \psi) = (\hat{\phi}, \hat{\psi})$ . Then by Lemma 5.2, there exists some  $C > 0$  (independent of the parameters) such that

$$\|T_t(\psi, \phi)\|_X = \|(\hat{\psi}, \hat{\phi})\|_X \leq C \left( R^p + \sup_{B_\delta} \{|\kappa_1|(x) + |\kappa_2|(x)|\} + \frac{1}{t^{\frac{p}{p-1}} |\delta|^{(\xi-1)\frac{p}{p-1}}} \right).$$

Hence, for  $(\hat{\phi}, \hat{\psi})$  to be in  $\mathcal{F}_R$ , it is sufficient to have

$$C \left( R^p + \sup_{B_\delta} \{|\kappa_1|(x) + |\kappa_2|(x)|\} + \frac{1}{t^{\frac{p}{p-1}} |\delta|^{(\xi-1)\frac{p}{p-1}}} \right) \leq R. \quad (5.29)$$

*Contraction.* Let  $0 < R < 1$ ,  $0 < \delta < 1$ , and  $t > 1$ , also  $\phi_i, \psi_i \in B_R$ . Set  $T_t(\phi_i, \psi_i) = (\hat{\phi}_i, \hat{\psi}_i)$ . Then by Lemma 5.3 we have

$$\|(\hat{\phi}_2, \hat{\psi}_2) - (\hat{\phi}_1, \hat{\psi}_1)\|_{X \times X} \leq C \left( \sup_{B_\delta} (\kappa_1 + \kappa_2) + 2R^{p-1} + \frac{1}{t\delta^{\xi-1}} \right) \|(\phi_2, \psi_2) - (\phi_1, \psi_1)\|_{X \times X}.$$

Hence, for  $T_t$  to be a contraction, it is sufficient to have

$$C \left( \sup_{B_\delta} (\kappa_1 + \kappa_2) + 2R^{p-1} + \frac{1}{t\delta^{\xi-1}} \right) \leq \frac{1}{2}. \quad (5.30)$$

We now choose the parameters. Note that we can satisfy both (5.29) and (5.30) by first taking  $R > 0$  sufficiently small, similarly we take  $\delta > 0$  sufficiently small and then we take  $t$  large enough. Thus we obtain that  $T_t$  is into and a contraction.

We can now apply Banach's Fixed Point Theorem to find the fixed point  $(\phi, \psi)$  of  $T_t$  on  $\mathcal{F}_R$  and hence  $(\phi, \psi)$  satisfies (3.4). Thus  $u_t = u(x) = w_t(x) + \phi(x)$ , and

$v_t = v(x) = w_t(x) + \psi(x)$  are solutions of (1.2) provided  $(\phi, \psi)$  satisfies (3.4). We need to verify that we can obtain a nonzero positive solution. Note that we have

$$\begin{aligned} |x|^{\sigma+1} |\nabla u_t(x)| &\geq r^{\sigma+1} |w'_t(r)| - |x|^{\sigma+1} |\nabla \phi(x)| \geq r^{\sigma+1} |w'_t(r)| - R, \\ |x|^{\sigma+1} |\nabla v_t(x)| &\geq r^{\sigma+1} |w'_t(r)| - |x|^{\sigma+1} |\nabla \psi(x)| \geq r^{\sigma+1} |w'_t(r)| - R \end{aligned}$$

since  $\phi, \psi \in B_R$ . Recall now by (5.11) for all positive  $t$ , we have  $(\lim_{r \rightarrow 0} r^{\sigma+1} w'_t(r) = -C_\beta)$  and hence by taking  $R > 0$  sufficiently small, we can see  $u_t, v_t$  are nonzero when for some small positive  $\epsilon$ , we have  $0 < |x| \leq \epsilon$ .

To show that both  $u_t$  and  $v_t$  are positive on  $B_1 \setminus \{0\}$ , note that by (3.1), we have

$$w_t(r) = \int_r^1 \frac{dy}{(ty^\xi + \beta y)^{\frac{1}{p-1}}}.$$

For all positive  $t$ , we can find some  $z \in (0, 1)$  such that we have  $ty^{\xi-1} < 1$  for any  $y$  that satisfies  $0 < y < z$ . Thus, since we have  $\frac{-1}{p-1} + 1 = \sigma$ , we get:

$$\begin{aligned} w_t &= \int_r^1 \frac{1}{(ty^\xi + \beta y)^{\frac{1}{p-1}}} dy \geq \int_r^z \frac{1}{(y(ty^{\xi-1} + \beta))^{\frac{1}{p-1}}} dy \geq \int_r^z \frac{1}{y^{\frac{1}{p-1}} (1 + \beta)^{\frac{1}{p-1}}} \\ &\geq C \left[ -\sigma y^{-\sigma} \right]_r^z = C \left[ \frac{1}{r^\sigma} - \frac{1}{z^\sigma} \right]. \end{aligned}$$

So there exists some positive constant  $C = C_{(\beta, p)}$  such that for all  $0 < r < z$ , we have

$$r^\sigma w_t \geq C \left( 1 - \left( \frac{r}{z} \right)^\sigma \right)$$

and we know

$$u_t = w_t + \phi \quad v_t = w_t + \psi$$

where  $\phi, \psi$  are in  $X$ . Thus near the origin when  $r$  goes to zero, we see that  $u_t, v_t$  are positive. Now for  $\epsilon < |x| < 1$ , note that since  $\kappa_i \geq 0$ , we have  $-\Delta u_t = (1 + \kappa_1(x)) |\nabla v|^p \geq 0$ , and  $-\Delta v = (1 + \kappa_2(x)) |\nabla u|^p \geq 0$ . Thus, by applying the

maximum principle, we can say that  $u_t, v_t$  are both positive on  $\epsilon < |x| < 1$ . It verifies that we have nonzero positive solutions.

□

# 6

## Conclusion

In this thesis, we studied the existence of our PDE problem

$$\begin{cases} -\Delta u = (1 + \kappa_1(x))|\nabla v|^p, & \text{in } B_1 \setminus \{0\}, \\ -\Delta v = (1 + \kappa_2(x))|\nabla u|^p, & \text{in } B_1 \setminus \{0\}, \\ u = v = 0, & \text{on } \partial B_1, \end{cases}$$

on  $B_1$  which is a unit ball centered at the origin in  $\mathbb{R}^N$ . We assumed  $N \geq 3$  and  $\frac{N}{N-1} < p < 2$ . Also,  $\kappa_1, \kappa_2 \geq 0$  are continuous functions such that  $\kappa_i(0) = 0$ . Our goal was to show that there exist an infinite number of solution  $u_t, v_t$  on the domain. With the approach we took in this thesis, we simplified our problem to two scalar problems and obtained two linear operators  $L_t^\pm$  where we could analyse on some function spaces  $X$ , and  $Y$ . We obtained an estimate which later we used to perform a fixed point argument in chapter 5 on the nonlinear operator  $T_t(\phi, \psi) = (\hat{\phi}, \hat{\psi})$  on a complete metric space to show the existence of positive singular solutions of this problem.

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