A COMPARATIVE STUDY OF ANALYSES OF NONLINEAR SYSTEMS WITH RANDOM EXCITATIONS

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ABSTRACT

A new technique, developed by Schultheiss, employing the use of a sampled data model and the theory of finite Markov Chains for analysing the response of nonlinear, open loop, low-pass, feedback systems, containing one zero-memory element, to wide-band, stationary, Gaussian, random signals, is applied to a particular system. The results from this technique applied to a particular example are then compared, obtaining a fair correlation with those predicted by the well-known quasi-linearization method of Booton applied to the same example.

PREFACE

The object of this thesis is to compare the two methods applicable to the analysis of the response of a certain class of nonlinear feedback systems to wide-band, stationary, Gaussian, random processes, and to determine which method if either, yields the correct solution. A method developed by Schultheiss, employing a sampled data model of the feedback system and the theory of finite Markov Chains, to predict the output probability density function of the system is applied to a low pass feedback system containing one zero-memory element in the forward path. The well-known quasi-linearization technique of Booton is then applied to the same system, and the results compared with those predicted by the sampled data model.

In Chapter 1, following a short introduction to the thesis subject, the theory of stationary random processes is outlined in brief, along with the theory of the quasi-linearization method of Booton. Chapter 2 gives a comprehensive review of the theory of finite Markov Chains as applied to the sampled data model of the feedback system. A linear example delimiting the scope of Schultheiss's technique is given. In Chapter 3 the Schultheiss method is applied to a nonlinear example. Chapter 4 includes Booton's technique applied to the same system, as well as an analog computer study of the system, together with a presentation and discussion of the results from the three techniques.

Chapter 5 further discusses the results, and suggestions for further research are cited.

The author is deeply grateful to Professor R. A. Johnson and Mr. W. H. Lehn for the many helpful suggestions that untangled the numerous problems that arose during the initial stages and throughout this work.

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CHAPTER 1

INTRODUCTION

The importance of random signal analysis is reviewed briefly. Mention is made of important contributions in the linear field; emphasis, however, is placed on developments for nonlinear systems. A very brief outline of the classification of random processes, the basic parameters used, and the concepts of generalized harmonic analysis is given. The quasi-linearization method of Booton for the analysis of Gaussian random processes in nonlinear feedback systems is outlined in detail.

1.1 THE PROBLEM

There are several sophisticated tools available for calculating the response of a nonlinear system to a prescribed random input, but due to the cumbersome mathematical techniques of these methods as applied to a particular example, new and more valuable methods are constantly being sought.

Schultheiss (1)[1], has recently developed a method that allows determination of the output probability density function of a nonlinear feedback system containing in the forward loop, one zero-memory nonlinear element and a low pass linear portion; the random input is restricted to that of a wide

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[1] The bracketed number refers to references in the bibliography.

band, stationary, Gaussian process (white noise).

It is the purpose of this thesis to apply this technique to a specific nonlinear system, using a sampled data feedback model as an approximation to the exact system, which in turn allows the use of Finite Markov Chain theory to calculate, by means of a digital computer, the output probability density function. The density functions so obtained are then compared with those predicted by the well known method of Booton (2).

1.2 REVIEW OF RANDOM SIGNAL ANALYSIS

Any electrical signal that can be determined only statistically is termed a random signal. Such signals often appear as spurious disturbances, and are then classified as random noise.

Random noise can be generated in numerous ways, the best known being the voltage developed across the terminals of a resistive sample due to the random motion of electrons within the material; shot noise in electron tubes provides another example of a random signal.

Because of the presence of these phenomena in electronic devices, Nyquist (3), in 1931, developed an equation giving the relationship between the noise voltage generated across a resistor and its temperature. This provided a technique for calculating the magnitude of the random noise present in electronic devices such as vacuum tubes, and their associated linear coupling networks. This did not yield the system response to the noise, but only the magnitudes of the disturbance present.

Due to the advent of automatic feedback systems, it became increasingly important to be able to calculate the actual response of a system to a random process. Since Nyquist's approach did not accomplish this, a new and more sophisticated technique was necessary.

Wiener (4), utilized the properties of stationary random processes, and the calculus of variations to develop an elegant linear filter theory for detection and prediction of signals in noise. Since Wiener's work, a large number of techniques have been presented in both the linear, and nonlinear fields.

Methods have been extended (5,9) to open loop systems containing a single zero-memory nonlinear element, mathematically separable from the linear portion of the open loop transfer function.

Chuang (6), has developed a workable technique for analysing the error response of a feedback system containing frequency independent nonlinear elements with nonstationary[2], white noise (wide band) inputs. This is accomplished by the

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[2] For stationary inputs it is a matter of reducing the more general nonstationary equations.

use of Markov process theory and the Fokker-Planck differential equation, which in special cases can be solved manually, but usually requires a digital computer. In general, the expediency of this method is restricted by the limited knowledge of the engineering user.

To date, the most useful tools developed dealing with the response of systems containing random signals may be classified as follows:

(i) Linear System Technique

(ii) Transform Methods

(iii) The Chuang Method, and

(iv) Error Criteria Methods.

The first method is limited strictly to linear systems; the main equations used are those given in the Wiener-Khinchin Theorem which employs correlation and generalized harmonic analyses. Satisfactory results are obtainable for all linear systems excited by stationary random processes.

The transform methods are due to the efforts of many[3], but because of the resulting complexity of the integral equations obtained, they are restricted entirely to open loop systems containing the simplest zero-memory nonlinearities, for instance: the ν^{th} Law Detector (10). The Gaussian process in all cases is the easiest to analyse, results being

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[3] See the bibliography of reference 5.

calculable for a wider variety of nonlinear functions.

Chuang's Method, as mentioned above, is useful for calculation of the error response of closed loop systems excited by either stationary or non-stationary wide-band[4] processes, but analytic solutions to the Fokker-Planck equation are not obtainable when the nonlinearity is not the zero-memory type.

Error Criteria Methods usually employ a particular error criterion to yield a quasi-linearized model of the zero-memory element present, Booton's technique being a prime example of this. The random inputs to these models are always restricted to the stationary Gaussian process. Results of these analyses may or may not be open to question, depending on the 'goodness' of the error criterion used.

The overall usefulness of the above techniques is severely restricted by the type of random process being dealt with; however, the Gaussian process is analysable in most cases. This covers a much broader range of interest than realized, since all random processes approach the Gaussian distribution if the sample space is large enough.[5]

In general we may conclude by saying that no single method of analysis has yet been developed which deals

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[4] Wide band here signifies "wide compared to the system bandwidth".

[5] This is justified by the Central Limit Theorem (11).

effectively, and accurately, with all types of random processes in linear and nonlinear systems.

1.3 CLASSIFICATION OF RANDOM PROCESSES

A random process is the set (ensemble) of time function records obtained from a large number of identical generators (the word generator denotes any mechanism that is able to produce an output that belongs to the class of functions being considered). The set, $\{x(t)\}$ composed of member functions $x_1(t)$, $x_2(t)$, ..., $x_n(t)$ is said to be stationary if the statistical properties are independent of the time origin of the process; further it is said to be ergodic if the ensemble average, $\{\overline{x(t)}\}$ is equal to the time average $\overline{x(t)}$, where the bar denotes an average in the statistical sense.

A Markov Process, upon which the calculations of this thesis are based, is defined as a random process in which the conditional probability that y lies in an interval y_n, y_n+dy_n at time t_n, given that y is equal to y_1, ..., y_{n-1} at times $t_1, \ldots t_{n-1}$, where $t_n > t_{n-1} > \ldots > t_1$, depends only (besides on y_n and t_n) on the value at time t_{n-1}. That is, $P_n(y_n t_n; y_{n-1}, t_{n-1}; \ldots; y_1, t_1)$ may be written as $P_n(y_n, t_n \mid y_{n-1}, t_{n-1})$, where P_n is the n - dimensional conditional probability density function.

1.4 PROBABILITY DISTRIBUTIONS AND BASIC STATISTICAL PARAMETERS.

For a random process consisting of functions, $\{x(t)\}$, the first probability distribution is defined as

$$f_1(X_1,t) = P\left[x(t) \leq X_1\right] \qquad 1.1$$

where $f_1(X_1 t)$ is the probability of finding x at time t less than some x equal to X_1. The first probability density function exists, and is equal to[6]

$$P_1(X_1,t) = \frac{\partial F_1}{\partial X_1} \qquad 1.2$$

If g(x) is a real variable, then the expectation of g(x) is defined as

$$E\left[g(x)\right] = \int_{-\infty}^{\infty} g(x) P_1(x,t)dx \qquad 1.3$$

If g(x) becomes identical to x then we have the ensemble average $\{\overline{x(t)}\}$ given by

$$\{\overline{x(t)}\} = \int_{-\infty}^{\infty} x P_1(x,t)dx \qquad 1.4$$

Further, if the process is ergodic, then the ensemble average becomes equal to the time average and gives

$$\overline{x(t)} = \{\overline{x(t)}\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt \qquad 1.5$$

In general, the n^{th} moment of a random process may be defined as

$$\{\overline{x^n(t)}\} = \int_{-\infty}^{\infty} x^n P_1 \,(x,t)dx \qquad 1.6$$

6 In the generalized case $\frac{\partial F_1}{\partial X_1}$ may contain a set of ~~measure zero.~~ impulses.

The most commonly used moments are the first and second, since for a Gaussian distribution these completely specify the process.

Given the second (bivariate) probability density function, $P_2(x,t_1;y,t_2)$ which specifies the probability of finding x in the range x, x+dx at time t_1, and finding y in the range y, y+dy at time t_2, a cross correlation between the variables x and y may be defined as

$$\phi_{xy}(t_1,t_2)=E\left[x(t_1)\cdot y(t_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1)y(t_2)P_2(x,t_1;y,t_2)\,dxdy \qquad 1.7$$

If the processes are stationary, then ϕ_{xy} becomes independent of the observation time, and the correlation function $\phi_{xy}(t_1,t_2)$ becomes a function of the difference of the two times of observation, (t_2-t_1) and may be written

$$\phi_{xy}(t_1 t_2) = \phi_{xy}(t_2-t_1) \qquad 1.8$$

Defining t_2-t_1 equal to τ, allows one to write equation 1.8 as

$$\phi_{xy}(t_2-t_1)=\phi_{xy}(\tau) \qquad 1.8a$$

Furthermore, if the process is ergodic then the crosscorrelation function may be written as

$$\phi_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)y(t+\tau)\,dt \qquad 1.9$$

If x(t) and y(t) are identical, then the so-called auto-correlation of the process is obtained

$$\phi_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt, \qquad 1.10$$

When τ is set to zero in equation 1.10, the mean square value, $\overline{x^2(t)}$ of the process is obtained.

1.5 POWER SPECTRAL DENSITIES AND THE WIENER-KHINCHIN THEOREM

The instantaneous power associated with a random current wave, x(t) passing through a one ohm resistor is $x^2(t)$ watts. The total energy over all time given by

$$\int_{-\infty}^{\infty} x^2(t)dt \qquad 1.11$$

In general the above integral does not exist, and the use of it is avoided by defining the average power of the random wave as

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^2(t)dt \qquad 1.12$$

When the limit in equation 1.12 exists, the power spectral density of x(t) may be defined as the limit over an infinite range of the magnitude squared of the Fourier transform of x(t), that is, as

$$\Phi(\omega) = \lim_{T\to\infty} \frac{1}{2T}\left|\int_{-T}^{T} x(t)e^{-j\omega t}\,dt\right|^2 \text{ watts/cps} \qquad 1.14$$

and is a real-valued, non-negative function of the radian frequency ω. If equation 1.14 is integrated between two values of ω the power in the band of frequencies so defined is obtained.

Using equation 1.14 as a starting point, it may be shown, (12, Chapter 1) that the following Fourier transform pair (the Wiener-Khinchin Theorem) exists between the spectral density and the autocorrelation of a random wave:

$$\phi_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_{xx}(\omega)\, e^{j\omega\tau}\, d\omega \qquad 1.15$$

$$\Phi_{xx}(\omega) = \int_{-\infty}^{\infty}\phi(\tau)\, e^{-j\omega\tau}\, d\tau \qquad 1.16$$

When dealing with the transformation of random processes through linear time-invariant systems it is frequently necessary to calculate the output spectral density function. Through the use of equation 1.15 and 1.16, it can be shown (13) that the output power spectrum for a linear time-invariant system having a transfer function $H(j\omega)$ is the modification of the input spectrum $\Phi_i(\omega)$:

$$\Phi_o(\omega) = \left| H(j\omega) \right|^2 \Phi_i(\omega) \qquad 1.17$$

in which $\Phi_o(\omega)$ is the output spectral density and $\Phi_i(\omega)$ the input spectral density, as shown in Figure 1.1

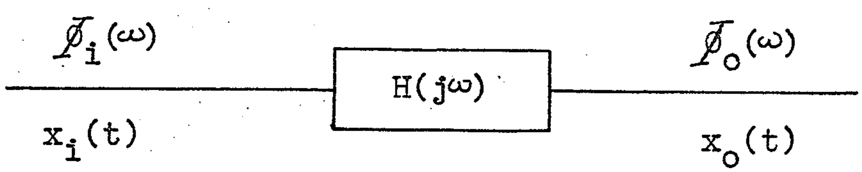

Figure 1.1

Linear System With a Random Input $x_i(t)$.

The output power is then easily obtained from equation 1.15 by setting τ to zero.

The above method is usually termed "Harmonic Analysis", and has become the most powerful tool for analyzing linear, time-invariant systems excited by stochastic signals.

1.6 BOOTON'S TECHNIQUE

Consider the block diagram of a nonlinear feedback system shown in Figure 1.2, in which N_1 represents the zero memory nonlinear element and $H(j\omega)$ the linear portion of the forward path.[7]

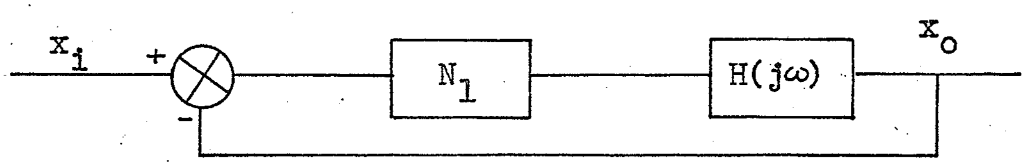

Figure 1.2
General Block Diagram of a Nonlinear Feedback System

When dealing with systems of the above type, the conventional sinusoidal describing function predicts fairly well the behaviour of the system with a deterministic input, the phase plane being used primarily for the regulatory case in low order systems. If there is a stochastic signal present at the input, neither the phase plane method nor the conventional sinusoidal describing function tells us anything of

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7 The above block diagram represents a class of nonlinear systems in which N_1 can be isolated from the linear portion of the transfer function.

the system's behaviour.

Booton, however, developed a criterion involving the properties of the stochastic signal present and those of the nonlinear element, to yield an equivalent gain corresponding to that nonlinearity.

Since this thesis compares an investigation of the Schultheiss technique with Booton's technique an outline of Booton's original work seems in order.

(i) Quasi-linearization of the nonlinear element

The nonlinear element is replaced by a linear gain K_{eq}, which is derived using a mean-square error criterion in the statistical sense. If f(x) is the characteristic of the nonlinear element, where x is the input, then the squared error for a particular x is

$$S.E. = \left[f(x)-K_{eq}x\right]^2 \qquad 1.18$$

If the input to the nonlinearity is random then the 'best' K_{eq} would be obtained by averaging the S.E. weighted by the first probability density function of x, and then minimizing with respect to K_{eq}.

This produces

$$0 = \frac{\partial M.S.E.}{\partial K_{eq}} = \frac{\partial}{\partial K_{eq}}\left[\int_{-\infty}^{\infty}\left[f(x)-K_{eq}x\right]^2 P(x)dx\right] \qquad 1.19$$

or

$$K_{eq} = \frac{\int_{-\infty}^{\infty} xf(x)P(x)dx}{\int_{-\infty}^{\infty} x^2P(x)dx} \qquad 1.20$$

If P(x) is Gaussian, K_{eq} is easily found as a function of the power at the input to the nonlinear element.

(ii) The use of K_{eq} in a particular example.

It now remains to be shown how the above criterion may be applied to a feedback system.

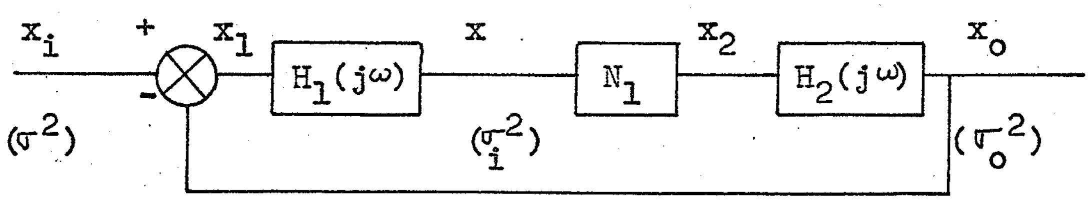

Figure 1.3

Nonlinear System Containing One Zero-Memory Element in the Forward Path.

Consider $x_i(t)$ to be a stationary Gaussian signal. The output of $H_1(j\omega)$ is Gaussian for a Gaussian input, (6, Chapter 1) but the output of N_1 is no longer such for a Gaussian input. However, if $H_2(j\omega)$ is 'sufficiently' low pass then x_o becomes approximately Gaussian (16,p.180); hence x_1, the difference between a Gaussian and a nearly Gaussian signal, is still nearly Gaussian. Due to this we may then introduce K_{eq} for N_1 without causing a significant change in the random processes present in the system. It is on this basis that the following analysis may be performed.

Consider x_i to have zero mean value, a standard deviation σ volts r.m.s., and a power spectrum $\Phi_i(\omega)$. It is desired to calculate the signal power, σ_i^2 of x at the input to N_1. From the block diagram, with N_1 replaced by K_{eq} we obtain

$$\bar{\varnothing}(\omega) = \frac{H_1(j\omega)\ \bar{\varnothing}_i(\omega)}{1+K_{eq}H_1(j\omega)H_2(j\omega)} \qquad 1.21$$

By equation 1.15 the power (mean square value of the process) σ_i^2 is given by

$$\sigma_i^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{H_1(j\omega)}{1+K_{eq}H_1(j\omega)H_2(j\omega)}\right|^2 \bar{\varnothing}_i(\omega)\ d\omega, \qquad 1.22$$

which may be expressed as a function of K_{eq}, and σ^2 the power contained in x_i, since

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{\varnothing}_i\ (\omega)\ d\omega \qquad 1.23$$

represents the input power of the random signal. From equations 1.20 and 1.22, K_{eq} may be written as

$$K_{eq} = F_1(\sigma_i^2, \sigma^2) \qquad 1.24$$

and

$$K_{eq} = F_2\ (\sigma_i^2) \qquad 1.25$$

In most cases the complexity of the functions F_1 and F_2 does not allow an analytical solution for K_{eq}, in which case F_1 and F_2 are plotted against σ_i for various values of σ^2. The intersection of the two curves (as shown in Figure 1.4) gives K_{eq} corresponding to a given signal power σ^2.

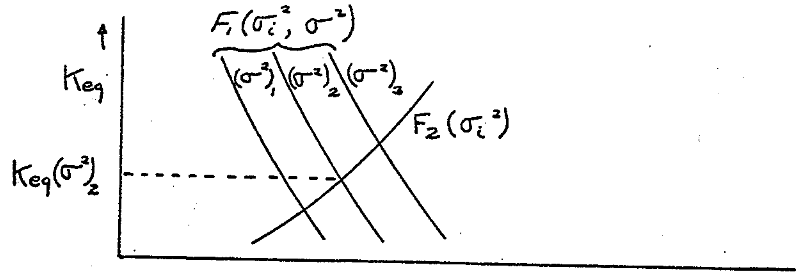

Figure 1.4

Determination of K_{eq} From the Functions F_1 and F_2

With K_{eq} now determined, the power, σ_o^2, which completely characterizes the output Gaussian process, may be easily calculated using the harmonic analysis method applied to the equation

$$\Phi_o(\omega) = \left| \frac{H_1(j\omega)\ H_2(j\omega)\ K_{eq}}{1+K_{eq}H_1(j\omega)H_2(j\omega)} \right|^2 \Phi_i(\omega) \qquad 1.26$$

The application of the method is straight forward enough once the camel of equivalent gain is swallowed.

CHAPTER 2

THE METHOD OF SCHULTHEISS

Since so few successful attempts have been presented which analyse random signal response in nonlinear feedback systems, it is thought that the method of Schultheiss, which accomplishes exactly this for a certain class of random processes, should be presented in detail with its background theory, and then an example delimiting the applicability of the technique given. The Finite Markov Chain model of a feedback system and its associated details will be referred to as the 'Schultheiss technique'.

In this chapter the theory of Finite State Markov Chains, that of the transition probability matrix, and the solution yielding the stationary absolute probabilities is given. The theory of Markov Chains is then applied to a linear, low pass, feedback system, enabling the calculation of the output probability density function when the system is excited by a wide-band, ergodic, Gaussian process.

2.1 MARKOV THEORY

As cited previously a Markov process obeys the following equation:

$$P_c\left[x(t_n)\middle|x(t_{n-1})\right] = P_c\left[x(t_n)\middle|x(t_{n-1}),x(t_{n-2}),\ldots,x(t_1)\right] \quad 2.1$$

where $P_c(A|B)$ denotes a conditional probability density function, which is defined as the probability of finding A in an interval A, A+ΔA subject to the hypothesis of B. The relation that defines P(A) in terms of the conditional function P(A|B) is written as follows:

$$P(A)=\int_{-\infty}^{\infty}P(B)P(A|B)dB \qquad 2.2$$

When equation 2.1 is substituted into equation 2.2 the result becomes

$$P\left[x(t_n)\right]=\int_{-\infty}^{\infty}P\left[x(t_{n-1})\right]P_c\left[x(t_n)|\,x(t_{n-1})\right]\,dx(t_{n-1}) \qquad 2.3$$

It is now apparent that if the conditional density function, $P_c\left[\ \right]$ is calculated, use of equation 2.3 yields the stationary output density function $P\left[x(t)\right]$. Fortunately in the Markov case, when applying the method to a feedback system the function $P_c\left[\ \right]$ is readily calculable from open-loop considerations. However, the complexity of the resulting function $P_c\left[x(t_n)|\,x(t_{n-1})\right]$, in all but the most trivial cases, presents a problem of integration, when substituted into equation 2.3. This difficulty may be avoided by restricting the output of the system to N quantized levels, accomplished by inserting, as shown in Figure 2.1, a sample and clamp device into the forward loop.

The sample and clamp device samples the output x_3 every T seconds, and holds x fixed at that value until it samples again, T seconds later. More concisely, it may be referred to as time sampling, holding, and amplitude quantizing.

The N quantized levels are now representable as a finite state Markov Chain of 'size' N[1], which allows the use of a transition matrix to calculate the absolute probabilities. In a particular case a judicious choice of the sampling interval, T, and the number of states N should yield satisfactory results.

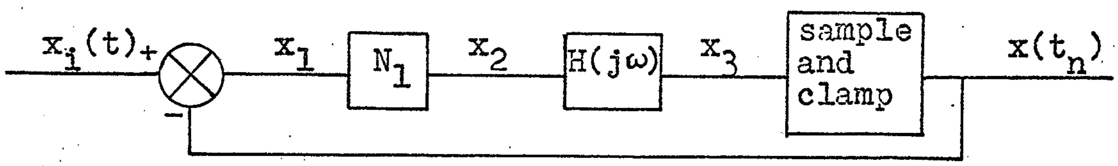

Figure 2.1

A Nonlinear System Showing the Insertion of a Sample and Clamp Circuit

2.2 THE TRANSITION PROBABILITY MATRIX

It is known that for a Markov process a particular event depends only on the preceding event. If the process is restricted to N states x_i, i=1,2, ...N, then there exists a fixed number $P_{ij} \geqslant 0$, which denotes the probability of state x_j appearing in next trial, (in the next sampling interval for the sampled data model) given that it is in state x_i. P_{ij} is defined as the one step transition probability; it is a conditional probability. For a system containing N possible states (levels) there will be an N by N

[1] See the section 'Linear Example' for a short proof of the Markov property of the sampled data model of the system.

matrix denoting the probability of transfering from and to every state in one step. This matrix is defined as a stochastic matrix, or, Transition Probability Matrix.

If one examines the rows of the matrix, one finds that $\sum_{j=1}^{N} P_{ij}$ is identically equal to one.

A little reflection shows that this must be true; that is, it is a certainty that there will be a transition from state x_i to some state x_j. If the stationary absolute probability is denoted for state x_i, by P_i, we have the discrete case for equation 2.3 which becomes

$$\sum_{i=1}^{N} P_i P_{ij} = P_j \qquad j=1,2,..N \qquad 2.4$$

Also, as for any probability distribution, the sum over all possible states is equal to one:

$$\sum_{i=1}^{N} P_i = 1 \qquad 2.5$$

2.3 HIGHER TRANSITION PROBABILITIES

A transition from x_i to x_j in exactly n steps (n sampling periods T) can occur in many different ways, one of which might be: $x_i \rightarrow x_{i_1} \rightarrow x_{i_2} \rightarrow \ldots x_{i_{n-1}} \rightarrow x_j$, where x_{i_k} is the state after k transitions. The conditional probability that the system passes through that particular path given that it was at x_i, is given by[2]

$$P_{i,i_1} \cdot P_{i_1,i_2} \cdot P_{i_2,i_3} \cdot \ldots \cdot P_{i_{n-1},j} \qquad 2.6$$

The sum of the expressions 2.6 over all possible paths then yields the probability of finding the system at time (r+n) in state x_j, given that at time r it was in state x_i.[3] Let us denote this by $P_{ij}^{(n)}$. It is easily seen that $P_{ij}^{(1)}$ is the one step transition probability P_{ij}. For $P_{ij}^{(2)}$ we sum over all possible paths including a transition in two steps from x_i to x_j. This yields for an N state process

$$P_{ij}^{(2)} = \sum_{k=1}^{N} P_{ik}P_{kj} \qquad 2.7$$

Similarly by induction there results $P_{ij}^{(3)} = \sum_{k=1}^{N} P_{ik}^{(2)}P_{kj}$ 2.8

and the recursion formula is easily written as

$$P_{ij}^{(n+1)} = \sum_{k=1}^{N} P_{ik}^{(n)}P_{kj} \qquad 2.9$$

It is also easily seen that for an N state process if all $P_{ij}^{(n)}$ are considered we are able to define an N by N matrix, P^n, corresponding to the totality of all the n step transition probabilities in terms of the recurrence relation defined above:

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[2] If we have a sample space of N independent events A_i i=1,2...N, then the probability of them all occurring simultaneously is the product of the probabilities of all events A_i.

[3] The n signifies a lapse of n sampling periods T, in the sampled-data model.

$$P^{n+1} = P^n P \qquad 2.10$$

2.4 STATIONARY ABSOLUTE PROBABILITIES

The equation 2.10 allows the calculation of the transition matrix P^n for n as large as desired. Since the ultimate aim is to calculate the stationary absolute probabilities of the nonlinear system output, a method of obtaining these is necessary.

Supposing that the matrix P has been previously calculated, let us define a row vector $\left[x_k^{(o)}\right]$ denoting the probability of occurence of states x_k at some point in time, r.[4] Then the (unconditional) probability of finding the system in state x_k at time r+n is, by using equations 2.4 and 2.9

$$x_k^{(n)} = \sum_j x_j P_{jk}^{(n)} \qquad 2.11$$

The vector $\left[x_k^{(n)}\right]$ may be written as $\left[x_k^{(o)}\right]P^n$ 2.12

representing the absolute probabilities of the N states after n transitions.

Instinctively we feel, that the influence of the initial state vector on the distribution 'wears off' as n becomes large; that is, the process, if it is to converge to a stationary distribution, 'forgets' from whence it started.

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[4] This actually represents the probability of finding the process during the interval r in the different intervals 'centered' at that x_k.

When this happens we have the stationary absolute probabilities of the process. For a proof of the above statement the reader is referred to (14), pp.356-7. It is there shown that for a stationary distribution, equation 2.12 converges to a fixed stationary distribution vector regardless of the initial vector $\left[x_k^{(o)}\right]$.

The problem at hand, that of finding the stationary absolute probabilities from the transition matrix, leads exactly to the above technique. To obtain the stationary distribution, it is necessary to perform the operation $\left[x_k^{(o)}\right] P^n$ until $\left[x_k^{(o)}\right] P^n$ becomes equal to $\left[x_k^{(o)}\right] P^{n+1}$.[5]

2.5 THE GAUSSIAN PROCESS

Since this thesis deals exclusively with Gaussian (normally) distributed random signals, a brief definition at this point is necessary.

The Gaussian probability density function is defined as

$$P(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-(x-m)^2 / 2\sigma^2\right] \qquad 2.13$$

with σ representing the standard deviation of the process, and m giving its mean value. The integral $\int_{-\infty}^{X} P(x)dx$ is defined as the normal distribution function, commonly called

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[5] This is akin to the eigenvalue problem concerning the eigenvector corresponding to the largest eigenvalue of a particular matrix (in this case P).

the Error Function, and is written as erf $[X]$ 2.14

2.6 LINEAR EXAMPLE

A linear example cited by Schultheiss (1) is believed to be sufficient to illustrate the use of a Markov Chain for analysing the random signal response of a feedback system.

Consider the simple feedback system shown in Figure 2.2.

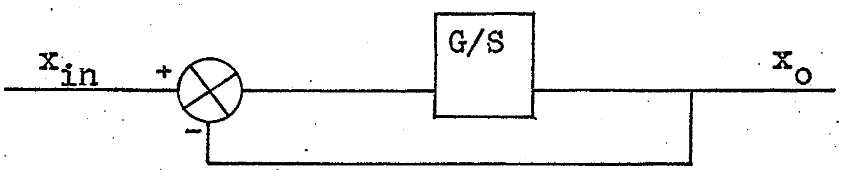

Figure 2.2

A Simple Low Pass Feedback System

The input is a stationary, (ergodic) Gaussian process with zero mean value, and a spectral level of N_o volts/cps.[6]

For comparison with the Markov model the output process is first calculated by the harmonic analysis method. From elementary block diagram manipulations, and use of equation 1.17 we obtain

$$\phi_o(\omega) = \frac{N_o G^2}{G^2 + \omega^2} \qquad 2.15$$

The output process will be Gaussian (6, pp.12-15) and of zero mean value. We know that such a process is completely characterized by its mean m, and mean square value σ_o^2.

[6] The dimensions of all variables defined in the remainder of this thesis will be consistent with those used when the system output has the dimensions of volts.

Therefore, from equation 1.15 we obtain for τ set to zero

$$\phi_o(0) = \sigma_o^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{N_oG^2}{G^2+\omega^2}\,d\omega \qquad 2.16$$

Equation 2.16 may be evaluated to yield $\sigma_o^2=N_oG/2$ 2.16a

$P(x_o)$ then becomes

$$P(x_o) = \left(\frac{N'_o\pi G}{2}\right)^{-\frac{1}{2}}\exp\left[\frac{-2x_o^2}{N'_oG}\right] \qquad 2.17$$

in which No has been replaced by $N'_o/2$.

When the sample and clamp technique is applied to the system in Figure 2.2, it becomes

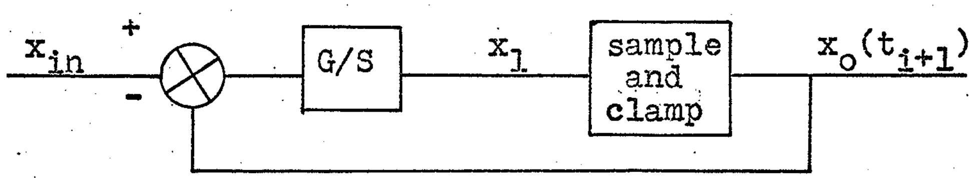

Figure 2.3

System With the Sampler Included

It is now necessary to show that the sampler loop is representable as a Markov Chain. This may be seen by the following argument. We are dealing with white noise at the input; therefore, successive values are uncorrelated. (This is essential for the Markov property to hold). Suppose the sampling interval is chosen to be T. Consider the interval $iT \leq t < (i+1)T$, where iT is defined as t_i. The output in the succeeding interval is given by

$$x_o(t_{i+1})=x_o(t_i)+\int_0^T G(x_{in}-x_o(t_i)\quad)\,dt \qquad 2.18$$

or $$x_o(t_{i+1})=(1-GT)x_o(t_i)+\Delta\left[x_{in}\right] \qquad 2.19$$

The term $-GTx_o(t_i)$ is the change due to the feedback voltage, the term $\Delta[x_{in}]$ is the change in the output due to the input; it is completely independent of $x_o(t_{i+1})$ and therefore a specification of $x_o(t_i)$ permits a calculation of $x_o(t_{i+1})$ through use of the statistics of $\Delta[x_{in}]$. Hence, the Markov property is shown.

It now remains to calculate the conditional probability density function $P_c[x_o(t_{i+1})|x_o(t_i)]$ for use in equation 2.3. To do this consider the term $\Delta[x_{in}]$. Because of the ergodicity of the input process the term $\Delta[x_{in}]$ over many sampling intervals, T may be represented as an ensemble. Therefore, ensemble methods are applicable to the problem. The output has zero mean value and so the calculation of its mean-square value characterizes the function

$P_c[\ ]$. Using equations 1.15 and 1.17 we obtain[7]

$$\sigma_o^2 = TG^2N_o \qquad 2.20$$

The conditional function may be written as

$$P_c\left[x_o(t_{i+1})\,\middle|\,x_o(t_i)\right] = (2\pi N_o G^2 T)^{-\frac{1}{2}} \cdot \exp\left[-\frac{\{x_o(t_{i+1})-(1-GT)x_o(t_i)\}^2}{2N_oG^2T}\right] \qquad 2.20a$$

After much tedious algebraic manipulation equation 2.20 becomes[8]

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7 See appendix A for actual calculation.

8 See appendix B for evaluation of the integral.

$$P(x_o)=\left[\frac{GN'_o}{2-GT}\right]^{-\frac{1}{2}} \cdot \exp\left[-\left(\frac{2-GT}{N'_oG}\right)x_o^{\,2}\right] \qquad 2.21$$

A comparison of equation 2.21 with equation 2.17 indicates that for $GT \ll 2$ the solutions agree. For T small then, perfect agreement should be obtained between the response of the sampled data model and the exact model.

2.7 CONCLUSIONS

The above technique has stated nothing of the linearity or nonlinearity of the system, and therefore, should be applicable to both when $P_{c_o}\left[x_o(t_{i+1})\middle|x_o(t_i)\right]$ is calculable, the only restriction being that the sampling interval be chosen judiciously enough to insure statistical independence between samples in successive sampling intervals.

CHAPTER 3

NONLINEAR EXAMPLE

A simple nonlinear feedback system containing a zero-memory, odd cubic nonlinearity in the forward loop is analysed for its response to a stationary (ergodic) Gaussian input. The calculations are presented in as much detail as believed necessary to show clearly the application of the Schultheiss Method to a nonlinear feedback system.

3.1 CHOICE OF THE NONLINEARITY TO BE USED

Quite frequently in nonlinear analysis (especially in feedback systems) the nonlinear element encountered has an odd characteristic (odd meaning: $f(x)=-f(-x)$). Generally, if the nonlinearity is not too violent, (as is, for example, the sgn (x) function), it may be approximated by a polynominal series. Soudack (15) approximated this type of function by use of a least-squares fit to a cubic polynomial of the form $Ax+Bx^3$. He applied this criterion to several grossly nonlinear, homogeneous, differential equations, (the regulating case in control system analysis) obtaining very good agreement between the exact solution and the polynomial fit solution.

Due to the presence of this type of nonlinearity in many feedback systems, the author feels[1] that Soudack's least-squares fit could be applied to any odd f(x) to yield the cubic equation 3.1. This, then, may be inserted into the system to be analysed as an approximation, and leads to a much more tractable analysis of the system's response to random signals.

3.2 THE NONLINEAR SYSTEM EXAMPLE

The Markov Chain method of analysis will now be applied to the feedback system shown in Figure 3.1.

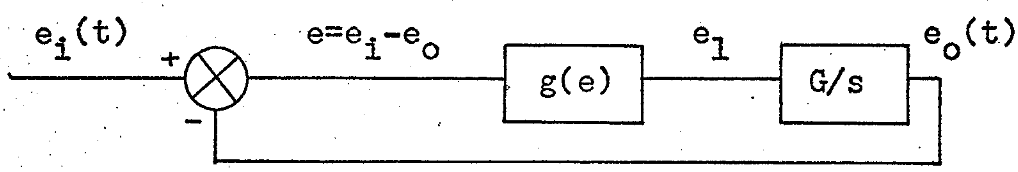

Figure 3.1
The Nonlinear Feedback System

The input, $e_i(t)$ is a stationary, wide band, Gaussian process[2], g(e) is any odd characteristic nonlinearity that is approximated by Soudack's least square technique to yield

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1 This opinion is based on the success of application of the cubic least-squares fit in developing the Elliptic Describing Function as applied to zero-memory odd characteristic nonlinearities. In reference to this see: Jopling,A.D.: A Study of Describing Function Techniques, Electrical Engineering M.Sc. Thesis, University of Manitoba, Feb. 1963.

2 The necessity for a wide band input is to ensure the validity of the Markov Model, that is, that the input crosscorrelation between successive samples be small.

$$g(e) \approx f(e) = Ae+Be^3 \qquad 3.1$$

G/s is the block diagram equivalent for an integrator of gain G; $e_o(t)$ is the output process as a function of time.

The Markov model then becomes as shown in Figure 3.2.

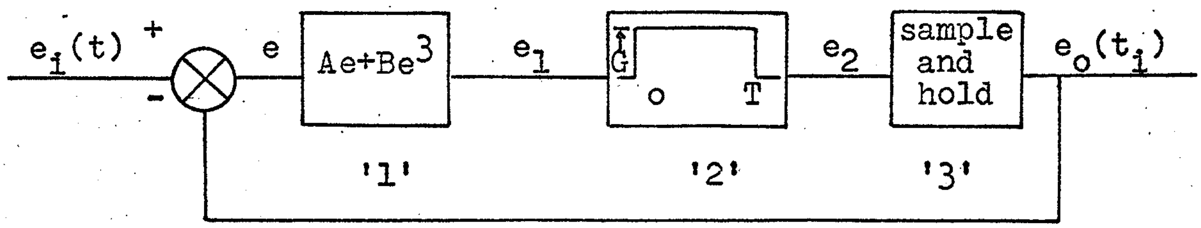

Figure 3.2

Markov Model of a Nonlinear Feedback System Containing a Zero-Memory, Odd Characteristic Nonlinearity

Block 1 represents the nonlinearity; block 2 is indicated by the weighting function equivalent of an integration for T seconds; block 3 represents the sample and hold circuit.

3.2.1 The Markov Property

First it must be shown that the sampled-data model represents a Markov Process. The output, $e_o(t)$ in the t_{i+1} interval may be written

$$e_o(t_{i+1}) = e_o(t_i) + \Delta[e_2] \qquad 3.2$$

It is clear that from a specification of $e_o(t_i)$ the output $e_o(t_{i+1})$ is obtainable if the correlation time of the input process is short compared to the sampling period, T (cf. footnote 2, Chapter 3). Also, the closed loop response must be slow compared to the correlation time of the input. Indeed, if this is not so then a different method of determining the input-output cross-correlation is necessary.

In order that the method be applicable, a stationary, wide-band Gaussian process was selected for the input, $e_i(t)$. The power spectrum was selected, for computational reasons, to be of the following form:

$$\Phi_{ii}(\omega) = \frac{2\beta\sigma^2}{\beta^2+\omega^2} \qquad 3.3$$

From the Wiener-Khinchin relation $\phi_{ii}(\tau)$ becomes

$$\phi_{ii}(\tau) = \sigma^2 e^{-\beta|\tau|} \qquad 3.4$$

3.2.2 Philosophy of Attack

It is now apparent from equation 3.2 that a specification of the statistical properties of $\Delta[e_2]$ would permit a complete analysis. However, due to the presence of the nonlinearity, the output from it is far from Gaussian. Fortunately when the process is passed through the low pass element, the output again becomes very nearly normally distributed.[3]

It is then only necessary to assume a Gaussian form for the output and proceed. We know that a complete characterization of any normally distributed variable is given by its mean and variance. Therefore a calculation of $\overline{\Delta e_2}$ and

[3] A good experimental proof of this is seen in Newton-Gould and Kaiser; Analytical Design of Linear Feedback Controls, p.194, John Wiley and Sons Inc., New York, 1957.

$\overline{\Delta e_2^2}$ will yield the proper conditional probability density function, $P_c \left[e_o(t_{i+1}) | e_o(t_i)\right]$. If the complexity of the above equation upon substitution into equation 2.3 does not allow an analytical integration, then the method outlined in sections 2.3, 2.4, and 2.5 must be used.

3.2.3 The Computations

The computation will proceed by calculating $\overline{\Delta e_2}$ and $\overline{\Delta e_2^2}$. Since we are dealing with an ergodic random process at the input, the average value of the integrator output may be written as

$$\overline{\Delta e_2} = \int_0^T G_{\overline{\Delta e_1}} \, dt \tag{3.5}$$

The variance of the input process is σ^2, as seen from equation 3.4. We may then write $P(e_i)$ as

$$P(e_i) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-e_i^{\,2}/2\sigma^2\right] \tag{3.6}$$

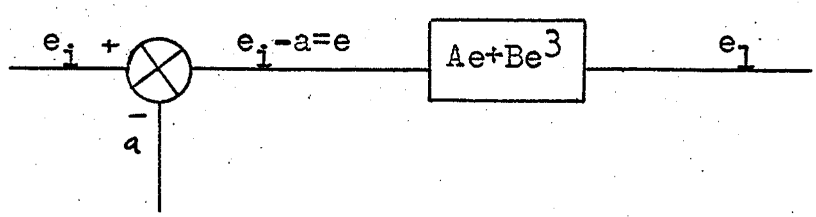

Figure 3.3
Block Diagram for Calculating $\overline{\Delta e_1}$

The output $e_o(t_i)$ is denoted by 'a' for ease in writing the mathematics.

From equation 1.3, $E\left[\Delta e_1\right]$ may be written as

$$E[\Delta e_1] = \int_{-\infty}^{\infty} e_1 P(e_1) de_1 \qquad 3.7$$

But e_1 may be written $e_1 = A(e_i - a) + B(e_i - a)^3$ 3.8

$P(e_1)$ may be defined in terms of the characteristic as

$$P(e_1) = P(e) \left| \frac{de}{de_1} \right| \qquad 3.9$$

When equation 3.9 is substituted into equation 3.7 the result is

$$E[\Delta e_1] = \int_{-\infty}^{\infty} (Ae + Be^3) P(e) de \qquad 3.10$$

since de and de_1 are always of the same sign.

By equation 3.6, P(e) becomes

$$P(e) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-(e+a)^2/2\sigma^2\right] \qquad 3.11$$

and $E[\Delta e_1]$ may be written as

$$E[e_1] = \overline{\Delta e_1} = \int_{-\infty}^{\infty} (Ae + Be^3) \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-(e+a)^2/2\sigma^2\right] de \qquad 3.12$$

Integration of equation 3.12 yields

$$\overline{\Delta e_1} = -\left[Ba^3 + (3B\sigma^2 + A)a\right] \qquad 3.13$$

Rewriting $e_o(t_i)$ for a, we obtain

$$\overline{\Delta e_1} = -\left[B(e_o(t_i))^3 + (3B\sigma^2 + A)e_o(t_i)\right] \qquad 3.14$$

From equation 3.5 we then calculate $\overline{\Delta e_2}$ to be

$$\overline{\Delta e_2} = -GT\left[B(e_o(t_i))^3 + (3B\sigma^2 + A)e_o(t_i)\right] \qquad 3.15$$

Proceeding now to the evaluation of $\overline{\Delta e_2^2}$, we find it is necessary to calculate the following integral expression:

$$\overline{\Delta e_2^2} = \overline{\left(\int_0^T G e_1 dt \right)^2} \qquad 3.16$$

where the bar over the integral denotes an ensemble average. The above expression may be calculated by evaluating the autocorrelation function $\phi_{22}(\tau)$ for τ set to zero. This in turn may be written

$$\overline{\Delta e_2^2} = G^2 \int_0^T d\sigma \int_0^T \phi_{11}(\sigma - u)\, du \qquad 3.17$$

where $\phi_{11}(\sigma - u)$ is the autocorrelation of the output[4] e_1.

It is now necessary to calculate the autocorrelation function of e_1. From equation 1.7 it is found that, $\phi_{11}(\tau)$ may be written as follows, (since the process is assumed ergodic):

$$\phi_{11}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1) f(x_2) P(x_1, x_2; \tau)\, dx_1 dx_2, \qquad 3.18$$

where f(x) is the characteristic of the nonlinearity, and $P(x_1, x_2; \tau)$ is the two-dimensional, Gaussian density function at the input to f(x).[5]

Due to the complex notation required to denote $\phi_{11}(\tau)$, let figure 3.3 be repeated with a different, and self-

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[4] See appendix C for a derivation of equation 3.17.

[5] See appendix D for the general Gaussian distribution.

explanatory notation.

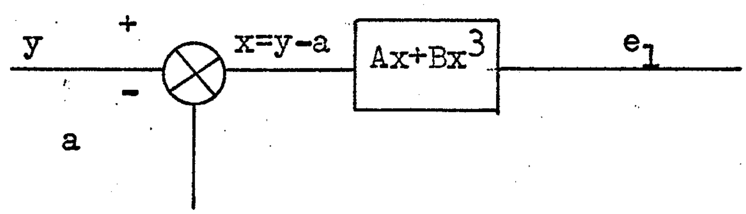


Figure 3.4

Figure 3.3 Repeated

Considering the two dimensional distribution at the input to the system, we have

$$P(y_1,y_2;\tau) = \frac{1}{2\pi(\sigma_1^2\sigma_2^2-\phi_{ii}^2(\tau))^{\frac{1}{2}}} \cdot$$

$$\exp\left\{-\tfrac{1}{2}\left[\frac{1}{\sigma_1^2\cdot\sigma_2^2-\phi_{ii}^2(\tau)}\right](\sigma_1^2y_1^2-2\phi_{ii}(\tau)y_1y_2+\sigma_2^2y_2^2)\right\} \qquad 3.19$$

In this case σ_1, σ_2 and σ are equal, and $\phi_{ii}(\tau)$ is the autocorrelation function of the input, e_i. The following relation of the variables is useful,

$$y=x+a \qquad 3.20$$

This transformation, along with the form of the nonlinear function f(x), may be substituted into equation 3.19 and subsequently into equation 3.18 to yield

$$\phi_{11}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(Ax_1+Bx_1^3)(Ax_2+Bx_2^3)\cdot\frac{1}{2\pi(\sigma^4-\phi_{ii}^2)^{\frac{1}{2}}}$$

$$\exp\left[-\tfrac{1}{2}\frac{1}{(\sigma^4-\phi_{ii}^2)}(\sigma^2(x_1+a)^2-2\phi_{ii}(x_1+a)(x_2+a)+\sigma^2(x_2+a)^2)\right]dx_1dx_2$$

3.21

Upon integration of equation 3.21 $\phi_{11}(\tau)$ becomes[6]

$$\phi_{11}(\tau) = (6B^2\sigma^6)\rho^3(\tau) + (18B^2\sigma^4 e_o^2(t_i))\rho^2(\tau) +$$
$$\left[(9B^2\sigma^6 + 4AB\sigma^4 + A^2\sigma^2) + (12B^2\sigma^4 + 6AB\sigma^2)e_o^2(t_i) + 9B^2\sigma^4 e_o^4(t_i)\right]\rho(\tau)$$
$$+ (9B^2\sigma^4 + 6AB\sigma^2 + A^2)e_o^2(t_i) + (6B^2\sigma^2 + 2AB)e_o^4(t_i) + B^2 e_o^6(t_i), \quad 3.22$$

in which $\rho(\tau)$ is the normalized autocorrelation of the input e_i.

More concisely this is expressed as $\phi_{11}(\tau) = b_3\rho^3(\tau) + b_2\rho^2(\tau) + b_1\rho(\tau) + b_o$

3.23

With $\phi_{ii}(\tau)$ as in equation 3.4, the substitution of equation 3.23 into equation 3.17 yields after integration[7]

$$\overline{\Delta^2 e_2} = \frac{2G^2}{\beta^2}\left\{\frac{b_3}{9}(\exp(-3\beta T) + 3\beta T - 1) + \frac{b_2}{4}(\exp(-2\beta T) + 2\beta T - 1) + b_1(\exp(-\beta T) + \beta T - 1) + \frac{\beta^2 b_o T^2}{2}\right\} \quad 3.24$$

It is now possible to write the conditional probability density function $P_c\left[e_o(t_{i+1})|e_o(t_i)\right]$. For ease of writing, $e_o(t_i)$ may be defined as 'a_i' from which $P_c[\ ]$ becomes

$$P_c(a_{i+1}|a_i) = \frac{1}{\sqrt{\pi\Delta e_2^2}} \exp\left[-\frac{\left(a_{i+1} - a_i - \overline{\Delta e_2}\right)^2}{\Delta e_2^2}\right] \quad 3.25$$

Upon substituting equation 3.25 into equation 2.3, it is

[6] See appendix E for details.

[7] See appendix F for details of integration.

found that due to the complexity of the functions $\overline{\Delta e_2}$ and $\overline{\Delta e_2^2}$ it is impossible to integrate the equation to obtain the stationary solution.

To avoid this difficulty the transition probability matrix method is applied in which the output e_o is quantized into N levels, which may be associated with the states of a finite Markov Chain.

The range of interest will be taken as $-E \leq e_o \leq E$ and an odd number of states will be used, each corresponding to an output interval of 2Δ. The first state will be given the range, $E-\Delta \leq e_o < \infty$, the second, $E-3\Delta \leq e_o < E-\Delta$ through to the Nth state, $-\infty < e_o < -E+\Delta$. The calculation of the one step transition probabilities for the stochastic matrix can be obtained from the conditional probability density function in the following manner: If N is sufficiently large, then the conditional function may be considered essentially constant over the range 2Δ and its value may be taken as that at the center of each initial state. One may write with little error then, taking as an example P_{32} (see Figure 3.5)

$$P_{32} = \int_{E-3\Delta}^{E-\Delta} P_c \left[e_o(t_{i+1}) \middle| e_o(t_i) = E-4\Delta \right] de_o(t_{i+1}) \qquad 3.26$$

P_{32} is represented by the cross-hatched area. For each hypothesized state we may write

$$e_o(t_i) = E-2k\Delta;\ k=0,1,2,\ \ldots,\ N-1$$

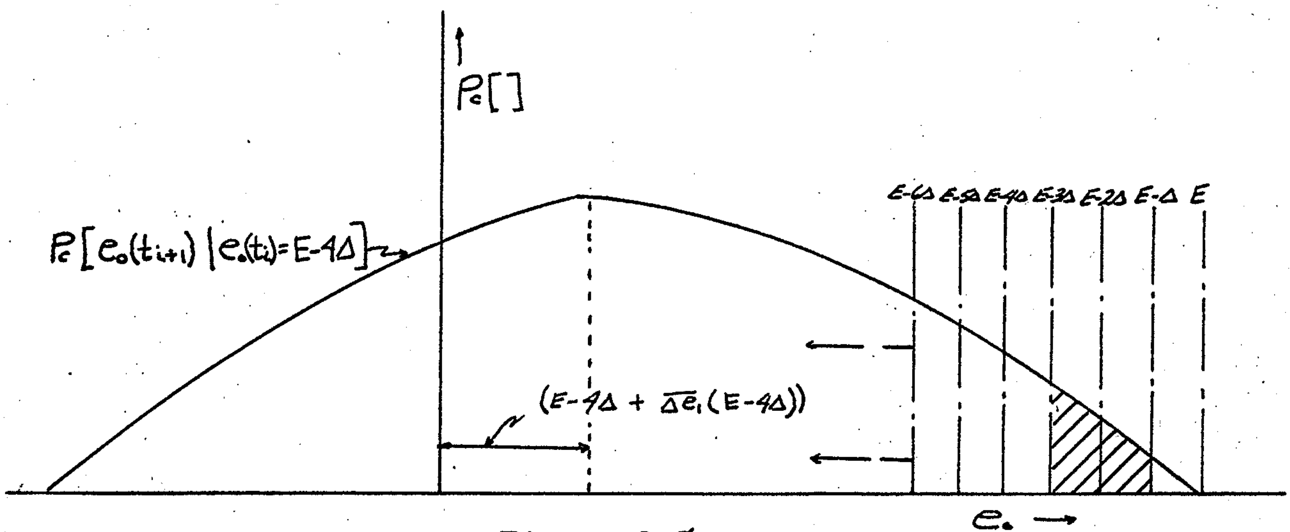


Figure 3.5

Showing Calculation of the One Step Transition Probabilities

A different conditional curve may be calculated for each k, the integrated quantized areas representing the transition probabilities corresponding to that k. When the above procedure is completed for all k, the total stochastic matrix is obtained.

Since the above matrix was derived from a stationary ergodic process, a solution for the stationary, absolute probabilities is possible by applying the method outlined in articles 2.3 and 2.4. Doing this, the stationary absolute probabilities are easily obtained by use of a digital computer. The above answer, however, represents the quantized output distribution P_j (the integrated output density function). To obtain the output probability density function, one must assume the probabilities constant over the interval 2Δ, corresponding to P_j, and then divide by 2Δ. If N, Δ, and T have been chosen with care and regard to the limitations of the method, the result should be completely satisfactory.

3.3 IMPLEMENTATION OF THE CALCULATIONS

In order to facilitate calculation of the problem, it was programmed on the I.B.M. 1620 Digital Computer. The program consisted of two phases:

(i) Calculation of the transition probability matrix,

(ii) Convergence of the matrix to the stationary absolute solution

Several cases of the problem were set up on the computer by varying the parameters A, B, σ^2, and N. The results are presented in Chapter 4.

CHAPTER 4

RESULTS AND COMPARISON WITH BOOTON'S TECHNIQUE

An outline of the computer study is given. Booton's method is applied to the cubic nonlinear example at hand. The results from Booton's and Schultheiss's method, along with those obtained from an analog computer study of the system are tabulated for comparison. A discussion of the results obtained from the three methods is given, with possible reasons for any discrepencies present.

4.1 THE COMPUTER STUDY

Since the basis of this thesis was to obtain a comparison of the Schultheiss technique and Booton's method for analysing random signal response in feedback control systems, it was thought that the range of values used in the example should encompass those found in an actual system.

With regard to this criterion, the subsequent parameter values were chosen for calculation of the output probability density function $P(e_o)$. A, the linear parameter in f(e) was chosen to range from 1.0 to 0.1; B the nonlinear gain constant was varied from zero to 5.0, depending on the r.m.s. value σ, of the input signal. In order to insure the validity of the Markov model, β, the effective bandwidth of the input signal, was fixed at 100 radians per second, approximately five times the linear, effective bandwidth, (equivalent to G) of the system, which was chosen to be 20 radians per second. Since the 'goodness' of the Markov model is also dependent upon the sampling interval T, the selection of the 'best' T was possible only after rough calculations were made based on the assumption that the statistics of each sampling interval T be as completely uncorrelated as possible, that is, that switching transients will essentially have died out by the end of each sampling period. This requires $1/\beta < T < 1/G$, the optimum value being 0.04 seconds for this example. The optimum (best) value of T was determined by setting B to zero and correlating the output function $P(e_o)$, predicted by the Markov model to that predicted by the exact, harmonic analysis method.

During these calculations it was noted that $P(e_o)$ was quite sensitive to changes in T, that is a change of 2:1 in T, correspondingly altered $P(e_o)$ from 10 to 15 percent.

The limitation on the number of quantizing intervals, N, was the size of the available storage space in the 20k memory of the IBM. 1620 computer; this fixed N to be 31. It is obvious that the larger N, the more accurate the results will be. To check the sensitivity of $P(e_o)$ on N, one case with N set to 21 was computed, with results shown in graph 4.2. However, it was noted on subsequent tests, that if B became large, $(0.3 \leq B < 0.5)$ there was a discrepancy between results for the two values of N.

The maximum value of the output, E (e_o max) was governed entirely by the r.m.s. value of the input process and the nonlinear gain constant B. Determination of the 'best' E corresponding to a given input was most easily accomplished after some experience was gained by testing a few cases on the computer. With N equal to 31, about forty five minutes were required to complete one example ('example' denoting a complete computation for one set of parameter values) of the two phase program. For N set to 21 computing time was cut down by at least one-half. Hence, results could be obtained fairly rapidly once the computer program had been compiled. However, the validity of the results remain in question due to the sensitivity of the output on the sampling interval, T.

4.2 BOOTON'S TECHNIQUE APPLIED TO THE CUBIC NONLINEAR FEEDBACK SYSTEM

The development presented in this section applies the method of Booton to the cubic nonlinearity feedback system of the previous section. The system is excited by the same input process.

The block diagram of the system is repeated here for convenience.

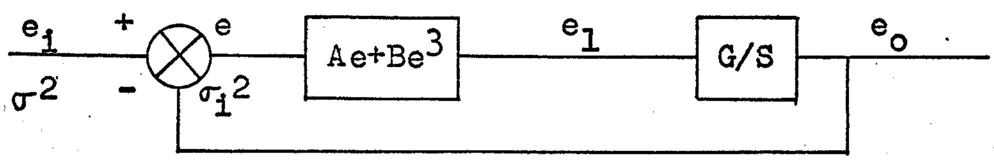


Figure 4.1

Block Diagram of the Original System

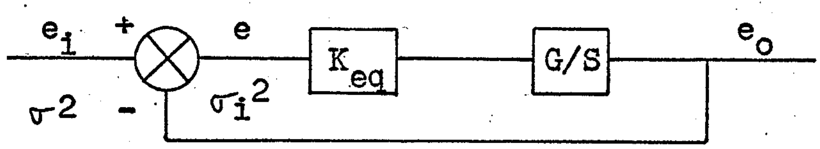


Figure 4.2

Booton's Equivalent System

Referring to Figure 1.3, we see that x_i becomes e_i, x_1 the input to the nonlinearity since $H_1(j\omega)$ is identically one; $H_2(j\omega)$ becomes $G/s \,\big|\, s=j\omega$ 4.1

and N_2 becomes $Ae+Be^3$ 4.2

The object is to calculate an expression for K_{eq}, given the input statistics of e. This may be accomplished by following the method outlined in section 1.5 of this thesis. From equation 1.20, K_{eq} is easily found (16, pp. 174-5) to be:

$$K_{eq}=A+3B\sigma_i^2 \qquad 4.3$$

The input signal power to N_1 may then be determined by use of equations 1.21 and 1.22 to be:

$$\sigma_i^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{1}{1+K_{eq}G/j\omega} \right|^2 \cdot \frac{2\beta\,\sigma^2}{\omega^2+\beta^2}\, d\omega \qquad 4.4$$

where the input spectrum is

$$\frac{2\beta\sigma^2}{\beta^2+\omega^2} \qquad 4.5$$

Equation 4.4 may be evaluated to yield

$$\frac{\beta\sigma^2}{\beta+K_{eq}G} \qquad 4.6$$

The two equations, 4.4 and 4.3 are then used to evaluate K_{eq} for the system in terms of σ^2, A, β, B and G.

$$K_{eq} = -\left\{(\beta-AG) \pm \sqrt{(\beta-AG)^2+4G\beta(A+3B\sigma^2)}\right\} \Big/ 2G \qquad 4.7$$

It is now necessary to determine σ_o^2 (mean-square value) of the output e_o. One notes from figure 4.2 that the system is now linear, and the method of harmonic analysis is applicable. Since the input process has zero mean value, then $\overline{e_o}$ is also zero. Calculating σ_o^2, we repeat equation 1.17 for convenience.

$$\Phi_o(\omega) = \left|H(j\omega)\right|^2 \Phi_i(\omega) \qquad 4.8$$

For the equivalent system the steady state transfer function becomes

$$H(j\omega) = \frac{K_{eq}G}{K_{eq}G+j\omega} \qquad 4.9$$

The use of equation 1.15 with τ set to zero yields the mean square value of the output to be

$$\phi_{oo}(0) = \sigma_o^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{K_{eq}^2 G^2}{K_{eq}^2 G^2 + \omega^2} \cdot \frac{2\beta\sigma^2}{\beta^2+\omega^2}\, d\omega \quad . \qquad 4.10$$

This integrates to $\sigma_o^2 = \dfrac{K_{eq} G \sigma^2}{\beta + K_{eq} G}$ 4.11[1]

Equation 4.11 completely characterizes the equivalent system output process since $\overline{e_o}$ is zero.

It is now apparent that an equivalent output process can be calculated. It is with this equivalent process that the results from Schultheiss's technique are compared for the ranges of A, B, and σ^2.

4.3 PRESENTATION OF TABLES AND GRAPHS

The results from both methods are presented for comparison in the following tables and graphs.

4.4 A DISCUSSION OF THE RESULTS OBTAINED FROM THE BOOTON AND SCHULTHEISS METHODS

A consideration of the resulting curves (Graph 4.1) for σ^2 equal to 0.1 and 0.05, shows that there is virtually no difference, in the linear case, between the Schultheiss method and the exact method. However, in order to obtain the calculated curves it was necessary in equation 3.25 to set B to zero. If there had been any errors in the assumptions made with respect to the nonlinearity, then setting B

[1] For the linear system (B=0) equation 4.11 reduces to $\dfrac{AG\sigma^2}{AG+\beta}$

TABLE 4.1

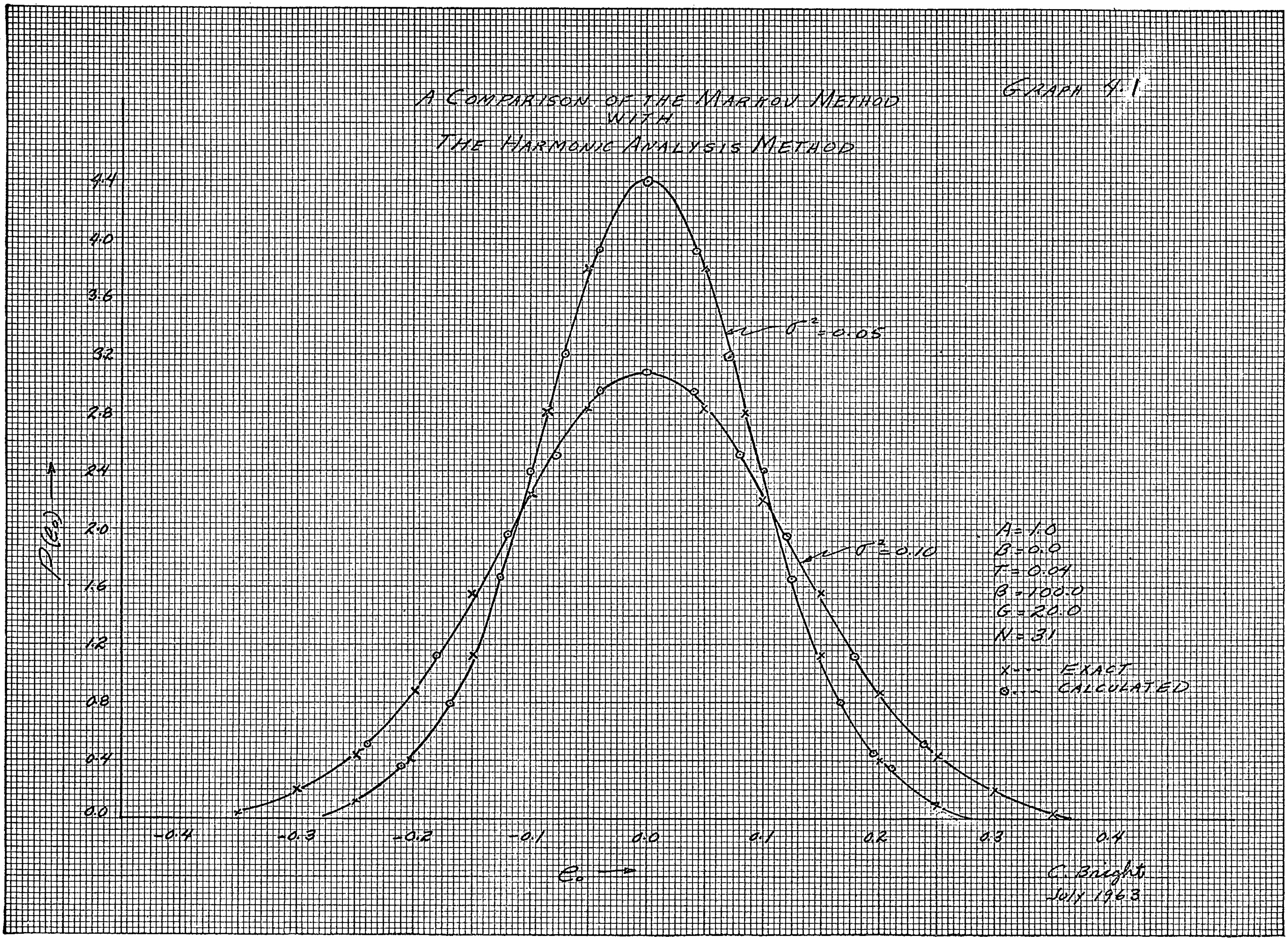
THE RESULTING R.M.S. OUTPUT σ_o FOR THE BOOTON AND SCHULTHEISS METHODS

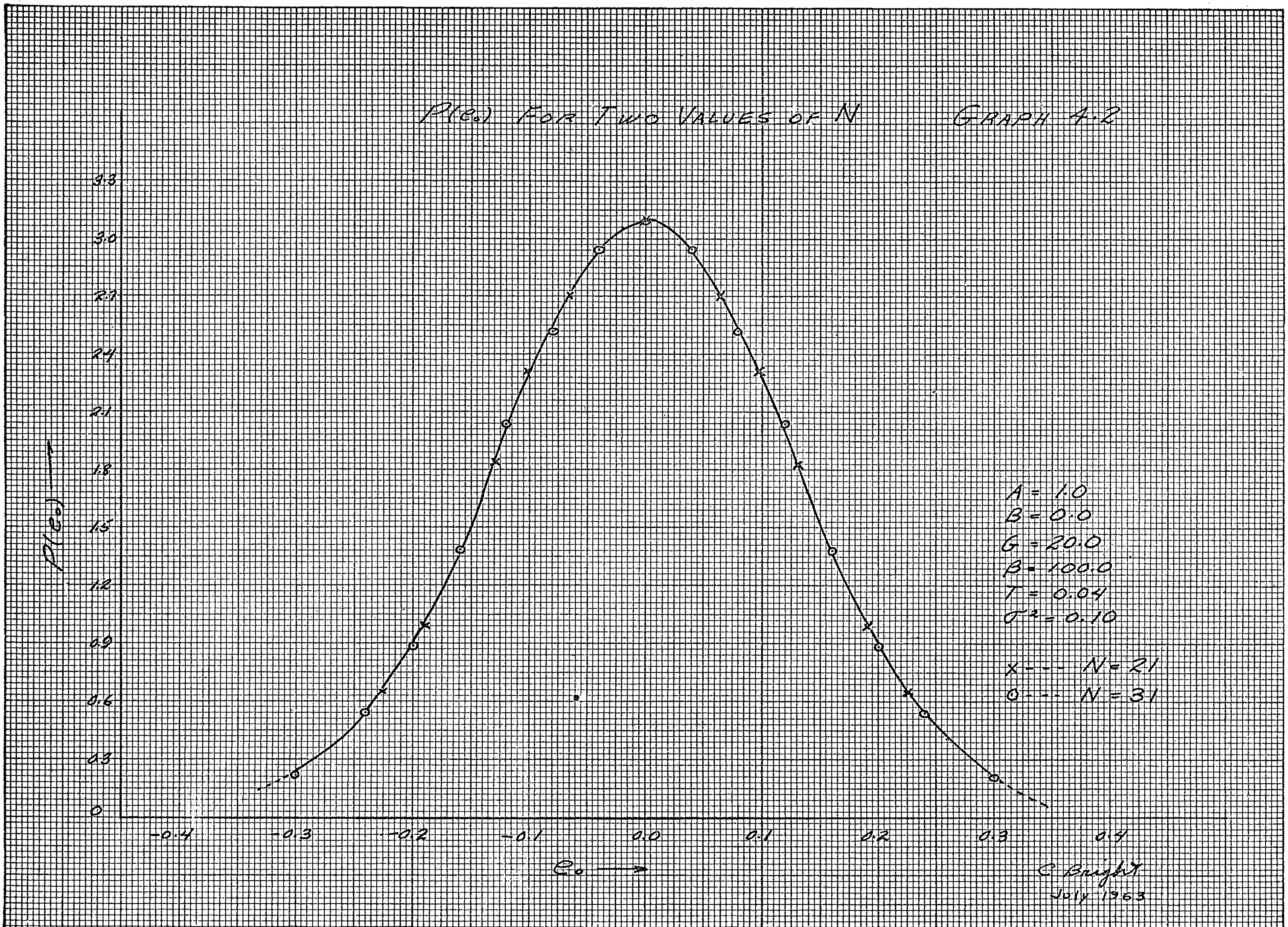
A	B	β	G	T	σ^2	K_{eq}	σ_o Booton	σ_o Schult.	σ_o Actual
1.0	0	100	20	0.04	0.05	1.0	0.0907	0.0905	0.0907
1.0	0	100	20	0.04	0.1	1.0	0.1287	0.1284	0.1287
0.1	0	100	20	0.04	1.0	0.1	0.129		0.129
0.1	0.1	100	20	0.04	1.0	0.365	0.261	0.194	
0.1	0.2	100	20	0.04	1.0	0.645	0.338	0.314	
0.1	0.3	100	20	0.04	1.0	0.883	0.388	0.505	
0.1	0.4	100	20	0.04	1.0	1.075	0.421	0.796	
0.1	0.0	100	20	0.04	0.5	0.1	0.091		0.091
0.1	0.1	100	20	0.04	0.5	0.232	0.149	0.100	
0.1	0.2	100	20	0.04	0.5	0.365	0.184	0.134	
0.1	0.3	100	20	0.04	0.5	0.492	0.215	0.178	
0.1	0.4	100	20	0.04	0.5	0.624	0.242	0.228	
0.1	0.0	100	20	0.04	0.3	0.1	0.0707		0.0707
0.1	0.1	100	20	0.04	0.3	0.175	0.100	0.067	
0.1	0.2	100	20	0.04	0.3	0.265	0.125	0.083	
0.1	0.3	100	20	0.04	0.3	0.337	0.141	0.101	
0.1	0.4	100	20	0.04	0.3	0.41	0.155	0.119	

TABLE 4.2

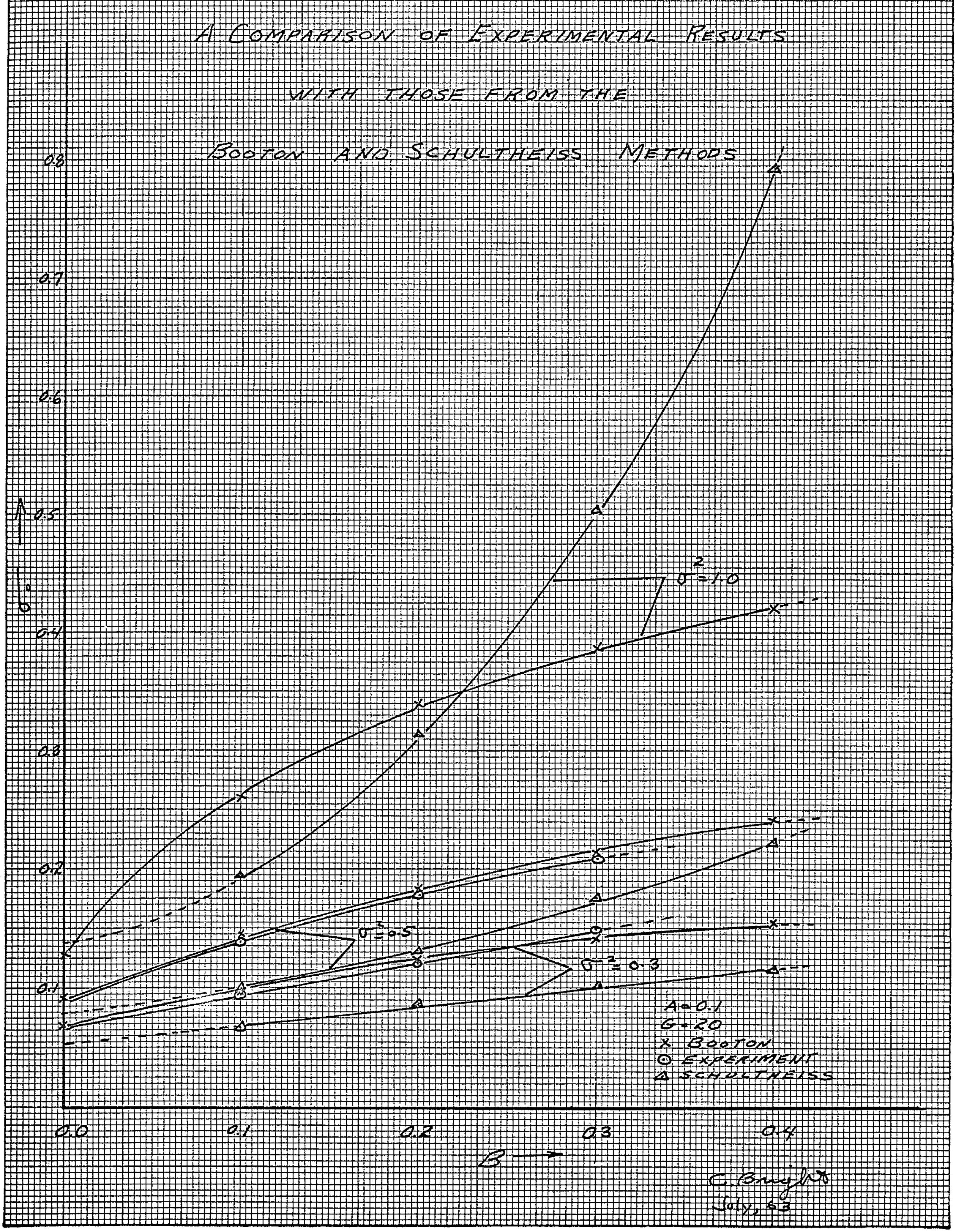
EXPERIMENTAL RESULTS

G	A	B	σ_i	σ_o
20	0.1	0.1	0.707	0.143
20	0.1	0.2	0.707	0.179
20	0.1	0.3	0.707	0.219
20	0.1	0.1	0.55	0.102
20	0.1	0.2	0.55	0.127
20	0.1	0.3	0.55	0.152
20	0.1	0.2	0.10	0.048
20	0.1	0.3	0.10	0.063
20	0.1	0.1	0.90	0.232


GRAPH 4.1
A COMPARISON OF THE MARKOV METHOD
WITH
THE HARMONIC ANALYSIS METHOD
P(e₀)
4.4
4.0
3.6
3.2
2.8
2.4
2.0
1.6
1.2
0.8
0.4
0.0
-0.4
-0.3
-0.2
-0.1
0.0
0.1
0.2
0.3
0.4
e₀
σ² = 0.05
σ² = 0.10
A = 1.0
B = 0.0
T = 0.04
β = 100.0
G = 20.0
N = 31
X --- EXACT
⊙ --- CALCULATED
C. Bright
July 1963


P(e₀) FOR TWO VALUES OF N
GRAPH 4.2
3.3
3.0
2.7
2.4
2.1
1.8
1.5
1.2
0.9
0.6
0.3
0
P(e₀)
-0.4
-0.3
-0.2
-0.1
0.0
0.1
0.2
0.3
0.4
e₀
A = 1.0
B = 0.0
G = 20.0
β = 100.0
T = 0.04
σ² = 0.10
X --- N = 21
O --- N = 31
C Bright
July 1963

GRAPH 4.3


A COMPARISON OF EXPERIMENTAL RESULTS
WITH THOSE FROM THE
BOOTON AND SCHULTHEISS METHODS
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.0
0.1
0.2
0.3
0.4
B →
σ₀ ↑
σ² = 1.0
σ² = 0.5
σ² = 0.3
A = 0.1
G = 20
× BOOTON
⊙ EXPERIMENT
Δ SCHULTHEISS
C. Bright
July, 63

to zero would eliminate them. Graph 4.1 does indicate, however, that the introduction of the sample and clamp circuit into the linear system yields precise results when N is large and T is set to the optimum value. Barring any errors in the mathematical manipulation the results for $B > 0$ should also be representative of the nonlinear system.

Graph 4.2 presents a check on the sensitivity of $P(e_o)$ with respect to the quantizing interval, that is N, the size of the stochastic matrix. As can be easily observed, for the two values of N, 31 and 21, there is no quantizing error existing for the input σ^2 less than or equal to 0.1.

It must be noted, however, that for larger values of σ^2 the quantizing interval itself will be greater for the same N, since it will then be necessary to choose the maximum value of e_o,(E) greater and hence Δ the quantizing interval, greater.

Since the sensitivity was only checked for one value of σ^2 it is not known what effect a larger Δ might have, but to be on the safe side the analyst should use N as large as possible. As cited in section 4.1, the time required for computation with N large is not prohibitive.

Graph 4.3 demonstrates very clearly the effect of varying the nonlinear gain constant, B on the r.m.s. value of the output e_o.

For values of σ^2 equal to 0.5, and 0.3, the curves comparing the two techniques agree to within an accuracy expected by statistical analysis, but in the results for σ^2 set to 1.0, we see that a very poor correlation is obtained. The reason for the poor correlation as B and σ^2 increases is not known, since the complexity of the functions involved render an analytical error analysis impossible. An intuitive justification for the discrepancies between the two methods will be attempted on the basis of the author's knowledge of the problem.

First, let us examine the type of nonlinear function used (as shown in figure 4.3) with respect to the parameters A and B.

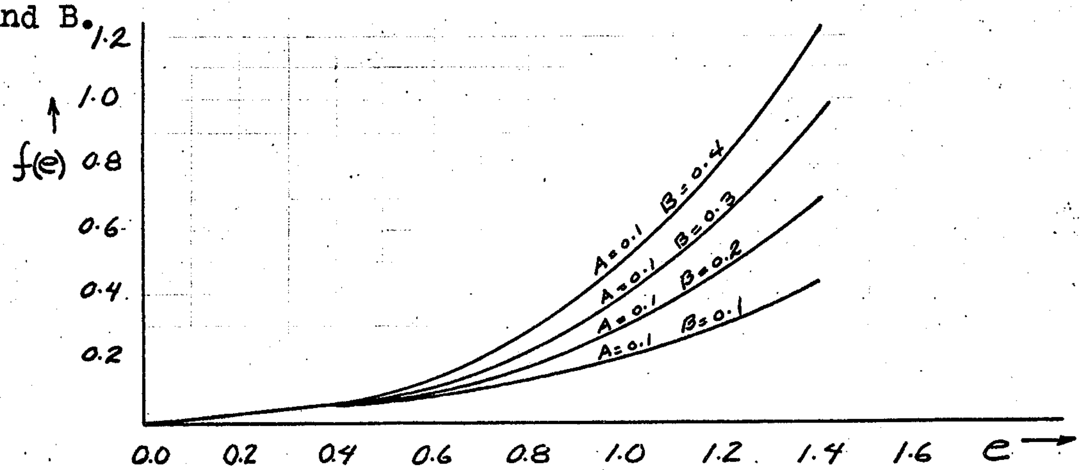


Figure 4.3

The Cubic Nonlinearity $f(e)=Ae+Be^3$

Upon examination of figure 4.3 it is easily seen that for e less than 0.8, the effect of B on the nonlinearity is small, that is, it could very easily be approximated by a corresponding straight line for each value of B. Evidence of this fact presents itself clearly, in the curve calculated by the Schultheiss technique for σ^2 equal to 0.3. It is almost a straight line, as it should be, since the system is operating in a linear manner.[2] The corresponding curve in Booton's case agrees fairly closely.

Considering the Schultheiss curve calculated for a value of σ^2 equal to 0.5, we see the effect of B as it is increased; the larger B the larger σ_o, which is again intuitively obvious, since as B increases so does the effective loop gain of the system.

For the largest σ^2 chosen, viz. 1.0, there is a rapid change in $P(e_o)$ as B increases. This is apparent in Figure 4.3.

It is now to be argued, which method is weighting the nonlinearity correctly? As previously stated, there is no known method of obtaining an exact solution to the problem. Booton's technique has been proven (17) experimentally for the example of a feedback system containing a linear gain with saturation. Does it not give satisfactory answers for the example in this thesis? This question is discussed more fully in section 4.5.

---

[2] For a Gaussian distribution 68 percent of the time the values are less than σ, that is, for σ^2 equal to 0.3, values are greater than 0.546 only 32 percent of the time.

We recall that K_{eq} is a linear gain, derived from a cubic equation of the form $Ae+Be^3$, by using a least-squares minimization. Recall that K_{eq} was derived to be

$$K_{eq}=A+3B\sigma_i^2 \qquad 4.12$$

We see K_{eq} as a function of A, B and σ_i^2 as it should be, but does a least squares minimization weight errors correctly when used in the statistical sense? Certainly the factor Be^3 has a very marked effect for $e \geq 1.0$ volts. In graph 4.3 ($\sigma^2=1.0$) 32 percent of the time the values of e_i are greater than 1.0. In this case, an increase in B greatly increases the overall system gain as is easily seen by the Schultheiss curve for σ equal to 1.0.

As for the validity of the Schultheiss technique the author sees it as exact except for the assumption that the output is Gaussian when the input is Gaussian. This, as mentioned before, is known to be a valid assumption, and results should, therefore, not be dependent on that assumption.

The sampling period T may introduce large errors if it is not chosen correctly but in this case it is believed chosen in the best possible way.

It would be extremely difficult to form a concrete conclusion regarding the applicability of the Schultheiss method to nonlinear systems without first computing, and checking, several other nonlinear examples.

4.5 EXPERIMENTAL RESULTS

To form a basis of comparison for the results obtained from the Booton and Schultheiss techniques, the system was simulated on an analog computer, and several tests run with A fixed at 0.1, and B ranging from zero to 0.3. The results are tabulated in Table 4.2, as well as plotted in Graph 4.3, for a direct comparison with the analytic results.

4.5.1 The Experiment

Implementation of the system required the use of operational amplifiers and a diode function generator, which relied on straight line segments to approximate the function $Ae+Be^3$. It proved to be difficult to set the segments in the range $o \leq f(e) < 0.3$. Errors up to 30 percent in f(e) for that range were possible if care was not taken in setting up the function.

For excitation, a variable level Gaussian noise generator was used. From previous tests performed on the generator,[3] the bandwidth of the output spectrum was known to be approximately 500 radians per second, as opposed to 100 radians per second used in the theoretical calculations. It is speculated that this had little or no effect on the results, due to the extreme low-pass nature of the system ($\simeq$ 10 radians per second).

[3] For more detail on the output spectrum of the noise generator see: Fieguth, W.: Noise Measurements and Detection of Signals in Noise, Graduate Engineering Physics Thesis, University of Manitoba, May, 1963.

The mean value of the input to the theoretical system was zero; the mean value of the output of the Gaussian generator could be adjusted to zero, but due to large (presumably very low frequency) variations in the generator output, readings were very difficult to obtain accurately.

A probability distribution analyser was used to measure the system's output distribution function, and from that, the standard deviation, σ_0 of the output was easily obtained. Several of the distributions measured were plotted on statistical paper, and it was noted that even for B large, ~~that~~ the distributions were close to Gaussian, bearing out the basic assumptions made in both the Schultheiss and Booton Analyses.

4.5.2 Discussion of Experimental Results

From an examination of the experimental curves obtained in Graph 4.3, it is observed that the experimental data agrees very closely to that calculated by Booton's technique. This indicates Booton's as the more accurate of the two analytic methods. However, one must keep in mind the possibility of compensating errors, such as:

(i) errors caused by poor simulation of f(e)

(ii) the wide input spectrum of the noise, and

(iii) inaccuracies caused by the very low frequency variations in the output of the noise generator.

This possibility was substantiated when the system was tested in its linear mode (B equal to zero), as the results obtained from the experiment did not at all agree with those calculated by Harmonic Analysis. The discrepancy was 2:1 ! In spite of all the evidence supporting the negative, it is believed by the author that the primary source of error was ~~from~~ the wide input spectrum. This could only be verified, or disclaimed by knowing the exact form of the output spectrum of the generator.

The experiment does, though, in the nonlinear case, support Booton's method of analysis, rather than the Schultheiss method.

CHAPTER 5

FINAL COMMENTS AND FUTURE INVESTIGATION

5.1 FINAL COMMENTS

A new technique developed by Schultheiss, for the analysis of the response to Gaussian random signals of nonlinear systems has been applied to a closed-loop system containing a nonlinearity of the zero-memory type. The results of the investigation have been compared with those of the method of Booton applied to the same example, but with only a limited degree of correlation.

A favourable correlation was obtained for input signals ranging from zero to 0.7 but for random signals of larger r.m.s. value there was little or no correlation when the nonlinear gain constant, B became more than three times the linear gain, A. Possible causes for this were cited in the last section of Chapter 4.

When applied to this example, it was found that the output probability density function was sensitive to the sampling interval T. For a change in T of 2:1, $P(e_o)$ correspondingly was altered by 10 to 15 percent. This in itself does not appear sensitive at first glance, but due to the means by which T is selected (Article 4.1) an error from the 'best' T of 400 percent is possible if some care is not taken. This property of T may, or may not, be a characteristic of the technique, and could only be proven so by selecting further examples to analyse. Hence, if T were chosen in the best possible manner, suitable results would be calculable regardless of the effect on the output.

It is impossible to form a concrete opinion regarding the Schultheiss method as applied to this example since so few calculations were performed that trends in results were not definitely established. Several more cases should be calculated with different values for β, the effective bandwidth of the input process, T the sampling interval, and G the gain of the system. The results from the experiment do indicate, however, that, barring errors in the experimental technique, the best correlation was obtained between the experiment and Booton's method.

The work involved in setting up a particular system would always be comparable to that in the example just completed. Booton's method, which yielded results of equal quality, in this case did not require as much computation, but for some examples it might prove to be much more tedious than the Schultheiss method. In spite of experiment supporting Booton's technique in this case, it is believed that the Schultheiss method is far more general than is implied by the example tested, since it may be applied to much more complicated systems (1).

One must keep in mind the following points when applying the technique:

(i) The open loop conditional probability density function, $P_c \left[e_o(t_{i+1}) \mid e_o(t_i) \right]$ must be calculable.

(ii) The input process must be Gaussian and have a short correlation time compared to the sampling interval T, as well as short compared to the effective response time of the system.

If one keeps in mind the above quite restrictive statements, it should be possible to apply the Schultheiss method to any nonlinear system for which $P_c\left[e_o(t_{i+1})\middle|e_o(t_i)\right]$ may be obtained, provided a digital computer with large enough capacity is available for computation.

APPENDIX A

CALCULATION OF MEAN SQUARE VALUE OF THE OUTPUT PROCESS FOR THE LINEAR EXAMPLE

We desire the variance of the output x_1 of the integrator excited by noise of spectral level N_o volts/cps. Since the integrator is sampled every T seconds, the variance must be computed over the interval T. This gives the following block diagram for the integrator operating in that mode.

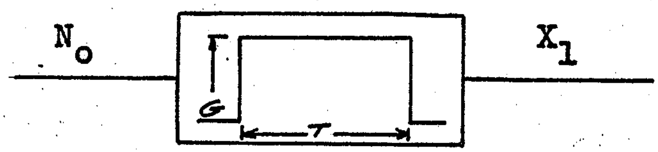


Figure A1
Equivalent Weighting Function Block for the Sampled Data Integrator

The Fourier transform (Laplace transform with s replaced by $j\omega$) of the weighting function may be written

$$H(j\omega) = \frac{G}{j\omega}(1-e^{-j\omega T}) \qquad \text{A.1}$$

By use of equation 1.15 and 1.17 we obtain the output spectral density at x_1 to be

$$\Phi_1(\omega) = N_o G^2 \left| \frac{1-e^{-j\omega T}}{j\omega} \right|^2 \qquad \text{A.2}$$

or

$$\Phi_1(\omega) = \frac{2N_o G^2}{\omega^2}(1-\cos\omega T) \qquad \text{A.3}$$

and the mean square value, σ_1^2 becomes

$$\sigma_1^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2N_oG^2}{\omega^2}(1-\cos\omega T)\,d\omega \qquad \text{A.4}$$

The integral A.4 may be evaluated using complex variable theory, to yield

$$\sigma_1^2 = N_oG^2T \qquad \text{A.5}$$

For purposes of comparison N_o is written as $N_o'/2$, where N_o' is a peak value. This is the desired result.

APPENDIX B

EVALUATION OF THE INTEGRAL RESULTING FROM EQUATION 2.20a

The substitution of equation 2.20a into equation 2.3 results in the following:

$$P_o\left[x(t_{n+1})\right] = \int_{-\infty}^{\infty} dx(t_n) P_o\left[x(t_n)\right] \; P_c\left[x(t_{n+1}) \mid x(t_n)\right] \qquad \text{B.1}$$

Since the process is stationary we may write $P_o\left[x(t_{n+1})\right]$ equal to $P_o\left[x(t_n)\right]$ for n large and therefore evaluate the above equation. The output $P_o\left[x(t_n)\right]$ has a Gaussian form and consequently may be written

$$P_o\left[x(t_n)\right] = (\pi K)^{-\frac{1}{2}} \exp\left[-x^2(t_n)/K\right] \qquad \text{B.2}$$

in which K is to be determined.

Then, equation B.1 becomes

$$P_o\left[x(t_{n+1})\right] = \int_{-\infty}^{\infty} (\pi K)^{-\frac{1}{2}} \exp\left[-x^2(t_n)/K\right] \cdot (\pi T G^2 N_o')^{-\frac{1}{2}}$$

$$\exp\left[-\left\{x(t_{n+1}) - (1-GT)\, x(t_n)\right\}^2 / TG^2 N_o'\right] dx(t_n) \qquad \text{B.3}$$

When N_o' is defined as $2N_o$, $x(t_n)$ as y, $x(t_{n+1})$ as x, (1-GT) as b, and TG^2N_o' as 'a', equation B.3 may be written in the simplified form

$$P_o(x) = \int_{-\infty}^{\infty} \frac{1}{\pi} (Ka)^{-\frac{1}{2}} \exp\left[-\; (x^2 - 2bxy + (b^2 + a/K)y^2)/\sqrt{a}\right] dy \qquad \text{B.4}$$

Also define (b^2+a/k) as q, and $\sqrt{q/a}\,y$ equal to z, then substitution of the change of variables, and completion of the square in the exponent yields

$$P_o(x) = \frac{1}{\pi}(Kq)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left[-x^2(1-b^2/q)/a\right] \cdot$$

$$\exp\left[-(z-bx/\sqrt{aq})^2\right] dz \qquad B.5$$

or

$$P_o(x) = \frac{1}{\sqrt{\pi}}(Kq)^{-\frac{1}{2}} \exp\left[-x^2(q-b^2)/aq\right] \qquad B.6$$

Solving for K by equating B.6 to B.2 we obtain

$$K = \frac{GN'_o}{2-GT} \;; \quad q=1 \qquad B.7$$

Therefore, the stationary output probability density function becomes

$$P(x_o) = \left(\frac{GN'_o}{2-GT}\right)^{-\frac{1}{2}} \cdot \exp\left[-\left(\frac{2-GT}{GN'_o}\right) x_o^2\right] \qquad B.8$$

where x has been replaced by the output x_o.

APPENDIX C

A DERIVATION OF EQUATION 3.17

Consider the following figure:

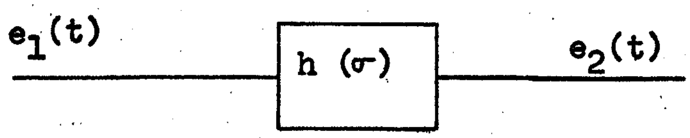


Figure C.1

Linear System with Weighting Function $h(\sigma)$.

By the superposition integral $e_2(t)$ may be written as

$$e_2(t) = \int_{-\infty}^{\infty} h(\sigma) e_1(t-\sigma)\, d\sigma \qquad \text{C.1}$$

or for t equal to $t+\tau$, $e_2(t+\tau)$ becomes

$$e_2(t+\tau) = \int_{-\infty}^{\infty} h(\sigma)\, e_1(t+\tau-\sigma)\, d\sigma \qquad \text{C.2}$$

Since we desire $\phi_{22}(\tau)\Big|_{\tau=0}$, then by equation 1.10 we may write $\phi_{22}(\tau)$ as

$$\phi_{22}(\tau) = \lim_{w\to\infty} \frac{1}{2w} \int_{-w}^{w} \left(\int_{-\infty}^{\infty} h(\sigma) e_1(t-\sigma)\, d\sigma \right) \left(\int_{-\infty}^{\infty} h(u) e_1(t+\tau-u)\, du \right) dt \qquad \text{C.3}$$

For deterministic signals, and stationary random processes the following operations may be performed:

$$\phi_{22}(\tau) = \int_{-\infty}^{\infty} h(\sigma)\, d\sigma \int_{-\infty}^{\infty} h(u)\, du \lim_{w\to\infty} \int_{-w}^{w} \frac{1}{2w}\; e_1(x) e_1(x+\sigma+\tau-u)\, dx$$

or

$$\phi_{22}(\tau) = \int_{-\infty}^{\infty} h(\sigma) \int_{-\infty}^{\infty} h(u) \quad \phi_{11}(\tau+\sigma-u)\, du\, d\sigma, \qquad \text{C.4}$$

or finally,

$$\phi_{22}(\tau)\Big|_{\tau=0} = \int_{-\infty}^{\infty} h(\sigma) \int_{-\infty}^{\infty} h(u)\phi_{11}(\sigma-u)\,du\,d\sigma \qquad \text{C.5}$$

Referring to figure 3.2, we see that $h(\sigma)$ in this case is equal to G over the interval 0 to T.

Therefore equation C.5 becomes

$$\overline{\Delta e_2^2} = \phi_{22}(0) = G^2 \int_0^T d\sigma \int_0^T \phi_{11}(\sigma-u)\,du \qquad \text{C.6}$$

This is the desired equation 3.17.

APPENDIX D

THE N-DIMENSIONAL GAUSSIAN DISTRIBUTION

For use in calculating the autocorrelation function of the output of a nonlinear device it is necessary to define the multidimensional Gaussian random variable, and specialize it for our needs. The general n-dimensional Gaussian probability density function is defined as follows (11):

$$P(x_1, x_2, \ldots, x_n) = (2\pi)^{-n/2} \cdot |M_n|^{-\frac{1}{2}}$$

$$\exp\left[-\tfrac{1}{2} \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{M_{rs}}{M_n} (x_r - \overline{x_r})(x_s - \overline{x_s})\right] \qquad \text{D.1}$$

where M_n is the nxn matrix of correlation coefficients ϕ_{rs}, $|M_n|$ is the determinant of M_n, with M_{rs} the cofactors of M_n. Also ϕ_{rs} equals ϕ_{sr} and may be written as $\overline{x_r x_s} - \overline{x_r}\ \overline{x_s}$, where the bar denotes a time average.

To find the autocorrelation of the cubic device in the nonlinear example, it is necessary to use only the bivariate distribution. Since we are dealing with a zero mean distribution with variance σ^2 the parameters of equation D.1 become

$$\overline{x_s} = \overline{x_r} = 0, \quad r, s = 1, 2,$$

$$\overline{x_1 x_1} = \sigma^2, \ \overline{x_1 x_2} = \phi_{12}(\tau)$$

$$\overline{x_2 x_1} = \phi_{21}(\tau) \text{ and } \overline{x_2 x_2} = \sigma^2$$

This yields for the bivariate probability density function, the following expression:

$$P(x_1,x_2) = \frac{1}{2\pi(\sigma^4 - \phi_{12}^2(\tau))^{\frac{1}{2}}} \exp\left[-\frac{1}{2(\sigma^4-\phi_{12}{}^2(\tau))} \cdot (\sigma^2 x_1^2 - 2\phi_{12}(\tau)x_1x_2 + \sigma^2 x_2^2)\right] \qquad D.2$$

APPENDIX E

DETERMINATION OF $\phi_{11}(\tau)$

Consider equation 3.21 from which $\phi_{11}(\tau)$ may be written

$$\phi_{11}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(Ax_1+Bx_1^3)(Ax_2+Bx_2^3)\cdot \frac{1}{2\pi(\sigma^4-\phi_{11}^2)^{\frac{1}{2}}}\cdot$$

$$\exp\left[-\frac{1}{2\pi(\sigma^4-\phi_{11}^2)}\left\{\sigma^2(x_1+a)^2-2\phi_{11}(x_1+a)(x_2+a)+\sigma^2(x_2+a)^2\right\}\right]dx_1dx_2$$

E.1

Let us define the normalized input autocorrelation function $\phi_{11}(\tau)/\sigma^2$ as $\rho(\tau)$, and apply the following transformations:

$Z_1 = (x_1+a)/\sigma$, and $Z_2 = (x_2+a)/\sigma$

We then obtain for $\phi_{11}(\tau)$

$$\phi_{11}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(Z_1-a)f(Z_2-a)}{2\pi(1-\rho^2)^{\frac{1}{2}}}\cdot$$

$$\exp\left[-\frac{1}{2(1-\rho^2)}(Z_1-\rho Z_2)^2+Z_2^2/2\right]dZ_1dZ_2 \qquad \text{E.2}$$

This, then may be written as

$$\int_{-\infty}^{\infty}f(Z_2-a)\cdot\exp\left[-Z_2^2/2\right]\cdot\int_{-\infty}^{\infty}\frac{f\,(Z_1-a)}{2\pi(1-\rho^2)^{\frac{1}{2}}}\cdot$$

$$\exp\left[-\frac{1}{2(1-\rho^2)^{\frac{1}{2}}}(Z_1-\rho Z_2)^2\right]dZ_1dZ_2 \qquad \text{E.3}$$

Let us define $Z_1-\rho Z_2$ as w_1; the inner integral in equation

E.3 then becomes

$$\int_{-\infty}^{\infty} \frac{f(Z_2-a)}{\sqrt{2\pi}} \exp\left[-\ Z_2^2/2\right] \cdot \int_{-\infty}^{\infty} \frac{f(\sigma[W_1+\rho Z_2]\ -a)}{\sqrt{2\pi}(1-\rho^2)^{\frac{1}{2}}} \cdot$$

$$\exp\left[-\frac{{W_1}^2}{2(1-\rho^2)}\right] dW_1 dZ_2 \qquad \text{E.4}$$

The integrations are easily performed, using any standard reference for the error function integral. After many tedious algebraic manipulations, the following answer was obtained for the output autocorrelation function of the cubic nonlinearity:

$$\begin{aligned} \phi_{11}(\tau) = {} & (6B^2\sigma^6)\rho^3 + (18B^2\sigma^4)a^2\rho^2 \\ & + \Big[(9B^2\sigma^6+4AB\sigma^4+A^2\sigma^2) + (12B^2\sigma^4+6AB\sigma^2)a^2 \\ & + (9B^2\sigma^4)a^4\Big]\rho + (9B^2\sigma^4+6AB\sigma^2+A^2)a^2 \\ & + (6B^2\sigma^2+2AB)a^4+B^2a^6 \end{aligned} \qquad \text{E.5}$$

This is the desired result.

APPENDIX F

CALCULATION OF $\overline{\Delta e_2^2}$ FROM $\phi_{11}(\tau)$

It is only a matter of substituting equation 3.23 into equation 3.17 and integrating. Doing this, we obtain

$$\overline{\Delta e_2^2} = G^2 \int_0^T d\sigma \int_0^T (b_3\rho^3(\sigma-u)+b_2\rho^2(\sigma-u)$$

$$+ b_1\rho(\sigma-u)+b_o)du \qquad \text{F.1}$$

The normalized input autocorrelation found from equation 3.4 may be written as

$$\rho(\tau) = \exp(-\beta|\tau|) \qquad \text{F.2}$$

The square and cube of the function are easily written as

$\rho^2(\tau) = \exp(-2\beta|\tau|)$, and $\rho^3(\tau) = \exp(-3\beta|\tau|)$

Considering the integral

$$\int_0^T du \int_0^T \exp\left[-b\beta\ |\sigma-u|\right] d\sigma \qquad \text{F.3}$$

with values of b equal to 1,2 and 3, we obtain, after integration, the result:

$$\frac{2}{\beta^2 b^2}\left[Tb\beta+\exp(-b\beta T)-1\right] \qquad \text{F.4}$$

For b equal to 3.2 and 1, the first three terms in the brackets of equation F.1 are easily obtained; the last term is evaluated to yield

$$\overline{\Delta e_2^2} = \frac{2G^2}{\beta^2}\left\{\frac{b_3}{9}\left(\exp(-3\beta T)+3\beta T-1\right)+\frac{b_2}{4}\left(\exp(-2\beta T)+2\beta T-1\right)\right.$$

$$\left. + b_1\left(\exp(-\beta T)+\beta T-1\right)+\beta^2 b_o T^2/2\right\} \qquad \text{F.5}$$

This is the desired result.

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