TOPOLOGICAL CRITERIA FOR THE CONTROLLABILITY AND OBSERVABILITY OF RLC NETWORKS

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by
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ABSTRACT

In this thesis the controllability and observability of RLC networks are investigated. The systematic formulation of the state equations for RLC networks, with capacitor voltages and inductor currents chosen as state variables, shows that controllability and observability must be extended to include improper systems if all cases are to be considered.

Sufficient conditions for controllability and observability, based upon the position of input and output ports are derived. It is shown that if a current source is placed in parallel with each capacitive twig and a voltage source in series with each inductive link, the network is always controllable. Similarly, if an output voltage is measured across each capacitive twig and an output current is measured in each inductive link, the network is always observable.

Sufficient conditions for uncontrollability and unobservability of networks with zero natural frequencies are also given. Specifically, if a network, in which edges have been included for the output variables, has capacitor-only cut-sets and/or inductor-only loops, it is both uncontrollable and unobservable.

A method of using transfer functions of a normal form system to test for controllability and observability is given. Topological formulae for a hybrid n-port are derived and used to examine the controllability and observability of an RLC network. An example is given to illustrate the procedure.

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desirable. Some work in this area has already been done by Narraway [6]. He has shown that certain network topologies, specifically, capacitor-only cut-sets and/or inductor-only loops are always uncontrollable. He has also proven that it is possible for CR networks containing capacitor-only cut-sets and LG networks containing inductor-only loops to be unobservable.

The main purpose of this thesis is to obtain topological criteria for the controllability and observability of RLC networks.

In Chapter II we introduce the basic concepts of controllability and observability and give the standard methods of testing normal form systems for these properties. A systematic formulation of the state equations for RLC networks is then given. This shows that normal form state equations are not always possible if the choice of state variables is limited to physical voltages and currents. The tests for controllability and observability are then extended to include the improper case.

We begin Chapter III by showing that it is always possible to make a network controllable and observable if complete freedom is given in the placement of input and output ports. A set of theorems then shows that systems with zero natural frequencies are uncontrollable or unobservable if the inputs or outputs are such that B and C, in the state equations, satisfy certain conditions. These theorems are then applied to RLC networks, and physical interpretations of the conditions imposed by B and C are given.

In Chapter IV we show that certain transfer functions may be used to test for controllability and observability. A theorem is given which shows that cancellations made in determining these transfer functions do not lead to incorrect results. A derivation of the topological formulae for the hybrid parameters of an n-port is also given. The chapter concludes with an example in which the controllability and observability of an RLC network is determined topologically.

CHAPTER II

CONTROLLABILITY AND OBSERVABILITY AND ITS EXTENSION TO IMPROPER SYSTEMS

This chapter introduces the concepts of controllability and observability of a linear, time-invariant system. The formulation of the state equations of an RLC network and some new theorems applicable to networks or systems whose state equations are not in normal form are also presented.

Concepts and terminology regarding linear system theory may be found in Chen [5] or Zadeh and Desoer [1], and for linear graph theory in Seshu and Reed [7] or Chan [8]. Standard mathematical symbolism is used throughout.

2.1 CONTROLLABILITY AND OBSERVABILITY

The state variable representation of a system is in itself a very powerful tool in systems analysis, but it also leads to the very interesting qualitative properties of linear systems known as controllability and observability. These dual concepts owe their origin to R.E. Kalman [2,3,4], who was the first to correctly answer the question: "Can any initial state of a given dynamical system be transferred to any desired state in a finite length of time by some control function?" Observability asks: "Can the state of the system be determined from a knowledge of the control and output functions over a finite length of time?" The study of controllability and observability answers these questions by giving necessary and sufficient conditions which are dependent upon the system parameters.

Consider the normal form state equations

$$\dot{x} = Ax + Bu \tag{2.1a}$$

$$y = Cx + Du \tag{2.1b}$$

where A,B,C and D are $n \times n$, $n \times m$, $q \times n$ and $q \times m$ constant matrices respectively, x is the $n \times 1$ state vector, u is the $m \times 1$ input vector and y is the $q \times 1$ output vector. The following basic definitions are adapted from Chen [5].

The state equation (2.1a) is said to be completely state controllable if, for any state x_o at time 0 in the state space S^n, there exists a finite time $t_1 > 0$ and an input $u_{[0,t_1]}$ that will transfer the state x_o to the zero state at the time t_1. Otherwise the equation is said to be uncontrollable.

The dynamical equation (2.1) is said to be completely state observable if, for any state x_o at time 0 in the state space S^n, there exists a finite time $t_1 > 0$ such that the knowledge of the input $u_{[0,t_1]}$ and the output $y_{[0,t_1]}$ over the time interval $[0,t_1]$ suffices to determine the state x_o. Otherwise the equation is said to be unobservable.

These definitions are of little use in actually testing a system. A complete set of theorems has therefore been developed to simplify the procedure. The most commonly used methods, which are given in Chen [5], are now stated.

The state equation (2.1a) is completely state controllable if and only if either of the following equivalent statements is true:

(a) The controllability matrix Q has rank n.

$$Q = [B, AB, \ldots, A^{n-1}B]$$

(b) The rows of $(sI_n - A)^{-1}B$ are linearly independent over the field of complex numbers.

The dynamical equation (2.1) is completely state observable if and only if either of the following equivalent statements is true:

(a) The observability matrix P has rank n.

$$P = [C^T, A^T C^T, \ldots, A^{T(n-1)} C^T]$$

(b) The columns of $C(sI_n - A)^{-1}$ are linearly independent over the field of complex numbers.

In most of the literature the adverbs "completely" and "state" are dropped and the properties are simply referred to as controllability and observability.

2.2 SYSTEMATIC FORMULATION OF THE STATE EQUATIONS FOR RLC NETWORKS

The assumption of a finite dimensional system for which state equations exist is implicit in this development of controllability and observability. It is therefore prudent to determine the conditions under which such equations exist. Bryant [9] has shown that the state variables for an RLC network with independent sources may be chosen with the aid of a normal tree. A normal tree is defined as a tree having as branches all of the independent voltage sources, the maximum possible number of capacitors, the minimum possible number of inductors, and none of the independent current sources.

Following a procedure similar to that given by Martens [10] and Balabanian and Bickart [11], the state equations can be constructed in the following way. Consider a normal tree. Kirchhoff's current and voltage laws partitioned with respect to the normal tree yield

$$\text{KCL: } QI = [Q_f, U] \begin{bmatrix} I_\ell \\ I_t \end{bmatrix} = 0 \qquad (2.2a)$$

$$\text{KVL: } BV = [U, B_f] \begin{bmatrix} V_\ell \\ V_t \end{bmatrix} = 0 \qquad (2.2b)$$

where the subscripts t and ℓ identify twig and link variables respectively. Because the subspaces associated with B and Q are orthogonal [7], that is, $QB^T = 0$, we have

$B_f = -Q_f^T$. Substitution of this orthogonality condition and the element relationships

$$V_\ell = Z_\ell I_\ell \qquad I_t = Y_t V_t \tag{2.3}$$

into equation (2.2) yields

$$\begin{bmatrix} Z_\ell & -Q_f^T \\ Q_f & Y_t \end{bmatrix} \begin{bmatrix} I_\ell \\ V_t \end{bmatrix} = 0 \tag{2.4}$$

If we partition this equation and use Martens' notation, where the subscripts V,C,G,Γ,I,L,R and S denote twig: voltage sources, capacitances, conductances, reciprocal inductances; link: current sources, inductances, resistances and elastances, we obtain

$$\left[\begin{array}{cccc|cccc} \frac{1}{p}S_\ell & 0 & 0 & 0 & -Q_{VS}^T & -Q_{CS}^T & 0 & 0 \\ 0 & R_\ell & 0 & 0 & -Q_{VR}^T & -Q_{CR}^T & -Q_{GR}^T & 0 \\ 0 & 0 & pL_\ell & 0 & -Q_{VL}^T & -Q_{CL}^T & -Q_{GL}^T & -Q_{\Gamma L}^T \\ 0 & 0 & 0 & 0 & -Q_{VI}^T & -Q_{CI}^T & -Q_{GI}^T & -Q_{\Gamma I}^T \\ \hline Q_{VS} & Q_{VR} & Q_{VL} & Q_{VI} & 0 & 0 & 0 & 0 \\ Q_{CS} & Q_{CR} & Q_{CL} & Q_{CI} & 0 & pC_t & 0 & 0 \\ 0 & Q_{GR} & Q_{GL} & Q_{GI} & 0 & 0 & G_t & 0 \\ 0 & 0 & Q_{\Gamma L} & Q_{\Gamma I} & 0 & 0 & 0 & \frac{1}{p}\Gamma_t \end{array}\right] \left[\begin{array}{c} I_S \\ I_R \\ I_L \\ I_I \\ \hline V_V \\ V_C \\ V_G \\ V_\Gamma \end{array}\right] = 0 \tag{2.5}$$

where p is the differential operator.

Elimination of the undesired variables V_G, V_Γ, I_S and I_R produces the state equations in the following form:

$$\begin{bmatrix} pI_L \\ pV_C \end{bmatrix} = -\begin{bmatrix} \mathcal{L}_{11}^{-1} & 0 \\ 0 & \mathcal{C}_{22}^{-1} \end{bmatrix} \begin{bmatrix} H_{11} & H_{14} \\ -H_{14}^T & H_{44} \end{bmatrix} \begin{bmatrix} I_L \\ V_C \end{bmatrix} - \begin{bmatrix} \mathcal{L}_{11}^{-1} & 0 \\ 0 & \mathcal{C}_{22}^{-1} \end{bmatrix} \begin{bmatrix} p\mathcal{L}_{12}+H_{12} & H_{13} \\ -H_{24}^T & p\mathcal{C}_{12}^T+H_{34}^T \end{bmatrix} \begin{bmatrix} I_I \\ V_V \end{bmatrix} . \tag{2.6a}$$

$$\begin{bmatrix} V_I \\ I_V \end{bmatrix} = \begin{bmatrix} p\mathcal{L}_{12}^T+H_{12}^T & H_{24} \\ -H_{13}^T & p\mathcal{C}_{12}+H_{34} \end{bmatrix} \begin{bmatrix} I_L \\ V_C \end{bmatrix} + \begin{bmatrix} p\mathcal{L}_{22}+H_{22} & H_{23} \\ -H_{23}^T & p\mathcal{C}_{11}+H_{33} \end{bmatrix} \begin{bmatrix} I_I \\ V_V \end{bmatrix} \tag{2.6b}$$

where

$$\mathcal{L}_{11} = L_\ell + Q_{\Gamma L}^T L_t Q_{\Gamma L} = \mathcal{L}_{11}^T \qquad \mathcal{L}_{12} = Q_{\Gamma L}^T L_t Q_{\Gamma I} \qquad L_t = \Gamma_t^{-1}$$

$$\mathcal{L}_{22} = Q_{\Gamma I}^T L_t Q_{\Gamma I} = \mathcal{L}_{22}^T$$

$$\mathcal{C}_{11} = Q_{VS} C_\ell Q_{VS}^T = \mathcal{C}_{11}^T \qquad \mathcal{C}_{12} = Q_{VS} C_\ell Q_{CS}^T \qquad C_\ell = S_\ell^{-1}$$

$$\mathcal{C}_{22} = C_t + Q_{CS} C_\ell Q_{CS}^T = \mathcal{C}_{22}^T$$

$$Y = G_t + Q_{GR} G_\ell Q_{GR}^T = Y^T \qquad G_\ell = R_\ell^{-1}$$

$$Z = R_\ell + Q_{GR}^T R_t Q_{GR} = Z^T \qquad R_t = G_t^{-1}$$

$$H_{11} = Q_{GL}^T Y^{-1} Q_{GL} = H_{11}^T$$

$$H_{22} = Q_{GI}^T Y^{-1} Q_{GI}^T = H_{22}^T$$

$$H_{33} = Q_{VR} Z^{-1} Q_{VR}^T = H_{33}^T$$

$$H_{44} = Q_{CR} Z^{-1} Q_{CR}^T = H_{44}^T$$

$$H_{12} = Q_{GL}^T Y^{-1} Q_{GI} \qquad H_{23} = -Q_{VI}^T + Q_{GI}^T Y^{-1} Q_{GR} G_\ell Q_{VR}^T$$

$$H_{13} = -Q_{VL}^T + Q_{GL}^T Y^{-1} Q_{GR} G_\ell Q_{VR}^T \qquad H_{24} = -Q_{CI}^T + Q_{GI}^T Y^{-1} Q_{GR} G_\ell Q_{CR}^T$$

$$H_{14} = -Q_{CL}^T + Q_{GL}^T Y^{-1} Q_{GR} G_\ell Q_{CR}^T \qquad H_{34} = Q_{VR} Z^{-1} Q_{CR}^T$$

Note that $\mathcal{G}_{22}$ and $\mathcal{L}_{11}$ are positive definite and hence nonsingular.

If we further define

$$\Lambda_1 = \begin{bmatrix} \mathcal{L}_{11} & 0 \\ 0 & \mathcal{G}_{22} \end{bmatrix} \qquad \Lambda_2 = \begin{bmatrix} \mathcal{L}_{12} & 0 \\ 0 & \mathcal{G}_{12}^T \end{bmatrix}$$

$$\Lambda_3 = \begin{bmatrix} \mathcal{L}_{22} & 0 \\ 0 & \mathcal{G}_{11} \end{bmatrix} \qquad H_1 = \begin{bmatrix} H_{11} & H_{14} \\ -H_{14}^T & H_{44} \end{bmatrix}$$

$$H_2 = \begin{bmatrix} H_{12} & H_{13} \\ -H_{24}^T & H_{34}^T \end{bmatrix} \qquad H_3 = \begin{bmatrix} H_{12}^T & H_{24} \\ -H_{13}^T & H_{34} \end{bmatrix}$$

$$H_4 = \begin{bmatrix} H_{22} & H_{23} \\ -H_{23}^T & H_{33} \end{bmatrix}$$

then equation (2.6) becomes

$$p\begin{bmatrix} I_L \\ V_C \end{bmatrix} = -\Lambda_1^{-1} H_1 \begin{bmatrix} I_L \\ V_C \end{bmatrix} - \Lambda_1^{-1} [p\Lambda_2 + H_2] \begin{bmatrix} I_I \\ V_V \end{bmatrix}$$

$$= -\Lambda_1^{-1} H_1 \begin{bmatrix} I_L \\ V_C \end{bmatrix} - \Lambda_1^{-1} H_2 \begin{bmatrix} I_I \\ V_V \end{bmatrix} - \Lambda_1^{-1} \Lambda_2 p \begin{bmatrix} I_I \\ V_V \end{bmatrix} \qquad (2.7a)$$

$$\begin{bmatrix} V_I \\ I_V \end{bmatrix} = [H_3 - \Lambda_2^T \Lambda_1^{-1} H_1] \begin{bmatrix} I_L \\ V_C \end{bmatrix} + [H_4 - \Lambda_2^T \Lambda_1^{-1} H_2] \begin{bmatrix} I_I \\ V_V \end{bmatrix}$$

$$+ [\Lambda_3 - \Lambda_2^T \Lambda_1^{-1} \Lambda_2] p \begin{bmatrix} I_I \\ V_V \end{bmatrix} \qquad (2.7b)$$

It may appear at first that, by choosing the outputs as the complementary variables of the inputs, we are not considering the general case. This, however, is not true. If, for example, we want a particular node-pair voltage as an output, we can simply put a current source between those two nodes, then write the state equations, and, finally, let the value of the current source be zero. The voltage response at those terminals is then the desired output variable. One must be careful, however, that controllability is not being investigated at the same time since the removal of some of the sources in the method described could lead to erroneous results.

The state equations of an RLC network with independent sources can, therefore, be associated with the general form

$$\dot{x} = Ax + B_1 u + B_2 \dot{u} \qquad (2.8a)$$

$$y = Cx + D_1 u + D_2 \dot{u} \qquad (2.8b)$$

This improper form reduces to the standard normal form when there are no inductor-current source cut-sets and no capacitor-voltage source loops. This can be seen by observing that, if $Q_{\Gamma I} = 0$ and $Q_{VS} = 0$, then $\mathbb{L}_{12} = 0$, $\mathbb{L}_{22} = 0$, $\mathbb{G}_{11} = 0$, and $\mathbb{G}_{12} = 0$, and therefore $\Lambda_2 = 0$ and $\Lambda_3 = 0$.

2.3 CONTROLLABILITY AND OBSERVABILITY OF IMPROPER EQUATIONS

The possible occurrence of the system state equations in improper form as in (2.8) is of some concern. A simple transformation

$$z = x - B_2 u \tag{2.9}$$

can be used to give a new set of equations

$$\dot{z} = Az + (AB_2 + B_1)u \tag{2.10a}$$

$$y = Cz + (CB_2 + D_1)u + D_2\dot{u} \tag{2.10b}$$

in a pseudo-normal form.

The source derivatives have been removed from the state equation but not from the output equation. This, however, is not restrictive since the output equation is only used in observability studies, in which case the inputs can conveniently be chosen to be identically zero and the equation can essentially be considered in normal form.

If we can show that controllability and observability are invariant with the transformation (2.9), then we can use (2.10) to determine the characteristics of the physical state variables of (2.8). The rest of this section is devoted to this issue.

Lemma 2.1:

The solution of $\dot{x} = Ax + B_1u + B_2\dot{u}$ is given by

$$x(t) = e^{At}(x(0)-B_2u(0)) + \int_0^t e^{A(t-\tau)}(AB_2+B_1)u(\tau)d\tau+B_2u(t)$$

PROOF:

It is well known [5] that the solution of equation (2.1a) is

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)d\tau \ .$$

The solution of (2.10a) is therefore

$$z(t) = e^{At}z(0) + \int_0^t e^{A(t-\tau)}(AB_2+B_1)u(\tau)d\tau \ .$$

However, $z = x - B_2u$, and substitution into the previous equation yields the desired result.

Lemma 2.2:

If $\dot{x} = Ax + B_1u + B_2\dot{u}$ (2.8a) is controllable, then there exists, for any x_1 in S^n, a $u_{[0,T]}$ with $u(0) = u(T) = 0$, such that $x(0) = x_1$ and $x(T) = 0$.

PROOF:

We will show how to construct a $u_{[0,T]}$ satisfying the required conditions. Let T_1 be an artibrary positive number and let $u_{[0,T_1]} = 0$. Then by Lemma 2.1, $x(T_1) = e^{AT_1}x(0) = e^{AT_1}x_1$. Next, since (2.8a) is controllable, there exists $\hat{u}_{[T_1,T_2]}$, $T_2 > T_1$ and finite, such that $x(T_2) = 0$ with $x(T_1) = e^{AT_1}x_1$. Finally, let $u_{[T_2,T]} = 0$, $T > T_2$ and finite, and the system will stay in the zero state thereafter. Therefore the input

$$u_{[0,T]} = \begin{cases} \hat{u}(t), & T_1 \leq t \leq T_2 \\ 0, & \text{otherwise} \end{cases}$$

transfers any initial state x_1 to the equilibrium state in finite time T with $u(0) = u(T) = 0$.

Note that the lemma is equally valid if $B_2 = 0$, that is, if we have a normal form system.

<u>Theorem 2.1</u>:

The controllability and observability of the dynamical equation (2.8) is invariant under the transformation (2.9).

PROOF:

We will first show the invariance of controllability. If (2.8a) is controllable, Lemma 2.2 shows that there exists, for any x_1 in S^n, a $u_{[0,T]}$, T finite and $u(0) = u(T) = 0$, such that $x(0) = x_1$ and $x(T) = 0$, but

$$z(0) = x(0) - B_2u(0) = x(0) = x_1$$

$$z(T) = x(T) - B_2u(T) = x(T) = 0$$

and hence $u_{[0,T]}$ transfers $z(0)=x_1$ to $z(T)=0$ proving that (2.10a) is also controllable. The converse follows by reversing the procedure.

The invariance of observability is easily established. Since observability is invariant under the specific input used, let $u(t) \equiv 0$. Then both equations are in normal form and in fact are identical. It then follows that, if one is observable, so is the other.

We can now define controllability and observability matrices Q' and P' respectively, for improper systems and give some appropriate theorems.

Theorem 2.2:

The state equation (2.8a) is completely state controllable if and only if the controllability matrix Q' has rank n.

$$Q' = [AB_2+B_1, A^2B_2+AB_1, \ldots, A^nB_2+A^{n-1}B_1]$$

PROOF:

From Theorem 2.1 we know that (2.8a) is controllable if and only if (2.10a) is controllable. However, (2.10a) is controllable if and only if the rank of Q is n.

$$Q = [AB_2+B_1, A^2B_2+AB_1, \ldots, A^nB_2+A^{n-1}B_1]$$

We immediately see that Q and Q' are identical and, therefore, (2.8a) is controllable if and only if the rank of Q' is n.

The dual theorem for observability is given next.

Theorem 2.3:

The dynamical equation (2.8) is completely state observable if and only if the observability matrix P' has rank n.

$$P' = [C^T, A^T C^T, \ldots, A^{T(n-1)} C^T]$$

PROOF:

The proof follows directly from the discussion in the proof of Theorem 2.1.

We have now developed all of the necessary theory to study the controllability and observability of both proper and improper systems.

CHAPTER III

TOPOLOGICAL CONDITIONS ON THE CONTROLLABILITY AND OBSERVABILITY OF RLC NETWORKS

Controllability and observability of an RLC network may be tested by use of the theorems given in Chapter II. In this chapter, we show that this approach is not always necessary if the topology of the network is known. Sufficient conditions for controllability and observability in terms of type and placement of input and output ports are derived. Sufficient conditions for uncontrollability and unobservability of networks having a zero eigenvalue are also given.

3.1 SUFFICIENT CONDITIONS FOR CONTROLLABILITY AND OBSERVABILITY OF RLC NETWORKS

If we are given an arbitrary system with complete freedom of access to all system variables, the question arises: "What are the conditions under which the system is always controllable and/or observable?" A close examination of the controllability and observability matrices of a proper system shows that, if B has rank n, the system is controllable, and, if C has rank n, it is observable. In the improper system the equivalent requirements are that AB_2+B_1 and C have rank n.

These controllability specifications imply that a particular source distribution will always make the system controllable. For RLC networks this situation has a simple solution as shown in the next theorem.

Theorem 3.1:

An RLC network with independent sources is controllable if there is a current source in parallel with each normal tree branch capacitor and a voltage source in series with each chord inductor.

PROOF:

Consider a network with some arbitrary source distribution. The state equation is then

$$\dot{x} = Ax + B_1 u + B_2 \dot{u} \tag{3.1}$$

or from Chapter II

$$px = -\Lambda_1^{-1} H_1 \; x - \Lambda_1^{-1} H_2 u - p\Lambda_1^{-1} \Lambda_2 u \; . \tag{3.2}$$

Alternatively, this may be expressed in the form

$$(p\Lambda_1 + H_1)x + H_2 u + p\Lambda_2 u = 0 \; . \tag{3.3}$$

If we now augment the network with an additional n-dimensional input vector

$$v = \begin{bmatrix} E_L \\ J_C \end{bmatrix} ,$$

by putting a current source in parallel with each branch capacitor, referenced so that it opposes the capacitor current in the cut-set equations, and a voltage source in series with each chord inductor, referenced so that it opposes the inductor voltage in the loop equations, the transpose of the right hand side of equation (2.5) is no longer zero but becomes $[0\ J_C^T\ 0000\ E_L^T\ 0]^T$. Equation (3.3) then becomes $(p\Lambda_1 + H_1)x + H_2 u + p\Lambda_2 u = v$ or equivalently

$$px = -\Lambda_1^{-1} H_1 x - [\Lambda_1^{-1} H_2, -\Lambda_1^{-1}] \begin{bmatrix} u \\ v \end{bmatrix} - p[\Lambda_1^{-1} \Lambda_2, 0] \begin{bmatrix} u \\ v \end{bmatrix} \quad (3.4)$$

We now have a new network, defined by equation (3.4) which meets the specifications outlined in the statement of the theorem. The first term in the controllability matrix Q' is

$$AB_2 + B_1 = -\Lambda_1^{-1} H_1 [-\Lambda_1^{-1} \Lambda_2, 0] + [-\Lambda_1^{-1} H_2, \Lambda_1^{-1}]$$
$$= [\Lambda_1^{-1} H_1 \Lambda_1^{-1} \Lambda_2 - \Lambda_1^{-1} H_2, \Lambda_1^{-1}]$$

Because Λ_1^{-1} is an n×n nonsingular matrix, the rank of $AB_2 + B_1$ is n, the rank of Q' is n, and the network is controllable.

The result given in this theorem ignores what would appear to be a simpler method of obtaining complete controllability. That would be to put a voltage source in parallel with each twig capacitor and a current source in series with each link inductor so that the capacitor voltages and inductor currents can be manipulated directly. Such direct control is indeed possible. However, the state equations previously written for the network are no longer valid, since the capacitor voltages and inductor currents no longer qualify as state variables. This is easily demonstrated by the fact that each new voltage source in the network must be put in the normal tree and hence the capacitor in parallel must be excluded. Similarly, all inductors in series with current sources are no longer chords of the normal tree. The control of these non-state variables is of little interest.

The observability specifications, on the other hand, imply that a special set of output variables will always make the system observable.

Theorem 3.2:

An RLC network with independent sources is observable if the output ports are chosen so that there is a voltage port in parallel with each normal tree branch capacitor and a current port in series with each chord inductor.

PROOF:

Since we are directly observing each state variable, the Theorem is obvious.

3.2 CONTROLLABILITY AND OBSERVABILITY OF THE ZERO NATURAL FREQUENCY

In the previous section we derived sufficient conditions for the controllability and observability of RLC networks. These conditions were in no way related to the structure of the source-free system. In this section we show that given an *a priori* knowledge of the existence of a zero natural frequency in the system, necessary conditions for controllability and observability can be derived. We shall, however, find it more convenient to obtain sufficient conditions for uncontrollability and unobservability rather than necessary conditions for controllability and observability. Consider now the following theorem:

Theorem 3.3:

The state equation (2.8a) is uncontrollable if there exists a non-zero vector a such that, $a^TA=0$, $a^TB_1=0$, or equivalently, if the matrix $[A,B_1]$ has rank less than n.

PROOF:

From Theorem 2.2 the equation is controllable if and only if Q' has rank n. If there exists $a \neq 0$ such that $a^TA = 0$ and $a^TB_1 = 0$ then ,

$$a^TQ' = [a^T(AB_2+ B_1), a^T(A^2B_2+AB_1),\ldots,a^T(A^nB_2+A^{n-1}B_1)] = 0$$

Therefore, the rows of Q' are dependent, proving that the rank of Q' is less than n and the equation is uncontrollable.

We now show the equivalence of the two conditions. If $a^TA = 0$, $a^TB_1 = 0$, $a \neq 0$, then $a^T[A,B_1] = 0$ and therefore the rows of $[A,B_1]$ are dependent, clearly showing that the rank of $[A,B_1]$ is less than n. The converse follows by reversing the procedure.

Corollary 3.1:

The state equation (2.1a) is uncontrollable if there exists a non-zero vector a such that, $a^TA=0$, $a^TB=0$, or equivalently, if the matrix [A,B] has rank less than n. Furthermore, another equivalent condition is that the $\dot{x}_i$, $i=1,2,\ldots,n$ are dependent.

PROOF:

The first part follows directly from Theorem 3.3 by noting that if $B_2\equiv0$ and $B_1=B$ equations (2.1a) and (2.8a) are identical.

To prove the second part, observe that if $a^TA=0$, $a^TB=0$, $a\neq0$, then $a^T\dot{x}=0$ and therefore the $\dot{x}_i$, $i=1,2,\ldots,n$ are dependent. The converse follows immediately by reversing the procedure after noting that u and x are independent.

Some comments on this theorem are now in order. First we see that A is singular and hence must have one or more zero eigenvalues. Furthermore, a is an eigenvector of A^T associated with the zero eigenvalue. It is well known in circuit theory that the observance of zero eigenvalues coincides with the occurrence of capacitor-current source cut-sets, capacitor only cut-sets, inductor-voltage source loops and/or inductor only loops [11]. Half of the conditions of the theorem are therefore satisfied in any of the preceding situations. The other condition is dependent upon the source distribution and, in fact, it is shown in the next theorem that the inclusion of capacitor-current source cut-sets and/or inductor-voltage source loops in the network is sufficient for $a^T B_1 = 0$.

Theorem 3.4:

An RLC network is uncontrollable if it contains any capacitor-only cut-sets and/or inductor-only loops.

PROOF:

If there is a capacitor-only cut-set the following relationship exists between the capacitor currents

$$[a_1^T, a_2^T] \begin{bmatrix} I_C \\ I_S \end{bmatrix} = 0$$

where $a_1 \neq 0$, since at least one of the capacitors must be in the normal tree. Using the fundamental cut-set equation from Chapter II we obtain

$$[a_1^T, a_2^T] \begin{bmatrix} -Q_{CS} & -Q_{CR} & -Q_{CL} & -Q_{CI} \\ U & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_S \\ I_R \\ I_L \\ I_I \end{bmatrix} = 0$$

However, since the chord currents of the normal tree are linearly independent variables,we must have

$$[a_1^T, a_2^T] \begin{bmatrix} -Q_{CS} & -Q_{CR} & -Q_{CL} & -Q_{CI} \\ U & 0 & 0 & 0 \end{bmatrix} = 0$$

or equivalently

$$a_1^T Q_{CR} = 0, \; a_1^T Q_{CL} = 0, \; a_1^T Q_{CI} = 0, \; a_1^T Q_{CS} = a_2^T .$$

Now consider $a^T = [0, a_1^T \theta_{22}]$. Because $a_1 \neq 0$ and θ_{22}

is nonsingular, $a \neq 0$. Then

$$a^T A = [0, a_1^T \mathcal{C}_{22}] \begin{bmatrix} -\mathcal{L}_{11}^{-1} & 0 \\ 0 & -\mathcal{C}_{22}^{-1} \end{bmatrix} \begin{bmatrix} H_{11} & H_{14} \\ -H_{14}^T & H_{44} \end{bmatrix}$$

$$= [a_1^T H_{14}^T, \ -a_1^T H_{44}]$$

$$= [a_1^T(-Q_{CL} + Q_{CR} G_\ell Q_{GR}^T Y^{-1} Q_{GL}), \ -a_1^T(Q_{CR} Z^{-1} Q_{CR}^T)]$$

$$= 0 \ .$$

Also

$$a^T B_1 = [0, a_1^T \mathcal{C}_{22}] \begin{bmatrix} -\mathcal{L}_{11}^{-1} & 0 \\ 0 & -\mathcal{C}_{22}^{-1} \end{bmatrix} \begin{bmatrix} H_{12} & H_{13} \\ -H_{24}^T & H_{34}^T \end{bmatrix}$$

$$= [a_1^T H_{24}^T, \ -a_1^T H_{34}^T]$$

$$= [a_1^T(-Q_{CI} + Q_{CR} G_\ell Q_{GR}^T Y^{-1} Q_{GI}), -a_1^T(Q_{CR} Z^{-1} Q_{VR}^T)]$$

$$= 0 \ .$$

Therefore, the network is uncontrollable by Theorem 3.3. The proof for inductor-only loops follows the same basic procedure but uses the loop equations instead of the cut-set equations.

Another way of stating this result is to say that controllability requires that there must be at least one current source in each capacitor cut-set and at least one voltage source in each inductor loop. The exact number of sources needed in either case is still undetermined. Theorem 3.1, however, places an upper bound on the number required.

An alternate proof of Theorem 3.4 has been given in a recent paper by Narraway [6]. The method used is not based on the general system theory results developed here, but follows from a physical argument depending on the conservation of charge and flux linkages.

We now turn our attention to observability and present the following theorem.

Theorem 3.5:

The dynamical equations (2.1) and (2.8) are unobservable if there exists a non-zero vector a such that, $a^T A^T = 0$, $a^T C^T = 0$, or, equivalently, if the matrix $[A^T, C^T]$ has rank less than n.

PROOF:

From Theorem 2.3 the equations are observable if and only if $P = P'$ has rank n. If there exists $a \neq 0$ such that $a^T A^T = 0$ and $a^T C^T = 0$, then

$$a^T P = a^T P' = [a^T A^T,\ a^T A^T C^T, \dots, a^T A^{T(n-1)} C^T] = 0 .$$

Therefore the rows of P are dependent proving that the rank of P is less than n and the equations are unobservable.

To show the equivalence of the two conditions, consider the following:

If $a^T A^T = 0$, $a^T C^T = 0$, $a \neq 0$ then $a^T [A^T, C^T] = 0$. Furthermore, the rows of $[A^T, C^T]$ are dependent and hence the rank of $[A^T, C^T]$ is less than n. The converse follows by reversing the procedure.

Application of this result to RLC networks yields the following theorem:

Theorem 3.6:

An RLC network, in which edges for the output variables are included in the network graph, is unobservable if it contains any capacitor-only cut-sets and/or inductor-only loops.

PROOF:

If there is a capacitor-only cut-set, Theorem 3.4 proves that there exists a_1, a_2, $a_1 \neq 0$, such that $a_1^T Q_{CR} = 0$, $a_1^T Q_{CL} = 0$, $a_1^T Q_{CI} = 0$, $a_1^T Q_{CS} = a_2^T$.
Now consider $a^T = [0, a_1^T]$

$$a^T A^T = [0, a_1^T] \begin{bmatrix} H_{11} & -H_{14} \\ H_{14}^T & H_{44} \end{bmatrix} \begin{bmatrix} -\mathcal{L}_{11}^{-1} & 0 \\ 0 & -\mathcal{C}_{22}^{-1} \end{bmatrix}$$

$$= [-a_1^T H_{14}^T \mathcal{L}_{11}^{-1} \;,\; -a_1^T H_{44} C_{22}^{-1}]$$

$$= [a_1^T (Q_{CL} - Q_{CR} G_\ell Q_{GR}^T Y^{-1} Q_{GL}) \mathcal{L}_{11}^{-1}, -a_1^T (Q_{CR} Z^{-1} Q_{CR}^T) \mathcal{C}_{22}^{-1}]$$

$$= 0 \;.$$

Also

$$a^T C^T = [0, a_1^T] \left\{ \begin{bmatrix} H_{12} & -H_{13} \\ H_{24}^T & H_{34}^T \end{bmatrix} - \begin{bmatrix} H_{11} & -H_{14} \\ H_{14}^T & H_{44} \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11}^{-1} & 0 \\ 0 & C_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathcal{L}_{12} & 0 \\ 0 & \mathcal{C}_{12}^T \end{bmatrix} \right\}$$

$$= [a_1^T H_{24}^T,\; a_1^T H_{34}^T] - [a_1^T H_{14}^T \mathcal{L}_{11}^{-1} \mathcal{L}_{12},\; a_1^T H_{44} \mathcal{C}_{22}^{-1} \mathcal{C}_{12}^T]$$

$$= [a_1^T(-Q_{CI}+Q_{CR}G_\ell Q_{GR}^T Y^{-1} Q_{GI}),\ a_1^T(Q_{CR}Z^{-1}Q_{VR}^T)]$$

$$- [a_1^T(-Q_{CL}+Q_{CR}G_\ell Q_{GR}^T Y^{-1} Q_{GL})\mathcal{L}_{11}^{-1}\ \mathcal{L}_{12},a_1^T(Q_{CR}Z^{-1}Q_{CR}^T)\mathcal{C}_{22}^{-1}\mathcal{C}_{12}^T]$$

$$= 0\ .$$

Therefore the network is unobservable.

The proof for inductor-only loops follows the same basic procedure but uses the loop equations instead of the cut-set equations.

An alternate statement of this theorem is as follows: A necessary condition for observability is that there is at least one output voltage in each capacitor cut-set and at least one output current in each inductor loop. Theorem 3.2 yields an upper bound on the number required.

Theorem 3.6 also represents a considerable generalization of a theorem given by Narraway [6]. He proved that there exists a $C \neq 0$ such that RC and LG networks containing zero eigenvalues are unobservable. He does not interpret the meaning of the C derived. We have shown that the restriction to two element type networks is not necessary and have essentially given a $C \neq 0$ which makes the network observable.

Alternate proofs of Theorems 3.4 and 3.6 are available in Appendix A. They are of interest since they follow from the basic system theory results developed here but do not require the symbolic formulation of the state equations. Instead, simple physical arguments are used. In addition, a method of constructing an eigenvector associated with a zero eigenvalue of A is given.

CHAPTER IV

TOPOLOGICAL CRITERIA FOR CONTROLLABILITY AND OBSERVABILITY VIA NETWORK TRANSFER FUNCTIONS

The formulation of the state equations of an RLC network is often a labourious task. On the other hand, the determination of network transfer functions is generally much easier. The establishment of controllability and observability would therefore be simpler if these transfer functions could be used instead of the conventional P and Q matrices. The purpose of this chapter is to develop such a method. Topological formulae are given for the appropriate transfer functions and some examples are presented.

4.1 CONTROLLABILITY AND OBSERVABILITY FROM NETWORK TRANSFER FUNCTIONS

The method to be developed in this section utilizes the normal form state equations of the system. The procedure is, therefore, restricted to those networks whose physical state variables occur in normal form state equations. This lack of generality is caused by the generation of non-physical state variables in the process of transforming an improper system into normal form. Such abstract state variables cannot be used in the transfer functions of the network. In the following discussions, we therefore limit our analysis to networks having no capacitor-voltage source loops, and no inductor- current source cut-sets.

The method for controllability uses the numerator matrix of the transfer function matrix from the inputs to the state variables. This transfer function matrix can be obtained from the state equations: $R_1(s) = (sI_n-A)^{-1}B = \frac{1}{d(s)}[adj(sI_n-A)]B$.

The following theorem is a consequence of the controllability matrix and the use of Fadeeva's Method [1]; (also called Souria-Frame Algorithm [1]).

Theorem 4.1:

The n-dimensional, linear, time-invariant dynamical equation

$$\dot{x} = Ax + Bu \qquad (4.1a)$$

$$y = Cx + Du \qquad (4.1b)$$

is completely state controllable if and only if the matrix

$$\hat{H} = [H_o, H_1, \ldots, H_{n-1}]$$

has rank n, where the set of matrices $\{H_i\}$, i=0,1,...,n-1 are obtained from

$$H(s) = H_o s^{n-1} + H_1 s^{n-2} + \ldots + H_{n-2} s + H_{n-1} = [adj(sI_n-A)]B.$$

PROOF:

From previous considerations we know that the equation is controllable if and only if the controllability matrix has rank n, that is $\rho[Q] = n$. We will prove that $\rho[Q] = \rho[\hat{H}]$.

Postmultiply Q by a nonsingular R of the form:

$$R = \begin{bmatrix} I_m & d_1 I_m & d_2 I_m & \cdots & & d_{n-1} I_m \\ 0 & I_m & d_1 I_m & \cdots & & d_{n-2} I_m \\ 0 & 0 & I_m & \cdots & & d_{n-3} I_m \\ \vdots & & & & & \vdots \\ 0 & \cdots & & 0 & I_m & d_1 I_m \\ 0 & \cdots & & & 0 & I_m \end{bmatrix}$$

where I_m is an $m \times m$ unit matrix and $\{d_i\}$, $i=1,2,\ldots,n-1$, is a set of real constants. Let $\bar{H} = QR$, then $\rho[\bar{H}] = \rho[QR] = \rho[Q]$ and the equation is controllable if and only if $\rho[\bar{H}] = n$.

Expansion of QR yields

$$\bar{H} = [B, (A+d_1 I_m)B, \{(A+d_1 I_m)A+d_2 I_m\}B, \ldots,$$

$$\{(\{(A+d_1 I_m)A+d_2 I_m\}A+\ldots)\ A+d_{n-1} I_m\}B]$$

$$= \hat{T} \begin{bmatrix} B & 0 & \cdots & 0 \\ 0 & B & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & B \end{bmatrix}$$

where $\hat{T} = [T_o, T_1, \ldots, T_{n-1}]$

and $T_o = I_n$

$T_1 = A+d_1 I_m = T_o A+d_1 I_m$

$T_2 = (A+d_1 I_m)A+d_2 I_m = T_1 A+d_2 I_m$

$\vdots$

$$T_{n-1} = (\{(A+d_1 I_m)A+d_2 I_m\}A+\ldots)A+d_{n-1}I_m = T_{n-2}A+d_{n-1}I_m \;.$$

Fadeeva's Algorithm for the expansion of $(sI_n-A)^{-1}$, shows that if we let the set $\{d_i\}$, $i=1,2,\ldots,n-1$, be the coefficients of the characteristic polynomial

$$\det(sI_n-A) = s^n+d_1 s^{n-1}+\ldots+d_{n-1}s+d_n$$

then

$$T(s) = T_o s^{n-1} + T_1 s^{n-2} +\ldots+ T_{n-2}s+T_{n-1} = \operatorname{adj}(sI_n-A)$$

and $H(s) = T(s)B$.

Furthermore, $\hat{H} = \bar{H}$ and $\rho[\hat{H}] = \rho[\bar{H}] = \rho[Q]$

The observability of a linear, time-invariant system, in terms of transfer functions, may be treated in a similar fashion to the previous theorem on controllability.

We first augment the system of state equations with the inclusion of an additional n-dimensional input vector v(t), chosen so that the state equations become

$$\dot{x} = Ax + [B,\hat{B}]\begin{bmatrix} u \\ v \end{bmatrix} \qquad (4.2a)$$

$$y = Cx + [D,\hat{D}]\begin{bmatrix} u \\ v \end{bmatrix} \qquad (4.2b)$$

and $\hat{B}$ is nonsingular.

If we set the original inputs equal to zero, and assume no direct transmission, the transfer function from the new inputs v to the outputs becomes

$$M(s) = C(sI_n - A)^{-1} \hat{B}$$

We can now give the dual to Theorem 4.1.

Theorem 4.2:

The n-dimensional, linear, time-invariant dynamical equation

$$\dot{x} = Ax + Bu + \hat{B}v \qquad \hat{B} \text{ nonsingular}$$

$$y = Cx + Du + \hat{D}v$$

is completely state observable if and only if the matrix

$$\hat{M} = [M_0^T, M_1^T, \ldots, M_{n-1}^T]$$

has rank n, where the set of matrices $\{M_i\}$, $i=0,1,\ldots,n-1$, are obtained from

$$M(s) = M_0 s^{n-1} + M_1 s^{n-2} + \ldots + M_{n-2} s + M_{n-1} = C[\mathrm{adj}(sI_n - A)]\hat{B}.$$

PROOF:

The proof follows the same basic steps as in Theorem 4.1. We shall simply point out the differences where they occur.

The equation is observable if and only if the rank of P is n. Consider $\bar{M} = R^T P^T \hat{B}$ where R is as defined in Theorem 4.1. Expansion of $\bar{M}$ yields $\bar{M} = C_d \hat{T}_1 \hat{B}$ where C_d is an n×n block diagonal matrix with all non-zero entries equal to C, and $\hat{T}_1^T = [T_0^T, T_1^T, \ldots, T_{n-1}^T]$ where the T_i, $i=0,1,\ldots,n-1$ are defined in Theorem 4.1. From Fadeeva's Algorithm $M(s) = CT(s)\hat{B}$ and therefore $\hat{M} = \bar{M}$. Finally, since $\hat{B}$ and R are nonsingular

$$\rho[\hat{M}] = \rho[\bar{M}] = \rho[P] .$$

This theorem establishes the use of a transfer function matrix in testing for observability. A physical interpretation is now needed if the method is to be of use in RLC networks. It was previously shown in Chapter III that if there is a current source in parallel with each capacitive twig, a voltage source in series with each inductive link, and no other sources, the state equation is $\dot{x} = Ax + \Lambda_1^{-1} v$. The transfer function matrix from these augmented inputs to the outputs is then $R_2(s) = C(sI_n - A)^{-1}\Lambda_1^{-1}$. Since Λ_1 is nonsingular, this transfer function meets the requirements of Theorem 4.2 and may therefore be used to test the observability of the network.

The method of determining the sub-matrices of $\hat{H}$ and $\hat{M}$ needs to be examined carefully. The correct entries can easily be computed by any one of several network theory methods.

The various procedures occasionally produce results that appear to be different, although they are essentially equivalent. This is due to unavoidable pole-zero cancellations inherent in the computational process [12].

The next theorem proves that these disappearing modes and therefore the actual method of computation is not important in determining the controllability or observability of the system.

Theorem 4.3:

If the numerator matrix G(s) of a transfer function of an n-dimensional, linear, time-invariant system, has a polynomial factor f(s) of degree r, that is,

$$G(s) = f(s)\, K(s)$$

then

$$\rho[G_o, G_1, \ldots, G_{n-1}] = \rho[K_o, K_1, \ldots, K_{n-r-1}]$$

PROOF:

$$G(s) = G_o s^{n-1} + G_1 s^{n-2} + \ldots + sG_{n-2} + G_{n-1}$$

$$= (f_o s^r + f_1 s^{r-1} + \ldots + f_{r-1} s + f_r)$$

$$(K_o s^{n-r-1} + K_1 s^{n-r-2} + \ldots + K_{n-r-2} s + K_{n-r-1}) ,$$

$$f_o \neq 0; \quad r > 0$$

then

$$[G_o, G_1, \ldots, G_{n-1}] = [K_o, K_1, \ldots, K_{n-r-1}]\, F^T$$

Or $\quad \hat{G} = \hat{K}\, F^T$

where:

$$F = \begin{bmatrix} f_oI_m & 0 & \cdots & & & & & & & 0 \\ f_1I_m & f_oI_m & 0 & \cdots & & & & & & 0 \\ \vdots & & & & & & & & & \vdots \\ f_rI_m & f_{r-1}I_m & f_{r-2}I_m & \cdots & f_oI_m & 0 & \cdots & & & 0 \\ 0 & f_rI_m & f_{r-1}I_m & \cdots & & f_oI_m & 0 & \cdots & & 0 \\ 0 & 0 & f_rI_m & \cdots & & & f_oI_m & 0 & \cdots & 0 \\ \vdots & & & & & & & \vdots & & \vdots \\ 0 & \cdots & & & & & & 0 & f_rI_m & f_{r-1}I_m \\ 0 & \cdots & & & & & & & 0 & f_rI_m \end{bmatrix}$$

is the mn×m(n-r), K_i and G_i are n×m and I_m is the m×m unit matrix.

Partition $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$, then $\hat{G}^T = F\hat{K}^T = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}\hat{K}^T$.

F_1 is m(n-r) square and nonsingular, thus $\rho[F] = m(n-r)$ and by elementary row operations

$$N\hat{G}^T = N \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}\hat{K}^T = \begin{pmatrix} NF_1 \\ 0 \end{pmatrix}\hat{K}_1^T .$$

$$\rho[N\hat{G}^T] = \rho[NF_1\hat{K}^T] \quad \text{and, therefore,}$$

$$\rho[\hat{G}] = \rho[\hat{K}]$$

since N and F_1 are nonsingular.

The method of construction of H or M is now clear. First compute all necessary transfer functions, then bring them to a common denominator, and finally, if desired, discard any common numerator terms. Removal of these common terms is not necessary, due to Theorem 4.3, however, dropping any obvious terms will reduce the order of the matrix and simplify the computation of the rank of $\hat{H}$ or $\hat{M}$. The test for controllability or observability then consists of finding the rank of $\hat{H}$ or $\hat{M}$, respectively, and comparing it to the dimension of the system.

It should be noted that this method is not restricted to passive networks. The theory is general, and may therefore be applied to any linear, time-invariant system for which state equations in normal form exist and the number of state variables can be found.

The dimension of the state space or the order of complexity [11], as it is called in circuit theory, is explicit if the state equations are known. However, if we choose to use transfer functions, we are faced with the computation of this quantity. The general problem has not yet been solved [13], however, partial solutions are available. The situation in RLC networks is known [11] and is repeated here.

The order of complexity of an RLC network with independent sources equals the total number of reactive elements, less the number of independent circuits consisting of capacitors only or capacitors and voltage sources, less the number of independent cut-sets consisting of inductors only or inductors and current sources.

This is equivalent to the construction of a normal tree as outlined in Chapter II.

The framework has now been provided to use transfer functions to determine both controllability and observability of a normal form system. This basic relationship has previously been noted in other forms [5,14]. Chen [5] states that a system is controllable (observable) if and only if the rows of $(sI_n-A)^{-1}B$ (columns of $C(sI_n-A)^{-1}$) are linearly independent over the field of complex numbers. We now offer alternate proofs of these theorems that are purely algebraic and are completely contained in the complex frequency domain. Chen's proofs are based upon the assumption that the Laplace transform is a one-to-one linear operator.

<u>Theorem 4.4</u>:

The state equation (4.1a) (dynamical equation (4.1)) is completely state controllable (observable) if and only if the rows of $(sI_n-A)^{-1}B$ (columns of $C(sI_n-A)^{-1}$) are linearly independent over the field of complex numbers.

PROOF:

We will first prove the contrapositive of the controllability part of the theorem. As a preliminary, note that given $\hat{H}$, as defined in Theorem 4.1, $x^T\hat{H} = 0$ is equivalent to the n simultaneous equations $x^TH_i = 0$, $i=0,1,\ldots,n$.

The rows of $(sI_n-A)^{-1}B$ are linearly dependent if and only if the rows of $H(s)=d(s)(sI_n-A)^{-1}B$ are linearly dependent or, equivalently, there exists $x \neq 0$ such that $x^T H(s) \equiv 0$. Furthermore, this is equivalent to $x^T H_i = 0$, $i=0,1,\ldots,n-1$ or $x^T \hat{H} = 0$. Finally, $x^T \hat{H} = 0$, $x \neq 0$ if and only if the rank of $\hat{H}$ is less than n, and from Theorem 4.1 the theorem is proved.

Using $\hat{M}$ from Theorem 4.2, the proof for observability follows in a similar manner after noting that the columns of $C(sI_n - A)^{-1}$ are dependent if and only if the columns of $C(sI_n - A)^{-1}\hat{B}$, $\hat{B}$ nonsingular, are dependent.

4.2 TOPOLOGICAL FORMULAE FOR A HYBRID n-PORT

Consider the linear, time-invariant n-port shown in Figure 4.1. If there exists a tree with V_V as branches and I_I as chords, then by superposition the network can be characterized by the system of hybrid parameters;

$$\begin{bmatrix} V_I \\ I_V \end{bmatrix} = \begin{bmatrix} H_{VI} & H_{VV} \\ H_{II} & H_{IV} \end{bmatrix} \begin{bmatrix} I_I \\ V_V \end{bmatrix} \tag{4.4}$$

where V_I, I_V, I_I and V_V are vectors of the port variables.

Topological formulae for the entries of this matrix can be derived from consideration of the topological formulae for two-ports; Martens [15] has given such formulae. However, since we choose to use simpler notation, the following discussion is necessary.

For the two-port shown in Figure 4.2, Seshu and

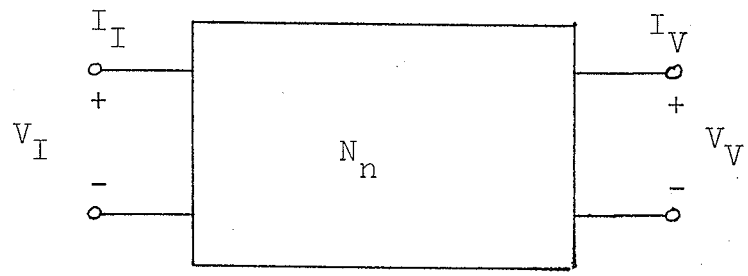

FIGURE 4.1

Reed [7] have given the following topological formulae for the open-circuit impedance and the short-circuit admittance parameters.

$$z_{rr} = \frac{W_{r,r'}}{V} \; ; \; z_{rs} = z_{sr} = \frac{W_{rs',r's} - W_{rs,r's'}}{V} \; ; \; z_{ss} = \frac{W_{s,s'}}{V}$$

$$y_{rr} = \frac{W_{s,s'}}{\Sigma U} \; ; \; y_{rs} = y_{sr} = \frac{W_{rs',r's} - W_{rs,r's'}}{\Sigma U} \; ; \; y_{ss} = \frac{W_{r,r'}}{\Sigma U} \qquad (4.5)$$

where, $W_{a,b}$ is the sum of all two-tree admittance products with the set of vertices a in one connected part and the set of vertices b in the other connected part, V is the sum of all tree admittance products and ΣU is the sum of all tree admittance products in the graph formed by short-circuiting vertices r and r' and s and s' [16].

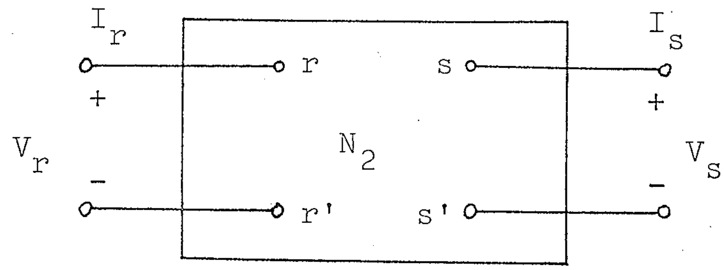

FIGURE 4.2

Equation 4.5 can be used to derive a topological formulae for every possible network function of the two-port. If we define $D_{rs} = W_{rs,r's'} - W_{rs',r's}$ then

$$\left.\frac{V_r}{V_s}\right|_{I_{s=o}} = \frac{D_{rs}}{W_{r,r'}} \; ; \; \left.\frac{V_s}{I_r}\right|_{I_{s=o}} = \frac{D_{rs}}{V}$$

$$\left.\frac{I_s}{V_r}\right|_{V_{s=o}} = \frac{-D_{rs}}{\Sigma U} \; ; \; \left.\frac{I_s}{I_r}\right|_{V_{s=o}} = \frac{-D_{rs}}{W_{s,s'}} \qquad (4.6)$$

$$\left.\frac{V_r}{I_r}\right|_{I_{s=o}} = \frac{W_{r,r'}}{V} \; ; \; \left.\frac{I_s}{V_s}\right|_{V_{r=o}} = \frac{W_{r,r'}}{\Sigma U}$$

This class of network functions used in conjunction with the Superposition Theorem allows us to find the entries of (4.4).

To make the notation less cumbersome, we define a set of standard terminations as follows: all V ports are short-circuited, all I ports are open-circuited. These terminations define a new network N_1. When we find it necessary to use a termination other than that defined as a standard termination, we shall append a superscript, denoting the port in question, to the topological formula. For example, if port k is a V port and is open-circuited instead of short-circuited, D_{rs} would become $D_{rs}^{(k)}$.

The formula for (4.4) can now be given:

$$h_{V_s I_r} = \begin{cases} \dfrac{D_{rs}}{V} & r \neq s \\ \dfrac{W_{r,r'}}{V} & r = s \end{cases} \qquad h_{I_s V_r} = \begin{cases} \dfrac{-D_{rs}^{(rs)}}{\Sigma U^{(rs)}} & r \neq s \\ \dfrac{V^{(r)}}{W_{r,r}^{(r)}} & r = s \end{cases} \tag{4.7}$$

$$h_{V_s V_r} = \frac{D_{rs}^{(r)}}{W_{r,r'}^{(r)}} \qquad h_{I_s I_r} = \frac{-D_{rs}^{(s)}}{W_{s,s'}^{(s)}}$$

where all topological formulae refer to network N_1.

Let us now examine the denominator terms in equation (4.7): V is the tree-admittance product of N_1; $W_{r,r'}^{(r)}$ and $W_{s,s'}^{(s)}$ are the two-tree admittance products of N_1 with r and r', s and s' in separate parts, respectively. It is well known [7] that these quantities can be computed by short-circuiting the two separate parts and finding the tree admittance products of the new network. Therefore, $W_{r,r'}^{(r)} = W_{s,s'}^{(s)} = V$. Similarly, it has been shown [16] that $\Sigma U^{(rs)}$ may be found by short-circuiting the ports and finding the tree admittance products, therefore, $\Sigma U^{(rs)} = V$.

The numerator term $V^{(r)}$ also needs some explanation. It is the tree product of the network formed by opening port r of N_1. This is the same as the two-tree product $W_{r,r'}$ of N_1.

Equation (4.7) can now be written as

$$h_{V_s I_r} = \begin{cases} \dfrac{D_{rs}}{V} & r \neq s \\ \dfrac{W_{r,r'}}{V} & r = s \end{cases} \qquad h_{I_s V_r} = \begin{cases} \dfrac{-D^{(rs)}}{V} & r \neq s \\ \dfrac{W_{r,r'}}{V} & r = s \end{cases} \tag{4.8}$$

$$h_{V_s V_r} = \frac{D^{(s)}_{rs}}{V} \qquad h_{I_s I_r} = \frac{-D^{(s)}_{rs}}{V}$$

where all topological formulae refer to network N_1.

Before the topological formulae developed in this section can be applied to Theorems 4.1 and 4.2, it must be shown that the type and form of the transfer functions needed are included in (4.4).

If we put a current source in parallel with each capacitor in the normal tree and a voltage source in series with each inductor in the normal cotree, a new normal tree can be defined. This tree has as branches all of the edges of the old normal tree plus the voltage sources just added. Similarly, the new normal cotree contains the original normal cotree plus the current sources just added. It is now clear that both the original sources and the new sources qualify as excitations in (4.4) and the transfer functions required by Theorems 4.1 and 4.2 are therefore included in equation (4.4).

Theorems 4.1 and 4.2 also require the transfer function matrix be rational. The topological formulae given do not give such a rational matrix directly. However, multiplication of both numerator and denominator by the appropriate power of s would yield the necessary result. The denominators of the topological formulae are all equal to the sum of all tree admittance products of N_1. Seshu and Reed [7] show that this is equivalent to the determinant of the node admittance matrix.

Martens [15] has shown that if this determinant is multiplied by s^{n_L}, where n_L is the number of inductors in the network, and an appropriate scale factor K, is included, the result is the characteristic polynomial. This is shown by $d(s) = K\, s^{n_L} \det Y_n = K\, s^{n_L}\, V$.

4.3 ILLUSTRATIVE EXAMPLE

Consider the network shown in Figure 4.3. The normal tree contains C_1, C_2 and R_2.

Describe a port on each capacitive branch and inscribe a port in the inductive chord. These additional ports, the input port and the output port are labelled as in Figure 4.4. This is the n-port from which we will compute

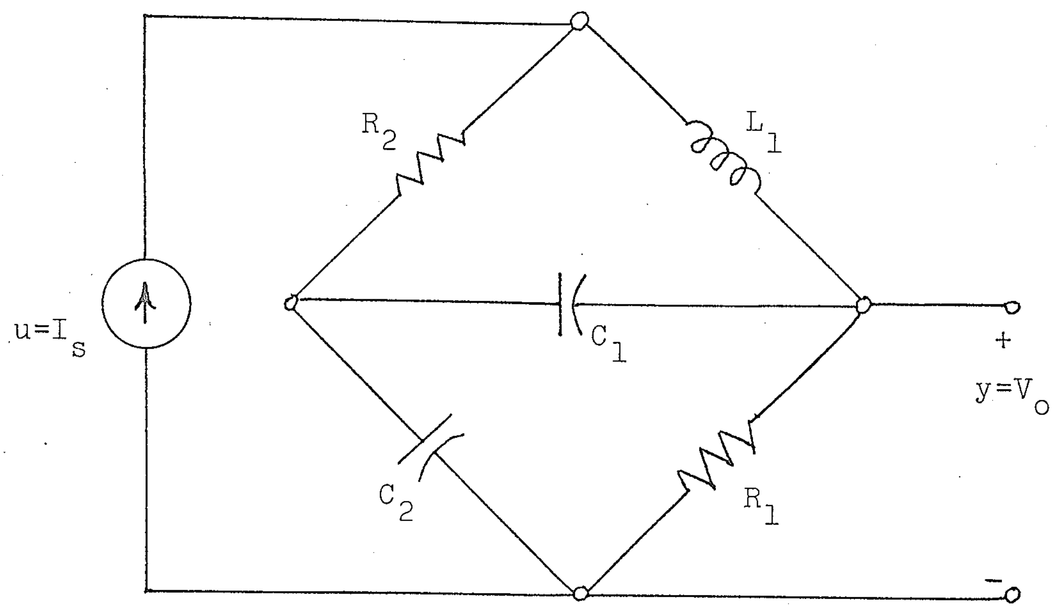

FIGURE 4.3

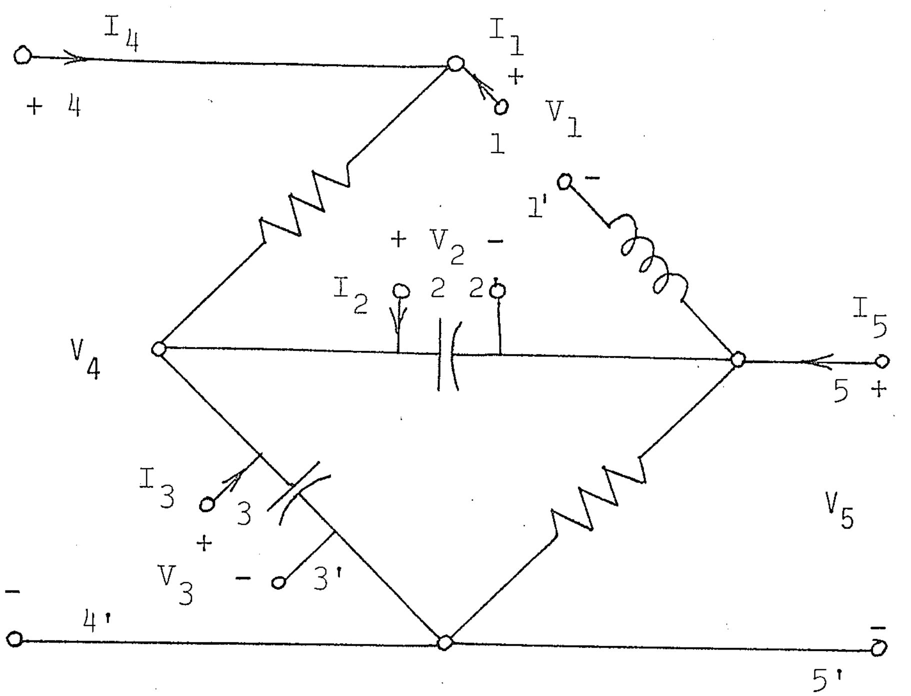

FIGURE 4.4

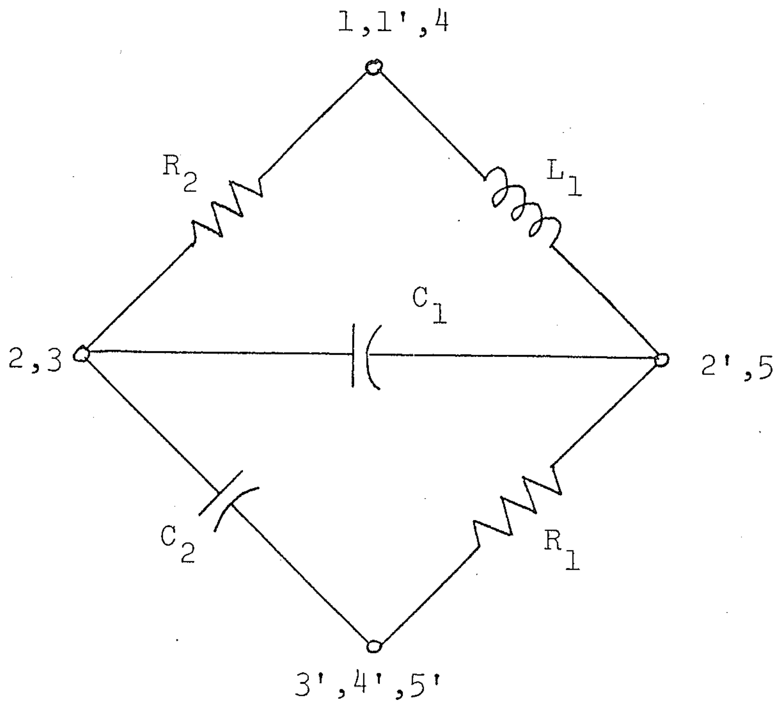

FIGURE 4.5

the transfer functions necessary to determine the controllability and observability of the network. Application of the topological formulae in (4.8) to network N_1, shown in Figure 4.5, yields:

$$h_{V_2 I_4} = \frac{D_{24}}{V} = \frac{W_{42,4'2'} - W_{42',4'2}}{V} = \frac{G_1 G_2 - C_2 \Gamma_1}{V}$$

$$h_{V_3 I_4} = \frac{D_{43}}{V} = \frac{W_{43,4'3'} - W_{43',4'3}}{V} = \frac{s^{-1}\Gamma_1 G_2 + G_1 G_2 + sC_1 G_2 + C_1 \Gamma_1}{V}$$

$$h_{I_1, I_4} = \frac{-D_{41}^{(1)}}{V} = \frac{W_{41',4'1}^{(1)} - W_{41,4'1'}^{(1)}}{V} = \frac{-(sC_1 C_2 \Gamma_1 + C_1 G_1 \Gamma_1 + C_2 G_1 \Gamma_1)}{V}$$

$$h_{V_5, I_2} = \frac{D_{25}}{V} = \frac{W_{25,2'5'} - W_{25',2'5}}{V} = 0$$

$$h_{V_5, I_3} = \frac{D_{35}}{V} = \frac{W_{35,3'5'} - W_{35',3'5'}}{V} = \frac{s^{-1} G_2 \Gamma_1 + sG_2 C_1 + C_1 \Gamma_1}{V}$$

$$h_{V_5, V_1} = \frac{D_{15}^{(1)}}{V} = \frac{W_{15,1'5'} - W_{15',1'5}}{V} = \frac{-G_2 C_2 \Gamma_1}{V}$$

Because there is only one inductor, multiply the numerator and denominator of each of the above formula by Ks. The H(s) and M(s) matrices defined in Theorems 4.1 and 4.2 are then obtained:

$$H(s) = K \begin{bmatrix} 0 \\ C_1 G_2 \\ -C_1 C_2 \Gamma_1 \end{bmatrix} s^2 + K \begin{bmatrix} G_1 G_2 - C_2 \Gamma_1 \\ G_1 G_2 + C_1 \Gamma_1 \\ -C_1 G_1 \Gamma_1 - C_2 G_1 \Gamma_1 \end{bmatrix} s + K \begin{bmatrix} 0 \\ \Gamma_1 G_2 \\ 0 \end{bmatrix}$$

$$M(s) = K \begin{bmatrix} 0 \\ G_2C_1 \\ 0 \end{bmatrix} s^2 + K \begin{bmatrix} 0 \\ C_1\Gamma_1 \\ -G_2C_2\Gamma_1 \end{bmatrix} s + K \begin{bmatrix} 0 \\ G_2\Gamma_1 \\ 0 \end{bmatrix}$$

$$\hat{H} = \begin{bmatrix} 0 & G_1G_2-C_2\Gamma_1 & 0 \\ C_1G_2 & G_1G_2+C_1\Gamma_1 & \Gamma_1G_2 \\ -C_1C_2\Gamma_1 & -C_1G_1\Gamma_1-C_2G_1\Gamma_1 & 0 \end{bmatrix}$$

$$\hat{M} = \begin{bmatrix} 0 & G_2C_1 & 0 \\ 0 & C_1\Gamma_1 & -G_2C_2\Gamma_1 \\ 0 & G_2\Gamma_1 & 0 \end{bmatrix}$$

$$\det \hat{H} = -\Gamma_1G_2(C_1C_2\Gamma_1)(G_1G_2-C_2\Gamma_1)$$

$$= -\Gamma_1^2G_2^2C_1C_2G_1+\Gamma_1^3G_2C_1C_2^2$$

$$\det \hat{M} = 0 \ .$$

The network is therefore always unobservable since $\rho[\hat{M}] \neq n$ and the controllability is dependent upon the element values. If we solve $\det \hat{H} = 0$ we obtain $R_1R_2 = \frac{L_1}{C_2}$. Thus the network is uncontrollable if and only if $R_1R_2 = \frac{L_1}{C_2}$.

CHAPTER V

CONCLUSIONS

The determination of the controllability and observability of a linear, time-invariant system is of major importance. The various theorems and techniques presented in this thesis augment the well known methods, especially for RLC networks.

The extension of controllability and observability to include improper systems allows the characteristics of any set of valid state variables to be examined. This extension is particularly applicable to RLC networks. In this case, the capacitor voltages and inductor currents commonly used as state variables do not always produce state equations in normal form. These new results therefore permit us to study the physical state variables instead of the abstract ones which must be used to obtain state equations in the normal form.

The topological restrictions for controllability and observability based upon placement of input and output ports are important, since they give instant information without calculations of any kind. If the network, with edges included for the output ports, has a capacitor-only cut-set and/or inductor-only loop, then it is immediately both uncontrollable and unobservable and the general testing procedures are not necessary. The method discussed is applicable only to networks having one or more zero natural frequencies. The extension to other modes appears to be difficult, since there are no topological formulae available for determination of the natural frequencies of a network.

The method developed forusing transfer functions to check for controllability and observability is of major interest. This procedure allows us to use topological formulae for the appropriate transfer functions, and thus obtain topological criteria for controllability and observability in RLC networks. The theory applies only to those networks having normal form state equations. However, since this is generally the case, the restriction is not great. The appropriate transfer functions necessary for the extension to improper networks can be derived, but they appear to be abstract and hence cannot be determined topologically.

APPENDIX A

Theorem A

An RLC network is uncontrollable if it contains any capacitor-only cut-sets and/or inductor-only loops.

PROOF:

We will first prove the theorem for capacitor only cut-sets. The capacitors in the cut-set can be put into the following three classifications:

1. All of the capacitors are in the normal tree.
2. Any capacitors not in the normal tree are in a capacitor-only loop.
3. Any capacitors not in the normal tree are in a capacitor-voltage source loop.

The existence and uniqueness of this three-way classification may be proved as follows.

The construction of the normal tree necessitates the inclusion of as many capacitors as possible. Clearly it may contain all capacitors; hence Case 1. If one or more capacitors are not in the normal tree chosen, then they must appear as links and, therefore, form loops with sets of tree branches. The twigs in such loops must be voltage sources or capacitors, for, if they were not, a new normal tree could be defined by including the capacitor link and deleting one of the original twigs that was not a capacitor or voltage source. This, however, is a contradiction that the original tree was a normal tree. Hence, any capacitors not in the normal tree must be in capacitor-only or capacitor-voltage source loops; hence Cases 2 and 3. This proves the three-way classification.

In each case Kirchhoff's current law for the cut-set yields $i_{c_1} + i_{c_2} + \ldots + i_{c_r} = 0$, where r equals the number of capacitors in the cut-set. This implies that the derivatives of the capacitor voltages are dependent. That is

$$C_1 \frac{dv_{c_1}}{dt} + C_2 \frac{dv_{c_2}}{dt} + \ldots + C_r \frac{dv_{c_r}}{dt} = 0 \qquad \text{(A.1)}$$

Also note that the state equations are generally in the improper form, $\dot{x} = Ax + B_1u + B_2\dot{u}$, in all cases.

We now consider each case separately.

1. Since all of the capacitors are in a normal tree, their voltages are valid state variables and (A.1) becomes

$$C_1\dot{x}_1 + C_2\dot{x}_2 + \ldots + C_r\dot{x}_r = 0$$

Therefore, a subset of the derivatives of the state variables is dependent and, hence, the whole set is dependent.

2. The existence of capacitor-only loops allows us to obtain, via Kirchhoff's voltage law, a linear dependence of the capacitor link voltages in terms of the capacitor twig voltages. Subsequently, from (A.1),

the linear dependence of a subset of the derivatives of the network state variables is obtained, and hence the whole set is dependent.

3. Kirchhoff's voltage law around the capacitor voltage source loop shows a dependence of the capacitor link voltages on the capacitor twig voltages and voltage sources. Substitution into (A.1) yields an equation involving the derivatives of the sources and a subset of the state variables. If we constrain the inputs such that $\dot{u} \equiv 0$, the state equations reduce to $\dot{x} = Ax + B_1 u$, and the derivatives of the state variables are dependent.

In each of the three cases we have shown that if $\dot{u} \equiv 0$, the derivatives of the state variables are dependent. Hence there exists $a \neq 0$ such that $a^T \dot{x} = 0$, and hence

$$a^T A x + a^T B_1 u = 0 \tag{A.2}$$

Since the state variables and inputs are linearly dependent, (A.2) requires $a^T A = 0$ and $a^T B_1 = 0$, and the network is uncontrollable by Theorem 3.1.

We should note that the use of $\dot{u} \equiv 0$ is not restrictive in any way. The equations are valid for all u. We simply choose this particular case in order to find a condition on the system matrices which are independent of the form of the input.

The inductor-only loop case follows in a similar manner by considering Kirchhoff's voltage law for the loop.

Theorem A.2:

An RLC network, in which edges for the output variables are included in the network graph, is unobservable if it contains any capacitor-only cut-sets and/or inductor-only loops.

PROOF:

Consider a network having a capacitor-only cut-set as shown in Figure A.1. It is sufficient to show that the network is unobservable for a particular input and a particular initial state. Choose the input $u(t) \equiv 0$. Then the capacitive twig voltages and inductive link currents of a normal tree yield state equations in the normal form.

We now consider a specific initial state and show that it is also a solution of the state equation. The initial state is specified as follows.

At least one of the cut-set capacitors is a twig in the normal tree. Assume C_1 with the cut-set reference defined by the voltage of C_1, let $v_{c_1} = K, K>0$, and let the other cut-set capacitive twig voltages be equal to K if their reference is the same as the cut-set, and equal to -K if the reference is opposite to the cut-set. Let all other capacitive twig voltages and inductive link currents be zero.

When we say that this initial state is also a solution, we require that the values of the state variables remain constant for all future time. We must therefore, modify the specification of the state previously given by replacing equal by identically equal. To show that this state is also a solution, we must prove that Kirchhoff's voltage and current laws and the element voltage-current relationships are satisfied.

From the proof of Theorem A.1, we know that all capactive links must be in capacitor-only loops and their voltages are linear combinations of the capacitive twig voltages. Capacitive link voltages in N_1 or N_2 which form loops entirely within these sub-networks must be identically zero by KVL. Capacitive link voltages in N_1 or N_2 which form loops containing any cut-set capacitive link voltages must also be identically zero since the loop reference agrees with and disagrees with the cut-set reference an equal number of times. Cut-set capacitive link voltages must be K if their reference agrees with the cut-set reference and -K if the reference is opposite to the cut-set since the loop, which has reference the same as the capacitor voltage, contains an odd number of the cut-set capacitor twig voltages. Similarly, since all inductive twigs are in inductor-only cut-sets, all inductor currents are identically zero.

We have shown that all capacitor voltages and inductor currents are constant. Hence, the element relationships demand that the complementary variables, that is, inductor voltages and capacitor currents, are identically zero.

Because the cut-set capacitor currents are identically zero, N_1 and N_2 can be considered as two separate unconnected networks. Furthermore, they essentially become completely resistive and, hence, all voltages and currents are identically zero, automatically satisfying Kirchhoff's voltage and current laws as well as Ohm's law for the resistors.

Since $x(0)$ is a constant non-zero solution , $\dot{x}(0) = Ax(0) = 0$ and, furthermore, any responses in N_1 or N_2 are identically zero. Thus $Cx(0) = y = 0$. The network is then unobservable by Theorem 3.5. A similar discussion applies for the case of inductor-only loops.

An interesting aspect of this theorem is that it gives a method of constructing an eigenvector of a zero eigenvalue of A.

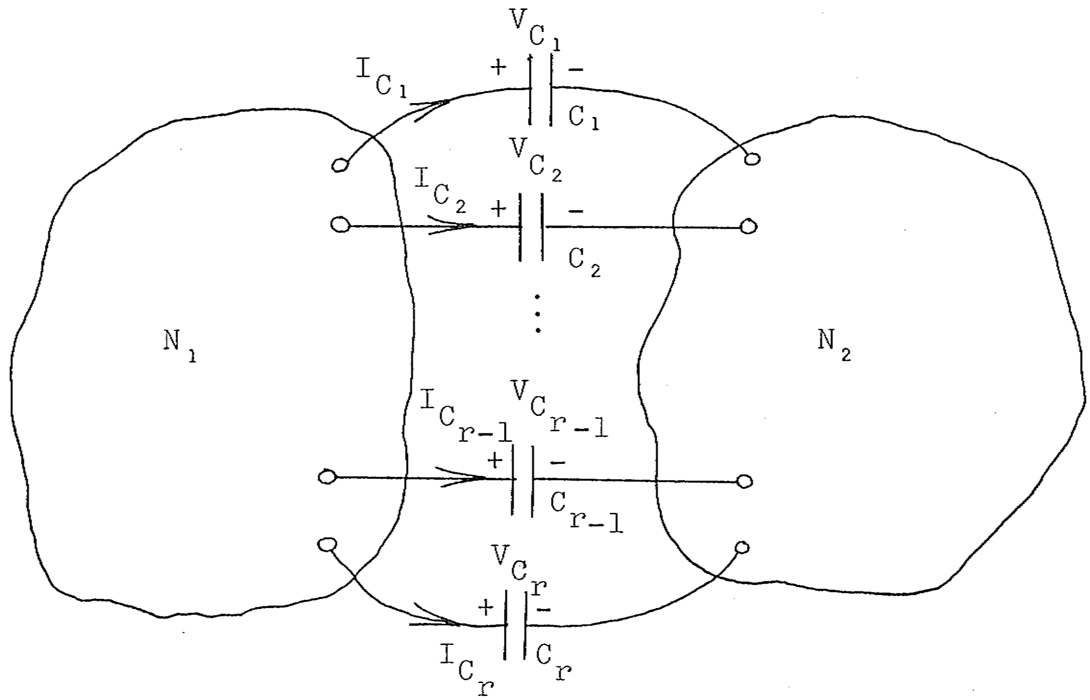

FIGURE A.1

NETWORK CONTAINING CAPACITOR ONLY CUT-SET

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