# Closed Ideals in C(X) and Related Algebraic Structures

by

Ross Stokke

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# Abstract

Given a topological space $X$, the ring $C(X)$ of continuous real-valued functions on $X$ is endowed with what is called the 'uniform metric'. The closed ideals of $C(X)$ in this metric are of much interest, and a new, purely algebraic characterization of these ideals is provided. The result is applied to describe the real maximal ideals of $C(X)$, and to characterize several types of topological spaces. A $\Phi$-algebra is an archimedian lattice-ordered algebra closely related to $C(X)$. $z$-ideals in $\Phi$-algebras are examined, and as an application to this study, several conditions equivalent to regularity in a $\Phi$-algebra are obtained. A uniform metric may also be placed upon a $\Phi$-algebra, and in this metric the closed ideals of a $\Phi$-algebra have received a fair amount of research attention. We give necessary and sufficient conditions to ensure that an ideal of a $\Phi$-algebra is closed, and for two broad classes of $\Phi$-algebras show that these conditions are equivalent, thus generalizing our characterization from the $C(X)$ case.

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# Contents

# Chapter 1

# Introduction

## 1.1 Introduction and Organization of the Thesis

This work divides itself naturally into two parts. The first part of the thesis is comprised of chapters two and three, which take place in a $C(X)$ setting. The final chapters make up part two of the thesis, in which the structure under consideration is a type of lattice-ordered algebra, called a $\Phi$-algebra.

Beginning with a topological space $X$, the ring $C(X)$ of all continuous real-valued functions on $X$ is endowed with what is called the 'uniform metric'. In [A1] the algebraic notion of a strongly divisible ideal was examined by F. Azarpanah, and in the third chapter of this thesis, we employ this concept to characterize the (uniformly) closed ideals in $C(X)$ as precisely the strongly divisible $z$-ideals in $C(X)$. This result is then applied to describe the real maximal ideals in $C(X)$, and to characterize realcompact and pseudocompact spaces. Beyond this we show that a space $X$ for which every countably generated ideal in $C(X)$ is principal is necessarily finite, and provide two new $C(X)$-type characterizations of weakly Lindelöf spaces. The theory of rings of continuous functions which is required for chapter three, is briefly outlined in the second chapter of this work.

A $\Phi$-algebra is an archimedian lattice-ordered algebra with properties much like $C(X)$, where $X$ is a topological space. A uniform metric may also be defined on a $\Phi$-algebra, and our principal motivation in studying $\Phi$-algebras is to generalize our characterization of the closed ideals in $C(X)$ to a $\Phi$-algebra setting. In the final chapter of this exposition, necessary and sufficient conditions for an ideal to be closed are obtained, and it is shown that for two relatively extensive classes of $\Phi$-algebras, these conditions agree; thus in these cases, our attempt to characterize the closed ideals of a $\Phi$-algebra is successful.

As in the $C(X)$ case, the $\Phi$-algebra results described above employ the notion of $z$-ideal, and in chapter five we take the opportunity to examine in some depth the role of $z$-ideals in $\Phi$-algebras. Using $z$-ideals, we derive and extend some of the results found in [HJ], and thereby show that $z$-ideals in $\Phi$-algebras are natural and useful. As an application to our study of $z$-ideals, we consider what we call a P-algebra - a $\Phi$-algebra in which every prime $l$-ideal is maximal - a generalization of $C(X)$ where $X$ is a P-space. Several equivalent conditions for a $\Phi$-algebra to be a P-algebra are given, for example if $\Phi$-algebra $A$ is uniformly closed, then $A$ is a P-algebra if and only if it is regular.

Chapter four provides a survey of the theory of $\Phi$-algebras to be used in the fifth and sixth chapters of the thesis. The symbol $\diamondsuit$ will be used to denote the end of a proof. We remark that attempts have been made to keep this work essentially self-contained.

# Chapter 2

# An Introduction to Rings of Continuous Functions

The intent of this chapter is to provide a brief exposition of the terminology and notation to be used in the ensuing portions of this work. All of the material in this chapter is well-known and can be found without exception in the textbooks [GJ] by L. Gillman and M. Jerison, and [PW] by J. Porter and R.G. Woods. A reader unfamiliar with the study of rings of continuous functions, who is perhaps looking for a sense of context, may find that in the prefaces of the aforementioned books, and the survey article [H2] by M. Henriksen.

## 2.1 The Ring $C(X)$

If $X$ is a topological space, then $C(X)$ denotes the ring (under pointwise operations of addition and multiplication) of continuous real-valued functions on $X$. $C^*(X)$ denotes the subring of bounded elements of $C(X)$. If $f, g \in C(X)$, then $f \vee g, f \wedge g \in C(X)$, where

$$(f \vee g)(x) = max\{f(x), g(x)\} \qquad (x \in X),$$

$$(f \wedge g)(x) = min\{f(x), g(x)\} \qquad (x \in X).$$

It is shown in Chapter three of [GJ] that if $X$ is any topological space, then there is a Tychonoff space, (that is a completely regular Hausdorff space), $X'$ such that $C(X)$ and $C(X')$ are ring isomorphic. Unless it is explicitly stated otherwise, henceforth all topological spaces will be assumed to be Tychonoff.

For $f \in C(X)$, the *zero − set* of $f$ is $Z(f) = f^{\leftarrow}(0) = \{x \in X : f(x) = 0\}$, and $coz f = X \backslash Z(f)$ is called the *cozero − set* of $f$. The collection of all zero-sets of $X$, $\mathcal{Z}(X) = \{Z(f) : f \in C(X)\}$ is closed under finite union and countable intersection; consequently $\mathcal{Z}(X)$ is a lattice under the set containment relation $\subset$. A filter on the lattice of zero-sets is called a *z − filter*. Thus a $z$-filter is a collection $\mathcal{F}$ of zero-sets of $X$ satisfying

(i) $\emptyset \notin \mathcal{F}$,

(ii) $Z_1, Z_2 \in \mathcal{F}$, implies $Z_1 \cap Z_2 \in \mathcal{F}$, and

(iii) $Z_1 \in \mathcal{F}$, $Z \in \mathcal{Z}(X)$ and $Z_1 \subset Z$, implies $Z \in \mathcal{F}$.

A maximal $z$-filter is called a *z − ultrafilter*.

A space $X$ is completely regular (but not necessily Hausdorff) if and only if its zero-sets comprise a base for the closed subsets of $X$; equivalently if and only if its cozero-sets comprise a base for the open subsets of $X$.

A function $f \in C(X)$ is a unit of the ring $C(X)$ if and only if $Z(f) = \emptyset$, and is a divisor of zero if and only if $Z(f)$ has non-empty interior.

The following which is [GJ, 1D] is used frequently in chapter three.

**2.1.1 Proposition** *Let $f, g \in C(X)$.*

*(1) If $Z(f)$ is a neighborhood of $Z(g)$, then $f$ is a multiple of $g$ - that is $f = hg$ for some $h \in C(X)$.*

*(2) If $|f| \leq |g|^r$ for some real $r > 1$, then $f$ is a multiple of $g$.*

Note that the association $f \mapsto Z(f)$ may be regarded as a surjective mapping from $C(X)$ onto $\mathcal{Z}(X)$, and as such the following notation is used. If $A \subset C(X)$ and $\mathcal{S} \subset \mathcal{Z}(X)$, then $Z[A] = \{Z(f) : f \in A\}$, and $Z^{\leftarrow}[\mathcal{S}] = \{f \in C(X) : Z(f) \in \mathcal{S}\}$. An ideal $I$ of $C(X)$ is called a $z-$*ideal* if $Z(f) \in Z[I]$ implies $f \in I$ - that is $I = Z^{\leftarrow}[Z[I]]$. It is easily seen that maximal ideals in $C(X)$ are $z$-ideals. The following theorem which is [GJ, 2.3] describes the relationship between ideals of $C(X)$ and $z$-filters on $X$.

**2.1.2 Theorem** *a) If $I$ is an ideal in $C(X)$, then $Z[I]$ is a z-filter on $X$.*
*b) If $\mathcal{F}$ is a z-filter on $X$, then $Z^{\leftarrow}[\mathcal{F}]$ is a z-ideal in $C(X)$.*

If $M$ is a maximal ideal of $C(X)$, and $\mathcal{U}$ is a $z$-ultrafilter on $X$, then $Z[M]$ is a $z$-ultrafilter on $X$ and $Z^{\leftarrow}[\mathcal{U}]$ is a maximal ideal in $C(X)$. Thus there is a one-to-one correspondence between the maximal ideals in $C(X)$ and the $z$-ultrafilters on $X$.

An ideal $I$ in $C(X)$ is *fixed* if $\bigcap Z[I] \neq \emptyset$, otherwise $I$ is *free*. The fixed maximal ideals of $C(X)$ are precisely the sets

$$M_p = \{f \in C(X) : p \in Z(f)\} \qquad (p \in X).$$

It is not difficult to see that for each $p \in X$, $\bigcap Z[M_p] = \{p\}$, [GJ, 4.6].

A space $X$ is $C-$*embedded* (resp. $C^*-$*embedded*) in a space $Y$ containing $X$ if for every $f \in C(X)$ (resp. $f \in C^*(X)$), there is a $F \in C(Y)$ (resp. $F \in C^*(Y)$ ) such that $F|X = f$.

## 2.2 The Stone-Čech Compactification

In this section we briefly describe the Stone-Čech compactification $\beta X$ of a space $X$ in the manner that will most benefit us in chapter three. Details of the construction may be found in chapter six of [GJ] and chapter four of [PW].

Adjoin to $X$ one new point for each free maximal ideal of $C(X)$. Thus an index set for the maximal ideals, denoted $\beta X$, is formed; the maximal ideals of $C(X)$ are precisely $\{M^p : p \in \beta X\}$. As noted earlier the fixed maximal ideals of $C(X)$ are $\{M_p : p \in X\}$, hence $X$ forms a ready-made index set for the fixed maximal ideals of $C(X)$, and for $p \in X$, $M^p$ and $M_p$ are used interchangeably.

$\beta X$ is topologized in such a way that it is a compact Hausdorff space containing (a homeomorphic copy of) $X$ densely, (that is $\beta X$ is a compactification of $X$), with the additional property that

$$(*) \qquad X \text{ is } C^* - \text{embedded in } \beta X.$$

As a compactification of $X$, $\beta X$ is unique in the sense that if $\gamma X$ is any other compactification of $X$ in which $X$ is $C^*$-embedded, then there is a homeomorphism from $\beta X$ onto $\gamma X$ that fixes $X$ pointwise. Equivalently one might say that $\beta X$ is the unique compactification of $X$ such that $C^*(X)$ and $C(\beta X)$ are ring isomorphic via the map $C(\beta X) \to C^*(X) : f \mapsto f|X$. $\beta X$ is called the *Stone-Čech compactification* of $X$.

Evidently,

$$X \text{ is compact if and only if } X = \beta X.$$

Hence $X$ is compact if and only if every maximal ideal of $C(X)$ is fixed.

To conclude this chapter we endeavor to define realcompactness and describe the Hewitt realcompactification of $X$. For more on realcompact spaces, the reader is referred to chapters five and eight of [GJ], and chapter five of [PW].

If $I$ is an ideal of $C(X)$, then for each $f \in C(X)$, $I(f)$ will denote the member of $C(X)/I$ for which $f$ is a coset representative. If $M$ is a maximal ideal of $C(X)$, then $C(X)/M$ is a totally ordered field containing a copy of the real field $\mathbf{R}$ via the embedding map $\mathbf{R} \to C(X)/M$ given by $r \mapsto M(r)$. $M$ is called *real* or *hyper-real* according as, with respect to the above map, $C(X)/M \cong \mathbf{R}$ or $C(X)/M$ contains $\mathbf{R}$ properly. The following theorem [GJ, 5.14] characterizes real ideals in terms of their corresponding $z$-ultrafilters.

**2.2.1 Theorem** *The following are equivalent for any maximal ideal $M$ in $C(X)$.*

*(1) $M$ is real.*

*(2) $Z[M]$ is closed under countable intersection.*

*(3) $Z[M]$ has the countable intersection property.*

Let $vX = \{p \in \beta X : M^p \text{ is real}\}$, and give $vX$ the subspace topology induced by $\beta X$. Every fixed maximal ideal is real, [GJ, 5.6], hence $X \subset vX \subset \beta X$, and $X$ is dense in $vX$. $X$ is called *realcompact* if every free maximal ideal of $C(X)$ is hyper-real- that is

$$X \text{ is realcompact if and only if } X = vX.$$

It can be shown that (see for example chapter eight of [GJ]) $vX$ is realcompact and is unique (up to a homeomorphism fixing $X$ pointwise) amongst those realcompact spaces containing $X$ densely with respect to the propery

$$(**) \qquad X \text{ is } C-\text{embedded in } vX.$$

Equivalently, $vX$ is the unique realcompact space containing $X$ densely such that the map $C(vX) \to C(X) : f \mapsto f|X$ is a ring isomormphism. $vX$ is called the *Hewitt realcompactification of* $X$.

A space $X$ is *pseudocompact* if every continuous real-valued continuous function on $X$ is bounded - that is $C(X) = C^*(X)$. In light of properties (*) of $\beta X$ and (**) of $vX$, it is evident that

$$X \text{ is pseudocompact if and only if } vX = \beta X.$$

Although this survey does not exhaust the theory of rings of continuous functions to be used in this exposition, we are now in a position to begin chapter three.

# Chapter 3

# Closed Ideals in $C(X)$

Beginning with a (Tychonoff) space $X$, a metric $\rho$ on $C(X)$ is defined by

$$\rho(f,g) = sup\{|f(x) - g(x)| \wedge 1 : x \in X\}, \qquad (f,g \in C(X)).$$

It is clear that sequence convergence in $(C(X), \rho)$ is uniform convergence in the ordinary sense, and as such $\rho$ is called the ***uniform metric*** on $C(X)$ and its induced topology is the ***uniform topology***, or ***u-topology*** on $C(X)$. For more information on the uniform topology the reader is referred to [Hew] in which $\rho$ was introduced by E. Hewitt. In the sequel, all topological properties of $C(X)$ will be with respect to the uniform topology.

If $K$ is a compact space then the uniform topology on $C(K)$ coincides with the supremum norm topology on $C(K)$. It is well-known that for compact $K$, $C(K)$ with the supremum norm is a Banach algebra. In contrast, if $X$ is an arbitrary space, then $C(X)$ is a complete topological vector space in the uniform metric, but is not in general a topological algebra. Indeed, in the absence of pseudocompactness, multiplication in $C(X)$ is not continuous, a fact easily derived from (3.1.6). Our chief concern will be with the closed ideals of $C(X)$.

If $K$ is a compact space then the closed ideals of $C(K)$ are precisely the intersections of maximal ideals of $C(K)$, [GJ, 4O]. In the case of an arbitrary space $X$, a characterization of the closed ideals of $C(X)$ was achieved by Nanzetta and Plank in [NP], however their their description, (unlike the one quoted above for closed ideals of $C(K)$, where $K$ is compact), is highly non-algebraic in nature. In the next section a new, purely algebraic characterization of the closed ideals of $C(X)$ is given. The result is then applied to describe the real maximal ideals of $C(X)$ and to characterize pseudocompact spaces. Where there is overlap with [NP], the proofs given are independent of, and perhaps simpler than those found in that paper.

The principal tool to be used in this chapter is the concept of strong divisibility which was introduced by F. Azarpanah in [$A_1$] and used there to characterize Lindelöf spaces. In section two, the results of 3.1 are used to show that a space $X$ for which every countably generated ideal of $C(X)$ is principal is necessarily finite. The final section of the chapter contains a characterization of weakly Lindelöf spaces in which a restricted type of strongly divisible ideal is employed.

## 3.1 Closed Ideals and Strong Divisibility

**3.1.1 Definition:** *An ideal $I$ of a commutative ring $R$ is called strongly divisible, (s.d.), if for every countable subset $\{a_n : n \in \mathbf{N}\}$ of $I$ there is an $a \in I$ and a subset $\{b_n : n \in \mathbf{N}\}$ of $R$ such that for each $n \in \mathbf{N}$, $ab_n = a_n$ .*

Thus $I$ is strongly divisible if for every countable subset $C$ of $I$, the elements of $C$ possess a common divisor in $I$. One immediately observes that any principal ideal is strongly divisible, and any countably generated strongly divisible ideal is principal. The first part of the following theorem is by F. Azarpanah and is found in [A1]. Its proof is not long and so for the sake of completeness we choose to include it.

**3.1.2 Theorem:** *Let $X$ be a Tychonoff space.*

*1. [Azarpanah] If $I$ is a z-ideal of $C(X)$ such that $Z[I]$ is closed under countable inter-*

*section, then $I$ is strongly divisible.*

*2. If $I$ is strongly divisible, then $Z[I]$ is closed under countable intersection.*

**Proof:** 1. Let $(f_n) \subset I$. By Weierstrass,

$$g = \sum_{n=1}^{\infty} 2^{-n} \frac{f_n^{2/3}}{1 + f_n^{2/3}}$$

belongs to $C(X)$ and clearly $Z(g) = \bigcap_{n=1}^{\infty} Z(f_n)$. $Z[I]$ is closed under countable intersection, so $Z(g) \in Z[I]$. $I$ is a $z$-ideal so $g \in I$. But for each $n$, $g \geq 2^{-n}\frac{f_n^{2/3}}{1+f_n^{2/3}}$, and therefore $|f_n| \leq |2^n(1 + f_n^{2/3})g|^{3/2}$. By (2.1.1), each $f_n$ is a multiple of $2^n(1 + f_n^{2/3})g$, hence each $f_n$ is a multiple of $g$. Therefore $I$ is strongly divisible.

2. Let $(f_n) \subset I$, so that $\{Z(f_n) : n \in \mathbf{N}\}$ is an arbitrary subset of $Z[I]$. $I$ is strongly divisible, so take $g \in I$ such that for each $n$, $g$ divides $f_n$. It is clear then that $Z(g) \subset \bigcap_{n=1}^{\infty} Z(f_n)$. But $Z[I]$ is a $z$-filter containing $Z(g)$, so $\bigcap_{n=1}^{\infty} Z(f_n) \in Z[I]$. $\diamond$

**3.1.3 Corollary** *1. A maximal ideal of $C(X)$ is real if and only if it is strongly divisible.*

*2.A space $X$ is pseudocompact if and only if every maximal ideal of $C(X)$ is strongly divisible.*

**Proof:** 1. A maximal ideal $M$ of $C(X)$ is real if and only if $Z[M]$ is closed under countable intersection (2.2.1) if and only if $M$ is strongly divisible (by the above theorem).
2. $X$ is pseudocompact if and only if $\upsilon X = \beta X$ if and only if every maximal ideal is real. $\diamond$

If $A$ is a subset of $C(X)$ then $\overline{A}$ will denote its (uniform) closure in $C(X)$.

**3.1.4 Theorem** *The following are equivalent for an ideal $I$ of $C(X)$.*

*1. $I$ is closed (in the uniform topology on $C(X)$).*

*2. $I$ is a strongly divisible $z$-ideal.*

**Proof:** 1. $\Rightarrow$ 2. Let $I$ be a closed ideal of $C(X)$. Suppose $f \in C(X)$ and $g \in I$ are such that $Z(f) = Z(g)$. For every positive integer $n$, let $h_n = [(f - 1/n) \vee 0] + [(f + 1/n) \wedge 0)]$. Then for each $n$, $Z(h_n) = f^{\leftarrow}[-1/n, 1/n]$, so $Z(g) = Z(f) \subset int Z(h_n)$. By (1.1.1) each $h_n$ is a multiple of g, hence each $h_n \in I$. But $|f - h_n| \leq 2/n \to 0$ as $n \to \infty$, so $h_n$ converges to $f$ uniformly (i.e. $h_n$ converges to $f$ in the uniform topology). Since $I$ is closed it follows that $f \in I$, proving that $I$ is a $z$-ideal. It remains to prove that $I$ is strongly divisible.

Let $(f_n)$ be a countable subset of $I$. By (3.1.2) , to prove strong divisibility it suffices to show that $\bigcap_{n=1}^{\infty} Z(f_n) \in Z[I]$. Let $g = \sum_1^{\infty} |f_n| \wedge 2^{-n}$, which by Weierstrass belongs to $C(X)$. If for each $n \in \mathbf{N}$, $g_n = \sum_{k=1}^{n} |f_k| \wedge 2^{-k}$, then it is clear that the sequence $(g_n)$ converges to $g$ uniformly. But for each $n$, $Z(g_n) = \bigcap_{k=1}^{n} Z(f_k) \in Z[I]$, and $I$ is a $z$-ideal, so each $g_n \in I$. $I$ is closed, so $g \in I$ and therefore $\bigcap_{n=1}^{\infty} Z(f_n) = Z(g) \in Z[I]$.

2. $\Rightarrow$ 1. Let $\bar{I}$ denote the closure of $I$ in $C(X)$ and let $f \in \bar{I}$. Then for every positive integer $n$, take $f_n \in I$ such that $|f - f_n| \leq 1/n$. If $x \in \bigcap_{n=1}^{\infty} Z(f_n)$, then for each $n$, $|f(x)| = |f(x) - f_n(x)| \leq 1/n$ and therefore $x \in Z(f)$. Hence $\bigcap_{n=1}^{\infty} Z(f_n) \subset Z(f)$. By (3.1.2), $\bigcap_{n=1}^{\infty} Z(f_n) \in Z[I]$ and thus $Z(f) \in Z[I]$. But $I$ is a $z$-ideal, so $f \in I$. Thus $I = \bar{I}$ and $I$ is closed. $\diamondsuit$

Earlier it was claimed that our characterization of closed ideals in $C(X)$ was of a purely algebraic nature. Although the definition of $z$-ideal given in chapter 2 may not have seemed especially algebraic, [GJ, 4A] asserts that the following algebraic condition is necessary and sufficient for an ideal $I$ in $C(X)$ to be a $z$-ideal:
Given $f \in C(X)$, if there exists $g \in I$ such that $f$ belongs to every maximal ideal containing $g$, then $f \in I$.

The following extends Corollary 4.3 of [NP]. We note that by a well-known theorem of algebra, if $I$ is an ideal in $C(X)$ such that $C(X)/I$ is isomorpic to the real field $\mathbf{R}$, then $I$ is necessarily maximal.

**3.1.5 Corollary** *The following are equivalent for an ideal $I$ of $C(X)$.*

1. *$I$ is real.*
2. *$I$ is a closed maximal ideal of $C(X)$.*
3. *$I$ is a maximal closed ideal of $C(X)$.*
4. *$I$ is a maximal strongly divisible ideal of $C(X)$.*
5. *$I$ is a strongly divisible maximal ideal of $C(X)$.*

**Proof:** The equivalence of 1., 2., and 5. is clear from (3.1.3) and (3.1.4), that 2. implies 3. is obvious.

3. $\Rightarrow$ 4. If $I \subset J$, where $J$ is strongly divisible, then $Z^{\leftarrow}[Z[J]]$ is a strongly divisible $z$-ideal (by 3.1.2), hence closed by (3.1.4), with $I \subset J \subset Z^{\leftarrow}[Z[J]]$. By 3. $I = Z^{\leftarrow}[Z[J]]$, hence $I = J$. Therefore $I$ is a maximal strongly divisible ideal of $C(X)$.

4. $\Rightarrow$ 3. Were $I \subset J$, with $J$ closed then $J$ is strongly divisible by (3.1.4), hence $I = J$ by 4.

3. $\Rightarrow$ 5. Supposing $I$ is not maximal, take $M$ a maximal ideal of $C(X)$ with $I$ properly contained in $M$. If $f \in M \backslash I$, then we claim that

$$\mathcal{B} = \{Z(f) \bigcap Z(g) : g \in I\}$$

is a base for a $z$-filter $\mathcal{A}$ on $X$ that is closed under countable intersection. To see this, observe that $\mathcal{A} \subset Z[M]$, $Z[M]$ a $z$-filter so $\emptyset \notin \mathcal{A}$. $I$ is closed, therefore strongly divisible, hence by (3.1.2) $Z[I]$ is closed under countable intersection; it follows that $\mathcal{B}$ is closed under countable intersection, whence $\mathcal{A}$ is closed under countable intersection.

From (3.1.2) $Z^{\leftarrow}[\mathcal{A}]$ is a strongly divisible $z$-ideal, hence a closed ideal of $C(X)$ containing $I$. But $f \in Z^{\leftarrow}[\mathcal{A}]$, $f \notin I$, so this containment is proper, a contradiction to 3. $\Diamond$

That an ideal is merely strongly divisible does not alone guarantee that it is closed, that is, strongly divisible ideals need not be $z$-ideals. For example, if $\mathbf{i}$ denotes the identity funtion on $\mathbf{R}$, then the principal, (hence strongly divisible) ideal $(\mathbf{i})$ is not a $z$-ideal in $C(\mathbf{R})$; see [GJ, 2.4].

The following is a slight extension of Theorem 2.1 of [NP]. A shorter though less interesting proof of their final implication is given.

**3.1.6 Corollary** *The following are equivalent:*

1. *$X$ is pseudocompact.*
2. *The closure of any ideal in $C(X)$ is an ideal.*
3. *Every ideal in $C(X)$ is contained in a strongly divisible ideal.*
4. *Every ideal in $C(X)$ is contained in a closed ideal.*

**Proof** 1. $\Rightarrow$ 2. If $X$ is pseudocompact, then the map $C(\beta X) \to C(X) : f \to f|X$ is an isometric isomorphism. Since $\beta X$ is compact, by [GJ,2M] closures of ideals of $C(\beta X)$ are ideals. 2. follows.

2. $\Rightarrow$ 3. Clear from (3.1.4).

3. $\Rightarrow$ 4. Let $I$ be a strongly divisible ideal of $C(X)$. Then $Z[I]$ is closed under countable intersection by (3.1.2). But $J = Z^{\leftarrow}[Z[I]]$ is a $z$-ideal, with $Z[J] = Z[I]$ closed under countable intersection. Hence $J$ is a strongly divisible $z$-ideal, therefore closed, and $I \subset J$. Thus 4. follows from 3.

4. $\Rightarrow$ 1. By 4., every maximal ideal is closed, therefore real by (3.1.5). Hence $vX = \beta X$ and $X$ is pseudocompact. $\diamond$

We remark that it is clear from the above corollary and proof of 3. $\Rightarrow$ 4. that if $X$ is pseudocompact and $I$ is a strongly divisible ideal of $C(X)$, then $\bar{I} = Z^{\leftarrow}[Z[I]]$.

We now attempt to determine which prime ideals of $C(X)$ are closed in the uniform topology. Recall that for each point $p \in \beta X$, $O^p$ denotes the ideal consisting of all functions $f$ in $C(X)$ for which $cl_{\beta X} Z(f)$ is a neighborhood of $p$. That is, for each $p \in \beta X$,

$$O^p = \{f \in C(X) : p \in int_{\beta X} cl_{\beta X} Z(f)\}.$$

If $p \in X$, $O^p$ is also denoted $O_p$, and may be described more simply as

$$O_p = \{f \in C(X) : p \in int_X Z(f)\}.$$

For each $p \in \beta X$, $O^p \subset M^p$, (an obvious consequence of the Gelfand-Kolmogoroff Theorem, [GJ, 7.3]). Details of these assertions may be found in chapter 7 of [GJ].

**3.1.7 Lemma** *For all $p \in \beta X$, $\overline{O^p} = \overline{M^p}$.*

**Proof:** It is enough to show that $M^p \subset \overline{O^p}$, so let $f \in M^p$ and let $\varepsilon > 0$. Let $g = [(f - \varepsilon) \vee 0] + [(f + \varepsilon) \wedge 0]$. Then $Z(g) = f^{\leftarrow}[-\varepsilon, \varepsilon]$. Now by [GJ,7D] $f^*(p) = 0$, where $f^* : \beta X \to \mathbf{R}^*$ is the (unique) continuous function from $\beta X$ into $\mathbf{R}^*$, (the one point compactification of $\mathbf{R}$), such that $f^*|X = f$. By [GJ, 7.12], $Z(g) \in Z[O^p]$ and therefore, since $O^p$ is a $z$-ideal, $g \in O^p$. But $|f - g| < \varepsilon$, hence $f \in \overline{O^p}$. ◇

[GJ, 7.15] asserts that for every prime ideal $P$ of $C(X)$, there is a unique $p \in \beta X$ such that $O^p \subset P \subset M^p$. It follows from the lemma that no non-maximal prime ideal of $C(X)$ is closed. However, as noted earlier, the closed maximal ideals of $C(X)$ are precisely the real ideals of $C(X)$, i.e. $M^p$ is closed if and only if $p \in \upsilon X$. Hence we have

**3.1.8 Corollary** *Let $X$ be a Tychonoff space.*

1. *If $p \in \beta X$ then $\overline{O^p} = M^p$ if and only if $p \in \upsilon X$.*
2. *Let $P$ be a prime ideal of $C(X)$, $M^p$ the unique maximal ideal containing $P$. Then $\overline{P}$ is a (necessarily maximal) ideal if and only if $p \in \upsilon X$.*

A space $X$ is pseudocompact if and only if $\beta X = \upsilon X$. Hence we get the following extension of (3.1.6).

**3.1.9 Corollary** *The following are equivalent for a Tychonoff space $X$.*

1. *$X$ is pseudocompact.*
2. *The closure of any prime ideal of $C(X)$ is a (maximal) ideal.*

Note that for each $p \in \upsilon X$, the $z$-ideal $O^p$ is not closed under countable intersection and therefore, by (3.1.2), is not strongly divisible, yet $\overline{O^p} = M^p$ is an ideal. Thus, if $I$ is a $z$-ideal of $C(X)$ such that $\bar{I}$ is an ideal of $C(X)$, it does not follow that $I$ is strongly divisible.

## 3.2 Spaces $X$ For Which Every Countably Generated Ideal of $C(X)$ is Principal

As an application to the above, in this section it is shown that those spaces with the property of the preceding title are precisely the finite (Tychonoff) spaces. The result is perhaps surprising given that *F-spaces* - spaces $X$ for which every finitely generated ideal in $C(X)$ is principal- may be non-discrete and of uncountably infinite cardinality, for example $\beta\mathbf{N}\setminus\mathbf{N}$, [GJ, 14.27]. Our theorem hinges upon the following observation.

**3.2.1 Proposition:** *The following are equivalent for a commutative ring $R$:*

*1. Every ideal in $R$ is strongly divisible.*

*2. Every countably generated ideal of $R$ is principal.*

**Proof:** 1. $\Rightarrow$ 2. That strongly divisible, countably generated ideals are principal is obvious.

2. $\Rightarrow$ 1. Let $I$ be an ideal of $R$ and let $\{a_n\}$ be any countable subset of $I$. Then $J = (a_n : n = 1, 2, ...)$ the ideal generated by $\{a_n\}$ is principal, say $J = (a)$, where $a \in R$. But $a$ divides each $a_n$, and since $J \subset I$, $a \in I$. Hence $I$ is strongly divisible. $\diamond$

In [A1], Azarpanah calls an ideal $I$ of a commutative ring $R$ divisible if for any finite subset $\{a_1, ..., a_n\}$ of $I$, there is an $a \in I$, and a subset $\{b_1, ..., b_n\}$ of $R$ such that for $1 \leq i \leq n$, $ab_i = a_i$. In other words, $I$ is divisible if every finite subset of $I$ has a divisor in $I$. Just as above, the following is true.

**3.2.2 Proposition:** *The following are equivalent for a commutative ring R:*

1. *Every ideal of R is divisible.*
2. *Every finitely generated ideal of R is principal.*

Thus we have the following

**3.2.3 Corollary:** *$X$ is an $F$-space if and only if every ideal of $C(X)$ is divisible.*

The following is a special case of a more general result of De Marco, see [D]. Our own proof is provided.

**3.2.4 Lemma** *If $X$ is compact, then the ideal $I = \bigcap_{p \in S} O_p$, where $S$ is a zero-set of $X$, is countably generated.*

**Proof:** Let $S = Z(g)$, where $g \in C(X)$, and consider the functions $g_n = [(g - 1/n) \vee 0] + [(g + 1/n) \wedge 0]$, $n = 1, 2, \ldots$ . Then for each $n$, $Z(g_n) = g^{\leftarrow}[-1/n, 1/n]$, and hence, $S \subset int(Z(g_n))$. It follows that each $g_n \in I$, and $\bigcap_n Z(g_n) = Z(g)$. Now take $f \in I$, so that $S \subset intZ(f)$. The increasing sequence of open sets $\{X \backslash Z(g_n) : n = 1, 2, \ldots\}$ covers the compact set $X \backslash intZ(f)$. Therefore a positive integer $k$ can be chosen such that $X \backslash intZ(f) \subset X \backslash Z(g_k)$, hence $Z(f)$ is a neighborhood of $Z(g_k)$, from which it follows that $g_k$ divides $f$, (2.1.1). Thus $I$ is generated by $\{g_n : n = 1, 2, \ldots\}$. $\diamond$

A topological space is a *P-space* if each of its zero-sets is open. (For more on P-spaces, see [GJ, 4JKL]). We will use a result from [GJ, 4K] in the next proof.

**3.2.5 Theorem:** *Every countably generated ideal of $C(X)$ is principal if and only if $X$ is finite.*

**Proof:** If every countably generated ideal of $C(X)$ is principal, then by (3.2.1), every ideal of $C(X)$ is strongly divisible, and consequently, $X$ is pseudocompact (3.1.6). Suppose for now that $X$ is compact and let $S$ be a zero-set of $X$. By the lemma, the ideal $I = \bigcap_{p \in S} O_p$ is countably generated and therefore principal by hypothesis, so take $f \in I$ such that $(f) = I$. Clearly then, $Z(f) = S$, but since $f \in I$, it is also true that $S \subset intZ(f)$. Hence $S = Z(f)$ is open and therefore $X$ is a P-space. But compact P-spaces are finite [GJ, 4K], so $X$ is finite. If $X$ is assumed only to be pseudocompact, then $C(X)$ is ring isomorphic to $C(\beta X)$, ($\beta X$ is the Stone-Cech compactification of $X$), so every countably generated ideal of $C(\beta X)$ is principal, and by the above argument $\beta X$ is finite, whence $X$ is finite.

Conversely suppose $X$ is finite, and let $I = (f_n : n = 1, 2, ...)$ be the countably generated, ( proper, non-trivial) ideal of $C(X)$ generated by $\{f_n : n = 1, 2, ...\}$. By the blanket assumption that all spaces are Tychonoff, $X$ is discrete (and compact). Therefore the zero-set $S = \bigcap Z[I] = \bigcap_n Z(f_n)$ is a non-empty proper subset of $X$, and so the characteristic function, call it $f$, on $X \backslash S$ is a non-unit in $C(X)$. Now $Z(f) = S$, so for each $n$, $Z(f) \subset Z(f_n)$, and clearly $f f_n = f_n$, hence $(f) \supset I$. On the other hand, using a compactness argument (similar to the one used in the proof of the previous lemma), it is easy to see that for some $n$, $\bigcap_{k=1}^{n} Z(f_k) = S$. Define $g$ to be $f_1^2 + ... + f_n^2$; then $g \in I$, and $Z(g) = Z(f)$. Define

$$h(x) = \frac{1}{g(x)}, \qquad if\ x \notin Z(g)$$

$$h(x) = 0, \qquad if\ x \in Z(g).$$

$X$ is discrete, so $h \in C(X)$, and $f = hg \in I$. It follows that $(f) = I$. ◇

**3.2.6 Corollary:** *Every ideal of $C(X)$ is closed if and only if $X$ is finite.*

**Proof:** Closed ideals are strongly divisible, which together with (3.2.1) and (3.2.5), provides necessity. If $X$ is finite, then by the above every countably generated ideal of $C(X)$ is principal, hence every ideal of $C(X)$ is strongly divisible. As a disrete space, $X$ is also a P-space, therefore every ideal of $C(X)$ is also a $z$-ideal, [GJ, 4J]. Thus every ideal of $C(X)$ is closed, by (3.1.4). ◇

## 3.3 Weakly Lindelöf Spaces

In F. Azarpanah's paper [A1], the notion of strongly divisible ideals is used to characterize (Tychonoff) Lindelöf spaces as those spaces $X$ such that every strongly divisible ideal of $C(X)$ is fixed. Here we give a similar characterization for weakly Lindelöf spaces.

Recall that a space $X$ is *weakly Lindelöf* if every open cover of $X$ contains a countable subfamily whose union is dense in $X$, [PW]. A collection $\mathcal{C}$ of subsets of a space $X$ will be said to have the *strong countable intersection property* (SCIP) if the intersection of any countable subfamily of $\mathcal{C}$ has non-empty interior. It is clear that a space $X$ is Lindelöf if and only if any collection of closed subsets of $X$ with the countable intersection property has non-empty intersection. The following is the analogue for weakly Lindelöf spaces. A subset of a topological space is *regular closed* if it is the closure of an open set.

**3.3.1 Theorem** *The following are equivalent for a topological space $X$*

1. *$X$ is weakly Lindelöf.*
2. *If $\mathcal{C}$ is a family of closed subsets of $X$ with SCIP then $\bigcap \mathcal{C} \neq \emptyset$.*
3. *If $\mathcal{C}$ is a family of regular closed subsets of $X$ with SCIP then $\bigcap \mathcal{C} \neq \emptyset$.*
4. *If $\mathcal{U}$ is any family of open subsets of $X$ with SCIP then $\bigcap\{cl(U) : U \in \mathcal{U}\} \neq \emptyset$.*
5. *If $\mathcal{C}$ is a family of basic closed subsets of $X$ with SCIP then $\bigcap \mathcal{C} \neq \emptyset$.*

**Proof:** 1. $\Rightarrow$ 2. Supposing 2. is false, let $\mathcal{C}$ be a family of closed subsets of $X$ with SCIP and $\bigcap \mathcal{C} = \emptyset$. Then $\{X \backslash C : C \in \mathcal{C}\}$ is an open cover of $X$. If $X$ is weakly Lindelöf then there is a countable subfamily $(C_n)$ of $\mathcal{C}$ such that $cl(\bigcup X \backslash C_n) = X$. But then $X = cl(\bigcup X \backslash C_n) = cl(X \backslash \bigcap C_n) = X \backslash int(\bigcap C_n)$, and therefore $int(\bigcap C_n) = \emptyset$, a contradiction. Therefore $X$ is not weakly Lindelöf.

2. $\Rightarrow$ 3. This is clear.

3. $\Rightarrow$ 4. If $\mathcal{U}$ is a family of open subsets of $X$ with SCIP then $\{cl(U) : U \in \mathcal{U}\}$ is a family of regular closed subsets of $X$ with SCIP, hence by 3., $\bigcap\{cl(U) : U \in \mathcal{U}\} \neq \emptyset$.

4. $\Rightarrow$ 1. Supposing 1. is false let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of $X$ such that the union of no countable subfamily of $\mathcal{U}$ is dense in $X$. Let $\mathcal{D} = \{D \subset A : D$ is countable$\}$. For each $D \in \mathcal{D}$ let $V_D = X \backslash cl[\bigcup_{\alpha\in D} U_\alpha]$. Then each $V_D$ is open and by the hypothesis on $\mathcal{U}$, each $V_D \neq \emptyset$. Let $\mathcal{V} = \{V_D : D \in \mathcal{D}\}$.

Claim: $\mathcal{V}$ has SCIP.

For let $(D_n)$ be a countable subfamily of $\mathcal{D}$. Then $D = \bigcup D_n \in \mathcal{D}$ and

$$\bigcap_n V_{D_n} = \bigcap_n X \backslash cl[\bigcup_{\alpha\in D_n} U_\alpha] = X \backslash \bigcup_n cl[\bigcup_{\alpha\in D_n} U_\alpha]$$

$$\supset X \backslash cl[\bigcup_n \bigcup_{\alpha\in D_n} U_\alpha] = X \backslash cl[\bigcup_{\alpha\in D} U_\alpha] = V_D.$$

But $V_D$ is open, non-empty and therefore $int(\bigcap_n V_{D_n}) \neq \emptyset$, proving the claim. But

$$\bigcap\{cl(V) : V \in \mathcal{V}\} = \bigcap\{cl(V_D) : D \in \mathcal{D}\} = \bigcap_{D\in\mathcal{D}} cl[X \backslash cl(\bigcup_{\alpha\in D} U_\alpha)]$$

$$= \bigcap_{D\in\mathcal{D}} X \backslash int[cl(\bigcup_{\alpha\in D} U_\alpha)] \subset \bigcap_{D\in\mathcal{D}} (X \backslash \bigcup_{\alpha\in D} U_\alpha) = X \backslash (\bigcup_{D\in\mathcal{D}} \bigcup_{\alpha\in D} U_\alpha)$$

$$= X \backslash \bigcup_{\alpha\in A} U_\alpha = X \backslash X = \emptyset,$$

since $\mathcal{U}$ covers $X$. Therefore 4. is false.

That 2. is equivalent to 5. is immediate. $\diamondsuit$

**3.3.2 Definition** *A strongly divisible ideal $I$ of a commutative ring $R$ comprised entirely of divisors of zero will be called neighborhood strongly divisible, or simply nsd.*

Recall from chapter 2 that a member $f$ of $C(X)$ is a divisor of zero if and only if $Z(f)$ has non-empty interior. Thus an ideal $I$ of $C(X)$ is nsd if given any countable subfamily $(f_n)$ of $I$ there is a $g \in I$ and $(h_n) \subset C(X)$ such that for each $n$, $f_n = gh_n$, and $Z(g)$ has non-empty interior.

**3.3.3 Lemma** *Let $X$ be Tychonoff, $I$ a z-ideal of $C(X)$. Then the following are equivalent.*

*1. $I$ is nsd.*

*2. $Z[I]$ has SCIP and is closed under countable intersection.*

**Proof:** The proof is similar to the proof of (3.1.2). ◇

**3.3.4 Theorem** *The following are equivalent for a Tychonoff space $X$:*

*1. $X$ is weakly Lindelöf.*

*2. Every nsd ideal of $C(X)$ is fixed.*

**Proof:** 1. ⇒ 2. Suppose $X$ is weakly Lindelöf, $I$ a nsd ideal of $C(X)$. Then, given any countable subset $(f_n)$ of $I$ there is a $g \in I$ and $(h_n) \subset C(X)$ such that for each $n$, $f_n = gh_n$, and $intZ(g) \neq \emptyset$. Therefore, $intZ(g) \subset Z(g) \subset \bigcap_n Z(f_n)$, hence $int[\bigcap_n Z(f_n)] \neq \emptyset$. Therefore $Z[I]$ has SCIP, so by (3.3.1), $\bigcap Z[I] \neq \emptyset$, that is $I$ is fixed.

2. ⇒ 1. Let $\mathcal{A}$ be a collection of zero-sets (basic closed in Tychonoff $X$), of $X$ with SCIP. Let $\mathcal{B}$ be the collection of all countable intersections of members of $\mathcal{A}$. Then $\mathcal{B}$ is a base for a $z$-filter $\mathcal{F}$ on $X$ which has SCIP and is closed under countable intersection. Therefore $Z^{\leftarrow}[\mathcal{F}]$ is a $z$-ideal and $Z[Z^{\leftarrow}[\mathcal{F}]] = \mathcal{F}$ has SCIP, and is closed under countable intersection. By the lemma, $Z^{\leftarrow}[\mathcal{F}]$ is nsd. Hence, by 2., $Z^{\leftarrow}[\mathcal{F}]$ is fixed, so $\bigcap \mathcal{F} \neq \emptyset$. But $\mathcal{A} \subset \mathcal{F}$ so $\bigcap \mathcal{A} \neq \emptyset$, and $X$ is weakly Lindelöf. ◇

As observed by Professor Henriksen, a corollary to the above is the following theorem which is part of [RW, 5.11].

**3.3.5 Corollary:** *Weakly Lindelöf almost P-spaces are Lindelöf.*

**Proof:** Zero-sets of almost P-spaces have non-empty interior by definition. The result follows immediately from the above and the Azarpanah characterization of Lindelöf spaces, which was quoted at the beginning of this section. ◇

As a final result we give another $C(X)$-type characterization of weakly Lindelöf spaces using essential ideals, a sort of ideal whose $C(X)$ properties have also been studied extensively by F. Azarpanah.

**3.3.6 Definition** *An ideal $I$ of a ring $R$ is said to be essential in $R$ if it meets every non-trivial ideal of $R$ non-trivially.*

Azarpanah showed that an ideal $I$ of $C(X)$ is essential in $C(X)$ if and only if $\bigcap Z[I]$ is nowhere dense in $X$, i.e. if and only if $int(\bigcap Z[I]) = \emptyset$, [A2]. We use this theorem to prove the following:

**3.3.7 Theorem** *The following are equivalent for a Tychonoff space $X$.*

*1. $X$ is weakly Lindelöf.*

*2. Every free ideal of $C(X)$ contains a countably generated essential ideal.*

**Proof:** 2. $\Rightarrow$ 1. If $C(X)$ contains no free ideals then $X$ is compact, hence weakly Lindelöf, so we assume that $C(X)$ contains free ideals. Let $F \subset C(X)$ such that $\mathcal{U} = \{X \backslash Z(f) : f \in F\}$ is a (basic) open cover of $X$. Let $I = (f : f \in F)$ be the ideal in $C(X)$ generated by $F$. If $I = C(X)$ then, in the following argument replace $I$ by any free (proper) ideal in $C(X)$. $\mathcal{U}$ covers $X$ so $I$ is free and therefore by 2. there is a countable subset $N$ of $I$ such that $E =< g : g \in N >$ is essential. $I$ is generated by $F$ so for each $g \in N$ there exist $f_1, ..., f_{n(g)} \in F$ and $h_1, ..., h_{n(g)} \in C(X)$ such that $g = \sum_1^{n(g)} h_k f_k$. Hence for each $g \in N$, $Z(g) \supset \bigcap_1^{n(g)} Z(f_k)$. Now let $M = \{f_k : 1 \leq k \leq n(g), g \in N\}$. Then $M$ is a countable subset of $F$. But $E$ is essential, and is generated by $N$, so

$$\emptyset = int(\bigcap Z[E]) = int[\bigcap_{g \in N} Z(g)] \supset int[\bigcap_{f \in M} Z(f)].$$

Hence $int[\bigcap_{f \in M} Z(f)] = \emptyset$, and therefore $cl[\bigcup_{f \in M} X \backslash Z(f)] = X$, proving that $X$ is weakly Lindelöf.

1. $\Rightarrow$ 2. Suppose $X$ is weakly Lindelöf, and $I$ is a free ideal of $C(X)$. Then $\bigcap Z[I] = \emptyset$, so $\bigcup\{X \backslash Z(f) : f \in I\} = X$ and therefore there is a countable subset $N$ of $I$ such that

$\{X \backslash Z(f) : f \in N\}$ is dense in $X$. But then $int[\bigcap_{f \in N} Z(f)] = \emptyset$, so letting $E = (f : f \in N)$, the ideal generated by $N$, $E$ is such that $\bigcap Z[E] = \bigcap_{f \in N} Z(f)$. Therefore $int(\bigcap Z[E]) = \emptyset$, hence $E$ is essential. ◇

# Chapter 4

# Introduction to $\Phi$-Algebras

In this chapter we provide the definition of a $\Phi$-algebra, and describe the Henriksen - Johnson Representation Theorem by means of which every $\Phi$-algebra may be regarded as an algebra of extended real-valued functions on a compact space. We also give several examples of $\Phi$-algebras and include some of the theory that will be used in the remaining chapters of this work. Although attempts are made to keep the $\Phi$-algebra portion of this thesis essentially self-contained, for further information the interested reader is referred to the seminal work on $\Phi$-algebras by M. Henriksen and D.G. Johnson [HJ] and the survey articles [H2], [H3] by M. Henriksen for an essentially complete history of the subject.

## 4.1 Preliminaries

An *$l$ − algebra* is an algebra $A$ over an ordered field $K$ which, under a partial ordering $\geq$, is a lattice which satisfies

1. $a \geq b$ implies $a + c \geq b + c$,
2. $a \geq 0$ and $b \geq 0$, implies $ab \geq 0$, and
3. $\alpha \geq 0$ and $a \geq 0$ implies $\alpha a \geq 0$.

for $a, b, c \in A$ and $\alpha \in K$.

An $l$-algebra $A$ is called an ***f − algebra*** if it satisfies

4. $a \wedge b = 0$ and $c \geq 0$ implies $ca \wedge b = ac \wedge b = 0$.

An $l$-algebra $A$ is said to be archimedian if, for $a, b \in A$, $a = 0$ whenever $na \leq b$ for all integers $n$. A real archimedean $f$-algebra with an identity is called a ***Φ-algebra***; a Φ-algebra is necessarily commutative, [BP].

Simple examples of Φ-algebras are the trivial Φ-algebra $\{0\}$ and the field of real numbers $\mathbf{R}$. $C(X)$, the ring of all continuous real-valued functions on a topological space $X$, is really the prototype Φ-algebra. Indeed much of the existing research on Φ-algebras is devoted to finding characterizations of $C(X)$ for various kinds of spaces $X$, an endeavour which, in its most general sense has yet to be completed, [$H_2$], [$H_3$]. Interesting examples of Φ-algebras which cannot be represented as $C(X)$ for any space $X$, are the respective algebras of Lebesgue measurable, and Baire functions on the real line $\mathbf{R}$, [HJ, 5.1]. We note that these algebras taken modulo their respective ideals of functions that vanish almost everywhere are also Φ-algebras, (definitions of Lebesgue measurable, and Baire functions may be found in most books on measure and integration, see for example [Mu]). A reader interested in seeing more exotic examples of Φ-algebras can find several in [HJ].

If $A$ is an $l$-algebra, then $A^+ = \{a \in A : a \geq 0\}$. For $a \in A$, let $a^+ = a \vee 0$, $a^- = (-a) \vee 0$, and $|a| = a \vee (-a)$. Then $a^+ \wedge a^- = 0$, and

(i) $a = a^+ - a^-$, and

(ii) $|a| = a^+ + a^-$.

If $A$ is in fact an $f$-ring, then

(iii) $a^2 \geq 0$ for each $a \in A$, and

(iv) $|ab| = |a||b|$ for all $a, b \in A$.

Proofs of these assertions may be found in [BP], though the reader is warned that there the authors define $a^-$ to be $-(-a) \vee 0$.

If $A$ and $B$ are real $l$-algebras then a mapping $\phi : A \to B$ which is both a lattice and algebra homomorphism will be called an *l-homomorphism*; similarly we define *l-monomorphism, l-epimorphism and l-isomorphism.*

If $A$ is a nontrivial real $f$-algebra with identity 1, then the mapping $r \mapsto r \cdot 1$ is an $l$-monomorphism from $\mathbf{R}$ into $A$; thus $\mathbf{R}$ is considered a sub-Φ-algebra of $A$ by identifying $r$ with $r \cdot 1$.

If $A$ is a Φ-algebra, then

$$A^* = \{a \in A : |a| \leq n \text{ for some } n \in \mathbf{N}\}$$

is a sub-Φ-algebra of $A$, and is called the *subalgebra of bounded elements of* $A$.

By an *ideal* in an $l$-algebra we shall mean a *proper* algebra ideal. If $I$ is an ideal of $l$-algebra $A$, then it is called *convex* if whenever $0 \leq a \leq b$, and $b \in I$, then $a \in I$. An ideal $I$ in an $l$-algebra $A$ is called an $l-$ *ideal* (or an absolutely convex ideal) if $a \in I$ whenever $a \in A$, $b \in I$ and $|a| \leq |b|$.

If $I$ is an $l$-ideal of the $l$-algebra $A$, and $a \in A$, then $I(a)$ will denote the image of $a$ under the canonical homomorphism from $A$ onto $A/I$, that is $I(a)$ denotes the residue class of $a$ modulo $I$. Thus $I$ denotes both the $l$-ideal and the canonical map $A \to A/I$.

Given a convex ideal $I$ of an $l$-algebra $A$, define the relation $\geq$ on $A/I$ by saying $I(a) \geq 0$ if there exists $x \in A$ such that $x \geq 0$ and $I(a) = I(x)$.
Under this relation $A/I$ becomes a partially ordered algebra, (those interested are referred to [GJ, 0.19] for the definition of a partially ordered ring, and [GJ. 5.2] for details of the assertion just made). Henceforth any reference to the order of a factor algebra will be to the order as defined above. The importance of $l$-ideals in $l$-algebras is apparent from the following theorem which is [GJ, 5.3]

**4.1.1 Theorem** *The following conditions on a convex ideal $I$ in an $l$-algebra $A$ are equivalent.*
*(1) $I$ is an $l$-ideal.*

*(2)* $x \in I$ *implies* $|x| \in I$.

*(3)* $x, y \in I$ *implies* $x \vee y \in I$.

*(4)* $I(a \vee b) = I(a) \vee I(b)$.

*(5)* $I(a) \geq 0$ *if and only if* $I(a) = I(|a|)$.

We note that in a partially-ordered algebra, if $a \vee b$ exists for all $a, b$, then $a \wedge b$ exists, and $a \wedge b = -(-a \vee -b)$, [GJ, 0.19]. Thus if $I$ is an $l$-ideal of the $l$-algebra $A$, then $A/I$ is also an $l$-algebra. Note that by condition (3) $I$ is a sublattice of $A$, and by condition (4) the canonical mapping from $A$ onto $A/I$ is a lattice homomorphism. Thus,

$$I(a \vee b) = I(a) \vee I(b), \qquad I(a \wedge b) = I(a) \wedge I(b), \qquad I(|a|) = |I(a)|.$$

Let $A$ be a $\Phi$-algebra. The collection of all maximal $l$-ideals in $A$ is denoted $\mathcal{M}(A)$. For $a \in A$, let

$$\mathcal{M}(a) = \{M \in \mathcal{M}(A) : a \in M\}.$$

It can be verified that the collection $\{\mathcal{M}(a) : a \in A\}$ is a base for the closed sets of a compact Hausdorff topology on $\mathcal{M}(A)$, called the *Stone* or *hull-kernel* topology on $\mathcal{M}(A)$; $\mathcal{M}(A)$ is also referred to as the *maximal l-ideal space of A*. For details the reader is referred to [HJ].

With regards to maximal $l$-ideals for reference we quote the following facts which are 1.6 and 1.7 of [HJ].

(i) If $M$ is a maximal $l$-ideal of $\Phi$-algebra $A$ then $M$ is a prime ideal, and $A/M$ is a totally ordered $f$-algebra without non-zero divisors of 0.

(ii) A maximal $l$-ideal of a $\Phi$-algebra $A$ need not be maximal as a ring ideal of $A$.

If $M$ is a maximal $l$-ideal of $A$, then the totally ordered algebra $A/M$ contains the real field $\mathbf{R}$ as a subfield via the embedding map $\mathbf{R} \rightarrow A/M : r \mapsto M(r \cdot 1)$. $M$ is called real if $A/M = \mathbf{R}$, otherwise $M$ is called hyper-real.

We now turn our attention to describing the representation theorem.

## 4.2 The Henriksen-Johnson Representation Theorem

Let $\gamma\mathbf{R} = \mathbf{R} \cup \{\pm\infty\}$ denote the two-point compactification of the real field $\mathbf{R}$. For a compact space $X$, let $D(X)$ denote the set of all continuous functions $f : X \to \gamma\mathbf{R}$ for which

$$\mathcal{R}(f) = \{x \in X : f(x) \in \mathbf{R}\}$$

is a dense (necessarily open) subset of $X$. The elements of $D(X)$ are called *extended (real-valued) functions*. Beginning with functions $f, g \in D(X)$, and $\lambda \in \mathbf{R}$, the functions $\lambda f$, $f \wedge g$, and $f \vee g$ defined pointwise are clearly also in $D(X)$. If there are functions $h, k \in D(X)$ satisfying

$$h(x) = f(x) + g(x), \qquad k(x) = f(x)g(x)$$

for all $x \in \mathcal{R}(f) \bigcap \mathcal{R}(g)$, then $h$ and $k$ are called the sum and product of $f$ and $g$, and we write $h = f + g$, $k = fg$. Note that since $\mathcal{R}(f) \cap \mathcal{R}(g)$ is dense in $X$ these operations are uniquely defined. While $\lambda f$, $f \wedge g$, and $f \vee g$ always exist in $D(X)$, [HJ, 2.1] provides an example where $D(X)$ is closed under neither addition nor multiplication. Indeed [HJ, 2.2] asserts that $D(X)$ is an algebra if and only if dense cozero-sets of $X$ are $C^*$-embedded in $X$.

We can now state the representation theorem which is [HJ, 2.3].

**4.2.1 Henriksen-Johnson Representation Theorem** *Every Φ-algebra $A$ is isomorphic to a sub-Φ-algebra $\overline{A}$ of $D(\mathcal{M}(A))$. Moreover, if $S$, $T$ are disjoint closed subsets of $\mathcal{M}(A)$, then there is an $\overline{a} \in \overline{A}$ such that $\overline{a}[S] = 0$, $\overline{a}[T] = 1$, and $0 \leq \overline{a} \leq 1$.*

Although it is not our intention to prove (4.2.1), (the interested reader is referred to [HJ] for the proof), it will be to our advantage in Chapter six to know the precise representation of an element $a \in A$, as an extended function on $\mathcal{M}(A)$.

To each $a \in A$, a function $\overline{a} : \mathcal{M}(A) \rightarrow \gamma\mathbf{R}$ is associated as follows.
If $a \in A^+$, and $M \in \mathcal{M}(A)$, take

$$\overline{a}(M) = inf\{\lambda \in \mathbf{R} : M(a) \leq \lambda\},$$

(where $\lambda$ is identified with $M(\lambda \cdot 1)$, and $inf\ \emptyset$ is understood to be $+\infty$).
If $a \in A$ is arbitrary, let

$$\overline{a}(M) = \overline{a^+}(M) - \overline{a^-}(M).$$

Note that since $a^+ \wedge a^- = 0$, $M(a^+) \wedge M(a^-) = 0$. But, as observed earlier, $A/M$ is totally ordered and hence $M(a^+) = 0$, or $M(a^-) = 0$. Thus $\overline{a^+}(M) = 0$ or $\overline{a^-}(M) = 0$, and $\overline{a}$ is well-defined. Let $\overline{A}$ denote the set of extended functions $\{\overline{a} : a \in A\}$

We remark that the theorem is proved by showing that $\overline{a}$ as defined above belongs to $D(\mathcal{M}(A))$ and $a \mapsto \overline{a}$ is an $l$-embedding of $A$ into $D(\mathcal{M}(A))$.

*Henceforth, wherever it is convenient to do so, a Φ-algebra A will be identified with its isomorphic copy, the algebra $\overline{A} \subset D(\mathcal{M}(A))$ of extended functions.*

Should explicit consideration of the construction of $\overline{a}$ from $a$ be necessary we shall refer to $\overline{a}$ as the *'Henriksen-Johnson representation of a'*, and the reader may wish to review the above at that time. This however should not be necessary until chapter six.

We presently outline some of the theory derived in [HJ] from the representation theorem. The first result is a Gelfand-Kolmogoroff type of characterization of the maximal $l$-ideals in a Φ-algebra $A$, [cf. GJ, 7.3], and will be of fundamental importance to this study. It is Theorem 2.5 of [HJ].

**4.2.2 Theorem** *A subset $M$ of a Φ-algebra $A$ is a maximal $l$-ideal of $A$ if and only if there is a unique $x \in \mathcal{M}(A)$ such that*

$$M = M_x = \{a \in A : (ab)(x) = 0 \ \textit{for all}\ b \in A\}.$$

In light of the above, an element of the maximal $l$-ideal space $\mathcal{M}(A)$ will be written as '$M$', or '$M_x$' if we wish to view it as a maximal $l$-ideal, and simply '$x$' if we wish to view it as a point of the topological space $\mathcal{M}(A)$.

An immediate consequence of (4.2.2), which is stated for reference is

**4.2.3 Corollary** *If $x \in \mathcal{M}(A)$, then $M_x$ is real if and only if $M_x = \{a \in A : a(x) = 0\}$.*

## 4.3 Uniformly Closed Φ-Algebras

Let $A$ be a nontrivial real $f$-algebra with 1. Given $a, b \in A$, define

$$\rho(a,b) = inf\{r \in \mathbf{R} : |a-b| \wedge 1 \leq r\}$$

where, as usual, $r \cdot 1$ and $r$ are identified. The verification that $\rho$ is a pseudometric on $A$ is routine. $\rho$ is defined on the trivial Φ-algebra in the obvious way. $\rho$ is called the ***uniform pseudometric*** on $A$ and the topology induced by $\rho$ is called the ***uniform topology***. Henceforth all references to topological properties of an $f$-algebra will be with respect to its uniform topology.

If $A$ is complete with respect to $\rho$, then $A$ is said to be ***uniformly closed***. Well-known examples of uniformly closed Φ-algebras are $\mathbf{R}$ and $C(X)$, where $X$ is any topological space. Indeed many Φ-algebras of interest, such as the respective algebras of Baire and Lebesgue functions on the real line are uniformly closed, and consequently the properties of uniformly closed Φ-algebras are of especial interest. We outline some of these properties now; the first is [HJ, 3.2] and [HJ, 3.7].

**4.3.1 Theorem** *A Φ-algebra $A$ is uniformly closed if and only if $A^*$ and $C(\mathcal{M}(A))$ are isomorphic.*

A Φ-algebra $A$ is ***closed under bounded inversion*** if $a \in A$, $a \geq 1$, implies $1/a \in A$. The principal ideal of a member $a$ of Φ-algebra $A$ will be denoted $(a)$, thus $(a) = \{ab : b \in A\}$.

The smallest $l$-ideal of $A$ containing $a$ will be denoted $(a)_l$ and will be called the *$l$-principal ideal of $A$ generated by $a$*, (as in [H]). It is easy to see that for any $a \in A$,

$$(a)_l = \{c \in A : |c| \leq |ab| \text{ for some } b \in A\}.$$

The following is [HJ, 3.3] and [HJ, 3.4].

**4.3.2 Theorem** *(1) Every uniformly closed Φ-algebra is closed under bounded inversion.* *(2) If $A$ is a Φ-algebra closed under bounded inversion, then for $a \in A$, $(a)_l = A$ if and only if $1/a \in A$.*

We conclude the present chapter by stating the following characterization theorem which is [HJ, 3.9]. Note that an element $a$ of a Φ-algebra $A$ of extended functions is a divisor of zero if and only if $a^{\leftarrow}(0)$ has non-empty interior, (see page 86, paragraph 2 of [HJ] for details).

**4.3.3 Theorem** *A Φ-algebra $A$ is isomorphic to $D(X)$ for some compact space $X$ if and only if*

*(1) $A$ is uniformly closed, and*

*(2) If $a \in A$, then either $a$ is a divisor of zero, or $(a)_l = A$.*

# Chapter 5

# $z$-Ideals in $\Phi$-Algebras

The main objective in our study of $\Phi$-algebras is to generalize the characterization of closed ideals of $C(X)$ as found in chapter three, to a $\Phi$-algebra setting. The notion of a $z$-ideal in $C(X)$ was central to our earlier theorem as it will be when we attempt to describe the closed $l$-ideals of $\Phi$-algebras. Because $z$-ideals are crucial to the study of rings of continuous functions, rather than simply using them for the specific purpose of describing closed $l$-ideals of $\Phi$-algebras, we take this opportunity to more broadly examine their role in $\Phi$-algebras. Using $z$-ideals we expand upon some of the results found in [HJ], thus illustrating the use of $z$-ideals in this more general context. Beyond this, as an application, we conclude the chapter with a study of what we call $P$-algebras - $\Phi$-algebras in which prime $l$-ideals are maximal - a generalization of $C(X)$, where $X$ is a P-space. We note that that $z$-ideals in the even more general settings of commutative rings and partially-ordered rings were studied by G. Mason in [M]. There, some of the results obtained in this chapter may also be found, (unbeknownst to the author at the time when this work was done!), though we remark that our notation is rather different from Mason's. Due to the fact that we shall remain only in the world of $\Phi$- algebras and therefore have access to the Henriksen-Johnson representation theorem, in those places where there is overlap with [M], the results that follow are sometimes slightly stronger than those found there.

Unless explicitly stated otherwise, throughout this chapter $A$ will denote a $\Phi$-algebra.

## 5.1 $z$-Ideals and $z$-Filters

**5.1.1 Definition** *An ideal $I$ of a $\Phi$-algebra $A$ will be called a $z$-ideal if $\mathcal{M}(f) \supset \mathcal{M}(g)$, $f \in A$ and $g \in I$ implies $f \in I$.*

Note that since $\mathcal{M}(f) \supset \mathcal{M}(g)$ if and only if $\mathcal{M}(f) = \mathcal{M}(fg)$, '$\supset$' may without loss of generality be replaced by '=' in the above definition. As noted in chapter 3, if $X$ is a topological space, then the above definition of $z$-ideal agrees with the usual notion of a $z$-ideal in $C(X)$, [GJ, 4A].

In [HJ] it is proved that the maximal $l$-ideals of $A$ are precisely the sets

$$M_x = \{f \in A : (fg)(x) = 0 \ for\ all\ g \in A\}, \qquad (x \in \mathcal{M}(A)).$$

It follows immediately that for each $f \in A$

$$\mathcal{M}(f) = \{x \in \mathcal{M}(A) : f \in M_x\}$$

$$= \{x \in \mathcal{M}(A) : (fg)(x) = 0 \ for\ all\ g \in A\}$$

These subsets of $\mathcal{M}(A)$ will be shown to play a similar role to that of zero-sets of a topological space $X$. Indeed, if $X$ were a compact topological space, and $\mathcal{M}(C(X))$ the maximal ideal space of $C(X)$, then for $f \in C(X)$, the sets $\mathcal{M}(f)$ and $Z(f)$ are equal, (up to the equivalence of $X$ and $\mathcal{M}(C(X))$).

We note that $\mathcal{M}$ can be regarded as a surjective map $\mathcal{M} : A \to \mathcal{M}[A] : f \mapsto \mathcal{M}(f)$, and as such the following notation is employed. If $S$ is a subset of $A$, then the collection of subsets of $\mathcal{M}(A)$, $\{\mathcal{M}(f) : f \in S\}$, will be denoted by $\mathcal{M}[S]$. Hence $\mathcal{M}[A] = \{\mathcal{M}(f) :$

$f \in A\}$ and $\mathcal{M}(A) = \{M : M \text{ is a maximal } l-\text{ideal of } A\}$; square and round brackets distinguish the difference. If $S$ is a subset of $\mathcal{M}[A]$, then $\mathcal{M}^{\leftarrow}[S] = \{f \in A : \mathcal{M}(f) \in S\}$. Using this notation we see that

$$I \text{ is a } z-\text{ideal in } A \text{ if and only if } I = \mathcal{M}^{\leftarrow}[\mathcal{M}[I]].$$

As a first connection between the zero-sets of a (completely regular) topological space and the collection $\mathcal{M}[A]$, observe that by the very definition of the Stone topology on $\mathcal{M}(A)$, $\mathcal{M}[A]$ comprises a base for the closed subsets of $\mathcal{M}(A)$. A second connection is the following

**5.1.2 Proposition:** $\mathcal{M}[A]$ *forms a lattice under set containment.*

**Proof:** Let $f, g \in A$. Then

$$\mathcal{M}(fg) = \{x \in \mathcal{M}(A) : fg \in M_x\}$$

$$= \{x \in \mathcal{M}(A) : f \in M_x \text{ or } g \in M_x\} \qquad (\text{maximal } l-\text{ideals are prime})$$

$$= \{x \in \mathcal{M}(A) : f \in M_x\} \bigcup \{x \in \mathcal{M}(A) : g \in M_x\} = \mathcal{M}(f) \bigcup \mathcal{M}(g),$$

and,

$$\mathcal{M}(f^2 + g^2) = \{x \in \mathcal{M}(A) : f^2 + g^2 \in M_x\}$$

$$= \{x \in \mathcal{M}(A) : f^2 \in M_x \text{ and } g^2 \in M_x\} \qquad , (\text{here the convexity of } M_x \text{ is used}),$$

$$= \mathcal{M}(f^2) \bigcap \mathcal{M}(g^2) = \mathcal{M}(f) \bigcap \mathcal{M}(g) \qquad (\text{maximal } l-\text{ideals are prime}).$$

◊

We remark that although for an arbitrary ring $A$, $\mathcal{M}[A]$ always forms a join semi-lattice, the fact that in this case $\mathcal{M}[A]$ forms a lattice is dependent upon the fact that $A$ has a partial-ordering.

What follows is a sequence of results which parallel the results of [GJ, chapter 2], stated in the context of Φ-algebras. Although the proofs that follow are very similar to those found in [GJ], in some cases we must be more careful than they are, due to the fact that although for any $g \in A$, $\mathcal{M}(g) \subset g^{\leftarrow}(0)$, the two sets need not be equal. To see this, note that if $M_x$ is a hyper-real ideal in $A$, then the totally-ordered $f$-algebra $A/M_x$ is non-archimedian. Thus an element $f \in A$ may be found such that $M_x(f)$ is 'infinitely small'; that is, $M_x(f)$ is positive, and for every natural number $n$, $M_x(f) \leq 1/n$, (see page 70 of [GJ] for the details of this assertion). It follows from the Henriksen-Johnson representation of $f$, that $f(x) = 0$, yet $x \notin \mathcal{M}(f)$.

A filter on the lattice $\mathcal{M}[A]$, will be called a *z-filter*.

**5.1.3 Proposition:** *a) If $I$ is an $l$-ideal of $A$, then $\mathcal{M}[I]$ is a z-filter.*
*b) If $\mathcal{F}$ is a z-filter on $\mathcal{M}[A]$, then $\mathcal{M}^{\leftarrow}[\mathcal{F}]$ is a z-ideal of $A$.*

**Proof:** a) If $M_x$ is a maximal $l$-ideal of $A$ containing $I$, then for any $f \in I$, $x \in \mathcal{M}(f)$ and hence $\emptyset \notin \mathcal{M}[I]$. If $f, g \in I$, so that $\mathcal{M}(f), \mathcal{M}(g) \in \mathcal{M}[A]$, then $f^2 + g^2 \in I$, and so $\mathcal{M}(f) \bigcap \mathcal{M}(g) = \mathcal{M}(f^2 + g^2) \in \mathcal{M}[I]$. Finally, if $f \in I$, $g \in A$, with $\mathcal{M}(g) \supset \mathcal{M}(f)$, then $fg \in I$; therefore $\mathcal{M}(f) \bigcup \mathcal{M}(g) = \mathcal{M}(fg) \in \mathcal{M}[I]$. Thus $\mathcal{M}[I]$ is a $z$-filter.
b) Let $I = \mathcal{M}^{\leftarrow}[\mathcal{F}]$, where $\mathcal{F}$ is a $z$-filter. $\emptyset \notin \mathcal{F}$, so $1 \notin I$. Let $f, g \in I$. Then $\mathcal{M}(f), \mathcal{M}(g) \in \mathcal{F}$, and it is clear that $\mathcal{M}(f-g) \supset \mathcal{M}(f) \bigcap \mathcal{M}(g)$; therefore $\mathcal{M}(f-g) \in \mathcal{F}$ and it follows that $f-g \in I$. If $f \in I$ and $g \in A$, then $\mathcal{M}(fg) = \mathcal{M}(f) \bigcup \mathcal{M}(g) \supset \mathcal{M}(f) \in \mathcal{F}$. But $\mathcal{F}$ is a filter, and hence $\mathcal{M}(fg) \in \mathcal{F}$. Therefore $fg \in I$, proving that $I$ is an ideal. Now as remarked above, $\mathcal{M}$ regarded as a mapping $A \to \mathcal{M}[A]$ is a surjection, so $\mathcal{M}[\mathcal{M}^{\leftarrow}[\mathcal{F}]] = \mathcal{F}$ and hence $\mathcal{M}^{\leftarrow}[\mathcal{M}[I]] = \mathcal{M}^{\leftarrow}[\mathcal{M}[\mathcal{M}^{\leftarrow}[\mathcal{F}]]] = \mathcal{M}^{\leftarrow}[\mathcal{F}] = I$. Therefore $I$ is a $z$-ideal. ◇

Each member of $\mathcal{M}(A)$ is an $l$-ideal, so by part (2) of (4.1.1), for each $f \in A$, $\mathcal{M}(f) = \mathcal{M}(|f|)$. It follows that *every z-ideal of $A$ is an $l$-ideal.*

An ultrafilter on the lattice $\mathcal{M}[A]$ will be called a *z-ultrafilter*. It is an easy corollary of the above proposition that the maximal *l*-ideals of $A$ and the $z$-ultrafilters of $A$ are in one-to-one corespondence via $\mathcal{M}$-imaging. Indeed the maximal *l*-ideals of $A$ are precisely

$$M_x = \{f \in A : f \in M_x\} = \{f \in A : x \in \mathcal{M}(f)\}, \qquad (x \in \mathcal{M}(A)),$$

so the $z$-ultrafilters of $A$ are precisely

$$\mathcal{U}_x = \mathcal{M}[M_x] = \{\mathcal{M}(f) : x \in \mathcal{M}(f)\} \qquad (x \in \mathcal{M}(A)).$$

Thus

$$\mathcal{U}_x = \mathcal{M}[M_x], \qquad \textit{and} \qquad M_x = \mathcal{M}^{\leftarrow}[\mathcal{U}_x].$$

## 5.2 Prime $z$-Ideals and Prime $z$-Filters

**5.2.1 Lemma:** *If $h, g \in A$, $x \in \mathcal{M}(h)$, and $h(x) \geq g(x) \geq 0$, then $x \in \mathcal{M}(g)$.*

Proof: If $x \in \mathcal{M}(h)$, and $k \in A^+$, then $(hk)(x) = 0$, and $(hk)(x) \geq (gk)(x)$, hence $(gk)(x) = 0$. It follows from (4.3.2) that $g \in M_x$, and therefore $x \in \mathcal{M}(g)$.

**5.2.2 Theorem:** *Let $A$ be a $\Phi$-algebra, $I$ a $z$-ideal of $A$. Then the following are equivalent.*

1. *$I$ is prime.*
2. *$I$ contains a prime ideal.*
3. *For all $g, h \in A$, if $gh = 0$, then $g \in I$, or $h \in I$.*
4. *For every $f \in A$, there is a member of $\mathcal{M}[I]$ on which $f$ does not change sign.*

**Proof:** 1. ⇒ 2. and 2. ⇒ 3. are obvious.

3. ⇒ 4. If $f \in A$, then $(f \vee 0)(f \wedge 0)$ is identically 0 on the dense subset $f^{\leftarrow}[\mathbf{R}]$ of $\mathcal{M}(A)$, hence $(f \vee 0)(f \wedge 0) = 0$. By 3., $f \vee 0 \in I$, or $f \wedge 0 \in I$; certainly $f$ does not change sign on either $\mathcal{M}(f \vee 0)$ or $\mathcal{M}(f \wedge 0)$.

4. $\Rightarrow$ 1. Suppose $gh \in I$ and consider $|h| - |g| \in A$. By hypothesis there is an $f \in I$ such that $|h| - |g|$ does not change sign on $\mathcal{M}(f)$, say $|h| - |g| \geq 0$ on $\mathcal{M}(f)$. By the lemma, if $x \in \mathcal{M}(f)$, then $x \in \mathcal{M}(|h|)$ implies $x \in \mathcal{M}(|g|)$. $\mathcal{M}(|h|) = \mathcal{M}(h)$, and $\mathcal{M}(|g|) = \mathcal{M}(g)$, so $\mathcal{M}(h) \bigcap \mathcal{M}(f) \subset \mathcal{M}(g) \bigcap \mathcal{M}(f) \subset \mathcal{M}(g)$. Therefore $\mathcal{M}(gh) \bigcap \mathcal{M}(f) = (\mathcal{M}(g) \bigcup \mathcal{M}(h)) \bigcap \mathcal{M}(f)$ $= (\mathcal{M}(g) \bigcap \mathcal{M}(f)) \bigcup (\mathcal{M}(h) \bigcap \mathcal{M}(f)) \subset \mathcal{M}(g)$. But $\mathcal{M}(gh) \bigcap \mathcal{M}(f) \in \mathcal{M}[I]$, a $z$-filter, so $\mathcal{M}(g) \in \mathcal{M}[I]$. Therefore $g \in I$, as $I$ is a $z$-ideal. Hence $I$ is prime. ◇

Recall that if $A$ is a commutative ring with unity, $I$ an ideal of $A$, and $\mathcal{P}$ the collection of all prime ideals of $A$ containing $I$, then $\bigcap \mathcal{P} = \{f \in A : f^n \in I \textit{ for some } n = 0, 1, 2, ...\}$, [GJ, 0.18]. Moreover, [HJ, 1.5] says that if $I$ is an $l$-ideal of $A$ disjoint from a multiplicative system $T$ of $A$, (that is $T \subset A$ is closed under multiplication, $1 \in T$, and $0 \notin T$), then $I$ is contained in a prime $l$-ideal of $A$ disjoint from $T$.

**5.2.3 Proposition:** *Every z-ideal is the intersection of the collection of all prime l-ideals containing it.*

**Proof:** Let $I$ be a $z$-ideal of Φ-algebra $A$, $\mathcal{P}$, $\mathcal{P}'$ respectively the collections of all prime ideals and prime $l$-ideals containing $I$, $J = \bigcap \mathcal{P}$, $J' = \bigcap \mathcal{P}'$. Suppose $f \in J$. Then for some power of $n$, $f^n \in I$. But maximal $l$-ideals are prime, so $\mathcal{M}(f) = \mathcal{M}(f^n)$, and since $I$ is a $z$-ideal, $f \in I$. Hence $I = J$. Now certainly $J \subset J'$. If $f \notin J$, then $T = \{f^n : n = 0, 1, 2, ...\}$ is a multiplicative system disjoint from $I$, hence $f \notin J'$. Therefore $J' \subset J$, hence $J' = I$. ◇

**5.2.4 Lemma:** *Intersections of z-ideals are z-ideals.*

**Proof:** Let $(I_\beta)$ be a collection of $z$-ideals of $A$. For every $\beta$, $\mathcal{M}^{\leftarrow}[\mathcal{M}[I_\beta]] = I_\beta$, so

$$\bigcap_\beta I_\beta = \bigcap_\beta \mathcal{M}^{\leftarrow}[\mathcal{M}[I_\beta]] = \mathcal{M}^{\leftarrow}[\bigcap_\beta \mathcal{M}[I_\beta]] \supset \mathcal{M}^{\leftarrow}[\mathcal{M}[\bigcap_\beta I_\beta]] \supset \bigcap_\beta I_\beta.$$

Therefore $\mathcal{M}^{\leftarrow}[\mathcal{M}[\bigcap_\beta I_\beta]] = \bigcap_\beta I_\beta$, hence $\bigcap_\beta I_\beta$ is a $z$-ideal. ◇

The following is [GJ, 2.11]; its proof is very short and so we choose to include it.

**5.2.5 Lemma:** *If $I$ and $J$ are ideals in a ring $A$, and neither is contained in the other, then $I \bigcap J$ is not prime.*

**Proof:** Let $a \in I \backslash J$, $b \in J \backslash I$. Then $ab \in I \bigcap J$, but $a \notin I \bigcap J$, and $b \notin I \bigcap J$. Therefore $I \bigcap J$ is not prime. ◊

**5.2.6 Theorem:** *Every prime l-ideal is contained in a unique maximal l-ideal.*

**Proof:** Every $l$-ideal is contained in at least one maximal $l$-ideal. If $M$, $M'$ were distinct maximal $l$-ideals containing prime $l$-ideal $P$, then $M \bigcap M'$ is not prime but is a $z$-ideal containing prime $l$-ideal $P$, a contradiction to (5.2.2). ◊

A $z$-filter $\mathcal{F}$ will be called *prime* if whenever the union of two sets from $\mathcal{M}[A]$ belongs to $\mathcal{F}$, then at least one of them belongs to $\mathcal{F}$.

**5.2.7 Theorem:** *1. If $P$ is a prime l-ideal in $A$, then $\mathcal{M}[P]$ is a prime z-filter.*
*2. If $\mathcal{F}$ is a prime z-filter, then $\mathcal{M}^{\leftarrow}[\mathcal{F}]$ is a prime z-ideal.*

**Proof:** 1. Let $Q = \mathcal{M}^{\leftarrow}[\mathcal{M}[P]]$. Then $Q$ is a $z$-ideal containing the prime $l$-ideal $P$, hence $Q$ is prime. Moreover, $\mathcal{M}[Q] = \mathcal{M}[P]$. Suppose $\mathcal{M}(f) \bigcup \mathcal{M}(g) \in \mathcal{M}[P]$. Then $\mathcal{M}(fg) \in \mathcal{M}[Q]$, hence $fg \in Q$, therefore $f \in Q$, or $g \in Q$. Therefore $\mathcal{M}(f) \in \mathcal{M}[Q] = \mathcal{M}[P]$, or $\mathcal{M}(g) \in \mathcal{M}[Q] = \mathcal{M}[P]$.
2. Suppose $\mathcal{F}$ is a prime $z$-filter and $fg \in \mathcal{M}^{\leftarrow}[\mathcal{F}]$. Then $\mathcal{M}(f) \bigcup \mathcal{M}(g) = \mathcal{M}(fg) \in \mathcal{M}[\mathcal{M}^{\leftarrow}[\mathcal{F}]] = \mathcal{F}$, so $\mathcal{M}(f) \in \mathcal{F}$, or $\mathcal{M}(g) \in \mathcal{F}$. Therefore $f \in \mathcal{M}^{\leftarrow}[\mathcal{F}]$, or $g \in \mathcal{M}^{\leftarrow}[\mathcal{F}]$, thus $\mathcal{M}^{\leftarrow}[\mathcal{F}]$ is a prime $z$-ideal of $A$. ◊

**5.2.8 Corollary** *Every prime z-filter is contained in a unique z-ultrafilter.*

## 5.3 $N_x$ and Prime $z$-ideals

In [HJ], the $l$-ideal

$$N_x = \{f \in A : f \text{ vanishes on a neighborhood of } x\}$$

is introduced. Expressed in our own notation, we claim that

$$N_x = \{f \in A : x \in int\mathcal{M}(f)\}.$$

To see this suppose that $f \in N_x$. Then a neighborhood $\mathcal{U}$ of $x$ may be found on which $f$ vanishes. If $g \in A$, then $\mathcal{R}(g) = g^{\leftarrow}[\mathbf{R}]$ is a dense open subset of $\mathcal{M}(A)$. Thus $\mathcal{R}(g) \cap \mathcal{U}$ is a dense subset of $\mathcal{U}$ on which $fg$ vanishes; it follows that $(fg)[\mathcal{U}] = \{0\}$. As $g$ was chosen arbitrarily, by (4.2.2), for every $y \in \mathcal{U}$, $f \in M_y$. Thus the open neighborhood $\mathcal{U}$ of $x$ is contained in $\mathcal{M}(f)$, showing that $N_x \subset \{f \in A : x \in int\mathcal{M}(f)\}$. The reverse containment is an immediate consequence of (4.2.2).

It follows easily from the second part of the Henriksen-Johnson Representation Theorem that $\bigcap \mathcal{M}[N_x] = \{x\}$, and so $N_x$ is contained in the unique maximal $l$-ideal $M_x$.

**5.3.1 Proposition** *For each $x \in \mathcal{M}(A)$,*

1. *$N_x$ is a $z$-ideal.*
2. *$f \in N_x$ if and only if $fg = 0$ for some $g \notin M_x$.*

**Proof:** 1. is clear.

2. If $f \in N_x$, then $x \in int\mathcal{M}(f)$. By the Henriksen-Johnson representation theorem, there is a $g \in A$, $0 \leq g \leq 1$ with $g[\mathcal{M}(A)\backslash int\mathcal{M}(f)] = 0$ and $g(x) = 1$. Then $g \notin M_x$ and $fg = 0$. Conversely suppose that $g \notin M_x$ and $fg = 0$. Then $x \notin \mathcal{M}(g)$, yet $\mathcal{M}(g) \bigcup \mathcal{M}(f) = \mathcal{M}(fg) = \mathcal{M}(A)$. Therefore $\mathcal{M}(A)\backslash\mathcal{M}(g)$ is an open set containing $x$ that is contained in $\mathcal{M}(f)$, hence $x \in int\mathcal{M}(f)$. Therefore $f \in N_x$. ◇

**5.3.2 Theorem:** *An $l$-ideal $I$ of $A$ is contained in a unique maximal $l$-ideal $M_x$ if and only if $I \supset N_x$.*

**Proof:** Since $M_x$ is the unique maximal $l$-ideal containing $N_x$, sufficiency is clear. Conversely suppose that $M_x$ is the unique maximal $l$-ideal containing $I$ and let $f \in N_x$. Take $g \notin M_x$ such that $fg = 0$. Since $g \notin M_x$, $(I, g)_l = A$, where $(I, g)_l$ denotes the smallest $l$-ideal containing both $I$ and $g$. Therefore there is an $h \in I$, and $s \in A^+$ such that $1 \leq h + s|g|$. Therefore $h \geq 1 - s|g|$, hence $h|f| \geq |f| - s|g||f| = |f| - s|gf| = |f|$. But $h|f| \in I$, and $I$ is an $l$-ideal, so $|f| \in I$, hence $f \in I$. ◇

The following corollary is [HJ, 2.10], where the result is obtained without the use of $z$-ideals. Nevertheless, the above illustrates that the notion of '$z$-ideal' is useful in the context of Φ-algebras, just as $z$-ideals are useful in studying $C(X)$.

**5.3.3 Corollary** *Let $P$ be a prime $l$-ideal of Φ-algebra $A$. Then there is a unique $x \in \mathcal{M}(A)$ such that $N_x \subset P \subset M_x$, and $N_x$ is the intersection of all prime $l$-ideals containing it.*

**Proof:** $N_x$ is a $z$-ideal, hence the last statement follows. It has already been shown that every prime $l$-ideal is contained in a unique maximal $l$-ideal, so by the above theorem the result follows. ◇

## 5.4 An Application: $P$-algebras

A Φ-algebra $A$ will be called a $P-$*algebra* if every prime $l$-ideal of $A$ is a maximal $l$-ideal. By the definition of a P-space, [GJ, 4J], $C(X)$ is a P-algebra if and only if $X$ is a P-space.

**5.4.1 Theorem** *The following are equivalent for a Φ-algebra $A$.*

1. *$A$ is a $P$-algebra.*
2. *Every $z$-ideal is an intersection of maximal $l$-ideals.*
3. *For each $x \in \mathcal{M}(A)$, $N_x = M_x$.*
4. *For each $f \in A$, $\mathcal{M}(f)$ is open.*

**Proof:** 1. ⇒ 2. Every $z$-ideal is an intersection of prime $l$-ideals.

2. ⇒ 3. $N_x$ is a $z$-ideal contained in precisely one maximal $l$-ideal, namely $M_x$. It follows from 2. that $N_x = M_x$.

3. ⇒ 4. Let $f \in A$. If $\mathcal{M}(f) = \emptyset$, then $\mathcal{M}(f)$ is open, otherwise take $x \in \mathcal{M}(f)$. Then $f \in M_x = N_x$, whence $x \in int(\mathcal{M}(f))$. Therefore $\mathcal{M}(f) = int(\mathcal{M}(f))$.

4. ⇒ 3. $M_x = \{f \in A : x \in \mathcal{M}(f)\} = \{f \in A : x \in int\mathcal{M}(f)\} = N_x$.

3. ⇒ 1. If $P$ is a prime $l$-ideal of A, then there is an $x \in \mathcal{M}(A)$ such that $N_x \subset P \subset M_x$ ; by 3., $P = M_x$. ◇

Combined with (4.3.3), the above yields the following immediate corollary, a result which in view of (5.4.3) is just [HJ, 3.10] in disguise.

**5.4.2 Corollary** *If $A$ is a uniformly closed P-algebra, then $A = D(\mathcal{M}(A))$.*

The rest of the chapter is devoted to proving (5.4.3) which characterizes uniformly closed P-algebras, and which, for the sake of the discussion that follows, we presently state. Recall that an algebra $A$ is called *regular* if for every $f \in A$ there is a $g \in A$ such that $f = fgf$. (5.4.3) should be compared with [GJ, 14.29], which is the theorem that it generalizes.

**5.4.3 Theorem** *If $A$ is a uniformly closed Φ-algebra, then the following are equivalent.*

*(1) $A$ is a P-algebra.*

*(2) For every $x \in \mathcal{M}(A)$, $N_x = M_x$.*

*(3) For every $f \in A$, $\mathcal{M}(f)$ is open.*

*(4) Every l-ideal is a z-ideal and every l-principal ideal is principal.*

*(5) Every ideal of $A$ is a z- ideal.*

*(6) For every $f, g \in A$, the ideal $(f, g)$ is the principal ideal $(f^2 + g^2)$.*

*(7) $A$ is a regular Φ-algebra.*

*(8) Every prime ideal in $A$ is a maximal ideal.*

Evidently, amongst uniformly closed Φ-algebras, the P-algebras are precisely the regular Φ-algebras, and as such interesting examples abound. As mentioned earlier, $C(X)$, where $X$ is a P-space, is one such example. Others include the Baire functions on **R** and the Lebesgue measurable functions on **R**, each of which if desired may be taken modulo its ideal of functions that vanish almost everywhere, [HJ, 3.10]; yet another example of a uniformly closed regular Φ-algebra is the epimorphic hull of $C(X)$ which is examined in the pre-print [RW], by Raphael and Woods. B. Brainerd in the late fifties studied regular F-rings, (F-rings are uniformly closed Φ-algebras, see [P] for references), and, as they pertain to $z$-ideals, regular rings were studied by Mason in [M]. Considering then the attention that regular Φ-algebras have received over the years, it is not surprising, (though admittedly the author who is rather naïve was quite surprised), that (5.4.3) is not entirely new. We do believe however that the implication (1) ⇒ (7) is new, and it is for this proof that the following lemmas are required. We comment that it would be interesting to know whether (1) ⇒ (7) holds if it is not assumed that $A$ is uniformly closed, (the author thinks not, although (7) ⇒ (1) does still hold). With no further commentary we now prove the theorem.

As noted in section 5.1, for $g \in A$, $\mathcal{M}(g) \subset g^{\leftarrow}(0)$, but in general equality does not hold, a phenomenon which can cause problems. The next lemma shows that in a uniformly closed P-algebra this problem may be corrected.

**5.4.4 Lemma** *Let $A$ be a uniformly closed P-algebra. For any $g \in A$ with $g \geq 0$, there is a unit $k \in A$ with $k \geq 1$, and $gk \geq 1$ on $\mathcal{M}(A) \backslash \mathcal{M}(g)$ such that $\mathcal{M}(g) = \mathcal{M}(kg) = (kg)^{\leftarrow}(0)$. Moreover, if $f \in A$ is such that $\mathcal{M}(f) = f^{\leftarrow}(0)$, then there is an $e \in A$ such that $ef = |f|$.*

**Proof:** By (5.4.1) $\mathcal{M}(g)$ is open, so $F = \mathcal{M}(A) \backslash \mathcal{M}(g)$ is a compact subset of $\mathcal{M}(A)$. Let $x \in F$. Then $x \notin \mathcal{M}(g)$, so $g \notin M_x$. Therefore there is a $k_x \in A$ such that $(gk_x)(x) > 0$; without loss of generality, $k_x \geq 1$ and $(gk_x)(x) > 1$. For each $x \in F$, choose such a $k_x$

and define the open subset $\mathcal{U}(x)$ of $\mathcal{M}(A)$ to be $\{y \in \mathcal{M}(A) : (gk_x)(y) > 1\}$. Then for every $x \in F$, $x \in \mathcal{U}(x)$, so by compactness of $F$, $x_1, ..., x_n$ can be chosen from $F$ such that $F \subset \bigcup_{i=1}^{n} \mathcal{U}(x_i)$. Let $k = k_{x_1} + ... + k_{x_n}$. Then $k \geq 1$ (hence $\mathcal{M}(k) = \emptyset$), and $gk \geq gk_{x_i}$, for $1 \leq i \leq n$. If $y \in F$, then $y \in \mathcal{U}(x_i)$ for some $i$, so $(gk_{x_i})(y) > 1$, therefore $(gk)(y) > 1$. Hence $(gk)(y) > 1$ for all $y \in F = \mathcal{M}(A) \backslash \mathcal{M}(g)$. Thus

$$(gk)^{\leftarrow}(0) \subset \mathcal{M}(g) = \mathcal{M}(g) \bigcup \mathcal{M}(k) = \mathcal{M}(kg) \subset (gk)^{\leftarrow}(0).$$

Therefore $\mathcal{M}(g) = \mathcal{M}(gk) = (gk)^{\leftarrow}(0)$.

To prove the last statement in the lemma, if $f \in A$ is such that $\mathcal{M}(f) = f^{\leftarrow}(0)$, then define $e : \mathcal{M}(A) \to \mathbf{R}$ by

$$e(x) = 1, \qquad if\ x \in f^{\leftarrow}[(0, \infty]] = \mathcal{V}_1$$

$$e(x) = 0, \qquad if\ x \in f^{\leftarrow}(0) = \mathcal{M}(f) = \mathcal{V}_2,$$

$$e(x) = -1, \qquad if\ x \in f^{\leftarrow}[[-\infty, 0)] = \mathcal{V}_3.$$

Then each $e|\mathcal{V}_i$ is continuous, and $\mathcal{M}(A)$ is the disjoint union of the open sets $\mathcal{V}_i$, whence $e$ is continuous. It is clear that $ef = |f|$. ◇

**5.4.5 Lemma** *Let $A$ be a uniformly closed P-algebra with $f, g \in A$, $g \geq 0$. If $\mathcal{M}(g) \subset \mathcal{M}(f)$, then there is an $h \in A$ such that $hg = f$.*

**Proof:** By (5.4.4), we may without loss of generality assume that $\mathcal{M}(g) = g^{\leftarrow}(0)$, and $g|[\mathcal{M}(A) \backslash \mathcal{M}(g)] \geq 1$. Using (4.3.2), we may take $f^* = (1 + |f|)^{-1} f$. Then $-1 \leq f^* \leq 1$ and since $f^*$ is a unit multiple of $f$, $f^*$ and $f$ are contained in precisely the same set of maximal $l$-ideals, hence $\mathcal{M}(f^*) = \mathcal{M}(f)$. Therefore $g^{\leftarrow}(0) = \mathcal{M}(g) \subset \mathcal{M}(f^*) \subset (f^*)^{\leftarrow}(0)$. Define $k : \mathcal{M}(A) \to \gamma\mathbf{R}$ by

$$k(x) = \frac{f^*(x)}{g(x)}, \qquad if\ x \notin g^{\leftarrow}(0) = \mathcal{M}(g)$$

$$k(x) = 0, \qquad if\ x \in g^{\leftarrow}(0)$$

[Here $\frac{f^*(x)}{g(x)}$ is taken to be 0 if $g(x) = \infty$. This is well-defined since $f^*$ is bounded] Then $k|[\mathcal{M}(A)\backslash\mathcal{M}(f)]$ is continuous, as is $k|\mathcal{M}(g) = 0$. Since M(g) is open, it follows that $k$ is itself continuous. In fact, because $g \geq 1$ on $\mathcal{M}(A)\backslash\mathcal{M}(g)$, it follows that $-1 \leq k \leq 1$, and so $k \in C(\mathcal{M}(A)) = A^* \subset A$. Also, for every $x \in \mathcal{R}(g)$, (the real part of g), $k(x)g(x) = f^*(x)$. But $kg \in A$ and $\mathcal{R}(g) = \mathcal{R}(kg)$ is dense in $\mathcal{M}(A)$ hence $kg = f^*$. Therefore $[(1+|f|)k]g = f$. ◊

**Proof of (5.4.3):** The equivalence of (1), (2), and (3) was established in (5.4.1).

(1) ⇒ (4) Let $I$ be an $l$-ideal of $A$ and suppose that $f \in A$, $g \in I$, and $\mathcal{M}(f) \supset \mathcal{M}(g)$. Then $|g| \in I$, and $\mathcal{M}(|g|) = \mathcal{M}(g) \subset \mathcal{M}(f)$. By (5.4.5) there is an $h \in A$ such that $h|g| = f$, therefore $f \in I$. Hence $I$ is a $z$-ideal of $A$.
Now let $f \in A$ and consider $(f)$ and $(f)_l = \{s \in A : |s| \leq |fg| \text{ } for\ some\ g \in A\}$. Certainly $(f) \subset (f)_l$. Suppose then that $s \in (f)_l$. Take $g \in A$ such that $|s| \leq |fg|$. Take $k \in A$ such that $k \geq 1$ and $\mathcal{M}(|fg|) = \mathcal{M}(k|fg|) = (k|fg|)^{\leftarrow}(0)$, (using 5.4.4). $k \geq 1$ so $k|fg| \geq |fg| \geq |s|$, and therefore $\mathcal{M}(k|fg|) \subset \mathcal{M}(|s|) = \mathcal{M}(s)$. By (5.4.5) there is an $h \in A$ such that $hk|fg| = s$. Now $k|fg| = |kfg|$ and $\mathcal{M}(kfg) = (kfg)^{\leftarrow}(0)$, so again by (5.4.4), there is an $e \in A$ such that $ekfg = |kfg| = k|fg|$. Thus $hekfg = s$, that is $(hekg)f = s$, hence $s \in (f)$. Thus $(f)_l = (f)$, proving that every $l$-principal ideal in $A$ is principal.

(4) ⇒ (5) Let $I$ be an ideal of $A$, and suppose $f \in I$, $g \in A$ with $|g| \leq |f|$. By (4), $(f)_l = (f) \subset I$. But $g \in (f)_l$, so $g \in I$, showing that $I$ is an $l$-ideal.

(5) ⇒ (6) Let $f, g \in A$, and consider the ideals $(f, g)$, $(f^2+g^2)$. Certainly $(f,g) \supset (f^2+g^2)$. Now $\mathcal{M}(f^2+g^2) = \mathcal{M}(f^2) \bigcap \mathcal{M}(g^2) = \mathcal{M}(f) \bigcap \mathcal{M}(g)$, the latter set contained in both $\mathcal{M}(f), \mathcal{M}(g)$. But $f^2+g^2 \in (f^2+g^2)$, and $(f^2+g^2)$ is a $z$-ideal by hypothesis, hence $f, g \in (f^2+g^2)$. Therefore $(f,g) \subset (f^2+g^2)$.

(6) ⇒ (7) Let $f \in A$. Taking $g = 0$ in statement (6), $(f^2) = (f)$. Therefore $f \in (f^2)$, hence there is a $g \in A$ such that $f = gf^2$. Thus $A$ is regular.

(7) ⇒ (1) Let $P$ be a prime $l$-ideal of $A$, and suppose that $M$ is an $l$-ideal of $A$ that properly contains $P$. Take then $f \in M \backslash P$, and let $g \in A$ be such that $f = f^2g$. Then $f - f^2g = 0$, so that $f(1 - fg)$ belongs to the prime ideal $P$, hence $1 - fg \in P \subset M$, and $fg \in M$. Therefore $1 = (1 - fg) + fg \in M$, and $M$ must equal $A$, proving that $P$ is maximal.

This establishes the equivalence of (1) through (7). Since every $z$-ideal is an $l$-ideal, that (1) (and (5)) imply (8) is clear; the converse, (8) implies (1), is obvious. ◇

**5.4.6 Corollary** *If $A$ is a uniformly closed $P$-algebra, then every ideal of $A$ is an $l$-ideal.*

# Chapter 6

# Closed ideals of $\Phi$-Algebras

Closed $l$-ideals of $\Phi$-algebras have been studied in [H] by M. Henriksen and [P] by D. Plank. Nice characterizations of closed $l$-ideals were attained in each of the aforementioned papers, but these descriptions are neither algebraic nor internal to the $l$-ideals themselves. In this chapter we endeavour to find such a characterization of the closed $l$-ideals of a $\Phi$-algebra using the concepts of strong divisibility and $z$-ideals. Admittedly the attempt is not entirely successful; however in section two both necessary and sufficient conditions on an $l$-ideal are found to ensure that it is closed. Moreover, it is shown that under certain constraints placed upon a $\Phi$-algebra, these two conditions agree, and hence an algebraic characterization of closed $l$-ideals is attained.

We begin by generalizing the results of chapter three, section one, concerning prime ideals and closure in $C(X)$ to arbitrary $\Phi$-algebras.

## 6.1 Closed Ideals and Prime Ideals

Let $A$ be a Φ-algebra and let $\mathcal{M}(A)$ denote the maximal ideal space of $A$. Recall from section 6, that for every $x \in \mathcal{M}(A)$ the set

$$N_x = \{a \in A : a \textit{ vanishes on a neighborhood of } x\}$$

is an $l$-ideal of $A$ contained in exactly one maximal $l$-ideal, namely $M_x$.

**6.1.1 Proposition** *For every $x \in \mathcal{M}(A)$, $\overline{N_x} = \overline{M_x}$.*

**Proof:** Let $x \in \mathcal{M}(A)$. Since $N_x \subset M_x$, $\overline{N_x} \subset \overline{M_x}$. To prove the reverse containment let $a \in M_x$ and let $\varepsilon > 0$. Then $a(x) = 0$ and since $a : \mathcal{M}(A) \to \gamma\mathbf{R}$ is continuous, $U = a^{\leftarrow}[(-\varepsilon, \varepsilon)]$ is an open neighborhood of $x$ in $\mathcal{M}(A)$. Let $b = [(a-\varepsilon) \vee 0] + [(a+\varepsilon) \wedge 0]$. Then $b \in A$, $|b - a| \leq 2\varepsilon$ and $b[U] = \{0\}$. Hence $b \in N_x$, and so $a \in \overline{N_x}$. Therefore $M_x \subset \overline{N_x}$, whence $\overline{M_x} \subset \overline{N_x}$. ◇

Now as shown in section 6, if $P$ is a prime ideal $l$-ideal of $A$, then there is a unique $x \in \mathcal{M}(A)$ such that $N_x \subset P \subset M_x$. By the proposition $\overline{N_x} = \overline{P} = \overline{M_x}$, from which it follows that non-maximal prime $l$-ideals of $A$ are never closed. Suppose however that $A$ is a uniformly closed Φ-algebra. Then by [P, 2.6] the closed maximal $l$-ideals of $A$ are precisely the real $l$-ideals of $A$. Thus if $M_x$ is a real $l$-ideal of $A$ with $P \subset M_x$, then $\overline{N_x} = \overline{P} = M_x$. Hence we have

**6.1.2 Corollary** *Let $A$ be a uniformly closed Φ-algebra.*

*1. If $M_x$ is a real $l$-ideal of $A$, then $\overline{N_x} = M_x$.*

*2. If $P$ is a prime $l$-ideal of $A$, then $\overline{P}$ is a (necessarily maximal) $l$-ideal if and only if the unique maximal $l$-ideal $M$ containing $P$ is real; in this case $\overline{P} = M$.*

## 6.2 $z$-Ideals, $Z$-ideals, and Strong Divisibility

At this point we attempt to generalize the characterization (3.1.4) of closed ideals of $C(X)$ to various types of Φ-algebras. To begin we consider another extension of the notion of $z$-ideal in the context of a Φ-algebra.

Let $A$ be a Φ-algebra. Let $\mathcal{R}(A)$ denote the space of real maximal $l$-ideals of $A$, and for each $a \in A$, let $\mathcal{S}(a) = \{M \in \mathcal{R}(A) : a \in M\}$. As before, let $\mathcal{M}(a) = \{M \in \mathcal{M}(A) : a \in M\}$. For the sake of comparison, we restate the definition of $z$-ideal.

**6.2.1 Definition** *Let $A$ be a Φ-algebra, and let $I$ be an ideal of $A$.*
*1. $I$ will be called a $z$-ideal if*
$(\mathcal{M}(b) \supset \mathcal{M}(a)$ *and* $a \in I) \Rightarrow b \in I$.
*2. $I$ will be called a $Z$-ideal if*
$(\mathcal{S}(b) \supset \mathcal{S}(a)$ *and* $a \in I) \Rightarrow b \in I$.

Clearly then, every $Z$-ideal is a $z$-ideal. It is a well-known result that if $X$ is a topological space, then the above definition of $z$-ideal agrees with the usual notion of a $z$-ideal in $C(X)$, [GJ, 4A]. In fact:

**6.2.2 Proposition** *If $X$ is a topological space, and $I$ is an ideal of $C(X)$, then $I$ is a $z$-ideal if and only if $I$ is a $Z$-ideal.*

**Proof:** Suppose $I$ is a $z$-ideal of $C(X)$ and suppose $\mathcal{S}(g) \supset \mathcal{S}(f)$, $f \in I$. Let $p \in Z(f)$. Then $f \in M_p$ and $M_p$ is real. Therefore $g \in M_p$, whence $p \in Z(g)$. Therefore $Z(f) \subset Z(g)$, and $Z(f) \in Z[I]$, $Z[I]$ a $z$-filter. It follows that $Z(g) \in Z[I]$. But $I$ is a $z$-ideal in $C(X)$ and therefore $g \in I$, showing that $I$ is a $Z$-ideal. ◇

Indeed in (6.2.7), we will show that if $A$ is any Φ-algebra of real-valued functions that is closed under inversion, (definitions of these concepts appear on page 50), then the

$z$-ideals of $A$ are $Z$-ideals. (6.2.9) provides an example of a $\Phi$-algebra in which $z$-ideals are not necessarily $Z$-ideals.

To prove the next few theorems, the following definitions and results from the papers [H] by M. Henriksen and [P] by D. Plank are needed.

In [H] an *ideal set* is defined to be a closed subset $\Delta$ of the space $\mathcal{M}(A)$ of maximal $l$-ideals of $\Phi$-algebra $A$ such that whenever $a \in A$ and $a[\Delta] = 0$, then $(ab)[\Delta] = 0$ for all $b \in A$. It is shown that

[H, 2.4] An $l$-ideal $I$ of a uniformly closed $\Phi$-algebra $A$ is closed if and only if there is an ideal set $\Delta$ of $M(A)$ such that $I = \{a \in A : a[\Delta] = 0\}$.

[P, 2.6] If $A$ is a uniformly closed $\Phi$-algebra, then a maximal $l$-ideal of $A$ is closed if and only if it is real.

[P, 3.7] If $I$ is either a closed or maximal ideal in the uniformly closed $\Phi$-algebra $A$, then $I$ is an $l$-ideal.

Recall also from chapter four, that if $A$ is uniformly closed then its sub-$\Phi$-algebra of bounded elements $A^*$ is $l$-isomorphic to $C(\mathcal{M}(A))$, and as such $C(\mathcal{M}(A))$ may be regarded as a sub-$\Phi$-algebra of $A$.

These results are used in the proofs that follow.

**6.2.3 Theorem** *Closed ideals of uniformly closed $\Phi$-algebras are strongly divisible $z$-ideals.*

**Proof:** Let $I$ be a closed ideal of uniformly closed $\Phi$-algebra $A$. Let $I^* = I \bigcap A^*$. Then $I^*$ is a closed ideal of $A^* = C(\mathcal{M}(A))$, and therefore is strongly divisible by (3.1.4). Let

$(a_n)$ be a countable subset of $I$. $A$ is closed under bounded inversion, so for each $n$, $(1+|a_n|)^{-1}$ exists in $A$. But each $\frac{a_n}{1+|a_n|} \in A^* \bigcap I = I^*$, so by the strong divisibility of $I^*$ there is a $b \in I^*$, and $(b_n) \subset A^*$ such that for each $n$, $b \cdot b_n = \frac{a_n}{1+|a_n|}$. Hence $b \in I$ and for each $n$, $b \cdot b_n \cdot (1+|a_n|) = a_n$, proving that $I$ is strongly divisible. As a closed ideal of uniformly closed $A$, $I$ is an $l$-ideal, [P, 3.7], so let $\Delta$ be an ideal set such that $I = \{a \in A : a[\Delta] = 0\}$. Suppose $\mathcal{M}(b) \supset \mathcal{M}(a)$ and $a \in I$. If $x \in \Delta$ then for all $c \in A$, $(ac)(x) = 0$, hence $a \in M_x$, by (4.2.2). Therefore $b \in M_x$, hence $b(x) = 0$. Thus $b[\Delta] = 0$ and therefore $b \in I$. This proves that $I$ is a $z$-ideal. ◇

**6.2.4 Theorem** *Any strongly divisible Z-ideal of a Φ-algebra A is closed in A.*

**Proof:** Let $I$ be a strongly divisible $Z$-ideal of Φ-algebra $A$, let $b \in \overline{I}$. Take $(b_n)$ a countable subset of $I$ such that for each $n \in \mathbf{N}$, $|b - b_n| < 1/n$. Then by the strong divisibility of $I$, there is an $a \in I$ and $(a_n) \subset A$ such that for each $n$, $a \cdot a_n = b_n$. Suppose $a \in M_x$, where $M_x$ is a real ideal of $A$. Then $a(x) = 0$ and therefore for each $n \in \mathbf{N}$, $a \cdot a_n(x) = 0$. Therefore for all $n$, $1/n > |a \cdot a_n(x) - b(x)| = |b(x)|$, hence $|b(x)| = 0$; therefore $b \in M_x$ by (4.2.3). This shows that $\mathcal{S}(a) \subset \mathcal{S}(b)$. But $a \in I$ and $I$ is a $Z$-ideal, so $b \in I$. Therefore $I = \overline{I}$, hence $I$ is closed. ◇

A natural question to ask is whether or not strongly divisible $z$-ideals in Φ-algebras are always closed. Certainly an argument similar to the one given above will not answer the question in the affirmative; however, if we assume that $A$ is a P-algebra, then the full converse to (6.2.3) holds. P-algebras were characterized in the fourth section of chapter five.

**6.2.5 Theorem** *If A is a P-algebra, then strongly divisible z-ideals in A are closed.*

**Proof:** Letting $I$ be a strongly divisible $z$-ideal in $A$, choose $b \in \overline{I}$. As in the proof of (6.2.4), we may find $a \in I$ and $(a_n) \subset A$ such that for each $n \in \mathbf{N}$, $|a \cdot a_n - b| < 1/n$.

Taking $x \in \mathcal{M}(a)$, for each $n$, $(a \cdot a_n)(x) = 0$, hence $b(x) = 0$, and therefore $\mathcal{M}(a) \subset b^{\leftarrow}(0)$. But $A$ is a P-algebra, so by (5.5.1), $\mathcal{M}(a)$ is open, hence $\mathcal{M}(a) \subset int(b^{\leftarrow}(0))$, and since $int(b^{\leftarrow}(0)) \subset \mathcal{M}(b)$, we have $\mathcal{M}(a) \subset \mathcal{M}(b)$. [We note that were $A$ not assumed to be a P-algebra, $\mathcal{M}(a) \subset \mathcal{M}(b)$ could not be shown in this manner]. But $I$ is a $z$-ideal in $A$ and $a \in I$, hence $b \in I$, proving that $I = \overline{I}$. ◇

Recall from [HJ] that a Φ-algebra $A$ is called an *algebra of real-valued functions* if its space of real maximal ideals $\mathcal{R}(A)$ is dense in $\mathcal{M}(A)$; in this case $A$ can be embedded as a sub-Φ-algebra of $C(\mathcal{R}(A))$. A Φ-algebra of real-valued functions $A$ is *closed under inversion* if, for all $a \in A$, $a^{\leftarrow}(0) \bigcap \mathcal{R}(A) = \emptyset$ implies $(a)_l = A$, where $(a)_l$ denotes the smallest $l$-ideal in $A$ containing $a$. The following is [HJ, 4.6].

**6.2.6** *If $A$ is a Φ-algebra of real-valued functions which is closed under inversion, then for each $x \in \mathcal{M}(A)$,*

$$M_x = \{a \in A : x \in (a^{\leftarrow}(0) \bigcap \mathcal{R}(A))^{-}\}.$$

We use this theorem to show that within the class of Φ-algebras of real-valued functions, that every $z$-ideal is a $Z$-ideal characterizes the property closed under inversion.

**6.2.7 Theorem** *Let $A$ be a Φ-algebra of real-valued functions. Then the following are equivalent.*

*(1) $A$ is closed under inversion.*

*(2) $z$-ideals in $A$ are $Z$-ideals.*

**Proof:** (1) ⇒ (2) Suppose $I$ is a $z$-ideal and $\mathcal{S}(b) \supset \mathcal{S}(a)$, $a \in I$. Let $x \in \mathcal{M}(A)$ and suppose $a \in M_x$. Then by the above, $x \in (a^{\leftarrow}(0) \bigcap \mathcal{M}(A))^{-}$. Now suppose $y \in a^{\leftarrow}(0) \bigcap \mathcal{R}(A)$. Then $a(y) = 0$ and $y \in \mathcal{R}(A)$, so $a \in M_y$. But $\mathcal{S}(b) \supset \mathcal{S}(a)$, so $b \in M_y$, hence $b(y) = 0$. Therefore $y \in b^{\leftarrow}(0) \bigcap \mathcal{R}(A)$, whence $(a^{\leftarrow}(0) \bigcap \mathcal{R}(A))^{-} \subset (b^{\leftarrow}(0) \bigcap \mathcal{R}(A))^{-}$. By the above theorem it follows that $b \in M_x$, hence $\mathcal{M}(b) \supset \mathcal{M}(a)$. But $a \in I$ and $I$ is a $z$-ideal, so $b \in I$. This shows that $I$ is a $Z$-ideal.

(2) ⇒ (1) Supposing $A$ is not closed under inversion, we may take $a \in A$ such that $a^{\leftarrow}(0) \cap \mathcal{R}(A) = \emptyset$, yet $(a)_l \neq A$. Then $\mathcal{S}(a) = \emptyset$, and a maximal $l$-ideal $M_x$ may be found such that $(a)_l \subset M_x$. Thus if $b \in A \backslash M$,then $\mathcal{S}(b) \supset \mathcal{S}(a)$, $a \in M_x$, yet $b \notin M_x$. It follows that the $z$-ideal $M_x$ is not a $Z$-ideal. ◇

As a consequence of the above we get the following characterization of the closed ideals in two broad classes of Φ-algbras.

**6.2.8 Corollary** *Let $A$ be a uniformly closed Φ-algebra that is either*
*(i) a Φ-algebra of real-valued functions that is closed under inversion, or*
*(ii) regular.*
*Then*
1. *An $l$-ideal $I$ of $A$ is closed if and only if it is a strongly divisible $z$-ideal.*
2. *A maximal $l$-ideal of $A$ is real if and only if it is strongly divisible.*

**6.2.9 Example** *A uniformly closed Φ-algebra of real-valued functions in which $z$-ideals need not be $Z$-ideals.*

An example of Henriksen and Johnson [HJ, 4.5], does the job. Let $A = \{f \in C(\mathbf{R}^+) : lim_{x \to \infty} f(x)e^{-ax} = 0 \; for \; all \; real \; a > 0\}$. Then $A$ is a uniformly closed Φ-algebra of real-valued functions. $A^*$ and $C(\mathbf{R}^+)$ are isomorphic, so $\mathcal{M}(A) = \mathcal{M}(A^*) = \mathcal{M}(C^*(\mathbf{R}^+)) = \beta\mathbf{R}^+$. Take $g(x) = e^{-x}$, $(x \in \mathbf{R}^+)$. Clearly then $g \in A$. Moreover, $g^{\leftarrow}(0) = \beta\mathbf{R}^+ \backslash \mathbf{R}^+ = \mathcal{M}(A) \backslash \mathcal{R}(A)$, therefore, $\mathcal{M}(g) \subset \mathcal{M}(A) \backslash \mathcal{R}(A)$, and hence $\mathcal{S}(g) = \emptyset$. But $1/g \notin A$, so $(g) \neq A$, and therefore there is a maximal $l$-ideal $M$ containing $g$. $M$ is then a $z$-ideal but is not a $Z$-ideal. For take any $h \notin M$. Then $\mathcal{S}(h) \supset \mathcal{S}(g)$, $g \in M$, but $h \notin M$.

The Henriksen-Johnson representation theorem has been used to characterize $C(X)$, for various homeomorphism classes of spaces $X$, algebraically within the class of Φ-algebras. Characterizations of several such classes of (realcompact) Tychonoff spaces may be found in [HJ] and many other papers. Nevertheless a longstanding open problem concerning Φ-algebras asks, 'find an internal characterization of $C(X)$ for $X$ a Tychonoff space within the class of Φ-algebras', [$H_3$, problem 5]. In the remainder of this section a discussion and modification of a Φ-algebra characterization of $C(X)$, for a Lindelöf space $X$, due to D. Plank, [P, 3.7], will be presented. We remark that Plank's result receives high praise in each of the survey articles [$H_2$] and [$H_3$] by M. Henriksen. Finally we propose by way of conjecture a Φ-algebra characterization of $C(X)$, for $X$ a weakly Lindelöf space.

In [P], Plank calls a Φ-algebra $A$ normal if every closed $l$-ideal of $A$ is contained in a closed maximal $l$-ideal of $A$, and proves that if $A$ is normal, then $\mathcal{R}(A)$ is a dense Lindelöf subspace of $\mathcal{M}(A)$, (and hence $A$ is a Φ-algebra of real-valued functions). He calls $A$ closed under inversion if $(a)_l = A$ whenever $a \in A$, $a$ is not a zero-divisor in $A$, and $a \notin M$ for all $M \in \mathcal{R}(A)$. This concept agrees with the definition of closed under inversion given earlier provided that $A$ is a Φ-algebra of real-valued functions. Since we shall only be concerned with this property in the context of Φ-algebras of real-valued functions, we will not distinguish between the two definitions. Plank proves the following theorem, which is itself a modification of a theorem found in the Henriksen-Johnson paper [HJ]. Note that $A$ is not assumed to be a Φ-algebra of real-valued functions; it follows from the assumption that $A$ is normal, [P, 3.1].

**6.2.10** [P, 3.7] **Theorem:** *A non-trivial Φ-algebra A is l-isomorphic to $C(X)$, for some Lindelöf space $X$, if and only if*

*(i) A is uniformly closed,*

*(ii) A is closed under inversion, and*

*(iii) A is normal.*

Using the vocabulary of the present section, we provide the following reformulation of Plank's result, which phrased in terms of strongly divisible, rather than closed ideals, perhaps seems slightly more algebraic in character.

**6.2.11 Theorem:** *A non-trivial Φ-algebra A is l-isomorphic to C(X) for some Lindelöf space X if and only if*
*(i) A is uniformly closed,*
*(ii) every z-ideal in A is a Z-ideal, and*
*(iii) every strongly divisible (z−)ideal of A is contained in a strongly divisible maximal l-ideal of A.*

The bracketed $z$ is meant to indicate that it is optional.

**Proof** Necessity is clear. Conversely assume conditions $(i),(ii),(iii)$ and let $I$ be a closed ideal of $A$. By (6.2.3) $I$ is a strongly divisible $z-$ideal in $A$, therefore contained in a strongly divisible maximal $l$-ideal of $A$ by (iii). But by (ii) $M$ is a strongly divisible $Z$-ideal, and therefore closed by (6.2.4), showing that $A$ is normal. Now by Plank's result (quoted earlier) $A$ is then a Φ-algebra of real-valued functions, hence by (ii) and (6.2.7), $A$ is closed under inversion, and sufficiency follows from (6.2.10). ◇

To conclude this section we offer the following conjecture, which despite persistent efforts we were unable to prove. Due to the fact that any $C(X)$ is $(l)$-isomorphic to $C(\upsilon X)$, we may as well assume that $X$ is realcompact. Recall that a strongly divisible ideal comprised entirely of zero-divisors is called neighborhood strongly divisible, or simply nsd.

**6.2.12 Conjecture** *A non-trivial Φ-algebra A is l-isomorphic to C(X) for some weakly Lindelöf (realcompact) space X if and only if*
*(i) A is uniformly closed,*
*(ii) every z-ideal in A is a Z-ideal, and*
*(iii) every nsd ideal in A is contained in a strongly divisible maximal l-ideal of A.*

We note by way of evidence to support the validity of this statement, Theorem 3.3.4, which under the additional hypothesis of realcompactness of $X$ equivalently reads:

$X$ is weakly Lindelöf if and only if every nsd ideal of $C(X)$ is contained in a strongly divisible maximal ideal of $C(X)$.

Hence the conditions (i), (ii), and (iii) of (6.2.12) are certainly necessary. We also note that arguing as in the proof of [P, 3.1], it follows from (i), (ii) and (iii) that $A$ is a $\Phi$-algebra of real-valued functions; unfortunately it seems that a proof of (6.2.12) would at this point require an alternative to Plank's methods.

## 6.3 Factor Algebras of Φ-algebras

In this section an attempt is made to describe the relationship between an element of a $\Phi$-algebra, and its corresponding coset in a factor algebra with respect to a given $l$-ideal. It is well-known that if $M$ is a maximal ideal of $C(X)$, then $M$ is real if and only if $Z[M]$ is closed under countable intersection. The motivation for the following was to obtain such a characterization of the real maximal ideals of a given $\Phi$-algebra, in terms of the subsets $\mathcal{M}(f)$, ($f \in A$). Unfortunately, such a nice result was not, in the general case obtained, though we did obtain some partial results. We develop the following as in chapter 5 of [GJ].

Recall from chapter four, that if $I$ is an $l$-ideal of $A$, and $f \in A$, then $I(f)$ denotes the image of $f$ under the canonical ($l$-)homomorphism from $A$ onto $A/I$. Before reading the proof of the next statement the reader may wish to review 1.1 of chapter four, and the discussion of the Henriksen-Johnson representation of an element of a given $\Phi$-algebra, which is found in the same chapter.

**6.3.1 Proposition:** *Let $A$ be a $\Phi$-algebra.*

1. *If $I$ is a z-ideal in $A$, then $I(f) \geq 0$ if and only if $f$ is non-negative on some member of $\mathcal{M}[I]$.*
2. *If $f$ is positive on some member of $\mathcal{M}[I]$, where $I$ is a z-ideal of $A$, then $I(f) > 0$.*

**Proof** 1. Supposing that $I(f) \geq 0$, it follows from (4.1.1) that $f - |f| \in I$, and thus $\mathcal{M}(f - |f|) \in \mathcal{M}[I]$. But $x \in \mathcal{M}(f - |f|)$ implies that $f - |f| \in M_x$, hence $M_x(f) = M_x(|f|)$. By the Henriksen-Johnson representation of $f, |f|$, $f(x) = |f|(x)$, showing that $f$ is non-negative on the member $\mathcal{M}(f - |f|)$ of $\mathcal{M}[I]$.

Conversely suppose that $g \in I$ and $f(x) \geq 0$,for every $x \in \mathcal{M}(g)$. Then for each $x \in \mathcal{M}(g)$, $M_x(f) \geq 0$, hence by (4.1.1), $f - |f| \in M_x$. Thus for each $x \in \mathcal{M}(g)$, $x \in \mathcal{M}(f - |f|)$ - that is $\mathcal{M}(g) \subset \mathcal{M}(f - |f|)$. But $I$ is a $z$-ideal, and $g \in I$, therefore $f - |f| \in I$, and hence $I(f) \geq 0$.

2. If $f$ is positive on $\mathcal{M}(g)$ for some $g \in I$, Then $\mathcal{M}(f) \bigcap \mathcal{M}(g) = \emptyset$, hence $f \notin I$. Therefore $I(f) \geq 0$ by part 1. and $I(f) \neq 0$. Therefore $I(f) > 0$. ◊

**6.3.2 Corollary:** *If $x \in \mathcal{M}(A)$, then $f(x) \geq 0$ if and only if $f$ is non-negative on a member of $\mathcal{M}[M_x]$.*

**Proof:** $f(x) \geq 0$ if and only if $M_x(f) \geq 0$, by the Henriksen-Johnson representation theorem. The result follows from part 1. above. ◊

As a second corollary we get the following extension of Theorem 5.2.2.

**6.3.3 Corollary:** *If $I$ is a z-ideal, then $A/I$ is totally ordered if and only if $I$ is prime.*

**Proof:** $A/I$ is totally ordered if and only if for every $f \in A$, $I(f) \geq 0$ or $I(f) \leq 0$, if and only if for every $f \in A$, $I(f) \geq 0$ or $I(-f) \geq 0$, if and only if for every $f \in A$, $f$ does not change sign on a member of $\mathcal{M}[I]$, (6.3.1), if and only if $I$ is prime, by (5.2.2). ◊

In fact assuming only that $P$ is a prime $l$-ideal, not necessarily a $z$-ideal we can show that $A/P$ is totally ordered. For take $f \in A$. Clearly, for every $x \in \mathcal{R}(f) = f^{\leftarrow}[\mathbf{R}]$, $(f - |f|)(f + |f|)(x) = 0$. But $\mathcal{R}(f)$ is dense in $\mathcal{M}(A)$, so $(f - |f|)(f + |f|) = 0$. $P$ is prime so $f - |f| \in P$, or $f + |f| \in P$, hence $P(f) = P(|f|)$, or $P(f) = P(-|f|)$ for every

$f \in A$. Thus $P(f) \geq 0$, or $P(f) \leq 0$ for every $f \in A$ (as $P : A \to A/P$ is order preserving, (4.1.1)), hence $A/P$ is totally ordered.

Note that since maximal ideals needn't be $l$-ideals, [HJ], it is also true that prime ideals needn't be $l$-ideals, unlike in a $C(X)$, [GJ, 5.5].

**6.3.4 Lemma:** *Let $A$ be a $\Phi$-algebra, $f \in A$, and let $\lambda$ be a positive real number. Let*

$$U_\lambda = \{x \in \mathcal{M}(A) : |f(x)| > \lambda\}, \qquad F_\lambda = \{x \in \mathcal{M}(A) : |f(x)| \geq \lambda\}.$$

*Then taking $g_\lambda = \lambda - (|f| \wedge \lambda)$, $U_\lambda \subset \mathcal{M}(g_\lambda) \subset F_\lambda$.*

**Proof:** $g_\lambda \geq 0$ and $g_\lambda(x) = 0$ if and only if $x \in F_\lambda$. Therefore if $x \in \mathcal{M}(g_\lambda)$, then $x \in F_\lambda$, that is $\mathcal{M}(g_\lambda) \subset F_\lambda$. Also $g_\lambda$ vanishes on the open set $U_\lambda$ so $U_\lambda \subset int(g^{\leftarrow}(0)) \subset \mathcal{M}(g_\lambda)$.
◇

**6.3.5 Proposition:** *Let $x \in \mathcal{M}(A)$, $f \in A$. For each $n = 1, 2, ...,$ let $g_n = n - (|f| \wedge n)$. Then the following are equivalent:*

1. *$|M_x(f)|$ is infinitely large.*
2. *$|f(x)| = \infty$.*
3. *$f$ is unbounded on every member of $\mathcal{M}[M_x]$.*
4. *For each $n$, $g_n \in M_x$.*

**Proof:** The equivalence of 1. and 2. is clear from the Henriksen-Johnson representation of $f$.

2. ⇔ 3. $\mathcal{M}[M_x] = \{\mathcal{M}(g) : x \in \mathcal{M}(g)\}$. Clearly then, if $|f(x)| = \infty$, then $f$ is unbounded on every member of $\mathcal{M}[M_x]$. Conversely suppose that $|f(x)| = \lambda < \infty$ and take $\mathcal{U} = f^{\leftarrow}[\lambda - 1, \lambda + 1]$. If $\mathcal{U} = \mathcal{M}(A)$, then $f$ is bounded, hence 3. is false. Otherwise, $\mathcal{U}$ is a closed neighborhood of $x$, and $\mathcal{U} \neq \mathcal{M}(A)$. Using the normality of $\mathcal{M}(A)$, take $\mathcal{V}$ a closed neighborhood of $x$ such that $\mathcal{V} \subset int\mathcal{U}$. Then choose $g \in A$ such that $g[\mathcal{V}] = 0$, $g[\mathcal{M}(A) \backslash int\mathcal{U}] = 1$. Then $x \in int g^{\leftarrow}(0) \subset \mathcal{M}(g) \subset \mathcal{U}$. Hence $\mathcal{M}(g) \in \mathcal{M}[M_x]$, and since

$f$ is bounded on $\mathcal{U}$, $f$ is bounded on $\mathcal{M}(g)$. Therefore 3. is false.

2. $\Rightarrow$ 4. $|f(x)| = \infty$, so by the lemma $x \in f^{\leftarrow}(n, \infty] \subset \mathcal{M}(g_n)$, hence $g_n \in M_x$.

4. $\Rightarrow$ 2. If for every $n$, $g_n \in M_x$, then for every $n$, $g_n(x) = 0$, hence $|f(x)| \geq n$. Therefore $|f(x)| = \infty$. ◇

**6.3.6 Proposition:** *Suppose that $A$ is a uniformly closed Φ-algebra, and $x \in \mathcal{R}(A)$. Then for any $(f_n) \subset M_x$ there is an $s \in M_x$ such that $\mathcal{M}(s) \subset \bigcap_n \mathcal{M}(f_n)$. In other words, every countable subfamily of $\mathcal{M}[M_x]$ has a lower bound in $\mathcal{M}[M_x]$.*

**Proof:** For each $n$, $\mathcal{M}(f_n) = \mathcal{M}(\frac{|f_n|}{1+|f_n|})$, so without loss of generality we may assume that $0 \leq f_n \leq 1$, for every $n$. Take for each $n$, $s_n = \sum_{k=1}^{n} 2^{-k} f_k$. Then $(s_n)$ is a Cauchy sequence in uniformly closed $A$, so there is an $s \in A$ such that $s_n \to s$. As a real ideal of uniformly closed $A$, $M_x$ is closed in $A$. Each $s_n \in M_x$, therefore $s \in M_x$. Also because each $s_n \leq s$, $\mathcal{M}(s) \subset \mathcal{M}(s_n)$, therefore $\mathcal{M}(s) \subset \bigcap_n \mathcal{M}(s_n)$. But for each $n$, $\mathcal{M}(s_n) = \bigcap_1^n \mathcal{M}(f_k)$, from which the proposition follows. ◇

In the context of uniformly closed regular Φ-algebras, the next Proposition generalizes (3.1.2).

**6.3.7 Proposition** *Let $A$ be a Φ-algebra, $I$ an $l$-ideal of $A$.*

*1. If $A$ is uniformly closed, regular and every countable subfamily of $\mathcal{M}[I]$ has a lower bound in $\mathcal{M}[I]$, then $I$ is strongly divisible.*

*2. If $I$ is strongly divisible, then every countable subfamily of $\mathcal{M}[I]$ has a lower bound in $\mathcal{M}[I]$.*

**Proof:** 1. Let $(f_n) \subset I$. By assumption there is a $g \in I$ such that $\mathcal{M}(g) \subset \mathcal{M}(f_n)$ for each $n$. By (5.4.5) each $f_n$ is a multiple of $g$, hence $I$ is strongly divisible.

2. Let $(f_n) \subset I$ so that $(\mathcal{M}(f_n))$ is a countable subfamily of $\mathcal{M}[I]$. Choosing $g \in I$ such that $g$ divides each $f_n$, it is clear that $\mathcal{M}(g)$ is a member of $\mathcal{M}[I]$ contained in each $\mathcal{M}(f_n)$. ◇

**6.3.8 Corollary:** *Let $A$ be a uniformly closed, regular Φ-algebra, and $M_x$ a maximal $l$-ideal of $A$. Then the following are equivalent:*

*(1) $M_x$ is real.*

*(2) $M_x$ is strongly divisible.*

*(3) Every countable subfamily of $\mathcal{M}[M_x]$ has a lower bound in $\mathcal{M}[M_x]$.*

**Proof:** The equivalence of (2) and (3) is immediate from the above proposition.
(1) $\Leftrightarrow$ (2) is (6.2.8). ◇

**Question:** Is $\mathcal{M}[A] = \{\mathcal{M}(f) : f \in A\}$ closed under countable intersection as is true with the lattice of zero-sets on a topological space ? Can it at least be shown that $\mathcal{M}[A]$ is closed under countable intersection under the additional hypotheses that $A$ is regular and/or uniformly closed ?

Recall that a Φ-algebra is normal if every closed $l$-ideal of $A$ is contained in a closed maximal $l$-ideal.

**6.3.9 Corollary:** *If $A$ is a uniformly closed regular Φ-algebra, then the following are equivalent:*

*1. $A$ is normal.*

*2. Every strongly divisible $z$-ideal of $A$ is contained in a strongly divisible maximal ideal of $A$.*

*3. Every strongly divisible ideal of $A$ is contained in a strongly divisible maximal ideal of $A$.*

**Proof:** 1. $\Leftrightarrow$ 2. is immediate from (6.2.8).
2. $\Rightarrow$ 3. If $I$ is a strongly divisible ideal, then by (6.3.7), $J = \mathcal{M}^{\leftarrow}[\mathcal{M}[I]]$ is a strongly divisible $z$-ideal containing $I$, by 2. then, $I$ is contained in a strongly divisible maximal ideal.
3. $\Rightarrow$ 2. is obvious. ◇

With the above in mind, the following theorem, which was first discovered by B. Brainerd, and later modified as stated below by D. Plank in [P], is in fact a purely algebraic characterization of $C(X)$, where $X$ is a Lindelöf P-space. Of course, condition 2, can be replaced with any of its equivalent forms as found in (5.4.3).

A $\Phi$-algebra is called $\sigma$-complete if every countable subset of $A$ that is bounded above in $A$ has a supremum in $A$. A $\sigma$-complete $\Phi$-algebra is necessarily uniformly closed, [P].

**6.3.10** [P, 4.2] **Theorem:** *A non-trivial $\Phi$-algebra $A$ is isomorphic to $C(X)$ for some Lindelöf P-space $X$ if and only if*

1. *$A$ is $\sigma$-complete,*
2. *$A$ is regular, and*
3. *$A$ is normal.*

# Bibliography


[A1] F. Azarpanah, *Algebraic properties of some compact spaces*, to appear.

[A2] F. Azarpanah, *Essential ideals in C(X)*, Priod. Math. Hungar, 31(2)(1995), 105-112.

[BP] G. Birkhoff and R.S. Pierce, *Lattice-ordered rings*, An. Acad. Brasil Ci., vol. 28 (1956), 42-69.

[D] G. De Marco, *On the countably generated z-ideals of C(X)*, PAMS, 31(2)(1972), 574-576.

[GJ] L. Gillman and M. Jerison, *Rings of continuous functions*, Springer-Verlag, 1976.

[H1] M. Henriksen, *Uniformly closed ideals of uniformly closed algebras of extended real-valued functions*, Symposia Math. 17 (1976), 49-53.

[H2] M. Henriksen, *Rings of continuous functions from an algebraic point of view*, Ordered Algebraic Structures, Kluwer Academic Publishers 1989, 144-174.

[H3] M. Henriksen, *A survey of f-rings and some of their generalizations*, Ordered Algebraic Structures, Kluwer Academic Publishers 1997, 1-26.

[Hew] E. Hewitt, *Rings of real-valued continuous functions.I*, Trans. Amer. Math. Soc. 64.(1948), 45-99.

[HJ] M. Henriksen and D.G. Johnson, *On the structure of a class of Archimedian lattice-ordered algebras*, Fund. Math., 50 (1961), 73-94.

[M] G. Mason, *z-ideals and prime ideals*, Journal of Algebra, 26, (1973), 280-297.

[Mu] M.E. Munroe, *Introduction to Measure Theory and Integration*, Addison-Wesely, 1953.

[NP] P. Nanzetta and D. Plank, *Closed ideals in C(X)*, 35, (1972), 601-606.

[P] D. Plank, *Closed l-ideals in a class of lattice-ordered algebras*, Illinois J. Math., 15, (1971), 515-224.

[PW] J. Porter and R.G. Woods, *Extensions and Absolutes of Hausdorff Spaces*, Springer-Verlag, New York 1987.

[RW] R.M. Raphael and R.G. Woods, *The epimorphic hull of C(X)*, to appear.