"REGIONS OF REAL MOTION FOR
JUPITER'S SATELLITES AND COMETS"

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INTRODUCTION

Mathematicians have not yet succeeded in obtaining the general solution of the "Problem of Three Bodies" though many of the ablest and most noted of them have devoted considerable attention to this intricate problem of Celestial Mechanics. A few particular solutions have been discovered, but these depend on special conditions of mass, distance, or motion in space.

Two of the bodies are assumed to be finite; the third infinitesimal, that is, it can be attracted by the finite bodies or masses, but it does not attract them or disturb their motion within an arbitrarily assigned time, however long.

One of the finite bodies chosen is the sun, the other is a planet, and the infinitesimal body is usually a satellite of the second finite body, or a comet, which is subject to the attractions of both the finite bodies. The masses of the known satellites and comets of the solar system are small in comparison with the sun and nearly all of the planets, and for the purpose of discussion they are assumed to be infinitesimal, though this assumption is not entirely warrantable, except in a few cases.

Granting that this assumption represents the true state of affairs, we proceed to ascertain those portions of relative space in which the infinitesimal body can move under the joint attraction of the two finite bodies, and in what portions of such space it can not move.

The object of this paper is to find approximately the nature of the confines of each of the satellites of Jupiter and a few of his family of comets moving under the joint attractions of the sun and Jupiter, but assuming that no disturbing influences other than those of the sun and Jupiter enter. The method employed to obtain such regions of relative space is the solution of a number of particular algebraical equations. Having found the solutions of these equations, the general principles of the "Problem of Three Bodies" are then applied, and the nature of the regions is inferred.

MOTION OF THE INFINITESIMAL BODY

Following the method and notation of Moulton (Celestial Mechanics p.278, et seq.) we take the sum of the masses of the sun and Jupiter as the unit of mass, so that the mass of the sun can be represented by $1-\mu$, and the mass of Jupiter by μ, the notation so chosen that μ is never greater than $\frac{1}{2}$. Now, the mass of the sun is 329,390 times, and the mass of Jupiter is 314.50 times the mass of the earth. It follows that.

$$1-\mu = \cdot 9990461154,$$

$$\mu = \cdot 0009538845.$$

For the sake of brevity we shall call the sun $1-\mu$, and Jupiter μ, and the satellite or comet, the infinitesimal body. The constant distance between $1-\mu$ and μ is 483.3 millions of miles, and we shall call this "unit distance". The velocity of Jupiter in his orbit is taken as unit velocity and by taking our origin at the centre of mass of $1-\mu$ and μ, and assuming that $1-\mu$ and μ revolve in circles about their centre of mass their mean angular motion is proportional to their linear velocities, and the gravitational constant k^2 becomes equal to unity. Let the direction of the axes of co-ordinates, with our origin at the centre of mass of $1-\mu$ and μ, be so chosen that the $\xi\eta$ - plane is the plane of their motion. Let the co-ordinates of $1-\mu$, μ, and the infinitesimal body be

$\xi_1, \eta_1, 0$; $\xi_2, \eta_2, 0$; and ξ, η, ζ respectively and let r_1 and r_2 denote the respective distances of $1-\mu$ and μ from the infinitesimal body, so that

$$r_1 = \sqrt{(\xi-\xi_1)^2 + (\eta-\eta_1)^2 + \zeta^2},$$

$$r_2 = \sqrt{(\xi-\xi_2)^2 + (\eta-\eta_2)^2 + \zeta^2}.$$

Then the differential equations of motion for the infinitesimal body are

$$(1) \quad \begin{cases} \dfrac{d^2\xi}{dt^2} = -(1-\mu)\dfrac{(\xi-\xi_1)}{r_1^3} - \mu\dfrac{(\xi-\xi_2)}{r_2^3}, \\[2ex] \dfrac{d^2\eta}{dt^2} = -(1-\mu)\dfrac{(\eta-\eta_1)}{r_1^3} - \mu\dfrac{(\eta-\eta_2)}{r_2^3}, \\[2ex] \dfrac{d^2\zeta}{dt^2} = -(1-\mu)\dfrac{\zeta}{r_1^3} - \mu\dfrac{\zeta}{r_2^3}. \end{cases}$$

Refer the motion of the bodies to a new system of axes with the same origin as the old, and rotating in the $\xi\eta$ - plane in the direction of motion of $1-\mu$ and μ. The equations of transformation are

$$(2) \quad \begin{cases} \xi = x\cos t - y\sin t, \\ \eta = x\sin t + y\cos t, \\ \zeta = z, \end{cases}$$

where x, y, z are the co-ordinates in the new system, with similar transformations for letters with the subscripts 1 and 2.

Equations (1) reduce to

$$
(3)\quad
\begin{cases}
\dfrac{d^2x}{dt^2} - 2\dfrac{dy}{dt} = x - (1-\mu)\dfrac{(x-x_1)}{r_1^3} - \mu\dfrac{(x-x_2)}{r_2^3}, \\[2ex]
\dfrac{d^2y}{dt^2} + 2\dfrac{dx}{dt} = y - (1-\mu)\dfrac{(y-y_1)}{r_1^3} - \mu\dfrac{(y-y_2)}{r_2^3}, \\[2ex]
\dfrac{d^2z}{dt^2} = -(1-\mu)\dfrac{z}{r_1^3} - \mu\dfrac{z}{r_2^3}.
\end{cases}
$$

Let us take the position of the x - axis at the origin of time so that it will continually pass through the centres of $1-\mu$ and μ.

Referred to this system of rotating axes, $y_1 = y_2 = 0$ and equations (3) become

$$
(4)\quad
\begin{cases}
\dfrac{d^2x}{dt^2} - 2\dfrac{dy}{dt} = x - (1-\mu)\dfrac{(x-x_1)}{r_1^3} - \mu\dfrac{(x-x_2)}{r_2^3}, \\[2ex]
\dfrac{d^2y}{dt^2} + 2\dfrac{dx}{dt} = y - (1-\mu)\dfrac{y}{r_1^3} - \mu\dfrac{y}{r_2^3}, \\[2ex]
\dfrac{d^2z}{dt^2} = -(1-\mu)\dfrac{z}{r_1^3} - \mu\dfrac{z}{r_2^3}.
\end{cases}
$$

By the introduction of the function U defined by the equation

$$U = \tfrac{1}{2}(x^2+y^2) + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2}$$

the right-hand members of (4) are of the forms $\frac{\delta U}{\delta x}$, $\frac{\delta U}{\delta y}$, and $\frac{\delta U}{\delta z}$, respectively, hence by multiplying the both members of (4) by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$, and $2\frac{dz}{dt}$, and adding, equations (4) admit the integral

(5) $$V^2 = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C,$$

where C, the constant of integration, is known as the <u>Constant of Relative Energy</u>, and whose value depends on the nature of the mass of, and its distribution in the infinitesimal body from which it arises.

In (5) we have expressed a relation between the square of the velocity of the infinitesimal body, its co-ordinates and distances from $1-\mu$ and μ at any instant, and its C. If, from initial conditions, we can determine the C, (5) will give us the velocity with which the infinitesimal body will move and its position in rotating space. If then we determine x, y, r_1 and r_2 such that V^2 is negative we have a position in space in which the infinitesimal body can <u>not</u> move. By assigning different values to x, y, r_1 and r_2 consistent with each other we can find the locus of all points in relative space where the infinitesimal body can be. If in (5) we put $V^2 = 0$ the equation becomes

$$x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C = 0,$$

which is the equation, for a particular C, of the surfaces of zero relative velocity. On one side of these surfaces, the infinitesimal body will have relative motion, on the other side it can not be, and on the surfaces defined by the equations

(6) $$\begin{cases} x^2 + y^2 + \dfrac{2(1-\mu)}{r_1} + \dfrac{2\mu}{r_2} - C = 0, \\ r_1 = \sqrt{(x-x_1)^2 + y^2 + z^2}, \\ r_2 = \sqrt{(x-x_2)^2 + y^2 + z^2}, \end{cases}$$

it will have no velocity relative to $1-\mu$ and μ, that is, for a single instant the three bodies will move in space as a rigid whole.

We can get an approximate idea of the nature of the Surfaces of Zero Relative Velocity from the shape of the curves of intersection of the surfaces with each of the planes of reference.

If we put $z = 0$ in the first equation of (6) we get the equations of the curves in the xy - plane for particular C's, viz,

$$x^2 + y^2 + \frac{2(1-\mu)}{\sqrt{(x-x_1)^2 + y^2}} + \frac{2\mu}{\sqrt{(x-x_2)^2 + y^2}} = C.$$

Put $y = 0$, and we get the curves in the xz - plane, viz,

$$x^2 + \frac{2(1-\mu)}{\sqrt{(x-x_1)^2 + z^2}} + \frac{2\mu}{\sqrt{(x-x_2)^2 + z^2}} = C.$$

Put $x = 0$ and we get the curves in the yz - plane, viz,

$$y^2 + \frac{2(1-\mu)}{\sqrt{x_1^2 + y^2 + z^2}} + \frac{2\mu}{\sqrt{x_2^2 + y^2 + z^2}} = C.$$

(See Moulton's "Celestial Mechanics" Art 155, and T.J.J.See's "Researches on the Evolution of the Stellar Systems" Vol 11 p.p. 166 - 175 for a discussion of the foregoing three equations.)

The following is a summary of the general character of the surfaces of constant relative energy, i.e. for different values of C .

" For large values of C they consist of two distinct parts :

(1) A closed fold somewhat resembling a Jacobian Ellipsoid of three unequal axis around each body, and pointed in each case end - on, with the extreme points tending to coalesce like the neck of an hour glass.

(2) A pair of curtains hanging down from the asymptotic cylinder and symmetrically arranged with respect to the xy-plane. With smaller values of C these two types of surfaces approach each other and finally coalesce in the xy-plane; the folds around the two bodies also unite into one surface enclosing both bodies. "

Consider the equation

$$x^2 + y^2 + \frac{2(1-\mu)}{\sqrt{(x-x_1)^2 + y^2}} + \frac{2\mu}{\sqrt{(x-x_2)^2 + y^2}} = C,$$

which is the equation of the curves in the xy - plane. If we put $y = 0$ we get

$$x^2 + \frac{2(1-\mu)}{\pm(x-x_1)} + \frac{2\mu}{\pm(x-x_2)} = C,$$

an equation whose solution gives the points at which the curves in the xy - plane cut the axis of x. The values of x so determined will depend on the value of C. If we know C which is a constant numerical quantity, we can determine the values of x which satisfy this equation.

Let us first consider the equation given by Jacobi's Integral, viz,

$$V^2 = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C,$$

which is the same as

$$C = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - V^2.$$

If we know x, y, r_1, r_2 and V, we can find C. Tisserand has shown that for a given infinitesimal body no matter how x, y, r_1 and r_2 may vary, and of course V^2, C always remains constant in value. Making use of Tisserand's Criterion we can determine C when the infinitesimal body moving in the xy -plane crosses the x - axis, i.e, we can find C by solving the equation

$$C = x^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - V^2 .$$

Having thus obtained C, we can substitute this value in

$$x^2 + \frac{2(1-\mu)}{\pm(x-x_1)} + \frac{2\mu}{\pm(x-x_2)} - C = 0 ,$$

the solution of which gives the points at which the curves of constant energy in the xy - plane cross the axis of x. Knowing C we can substitute in

$$\pm\frac{2(1-\mu)}{\sqrt{x_1^2+z^2}} \pm \frac{2\mu}{\sqrt{x_2^2+z^2}} = C,$$

the solution of which, for Z, (x = o) gives the points at which the surfaces cut the axis of Z. An approximate solution could be found by trial, and then a more accurate one obtained by differential corrections.

Then, knowing C, the substitution of its value in

$$y^2 \pm \frac{2(1-\mu)}{\sqrt{x_1^2+y^2}} \pm \frac{2\mu}{\sqrt{x_2^2+y^2}} = C,$$

gives the solution for y, (z = o), i.e, for the points at which the surfaces cut the axis of y. As in the case of z, an approximate solution could first be found by trial, and a more accurate value found by differential corrections. The intercepts on the x - axis will indicate if the intercepts on the y - and z -axis are real or imaginary.

x_1 and x_2 are constant in value and are equal numerically to $-\mu$ and $1-\mu$ respectively.

Let us apply these principles to determine the approximate form of the surfaces of zero relative velocity, hence the regions