PERMANENTS OF DOUBLY STOCHASTIC MATRICES

by

Laurentiu Ioan Troanca

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled *"Permanents of Doubly Stochastic Matrices"* by *Laurentiu Ioan Troanca* in partial fulfillment of the requirements for the degree of *Master of Science.*

Dated: _____

Research Supervisor: _____
K. A. Kopotun

Examining Committee: _____
D. Gunderson

Examining Committee: _____
R. Mason

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Author: *Laurentiu Ioan Troanca*

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Abstract

If A is an $n \times n$ matrix, then the permanent of A is the sum of all products of entries on each of $n!$ diagonals of A. Also, A is called doubly stochastic if it has non-negative entries and the row and column sums are all equal to one.

A conjecture on the minimum of the permanent on the set of doubly stochastic matrices was stated by van der Waerden in 1926 and became one of the most studied conjectures for permanents. It was open for more than 50 years until, in 1981, Egorychev and Falikman independently settled it.

Another conjecture (which, if it were true, would imply the van der Waerden conjecture) was originally stated by Holens in 1964 in his M.Sc. thesis at the University of Manitoba. Three years later, Doković independently introduced an equivalent conjecture. This conjecture is now known as the Holens-Doković conjecture, and while known not to be true in general, it still remains unresolved for some specific cases.

This thesis is devoted to the study of these and other conjectures on permanents.

Contents

Chapter 1

Preliminaries

1.1 Notation

A^T : the transpose of matrix A.

A^* : the complex conjugate transpose of matrix A.

$A * B$: the Hadamard product of matrices A and B, *i.e.*, if $A = (a_{ij})$, $B = (b_{ij})$, then $A * B = (a_{ij}b_{ij})$.

$A(i|j)$: the $(n-1) \times (n-1)$ submatrix obtained from an $n \times n$ matrix A after deleting the i-th row and j-th column.

$\det(A)$: determinant of a matrix A.

Δ_n^k : the set of $n \times n$ matrices of non-negative integers which have each row and column sum equal to k.

I_n : the identity matrix.

J_n : represents $(1/n)_{n \times n}$

Λ_n^k : the set of $(0,1)$ $n \times n$ matrices with row and column sum equal to k.

$\|A\|$: the Euclidean norm of A, *i.e.*, $\|A\| = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$.

$\text{per}(A)$: permanent of A.

$\oplus$: direct sum of matrices, *i.e.*, $\sum_{i=1}^{n} A_i = diag(A_1 A_2 ... A_n)$

Ω_n : the set of $n \times n$ doubly stochastic matrices, *i.e.*,

$$\Omega_n := \left\{ (a_{ij})_{n \times n} \big| a_{ij} \geq 0, 1 \leq i, j \leq n, \ \sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = 1 \right\}.$$

Ω_n^0 : the set of all matrices from Ω_n with zero main diagonal, *i.e.*, $A \in \Omega_n^0$ if and only if $A \in \Omega_n$ and $a_{ii} = 0$, $1 \leq i \leq n$.

$\sigma_k(A)$: the sum of all permanents of order k of a matrix A (subpermanents).

$\rho(A)$: the rank of a matrix A.

$tr(A)$: the trace of a matrix A, *i.e.*, $tr(A) = \sum_{i=1}^{n} a_{ii}$.

1.2   Permanent of a matrix

1.2.1   History

According to Minc (1978) [104], the permanent function first appeared in 1812 in two articles written by Binet [9] and Cauchy [18]. Binet offered identities for determining the permanents of $m \times n$ matrices with $m \leq 4$. Cauchy, in the same article dealing mostly with determinants, used the permanent as a type of ordinary symmetric function and named this function "fonction symétrique permanents." The first author who called this function "permanent" was Muir [111] in 1882. There were 20 articles

about permanents, published by 14 different authors, during the period from 1812 to 1900. In the majority of these, the permanent appeared in formulas regarding determinants. In 1881, Scott (1881) [125] stated (without proof) an interesting identity for a specific matrix which remained a conjecture until Minc (1979) [105] and Kittappa (1981) [67] obtained partial results and, finally, Svrtan (1983) [128] proved it in general. In 1978, Minc [104] did a survey in which 20 unsolved conjectures and 10 open problems were mentioned. Eight years later, the same author published the second survey regarding the permanent function and stated a total of 44 conjectures and 18 open problems in the history of permanents. A few of these conjectures and open problems were solved prior to 1986. About 20 years later, in 2005, Cheon and Wanless [25] listed the same number of 44 conjectures and 18 open problems with new results in a few of these.

1.2.2 Definitions and examples

If $A = (a_{ij})$ is an $m \times n$ matrix with $m \leq n$, then

$$\text{per}(A) := \sum_{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{m\sigma(m)} = \sum_{\sigma} \prod_{i=1}^{m} a_{i\sigma(i)}, \tag{1.1}$$

where summation is over all one-to-one mappings σ from $\{1, \ldots, m\}$ into $\{1, \ldots, n\}$. The vector $(a_{1\sigma(1)}, a_{2\sigma(2)}, \ldots, a_{m\sigma(m)})$ is called a diagonal of the matrix A, and the product $a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{m\sigma(m)}$ is a diagonal product of A. In other words, the permanent of A is the sum of all diagonal products of A. For example, if

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

then $\text{per}(A) = 1 \times 5 + 1 \times 6 + 2 \times 4 + 2 \times 6 + 3 \times 4 + 3 \times 5 = 58$.

The permanent function is widely used in combinatorics. We mention just two examples (see Minc [104]). The first one is the dance problem. In how many ways can a dance be arranged for n married couples in such a way that no wife dances with her husband? The problem is to determine the number D_n of permutations of n elements that fix no element (derangements of n elements). If J is the $n \times n$ matrix with all entries ones, then it is well known that

$$D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}) = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

However one can verify also that $\text{per}(J - I_n) = D_n$.

Another combinatorial problem is to study in how many ways can n couples be placed at a round table so that men and women are sitting in alternate places and no wife is sitting on either side of her husband. This problem is due to Lucas (1891) [78] who wrote an article called "Théorie des Nombres" in 1891. Let P be the $n \times n$ permutation matrix with 1's in the positions $(1,2), (2,3), \ldots, (n-1,n), (n,1)$, and the husbands are seated in alternate places. For each such seating the wives may be arranged in

$$U_n = \text{per}(J - I_n - P) = \text{per}\left(\sum_{i=2}^{n-1} P^i\right)$$

ways. The numbers U_n are called *menage* numbers.

Binet (1881) [9] obtained a formula for the permanent of a $2 \times n$ matrix A by writing the product of the sums of elements in two rows in A in the following way:

$$\prod_{i=1}^{2} \sum_{j=1}^{n} a_{ij} = \sum_{p,q=1}^{n} a_{1p} a_{2q} = \sum_{p \neq q} a_{1p} a_{2q} + \sum_{p=1}^{n} a_{1p} a_{2p}$$

$$= \text{per}(A) + \sum_{p=1}^{n} a_{1p} a_{2p},$$

and so

$$\text{per}(A) = \prod_{i=1}^{2} \sum_{j=1}^{n} a_{ij} - \sum_{p=1}^{n} a_{1p} a_{2p}.$$

Using a similar approach, Binet also found formulas for the permanents of $m \times n$ matrices with $m = 3$ and 4. Following Binet and Cauchy, the authors Borchardt (1855) [11], Cayley (1859) [19] and Muir (1882) [111], (1897) [112], (1899) [113] devoted their work to identities involving determinants and permanents. Some of the results were later generalized by Levine (1859) [73], (1860) [17] and Carlitz (1860) [17]. Minc (1978) [104] summarized the results in the following way. If A is an $n \times n$ matrix, and E and O are the sets of even and odd permutations of $n!$ elementary products, respectively (where below, $f(A)$ represents the undefined remaining terms) then

$$
\begin{aligned}
\mathrm{per}(A) \cdot \det(A) &= \left(\sum_{\sigma \in E} \prod_{i=1}^{n} a_{i,\sigma(i)} \right)^2 - \left(\sum_{\sigma \in O} \prod_{i=1}^{n} a_{i,\sigma(i)} \right)^2 \\
&= \sum_{\sigma \in E} \prod_{i=1}^{n} a_{i,\sigma(i)}^2 - \sum_{\sigma \in O} \prod_{i=1}^{n} a_{i,\sigma(i)}^2 + f(A) \\
&= \det(A * A) + f(A).
\end{aligned}
$$

Permanents and determinants have similar definitions which might imply that many properties for determinants are analogous to those properties for permanents. There are two major differences between permanents and determinants: multiplicativity and invariance.

- Multiplicativity: $\det(AB) = \det(A)\det(B)$ but $\mathrm{per}(AB) \neq \mathrm{per}(A)\,\mathrm{per}(B)$, in general.

Brualdi (1966) [15] proved multiplicativity for a special case.

Theorem 1.1 (Brualdi)**.** *If A is a $n \times n$ non-negative fully indecomposable matrix (see Definition 4) and B is a $n \times n$ non-negative matrix with nonzero permanent,*

then $\mathrm{per}(AB) = \mathrm{per}(A)\,\mathrm{per}(B)$ *if and only if there are permutation matrices* P *and* Q *such that* PAQ *is a diagonal matrix.*

- The permanent fails in the invariance under a few elementary operations on matrices.

If A is an $m \times n$ matrix, D is an $m \times m$ diagonal matrix, and G is an $n \times n$ diagonal matrix, then in general, $\mathrm{per}(DAG) \neq \mathrm{per}(D)\,\mathrm{per}(A)\,\mathrm{per}(G)$.

A few properties for permanents are immediate consequences of the definition:

- If A is a $n \times n$ matrix, then $\mathrm{per}(A^T) = \mathrm{per}(A)$.

- The permanent of an $m \times n$ matrix with $m \leq n$ is a multilinear function of the rows of each matrix. In the case when $m = n$, the permanent is also a multilinear function of the columns.

- Let A be an $m \times n$ matrix, $m \leq n$ and suppose that P and Q are permutation matrices of order m, and n respectively. Then $\mathrm{per}(PAQ) = \mathrm{per}(A)$.

Many other interesting properties can be found in Minc [104]. For $1 \leq k \leq n$, let $Q_{k,n}$ be the set of strictly increasing sequences of k integers taken from 1 to n, and let $G_{k,n}$ be the set of non-decreasing sequences of k integers taken from 1 to n. Let A be an $m \times n$ matrix, and let k, r be positive integers such that $1 \leq k \leq m$, $1 \leq r \leq n$, and let $\alpha = (i_1, \ldots, i_k) \in Q_{k,m}$, and $\beta = (j_1, \ldots, j_r) \in Q_{r,n}$.

Define $A[\alpha|\beta]$ to be the $k \times r$ submatrix of A "lying" in rows α and columns β and whose (i, j) entry is $a_{\alpha_i \beta_j}$. Similarly, $A(\alpha|\beta)$ is the submatrix of A complementary to $A[\alpha|\beta]$.

- Laplace expansion (see Minc [104]) for permanents: if $1 \leq r \leq n$ and $\beta \in Q_{r,n}$ is a fixed sequence, then

$$\mathrm{per}(A) = \sum_{\alpha \in Q_{r,n}} \mathrm{per}\, A[\alpha|\beta]\, \mathrm{per}\, A(\alpha|\beta).$$

- In particular for $r, s = 1, 2, \ldots, n$, the following expansion of the permanent with respect to the r-th row or the s-th column is valid:

$$\mathrm{per}(A) = \sum_{j=1}^{n} a_{rj}\, \mathrm{per}(A(r|j)) = \sum_{i=1}^{n} a_{is}\, \mathrm{per}(A(i|s)).$$

Recall that $|A| = (|a_{ij}|)$.

- Triangular inequality: $|\mathrm{per}(A)| \leq \mathrm{per}\, |A|$.

- If A is an $n \times n$ matrix and c is a real number, then $\mathrm{per}(cA) = c^n\, \mathrm{per}(A)$.

- If D and E are diagonal matrices, and A is an $n \times n$ matrix, then

$$\mathrm{per}(DA) = \mathrm{per}(AD) = \mathrm{per}(A) \cdot \mathrm{per}(D)$$

and

$$\mathrm{per}(DAE) = \left(\prod_{i=1}^{n} D_{ii} E_{ii} \right) \mathrm{per}(A).$$

- If A, B are non-negative matrices, then $\mathrm{per}(A + B) \geq \mathrm{per}(A) + \mathrm{per}(B)$.

In this thesis, we deal mostly with square matrices, *i.e.*, with $m = n$. Many results in the area of permanents regarding matrices consisting of zero and ones, matrices with complex entries, and matrices with non-negative real entries, in particular, doubly stochastic matrices. The focus in this thesis is on the permanents of doubly stochastic matrices.

Following are several pertinent definitions with examples.

Definition 1. *A non-negative matrix with all row sums (or column sums) exactly 1 is called stochastic. If both row and column sums are exactly 1, then the matrix is called doubly stochastic.*

Definition 2. *A non-negative matrix with all row sums (or column sums) not exceeding 1 is called a sub-stochastic matrix. If both row and column sums are at most 1, then the matrix is called doubly sub-stochastic.*

Definition 3. *An $n \times n$ non-negative matrix is called partly decomposable if it contains a $k \times (n - k)$ zero submatrix.*

Note that a matrix A is partly decomposable if and only if there exists P, Q permutation matrices and two square matrices X, Z, with $PAQ = \begin{pmatrix} X & Y \\ O & Z \end{pmatrix}$.

Definition 4. *A matrix which does not contain a $k \times (n - k)$ zero submatrix for $k \in [1, n - 1]$ is called fully indecomposable.*

Let $E_{ij} = (e_{kl})_{k,l=1}^{n}$, where $e_{kl} = 1$ if $k = i, \quad l = j$ and $e_{kl} = 0$, otherwise.

Definition 5. *A non-negative matrix $A = (a_{ij})$ is called nearly decomposable if it is fully indecomposable and if, for each entry $a_{ij} \neq 0$, the matrix $A - a_{ij}E_{ij}$ is partly decomposable.*

For example,
$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 0 & 5 & 6 \end{pmatrix}$$
is fully indecomposable matrix. In order for A to be nearly decomposable the matrix $A - a_{ij}E_{ij}$ has to be partly decomposable, with E_{ij} as the 3×3 matrix with zero everywhere excluding position (i, j) where the entry is 1.

Definition 6. *A non-negative matrix A has a doubly stochastic pattern if and only if there exists a doubly stochastic matrix with the same zero pattern (zeros in the same positions).*

Definition 7. *A square complex matrix A is normal if and only if $A^*A = AA^*$, where A^* is the conjugate transpose of A (for real matrices, $A^* = A^T$).*

For example,
$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$
is normal, since
$$A^* = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$
and the condition $A^*A = AA^*$ is satisfied:
$$AA^* = A^*A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Definition 8. *A Hermitian matrix is a square matrix with complex entries which is equal to its own conjugate transpose.*

For example,
$$A = \begin{pmatrix} 1 & 1+i & 3-i \\ 1-i & 3 & 3-2i \\ 3+i & 3+2i & 4 \end{pmatrix}$$
is a Hermitian matrix.

In any Hermitian matrix, the main diagonal entries are real. While not all normal matrices are Hermitian, every Hermitian matrix is (trivially) normal.

Definition 9. *An $n \times n$ matrix A is positive definite if for all non-zero vectors x, $x^T A x > 0$. If $x^T A x \geq 0$, then the matrix is positive semidefinite.*

The next proposition is a standard result in linear algebra (see, *e.g.*, [59]):

Proposition 1.1. *A Hermitian matrix is positive definite if all eigenvalues are positive. If the eigenvalues are all non-negative, then the matrix is positive semidefinite.*

For example, if $A = \begin{pmatrix} 2 & 1+i \\ 1-i & 4 \end{pmatrix}$, then $A - \lambda I = \begin{pmatrix} 2-\lambda & 1+i \\ 1-i & 4-\lambda \end{pmatrix}$ and the characteristic equation is $\lambda^2 - 6\lambda + 6 = 0$ with the roots $\lambda_1 = 3 + \sqrt{3}$, $\lambda_2 = 3 - \sqrt{3}$. Since each $\lambda_i > 0$, $i = 1, 2$, A is positive definite.

For $1 \leq i, j \leq 2$, let A_{ij} be $n_i \times n_j$, $\sum_{i=1}^{2} n_i = n$. Recall that a matrix A is block upper triangular if has the following form $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$.

Definition 10. *An $n \times n$ matrix A is called reducible if and only if, for some permutation matrix P, the matrix $P^T A P$ is block upper triangular. If a square matrix is not reducible, then the matrix is called irreducible.*

1.2.3 Properties

We now look at the properties of permanents in historical development. The first few properties mentioned in this section are combined with the determinants.

Schur (1918) [124] proved that, if A is a positive semidefinite Hermitian matrix, then $\text{per}(A) \geq \det(A)$ with quality if and only if A is diagonal or has a zero row.

An interesting result for doubly stochastic matrices is due to Hardy, Littlewood and Polya (1934) [53] who continued Muirhead's (1903) [114] research. This result can be summarized as follows. For two n-tuples of non-negative integers α and β, there exists a doubly stochastic $n \times n$ matrix A such that $\alpha^T = A\beta^T$.

Polya (1913) [119] showed that, for an 3×3 matrix, there is no linear transformation that would convert permanent into determinant. Marcus and Minc (1961) [88] generalized Polya's result for $n > 3$.

It is obvious that, if A is a non-negative $n \times n$ matrix, then

$$|\det(A)| \leq \operatorname{per}(A) \leq \prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij}.$$

Gibson (1968) [46] showed another inequality between the permanent and determinant. If A is an $n \times n$ sub-stochastic matrix, then $\operatorname{per}(I - A) \geq \det(I - A) \geq 0$.

A year later, having the same goal to find identities between determinants and permanents, Gibson (1969) [47] considered the following problem. Let A, B be $n \times n$ matrices. If $a_{ij} = 0$ for $j > i + 1$, and if $b_{ij} = a_{ij}$, for $i \geq j$, and $b_{ij} = -a_{ij}$, for $i < j$, then $\operatorname{per}(A) = \det(B)$. In other words, if two square matrices A, B are lower-triangular and B is related to A by the last two equalities above, then $\operatorname{per}(A) = \det(B)$. A *tri-diagonal* matrix has nonzero elements only in the main diagonal, the first diagonal below this, and the first diagonal above the main diagonal. The similar result holds for tri-diagonal matrices.

Engel and Schneider (1973) [33] showed that, if $A = (a_{ij})$ is a real $n \times n$ matrix with $a_{ii} \geq 0$ and $a_{ij} \leq 0$ for $i \neq j$, $i, j = 1, \ldots, n$, then

$$\det(A) + \operatorname{per}(A) \geq 2 \prod_{i=1}^{n} a_{ii}.$$

Also, if $A \in \Omega_n$ with $a_{ii} \geq \frac{1}{2}$, then $\operatorname{per}(A) \geq \frac{1}{2^{n-1}}$.

Recall that $A * B$ is the Hadamard product. Chollet (1982) [26] asked if there might be any permanental relation analogue to Oppenheim's inequality for determinants which states that, for $m \times m$ positive semidefinite Hermitian matrices A, B, $\det(A * B) \geq \det(A) \cdot \det(B)$. Indeed, he proved that $\operatorname{per}(A * A^*) \leq (\operatorname{per}(A))^2$.

Five years later, Gregorac and Hentzel (1987) [50] demonstrated that for non-negative 2×2 and 3×3 Hermitian matrices, and for $B = A^*$,

$$\operatorname{per}(A * B) \leq \operatorname{per}(A) \cdot \operatorname{per}(B).$$

Among the elementary properties stated in Section 1.2.2 is Binet-Cauchy's [104] theorem for permanents. Recall that $\mu(\alpha)$ (see Minc [104]) is the product of the factorials of the multiplicities of the distinct integers in the sequence α. For example $\mu(1, 2, 2, 4, 4, 5, 5, 5) = 1!2!2!3!$. For the next theorem, also recall that $G_{m,n}$ was defined in Section 1.2.2 as the set of non-decreasing sequences of k integers taken from 1 to n.

Theorem 1.2. *If A is an $m \times n$ and B is an $n \times m$ matrices with $m \leq n$. Then*

$$\operatorname{per}(AB) = \sum_{\alpha \in G_{m,n}} \frac{1}{\mu(\alpha)} \operatorname{per}(A[1, \ldots, m|\alpha]) \operatorname{per}(B[\alpha|1, \ldots, m]).$$

Marcus and Newman (1962) [91] proved that if A and B are two real doubly stochastic matrices then

$$|\operatorname{per}(AB)|^2 \leq \operatorname{per}(AA^T) \operatorname{per}(B^T B).$$

If equality holds, then one of the following must occur:

1. A row of A or a column of B is zero,

2. No row of A and no column of B is zero and there exists a diagonal matrix D and a permutation matrix P, both doubly stochastic such that $A^T = BDP$.

A necessary and sufficient condition for the permanent of a square non-negative matrix to be zero was provided independently by Frobenius (1917) [45] and König (1936) [69].

Theorem 1.3 (Frobenius-König). *Let A be an $n \times n$ non-negative matrix. Then* $\operatorname{per}(A) = 0$ *if and only if A contains an $s \times t$ zero submatrix such that $s + t = n + 1$.*

Marcus and Nikolai (1969) [93] showed that if A and B are positive semidefinite matrices, then $\operatorname{per}(A + B) \geq \operatorname{per}(A)$.

Wen and Wang (2007) [140] proved two relations involving the permanent function. Their starting point was Chebyshev's sum inequality (see Hardy, Littlewood and Polya (1952) [54], pp. 44–45) which states that if for $1 \leq i \leq n$ $a_i, b_i \in \mathbb{R}$, $a_1 \leq a_2 \leq \cdots \leq a_n$, $b_1 \leq b_2 \leq \cdots \leq b_n$, then

$$\frac{1}{n} \sum_{i=1}^{n} a_i b_i \geq \left(\frac{1}{n} \sum_{i=1}^{n} a_i \right) \left(\frac{1}{n} \sum_{i=1}^{n} b_i \right).$$

Their first result confirmed the following version of Oppenheim's inequality for permanents. If $A = (a_{ij})$ and $B = (b_{ij})$ are two $n \times n$ matrices with positive entries, and

$$\frac{a_{i,1}}{a_{i+1,1}} \leq \frac{a_{i,2}}{a_{i+1,2}} \leq \cdots \leq \frac{a_{i,n}}{a_{i+1,n}}, \quad i = 1, 2, \ldots, n-1,$$

$$\frac{b_{i,1}}{b_{i+1,1}} \leq \frac{b_{i,2}}{b_{i+1,2}} \leq \cdots \leq \frac{b_{i,n}}{b_{i+1,n}}, \quad i = 1, 2, \ldots, n-1,$$

then

$$\frac{\operatorname{per}(A * B)}{n!} \geq \frac{\operatorname{per}(A)}{n!} \frac{\operatorname{per}(B)}{n!}.$$

Equality holds if and only if $\rho(A) = 1$ or $\rho(B) = 1$.

Another result from Wen and Wang [140] was that if $A = (a_{ij})$, and $B = (b_{ij})$ are two $n \times n$ matrices with positive entries with $b_{i1} \leq b_{i2} \leq \cdots \leq b_{in}$ and $\frac{a_{i1}}{b_{i1}} \leq \frac{a_{i2}}{b_{i2}} \leq \cdots \leq \frac{a_{in}}{b_{in}}$, $i = 1, 2, \ldots, n$, then

$$\frac{\mathrm{per}(A)}{\prod\limits_{i=1}^{n} \sum\limits_{j=1}^{n} a_{ij}} \leq \frac{\mathrm{per}(B)}{\prod\limits_{i=1}^{n} \sum\limits_{j=1}^{n} b_{ij}}.$$

If all the signs in the original inequalities are reversed, then the conclusion still holds. Equality holds if and only if $a_{i1}/b_{i1} = a_{i2}/b_{i2} = \cdots = a_{in}/b_{in}$, $i = 1, 2, \ldots, n$.

1.2.4 Lower bounds

Motivated by van der Waerden's conjecture (see Chapter 3), many authors tried to find lower bounds for permanents of various kinds of matrices. In this section, we discuss lower bounds for the following kinds of matrices: $(0, 1)$, non-negative, positive semidefinite, fully indecomposable, and doubly stochastic.

(a) (0,1) matrices

The first important formula which was a starting point for many lower bound results was Frobenius-König Theorem (see Theorem 1.3). Minc (1974) [102] used Theorem 1.3 to prove that the permanent was strictly positive for a special kind of matrices.

Theorem 1.4. *If A is an $m \times n$ $(0, 1)$ matrix, $m \leq n$, and if each row sum is greater than or equal to m, then $\mathrm{per}(A) > 0$.*

Proof. If, in a $(0, 1)$ matrix every row sum is greater than or equal to m, the implication is that every row has at least m ones, and $n - m$ is greater than or equal to the

number of zeros in each row. If A has an $s \times t$ zero submatrix, then $t \leq n - m$ and $s \leq m$. Adding these inequalities we get $t + s \leq n - m + m = n$. Now, Theorem 1.3 states that a necessary and sufficient condition for $\text{per}(A)$ to be 0 is that A contain an $s \times t$ submatrix such that $s + t = n + 1$. Now, using $s + t \leq n$, we get $s < m$, and so $\text{per}(A) > 0$. $\qquad\square$

Hall (1948) [52] showed that, if A is an $m \times n$ $(0, 1)$ matrix with nonzero permanent and row sums at least k, $k \leq m$, then

$$\text{per}(A) \geq k!.$$

Mann and Ryser (1953) [81] extended Hall's result for the case when $k \geq m$. Let A be an $m \times n$ $(0, 1)$ matrix, $m \leq n$, with at least k one's in each row. If $k \geq m$, then

$$\text{per}(A) \geq \frac{k!}{(k - m)!}.$$

Jurkat and Ryser (1966) [64] provided a formula which expressed the permanent as a product of matrices formed from rows of the initial matrix. If $a = (a_1, a_2, \ldots, a_n)$ is a real n-tuple, let $a^* = (a_1^*, a_2^*, \ldots, a_n^*)$ be the n-tuple a rearranged in non-increasing order $a_1^* \geq a_2^* \geq \cdots \geq a_n^*$, and let $a' = (a_1', a_2', \ldots, a_n')$ be the n-tuple a rearranged in non-decreasing order $a_1' \leq a_2' \leq \cdots \leq a_n'$. If A is an $n \times n$ $(0, 1)$ matrix with row sums $r_1, r_2, \ldots, r_n$, then $\bar{A}$ is the $(0, 1)$ matrix whose i-th row has the first r_i^* entries equal to 1 and the remaining entries equal to 0, with $i = 1, \ldots, n$.

Theorem 1.5 (Jurkat-Ryser). *If A is an $n \times n$ $(0, 1)$ matrix with row sums r_i, $i = 1, 2, \ldots, n$, and with $\{r_i + i - n\} = max(0, r_i + i - n)$, then*

$$\text{per}(A) \geq \prod_{i=1}^{n} \{r_i + i - n\}.$$

If $\mathrm{per}(A) \neq 0$, *then equality holds if and only if there exists a permutation matrix P*

such that $AP = \bar{A}$.

Recall that Λ_n^k is the set of $(0,1)$ $n \times n$ matrices with row and column sums equal to k. Minc (1969) [100] also showed that, if $A \in \Lambda_n^k$, then

$$\mathrm{per}(A) \geq n(k-2) + 2.$$

Hall (1948) [52] proved that, if $A \in \Lambda_n^3$ (note that this implies that $\frac{1}{3}A \in \Omega_n$), then $\mathrm{per}\left(\frac{1}{3}A\right) \geq \frac{n!}{n^n}$, which verified van der Waerden's conjecture for doubly stochastic matrices A such that $A \in \Lambda_n^3$.

Sinkhorn (1969) [126] stated that if A is a nearly decomposable $n \times n$ $(0,1)$ matrix with m rows containing exactly three ones and $n - m$ rows containing exactly two ones, then $\mathrm{per}(A) \geq m$. If A is a fully indecomposable member of Λ_n^3, then $\mathrm{per}(A) \geq n$.

Hartfiel (1970) [55] demonstrated that for any $A \in \Lambda_n^3$, $\mathrm{per}(A) \geq n + 3$, and Hartfiel and Crosby (1972) [58] proved that, for any $A \in \Lambda_n^k$, $\mathrm{per}(A) \geq (k-1)(k-2)n/2$. Voorhoeve (1979) [135] showed that for any $A \in \Lambda_n^3$, $\mathrm{per}(A) \geq 6\,(4/3)^{n-3}$.

Wanless (2006) [139] applied a result by Schrijver (the inequality (1.2) below) to matrices from Λ_n^k to show that

$$\lim_{n \to \infty} \left(\min_{A \in \Lambda_n^k} \mathrm{per}(A) \right)^{\frac{1}{n}} = \frac{(k-1)^{k-1}}{k^{k-2}}.$$

(b) Positive semidefinite matrices

Consider the Hadamard determinant theorem (see Marcus and Minc (1962) [90]) in order to find a similar relation for permanent function: "If $A = (a_{ij})$ is a positive semidefinite Hermitian $n \times n$ matrix, then $\det(A) \leq \prod_{i=1}^{n} a_{ii}$ with equality if and

only if A has a zero row or A is diagonal." Marcus and Minc (1962) [90] found the corresponding permanent relation for a positive semidefinite Hermitian $n \times n$ matrix:

$$\text{per}(A) \geq \frac{n!}{n^{2n}} \prod_{i=1}^{n} a_{ii},$$

where the inequality is strict unless A has a zero row.

Marcus (1963) [82] showed that for any positive semidefinite Hermitian $n \times n$ matrix A, $\text{per}(A) \geq \prod_{i=1}^{n} a_{ii}$.

Marcus and Minc (1965) [86] offered another lower bound for a positive semidefinite Hermitian n-square matrix A, with row sums r_i,

$$\text{per}(A) \geq \frac{n!}{s(A)^n} \prod_{i=1}^{n} |r_i|^2,$$

where $s(A)$ denotes the sum of all entries in A. Moreover, equality holds if and only if A has a zero row or the rank of A is 1.

Doković (1967) [28] showed that if A is a positive semidefinite $n \times n$ matrix and $G_{r,n}$ is the set of non-decreasing sequences of r integers taken from 1 to n and $\alpha \in G_{r,n}$, then $\text{per}(A[\alpha|\alpha]) \geq \frac{r!}{n^r}$.

(c) non-negative matrices

Marcus (1964) [83] demonstrated that, if $A = (a_{ij})$ is an $n \times n$ non-negative Hermitian matrix, then

$$\text{per}(A) \geq \prod_{i=1}^{n} a_{ii}$$

with equality if and only if A has a zero row or A is a diagonal matrix.

Recall that Δ_n^k represents the set of $n \times n$ matrices with row and column sum equal to k. Also, if $A_{(i)} = (a_{i1}, a_{i2}, \ldots, a_{in})$ is the i-th row of A, then let $(a_{i1}^*, a_{i2}^*, \ldots, a_{in}^*)$ represents the n-tuple $A_{(i)}$ arranged in a non-increasing order, and $(a_{i1}', a_{i2}', \ldots, a_{in}')$

the same n-tuple arranged in non-decreasing order. We advice the reader that in this context a_{ij}^* is not the complex conjugate.

Minc (1969) [99] provided the following bounds for permanents of non-negative matrices.

Theorem 1.6. *If $A = (a_{ij})$ is a non-negative $n \times n$ matrix, then*

$$\prod_{i=1}^{n} \sum_{t=1}^{i} a_{it}' \leq \mathrm{per}(A) \leq \prod_{i=1}^{n} \sum_{t=1}^{i} a_{it}^*.$$

Moreover, if A is positive, the equality can occur if and only if the first $n - 1$ rows of A are multiples of $(1, 1, \ldots, 1)$.

Levow (1971) [74] showed that for any $A \in \Delta_n^k$ the following inequality holds

$$\mathrm{per}(A) \geq \binom{k}{2} \frac{n}{2} + k.$$

Schrijver and Valiant (1980) [122] provided a lower bound for permanents on Δ_n^k. Their main result is

$$\min_{A \in \Delta_n^k} \mathrm{per}(A) \leq \frac{k^{2n}}{\binom{nk}{n}}.$$

Schrijver (1998) [123] proved that, for any n, k integers with $n \geq k \geq 1$ and any $A \in \Delta_n^k$,

$$\mathrm{per}(A) > \left(\frac{(k-1)^{k-1}}{k^{k-2}} \right)^n. \tag{1.2}$$

Also, for any given k, the fraction $\frac{k-1}{k^{k-2}}$ is the best possible because

$$\lim_{n \to \infty} \left(\min_{A \in \Delta_n^k} \mathrm{per}(A) \right)^{\frac{1}{n}} = \frac{(k-1)^{k-1}}{k^{k-2}}.$$

(d) Fully indecomposable matrices

Minc (1969) [100] demonstrated that, if A is a fully indecomposable $n \times n$ $(0, 1)$ matrix, and $s(A)$ is the sum of all entries of A, then

$$\mathrm{per}(A) \geq s(A) - 2n + 2.$$

Minc (1973) [101] proved the following theorem for a lower bound of a fully indecomposable $n \times n$ $(0,1)$ matrix with row sums r_i, $i = 1, \ldots, n$.

Theorem 1.7. *Let A be a fully indecomposable $n \times n$ $(0,1)$ matrix with row sums $r_1, r_2, \ldots, r_n$, then*

$$\mathrm{per}(A) \geq \max_i r_i.$$

Equality holds if and only if at least $n - 1$ of the row sums equal 2.

The same year, Hartfiel [57] improved the lower bound for a fully indecomposable matrix. Recall the following notations: if $0 \leq R - 1 < n$ set $R_1 = R$ and $R_2 = 1$. If $n \leq R - 1$ set $R_1 = n$ and $R_2 = R - n + 1$.

Theorem 1.8. *If A is a fully indecomposable $n \times n$ $(0,1)$ matrix with $k \geq 3$ ones in each row, then*

$$\mathrm{per}(A) \geq s(A) - 2n + 2 + \sum_{i=2}^{k-3}(i! - 1)n + [(k-2)! - 1]R_1 + [(k-1)! - 1]R_2.$$

(e) Doubly stochastic matrices

Recall that Ω_n denotes the set of all doubly stochastic $n \times n$ matrices. Marcus and Newman (1959) [92] gave a weak lower bound for the permanent function on Ω_n by showing that for any $A \in \Omega_n$

$$\mathrm{per}(A) \geq (n^2 - n + 1)^{1-n}. \tag{1.3}$$

Holens (1964) [60] demonstrated that if $A \in \Omega_n$, then

$$\mathrm{per}(A) \geq \left(n^2 - 2n + 2\right)^{1-n},$$

which is slightly better than (1.3). It is likely that Holens was unaware of the following result. Marcus and Minc (1962) [89] also improved (1.3) by showing that,

if $A \in \Omega_n$, then

$$\mathrm{per}(A) \geq \frac{1}{n^n}. \tag{1.4}$$

Rothaus (1972) [120] showed that if $A \in \Omega_n$, then

$$\mathrm{per}(A) \geq \frac{1}{n^{n-1}}.$$

He also showed that there exists an r depending on n such that $\mathrm{per}(A^r)$ for $A \in \Omega_n$ achieves its minimum uniquely at the matrix with equal entries which is J_n.

Marcus and Minc (1962) [89] verified the van der Waerden conjecture (see Chapter 3) for a particular class of doubly stochastic matrices by showing that, if $A \in \Omega_n$ is a positive semidefinite Hermitian $n \times n$ matrix, then

$$\mathrm{per}(A) \geq \frac{n!}{n^n},$$

where the equality holds if and only if $A = J_n$.

Marcus and Minc (1965) [84] demonstrated that if $A \in \Omega_n$ with at least m eigenvalues of modulus 1, then

$$\mathrm{per}(A) \geq (n - m + 1)^{-(n-m+1)}.$$

Moreover, if A is irreducible, then $\mathrm{per}(A) \geq (m/n)^n$.

London (1971) [76] established the following two properties for a minimizing matrix. If A is a minimizing matrix for $\mathrm{per}(S)$, $S \in \Omega_n$, then:

1. If $a_{ij} > 0$, then $\mathrm{per}(A(i|j)) = \mathrm{per}(A)$.

2. If $a_{ij} = 0$, then $\mathrm{per}(A(i|j)) = \mathrm{per}(A) + \beta$, $\beta \geq 0$.

Merris (1973) [95] proposed the following conjecture: if $A \in \Omega_n$, then

$$n \cdot \mathrm{per}(A) \geq \min_i \sum_{j=1}^{n} \mathrm{per}(A(j|i)),$$

and provided a counter-example with "max" instead of "min".

Friedland (1978) [41] improved the lower bound in (1.4) for $A \in \Omega_n$, namely

$$\operatorname{per}(A) \geq \frac{1}{n!}. \tag{1.5}$$

Friedland (1979) [42] gave a further improvement by showing that, for any $A \in \Omega_n$,

$$\operatorname{per}(A) \geq \frac{1}{e^n}. \tag{1.6}$$

Bang (1976) [3] outlined a proof for (and, in (1979) [4], provided a complete proof of) the following lower bound:

$$\operatorname{per}(A) \geq \frac{1}{e^{n-1}}, \quad A \in \Omega_n. \tag{1.7}$$

This was the best lower bound proved for Ω_n before the final resolution of the van der Waerden conjecture in 1981.

Foregger (1980) [36] showed that, for $2 \leq n \leq 9$, the minimum value of the permanent of a nearly decomposable $A \in \Omega_n$ is $\frac{1}{2^{n-1}}$.

After 1981, when Falikman [34] and Egorychev [32] obtained the exact lower bound for the permanents on the set of all doubly stochastic matrices, the efforts shifted to determining the minimum of the permanent on various subsets of Ω_n. For example, a subset Z of $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ defines a set $\Omega_n(Z) = \{A = (a_{ij}) \in \Omega_n | a_{ij} = 0 \text{ if } (i,j) \in Z\}$. In other words, $\Omega_n(Z)$ is defined by specifying the zero pattern.

Recall that $\|A\| = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \right)^{1/2}$. Achilles (1977) [1] showed that if Ω_n^k is the set of $n \times n$ doubly stochastic matrices having k zero entries in the same row or the same column (without loss of generality, in the first k positions in the first row) and, if $A \in \Omega_n^k$, then the minimum of $\|A - J_n\|$ is achieved at

$$
A_{min} = \begin{pmatrix}
0 & \dots & 0 & \frac{1}{n-k} & \dots & \frac{1}{n-k} \\
\frac{1}{n-1} & \dots & \frac{1}{n-1} & & & \\
.. & .. & .. & & & \\
.. & .. & .. & & \frac{n-k-1}{(n-1)(n-k)} & \\
.. & .. & .. & & & \\
\frac{1}{n-1} & \dots & \frac{1}{n-1} & & &
\end{pmatrix}
$$

and is equal to $\dfrac{\sqrt{k}}{\sqrt{(n-1)(n-k)}}$.

Another Achilles's [1] result is that, on the set Ω_n^k, then the permanent function

has a local minimum at A_{min}.

Define $\arg\min\limits_{X\in\Omega_n} \mathrm{per}(X)$ to be the set of those $X \in \Omega_n$ that have minimum

permanent. Let Ω_n^* be the subset of Ω_n with a zero entry in the $(1,1)$-position.

Knopp and Sinkhorn (1982) [68] studied the minimum of the permanent on Ω_n^*. If

$T_n \in \arg\min\limits_{X\in\Omega_n^*} \mathrm{per}(X)$, and $n > 3$, then

$$
T_n = \begin{pmatrix}
0 & \frac{1}{n-1} & \dots & \frac{1}{n-1} \\
\frac{1}{n-1} & & & \\
\dots & & \frac{n-2}{n-1}J_{n-1} & \\
\frac{1}{n-1} & & &
\end{pmatrix}.
$$

Also, for $n = 2$, the minimum value of a permanent of $A \in \Omega_2$ is $\mathrm{per}(T_2) = 1$.

For $n = 3$, the minimum value of a permanent of $A \in \Omega_3$ is achieved at T_3 and its

permutations.

For $s + t \le n - 1$, let $Z \subseteq \{(i,j)|1 \le i \le s, 1 \le j \le t\}$ be fixed, and let $\Omega_n(Z)$ be

the set of all doubly stochastic matrices with 0's in every $(i,j) \in Z$; i.e., $\Omega_n(Z)$ is a

subpolytope (see Section 2.1) of Ω_n. If $s = t = 2$, then Minc (1984) [109] found

$$\min\{\text{per}(A)|A \in \Omega_n(Z)\} = \frac{(n-s)!}{(n-s)^{n-s}} \cdot \frac{(n-t)!}{(n-t)^{n-t}} \cdot \frac{(n-s-t)^{n-s-t}}{(n-s-t)!}.$$

Chang (1984) [21] found the minimum value of the permanent on sets of doubly stochastic matrices with at least one fixed entry.

Chang (1988) [23] demonstrated that, for $0 \le a \le \frac{1}{2}$,

$$\begin{pmatrix} 0 & a & 1-a \\ d & 1-d & 0 \\ 1-d & d-a & a \end{pmatrix} \in \arg\min\{\text{per}(B)|B = (b_{ij}) \in \Omega_3, b_{11} = 0, b_{12} = a\},$$

where $d = \frac{2-a-2a^2}{4(1-a)}$.

Brualdi (1985) [14] showed that, for $Z = \{(i,j)|i+j \le n-1\}$ and any $A \in \Omega_n(Z)$, $\text{per}(A) \ge 1/2^{n-1}$, with equality if and only if $a_{ij} = 1/2$, for all (i,j) with $i+j = n$ and $a_{1n} = a_{n1} = 1/2$.

1.2.5 Upper bounds

Since every term in the expansion of $\text{per}(A)$ is included in $\prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$, which is equal to 1 for all doubly stochastic matrices A, we conclude that $\text{per}(A) \le 1$, for $A \in \Omega_n$. Equality is achieved only if A is a permutation matrix (see also Marcus and Newman (1959) [92]).

Ryser (1960) [121] conjectured that for a $A \in \Lambda_{mk}^{k}$, with $1 < k < n$, the permanent takes its maximum on the direct sum (see Section 1.1) of $k \times k$ matrices of 1's.

Minc (1963) [97] showed that, if A is an $n \times n$ $(0,1)$ matrix with the row sums $r_i, \quad i = 1, \ldots, n$, then

$$\text{per}(A) \le \prod_{i=1}^{n} \frac{r_i + 1}{2}, \tag{1.8}$$

with equality if and only if A is a permutation matrix, and conjectured that

$$\operatorname{per}(A) \leq \prod_{i=1}^{n} (r_i!)^{\frac{1}{r_i}}. \tag{1.9}$$

The estimate (1.9 was proved by Bregman (1973) [12].) In 1967, Minc [98] improved (1.8) by showing that, if A is a $(0,1)$ matrix with row sums r_i, $i = 1, \ldots, n$, then

$$\operatorname{per}(A) \leq \prod_{i=1}^{n} \frac{r_i + \sqrt{2}}{1 + \sqrt{2}}.$$

Nijenhuis and Wilf (1970) [115] showed that if A is an $n \times n$ $(0,1)$ matrix with row sums r_i, $i = 1, \ldots, n$, and where $\tau \approx 0.1367\ldots$ is a universal constant, then

$$\operatorname{per}(A) \leq \prod_{i=1}^{n} ((r_i!)^{\frac{1}{r_i}} + \tau).$$

Ostrand (1970) [116] provided an upper bound for an $m \times n$ $(0,1)$ matrix with row sums $r_1 \leq \cdots \leq r_m$:

$$\operatorname{per}(A) \leq \prod_{i=1}^{n} \max(1, r_i - i + 1).$$

Foregger (1975) [35] proved that, if A is an $n \times n$ $(0,1)$ matrix with all row sums greater or equal to 3, and N is the number of positive entries in A, then

$$\operatorname{per}(A) < 2^{N-2n}.$$

Merriell (1980) [94] established the maximum of the permanent on Λ_n^k for some special cases. For example, the maximum on Λ_{3t+1}^3 is $6^t 9$, and the maximum on Λ_{3t+2}^3 is $6^{t-2} 9^2$.

Baum and Eagon (1967) [7] were interested in applications in a statistical estimation for probabilistic functions and in a mathematical model for ecology. In particular, one of their inequalities when applied to permanents yields the following

estimate. Let S be the set of $m \times n$ row stochastic matrix with permanent different from 0. If $f : S \to S$ with $f(A)_{ij} = a_{ij} \operatorname{per}(A(i|j))/\operatorname{per}(A)$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, for all $A = (a_{ij})$ from S, then

$$\operatorname{per}(A) < \operatorname{per}(f(A)), \text{ unless } f(A) = A. \tag{1.10}$$

Baum and Sell (1968) [8] improved (1.10):

$$\operatorname{per}(A) \leq \operatorname{per}((1 - t)A + tf(A)), \quad t \in (0, 1],$$

with equality if and only if $f(A) = A$.

Foregger (1975) [35] showed that, if A is a fully indecomposable matrix with non-negative integers entries, and $s(A)$ represents the sum of all entries from A, then

$$\operatorname{per}(A) \leq 2^{s(A)-2n} + 1. \tag{1.11}$$

For $2 \leq m \leq n$, let $A = (a_{ij})$ be a positive $m \times n$ matrix. Luo (1980) [79] proved that

$$\operatorname{per}(A) \leq \frac{(n-1)!}{(n-m)!} \prod_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij}^{m} \right)^{\frac{1}{m}}.$$

Using graph methods, Donald, Elwin, Hagar and Salamon (1984) [29] proved that if A is a fully indecomposable non-negative $n \times n$ matrix with row sums r_i and all entries integers, then

$$\operatorname{per}(A) \leq 1 + \prod_{i=1}^{n} (r_i - 1).$$

Cheon and Wanless (2005) [25] published an update on Minc's surveys of open problems in the field of permanents. One of the chapters in their paper contains two tables (with $n = 1, \ldots, 11$ and $k = 1, \ldots, 11$) for minimum and maximum values of the permanent for non-negative $n \times n$ $(0, 1)$ matrices with row and column sums equal to k.

Chapter 2

Doubly stochastic matrices

2.1 Properties and more inequalities

In this section, only properties of doubly stochastic matrices (recall that the set of all such $n \times n$ matrices is denoted by Ω_n), and the permanents of matrices from Ω_n will be discussed. Since a few of the properties describing permanents of doubly stochastic matrices were presented in Section 1.2.3, these will not be described again.

We start with the following observation (see Minc [104]).

Proposition 2.1. *If a matrix is doubly stochastic, then the matrix is square.*

Proof. Let $A = (a_{ij})$ be an $n \times m$ matrix such that $\sum_{i=1}^{n} a_{ij} = 1$, $1 \leq j \leq m$, and $\sum_{j=1}^{m} a_{ij} = 1$, $1 \leq i \leq n$. Summing these equations we get the sum of all entries from A in two different ways: $\sum_{j=1}^{m}(a_{1j}+a_{2j}+\cdots+a_{nj}) = n$, $\sum_{i=1}^{n}(a_{i1}+a_{i2}+\cdots+a_{im}) = m$. Hence, $n = m$. $\qquad\square$

Proposition 2.2 (Minc [104], 1978)**.** *The permanent of doubly stochastic matrix is positive.*

Proof. Since all entries of $A \in \Omega_n$ are non-negative, the permanent of A can not be less than zero. Let's suppose $\text{per}(A) = 0$, then using Theorem 1.3 (which is Frobenius-König theorem) there exist permutation matrices P and Q with $PAQ = \begin{pmatrix} X & Y \\ O & Z \end{pmatrix}$, and let O be the zero $h \times k$ matrix with $h + k = n + 1$.

Let $s(A)$ be the sum of all entries in the matrix A. Applying this summation to PAQ matrix: $n = s(PAQ) \geq s(X) + s(Z)$.

If O is an $h \times k$ matrix, then all the nonzero entries in the first k columns are included in B, therefore $s(X) = k$. In the same way for Z, hence $s(Z) = h$. Therefore, $n \geq s(X) + s(Z) = k + h$. By our supposition $n = k + h - 1$, which is a contradiction. Hence, the hypothesis is false. Therefore, $\text{per}(A) > 0$. $\qquad \square$

Proposition 2.2 implies that every doubly stochastic matrix must have a positive diagonal (see Minc [104]). Consider the following version of the dance problem. In a school, there are n boys and n girls. The question is, if each boy has previously met exactly k girls and each girl has previously met exactly k boys, is it possible to make pairs of boys and girls into dance partners who have met before. If $A = (a_{ij})$ is the $n \times n$ adjacency matrix, *i.e.*, $a_{ij} = 1$ if the i-th boy met the j-th girl and $a_{ij} = 0$, otherwise. In each row and column of this matrix, A contains exactly k ones, and so $\frac{1}{k}A \in \Omega_n$. Proposition 2.2 implies that there is a positive diagonal and, hence, it is possible to pair boys and girls in this dance problem in k completely different ways.

Another way to view this dance problem is with graphs. The following is an old theorem (likely due to König, and also follows from Hall's matching theorem—see nearly any text on graph theory, *e.g.* West (1996) [141]).

Theorem 2.3. *In any k-regular subgraph of $K_{n,n}$ (the complete bipartite graph on two partite sets, each with n vertices), there exists perfect matching.*

As a consequence, (applying this theorem k times) in the adjacency matrix A there are k disjoint diagonals, each with all positive entries.

The following theorem was discovered by John von Neumann [134] in 1928, and independently proved by Birkhoff (1946) [10].

Theorem 2.4 (Neumann (1928) [134], Birkhoff (1946) [10]). *If $A \in \Omega_n$, then $A = \sum_{i=1}^{s} \alpha_i P_i$, where the P_i's are permutation matrices and α_i's are non-negative numbers with $\sum_{i=1}^{s} \alpha_i = 1$.*

In other words, every doubly stochastic matrix is a convex combination of permutation matrices.

Some of the additional elementary properties of doubly stochastic matrices are:

- If $A \in \Omega_n$ and P, Q are permutation matrices, then $PAQ \in \Omega_n$.

- The product of two doubly stochastic matrices of order n is also a doubly stochastic matrix of order n.

- If $A, B \in \Omega_n$ and $\alpha \in (0, 1)$, then the convex combination $\alpha A + (1 - \alpha)B$ is in Ω_n.

Brualdi (1966) [15] showed that if $A \in \Omega_n$, then $\text{per}(AA^T) = \text{per}(A^2)$ if and only if A is a permutation matrix.

London (1973) [77] presented the following two interesting results regarding doubly stochastic matrices. Recall that $\rho(A)$ represents the rank of the matrix A.

Proposition 2.5. *If $A \in \Omega_n$, then $\rho(A - J_n) = \rho(A) - 1$.*

As an observation the above property does not hold for any $n \times n$ matrices.

Proposition 2.6. *Let $i, j = 1, \ldots, n$, and suppose that α_i and β_j are real numbers such that there exist i and j for which $\alpha_i \beta_j \neq 0$, $-\frac{1}{n} \leq \alpha_i \beta_j$, $\sum_{i=1}^{n} \alpha_i = 0$, $\sum_{j=1}^{n} \beta_j = 0$. If $A \in \Omega_n$, then $\rho(A) = 2$ if and only if $A = J_n + P$, for some $P = (p_{ij})$ an $n \times n$ matrix with $p_{ij} = \alpha_i \beta_j$.*

Let $A = (\alpha_{ij})$ and $B = (b_{ij})$ be $n \times n$ non-negative doubly stochastic matrices. Let $\beta_1, \beta_2, \ldots, \beta_k$ be the column partition of A corresponding to its fully indecomposable components, and let $\gamma_1, \gamma_2, \ldots, \gamma_l$ be the row partition of B corresponding to its fully indecomposable components. Recall that $|\beta_i \cap \gamma_i|$ represents the number of common components among β_i and γ_i. Brualdi (1966) [15] stated that $\operatorname{per}(AB) = \operatorname{per}(A) \operatorname{per}(B)$ if and only if the following are true:

- For $1 \leq i \leq k, 1 \leq j \leq l$, $|\beta_i \cap \gamma_i| \leq 1$, and

- If G is the graph whose vertices are the β_i with $|\beta_i| > 1$ and the γ_j with $|\gamma_j| > 1$ such that there is an edge joining β_i and γ_j provided $|\beta_i \cap \gamma_j| = 1$ and these are the only edges, then G has no circuits.

Marcus and Minc (1965) [86] proved that if N is $m \times m$, normal, and with eigenvalues $r_1, \ldots, r_m$, then

$$|\operatorname{per}(N)| \leq \frac{1}{m} \sum_{i=1}^{m} |r_i|^m.$$

If, in addition, $N \in \Omega_m$, then $|\operatorname{per}(N)| \leq \frac{\rho(N)}{m}$. The last inequality is strict unless either N is a permutation matrix or $m = 2$ and $N = J_2$.

If A is an arbitrary $m \times m$ doubly stochastic matrix, then

$$\operatorname{per}(A) \leq \left(\frac{\rho(A)}{m} \right)^{\frac{1}{2}},$$

equality holds if and only if $\rho(A) = m$ and A is a permutation matrix.

Wilf (1966) [142] calculated the "average" permanent of a class of doubly stochastic matrices. Let n, s be fixed positive integers, $(P_1, \ldots, P_s)$ be an ordered set of $n \times n$ permutation matrices, and $K_{n,s}$ be the set of $(n!)^s$ matrices that result from calculating $A = s^{-1}(P_1 + \cdots + P_s)$ for each possible ordered set $(P_1, \ldots, P_s)$. The matrices from $K_{n,s}$ are doubly stochastic. If

$$\gamma_{n,s} := (n!)^{-s} \sum_{P_1,\ldots,P_s} \mathrm{per}\left(\frac{1}{s}(P_1 + \cdots + P_s)\right),$$

then, for $n \to \infty$,

$$\gamma_{n,s} \sim \sqrt{2\pi n}\left(1 - \frac{2}{s}\right)^{-\frac{1}{2}(s-1)}\left(1 - \frac{1}{s}\right)^{ns-n+s-\frac{1}{2}}.$$

Eberlein (1969) [30] established a condition for minimizing permanent function over the set of doubly stochastic matrices. If $A \in \Omega_n$ and $\mathrm{per}(A) \leq \mathrm{per}(S)$, for any S doubly stochastic matrix, then any two lines (rows or columns) of A are either with different zero patterns or equal (one has a zero component where the other has not.)

Gyires (1976) [51] proved that if $A \in \Omega_n$, then

$$\frac{\mathrm{per}(A^2) + \sqrt{\mathrm{per}(AA^T \mathrm{per}(A^T A))}}{2} \geq \frac{n!}{n^n}.$$

Minc (1975) [103] showed that for any $A \in \Omega_n$, and $t \leq n - 2$, with t positive integer, if all permanental minors from A of order t are equal, then $A = J_n$.

Marcus and Minc (1967) [87] asked the following question.

Proposition 2.7. *For fixed $A, B \in \Omega_n$, if for all $\alpha \in [0, 1]$, $\mathrm{per}(\alpha A + (1 - \alpha)B)$ is constant, then is it true that $A = B$?*

Definition 2.8 (Wang [136], 1977)**.** *Two matrices $A, B \in \Omega_n$ where $A \neq B$ are said to form a permanental pair if, for any $\alpha \in [0, 1]$, then $\mathrm{per}(\alpha A + (1 - \alpha)B)$ is constant.*

Wang proved the following: (i) for every $n \geq 3$, there exists infinitely many permanental pairs; (ii) no permutation matrix can form permanental pairs with any doubly stochastic matrix; (iii) the matrix J_n does not form a permanental pair with any doubly stochastic matrix. In 1979, Wang and Brenner [13] showed that there exist no permanental pairs which contain a direct sum (see Section 1.1) $A = J_{n_1} \oplus J_{n_2} \oplus \cdots \oplus J_{n_k}$.

Recall that a *convex hull* is the set of convex combinations $\{\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_k x_k | k \geq 1, x_i \in \mathbf{R}^n, \lambda_i x_i \geq 0,$ and $\sum_{i=1}^{k} \lambda_i = 1.\}$

Let V be the set of all $n \times n$ permutation matrices and let P be the polytope formed by the convex hull of V in $M_{n \times n}(\mathbf{R})$, an n^2 dimensional space. (Then P is said to have dimension n^2.) Observing that by Theorem 2.4, P contains Ω_n.

A *subpolytope Q* of P is the intersection of P with some affine subspace of $M_{n \times n}(\mathbf{R})$ (and the dimension of Q is the dimension of this subspace).

Gibson (1980) [48] define a subpolytope Q of P to be permanental if the permanent function is constant in Q, and proved that such a Q exists with dimension at least $\frac{n^2 - 3n + 4}{2}$.

It is known that the permanent function is not convex on the set of doubly stochastic matrices (see Minc[107]). In other words, it is not true that $\mathrm{per}(\alpha A + (1 - \alpha) B) \leq \alpha \, \mathrm{per}(A) + (1 - \alpha) \, \mathrm{per}(B)$, for all $A, B \in \Omega_n$ and $\alpha \in [0, 1]$. At the same time, it was shown by Perfect (1964) [118] that, for every $A \in \Omega_n$,

$$\mathrm{per}\left(\frac{1}{2}(I_n + A)\right) \leq \frac{1}{2}(1 + \mathrm{per}(A)).$$

2.2   Conjectures and open problems

This section is devoted to unresolved conjectures and open problems on the permanent function on the set of doubly stochastic matrices. The numbers assigned to the conjectures and open problems are the same as in Minc (1978) [104] and in Cheon and Wanless (2005) [25].

Conjecture 3 (Marcus and Minc [87], 1967). *If $S \in \Omega_n$, $n \geq 2$, then $\mathrm{per}(S) \geq \mathrm{per}(\frac{nJ_n - S}{n-1})$. For $n \geq 4$, equality holds if and only if $S = J_n$.*

Marcus and Minc (1967) [87] proved this conjecture for positive semidefinite symmetric matrices and for matrices in a small neighborhood of J_n.

If $n = 2$, then the above relation is an equality. The case when $n = 3$ was resolved by Wang (1977) [136] who, in particular, showed that if

$$S = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$$

then $\mathrm{per}(S) = \mathrm{per}\left(\frac{1}{2}(3J_3 - S)\right) = \frac{1}{4}$. For $n = 4$, Foregger (1979) [39] showed that Conjecture 3 is true. Hwang (1989) [61] verified the validity of Conjecture 3 for partly decomposable matrices. Malek (1989) [80] proved a stronger version of this conjecture for a special case of doubly stochastic matrices.

Conjecture 4 (Wang [136], 1977). *If $S \in \Omega_n$, $n \geq 2$, then $\mathrm{per}(S) \geq \mathrm{per}(\frac{nJ_n + S}{n+1})$ and if $n \geq 3$ equality holds if and only if $S = J_n$.*

Wang (1977) [136] proved this conjecture for $n = 3$. Chang (1983) [20] showed

that Conjecture 4 is true for matrices in the complement of sufficiently large neighborhood of J_n. Five years later, Chang (1988) [23] and Foregger (1988) [38] independently showed that the conjecture is true for $n = 4$. Hwang (1989) [61] provided a proof for the case of partly decomposable matrices.

Recall that $\sigma_k(A)$ represents the sum of all permanents of order k of a matrix A.

Conjecture 10 (Tverberg [129], 1963). *If $A \in \Omega_n$ and $A \neq J_n$, then for any t with $2 \leq t \leq n$, $\sigma_t(A) > \sigma_t(J_n) = \binom{n}{t}^2 \frac{t!}{n^t}$.*

This conjecture was proved by Friedland (1982) [43] (for more details, see Section 3.3).

Conjecture 12 (Holens [60], 1964 and Doković [27], 1967). *If $A \in \Omega_n$, and $k \in \mathbb{Z}$, $k \in [2, n]$, then $\sigma_k(A) \geq \frac{(n-k+1)^2}{nk}\sigma_{k-1}(A)$, with equality in the case $2 \leq k \leq n-1$ only if $A = J_n$.*

More details on this conjecture are in Chapter 3.

Conjecture 13 (Sinkhorn [127], 1977). *If $A \in \Omega_n$ and if $\mathrm{per}(A(i|j)) \geq \mathrm{per}(A)$, for all i, j, then either $A = J_n$ or up to permutation of rows and columns, $A = \frac{1}{2}(I_n + P_n)$, where P_n is a full cycle permutation matrix.*

This conjecture was proved by Bapat (1983) [6].

Conjecture 15 (Foregger [36], 1980). *If A is a nearly decomposable doubly stochastic matrix, then $\mathrm{per}(A) \geq 1/2^{n-1}$, with equality if and only if $A = (P + I_n)/2$ up to permutations of rows and columns.*

Foregger (1980) [36] proved a particular case of this conjecture, when $2 \leq n \leq 8$.

Conjecture 17 (Foregger [104], 1978). *For any positive integer n, there exists an integer $k = k(n)$ such that $\mathrm{per}(A^k) \leq \mathrm{per}(A)$, for all $A \in \Omega_n$.*

Chang (1984) [22] produced a partial result: for any positive integer n and $c \in (0, \frac{1}{n}]$, there exists an integer $N = N(n,c) > 1$ such that if $S = (s_{ij}) \in \Omega_n$ satisfies $c \leq min\{s_{ij}, 1 \leq i, j \leq n\}$, for all $k \geq N$, then $\mathrm{per}(S^{2^k}) \leq \mathrm{per}(S)$. In 1990, Chang [24] showed that the conjecture is true for $n = 3$.

Conjecture 18 (Merris [95], 1973). *If $A \in \Omega_n$, then*

$$n\,\mathrm{per}(A) \quad \geq \quad \min_i \sum_{j=1} \mathrm{per}(A(j|i)).$$

As far as we know, this is one of the few conjectures in which no one has made any progress so far.

Conjecture 20 (Gyires [51], 1978). *If $A \in \Omega_n$, then*

$$\frac{4(\mathrm{per}(A))^2}{\mathrm{per}(AA^*) + \mathrm{per}(A^*A) + 2\,\mathrm{per}(A^2)} \geq \frac{n!}{n^n}, \tag{2.1}$$

with equality if and only if $A = J_n$.

Chang (1990) [24] proved this conjecture for $n = 3$. He also showed that for $A = (a_{ij}) \in \Omega_n$ and $y := \min\{a_{ij}, 1 \leq i, j \leq n\}$, if $y \geq \frac{n-2}{(n-1)^2}$, then inequality (2.1) holds.

Conjecture 29 (Wang [137], 1979). *If $B \in \Omega_n$, $n \geq 3$, and for all $A \in \Omega_n$ and any $\theta \in [0, 1]$,*

$$\mathrm{per}(\theta B + (1 - \theta)A) \leq \theta\,\mathrm{per}(B) + (1 - \theta)\,\mathrm{per}(A), \tag{2.2}$$

then B is a permutation matrix.

For all $A \in \Omega_n$ and any $\theta \in [0, 1]$, the matrix B is called a star if $\mathrm{per}(\theta B + (1 - \theta)A) \leq \theta \, \mathrm{per}(B) + (1 - \theta) \, \mathrm{per}(A)$. Wang [137] proved that if B is a star, then $\mathrm{per}(B) \geq \frac{1}{2^{n-1}}$.

By definition, $\sigma_n(A) = \mathrm{per}(A)$, and $\sigma_1(A)$ is the sum of all entries of A. Kopotun (1996) [70] conjectured that a modification of the inequality (2.2) holds for σ_k instead of the permanent function. If $A \in \Omega_n$, then for every $k \geq 3$, there exists $n_k \geq k + 1$ such that for all $n \geq n_k$ and $\theta \in [0, 1]$, the inequality

$$\sigma_k(\theta J_n + (1 - \theta)A) \leq \theta \sigma_k(J_n) + (1 - \theta)\sigma_k(A).$$

A partial result has been obtained for $k \leq 3$ and $B = J_n$. Also, Kopotun showed that $\sigma_k(A)$ is convex for $n \geq 2$ and $\sigma_3(A)$ is convex for $n \geq 4$.

Karuppan and Arulraj (1998) [65] provided a counterexample showing that Conjecture 29 fails, for $n = 3$. They modified the conjecture as follows.

Conjecture 29' (Karuppan and Arulraj [65], 1998). *The inequality*

$$\mathrm{per}(\theta B + (1 - \theta)A) \leq \theta \, \mathrm{per}(B) + (1 - \theta) \, \mathrm{per}(A)$$

is true for all $A \in \Omega_n$ and any $\theta \in [0, 1]$ if and only if B is permutation equivalent to the direct sum (see Section 1.1) of I_n and some number of matrices from Ω_2.

Conjecture 34 (Lih and Wang [75], 1982). *If $A \in \Omega_n$ and $\alpha \in [\frac{1}{2}, 1]$, then*

$$\mathrm{per}(\alpha J_n + (1 - \alpha)A) \leq \alpha J_n + (1 - \alpha) \, \mathrm{per}(A).$$

Lih and Wang proved the conjecture for $n = 3$. Foregger (1988) [38] verified it for $n = 4$ by showing that, if $A \in \Omega_4$ and $t_1 < t \leq 1$, where t_1 is the unique real root of $106t^3 - 418t^2 + 465t - 153$, then

$$\mathrm{per}(tJ_4 + (1 - t)A) \leq t \, \mathrm{per}(J_4) + (1 - t) \, \mathrm{per}(A),$$

with equality if and only if $A = J_4$.

Hwang (1991) [62] conjectured and proved for $n = 3$ that if $A \in \Omega_n$, $n \geq 2$, $A \neq J_n$, then the permanent function is strictly convex on the straight line segment joining J_n and $(J_n + A)/2$. Hwang also wondered if his conjecture is equivalent to Conjecture 34.

Conjecture 35 (Kim and Roush [66], 1981). *The maximum value of* $\operatorname{per}(I - A)$ *for* $A \in \Omega_{2k+1}$ *is* $3 \cdot 2^{k-2}$. *This value is obtained for the direct sum (see Section 1.1) of*

$$\frac{1}{2} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and $k - 1$ *copies of* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

This is another conjecture without even partial results according to our knowledge.

Let $Z = \{(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)\}$ where $1 \leq i_k, j_k \leq n$, $1 \leq k \leq n$, and let $\Omega_n(Z)$ be the subset of $\{S = (s_{ij})_{n \times n} \in \Omega_n | s_{ij} = 0$ for all $(i, j) \in Z\}$.

The concept of a "tie point" was introduced by Hartfiel (1971) [56] for a nearly decomposable matrix. The point (i, j) is called a tie point for A if $a_{ij} = 0$ and replacing a_{ij} with a positive entry creates $\bar{A}$ with the property that if any other positive entry of $\bar{A}$ is replaced with a 0, then the resulting matrix is partly decomposable.

Conjecture 41 (Foregger and Sinkhorn [40], 1986). *If A is a nearly decomposable matrix minimizing the permanent in $\Omega_n(Z)$ and $(i, j) \in Z$, then $\operatorname{per} A(i|j) > \operatorname{per}(A)$ implies that (i, j) is a tie point for A.*

The only progress regarding this conjecture was made by Foregger (1987) [37] who proved Conjecture 41 for a special matrix associated with a certain type of bipartite graph.

Conjecture 44 (Minc [110], 1987)**.** *The permanent function on the set of $n \times n$ doubly stochastic matrices with zero trace achieves its minimum uniquely at the matrix all of whose off-diagonal entries are $1/(n-1)$.*

This conjecture seems to be well known and appeared in Minc (1987) [110]. There appears to be no progress toward resolution of this conjecture.

In his book devoted exclusively to permanents, Minc [104] provided ten open problems. Later, he added eight more problems in two different surveys. Here, we discuss only problems dealing with permanents of doubly stochastic matrices.

Problem 8 (Friedland and Minc [44], 1978)**.** *Find matrices A on the boundary of Ω_n so that the permanent is monotone increasing on the segment $(1-\theta)J_n + \theta A$, $\theta \in [0, 1]$.*

Foregger(1988) [38] constructed classes of doubly stochastic 4×4 matrices with the above property.

Problem 13 (Minc [104], 1978)**.** *Determine the largest number $b = b(n)$ such that $\mathrm{per}(A) \geq \frac{n!}{n^n}$, for any real $n \times n$ matrix A all of whose row and column sums are equal to 1 and which satisfy $\|A - J_n\| \leq b$.*

In order to state the last three open problems we need some additional definitions.

Let $A = (a_{ij})$ be an $n \times n$ $(0,1)$ matrix and define $\Omega(A)$ to be the subset of Ω_n determined by A such that $\Omega(A) = \{S = (s_{ij}) \in \Omega_n | s_{ij} = 0 \text{ if } a_{ij} = 0\}$. In other

words, $\Omega(A)$ represents the set of all doubly stochastic matrices with the same zero pattern as A.

An $n \times n$ $(0, 1)$ matrix A is called *cohesive* if there exists a matrix B in the interior of $\Omega(A)$ for which $\operatorname{per}(B) = min\{\operatorname{per}(S)|S \in \Omega(A)\}$.

If $A = (a_{ij})$ is an $n \times n$ $(0, 1)$ matrix and P represents a permutation matrix, then

$$b(A) = \frac{1}{\operatorname{per}(A)} \sum_{P \in \Omega(A)} P \in \Omega(A)$$

is called the *barycenter* of $\Omega(A)$. An $n \times n$ $(0, 1)$ matrix A is said to be *barycentric* if the minimum permanent over $\Omega(A)$ takes place at the barycenter of $\Omega(A)$.

Problem 14 (Brualdi (1985) [14]). *Characterize cohesive matrices.*

Problem 15 (Brualdi (1985) [14]). *Characterize barycentric matrices.*

Chapter 3

Van der Waerden's conjecture

What is the minimum of the permanent on the set of all doubly stochastic matrices? This question was posed van der Waerden [130] in 1926, and became one of the most famous and most hunted questions in the theory of permanents; it remained unresolved for more than 50 years.

Conjecture (van der Waerden's (1926) [130]). *If $A \in \Omega_n$, then*

$$\text{per}(A) \geq \frac{n!}{n^n},$$

and equality holds if and only if and $A = J_n = (1/n)_{n \times n}$.

3.1 History and earlier results

Between 1926 and 1959, no progress in the van der Waerden conjecture was registered. In 1959, Marcus and Newman [92] investigated properties of minimizing matrices in the set of all doubly stochastic matrices. A matrix $A \in \Omega_n$ is called minimizing if

$$A = \arg \min_{S \in \Omega_n} \text{per}(S).$$

The authors' goal was to show that the only matrix which satisfies those properties is J_n. Although this project was not successful, their achievement represented a major step forward.

It was shown in [92] that (i) if A is a minimizing matrix, then it is fully in-decomposable; and (ii) for any $1 \le s, p \le n$, if $A = (a_{ij}) \in \Omega_n$ is a minimizing matrix with $a_{sp} > 0$, then $\text{per}(A(s|p)) = \text{per}(A)$. In order to prove (ii), Marcus and Newman showed that if A was a minimizing doubly stochastic matrix in a suf-ficiently small neighborhood of J_n (but different from J_n) with all positive entries, then $\text{per}(A) > \text{per}(J_n)$. In particular, this implies that if $A \in \Omega_n$ is a minimizing matrix with all of its entries positive, then $A = J_n$.

Marcus and Newman (1962) [91] proved the conjecture for symmetric positive semidefinite doubly stochastic matrices. Minc (1963) [96] slightly generalized (and offered a new proof of) the main result in [91]. Let Φ_n be the set of positive semidef-inite Hermitian $n \times n$ matrices which have $e = (1, 1, \ldots, 1)$ as a characteristic vector. If $H \in \Phi_n$ and the row sums of H are all equal to λ_1, then $\text{per}(H) \ge n! \left(\frac{\lambda_1}{n}\right)^n$, with equality if and only if either a row of H is zero or H is a non-negative multiple of J_n.

Marcus and Minc (1968) [85] proved that if A is an $n \times n$ positive definite Her-mitian doubly stochastic matrix with least eigenvalue λ_n, then

$$\text{per}(A) \ge \text{per}(J_n) + \lambda_n^n(1 - \text{per}(J_n)),$$

and that if A is a normal doubly stochastic matrix $n \times n$ whose eigenvalues lie in the sector $\left[\frac{-\pi}{2m}, \frac{\pi}{2m}\right]$ of the complex plane, then

$$\sigma_m(A) \ge \sigma_m(J_m) + \frac{1}{2}\frac{(m-2)!}{n^{m-2}}\binom{n-2}{m-2}^2 \|A - J_n\|^2,$$

where $\|A - J_n\|$ denotes the Euclidean norm of $A - J_n$.

Eberlein and Mudholkar (1968) [31] proved the conjecture for $n = 3$ and $n = 4$, and Eberlein (1969) [30] proved it for $n = 5$. For $n = 6, 7, 8$, Graf (1971) [49] determined regions in the set of doubly stochastic matrices such that the inequality in the conjecture was satisfied for all matrices from these regions. Butler (1975) [16] verified the conjecture for regular doubly stochastic matrices, *i.e.,* matrices A such that $A = AXA$ for some $X \in \Omega_n$.

3.2   Resolution

In 1981, Egorychev [32] and Falikman [34] independently proved the van der Waerden Conjecture (uniqueness of the minimizing matrix was not discussed in Falikman's paper).

According to Egorychev's [32] paper one main idea was an inequality for "mixed discriminants" or "mixed volumes of convex bodies" by Aleksandrov.

Theorem 3.2.1 (Egorychev). *If $A = (a_{ij})$ is an $n \times n$ real matrix with its first $n-1$ columns being non-negative. Then,*

$$(\mathrm{per}(A))^2 \geq \mathrm{per}(a_1, \ldots, a_{n-2}, a_{n-1}, a_{n-1}) \cdot \mathrm{per}(a_1, \ldots, a_{n-2}, a_n, a_n).$$

If $a_1, a_2, \ldots, a_{n-1}$ are positive, then equality can hold if and only if a_n is a multiple of a_{n-1}.

The following theorem became the key result in Egorychev's proof.

Theorem 3.2.2 (Egorychev). *If A is a column (or row) stochastic $n \times n$ matrix, with $k, l = 1, 2, \ldots, n$, satisfying $0 < \mathrm{per}(A) \leq \mathrm{per}(A(k|l))$, then $\mathrm{per}(A) = \mathrm{per}(A(k|l))$.*

Falikman's proof was based on minimizing the function, for $\epsilon \in \mathbb{R}$,

$$F_\epsilon(X) = \text{per}(X) + \epsilon \left(\prod_{1 \leq i,j \leq n} x_{ij} \right)^{-1}$$

over the set $\Omega_n^* = \{X = (x_{ij}) \in \Omega_n | x_{ij} \neq 0, \quad 1 \leq i,j \leq n\}$.

3.3   Generalizations and post resolution

Recall that $\sigma_k(A)$ denotes the sum of all subpermanets of order k of a matrix A. Tverberg (1963) [129] conjectured that, if $A \in \Omega_n$ with $A \neq J_n$ and $2 \leq k \leq n$, then

$$\sigma_k(A) > \sigma_k(J_n), \tag{3.1}$$

and proved this conjecture for $k = 2$ and $k = 3$. When $k = n$, (3.1) is a generalization of the van der Waerden conjecture. Tverberg conjecture was settled by Friedland [43] in 1982 by relating the problem of minimizing σ_k on Ω_n to that of minimizing the permanent function on a particular subsets of Ω_{2n-k}. (These particular subsets corresponded to faces in the polytope associated with Ω_n as defined in Section 2.1.)

The van der Waerden conjecture was so popular that, even after its resolution, there were many papers published discussing, analyzing and simplifying its proofs. In particular, Pach (1981) [117] showed a detailed exposition of Egorychev's proof. Lagarias (1982) [72] gave van der Waerden's conjecture background and provided an outlines of Egorychev's proof. Ando (1982) [2] explained Egorychev's proof. Jungnickel included the proof of van der Waerden's conjecture in his book [63] published in 1982. Lint (1981) [131], (1982) [132], (1983) [133] produced three easily readable articles devoted to its resolution. Minc (1982) [106] provided two versions of

Egorychev's proof. Laffey (1983) [71] presented a detailed proof using Egorychev's techniques. Bang (1983) [5] polished Falikman's proof and included the proof of the uniqueness of the minimizing matrix. Minc (1983) [108] presented a short history and both versions of the proofs.

3.4 The Holens-Doković conjecture

In 1964, Thomas Frederick Holens (1964) [60] defended his M.Sc. thesis at the University of Manitoba, in the third chapter of which he dealt with the permanent function of doubly stochastic matrices. Let $A \in \Omega_n$ and let A_k be the matrix formed by replacing all the entries of k columns of A by $1/n$. Holens proposed the following conjecture and proved it in the case $n = 2$.

Conjecture (Holens (1964) [60]). *If A is doubly stochastic, then*

$$\operatorname{per}(A) \geq \frac{\sum \operatorname{per}(A_1)}{\binom{n}{1}} \geq \cdots \geq \frac{\sum \operatorname{per}(A_k)}{\binom{n}{k}} \geq \cdots \geq \frac{\sum \operatorname{per}(A_{n-1})}{\binom{n}{n-1}} \geq \operatorname{per}(A_n).$$

In 1967, Doković [27] proposed the following conjecture, which he proved for $k = 3$ (if $k = 1$ or 2 then the statement is trivial).

Conjecture (Doković (1967) [27]). *If $A \in \Omega_n$, and $k = 1, 2, \ldots, n$, then*

$$\sigma_k(A) \geq \frac{(n-k+1)^2}{nk} \sigma_{k-1}(A). \tag{3.2}$$

As it turns out, these two conjectures are equivalent (verification is direct and is left to the reader) and are now known as *the Holens-Doković conjecture*.

Since (see Minc [104])

$$\left. \frac{\partial}{\partial \theta} \sigma_k(\theta J_n + (1-\theta)A) \right|_{\theta=0} = \sum_{\alpha, \beta \in Q_{k,n}} \sum_{i \in \alpha, j \in \beta} \left(\frac{1}{n} - a_{ij} \right) \operatorname{per}((A[\alpha|\beta])(i|j)) =$$

$$= \frac{1}{n} \sum_{\alpha,\beta\in Q_{k,n}} \sum_{i\in\alpha,j\in\beta} \mathrm{per}((A[\alpha|\beta])(i|j)) - k\sigma_k(A) =$$

$$= \frac{(n-k+1)^2}{n}\sigma_{k-1}(A) - k\sigma_k(A),$$

the Holens-Doković inequality states that the derivative of $\sigma_k(\theta J_n + (1-\theta)A)$ is non-positive at $\theta = 0$, which implies that the function $\sigma_k(A)$ is non-increasing on the line segment joining A and J_n.

The Holens-Doković conjecture is an attractive conjecture which, unfortunately, turned out to be false in general. In 1996, Wanless [138] published the paper "The Holens-Doković's conjecture on permanents fails!" with the title saying it all. There is still hope for positive results though since n and k were not independent (e.g. $n = k = 4$) in Wanless' counterexamples, and, in fact, the smallest counterexample involved a 22×22 matrix.

While the Holens-Doković conjecture is dead, in general, many cases are still open.

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