A STUDY OF WAVE PROPAGATION

ON

ELASTIC MEMBRANES

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ABSTRACT

WAVE PROPAGATION IN ELASTIC SURFACES IS TREATED USING THE CONCEPT OF A WAVE CURVE IN THE SURFACE. THE BASIC GEOMETRIC AND KINEMATIC DISCONTINUITY RELATIONS ACROSS WAVE CURVES ARE DERIVED. EXPRESSIONS FOR THE SPEED OF PROPAGATION OF EQUIVOLUMINAL AND IRROTATIONAL WAVES ON THE SURFACE ARE GIVEN ALONG WITH A DIFFERENTIAL EQUATION GOVERNING WAVE STRENGTH DURING PROPAGATION. THE WAVE CURVES ARE SHOWN TO FORM A SYSTEM OF GEODESIC PARALLELS ON THE SURFACE AS THEY PROPAGATE. THE GENERAL SOLUTION AND SOME EXAMPLES OF THE STRENGTH EQUATION ARE GIVEN.

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To Hea Ryun

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NOTATIONS

$a_{\alpha\beta}$, $a^{\alpha\beta}$	covariant and contravariant components of the surface metric tensor
$A_{\alpha\beta}$	a surface tensor
C(t) or C	a wave curve on S
κ	principal curvature of a curve in three dimensional space
κ_g	geodesic curvature of a curve on S
K	Gaussian curvature of S
G	normal velocity of C
p	a scalar on S
$R^{\eta}_{\alpha\beta\gamma}$	components of the Riemanian curvature tensor
s	arc length of C
S	an arbitrary surface
t	time
$\underline{t}$	a surface vector or a first order surface tensor
t_α, t^α, (or $\bar{t}_\alpha$, $\bar{t}^\alpha$)	components of covariant and contravariant surface vectors in u^α (or $\bar{u}^\alpha$) system
$\underset{\sim}{t} = T^{\alpha\cdots\gamma}_{\sigma\cdots\tau}$	an absolute tensor field on S
u^α, $\bar{u}^\alpha$	curvilinear coordinate system on S
$\underline{v}$	velocity vector on S
v_α, v^α	covariant and contravariant components of velocity vector on S
v_n	normal velocity of a particle on S

$\underline{w}$	displacement vector on S
w_α, w^α	covariant and contravariant components of displacement vector on S
W	wave strength
$\frac{D}{Ds}$	operator for absolute differentiation along C
$\frac{D}{Dt}$	operator for absolute time differentiation
$\frac{\delta}{\delta t}$	operator for δ-time differentiation
$\frac{d_c}{dt}$	operator for convective differentiation
$\alpha, \beta, \gamma, \ldots$	coordinate indices ranging 1,2
δ_α^β	the Kronecker deltas
$e_{\alpha\beta}$	covariant components of strain tensor
$\epsilon_{\alpha\beta}$	covariant components of rate of strain tensor
$\varepsilon_{\alpha\beta}$	covariant components of rate of strain tensor in convective coordinate system
γ	surface density
$\gamma_{\alpha\beta}$	covariant components of surface metric tensor in convective coordinate system
$\tau_{\alpha\beta}$	covariant components of stress tensor on S
Ω	mean curvature of a surface S
σ	distance along a geodesic normal to C
λ, μ	Lame constants

ν_α, ν^α	covariant and contravariant components of geodesic unit normal vector to C
$\Gamma_{\alpha\beta\gamma}, \Gamma_{\alpha\beta}^{\gamma}$	Christoffel symbols of the first and second kind
$B, \zeta, \bar{\zeta}, \omega$, $\lambda_\alpha, \omega_\alpha, \hat{\omega}_\alpha$, $\lambda_{\alpha\beta}$	scalar, vector or tensor quantities defined over C on S
$[\quad]$	denotes jump or discontinuity in the quantities enclosed

1. Introduction

Considerable attention has been given recently to the problem of wave propagation in elastic sheets, thin plates, and shells. [1, 2,3,4,5] * Jahsman [6] derived an expression for the velocity of circular waves in elastic sheets and plates and also showed that wave strength diminished as $r^{-\frac{1}{2}}$ where r is the radius of a circular wave. The problem of wave propagation in cylindrical and spherical shells has also been treated by several authors. [7,8,9]

Thomas [10,11] employed Hadamard's idea of singular surface as a wave surface in three dimensional space. He established basic discontinuity or so called compatibility conditions across a wave surface . Applications of the compatibility conditions were made to deduce speed and directions of wave propagation as well as decay of wave strength equations in elastic, plastic and crystalline solids.

In this thesis, analogous to the wave surface in three dimensional space, a singular curve is introduced as a wave curve to develop a theory of wave propagation in a surface. Closely following Thomas's method, basic compatibility conditions are formulated across a wave curve on a surface. These conditions are used to derive wave strength and speed equations of wave curves in homogeneous isotropic surfaces.

* Numbers in square brackets refer to the bibliography.

Specifically, in chapter 2 is given a brief review of the intrinsic geometry of a surface along with the definitions of surface deformation tensor, wave curve, absolute derivative and absolute time derivative of a surface tensor. Also the basic compatibility conditions are derived.

In chapter 3, the first and second order geometrical and kinematical compatibility conditions across a wave curve are deduced in appropriate form for a scalar and a surface vector.

In chapter 4, using Reynold's transport theorem and the principles of mass and momentum conservation, the basic field equations are derived for an elastic surface. The dynamical discontinuity relations associated with these equations are derived.

In chapter 5, assuming that a wave curve is moving into an elastic medium at rest, wave speeds as well as wave strength equation of equivoluminal and irrotational wave during propagation are formulated.

In chapter 6, applications of wave strength equation are made for several surfaces. It is shown that wave strength equation of wave curves in a plane agree with the corresponding results of Thomas and Jahsman.

Appendices are given as chapter 7.

2. Surface Geometry and Kinematics

In this chapter, a brief review of the intrinsic geometry of surfaces is discussed. Absolute derivative, δ-time derivative as well as absolute time derivative of an absolute tensor field are defined. Basic relations for the compatibility conditions of discontinuities across a curve in the surface are defined.

Let S be a surface defined by the first fundamental form

$$ds^2 = a_{\alpha\beta}\, du^\alpha du^\beta \qquad (\alpha = 1, 2) \tag{2.1}$$

where $a_{\alpha\beta}$ are the covariant components of the surface metric tensor, s is arc length and u^α are the curvilinear coordinates of the surface. A moving curve C(t) on S is given by

$$u^\alpha = u^\alpha(s, t) \tag{2.2}$$

where t is time. Functions u^α are assumed to be as smooth as desired.

Let

$$\bar{u}^\alpha = \bar{u}^\alpha(u^\beta) \tag{2.3}$$

*Greek indices have the range 1,2 throughout this discussion.

be the equation of an arbitrary differentiable curvilinear coordinate transformation on the surface S. It is assumed that $\bar{u}^\alpha$ are single valued functions whose functional determinant $\left|\frac{\partial \bar{u}^\alpha}{\partial u^\beta}\right|$ is not equal to zero.

A surface vector or first order tensor t_α or t^α in u^α and $\bar{t}_\alpha$ or $\bar{t}^\alpha$ in $\bar{u}^\alpha$ is related by

$$t_\alpha = \bar{t}_\beta \frac{\partial \bar{u}^\beta}{\partial u^\alpha}, \qquad \bar{t}_\alpha = t_\beta \frac{\partial u^\beta}{\partial \bar{u}^\alpha}$$
$$t^\alpha = \bar{t}^\beta \frac{\partial u^\alpha}{\partial \bar{u}^\beta}, \qquad \bar{t}^\alpha = t^\beta \frac{\partial \bar{u}^\alpha}{\partial u^\beta} \tag{2.4}$$

The covariant derivative of a surface vector with components t_α is defined by

$$t_{\alpha;\beta} \equiv t_{\alpha,\beta} - t_\gamma \Gamma_{\alpha\beta}^{\gamma} \tag{2.5}$$

and absolute derivative of the vector along a curve C as

$$\frac{Dt_\alpha}{DS} \equiv t_{\alpha;\beta}\frac{du^\beta}{ds} = (t_{\alpha,\beta} - t_\gamma \Gamma_{\alpha\beta}^{\gamma})\frac{du^\beta}{ds} = t_{\alpha,s} - t_\gamma \Gamma_{\alpha\beta}^{\gamma}\frac{du^\beta}{ds} \tag{2.6}$$

where the semicolon denotes covariant differentiation and the comma ordinary (partial) differentiation with respect to the indicated index. Analogously, definitions for the higher order tensors and their derivatives can be made. These notations and the summation convention for the repeated indices have been adopted throughout this discussion. The quantities $\Gamma_{\alpha\beta\gamma}$ and $\Gamma_{\alpha\beta}^{\gamma}$ are the Christoffel symbols of the first and second kind respectively, defined by

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial a_{\alpha\gamma}}{\partial u^{\beta}} - \frac{\partial a_{\alpha\beta}}{\partial u^{\gamma}} + \frac{\partial a_{\gamma\beta}}{\partial u^{\alpha}}\right), \quad \Gamma_{\alpha\beta}^{\sigma} = a^{\gamma\sigma}\Gamma_{\alpha\beta\gamma} . \tag{2.7}$$

Similarly, we have

$$\frac{Dt^{\alpha}}{Ds} \equiv t^{\alpha}_{;\beta}\frac{du^{\beta}}{ds} = t^{\alpha}_{,s} + t^{\gamma}\Gamma^{\alpha}_{\gamma\beta}\frac{du^{\beta}}{ds} \tag{2.8}$$

for the contravariant components of a surface vector $\underline{t}$. If $\frac{Dt^{\alpha}}{Ds}$ or $\frac{Dt_{\alpha}}{Ds}$ vanish, the vector $\underline{t}$ is said to undergo a parallel displacement along a curve C.

Furthermore, we have the following relations

$$a_{\alpha\beta}a^{\beta\gamma} = \delta_{\alpha}^{\gamma} , \qquad \delta_{\alpha}^{\gamma} = \begin{cases} 1 & ; \alpha = \gamma \\ 0 & ; \alpha \neq \gamma \end{cases}, \tag{2.9}$$

$$a_{\alpha\beta}u^{\alpha}_{,s}u^{\beta}_{,s} = 1 , \tag{2.10}$$

$$a_{\alpha\beta}\nu^{\alpha}\nu^{\beta} = 1 , \qquad a^{\alpha\beta}\nu_{\alpha}\nu_{\beta} = 1, \tag{2.11}$$

$$a_{\alpha\beta}u^{\alpha}_{,s}\nu^{\beta} = 0, \qquad a^{\alpha\beta}u_{\alpha,s}\nu^{\beta} = 0, \tag{2.12}$$

in which $a^{\alpha\beta}$ are the contravariant components of the surface metric tensor, δ^{γ}_{α} are the Kronecker deltas and ν^{α}, ν_{α} being the contravariant and covariant components of geodesic unit normal vector, respectively, to a curve C at the point in question. Equation (2.10) shows us that $u^{\alpha}_{,s} \equiv \frac{du^{\alpha}}{ds}$ are the components of the unit tangent vector along the curve C on the surface S.

We also have [12,14]

$$\frac{Du^{\alpha}_{,s}}{Ds} = \frac{d^2u^{\alpha}}{ds^2} + \Gamma^{\alpha}_{\gamma\beta}\frac{du^{\beta}}{ds}\frac{du^{\gamma}}{ds} = K_g\nu^{\alpha}, \tag{2.13}$$

$$\frac{D\nu^{\alpha}}{Ds} = \nu^{\alpha}_{;\beta}\frac{du^{\beta}}{ds} = -K_g u^{\alpha}_{,s} \tag{2.14}$$

for the absolute derivatives of $u^{\alpha}_{,s}$ and ν^{α} where K_g * is the geodesic curvature at the point in question of C. Geodesic curvature K_g of C can be interpreted as the arc rate of change of the angle which a vector makes with the tangent to a curve C as the vector undergoes a parallel displacement along C. The curves $u^{\alpha} = u^{\alpha}(s)$, which are solutions of the equations

$$\frac{Du^{\alpha}_{,s}}{Ds} = \frac{d^2u^{\alpha}}{ds^2} + \Gamma^{\alpha}_{\gamma\beta}\frac{du^{\beta}}{ds}\frac{du^{\gamma}}{ds} = 0 \tag{2.15}$$

* Geodesic curvature K_g is the same as the intrinsic curvature K of a curve on the surface. Viewed from three dimensional Euclidean space the geodesic curvature K_g is the projection of the principal curvature of the curve in space onto the tangent plane to the surface at the point in question.

are called geodesics. The tangent vector to geodesics are in parallel propagation, hence $K_g = 0$ along geodesics. We set up geodesics at each point to the curve C and perpendicular to C. When equal length are measured from C along these geodesics, the loci of their end points are orthogonal trajectories of the geodesics and these orthogonal trajectories are called geodesic parallels to the curve C. [14]

We are interested in situations where the surface S consists of particles which can undergo motion. If w and v denote the infinitesimal surface displacement and surface velocity vectors respectively, then the infinitesimal strain $e_{\alpha\beta}$ and rate of strain $\epsilon_{\alpha\beta}$ are given by (see appendix 2)

$$e_{\alpha\beta} = \frac{1}{2}(w_{\alpha;\beta} + w_{\beta;\alpha}) \tag{2.16}$$

$$\epsilon_{\alpha\beta} = \frac{1}{2}(v_{\alpha;\beta} + v_{\beta;\alpha}) \tag{2.17}$$

Let C(t) divide S into two regions S_1 and S_2. (Fig.1) Define $t(u^\alpha, t)$ to be a continuously differentiable absolute tensor field in regions S_1 and S_2 obtained by excluding points on the curve C(t) from the immediate neighborhood of this curve in S. Assume that the tensor field $t(u^\alpha, t)$ and its partial derivatives to the order of consideration have finite limiting values which are continuous functions of s and t as we approach the curve C from both regions S_1 and S_2. [10]

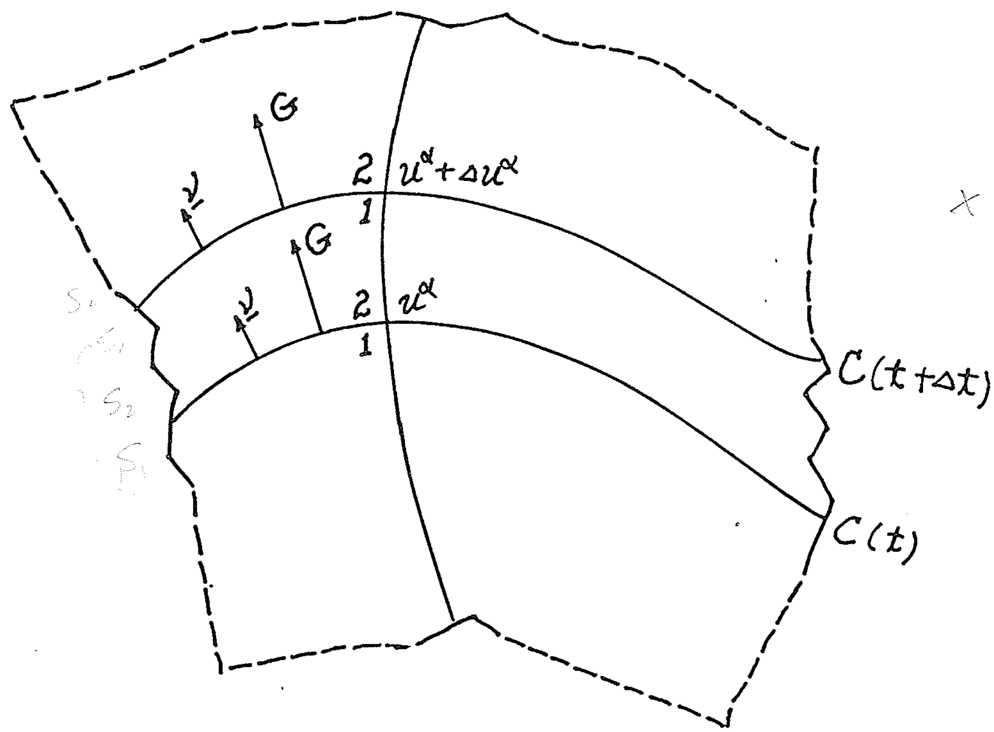

Fig. 1. Wave curve C(t), region 1, 2, geodesic unit normal $\underline{\nu}$, and normal speed G on S

By Hadamard's lemma, we have[17]

$$\frac{D[\underset{\sim}{t}]}{Ds} = [\underset{\sim}{t}_{;\alpha}]\, u^{\alpha}_{,s} \tag{2.18}$$

where

$$[\underset{\sim}{t}] = \underset{\sim}{t}_2 - \underset{\sim}{t}_1 \;, \qquad [\underset{\sim}{t}_{;\alpha}] = \underset{\sim}{t}_{2;\alpha} - \underset{\sim}{t}_{1;\alpha} \tag{2.19}$$

over the curve C(t), $\underset{\sim}{t}_1$ and $\underset{\sim}{t}_2$ denoting the tensor in S_1 and S_2. A curve which carries discontinuities will be called a wave curve. Multiplying equation (2.18) by $u^{\beta}_{,s}$ and $a_{\beta\gamma}$ and making use of equation (7.10, see appendix 1) we obtain

$$[\underset{\sim}{t}_{;\gamma}] = \underset{\sim}{B}\,\nu_{\gamma} + \frac{D[\underset{\sim}{t}]}{Ds}\, u_{\gamma,s} \tag{2.20}$$

where

$$\underset{\sim}{B} = [\underset{\sim}{t}_{;\alpha}]\,\nu^{\alpha} \tag{2.21}$$

as the basic equations for the geometrical compatibility conditions of discontinuities across the wave curve C on S.

Define δ-time derivative as the rate of change of $\underset{\sim}{t}$ along geodesic normal to a wave curve C in the following manner.

$$\frac{\delta \underset{\sim}{t}}{\delta t} = \lim_{\Delta t\to 0}\frac{\Delta \underset{\sim}{t}}{\Delta t} = \lim_{\Delta t\to 0}\left(\frac{\Delta \underset{\sim}{t}}{\Delta u^{\alpha}}\frac{\Delta u^{\alpha}}{\Delta t}\right) + \frac{\partial \underset{\sim}{t}}{\partial t} \tag{2.22}$$

Thus, speed of normal velocity along the geodesic normal of a wave curve C can be defined by

$$G = \lim_{\Delta t \to 0} \frac{\Delta \sigma}{\Delta t} \tag{2.23}$$

where $\Delta\sigma$ is small distance along the normal. It also follows that

$$\frac{\delta u^\alpha}{\delta t} = \lim_{\Delta t \to 0} \frac{\Delta u^\alpha}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{\Delta \sigma}{\Delta t} \frac{\Delta u^\alpha}{\Delta \sigma} \right) = G \nu^\alpha \tag{2.24}$$

and we can show that [11]

$$\left[\!\left[\frac{\delta \mathbf{t}}{\delta t} \right]\!\right] = \frac{\delta [\![\mathbf{t}]\!]}{\delta t} \tag{2.25}$$

where

$$\left[\!\left[\frac{\delta \mathbf{t}}{\delta t} \right]\!\right] = \frac{\delta \mathbf{t}_2}{\delta t} - \frac{\delta \mathbf{t}_1}{\delta t} . \tag{2.26}$$

For an absolute tensor field, by definition, we have

$$\begin{aligned} \frac{\delta}{\delta t}(\mathbf{t}) &= \frac{\delta}{\delta t}(t^{\alpha \cdots \gamma}_{\sigma \cdots \tau}) = \frac{\partial \mathbf{t}}{\partial u^\lambda} G \nu^\lambda + \frac{\partial \mathbf{t}}{\partial t} \\ &= (\mathbf{t}_{;\lambda}) G\nu^\lambda - \left\{ (t^{\delta \cdots \gamma}_{\sigma \cdots \tau}) \Gamma^\alpha_{\delta \epsilon} + \cdots + (t^{\alpha \cdots \delta}_{\sigma \cdots \tau}) \Gamma^\delta_{\mu \epsilon} \right\} G \nu^\epsilon \\ &\quad + \left\{ (t^{\alpha \cdots \gamma}_{\mu \cdots \tau}) \Gamma^\mu_{\sigma \gamma} + \cdots + (t^{\alpha \cdots \gamma}_{\sigma \cdots \mu}) \Gamma^\mu_{\tau \gamma} \right\} G \nu^\gamma \\ &\quad + \frac{\partial \mathbf{t}}{\partial t} \end{aligned} \tag{2.27}$$

for the δ-time derivative of an absolute surface tensor field $\underset{\sim}{t}$.

We now can define absolute time derivative by

$$\frac{D\underset{\sim}{t}}{Dt} = \frac{\delta \underset{\sim}{t}}{\delta t} - \{(t^{\alpha\cdots\gamma}_{\mu\cdots\tau})\Gamma^{\mu}_{\sigma\gamma} + \cdots + (t^{\alpha\cdots\gamma}_{\sigma\cdots\mu})\Gamma^{\mu}_{\tau\gamma}\}G\nu^{\gamma} \tag{2.28}$$

$$+\{(t^{\delta\cdots\gamma}_{\sigma\cdots\tau})\Gamma^{\alpha}_{\delta\epsilon} + \cdots + (t^{\alpha\cdots\delta}_{\sigma\cdots\tau})\Gamma^{\delta}_{\mu\epsilon}\}G\nu^{\epsilon}$$
$$= \frac{\partial \underset{\sim}{t}}{\partial t} + \underset{\sim}{t}_{;\lambda}G\nu^{\lambda}$$

which will behave as a tensor by an arbitrary coordinate transformation $u^{\alpha} \to \bar{u}^{\alpha}$. As a basic kinematical compatibility condition of discontinuities of a wave curve C for an absolute surface tensor field $\underset{\sim}{t}$, we obtain the following equation from relations (2.28) and (2.27)

$$\left[\frac{\partial \underset{\sim}{t}}{\partial t}\right] = -[\underset{\sim}{t}_{;\lambda}]G\nu^{\lambda} + \frac{D[\underset{\sim}{t}]}{Dt} \tag{2.29}$$

in which the last term follows from equation (2.25).

3. First and Second Order Geometrical and Kinematical Compatibility Conditions

This chapter, closely following Thomas's idea for the wave surface in three dimensional space [11], presents detailed derivations of the first and second order geometrical and kinematical compatibility conditions across a wave curve C for a scalar p and a surface vector $\underset{\sim}{t}$. The compatibility conditions which will follow are of the appropriate form in the sense that the various terms in the conditions will behave as scalars, vectors or tensors under arbitrary coordinate transformation $u^{\alpha} \rightarrow \bar{u}^{\alpha}$.

Putting p, any scalar quantity in place of $\underset{\sim}{t}$ in equation (2.20) we obtain the first order geometrical compatibility conditions

$$[\![p_{,r}]\!] = \zeta \nu_r + [\![p]\!]_{,s} u_{r,s} \qquad (3.1)$$

since $p_{;r} = p_{,r}$ for a scalar quantity p where

$$\zeta = [\![p_{,r}]\!] \nu^r \qquad (3.2)$$

is a function defined over the curve C(t). The terms in equations (3.1) are of vector character and these equations are the appropriate form of the geometrical compatibility conditions of the first order for a scalar p.

*$[\![\ ]\!]$ in equations denote the jump of quantities in the brackets across a wave curve as in the previous discussion.

When we set $\underset{\sim}{t} = \underline{t}$, a surface vector, in equation (2.20) we obtain the relations

$$[\![t_{\alpha;\gamma}]\!] = \lambda_\alpha \nu_\gamma + \frac{D[\![t_\alpha]\!]}{DS} u_{\gamma,s} \tag{3.3}$$

where

$$\lambda_\alpha = [\![t_{\alpha;\gamma}]\!] \nu^\gamma \tag{3.4}$$

are components of a surface vector defined on C. Equations (3.3) are the required form of geometrical compatibility conditions of the first order for a surface vector $\underline{t}$ since all the terms are of tensorial character under the coordinate transformation $u^\alpha \to \bar{u}^\alpha$.

If we put $\underset{\sim}{t} = p$ in equation (2.29), appropriate form of kinematical compatibility conditions of the first order for a scalar p is obtained in the form

$$\left[\!\!\left[\frac{\partial p}{\partial t}\right]\!\!\right] = -\zeta G + \frac{\delta[\![p]\!]}{\delta t} \tag{3.5}$$

where ζ is defined over the curve C as in equations (3.1).

Again, putting $\underset{\sim}{t} = t_\alpha$ in equation (2.29), we have

$$\left[\!\!\left[\frac{\partial t_\alpha}{\partial t}\right]\!\!\right] = -\lambda_\alpha G + \frac{D[\![t_\alpha]\!]}{Dt} \tag{3.6}$$

in which

$$\frac{D[\![t_\alpha]\!]}{Dt} = \frac{\delta[\![t_\alpha]\!]}{\delta t} - G[\![t_\sigma]\!]\Gamma^\sigma_{\alpha\gamma}\nu^\gamma \tag{3.7}$$

as the appropriate form of the kinematical compatibility conditions of the first order for a surface vector $\underline{t}$. We consider now the derivation of the second order compatibility conditions across a moving curve C.

Putting $\underset{\sim}{t} = p_{,\alpha}$ in equation (2.20), we can obtain the relations

$$[\![P_{;\alpha\gamma}]\!] = \zeta_\alpha \nu_\gamma + \frac{D[\![P_{,\alpha}]\!]}{DS} u_{\gamma;S} \tag{3.8}$$

where

$$\zeta_\alpha = [\![P_{;\alpha\beta}]\!]\nu^\beta \tag{3.9}$$

are values defined over the curve C. Since the left members of equations (3.8) are symmetric in α and γ, we can write

$$\zeta_\alpha \nu_\gamma + \frac{D[\![P_{,\alpha}]\!]}{DS} u_{\gamma;S} = \zeta_\gamma \nu_\alpha + \frac{D}{DS}[\![P_{,\gamma}]\!] u_{\alpha;S} \tag{3.10}$$

Multiplying equations (3.10) by ν^γ and sum on the repeated index γ, it follows that

$$\zeta_\alpha = \tilde{\zeta}\nu_\alpha + \frac{D[\![P_{,\gamma}]\!]}{DS}\nu^\gamma u_{\alpha;S} \quad ; \quad \tilde{\zeta} = \zeta_\gamma \nu^\gamma \tag{3.11}$$

Substituting equations (3.1) and (2.14) successively in the identity

$$\frac{D}{Ds}[\![P_{,r}]\!]\nu^r = \frac{D}{Ds}\left([\![P_{,r}]\!]\nu^r\right) - [\![P_{,r}]\!]\frac{D\nu^r}{Ds} \tag{3.12}$$

we obtain

$$\frac{D}{Ds}[\![P_{,r}]\!]\nu^r = \zeta_{,s} + [\![P]\!]_{,s}\,K_g \tag{3.13}$$

since $\nu_r \frac{D\nu^r}{Ds} = 0$ by differentiating equation (2.11). Equations (3.10) become of the form

$$\zeta_\alpha = \tilde{\zeta}\,\nu_\alpha + \zeta_{,s}u_{\alpha,s} + [\![P]\!]_{,s}u_{\alpha,s}K_g \tag{3.14}$$

upon introducing equation (3.13). Substituting equations (3.1) and (3.14) in (3.8) it finally follows that

$$[\![P_{;\alpha\gamma}]\!] = \tilde{\zeta}\nu_\alpha\nu_\gamma + \left(\frac{D}{Ds}[\![P]\!]_{,s} - \zeta K_g\right)u_{\alpha,s}u_{\gamma,s} \tag{3.15}$$

$$+\left(\zeta_{,s} + [\![P]\!]_{,s}K_g\right)\left(\nu_\gamma u_{\alpha,s} + \nu_\alpha u_{\gamma,s}\right)$$

as the appropriate form of the geometrical compatibility conditions of the second order for a scalar p.

To deduce the second order geometrical compatibility conditions for a surface vector $\underset{\sim}{t}$, put $\underset{\sim}{t} = t_{\alpha;\beta}$ in equations (2.20) to obtain

$$[\![t_{\alpha;\beta\gamma}]\!] = \lambda_{\alpha\beta}\nu_{\gamma} + \frac{D[\![t_{\alpha;\beta}]\!]}{Ds} u_{\gamma,s} \tag{3.16}$$

in which

$$\lambda_{\alpha\beta} = [\![t_{\alpha;\beta\gamma}]\!] \nu^{\gamma} \tag{3.17}$$

are quantities defined over the curve C. To find the relations between $t_{\alpha;\beta\gamma}$ and $t_{\alpha;\gamma\beta}$, we can introduce the Ricci identities [14]

$$[\![t_{\alpha;\beta\gamma}]\!] = [\![t_{\alpha;\gamma\beta}]\!] + [\![t_{\eta}]\!] R^{\eta}_{\alpha\beta\gamma} \tag{3.18}$$

where $R^{\eta}_{\alpha\beta\gamma}$ are components of the Riemanian curvature tensor. From equations (3.16) and (3.18) we have

$$\lambda_{\alpha\beta}\nu_{\gamma} + \frac{D}{Ds}[\![t_{\alpha;\beta}]\!] u_{\gamma,s} = \lambda_{\alpha\gamma}\nu_{\beta} + \frac{D}{Ds}[\![t_{\alpha;\gamma}]\!] u_{\beta,s} + [\![t_{\eta}]\!] R^{\eta}_{\alpha\beta\gamma} \tag{3.19}$$

Following the same procedure as for the derivation of equations (3.15) the geometrical compatibility conditions of the second order for a surface vector $\underline{t}$ will be given by

$$[\![t_{\alpha;\beta\gamma}]\!] = \tilde{\lambda}_\alpha \nu_\beta \nu_\gamma + \left(\frac{D^2}{Ds^2}[\![t_\alpha]\!] - \lambda_\alpha K_g\right) u_{\beta,s} u_{\gamma,s}$$
$$+ \left\{ \frac{D}{Ds}(\lambda_\alpha) + \frac{D}{Ds}[\![t_\alpha]\!] K_g \right\} (u_{\beta,s}\nu_\gamma + u_{\gamma,s}\nu_\beta) \tag{3.20}$$
$$+ [\![t_\eta]\!] R^\eta_{\alpha\beta\gamma}$$

where

$$\tilde{\lambda}_\alpha = [\![t_{\alpha;\beta\gamma}]\!] \nu^\beta \nu^\gamma . \tag{3.21}$$

Consider the following identities

$$\frac{D}{Dt}[\![P_{,\alpha}]\!] = [\![P_{;\alpha\beta}]\!] G \nu^\beta + \left[\!\!\left[\frac{\partial^2 P}{\partial u^\alpha \partial t} \right]\!\!\right] , \tag{3.22}$$

$$\frac{D}{Dt}[\![P_{,\alpha}]\!] = \frac{D}{Dt}\left(\zeta \nu_\alpha + [\![P]\!]_{,s} u_{\alpha,s}\right) . \tag{3.23}$$

Equating (3.22) and (3.23) and then carrying out the time differentiation in equations (3.23) it follows that

$$\left[\!\!\left[\frac{\partial^2 P}{\partial u^\alpha \partial t} \right]\!\!\right] = -[\![P_{;\alpha\beta}]\!] G \nu^\beta + \frac{D}{Dt}(\zeta)\nu_\alpha + \zeta \frac{D\nu_\alpha}{Dt} \tag{3.24}$$
$$+ \frac{D}{Dt}(u_{\alpha,s})[\![P]\!]_{,s} + \left\{ \frac{D}{Ds}([\![P]\!]_{,s})\frac{\delta s}{\delta t} + \frac{\partial^2 [\![P]\!]}{\partial s \partial t} \right\} u_{\alpha,s}$$

over a curve C. For the time derivative of geodesic unit normal vector we have (see appendix 3)

$$\frac{D\nu_\alpha}{Dt} = - G_{,s}\, u_{\alpha,s} \tag{3.25}$$

and also for the variation of unit tangent vector

$$\frac{D}{Dt}(u_{\alpha,s}) = \nu_\alpha G_{,s} - u_{\alpha,s}\, u_{\gamma,s} \frac{\partial^2 u^\gamma}{\partial s \partial t} . \tag{3.26}$$

When equations (7.23, see appendix 3), (3.15), (3.25), (3.26) are substituted in the right members of equations (3.24) successively, it follows that

$$\left[\!\left[\frac{\partial^2 p}{\partial u^\alpha \partial t}\right]\!\right] = -\left(\zeta G - \frac{\delta \zeta}{\delta t} - G_{,s}[\![P]\!]_{,s}\right)\nu_\alpha + \Big\{(\zeta G)_{,s} + G\kappa_g [\![P]\!]_{,s} \tag{3.27}$$

$$+ \frac{D}{Ds}[\![P]\!]_{,s}\, u_{\beta,s} \frac{\partial u^\beta}{\partial t} + [\![P]\!]_{,s}\, u_{\gamma,s} \frac{\partial^2 u^\gamma}{\partial s \partial t} - \frac{\partial^2 [\![P]\!]}{\partial s \partial t}\Big\} u_{\alpha,s}$$

as a part of kinematical compatibility conditions of the second order for a scalar p.

We can derive the appropriate form of the kinematical compatibility conditions of the second order for a vector $\underline{t}$ in the form

$$\left[\frac{\partial t_{\alpha;\beta}}{\partial t}\right] = -\left\{\tilde{\lambda}_\alpha G - \frac{D\lambda_\alpha}{Dt} - G_{,s}\frac{D}{Ds}[[t_\alpha]]\right\}\nu_\beta - \left\{\frac{D}{Ds}(\lambda_\alpha G)\right. \tag{3.28}$$

$$+ K_g G\frac{D}{Ds}[t_\alpha] + \frac{D^2}{Ds^2}[t_\alpha]\, u_{\gamma,s}\frac{\partial u^\gamma}{\partial t} + \frac{D}{Ds}[t_\alpha]\, u_{\gamma,s}\frac{\partial^2 u^\gamma}{\partial s\partial t}$$

$$\left. - \frac{D}{Ds}\left(\frac{\partial[t_\alpha]}{\partial t}\right)\right\} u_{\beta,s} + [t_\eta] R^\eta_{\alpha\beta\gamma} G \nu^\gamma$$

where

$$\frac{D}{Ds}\left(\frac{\partial[t_\alpha]}{\partial t}\right) = \frac{\partial^2[t_\alpha]}{\partial t\,\partial s} - \frac{\partial[t_\gamma]}{\partial t}\Gamma^\gamma_{\alpha\beta} u^\beta_{,s} \tag{3.29}$$

when we follow the same procedures used in the derivation of equations (3.27).

To find the suitable form for $\left[\frac{\partial^2 P}{\partial t^2}\right]$, consider the identities

$$\frac{D}{Dt}\left[\frac{\partial P}{\partial t}\right] = \left[\frac{\partial^2 P}{\partial u^\alpha \partial t}\right] G\nu^\alpha + \left[\frac{\partial^2 P}{\partial t^2}\right] \tag{3.30}$$

$$\frac{D}{Dt}\left[\frac{\partial P}{\partial t}\right] = \frac{D}{Dt}\left(-\zeta G + \frac{\delta P}{\delta t}\right) . \tag{3.31}$$

From both of the right members of equations (3.30) and (3.31), it follows that

$$\left[\frac{\partial^2 p}{\partial t^2}\right] = \tilde{\zeta} G^2 - 2G\frac{\delta\zeta}{\delta t} - \zeta\frac{\delta G}{\delta t} - G G_{,s}[p]_{,s} + \frac{\delta^2 p}{\delta t^2} \tag{3.32}$$

by substituting equations (3.27), as the required form of the remaining kinematical compatibility conditions for a scalar p.

Finally, consider the identities

$$\frac{D}{Dt}\left[\frac{\partial t_\alpha}{\partial t}\right] = \left[\frac{\partial t_{\alpha;\beta}}{\partial t}\right] G\nu^\beta + \left[\frac{\partial^2 t_\alpha}{\partial t^2}\right] \tag{3.33}$$

$$\frac{D}{Dt}\left[\frac{\partial t_\alpha}{\partial t}\right] = \frac{D}{Dt}\left\{-\lambda_\alpha G + \frac{D[t_\alpha]}{Dt}\right\} \tag{3.34}$$

In a similar manner as for the derivation of equation (3.32), we obtain

$$\left[\frac{\partial^2 t_\alpha}{\partial t^2}\right] = \tilde{\lambda}_\alpha G^2 - 2G\frac{D\lambda_\alpha}{Dt} - \lambda_\alpha\frac{\delta G}{\delta t} - G G_{,s}\frac{D}{Ds}[t_\alpha] \tag{3.35}$$

$$+ [t_\eta] R^\eta_{\alpha\beta\gamma} G^2 \nu^\gamma \nu^\beta + \frac{D^2[t_\alpha]}{Dt^2}$$

as the appropriate form for the remaining compatibility conditions for a surface vector $\underline{t}$.

The appropriate form of the compatibility conditions for higher order tensors across a moving curve $C(t)$ are easily obtained from the basic relations in the same way as above and the same form as the above equations.

4. First and Second Order Dynamical Conditions

In this chapter, Reynold's transport theorem is introduced to deduce the basic dynamical field equations and discontinuity relations for a surface undergoing infinitesimal deformation. These relations comprise the principles of conservation of mass and momenta.

Reynold's transport theorem [15] is given by

$$\frac{d}{dt}\int_S \underset{\sim}{t}(u,t)\, dS = \int_S \frac{\partial \underset{\sim}{t}}{\partial t}\, dS + \int_C v_n \underset{\sim}{t}\, ds \tag{4.1}$$

where $\underset{\sim}{t}$ is an absolute surface tensor as the preceeding discussions and $v_n = v_\alpha \nu^\alpha$, the normal component of the velocity of material particle of S at points of moving curve C.

Let γ be the density of the surface material under consideration. We assume the law of mass conservation, that is, the mass of the moving surface S remains unchanged under deformation. This condition is given by

$$\frac{d}{dt}\int_S \gamma\, dS = 0 \tag{4.2}$$

If we put $\underset{\sim}{t} = \gamma$ and $v_n = v_\alpha \nu^\alpha$ in equation (4.1) and make use of

Green's theorem [15] given by

$$\int_S t_{\alpha;}{}^{\alpha}\, dS = \int_C t_\alpha \nu^\alpha\, ds \tag{4.3}$$

it follows that

$$\int_S \frac{\partial \gamma}{\partial t} dS + \int_C \gamma v_\alpha \nu^\alpha ds = \int_S \frac{\partial \gamma}{\partial t} dS + \int_S (\gamma v_\alpha)_;{}^\alpha dS = 0 \tag{4.4}$$

by applying equation (4.2). Since S is arbitrary, we have the equation of continuity in the form

$$\frac{\partial \gamma}{\partial t} + (\gamma v_\alpha)_;{}^\alpha = \frac{d\gamma}{dt} + \gamma v_{\alpha;}{}^\alpha = 0 \tag{4.5}$$

It can be shown that the stress on a curve in S is given by [15]

$$t_\alpha = \tau_{\alpha\beta} \nu^\beta \tag{4.6}$$

where t_α are components of force per unit arc length, $\tau_{\alpha\beta}$ components of stress tensor, and ν^β components of geodesic unit normal vector to the curve. We also assume the principle of conservation of linear momentum, that is, the rate of change of the momentum of S in any fixed direction is equal to the component, in this direction, of the total external force acting on S, expressed by

$$\frac{d}{dt}\int_{S} \gamma v_\alpha \, dS = \int_{C} t_\alpha \, ds \tag{4.7}$$

Substituting equation (4.6) in (4.7) we have

$$\frac{d}{dt}\int_{S} \gamma v_\alpha \, dS = \int_{C} \tau_{\alpha\beta} \nu^\beta \, ds \, . \tag{4.8}$$

From equation (4.1) and (4.5) we can show that

$$\frac{d}{dt}\int_{S} \gamma v_\alpha \, dS = \int_{S} \gamma \frac{dv_\alpha}{dt} \, dS \tag{4.9}$$

by putting $\underset{\sim}{t} = v_\alpha$ in equation (4.1). Hence we obtain the relations

$$\frac{d}{dt}\int_{S} \gamma v_\alpha \, dS = \int_{S} \gamma \frac{dv_\alpha}{dt} \, dS = \int_{C} \tau_{\alpha\beta} \nu^\beta \, ds \tag{4.10}$$

and applying Green's theorem for the last term of equation (4.10), the equation of motion will follow in the form

$$\gamma \frac{dv_\alpha}{dt} = \tau_{\alpha\beta;}{}^{\beta} \, . \tag{4.11}$$

As usual, we can show that $\tau_{\alpha\beta}$ are the components of a symmetric tensor by making use of principle of conservation of angular momentum [15] .

If $\underline{w}$ denotes the displacement vector in S, we have

$$v_\alpha = \frac{\partial w_\alpha}{\partial t} + w_{\alpha;\beta} v^\beta, \tag{4.12}$$

but for infinitesimal deformation we can put

$$v_\alpha = \frac{\partial w_\alpha}{\partial t} \tag{4.13}$$

approximately, neglecting the second term in the right side of equation (4.12). Similarly, we have

$$\frac{d v_\alpha}{d t} = \frac{\partial^2 w_\alpha}{\partial t^2} \tag{4.14}$$

approximately, thus equations (4.11) can be written as

$$\tau_{\alpha\beta;}{}^{\beta} = \gamma \frac{\partial^2 w_\alpha}{\partial t^2}. \tag{4.15}$$

Having derived the basic field equations, we now turn to the question of discontinuity relations in the dynamical variables across a wave curve C. As in the earlier discussions, assume that the surface is divided by a wave curve C into regions S_1 and S_2. If we apply equation (4.1) for regions S_1 and S_2, and denote $\underset{\sim}{t}_1$, and $\underset{\sim}{t}_2$ as the value of $\underset{\sim}{t}$ on sides of C bodering S_1 and S_2, respectively, it follows that

$$\frac{d}{dt}\int_S \underset{\sim}{t}\, dS = \int_S \frac{\partial \underset{\sim}{t}}{\partial t} dS + \int_\Sigma \underset{\sim}{t}\, v_n\, ds + \int_C (\underset{\sim}{t}_1 - \underset{\sim}{t}_2) G\, ds \tag{4.16}$$

where Σ is the curve bounding the surface S and G being the normal speed of the curve C. (Fig.2) Putting $\underset{\sim}{t} = \gamma$ in equation (4.16) we have the following relation from equation (4.2)

$$\frac{d}{dt}\int_S \gamma\, dS = \int_S \frac{\partial \gamma}{\partial t} dS + \int_{\Sigma_1} \gamma v_n\, ds + \int_{\Sigma_2} \gamma v_n\, ds \tag{4.17}$$

$$+ \int_C (\gamma_1 - \gamma_2) G\, ds = 0$$

where γ_1 and γ_2 are the values of the density on the side 1 and 2 of the curve C and Σ_1 and Σ_2 are the portions of the curve bounding the surface S_1 and S_2. Let S approaches zero at a fixed time so that S will shrink into a part of C. Then equation (4.17) will become

$$\gamma_1 (v_{1n} - G) = \gamma_2 (v_{2n} - G) \tag{4.18}$$

as the first dynamical relation required across the curve where v_{1n} and v_{2n} are the normal components of the particle velocity on the side 1 and 2 of the curve C since the surface integral approaches zero but

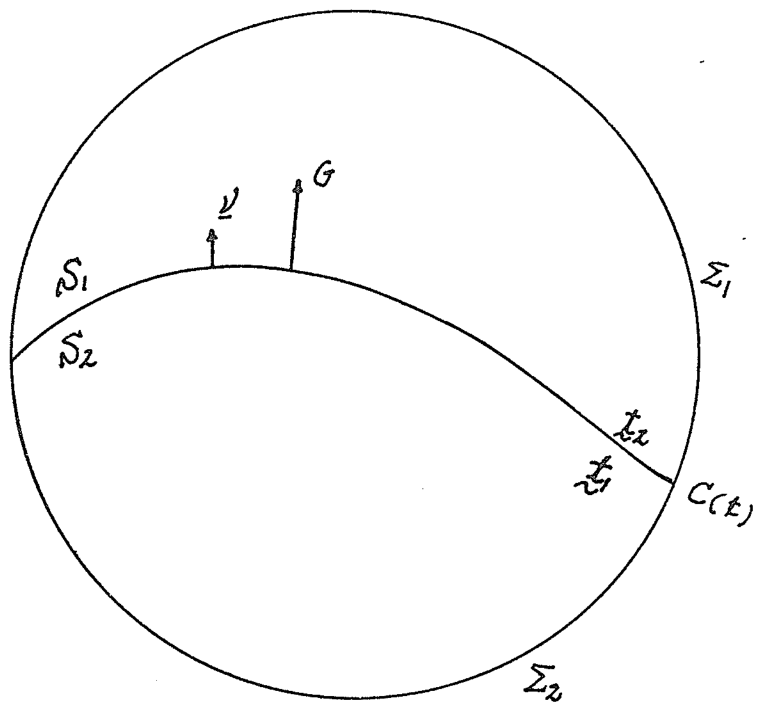

Fig. 2. Wave curve C(t) and the bounding curve Σ of S

$$\int_{\Sigma_1} \gamma v_n \, ds \longrightarrow -\int_c \gamma_1 v_{1n} \, ds$$

$$\int_{\Sigma_2} \gamma v_n \, ds \longrightarrow \int_c \gamma_2 v_{2n} \, ds \tag{4.19}$$

when we assign positive sign to the outward normal.

Similarly from the equation of principles of linear momentum (4.7) when the surface S approaches zero, we obtain the following relation

$$\int_c \tau_{\alpha\beta} \nu^\beta \, ds \longrightarrow \int_c (\tau_{2\alpha\beta} - \tau_{1\alpha\beta}) \nu^\beta \, ds \tag{4.20}$$

for the stress difference in the normal direction between the region S_1 and S_2. Also in the same way as the above, putting $\underset{\sim}{t} = \gamma v_\alpha$ in equation (4.16) and taking the limit S $\longrightarrow$ 0, it follows that

$$\int_c \gamma_1 v_{1\alpha} (G - v_{1n}) \, ds - \int_c \gamma_2 v_{2\alpha} (G - v_{2n}) \, ds \tag{4.21}$$

Finally the equation

$$\gamma_1 (v_{1n} - G) [\![v_\alpha]\!] = [\![\tau_{\alpha\beta}]\!] \nu^\beta \tag{4.22}$$

follows from equations (4.10), (4.20) and (4.21) as the required second order dynamical discontinuity relation in which

$$[\![v_\alpha]\!] = v_{2\alpha} - v_{1\alpha} \tag{4.23}$$

and

$$[\![\tau_{\alpha\beta}]\!] = \tau_{2\alpha\beta} - \tau_{1\alpha\beta} \,. \tag{4.24}$$

5. Waves in Elastic Surfaces

In this chapter, the propagation speeds of equivoluminal and irrotational wave curves are obtained by making use of the compatibility conditions in the previous chapters. It is also shown that these waves propagate along the geodesic normals of the wave curve, that is, successive wave positions form the geodesic parallels of the original wave curve. Finally, a differential equation for the variation of wave strength is established.

Assume stress-strain relations for the infinitesimal deformation of the isotropic homogeneous surface elastic medium as [16]

$$\tau_{\alpha\beta} = 2\mu e_{\alpha\beta} + \lambda\theta\delta_{\alpha\beta} \tag{5.1}$$

where μ, λ are material constants and $\theta = e_{\alpha;}^{\ \alpha}$, analogous to the three dimensional case. We list here the basic field equations for reference purposes:

$$\tau_{\alpha\beta} = 2\mu e_{\alpha\beta} + \lambda\theta\delta_{\alpha\beta} \quad ; \text{ stress-strain relations} \tag{5.1a}$$

$$e_{\alpha\beta} = \frac{1}{2}(w_{\alpha;\beta} + w_{\beta;\alpha}) \quad ; \text{ strain-displacement relations} \tag{5.2}$$

$$\tau_{\alpha\beta;}{}^{\beta} = \gamma \frac{\partial^2 w_\alpha}{\partial t^2} . \tag{5.3}$$

$$\frac{\partial \gamma}{\partial t} + (\gamma v_\alpha)_;{}^{\alpha} = 0 \tag{5.4}$$

Consider a moving curve C(t) on S propagating from region S_1 into region S_2. The wave curve is such that across it there may occur discontinuities in density, stress or particle velocity. Particles in the region S_2 are at rest and unstrained so that we can put

$$\gamma_2 = \gamma,\ v_{2\alpha} = 0,\ \tau_{2\alpha\beta} = 0,\ v_{2n} = 0,\ w_{2\alpha} = 0 \tag{5.5}$$

while in S_1,

$$\gamma_1 = \bar{\gamma},\ v_{1\alpha} = \bar{v}_\alpha,\ \tau_{1\alpha\beta} = \bar{\tau}_{\alpha\beta},\ v_{1n} = \bar{v}_n,\ w_{1\alpha} = \bar{w}_\alpha . \tag{5.6}$$

For this case, equations (4.17) and (4.21) will give the first and the second dynamical relations of discontinuity in the form, respectively,

$$\gamma G = \bar{\gamma}(G - \bar{v}_n) \tag{5.7}$$

$$[\![\tau_{\alpha\beta}]\!]\nu^{\beta} = -\gamma G [\![v_{\alpha}]\!] . \tag{5.8}$$

Assume that $[\![w_{\alpha}]\!] = 0$, that is, deformation is continuous but that there are discontinuities in the derivatives $w_{\alpha;\beta}$ across C(t). Such a curve will be a wave curve of order 1. Since $[\![w_{\alpha}]\!] = 0$, equations (3.2) and (3.5) the geometrical and kinematical compatibility conditions of the first order will become

$$[\![w_{\alpha;\beta}]\!] = -\bar{w}_{\alpha;\beta} = \omega_{\alpha}\nu_{\beta} \quad ; \quad \left[\!\!\left[\frac{\partial w_{\alpha}}{\partial t}\right]\!\!\right] = -G\omega_{\alpha} \tag{5.9}$$

where

$$\omega_{\alpha} = [\![w_{\alpha;\beta}]\!]\nu^{\beta} \tag{5.10}$$

are functions defined over the curve C(t).

As it is assumed that C(t) is a wave curve of order 1, not all of the functions ω_{α} can vanish. Hence we define the wave strength by

$$\overline{W} = \sqrt{\omega_\alpha \omega^\alpha}\,. \tag{5.11}$$

From equations (5.1), (5.6) and (5.9), it follows that

$$\overline{\tau}_{\alpha\beta} \nu^\beta = -\mu \omega_\alpha - (\lambda+\mu)(\omega_\beta \nu^\beta)\nu_\alpha\,. \tag{5.12}$$

Also combining equations (5.8), (5.9) and (5.12) we have the relations

$$\mu \omega_\alpha + (\lambda+\mu)(\omega_\beta \nu^\beta)\nu_\alpha = \Upsilon\, \sigma^2 \omega_\alpha\,. \tag{5.13}$$

Assuming that $\omega_\alpha \nu^\alpha$ does not vanish on C(t) and multiplying equation (5.12) by ν^α we have

$$\mu \omega + (\lambda+\mu)\omega = \Upsilon \sigma^2 \omega \tag{5.14}$$

where

$$\omega = \omega_\alpha \nu^\alpha \tag{5.15}$$

and finally it follows from equation (5.14) that

$$G^2 = \frac{\lambda + 2\mu}{\gamma} . \tag{5.16}$$

Since $\omega = \omega_\alpha \nu^\alpha \neq 0$, we have

$$[\![w_{\alpha;\beta}]\!] = \omega \nu_\alpha \nu_\beta \tag{5.17}$$

from the first part of the equation (5.9) by putting

$$\omega_\alpha = \omega \nu_\alpha . \tag{5.18}$$

Hence it follows that

$$\bar{w}_{\alpha;\beta} - \bar{w}_{\beta;\alpha} = -\omega(\nu_\alpha \nu_\beta - \nu_\beta \nu_\alpha) = 0 \tag{5.19}$$

that is, the rotation [10] vanishes immediately behind the wave front. Thus we can say that the speed given by equation (5.16) is that of irrotational wave.

If we put $\omega = 0$ but $\omega_\alpha \neq 0$, we have

$$G^2 = \frac{\mu}{\gamma} \tag{5.20}$$

from equation (5.14). For this case, from the first part of equation

(5.9), we have

$$\bar{w}_{\alpha;}{}^{\alpha} = 0 \qquad (5.21)$$

Also we have

$$\bar{v}_n = \bar{v}_\alpha \nu^\alpha = -G\omega_\alpha \nu^\alpha = 0 \qquad (5.22)$$

from the second part of equation (5.9). Finally it follows from equation (5.7) that

$$\gamma = \bar{\gamma}, \qquad (5.23)$$

that is, density remains constant immediately behind the wave front so that equation (5.20) can be called as the speed of equivoluminal wave. It is shown that expressions for the speed of irrotational and equivoluminal wave are analogous to those of the corresponding waves in the three dimensional space [11].

Since the speed G is constant, we have

$$\frac{D\nu^\alpha}{Dt} = 0 \qquad (5.24)$$

Consider a system of curves $u^\alpha = u^\alpha(\sigma)$ having as tangents ν^α the normals to the wave curve C(t) at some time t. We can write equation (5.24) as

$$\frac{D\nu^\alpha}{D\sigma}\frac{D\sigma}{Dt} = 0 . \tag{5.25}$$

If $G \neq 0$, we have

$$\frac{D\nu^\alpha}{D\sigma} = 0 \tag{5.26}$$

for all σ. Since equation (5.26) satisfies condition (2.15), we see that the system of curves $u^\alpha = u^\alpha(\sigma)$ are geodesics on S normal to C(t). Since G is a constant for both equivoluminal and irrotational waves in S, the wave curve C propagates the same distance from the initial position in S in a time interval. Thus we can say that the successive positions of the irrotational and equivoluminal wave curve C(t) form a family of geodesic parallel curves.

To find the expression for the variation of wave strength, we utilize the second order geometrical and kinematical compatibility conditions. Since $[\![w^i_\alpha]\!] = 0$, that is, deformation is continuous over the curve C(t), and speeds are constants for the waves to be investigated, it follows that

$$[w_{\alpha;\beta\gamma}] = -\bar{w}_{\alpha;\beta\gamma} = \hat{\omega}_\alpha \nu_\beta \nu_\gamma - \omega_\alpha \kappa_g u_{\beta,s} u_{\gamma,s} \tag{5.27}$$

$$+\omega_{\alpha;s}(u_{\beta,s}\nu_\gamma + u_{\gamma,s}\nu_\beta)$$

$$\left[\!\left[\frac{\partial w_{\alpha;\beta}}{\partial t}\right]\!\right] = -\frac{\partial \bar{w}_{\alpha;\beta}}{\partial t} = -\left(\hat{\omega}_\alpha G - \frac{D\omega_\alpha}{Dt}\right)\nu_\beta - \omega_{\alpha;s} G u_{\beta,s} \tag{5.28}$$

$$\left[\!\left[\frac{\partial^2 w_\alpha}{\partial t^2}\right]\!\right] = -\frac{d\bar{v}_\alpha}{dt} = \hat{\omega}_\alpha G^2 - 2G\frac{D\omega_\alpha}{Dt} \tag{5.29}$$

from relations (3.19), (3.27) and (3.34) where

$$\hat{\omega}_\alpha = [\![w_{\alpha;\beta\gamma}]\!]\nu^\beta\nu^\gamma \tag{5.30}$$

are new functions defined over the curve C. If we contract on the indices β and γ : and α and β respectively, in equations (5.27) and then take scalar products with ν^α and ν^γ for the resulting equations, respectively, we can show that

$$[\![w_{\alpha;\beta}{}^{\beta}]\!]\nu^\alpha = -\bar{w}_{\alpha;\beta}{}^{\beta}\nu^\alpha = \hat{\omega} - \omega\kappa_g \tag{5.31}$$

$$[\![w_{\alpha;\gamma}{}^{\alpha}]\!]\nu^\gamma = -\bar{w}_{\alpha;}{}^{\alpha}{}_{\gamma}\nu^\gamma = \hat{\omega} + \omega_{\alpha;s}u^\alpha_{,s} \tag{5.32}$$

where

$$\hat{\omega} = \hat{\omega}_\alpha \nu^\alpha \tag{5.33}$$

From the stress-strain relations (5.1) together with equations (5.31) and (5.33), we obtain

$$[\tau_{\alpha\beta;}{}^{\beta}] = (\mu+\lambda)[w_{\beta;\alpha}{}^{\beta}] + \mu[w_{\alpha;\beta}{}^{\beta}] \tag{5.34}$$

The equations of motion will give the relation

$$[\tau_{\alpha\beta;}{}^{\beta}]\nu^\alpha = [\gamma \frac{dw_\alpha}{dt}]\nu^\alpha \tag{5.35}$$

after multiplying by ν^α. Substituting equations (5.29), (5.31), (5.32) and (5.34) in (5.35) we are led to the relation

$$(\lambda+2\mu)(\hat{\omega} - K_g \omega) = (\hat{\omega}_\alpha G^2 - 2G \frac{D\omega_\alpha}{Dt})\nu^\alpha \bar{\gamma} \tag{5.36}$$

by introducing the relations

$$\omega_{\beta;s} = \omega_{;s}\nu_\beta - K_g \omega u_{\beta s} \tag{5.37}$$

which is obtained by differentiating the equation (5.18). From equations (5.7) and the second of equation (5.9), we have the relation

$$\Upsilon = \bar{\Upsilon}(1-\omega), \tag{5.38}$$

Finally, if we substitute equations (5.16), (5.18), (5.38) and the identity

$$\frac{d\omega}{dt} = G\frac{d\omega}{d\sigma} \tag{5.39}$$

in the equation (5.36), we obtain a differential equation

$$2\frac{d\bar{W}}{\bar{W}} = K_g\, d\sigma \tag{5.40}$$

upon neglecting the small quantity $\omega(\ll 1)$.

The above equation (5.40) is the required differential equation for the variation of the strength W of an irrotational wave during its propagation in the surface elastic medium. Similarly exactly the same expression can be obtained for the equivoluminal wave.

6. Applications

In this chapter, a general solution of equation (5.40), the differential equation for the variation of wave strength is given. Several examples for specific surfaces of interest are given. Furthermore, it is shown that the decay equation for waves in the plane agree with Thomas's results for the corresponding cylindrical surface waves in a three dimensional elastic medium [10] . Also the solution of equation (5.40) for the circular plane wave agrees with Jahsman's result [6] .

Consider equation (5.40) first in an arbitrary surface S. We can choose any wave curve C on S to be a coordinate curve $u^1 = 0$, and take geodesic normals to C as coordinate curves $u^2 =$ constant. If we put geodesic parallels to C as curves $u^1 =$ constant, then $u^1 =$ constant and $u^2 =$ constant form a geodesic parallel coordinate system. [13] If K_{g1} and K_{g2} denote the geodesic curvature of the coordinate curves $u^2 =$ constant and $u^1 =$ constant, respectively, we have

$$K_{g2} = \frac{1}{\sqrt{a}}\left(\frac{\partial}{\partial u^2}\left(\frac{a_{21}}{\sqrt{a_{22}}}\right) - \frac{\partial}{\partial u^1}\sqrt{a_{22}}\right) \tag{6.1}$$

where we have used the general expression for geodesic curvature [14]

$$K_{g\alpha} = \frac{1}{\sqrt{a}}\left(\frac{\partial}{\partial u^\alpha}\frac{a_{\alpha\beta}}{\sqrt{a_{\alpha\alpha}}} - \frac{\partial}{\partial u^\beta}\sqrt{a_{\alpha\alpha}}\right), \quad (\alpha \neq \beta) \tag{6.2}$$

in which $a = det(a_{\alpha\beta})$.

Introducing equation (6.1) in equation (5.40), we have

$$2\frac{dW}{W} = \frac{1}{\sqrt{a}}\left(\frac{\partial}{\partial u^2}\frac{a_{21}}{\sqrt{a_{22}}} - \frac{\partial}{\partial u^1}\sqrt{a_{22}}\right)du^1 \tag{6.3}$$

for the curve $u^1 =$ constant since $d\sigma = du^1$ for this case. The first fundamental form corresponding to the geodesic parallel coordinate system is given by [13, 14]

$$ds^2 = (du^1)^2 + a_{22}(u^1, u^2)(du^2)^2 . \tag{6.4}$$

Thus equation (6.3) is reduced to the following

$$2\frac{dW}{W} = -\frac{1}{\sqrt{a_{22}}}\frac{\partial\sqrt{a_{22}}}{\partial u^1}du^1 \tag{6.5}$$

since $a_{21} = 0$ and $a = a_{22}$. When we consider u^2 as a parameter, equation (6.5) is easily integrated to give the wave strength during its propagation by

$$W = \frac{W_0}{(a_{22})^{1/4}} \tag{6.6}$$

for the curve $u^1 =$ constant, where W_0, determined by initial wave strength and form, can be a function of the coordinate u^2. Equation (6.6) reveals that if we have proper expression for a_{22} in a geodesic parallel coordinate system, we can always find a solution for the differential equation (5.40) on the surface in question. Furthermore, when we introduce a geodesic polar coordinate system,* according to the shape of wave curve and surface in question, the same equation (6.6) will hold true.

In the following, several applications of equation (6.6) are given for specific surfaces.

* If the linear element can be expressed on S as

$$ds^2 = (du^1)^2 + a_{22}(du^2)^2$$

which satisfies the conditions

$$(a_{22})_{u^1=0} = 0, \quad \left(\frac{\partial\sqrt{a_{22}}}{\partial u^1}\right)_{u^1=0} = 1,$$

the coordinate u^1, u^2 are called a geodesic polar coordinate system.

Example 1.

For the surface of Liouville, the first fundamental form is given by [14]

$$ds^2 = (A_1 + A_2)\{B_1^2 (du^1)^2 + B_2^2 (du^2)^2\} \tag{6.7}$$

where A_α and B_α are function of u^α. If we introduce the following coordinate transformation,

$$u^{1*} = \int_0^{u^1} \sqrt{(A_1 + A_2) B_1^2}\, du^1, \quad u^{2*} = u^2, \tag{6.8}$$

equation (6.7) will be reduced to

$$ds^2 = (du^{1*})^2 + a_{22} (du^{2*})^2 \tag{6.9}$$

where

$$a_{22} = \{A_1(u^{\alpha*}) + A_2(u^{\alpha*})\} B_2^2(u^{\alpha*}). \tag{6.10}$$

The wave strength equation for the wave curve $u^1 =$ constant follows, formally, as

$$\overline{W} = \frac{\overline{W}_0}{\left[\left\{A_1(u^{\alpha *}) + A_2(u^{\alpha *})\right\} B_2^2(u^{\alpha *})\right]^{1/4}} \tag{6.11}$$

from equation (6.6)

<u>Example 2.</u>

For a surface of revolution whose linear element can be written in the form

$$ds^2 = (du^1)^2 + \{\psi(u^1)\}^2 (du^2)^2 \tag{6.12}$$

we can immediately write the wave strength equation for the wave curve $u^1 =$ constant as

$$\overline{W} = \frac{\overline{W}_0}{\sqrt{\psi(u^1)}} . \tag{6.13}$$

<u>Example 3.</u>

In this example, we will present simple but useful wave strength equations for the wave curves on surfaces of revolution with constant Gaussian curvature K. Gaussian curvature K is given by

$$K = -\frac{1}{\sqrt{a_{22}}}\frac{\partial^2\sqrt{a_{22}}}{(\partial u^1)^2} \qquad (6.14)$$

in the geodesic polar coordinate system, which is an intrinsic property of the surface, namely, a measure of the departure of the surface in question from being a Euclidean two dimensional space.

Before considering further the surfaces of constant K, it is interesting to notice that if a geodesic polar coordinate system is introduced on a surface, $\sqrt{a_{22}}$ can be expressed as [14]

$$\sqrt{a_{22}} = u^1 - \frac{1}{3!}K_0(u^1)^2 + \cdots \qquad (6.15)$$

where K_0 is the value of K at the origin of geodesic polar coordinate system. Hence, once the value of K at a point is known it is possible to calculate, approximately, the strength of wave propagation in the neighborhood of the point.

Surfaces of constant Gaussian curvature K can be classified into three types, that is, for $K = 0$, $K > 0$ and $K < 0$.
We will consider the equation (6.6) for each case separately.

Case 1. K $= 0$

When Gaussian curvature of a surface vanishes, it is called a surface isometric with the plane. Solving the differential equation (6.14) in a geodesic polar coordinates, we have

$$a_{22} = (u^1)^2, \tag{6.16}$$

hence

$$ds^2 = (du^1)^2 + (u^1)^2(du^2)^2 \tag{6.17}$$

for the linear element of the surface. The wave strength equation (6.6) results in the form

$$\mathbb{W} = \frac{\mathbb{W}_0}{\sqrt{u^1}} \tag{6.18}$$

for the wave curve $u^1 =$ constant. We can see that wave strength diminishes as $\mathbb{W} \to 0$ when $u^1 \to \infty$ where $u^1 > 0$.
In particular, for a circular wave of radius r on a plane, we have

$$\mathbb{W} = \frac{\mathbb{W}_0}{\sqrt{r}}\,. \tag{6.19}$$

Case 2. K $=$ constant > 0

These type of surfaces are called the spherical surface of revolution. Again, solving differential equation (6.14), we have, in a geodesic polar coordinate system,

$$a_{22} = \frac{1}{K} \sin^2(\sqrt{K}\, u^1), \quad (K > 0) \tag{6.20}$$

and the first fundamental form will be

$$ds^2 = (du^1)^2 + \frac{1}{K} \sin^2(\sqrt{K}\, u^1)\, (du^2)^2. \tag{6.21}$$

From equation (6.6), the wave strength equation for the curve $u^1 =$ constant can be written as

$$\overline{W} = \frac{\overline{W}_0}{\sqrt{\sqrt{\frac{1}{K}} \sin(\sqrt{K}\, u^1)}} . \tag{6.22}$$

Note that wave strength diminishes as u^1 increases within the range $0 < \sqrt{K}\, u^1 \leqslant \frac{\pi}{2}$, In particular for the sphere of radius r, we have

$$K = \frac{1}{r^2}, \tag{6.23}$$

so that the wave strength equation for the wave curve $u' =$ constant is

$$\overline{W} = \frac{\overline{W_0}}{\sqrt{r \sin \frac{u'}{r}}}, \quad \left(0 < \frac{u'}{r} \leq \frac{\pi}{2}\right). \tag{6.24}$$

The wave curves are a system of parallel circles on the sphere.

Case 3. K $=$ constant < 0.

A surface of negative constant Gaussian curvature is called pseudospherical surface. For these surfaces, we have

$$a_{22} = \left(-\frac{1}{K}\right) \sinh^2(\sqrt{-K}\, u'), \quad (K < 0) \tag{6.25}$$

by solving equation (6.14) and thus

$$ds^2 = (du')^2 + \left(-\frac{1}{K}\right) \sinh^2(\sqrt{-K}\, u')(du^2)^2 \tag{6.26}$$

for the first fundamental form. The wave strength equation will be expressed by

$$\overline{W} = \frac{\overline{W_0}}{\sqrt{\sqrt{\frac{1}{K}} \sinh(\sqrt{-K}u')}} \tag{6.27}$$

We can easily see that $\overline{W}$ diminishes as u' increases where $u' > 0$.

In the following, a brief comparision of this results with Thomas's [10] and Jahsman's result [6] is given. For a plane, equation (5.40) can be written as

$$\frac{d\overline{W}}{\overline{W}} = \frac{1}{2} K \, d\sigma \tag{6.28}$$

since the geodesic curvature $K_g = K$ where K is the curvature of a curve in the plane. If we cut a cylinder orthogenal to its generators by a plane, the propagation of a cylindrical surface wave in three dimensional space can be seen as the case of wave curve in the plane whose curvature is the same as that of meridian of the cylinder. From this view point, we can compare equation (6.28) with the corresponding result of Thomas's theory. He established the differential equation for the variation of the strength W of the wave surface during its propagation in three dimensional elastic medium in the form

$$\frac{d\overline{W}}{\overline{W}} = \Omega \, d\sigma \tag{6.29}$$

where Ω denote the mean curvature of the surface and the distance measured along the normals to the wave surface. For a cylindrical surface, we have

$$\Omega = \frac{1}{2}K \tag{6.30}$$

where K is the curvature of a meridian of the cylinder and it follows that

$$\frac{d\overline{W}}{\overline{W}} = \frac{1}{2}K\,dr \tag{6.31}$$

from equation (6.29). Thus we can see that equation (6.28) is the same as (6.31), the corresponding equations for the comparison.

The wave strength equation for the propagation of a circular wave of radius r in the plane, is given by

$$\overline{W} = \frac{\overline{W}_0}{\sqrt{r}} \tag{6.32}$$

upon substitution $K = -\frac{1}{r}$ in equation (6.28). Equation (6.32) which is already given by equation (6.19) agrees with Jahsman's result. Also, using equation (6.31), we have the same expression for the strength equation of a circular cylindrical surface wave of radius r in the three dimensional case.

7. Appendices.

Appendix 1. Derivation of equation

$$u^{\alpha}_{,s} u^{\beta}_{,s} = a^{\alpha\beta} - \nu^{\alpha}\nu^{\beta}. \qquad [12]$$

From equation (2.11), (2.12) we have relations

$$a_{\alpha\beta}\,\nu^{\alpha}\nu^{\beta} = 1, \qquad a_{\alpha\beta}\,u^{\alpha}_{,s}\nu^{\beta} = 0. \tag{7.1}$$

Formally put

$$u^{\beta}_{,n} = \nu^{\beta} \tag{7.2}$$

and define $G_{\sigma\tau}$ such that

$$G_{ss} = 1, \qquad G_{sn} = 0, \qquad G_{nn} = 1, \tag{7.3}$$

and set

$$a_{\alpha\beta}\, u^{\alpha}_{,\sigma} u^{\beta}_{,\tau} = G_{\sigma\tau}. \tag{7.4}$$

Define quantities d^{α}_{τ} and symmetric quantities $G^{\sigma\tau}$ such that

$$u^{\alpha}_{,\sigma} d^{\sigma}_{\tau} = \delta^{\alpha}_{\tau} \quad , \qquad G_{\sigma\tau} G^{\sigma\gamma} = \delta^{\gamma}_{\tau} \tag{7.5}$$

where δ is the Kronecker delta. Multiply (7.4) by d^{τ}_{μ}, and obtain relations

$$a_{\alpha\mu} u^{\alpha}_{,\sigma} = G_{\sigma\tau} d^{\tau}_{\mu} \, . \tag{7.6}$$

Multiply (7.6) by $a^{\mu\nu}$ and obtain relations

$$u^{\nu}_{,\sigma} = G_{\sigma\tau} d^{\tau}_{,\mu} a^{\mu\nu} \, . \tag{7.7}$$

Multiply (7.7) by $G^{\sigma\gamma}$, thus it follows

$$G^{\sigma\tau} u^{\nu}_{,\sigma} = a^{\mu\nu} d^{\gamma}_{,\mu} \, . \tag{7.8}$$

Multiply (7.8) by u^{ω}_{γ}, then it follows

$$a^{\alpha\beta} = G^{\sigma\tau} u^{\alpha}_{,\sigma} u^{\beta}_{,\tau} \, . \tag{7.9}$$

Finally, expanding equation (7.9) we obtain required relations

$$u^{\alpha}_{,s} u^{\beta}_{,s} = a^{\alpha\beta} - \nu^{\alpha}\nu^{\beta} \, . \tag{7.10}$$

Appendix 2. Deformation tensor in a surface [16]

The equations for convective derivative $\frac{d_c}{dt}A_{\alpha\beta}$ of any second order surface tensor $A_{\alpha\beta}$ in the spatial (or fixed) coordinate system are given by [16]

$$\frac{d_c}{dt}A_{\alpha\beta} = \frac{\partial A_{\alpha\beta}}{\partial t} + A_{\alpha\beta;\delta}v^{\delta} + A_{\alpha\gamma}v^{\gamma}_{;\beta} + A_{\tau\beta}v^{\tau}_{;\alpha} \tag{7.11}$$

where $\underline{v}$ denotes particle velocity in the surface.

Equation (2.1) gives

$$ds^2 = a_{\alpha\beta}\,du^{\alpha}du^{\beta} \tag{7.12}$$

in the spatial coordinate system u^{α} but in the convective coordinate system ξ^{α}, we have

$$ds^2 = \gamma_{\alpha\beta}\,d\xi^{\alpha}d\xi^{\beta} \tag{7.13}$$

where

$$\gamma_{\alpha\beta} = \frac{\partial u^{\delta}}{\partial \xi^{\alpha}}\frac{\partial u^{\sigma}}{\partial \xi^{\beta}}a_{\delta\sigma}. \tag{7.14}$$

If ds_0 denotes the linear element at time $t = t_0$, we can write

$$ds^2 - ds_0^2 = \{\gamma_{\alpha\beta}(\xi^\delta, t) - \gamma_{\alpha\beta}(\xi^\delta, t_0)\}\, d\xi^\alpha d\xi^\beta$$

$$= 2\eta_{\alpha\beta}(\xi^\delta, t, t_0)\, d\xi^\alpha d\xi^\beta \tag{7.15}$$

where

$$\eta_{\alpha\beta} = \frac{1}{2}\{\gamma_{\alpha\beta}(\xi^\delta, t) - \gamma_{\alpha\beta}(\xi^\delta, t_0)\}\ . \tag{7.16}$$

Derivatives of $\eta_{\alpha\beta}$ with respect to time is the rate of deformation tensor given by

$$\varepsilon_{\alpha\beta}(\xi^\delta, t) = \frac{d}{dt}\eta_{\alpha\beta}(\xi^\delta, t) = \frac{1}{2}\frac{d}{dt}\gamma_{\alpha\beta}(\xi^\delta, t) \tag{7.17}$$

in the convective coordinates.

From equation (7.11) and the transformation equation (7.14) the rate of deformation tensor in the spatial coordinate system is given by

$$\epsilon_{\alpha\beta} = \frac{1}{2}\frac{dc}{dt}a_{\alpha\beta} = \frac{1}{2}(v_{\alpha;\beta} + v_{\beta;\alpha}) \tag{7.18}$$

since $a_{\alpha\beta;\gamma}$ is always zero. The deformation w produced in the medium in a small time interval Δt has components given by $v_\alpha \Delta t$ to a first approximation so that we have

$$e_{\alpha\beta} = \frac{1}{2}(w_{\alpha;\beta} + w_{\beta;\alpha}) \tag{7.19}$$

for the strain tensor $e_{\alpha\beta}$.

Appendix 3. Derivation of absolute time derivative of unit geodesic normal and unit tangent vector.

From equation (2.25), we have

$$\frac{\delta u^\alpha}{\delta t} = G\nu^\alpha. \tag{7.20}$$

Since $u^\alpha_{;s} = u^\alpha_{,s}$, we have the identity, by definition,

$$\frac{\delta u^\alpha}{\delta t} = \frac{D u^\alpha}{D t} = u^\alpha_{,s}\frac{\delta s}{\delta t} + \frac{\partial u^\alpha}{\partial t}. \tag{7.21}$$

It follows from equation (7.20) and (7.21) that

$$u^\alpha_{,s}\frac{\delta s}{\delta t} + \frac{\partial u^\alpha}{\partial t} = G\nu^\alpha \tag{7.22}$$

Multiplying (7.22) by $a_{\alpha\beta}u^\alpha_{,s}$ it follows

$$\frac{\delta s}{\delta t} + a_{\alpha\beta}u^\alpha_{,s}\frac{\partial u^\beta}{\partial t} = 0. \tag{7.23}$$

Consider the following identity

$$\frac{D u^\alpha_{,s}}{D t} = \frac{D u^\alpha_{,s}}{D s}\frac{\delta s}{\delta t} + \frac{\partial^2 u^\alpha}{\partial s \partial t}. \tag{7.24}$$

Multiplying equation (7.24) by ν_α and using equations (2.13) and (2.14), we have

$$\nu_\alpha \frac{D u^\alpha_{,s}}{D t} = K_g\left(\frac{\delta s}{\delta t} + a_{\alpha\beta}u^\alpha_{,s}\frac{\partial u^\beta}{\partial t}\right) + G_{,s} \tag{7.25}$$

If we introduce equation (7.23), we have

$$\nu_\alpha \frac{D u^\alpha_{,s}}{Dt} = \sigma_{,s} \,. \tag{7.26}$$

Differentiating equation (2.11) and (2.12), and making use of equation (7.26), we obtain

$$\nu^\alpha \frac{D \nu_\alpha}{Dt} = 0 \,, \tag{7.27}$$

$$u^\alpha_{,s} \frac{D \nu_\alpha}{Dt} = - \sigma_{,s} \,. \tag{7.28}$$

Multiplying equation (7.28) by $u^\beta_{,s}$ and introducing equation (7.10) and (7.27), we finally have

$$\frac{D \nu^\beta}{Dt} = - \sigma_{,s} u^\beta_{,s} \tag{7.29}$$

as the absolute time derivatives of unit geodesic normal vector.

Analogous to equation (7.10) we have

$$u_{\alpha,s} u_{\beta,s} = a_{\alpha\beta} - \nu_\alpha \nu_\beta \tag{7.30}$$

Differentiating equation (7.30), it follows that

$$\frac{D}{Dt}(u_{\alpha,s})u_{\beta,s} = -u_{\alpha,s}\frac{D}{Dt}(u_{\beta,s}) - \nu_\alpha\frac{D\nu_\beta}{Dt} - \nu_\beta\frac{D\nu_\alpha}{Dt}. \tag{7.31}$$

Multiply equation (7.31) by $u^\beta_{,s}$ and introducing equation (7.29), we have

$$\frac{D}{Dt}(u_{\alpha,s}) = G_{,s}\nu_\alpha - u_{\alpha,s}u_{\beta,s}\left(\frac{Du^\beta_{,s}}{Dt}\right). \tag{7.32}$$

Substituting equations (7.24) and (7.26) in equation (7.32) as the expression

$$\frac{D}{Dt}(u_{\alpha,s}) = \nu_\alpha G_{,s} - u_{\alpha,s}u_{\beta,s}\left(\frac{\partial^2 u^\beta}{\partial s\partial t}\right) \tag{7.33}$$

as the required absolute time derivatives of unit tangent vector along a wave curve C.

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