

A STUDY OF WAVE PROPAGATION

ON

ELASTIC MEMBRANES

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ABSTRACT

WAVE PROPAGATION IN ELASTIC SURFACES IS TREATED USING THE CONCEPT OF A WAVE CURVE IN THE SURFACE. THE BASIC GEOMETRIC AND KINEMATIC DISCONTINUITY RELATIONS ACROSS WAVE CURVES ARE DERIVED. EXPRESSIONS FOR THE SPEED OF PROPAGATION OF EQUIVOLUMINAL AND IRROTATIONAL WAVES ON THE SURFACE ARE GIVEN ALONG WITH A DIFFERENTIAL EQUATION GOVERNING WAVE STRENGTH DURING PROPAGATION. THE WAVE CURVES ARE SHOWN TO FORM A SYSTEM OF GEODESIC PARALLELS ON THE SURFACE AS THEY PROPAGATE. THE GENERAL SOLUTION AND SOME EXAMPLES OF THE STRENGTH EQUATION ARE GIVEN.

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To Hea Ryun

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NOTATIONS

$a_{\alpha\beta}, a^{\alpha\beta}$	covariant and contravariant components of the surface metric tensor
$A_{\alpha\beta}$	a surface tensor
$C(t)$ or C	a wave curve on S
κ	principal curvature of a curve in three dimensional space
κ_g	geodesic curvature of a curve on S
K	Gaussian curvature of S
G	normal velocity of C
p	a scalar on S
$R_{\alpha\beta\gamma\delta}$	components of the Riemannian curvature tensor
s	arc length of C
S	an arbitrary surface
t	time
\underline{t}	a surface vector or a first order surface tensor
$t_\alpha, t^\alpha,$ (or $\bar{t}_\alpha, \bar{t}^\alpha$)	components of covariant and contravariant surface vectors in u^α (or \bar{u}^α) system
$\underline{t} = \underline{t}_{\sigma \dots \tau}^{\alpha \dots \gamma}$	an absolute tensor field on S
u^α, \bar{u}^α	curvilinear coordinate system on S
\underline{v}	velocity vector on S
v_α, v^α	covariant and contravariant components of velocity vector on S
\underline{v}_n	normal velocity of a particle on S

\underline{w}	displacement vector on S
w_{α}, w^{α}	covariant and contravariant components of displacement vector on S
W	wave strength
$\frac{D}{Ds}$	operator for absolute differentiation along C
$\frac{D}{Dt}$	operator for absolute time differentiation
$\frac{\delta}{\delta t}$	operator for δ -time differentiation
$\frac{dc}{dt}$	operator for convective differentiation
$\alpha, \beta, \gamma, \dots$	coordinate indices ranging 1,2
δ_{α}^{β}	the Kronecker deltas
$E_{\alpha\beta}$	covariant components of strain tensor
$\dot{E}_{\alpha\beta}$	covariant components of rate of strain tensor
$\tilde{E}_{\alpha\beta}$	covariant components of rate of strain tensor in convective coordinate system
γ	surface density
$\gamma_{\alpha\beta}$	covariant components of surface metric tensor in convective coordinate system
$\tau_{\alpha\beta}$	covariant components of stress tensor on S
Ω	mean curvature of a surface S
σ	distance along a geodesic normal to C
λ, μ	Lame constants

v_α, v^α

covariant and contravariant components of geodesic
unit normal vector to C

 $\Gamma_{\alpha\beta\gamma}, \Gamma_{\alpha\beta}^{\gamma}$

Christoffel symbols of the first and second kind

$$\left. \begin{array}{l} B, \xi, \bar{\xi}, \omega \\ \kappa_\alpha, \omega_\alpha, \hat{\omega}_\alpha \end{array} \right\}$$

scalar, vector or tensor quantities defined over C on S

$$\left[\begin{array}{l} \kappa_{\alpha\beta} \end{array} \right]$$

denotes jump or discontinuity in the quantities enclosed

1. Introduction

Considerable attention has been given recently to the problem of wave propagation in elastic sheets, thin plates, and shells. [1, 2,3,4,5] * Jahsman [6] derived an expression for the velocity of circular waves in elastic sheets and plates and also showed that wave strength diminished as $r^{-\frac{1}{2}}$ where r is the radius of a circular wave. The problem of wave propagation in cylindrical and spherical shells has also been treated by several authors. [7,8,9]

Thomas [10,11] employed Hadamard's idea of singular surface as a wave surface in three dimensional space. He established basic discontinuity or so called compatibility conditions across a wave surface. Applications of the compatibility conditions were made to deduce speed and directions of wave propagation as well as decay of wave strength equations in elastic, plastic and crystalline solids.

In this thesis, analogous to the wave surface in three dimensional space, a singular curve is introduced as a wave curve to develop a theory of wave propagation in a surface. Closely following Thomas's method, basic compatibility conditions are formulated across a wave curve on a surface. These conditions are used to derive wave strength and speed equations of wave curves in homogeneous isotropic surfaces.

* Numbers in square brackets refer to the bibliography.

Specifically, in chapter 2 is given a brief review of the intrinsic geometry of a surface along with the definitions of surface deformation tensor, wave curve, absolute derivative and absolute time derivative of a surface tensor. Also the basic compatibility conditions are derived.

In chapter 3, the first and second order geometrical and kinematical compatibility conditions across a wave curve are deduced in appropriate form for a scalar and a surface vector.

In chapter 4, using Reynold's transport theorem and the principles of mass and momentum conservation, the basic field equations are derived for an elastic surface. The dynamical discontinuity relations associated with these equations are derived.

In chapter 5, assuming that a wave curve is moving into an elastic medium at rest, wave speeds as well as wave strength equation of equivoluminal and irrotational wave during propagation are formulated.

In chapter 6, applications of wave strength equation are made for several surfaces. It is shown that wave strength equation of wave curves in a plane agree with the corresponding results of Thomas and Jahsman.

Appendices are given as chapter 7.

2. Surface Geometry and Kinematics

In this chapter, a brief review of the intrinsic geometry of surfaces is discussed. Absolute derivative, δ -time derivative as well as absolute time derivative of an absolute tensor field are defined. Basic relations for the compatibility conditions of discontinuities across a curve in the surface are defined.

Let S be a surface defined by the first fundamental form

$$ds^2 = a_{\alpha\beta} du^\alpha du^\beta \quad (\alpha = 1, 2) \quad (2.1)$$

where $a_{\alpha\beta}$ are the covariant components of the surface metric tensor, s is arc length and u^α are the curvilinear coordinates of the surface.

A moving curve $C(t)$ on S is given by

$$u^\alpha = u^\alpha(s, t) \quad (2.2)$$

where t is time. Functions u^α are assumed to be as smooth as desired.

Let

$$\bar{u}^\alpha = \bar{u}^\alpha(u^\beta) \quad (2.3)$$

*Greek indices have the range 1,2 throughout this discussion.

be the equation of an arbitrary differentiable curvilinear coordinate transformation on the surface S. It is assumed that \bar{u}^α are single valued functions whose functional determinant $\left| \frac{\partial \bar{u}^\alpha}{\partial u^\beta} \right|$ is not equal to zero.

A surface vector or first order tensor t_α or t^α in u^α and \bar{t}_α or \bar{t}^α in \bar{u}^α is related by

$$\begin{aligned}
 t_\alpha &= \bar{t}_\beta \frac{\partial \bar{u}^\beta}{\partial u^\alpha}, & \bar{t}_\alpha &= t_\beta \frac{\partial u^\beta}{\partial \bar{u}^\alpha} \\
 t^\alpha &= \bar{t}^\beta \frac{\partial u^\alpha}{\partial \bar{u}^\beta}, & \bar{t}^\alpha &= t^\beta \frac{\partial \bar{u}^\alpha}{\partial u^\beta}
 \end{aligned}
 \tag{2.4}$$

The covariant derivative of a surface vector with components t_α is defined by

$$t_{\alpha;\beta} \equiv t_{\alpha,\beta} - t_\gamma \Gamma_{\alpha\beta}^\gamma
 \tag{2.5}$$

and absolute derivative of the vector along a curve C as

$$\frac{Dt_\alpha}{Ds} \equiv t_{\alpha;\beta} \frac{du^\beta}{ds} = (t_{\alpha,\beta} - t_\gamma \Gamma_{\alpha\beta}^\gamma) \frac{du^\beta}{ds} = t_{\alpha,s} - t_\gamma \Gamma_{\alpha\beta}^\gamma \frac{du^\beta}{ds}
 \tag{2.6}$$

where the semicolon denotes covariant differentiation and the comma ordinary (partial) differentiation with respect to the indicated index. Analogously, definitions for the higher order tensors and their derivatives can be made. These notations and the summation convention for the repeated indices have been adopted throughout this discussion. The quantities $\Gamma_{\alpha\beta}^\gamma$ and $\bar{\Gamma}_{\alpha\beta}^\gamma$ are the Christoffel

symbols of the first and second kind respectively, defined by

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2} \left(\frac{\partial a_{\alpha\gamma}}{\partial u^\beta} - \frac{\partial a_{\alpha\beta}}{\partial u^\gamma} + \frac{\partial a_{\beta\gamma}}{\partial u^\alpha} \right), \quad \Gamma_{\alpha\beta}^\gamma = a^{\delta\gamma} \Gamma_{\alpha\beta\delta} \quad (2.7)$$

Similarly, we have

$$\frac{Dt^\alpha}{Ds} \equiv t_{j\beta}^\alpha \frac{du^\beta}{ds} = t_{j\beta}^\alpha + t_{r\beta}^\alpha \frac{du^\beta}{ds} \quad (2.8)$$

for the contravariant components of a surface vector \underline{t} . If $\frac{Dt^\alpha}{Ds}$ or $\frac{Dt_\alpha}{Ds}$ vanish, the vector \underline{t} is said to undergo a parallel displacement along a curve C.

Furthermore, we have the following relations

$$a_{\alpha\beta} a^{\beta\gamma} = \delta_\alpha^\gamma, \quad \delta_\alpha^\gamma = \begin{cases} 1 & ; \alpha = \gamma \\ 0 & ; \alpha \neq \gamma \end{cases} \quad (2.9)$$

$$a_{\alpha\beta} u_{,s}^\alpha u_{,s}^\beta = 1, \quad (2.10)$$

$$a_{\alpha\beta} \nu^\alpha \nu^\beta = 1, \quad a^{\alpha\beta} \nu_{,\alpha} \nu_{,\beta} = 1, \quad (2.11)$$

$$a_{\alpha\beta} u_{,s}^\alpha \nu^\beta = 0, \quad a^{\alpha\beta} u_{,\alpha s} \nu_\beta = 0, \quad (2.12)$$

in which $a^{\alpha\beta}$ are the contravariant components of the surface metric

tensor, δ_{α}^{γ} are the Kronecker deltas and $\nu^{\alpha}, \nu_{\alpha}$ being the contravariant and covariant components of geodesic unit normal vector, respectively, to a curve C at the point in question. Equation (2.10) shows us that $u_{,s}^{\alpha} \equiv \frac{du^{\alpha}}{ds}$ are the components of the unit tangent vector along the curve C on the surface S.

We also have [12,14]

$$\frac{Du^{\alpha}}{Ds} = \frac{d^2u^{\alpha}}{ds^2} + \Gamma_{\gamma\beta}^{\alpha} \frac{du^{\beta}}{ds} \frac{du^{\gamma}}{ds} = K_g \nu^{\alpha}, \quad (2.13)$$

$$\frac{D\nu^{\alpha}}{Ds} = \nu_{i\beta}^{\alpha} \frac{du^{\beta}}{ds} = -K_g u_{,s}^{\alpha} \quad (2.14)$$

for the absolute derivatives of $u_{,s}^{\alpha}$ and ν^{α} where K_g^* is the geodesic curvature at the point in question of C. Geodesic curvature

K_g of C can be interpreted as the arc rate of change of the angle which a vector makes with the tangent to a curve C as the vector undergoes a parallel displacement along C. The curves $u^{\alpha} = u^{\alpha}(s)$, which are solutions of the equations

$$\frac{Du^{\alpha}}{Ds} = \frac{d^2u^{\alpha}}{ds^2} + \Gamma_{\gamma\beta}^{\alpha} \frac{du^{\beta}}{ds} \frac{du^{\gamma}}{ds} = 0 \quad (2.15)$$

* Geodesic curvature K_g is the same as the intrinsic curvature K of a curve on the surface. Viewed from three dimensional Euclidean space the geodesic curvature K_g is the projection of the principal curvature of the curve in space onto the tangent plane to the surface at the point in question.

are called geodesics. The tangent vector to geodesics are in parallel propagation, hence $Kg = 0$ along geodesics. We set up geodesics at each point to the curve C and perpendicular to C. When equal length are measured from C along these geodesics, the loci of their end points are orthogonal trajectories of the geodesics and these orthogonal trajectories are called geodesic parallels to the curve C. [14]

We are interested in situations where the surface S consists of particles which can undergo motion. If \underline{w} and \underline{v} denote the infinitesimal surface displacement and surface velocity vectors respectively, then the infinitesimal strain $e_{\alpha\beta}$ and rate of strain $\epsilon_{\alpha\beta}$ are given by (see appendix 2)

$$e_{\alpha\beta} = \frac{1}{2} (w_{\alpha;\beta} + w_{\beta;\alpha}) \quad (2.16)$$

$$\epsilon_{\alpha\beta} = \frac{1}{2} (v_{\alpha;\beta} + v_{\beta;\alpha}) \quad (2.17)$$

Let C(t) divide S into two regions S_1 and S_2 . (Fig.1) Define $\underline{t}(u^\alpha, t)$ to be a continuously differentiable absolute tensor field in regions S_1 and S_2 obtained by excluding points on the curve C(t) from the immediate neighborhood of this curve in S. Assume that the tensor field $\underline{t}(u^\alpha, t)$ and its partial derivatives to the order of consideration have finite limiting values which are continuous functions of s and t as we approach the curve C from both regions S_1 and S_2 . [10]

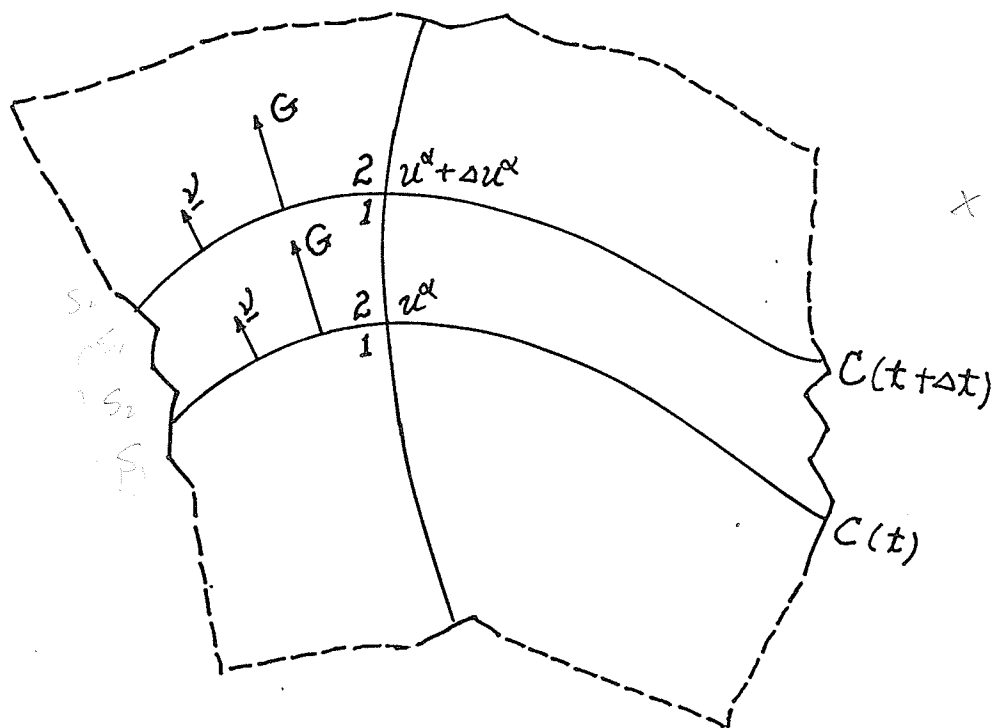


Fig. 1. Wave curve $C(t)$, region 1, 2, geodesic
unit normal \underline{n} , and normal speed G on S

By Hadamard's lemma, we have [17]

$$\frac{D[\underline{t}]}{Ds} = [\underline{t}_{; \alpha}] u_{; \alpha}^{\alpha} \quad (2.18)$$

where

$$[\underline{t}] = \underline{t}_2 - \underline{t}_1, \quad [\underline{t}_{; \alpha}] = \underline{t}_{2; \alpha} - \underline{t}_{1; \alpha} \quad (2.19)$$

over the curve $C(t)$, $\underline{t}_{; \alpha}$ and $\underline{t}_{; \alpha}$ denoting the tensor in S_1 and S_2 .

A curve which carries discontinuities will be called a wave curve.

Multiplying equation (2.18) by $u_{; \alpha}^{\beta}$ and $a_{\beta \gamma}$ and making use of equation (7.10, see appendix 1) we obtain

$$[\underline{t}_{; \alpha}] = B_{\alpha}^{\gamma} + \frac{D[\underline{t}]}{Ds} u_{; \alpha}^{\alpha} \quad (2.20)$$

where

$$B_{\alpha} = [\underline{t}_{; \alpha}] \nu^{\alpha} \quad (2.21)$$

as the basic equations for the geometrical compatibility conditions of discontinuities across the wave curve C on S .

Define δ -time derivative as the rate of change of \underline{t} along geodesic normal to a wave curve C in the following manner.

$$\frac{\delta \underline{t}}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{\Delta \underline{t}}{\Delta t} = \lim_{\delta t \rightarrow 0} \left(\frac{\Delta \underline{t}}{\Delta u^{\alpha}} \frac{\Delta u^{\alpha}}{\Delta t} \right) + \frac{\partial \underline{t}}{\partial t} \quad (2.22)$$

Thus, speed of normal velocity along the geodesic normal of a wave curve C can be defined by

$$G = \lim_{\Delta\sigma \rightarrow 0} \frac{\Delta\sigma}{\Delta t} \quad (2.23)$$

where $\Delta\sigma$ is small distance along the normal. It also follows that

$$\frac{\delta u^\alpha}{\delta t} = \lim_{\Delta\sigma \rightarrow 0} \frac{\Delta u^\alpha}{\Delta t} = \lim_{\Delta\sigma \rightarrow 0} \left(\frac{\Delta\sigma}{\Delta t} \frac{\Delta u^\alpha}{\Delta\sigma} \right) = G v^\alpha \quad (2.24)$$

and we can show that [11]

$$\left[\frac{\delta t}{\delta t} \right] = \frac{\delta [t]}{\delta t} \quad (2.25)$$

where

$$\left[\frac{\delta t}{\delta t} \right] = \frac{\delta t_2}{\delta t} - \frac{\delta t_1}{\delta t} \quad (2.26)$$

For an absolute tensor field, by definition, we have

$$\begin{aligned} \frac{\delta}{\delta t} (t) &= \frac{\delta}{\delta t} (t^\alpha \dots r) = \frac{\partial t}{\partial u^\alpha} G v^\alpha + \frac{\partial t}{\partial t} \\ &= (t_{;\alpha}) G v^\alpha - \left\{ (t^\alpha \dots r) \Gamma_{\sigma\tau}^\alpha + \dots + (t^\alpha \dots s) \Gamma_{\mu\epsilon}^\delta \right\} G v^\epsilon \\ &\quad + \left\{ (t^\alpha \dots r) \Gamma_{\sigma r}^\mu + \dots + (t^\alpha \dots s) \Gamma_{\tau r}^\mu \right\} G v^r \\ &\quad + \frac{\partial t}{\partial t} \end{aligned} \quad (2.27)$$

for the δ -time derivative of an absolute surface tensor field t .

We now can define absolute time derivative by

$$\frac{D\tilde{t}}{Dt} = \frac{\delta t}{\delta t} - \left\{ (t^{\alpha \dots \delta})_{\mu \dots \tau} \Gamma_{\sigma}^{\mu} + \dots + (t^{\alpha \dots \delta})_{\sigma \dots \mu} \Gamma_{\tau}^{\mu} \right\} G^{\nu \tau} \quad (2.28)$$

$$+ \left\{ (t^{\delta \dots \delta})_{\sigma \dots \tau} \Gamma_{\delta}^{\alpha} + \dots + (t^{\alpha \dots \delta})_{\sigma \dots \tau} \Gamma_{\mu \epsilon}^{\delta} \right\} G^{\nu \epsilon}$$

$$= \frac{\partial \tilde{t}}{\partial t} + \tilde{t}_{;\lambda} G^{\nu \lambda}$$

which will behave as a tensor by an arbitrary coordinate transformation

$u^{\alpha} \rightarrow \bar{u}^{\alpha}$. As a basic kinematical compatibility condition of discontinuities of a wave curve C for an absolute surface tensor

field \tilde{t} , we obtain the following equation from relations (2.28) and (2.27)

$$\left[\frac{\partial \tilde{t}}{\partial t} \right] = -[\tilde{t}_{;\lambda}] G^{\nu \lambda} + \frac{D[\tilde{t}]}{Dt} \quad (2.29)$$

in which the last term follows from equation (2.25).

3. First and Second Order Geometrical and Kinematical Compatibility Conditions

This chapter, closely following Thomas's idea for the wave surface in three dimensional space [11], presents detailed derivations of the first and second order geometrical and kinematical compatibility conditions across a wave curve C for a scalar p and a surface vector \underline{t} . The compatibility conditions which will follow are of the appropriate form in the sense that the various terms in the conditions will behave as scalars, vectors or tensors under arbitrary coordinate transformation $u^\alpha \rightarrow \bar{u}^\alpha$.

Putting p , any scalar quantity in place of \underline{t} in equation (2.20) we obtain the first order geometrical compatibility conditions

$$[P_{,r}] = \zeta \nu_r + [P]_{,s} u_{r,s} \quad (3.1)$$

since $P_{;r} = P_{,r}$ for a scalar quantity p where

$$\zeta = [P_{,r}] \nu^r \quad (3.2)$$

is a function defined over the curve $C(t)$. The terms in equations (3.1) are of vector character and these equations are the appropriate form of the geometrical compatibility conditions of the first order for a scalar p .

* [] in equations denote the jump of quantities in the brackets across a wave curve as in the previous discussion.

When we set $\underline{\tilde{t}} = \underline{t}$, a surface vector, in equation (2.20) we obtain the relations

$$[t_{\alpha;r}] = \lambda_{\alpha} \nu_r + \frac{D[t_{\alpha}]}{Ds} u_{r,s} \quad (3.3)$$

where

$$\lambda_{\alpha} = [t_{\alpha;r}] \nu^r \quad (3.4)$$

are components of a surface vector defined on C . Equations (3.3) are the required form of geometrical compatibility conditions of the first order for a surface vector \underline{t} since all the terms are of tensorial character under the coordinate transformation $u^{\alpha} \rightarrow \bar{u}^{\alpha}$.

If we put $\underline{\tilde{t}} = p$ in equation (2.29), appropriate form of kinematical compatibility conditions of the first order for a scalar p is obtained in the form

$$\left[\frac{\partial p}{\partial t} \right] = -\zeta G + \frac{\delta[p]}{\delta t} \quad (3.5)$$

where ζ is defined over the curve C as in equations (3.1).

Again, putting $\underline{\tilde{t}} = t_{\alpha}$ in equation (2.29), we have

$$\left[\frac{\partial t_{\alpha}}{\partial t} \right] = -\lambda_{\alpha} G + \frac{D[t_{\alpha}]}{Dt} \quad (3.6)$$

in which

$$\frac{D[t_\alpha]}{Dt} = \frac{\delta[t_\alpha]}{\delta t} - G[t_\gamma] \Gamma_{\alpha\gamma}^\sigma \nu^\gamma \quad (3.7)$$

as the appropriate form of the kinematical compatibility conditions of the first order for a surface vector \underline{t} . We consider now the derivation of the second order compatibility conditions across a moving curve C.

Putting $\underline{t} = p_{,\alpha}$ in equation (2.20), we can obtain the relations

$$[P_{;\alpha\gamma}] = \zeta_\alpha \nu_\gamma + \frac{D[P_{;\alpha}]}{Ds} u_{\gamma s} \quad (3.8)$$

where

$$\zeta_\alpha = [P_{;\alpha\beta}] \nu^\beta \quad (3.9)$$

are values defined over the curve C. Since the left members of equations (3.8) are symmetric in α and γ , we can write

$$\zeta_\alpha \nu_\gamma + \frac{D[P_{;\alpha}]}{Ds} u_{\gamma s} = \zeta_\gamma \nu_\alpha + \frac{D[P_{;\gamma}]}{Ds} u_{\alpha s} \quad (3.10)$$

Multiplying equations (3.10) by ν^γ and sum on the repeated index

γ , it follows that

$$\zeta_\alpha = \tilde{\zeta}_\alpha + \frac{D[P_{;r}]}{Ds} \nu^r u_{\alpha s} ; \quad \tilde{\zeta} = \zeta_r \nu^r \quad (3.11)$$

Substituting equations (3.1) and (2.14) successively in the identity

$$\frac{D}{Ds} [P_{,r}] \nu^r = \frac{D}{Ds} ([P_{,r}] \nu^r) - [P_{,r}] \frac{D \nu^r}{Ds} \quad (3.12)$$

we obtain

$$\frac{D}{Ds} [P_{,r}] \nu^r = \zeta_{,s} + [P]_{,s} k_g \quad (3.13)$$

since $\nu_r \frac{D \nu^r}{Ds} = 0$ by differentiating equation (2.11). Equations (3.10) become of the form

$$\zeta_\alpha = \tilde{\zeta} \nu_\alpha + \zeta_{,s} u_{\alpha,s} + [P]_{,s} u_{\alpha,s} k_g \quad (3.14)$$

upon introducing equation (3.13). Substituting equations (3.1) and (3.14) in (3.8) it finally follows that

$$[P_{,\alpha\beta}] = \tilde{\zeta} \nu_\alpha \nu_\beta + \left(\frac{D}{Ds} [P]_{,s} - \zeta k_g \right) u_{\alpha,s} u_{\beta,s} \quad (3.15)$$

$$+ (\zeta_{,s} + [P]_{,s} k_g) (\nu_\beta u_{\alpha,s} + \nu_\alpha u_{\beta,s})$$

as the appropriate form of the geometrical compatibility conditions of the second order for a scalar p .

To deduce the second order geometrical compatibility conditions

for a surface vector \underline{t} , put $\underline{t} = t_{\alpha;\beta}$ in equations (2.20) to obtain

$$[t_{\alpha;\beta\gamma}] = \lambda_{\alpha\beta}\nu_{\gamma} + \frac{D[t_{\alpha;\beta}]}{Ds} u_{\gamma,s} \quad (3.16)$$

in which

$$\lambda_{\alpha\beta} = [t_{\alpha;\beta\gamma}]\nu^{\gamma} \quad (3.17)$$

are quantities defined over the curve C . To find the relations between $t_{\alpha;\beta\gamma}$ and $t_{\alpha;\gamma\beta}$, we can introduce the Ricci identities [14]

$$[t_{\alpha;\beta\gamma}] = [t_{\alpha;\gamma\beta}] + [t_{\gamma}] R_{\alpha\beta}^{\gamma} \quad (3.18)$$

where $R_{\alpha\beta}^{\gamma}$ are components of the Riemannian curvature tensor.

From equations (3.16) and (3.18) we have

$$\lambda_{\alpha\beta}\nu_{\gamma} + \frac{D}{Ds} [t_{\alpha;\beta\gamma}] u_{\gamma,s} = \lambda_{\alpha\gamma}\nu_{\beta} + \frac{D}{Ds} [t_{\alpha;\gamma\beta}] u_{\beta,s} + [t_{\gamma}] R_{\alpha\beta}^{\gamma} \quad (3.19)$$

Following the same procedure as for the derivation of equations (3.15) the geometrical compatibility conditions of the second order for a surface vector \underline{t} will be given by

$$\begin{aligned}
[t_{\alpha;\beta\gamma}] &= \tilde{\lambda}_{\alpha} \nu_{\beta} \nu_{\gamma} + \left(\frac{D^2}{DS^2} [t_{\alpha}] - \lambda_{\alpha} K_{\gamma} \right) u_{\beta,s} u_{\gamma,s} \\
&+ \left\{ \frac{D}{Ds} (\lambda_{\alpha}) + \frac{D}{Ds} [t_{\alpha}] K_{\gamma} \right\} (u_{\beta,s} \nu_{\gamma} + u_{\gamma,s} \nu_{\beta}) \\
&+ [t_{\gamma}] R_{\alpha\beta\gamma}^{\eta}
\end{aligned} \tag{3.20}$$

where

$$\tilde{\lambda}_{\alpha} = [t_{\alpha;\beta\gamma}] \nu^{\beta} \nu^{\gamma} \tag{3.21}$$

Consider the following identities

$$\frac{D}{Dt} [P_{,\alpha}] = [P_{;\alpha\beta}] G \nu^{\beta} + \left[\frac{\partial^2 P}{\partial u^{\alpha} \partial t} \right], \tag{3.22}$$

$$\frac{D}{Dt} [P_{,\alpha}] = \frac{D}{Dt} (\zeta \nu_{\alpha} + [P]_{,s} u_{\alpha,s}) \tag{3.23}$$

Equating (3.22) and (3.23) and then carrying out the time differentiation in equations (3.23) it follows that

$$\begin{aligned}
\left[\frac{\partial^2 P}{\partial u^{\alpha} \partial t} \right] &= -[P_{;\alpha\beta}] G \nu^{\beta} + \frac{D}{Dt} (\zeta) \nu_{\alpha} + \zeta \frac{D \nu_{\alpha}}{Dt} \\
&+ \frac{D}{Dt} (u_{\alpha,s}) [P]_{,s} + \left\{ \frac{D}{Ds} ([P]_{,s}) \frac{\delta s}{\delta t} + \frac{\partial^2 [P]}{\partial s \partial t} \right\} u_{\alpha,s}
\end{aligned} \tag{3.24}$$

over a curve C. For the time derivative of geodesic unit normal vector

we have (see appendix 3)

$$\frac{Dk}{Dt} = -G_{,s} u_{\alpha,s} \quad (3.25)$$

and also for the variation of unit tangent vector

$$\frac{D}{Dt}(u_{\alpha,s}) = \nu_{\alpha} G_{,s} - u_{\alpha,s} u_{\gamma,s} \frac{\partial^2 u^{\gamma}}{\partial s \partial t} \quad (3.26)$$

When equations (7.23, see appendix 3), (3.15), (3.25), (3.26) are substituted in the right members of equations (3.24) successively, it follows that

$$\begin{aligned} \left[\frac{\partial^2 p}{\partial u^{\alpha} \partial t} \right] &= -(\zeta G - \frac{\delta \zeta}{\delta t} - G_{,s} [P]_{,s}) \nu_{\alpha} + \{ (\zeta G)_{,s} + G \nu_{\gamma} [P]_{,s} \\ &+ \frac{D}{Ds} [P]_{,s} u_{\beta s} \frac{\partial u^{\beta}}{\partial t} + [P]_{,s} u_{\gamma s} \frac{\partial^2 u^{\gamma}}{\partial s \partial t} - \frac{\partial^2 [P]}{\partial s \partial t} \} u_{\alpha,s} \end{aligned} \quad (3.27)$$

as a part of kinematical compatibility conditions of the second order for a scalar p .

We can derive the appropriate form of the kinematical compatibility conditions of the second order for a vector \underline{t} in the form

$$\left[\frac{\partial t_{\alpha\beta}}{\partial t} \right] = - \left\{ \lambda_{\alpha} G - \frac{D\lambda_{\alpha}}{Dt} - G_{,s} \frac{D}{Ds} [t_{\alpha}] \right\} \nu^{\beta} - \left\{ \frac{D}{Ds} (\lambda_{\alpha} G) \right. \quad (3.28)$$

$$+ K_{\gamma} G \frac{D}{Ds} [t_{\alpha}] + \frac{D^2}{Ds^2} [t_{\alpha}] u_{\beta s} \frac{\partial u^{\gamma}}{\partial t} + \frac{D}{Ds} [t_{\alpha}] u_{\beta s} \frac{\partial^2 u^{\gamma}}{\partial s \partial t} - \frac{D}{Ds} \left(\frac{\partial [t_{\alpha}]}{\partial t} \right) \left. \right\} u_{\beta s} + [t_{\gamma}] R_{\alpha\beta\gamma}^{\eta} G \nu^{\gamma}$$

where

$$\frac{D}{Ds} \left(\frac{\partial [t_{\alpha}]}{\partial t} \right) = \frac{\partial^2 [t_{\alpha}]}{\partial t \partial s} - \frac{\partial [t_{\alpha}]}{\partial t} \Gamma_{\alpha\beta}^{\gamma} u_{,s}^{\beta} \quad (3.29)$$

when we follow the same procedures used in the derivation of equations (3.27).

To find the suitable form for $\left[\frac{\partial^2 P}{\partial t^2} \right]$, consider the identities

$$\frac{D}{Dt} \left[\frac{\partial P}{\partial t} \right] = \left[\frac{\partial^2 P}{\partial u^{\alpha} \partial t} \right] G \nu^{\alpha} + \left[\frac{\partial^2 P}{\partial t^2} \right] \quad (3.30)$$

$$\frac{D}{Dt} \left[\frac{\partial P}{\partial t} \right] = \frac{D}{Dt} \left(-\zeta G + \frac{\delta P}{\delta t} \right) \quad (3.31)$$

From both of the right members of equations (3.30) and (3.31), it follows that

$$\left[\frac{\partial^2 p}{\partial t^2} \right] = \tilde{\zeta} G^2 - 2G \frac{\delta \zeta}{\delta t} - \zeta \frac{\delta G}{\delta t} - G G_{,s} [P]_{,s} + \frac{\delta^2 p}{\delta t^2} \quad (3.32)$$

by substituting equations (3.27), as the required form of the remaining kinematical compatibility conditions for a scalar p .

Finally, consider the identities

$$\frac{D}{Dt} \left[\frac{\partial t_\alpha}{\partial t} \right] = \left[\frac{\partial t_{\alpha,\beta}}{\partial t} \right] G^{\nu\beta} + \left[\frac{\partial^2 t_\alpha}{\partial t^2} \right] \quad (3.33)$$

$$\frac{D}{Dt} \left[\frac{\partial t_\alpha}{\partial t} \right] = \frac{D}{Dt} \left\{ -\lambda_\alpha G + \frac{D[t_\alpha]}{Dt} \right\} \quad (3.34)$$

In a similar manner as for the derivation of equation (3.32), we obtain

$$\begin{aligned} \left[\frac{\partial^2 t_\alpha}{\partial t^2} \right] &= \tilde{\lambda}_\alpha G^2 - 2G \frac{D\lambda_\alpha}{Dt} - \lambda_\alpha \frac{\delta G}{\delta t} - G G_{,s} \frac{D}{Ds} [t_\alpha] \\ &+ [t_\alpha] R_{\alpha\beta\gamma}^{\gamma} G^{\nu\gamma} G^{\nu\beta} + \frac{D^2[t_\alpha]}{Dt^2} \end{aligned} \quad (3.35)$$

as the appropriate form for the remaining compatibility conditions for a surface vector \underline{t} .

The appropriate form of the compatibility conditions for higher order tensors across a moving curve $C(t)$ are easily obtained from the basic relations in the same way as above and the same form as the above equations.

4. First and Second Order Dynamical Conditions

In this chapter, Reynold's transport theorem is introduced to deduce the basic dynamical field equations and discontinuity relations for a surface undergoing infinitesimal deformation. These relations comprise the principles of conservation of mass and momenta.

Reynold's transport theorem [15] is given by

$$\frac{d}{dt} \int_S \underline{t}(u, t) dS = \int_S \frac{\partial \underline{t}}{\partial t} dS + \int_C v_n \underline{t} ds \quad (4.1)$$

where \underline{t} is an absolute surface tensor as the preceding discussions and $v_n = v_\alpha \nu^\alpha$, the normal component of the velocity of material particle of S at points of moving curve C.

Let γ be the density of the surface material under consideration. We assume the law of mass conservation, that is, the mass of the moving surface S remains unchanged under deformation. This condition is given by

$$\frac{d}{dt} \int_S \gamma dS = 0 \quad (4.2)$$

If we put $\underline{t} = \gamma$ and $v_n = v_\alpha \nu^\alpha$ in equation (4.1) and make use of

Green's theorem [15] given by

$$\int_S t_{\alpha i}^{\alpha} dS = \int_C t_{\alpha} \nu^{\alpha} ds \quad (4.3)$$

it follows that

$$\int_S \frac{\partial r}{\partial t} dS + \int_C r v_{\alpha}^{\alpha} ds = \int_S \frac{\partial r}{\partial t} dS + \int_S (r v_{\alpha}^{\alpha})_{; i}^{\alpha} dS = 0 \quad (4.4)$$

by applying equation (4.2). Since S is arbitrary, we have the equation of continuity in the form

$$\frac{\partial r}{\partial t} + (r v_{\alpha}^{\alpha})_{; i}^{\alpha} = \frac{dr}{dt} + r v_{\alpha}^{\alpha} = 0 \quad (4.5)$$

It can be shown that the stress on a curve in S is given by [15]

$$t_{\alpha} = \tau_{\alpha\beta} \nu^{\beta} \quad (4.6)$$

where t_{α} are components of force per unit arc length, $\tau_{\alpha\beta}$ components of stress tensor, and ν^{β} components of geodesic unit normal vector to the curve. We also assume the principle of conservation of linear momentum, that is, the rate of change of the momentum of S in any fixed direction is equal to the component, in this direction, of the total external force acting on S , expressed by

$$\frac{d}{dt} \int_S \gamma \vec{v}_\alpha dS = \int_C t_\alpha ds \quad (4.7)$$

Substituting equation (4.6) in (4.7) we have

$$\frac{d}{dt} \int_S \gamma \vec{v}_\alpha dS = \int_C \tau_{\alpha\beta} \nu^\beta ds \quad (4.8)$$

From equation (4.1) and (4.5) we can show that

$$\frac{d}{dt} \int_S \gamma \vec{v}_\alpha dS = \int_S \gamma \frac{d\vec{v}_\alpha}{dt} dS \quad (4.9)$$

by putting $\vec{t} = \vec{v}_\alpha$ in equation (4.1). Hence we obtain the relations

$$\frac{d}{dt} \int_S \gamma \vec{v}_\alpha dS = \int_S \gamma \frac{d\vec{v}_\alpha}{dt} dS = \int_C \tau_{\alpha\beta} \nu^\beta ds \quad (4.10)$$

and applying Green's theorem for the last term of equation (4.10), the equation of motion will follow in the form

$$\gamma \frac{d\vec{v}_\alpha}{dt} = \tau_{\alpha\beta} \nu^\beta \quad (4.11)$$

As usual, we can show that $\tau_{\alpha\beta}$ are the components of a symmetric tensor by making use of principle of conservation of angular momentum [15].

If \underline{w} denotes the displacement vector in S, we have

$$v_{\alpha} = \frac{\partial w_{\alpha}}{\partial t} + w_{\alpha;\beta} v^{\beta}, \quad (4.12)$$

but for infinitesimal deformation we can put

$$v_{\alpha} = \frac{\partial w_{\alpha}}{\partial t} \quad (4.13)$$

approximately, neglecting the second term in the right side of equation (4.12). Similarly, we have

$$\frac{dw_{\alpha}}{dt} = \frac{\partial^2 w_{\alpha}}{\partial t^2} \quad (4.14)$$

approximately, thus equations (4.11) can be written as

$$\tau_{\alpha\beta} = \gamma \frac{\partial^2 w_{\alpha}}{\partial t^2}. \quad (4.15)$$

Having derived the basic field equations, we now turn to the question of discontinuity relations in the dynamical variables across a wave curve C . As in the earlier discussions, assume that the surface is divided by a wave curve C into regions S_1 and S_2 . If we apply equation (4.1) for regions S_1 and S_2 , and denote $t_{\sim 1}$, and $t_{\sim 2}$ as the value of t on sides of C bordering S_1 and S_2 , respectively, it follows that

$$\frac{d}{dt} \int_S \rho \, dS = \int_S \frac{\partial \rho}{\partial t} \, dS + \int_{\Sigma} \rho v_n \, ds + \int_C (\rho_1 - \rho_2) G \, ds \quad (4.16)$$

where Σ is the curve bounding the surface S and G being the normal speed of the curve C . (Fig.2) Putting $\rho = \gamma$ in equation (4.16) we have the following relation from equation (4.2)

$$\frac{d}{dt} \int_S \gamma \, dS = \int_S \frac{\partial \gamma}{\partial t} \, dS + \int_{\Sigma_1} \gamma v_n \, ds + \int_{\Sigma_2} \gamma v_n \, ds \quad (4.17)$$

$$+ \int_C (\gamma_1 - \gamma_2) G \, ds = 0$$

where γ_1 and γ_2 are the values of the density on the side 1 and 2 of the curve C and Σ_1 and Σ_2 are the portions of the curve bounding the surface S_1 and S_2 . Let S approaches zero at a fixed time so that S will shrink into a part of C . Then equation (4.17) will become

$$\gamma_1 (v_{1n} - G) = \gamma_2 (v_{2n} - G) \quad (4.18)$$

as the first dynamical relation required across the curve where

v_{1n} and v_{2n} are the normal components of the particle velocity on the side 1 and 2 of the curve C since the surface integral approaches zero but

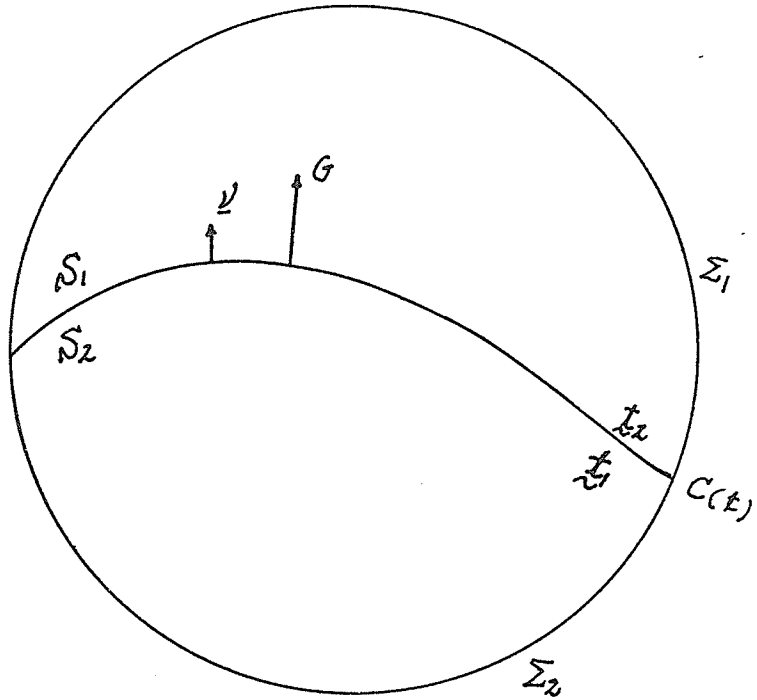


Fig. 2. Wave curve $C(t)$ and the bounding
curve Σ of S

$$\int_{\Sigma_1} r v_n ds \longrightarrow - \int_c r_1 v_{1n} ds \quad (4.19)$$

$$\int_{\Sigma_2} r v_n ds \longrightarrow \int_c r_2 v_{2n} ds$$

when we assign positive sign to the outward normal.

Similarly from the equation of principles of linear momentum (4.7) when the surface S approaches zero, we obtain the following relation

$$\int_c \tau_{\alpha\beta} v^\beta ds \longrightarrow \int_c (\tau_{2\alpha\beta} - \tau_{1\alpha\beta}) v^\beta ds \quad (4.20)$$

for the stress difference in the normal direction between the region S_1 and S_2 . Also in the same way as the above, putting $\underline{t} = r \underline{v}_\alpha$ in equation (4.16) and taking the limit $S \rightarrow 0$, it follows that

$$\int_c r_1 v_{1\alpha} (G - v_{1n}) ds - \int_c r_2 v_{2\alpha} (G - v_{2n}) ds \quad (4.21)$$

Finally the equation

$$r_1 (v_{1n} - G) [v_2] = [[\tau_{\alpha\beta}]] v^\beta \quad (4.22)$$

follows from equations (4.10), (4.20) and (4.21) as the required second order dynamical discontinuity relation in which

$$[[U_{\alpha}]] = U_{2\alpha} - U_{1\alpha} \quad (4.23)$$

and

$$[[\tau_{\alpha\beta}]] = \tau_{2\alpha\beta} - \tau_{1\alpha\beta}. \quad (4.24)$$