THE COLLINEATION GROUP

OF A

VEBLEN-WEDDERBURN NON-DESARGUESIAN PROJECTIVE GEOMETRY

BY

DIANE JOHNSON

A THESIS

PRESENTED TO THE

FACULTY OF GRADUATE STUDIES AND RESEARCH

OF THE

UNIVERSITY OF MANITOBA

IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

APRIL 1956

THE WRITER WISHES TO EXPRESS SINCERE APPRECIATION TO DR. N.S. MENDELSOHN OF THE DEPARTMENT OF MATHEMATICS FOR HIS INTEREST AND DIRECTION AND TO THE NATIONAL RESEARCH COUNCIL FOR THEIR ASSISTANCE IN THE FORM OF A BURSARY FOR THE YEAR 1955-1956

TABLE OF CONTENTS

INTRODUCTION

In the latter part of the last century it was discovered that Desargues's theorem could not be proved in the plane from the axioms of incidence. Subsequently several non-Desarguesian plane projective geometries have been constructed.

It is the purpose of this report to investigate one such geometry from the point of view of collineation theory. The report is in four parts. Part I deals with the more important properties which distinguish Desarguesian from non-Desarguesian geometries. Part II deals with the special geometry discovered by Veblen and Wedderburn in [1] . Part III is concerned with the finding of the collineation group of this geometry and with some geometrical consequences. These geometrical consequences are special cases of some unpublished theorems of N. Mendelsohn. Part IV discusses the collineation group from the point of view of its abstract properties.

PART I.

DISTINCTION BETWEEN DESARGUESIAN AND NON-DESARGUESIAN GEOMETRY

A projective plane geometry is a mathematical system consisting of elements called points and lines respectively. These elements which are finite or infinite in number are connected by a relationship called "on" subject to the following conditions.

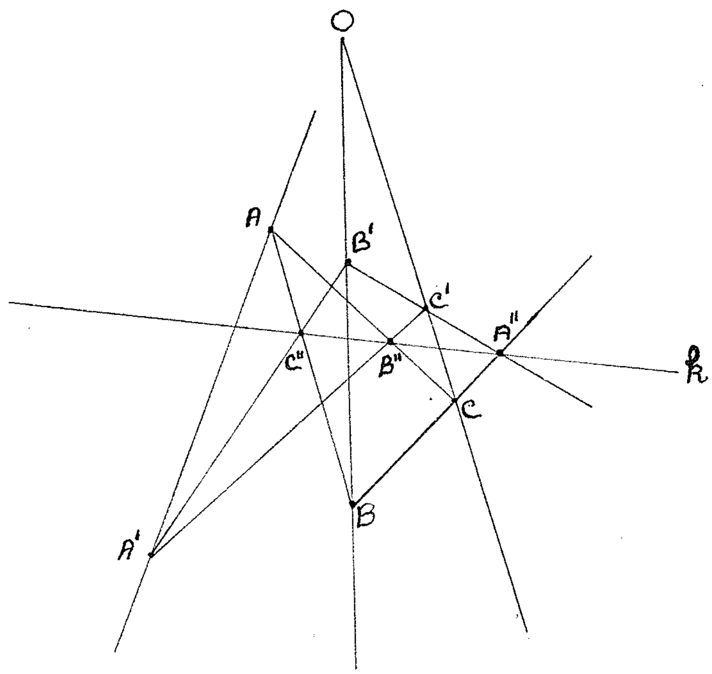
1. There is one and only one line on any two distinct points.
2. There is one and only one point which is on both of two distinct lines.
3. There are four points no three of which are on one line.

In projective geometry it is customary to denote lines by small letters of the alphabet and the points by capital letters of the alphabet. In cases where points P,Q,R are all on the same line "m" the expression 'P,Q,R are collinear' may be used. Similarly, if lines p,q,r are all on the same point P the expression 'p,q,r are concurrent' may be used. Furthermore the statement 'the line p passes through the point P' may be taken as a paraphrase of 'the point P is on the line p'.

Two point figures A,B,C,... and A',B',C',... are said to be centrally perspective from a point O (called the centre of perspectivity) if the lines AA',BB',CC',... are all on the point O. Two line figures a,b,c,... and a',b',c',... are said to be axially perspective on the line p (called the axis of perspectivity) if the points aa',bb',cc',... are all on p.

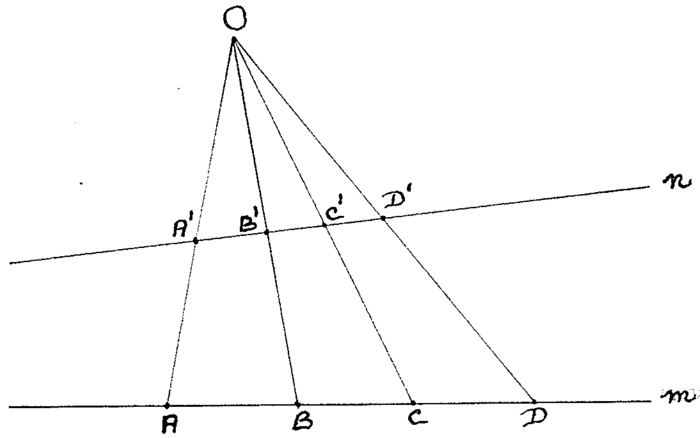
One of the most important theorems in projective geometry is that of Desargues which states that if two triangles are centrally perspective then they are axially perspective. At this point it is very important to note that for a projective geometry of two dimensions, Desargues's theorem is not an immediate consequence of the conditions of the first paragraph. In fact, there exist systems of points and lines for which all these conditions hold but for which Desargues's theorem is not generally true. Such a geometrical system is called non-Desarguesian. There are many examples of non-Desarguesian geometry, one of which, the special Veblen-Wedderburn system, will be described in part II of this paper.

It is a known fact that Desargues's theorem in the plane is equivalent to its converse. In other words, if Desargues's theorem is true for all centrally perspective pairs of triangles, its converse is true for all axially perspective pairs, and if Desargues's theorem fails for one pair of centrally perspective triangles, its converse fails for at least one pair of axially perspective triangles. A proof that Desargues's theorem implies its converse will now be given.



Consider the triangles ABC and A'B'C' which are axially perspective on the line k. i.e. AB meets A'B' at C", BC meets B'C' at A", CA meets C'A' at B", and A",B",C" are on k. Let the line CC' and the line BB' intersect in O. Apply Desargues's theorem to the triangles BB'C" and CC'B". The line BC" meets the line CB" in the point A. The line B'C" meets the line C'B" in the point A'. The line BB' meets the line CC' in the point O. Hence, since Desargues's theorem holds for the triangles BB'C" and CC'B", the points A,A', and O are collinear. Therefore the triangles ABC and A'B'C' are centrally perspective.

Now, suppose that in the plane there exists a set of points A,B,C,... on a line m, a set of points A',B',C',... on a line n distinct from m, and a point O not on m or n. If the lines AA',BB',CC',DD',... are all on the point O, the



relationship between the two figures is said to be a central perspectivity. This relationship can be expressed symbolically as follows:

$$m(A,B,C,\ldots) \overset{O}{\barwedge} n(A',B',C',\ldots).$$

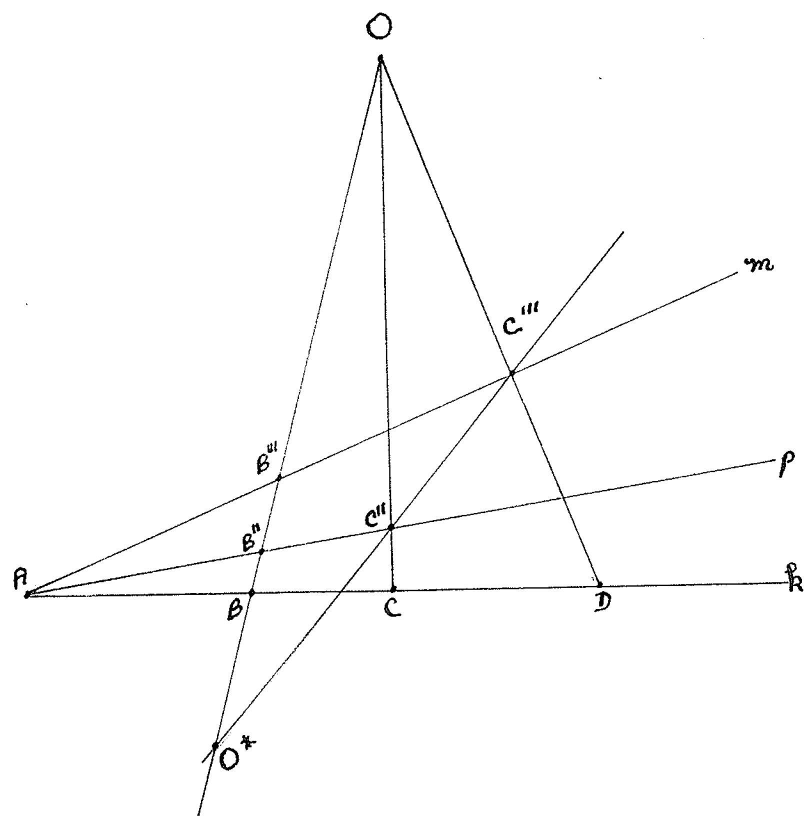
The points of a line m are said to be projectively related to those of a line n if the points of n are obtained from those of m as a result of a finite sequence of perspectivities. In other words

$$m(A,B,C,\ldots) \overset{O_1}{\barwedge} m_1(A_1,B_1,C_1,\ldots)\ldots\ldots\ldots \overset{O'}{\barwedge} n(A',B',C',\ldots).$$

In this case we will say

$$m(A,B,C,\ldots) \sim n(A',B',C',\ldots) .$$

Given any three points A,B,C on a line and any other three points A',B',C' on the same line there always exists a projective transformation (in this case a sequence consisting of two central perspectivities) which sends A into A', B into B', and C into C'. Further, if A,B,C,D are any four points on a line k, there exists a projective transformation which sends A into A, B into B, and C into D. For this latter case the following proof is given.



Let p,m be two arbitrary lines concurrent with k at the point A. Let O be any other point not on any of the lines m,p,k. Let OB and OC meet p at B" and C" respectively. Let OD meet m at C''' and OB meet m at B'''.

Let C''',C" meet OB at O^*. Then

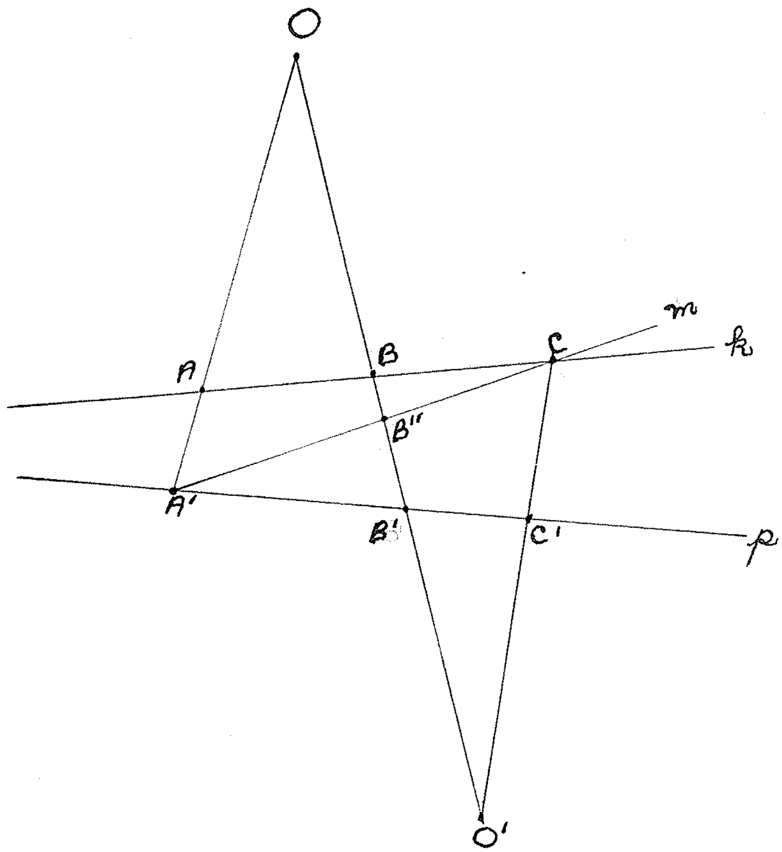
$$k(A,B,C) \overset{O}{\wedge} p(A,B'',C'') \overset{O^*}{\wedge} m(A,B''',C''') \overset{O}{\wedge} k(A,B,D) .$$

Hence

$$k(A,B,C) \sim k(A,B,D) .$$

i.e. There exists a projective transformation which sends A into A, B into B, and C into D. By iteration of transformations of this type, any three distinct points on a line may be mapped into any other set of three points in the same line.

In the case of two distinct lines, a simpler proof can be given of the fact that any three points of the first line can be projectively related to any three points of the second in the following way.



Let A,B,C and A',B',C' be two sets of points on the lines k and p respectively. Let AA' and BB' meet at the point O, and let BB' and CC' meet in the point O'. Let m be the line A'C and let BB' meet m in B".

Then

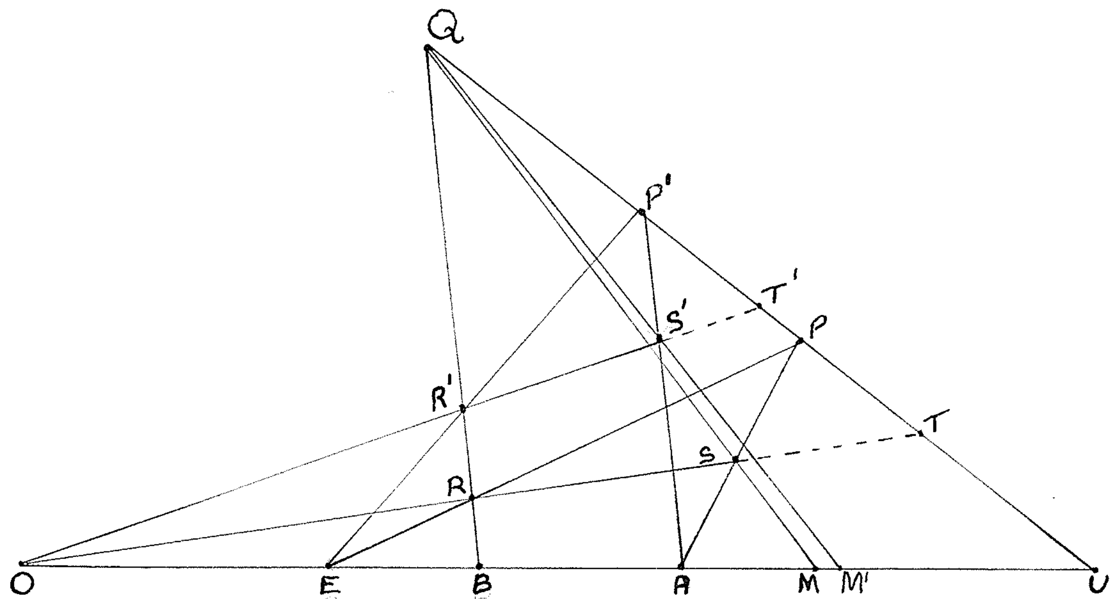
$$k(A,B,C) \overset{O}{\underset{\wedge}{-}} m(A',B'',C) \overset{O'}{\underset{\wedge}{-}} p(A',B',C') \quad .$$

Hence

$$k(A,B,C) \sim p(A',B',C') \quad .$$

i.e. There exists a projective transformation between the lines k and p which sends the points A into A', B into B', and C into C'.

It is a known property of plane projective geometries for which Desargues's and Pappus's[1] theorems are true that a projective transformation in a line reduces to the identity if three points are fixed. It will now be shown that if Desargues's theorem (and hence its converse) fails that there exists a line and four points on it, together with a projective transformation, which keeps three of these points fixed but which does not keep the fourth fixed.



1. Pappus's theorem states:

If A,B,C are any three distinct points of a line m, and A',B',C' any three distinct points of another line m', the three points of intersection of pairs of lines AB' and A'B, BC' and B'C, CA' and C'A are collinear.

For let PRS and P'R'S' be two triangles for which the converse of Desargues's theorem fails. Let the line PS and the line P'S' meet at A; the line PR and the line P'R' at E; the line RS and the line R'S' at O. The points O,E,A are collinear. Let the line RR' and the line PP' meet at Q. Since the converse of Desargues's theorem fails, Q,S, and S' are not collinear. Let QS meet OE at M and QS' meet OE at M' where M is not equal to M'. Let R'S',RS and OE meet PP' at T',T and U respectively. Then

$$OMAU \overset{S}{\underset{\wedge}{\rightarrow}} TQPU \overset{R}{\underset{\wedge}{\rightarrow}} OBEU \overset{R'}{\underset{\wedge}{\rightarrow}} T'QP'U \overset{S'}{\underset{\wedge}{\rightarrow}} OM'AU \quad .$$

Hence

$$OMAU \sim OM'AU$$

Therefore there exists a projective transformation which sends O into O, A into A, U into U, but which sends M into M'.

A collineation of a plane is any one to one correspondence between the points of a plane which has the following properties.

1. The transformation has an inverse.
2. The images of collinear points are collinear points.
3. Any three collinear points are the images of three points which are collinear.

For geometries in which Desargues's theorem is true the transformations $\rho\, x_i' = \sum a_{ij}\, x_j$ for which the determinant

(a_{ij}) is not equal to zero are all collineations called projective collineations. In general, if the co-ordinate field has an automorphism φ the transformation $\rho x'_i = \sum a_{ij} \varphi(x_j)$ is also a collineation. For geometries in which Desargues's theorem fails, it is not always advantageous to distinguish between projective and non-projective collineations.

In non-Desarguesian geometries, it is useful to distinguish various special cases of Desargues's theorem. In the case where the centre of perspectivity is on the axis of perspectivity the theorem is usually referred to as the little Desargues's theorem. Other cases are of interest but these are not needed in the subsequent development.

PART II.

THE SPECIAL VEBLEN-WEDDERBURN NON-DESARGUESIAN GEOMETRY

This special geometry is defined in terms of a system of co-ordinates. These co-ordinates form a "near-field" which is a system of elements satisfying all the axioms of a field except for the commutative law of multiplication and the distributive law $(b+c)a = ba+ca$. A special near-field can be constructed as follows. Each element is an expression of the form $a+bj$ where a and b are elements of the Galois field of order three, and where $j^2 = 2$. If $b = 0$, then for all a, $a+bj$ is called a scalar element. If $b \neq 0$, the element $a+bj$ will be referred to as a non scalar element. The right distributive law will be replaced by the law $(a + bj)j = -j(a + bj)$ provided a and b are both distinct from zero. Any scalar element commutes with any other element under multiplication and hence also is distributive on the right. In this particular near-field, although multiplication is not commutative, it can be verified that either $\alpha\beta = \beta\alpha$ or $\alpha\beta = -\beta\alpha$. In what follows, the development will be restricted to this special near-field since in this case, computation is relatively simple because of the laws which hold. The multiplication table for the elements of the near-field (distinct from the elements zero and one) appears below.

	2	j	2j	1 + j	1 + 2j	2 + j	2 + 2j
2	1	2j	j	2 + 2j	2 + j	1 + 2j	1 + j
j	2j	2	1	2 + j	1 + j	2 + 2j	1 + 2j
2j	j	1	2	1 + 2j	2 + 2j	1 + j	2 + j
1 + j	2 + 2j	1 + 2j	2 + j	2	2j	j	1
1+2j	2 + j	2 + 2j	1 + j	j	2	1	2j
2 + j	1 + 2j	1 + j	2 + 2j	2j	1	2	j
2+2j	1 + j	2 + j	1 + 2j	1	j	2j	2

A point of this geometry is defined as one of the following systems of three co-ordinates:

(α) (a,b,1)

(β) (a,1,0)

(γ) (1,0,0)

where a and b are arbitrary elements of the near-field.

A line is defined as all points which satisfy an equation of one of the following forms:

(1) $x + ya + zb = 0$

(2) $y + zc = 0$

(3) $z = 0$

Since the left distributive law holds, a general point (x,y,z) can be represented by (kx,ky,kz) where $k \neq 0$. In other words, if a set of numbers (x,y,z) satisfies the equation of a line, the set (kx,ky,kz) will satisfy the same equation of a line. It is however, important to note that multiplication on the right of the co-ordinates of a point by a non-zero constant k is, in general, not valid. That this leads to a result which is not valid is shown by the following example. Consider the point with co-ordinates (1+j,j,1). Multiplying this set of co-ordinates on the right by 2+j, we get the co-ordinates (j,2+2j,2+j) and multiplying this last set on the left by 1+2j, we get (2+2j,2j,1) which is not the same as the first set of co-ordinates (1+j,j,1).

In order to show that the points and lines of this Veblen-Wedderburn geometry form a projective plane geometry, it is necessary to prove that

1. Any two distinct lines have one and only one point in common.
2. Any two distinct points have one and only one line in common.
3. There are four points A,B,C,D no three of which are on a line.

Proof of 1.

In order to show that any two distinct lines have one and only one point in common it is necessary to consider several possibilities. Hence the proof will be divided up into the following five cases.

a) Two distinct lines of type (1).

b) A line of type (1) and a line of type (2).

c) A line of type (1) and a line of type (3).

d) Two distinct lines of type (2).

e) A line of type (2) and a line of type (3).

a) Two distinct lines of type (1).

$$x + ya_1 + zb_1 = 0$$

$$x + ya_2 + zb_2 = 0$$

Upon eliminating x, the equations reduce to the form

$$y(a_1 - a_2) + z(b_1 - b_2) = 0$$

If $b_1 = b_2 = b$, then $a_1 \neq a_2$, and hence $y = 0$. Taking $z = 1$, $x = 2b$. Therefore the point common to these lines has co-ordinates of type (α).

If $a_1 = a_2 = a$, then $b_1 \neq b_2$, and hence $z = 0$. Therefore taking $y = 1$, $x = 2a$. The point common to these lines has co-ordinates of type (β).

If $a_1 \neq a_2$ and $b_1 \neq b_2$, take $z = 1$. Then $y = -(b_1 - b_2)(a_1 - a_2)^{-1}$ and $x = \{-(b_1 - b_2)(a_1 - a_2)^{-1}\}(a_1 + a_2) + (b_1 + b_2)$. Therefore the point common to these lines has co-ordinates of type (α).

b) A line of type (1) and a line of type (2).

$$x + ya_1 + zb_1 = 0$$

$$y + zb_2 = 0 \qquad \text{where } z \neq 0.$$

Take $z=1$, then $y=-b_2$, and hence $x=b_2a_1-b_1$. Therefore the point common to these two equations is of type (α).

c) A line of type (1) and a line of type (3).

$$x+ya_1+zb_1=0$$
$$z=0$$

Since $z=0$, take $y=1$, and hence $x=2a_1$. The common point then has co-ordinates of type (β).

d) Two lines of type (2).

$$y+zc_1=0$$
$$y+zc_2=0 \qquad \text{where } c_1 \neq c_2.$$

Solving for z, the equation becomes

$$z(c_1-c_2)=0 \qquad \text{i.e.} \quad z=0.$$

Hence $y=0$ and $x=1$. Therefore the point common to these lines has co-ordinates of type (γ).

e) A line of type (2) and a line of type (3).

$$y+zc=0$$
$$z=0$$

The only solution is (1,0,0). Therefore the point common to these lines has co-ordinates of type (γ).

Proof of 2.

To show that any two distinct points have one and only one line in common, it is necessary to divide the proof into the following five cases.

a) Two points of type (α).

b) One point of type (α) and one point of type (β).

c) One point of type (α) and one point of type (γ).

d) Two points of type (β).

e) One point of type (β) and one point of type (γ).

a) Two points of type (α).

$(a_1, b_1, 1)$

$(a_2, b_2, 1)$

Consider the equation $x + y\alpha + z\beta = 0$. If this line is to be on $(a_1, b_1, 1)$ and $(a_2, b_2, 1)$ then

$$\left.\begin{aligned} a_1 + b_1\alpha + \beta &= 0 \\ \text{and} \quad a_2 + b_2\alpha + \beta &= 0 \end{aligned}\right\} \qquad \text{(i)}$$

Upon eliminating β the equations reduce to

$$b_1\alpha - b_2\alpha = a_2 - a_1 . \qquad \text{(ii)}$$

Since the right distributive law is not valid the left side of (ii) can not be further simplified. To establish whether there exists an α for which (ii) is valid, it is necessary to prove the following lemma.

<u>Lemma</u>: If $a \neq b$ then the equation $ax-bx = c$, where a,b,c are elements of the Veblen-Wedderburn near-field, has exactly one solution.

Proof-

Consider the nine elements $x_1, x_2, x_3, \ldots, x_9$ where these x_i are the elements of the Veblen-Wedderburn near-field F.

Form $ax_1-bx_1, ax_2-bx_2, \ldots, ax_9-bx_9$ which are now shown to be distinct.

For, suppose they are not distinct. Then for at least one pair of elements x_i and x_j with $i \neq j$

$$ax_i-bx_i = ax_j-bx_j \text{ .}$$

i.e. $$a(x_i-x_j) = b(x_i-x_j).$$

Since $x_i \neq x_j$, then $a = b$ which contradicts the hypothesis that $a \neq b$. Hence the quantities ax_i-bx_i, where $i = 1,2,\ldots,9$ are all distinct.

Since there are nine elements of the form ax-bx in F, all the latter elements appear amongst those of the form ax-bx. Hence, there will be some x, say $x = \bar{x}$, such that $a\bar{x}-b\bar{x} = c$. This completes the proof of the lemma.

Hence in equation (ii) of section a) there exists an α say $\alpha = k$ such that $b_1k-b_2k = a_2-a_1$.

Solving for β in (i),

$$\beta = (a_1 + a_2) + (b_1 k + b_2 k).$$

Therefore the line on these points has the equation

$$x + yk + z\left\{(a_1 + a_2) + (b_1 k + b_2 k)\right\} = 0$$

which is an equation of type (1).

b) One point of type (α) and one point of type (β).

$$(a_1, b, 1)$$

$$(a_2, 1, 0)$$

Consider the equation $x + y\alpha + z\beta = 0$. If this line is to be on the above points then

$$a_1 + b\alpha + \beta = 0 \qquad \text{(i)}$$

and $$a_2 + \alpha = 0 . \qquad \text{(ii)}$$

Hence $\alpha = 2a_2$ and $\beta = 2a_1 + ba_2$.

The line on these points has the equation

$$x + y(2a_2) + z(2a_1 + ba_2) = 0 .$$

i.e. An equation of type (1).

c) One point of type (α) and one point of type (γ).

$$(a, b, 1)$$

$$(1, 0, 0)$$

It is immediately verified that these points lie on the line $y + z(2b) = 0$. Therefore the line on these points has the equation of type (2).

d) Two points of type (β).

$$(a_1,1,0)$$

$$(a_2,1,0)$$

It is immediately verified that these points lie on the line $z = 0$. Therefore the line on these points has an equation of type (3).

e) One point of type (β) and one point of type (γ).

$$(a,1,0)$$

$$(1,0,0)$$

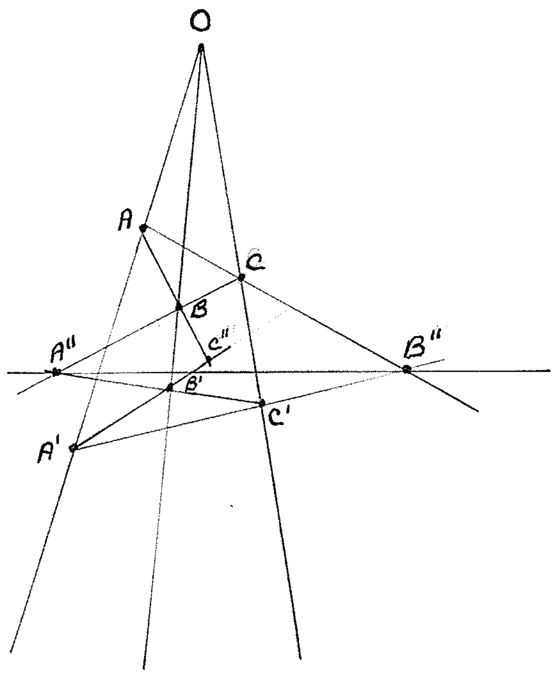
It is immediately verified that these points lie on the line $z = 0$. Therefore the line on these points has an equation of type (3).

Proof of 3.

The four points (1,0,0),(0,1,0),(0,0,1), and (1,1,1) satisfy the required condition.

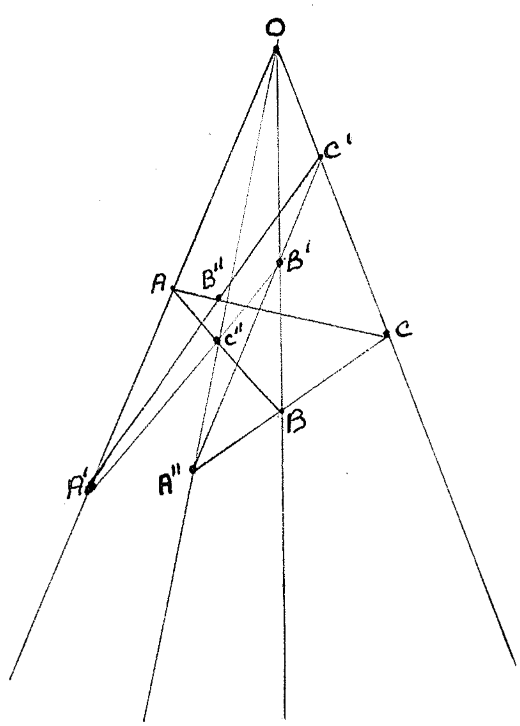
This completes the proof that the points and lines of the Veblen-Wedderburn geometry form a projective plane geometry.

That the geometry is non-Desarguesian is shown by the following example.



Consider the two triangles ABC and A'B'C' which are centrally perspective from O. Let O,A,A', B,B',C,C' have co-ordinates (1,j,1), (j,1,1),(2+2j,2+2j,1), (2j,2+2j,1),(2+2j,0,1), (2+j,1+j,1), and (1,j,0) respectively. The line AC with equation $x+y(2+j)+z(1+j)=0$ and the line A'C' with equation $x+yj+z2=0$ meet in the point B'' which has co-ordinates (2j,2+j,1); the line BC with equation $x+y2+z2=0$ and the line B'C' with equation $x+yj+z(1+j)=0$ meet in the point A'' which has co-ordinates (1+2j,2j,1); the line AB with equation $x+y(1+j)+z(2+j)=0$ and the line A'B' with equation $x+z(1+j)=0$ meet in the point C'' which has co-ordinates (2+2j,1+j,1). The line with equation $x+y(2+2j)=0$ is on the points A'' and B''. However, the co-ordinates of the point C'' do not satisfy this equation. Hence Desargues's theorem does not always hold for this geometry. However, if a so-called "sub-geometry is considered namely the one in which the co-ordinates of the points are the scalar elements only then for this geometry Desargues's theorem is definitely valid.

That the little Desargues's theorem does not always hold in the Veblen-Wedderburn geometry is shown by the following example.



Consider the triangles ABC and A'B'C' which are centrally perspective from O. Let the points O,A,A',B,B', C,C' have co-ordinates $(1+j,j,1)$, $(j,1,0)$, $(2+2j,1,1)$, $(j,1+2j,1)$, $(1,2,1)$, $(j,2+2j,1)$, and $(1+2j,1,0)$ respectively. The line AB with equation $x+y(2j)+z(2+j)=0$ and the line A'B' with equation $x+y(1+2j)+z(2j)=0$ meet in the point C'' which has co-ordinates $(0,2+2j,1)$; the line BC with equation $x+z(2j)=0$ and the line B'C' with equation $x+y(2+j)+z(1+j)=0$ meet on the point A'' which has co-ordinates $(j,1,1)$; the line AC with equation $x+y(2j)+z2=0$ and the line A'C' with equation $x+y(2+j)+z2=0$ meet on the point B'' which has co-ordinates

(1,0,1). The line with equation $x + y(2 + 2j) + z = 0$ is on the points A",O, and C". However the co-ordinates of the point B" do not satisfy this equation. Hence the little Desargues's theorem does not always hold for this geometry. It will later be shown that the little Desargues's theorem holds for all triangles in central perspectivity from (1,0,0) and that (1,0,0) is the only point with this property.

In order to obtain the collineation group of the geometry it is desirable to investigate whether or not the lines of this geometry can be represented in any other forms save those of types (1),(2), and (3) previously mentioned. With this in mind, the following question is posed - Is $xa + yb + zc = 0$ an equation of a line of this geometry when a is non scalar ? This question is answered by the following lemma and theorem.

<u>Lemma</u>: If, for all a,b,c with $a \neq 0$, $xa + yb + zc = 0$ is a line then it is the line $x + yba^{-1} + zca^{-1} = 0$.

Proof:

For $xa + yb + zc = 0$ to be a straight line it must be equivalent to $x + y\alpha + z\beta = 0$ for some α and β.

Suppose $x + y\alpha + z\beta = 0$.

Then $x = -y\alpha - z\beta$.

Substituting for x in $xa + yb + zc = 0$ we get

$$xa + yb + zc = -(y\alpha + z\beta)a + yb + zc.$$

Since $xa + yb + zc = 0$ and $x + y\alpha + z\beta = 0$ are equivalent $-(y\alpha + z\beta)a + yb + zc = 0$ whenever (x, y, z) are co-ordinates of a point on $x + y\alpha + z\beta = 0$.

Take $(-\alpha, 1, 0)$ as a point on $x + y\alpha + z\beta = 0$.

Then $-\alpha a + b = 0$.

Therefore $\alpha = ba^{-1}$.

Take $(-\beta, 0, 1)$ as another point on $x + y\alpha + z\beta = 0$.

Then $-\beta a + c = 0$.

Therefore $\beta = ca^{-1}$.

Hence if $xa + yb + zc = 0$ is a line then it is the line $x + yba^{-1} + zca^{-1} = 0$.

<u>Theorem:</u> If "a" is non scalar $xa + yb + zc = 0$ is the equation of a line if and only if $bc = 0$.

Proof:

<u>Only If:</u> Suppose neither b nor c is zero. From the lemma, it follows that any point which satisfies $x + yba^{-1} + zca^{-1} = 0$ must satisfy $xa + yb + zc = 0$ if the latter equation is to be a line of the geometry.

Consider the equation

$$x + yba^{-1} + zca^{-1} = 0.$$

Now $x = -yba^{-1} - zca^{-1}$.

Substituting for x in $xa + yb + zc = 0$ we get

$(yba^{-1} + zca^{-1})a = yb + zc.$

Then $yba^{-1} + zca^{-1} = (yb + zc)a^{-1}$.

Let $a^{-1} = p + qj$ where p and q are scalars.

Then $yb(p + qj) + zc(p + qj) = (yb + zc)(p + qj)$

and $p(yb + zc) + q(ybj + zcj) = p(yb + zc) + q(yb + zc)j.$

Then for all y and z

$$ybj + zcj = (yb + zc)j. \qquad \text{(i)}$$

Take $z = c^{-1}$, $y = (1 + j)b^{-1}$.

Substituting in (i)

$$j + 1 = 1. \qquad \text{Impossible.}$$

i.e. The point $(c\{[2 + 2j]a^{-1} - a^{-1}\}, c(1 + j)b^{-1}, 1)$

satisfies $x + yba^{-1} + zca^{-1} = 0$ but not $xa + yb + zc = 0.$

The supposition is therefore contradicted and

hence if a is non scalar, and $bc \neq 0$,

$xa + yb + zc = 0$ is not the equation of a line.

<u>If</u>

(i) Suppose $b = 0$ and $c \neq 0$, then $xa + yb + zc = 0$ becomes $xa + zc = 0.$ From the lemma, if $xa + zc = 0$ is a line, then it is the line $x + zca^{-1} = 0.$

It is now shown that these equations are equivalent.

Suppose $x + zca^{-1} = 0$.

Then $x = -zca^{-1}$.

Therefore $xa + zc = (-zca^{-1})a + zc = 0$.

i.e. For a non scalar and $c \neq 0$, $xa + zc = 0$ is an equation of a line.

Similarly, for a non scalar, $c = 0$, and $b \neq 0$, it follows that $xa + yb = 0$ is an equation of a line.

(ii) Suppose $b = 0$ and $c = 0$. Then $xa + yb + zc = 0$ becomes $x = 0$ which is an admissible line of the geometry.

In a similar manner, a more general result can be obtained: viz. If $xa + yb + zc + \sum_{i=1}^{n}(xa_i + yb_i + zc_i)\alpha_i = 0$ is a straight line for every α_i then two of a_i, b_i, c_i must be zero for all i. Hence the expression can be reduced to the form $xA + yB + zC = 0$ and the above theorem will again apply. Hence the lines of this geometry can only be represented by equations which have forms similar to types (1), (2), and (3).[2]

---

2. Veblen and Wedderburn [I] state in a footnote that under the particular transformation which they consider the points $(1,0,0), (0,j,1), (j,j,1)$ on the line $yj + z = 0$ are sent into the points $(1,2,0), (1,0,2+2j), (1+j,2j,2+2j)$ of which the first two but not the third lie on $x(1+j) + y(1+j) + z = 0$. However this is not a line. All three points lie on $x + y + z(2+2j) = 0$.

PART III.

DETERMINATION OF ALL ANALYTIC COLLINEATIONS

In the classical (Desarguesian) geometry it can be shown that the most general projective collineation is a transformation of the form $\rho x'_i = \sum_{j=1}^{3} a_{ij} x_j$ where the determinant $(a_{ij}) \neq 0$. However, the most general collineation is given by $\rho x'_i = \sum_{j=1}^{3} a_{ij} \varphi(x_j)$ where the mapping x into $\varphi(x)$ is an automorphism of the co-ordinate field.

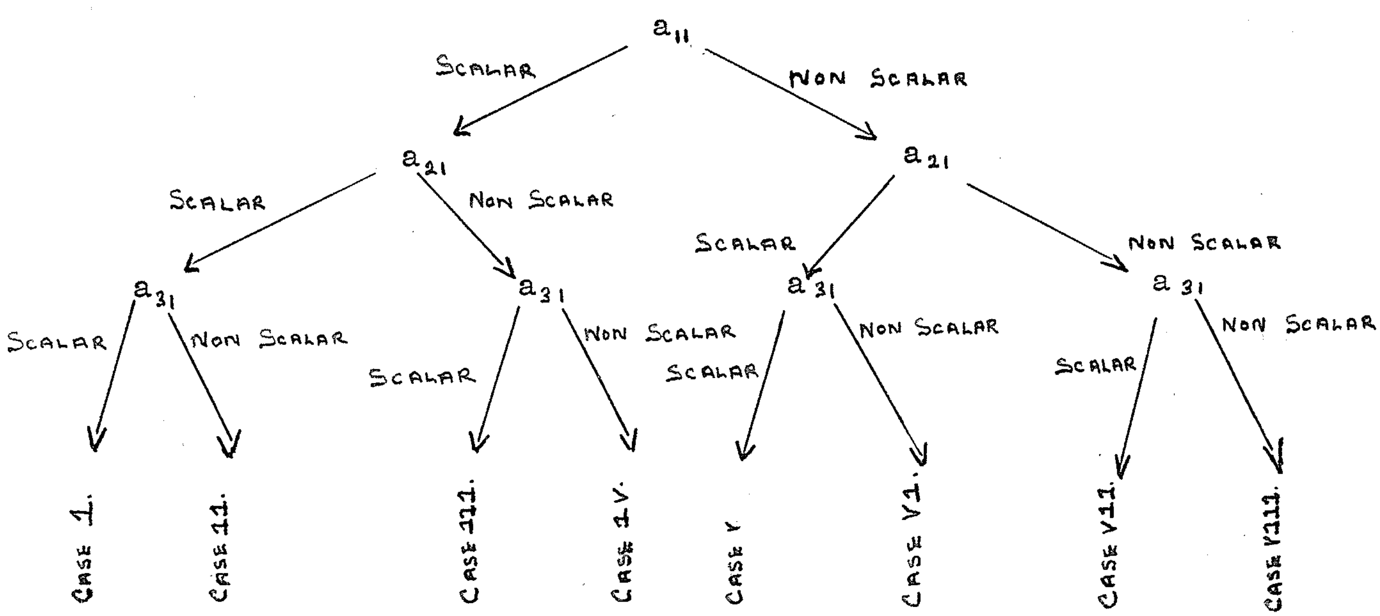
In a general non-Desarguesian geometry it is not generally worth while to attempt to distinguish a projective collineation from a non projective one. In the case considered here, it is easily seen that the mapping $\rho x'_i = \varphi(x_i)$ where $i = 1,2,3$ is a collineation if x into $\varphi(x)$ is an automorphism of the near-field. Furthermore, if the transformation $\rho x'_i = \sum_{j=1}^{3} x_j a_{ij}$ where $i = 1,2,3$ is a collineation then so also is the transformation $\rho x'_i = \sum_{j=1}^{3} \varphi(x_j) a_{ij}$. Hence we consider only transformations of the form $\rho x'_i = \sum_{j=1}^{3} x_j a_{ij}$ where $i = 1,2,3$ which will be termed analytic collineations when the mappings actually are collineations. In this connection, it may be stated that not all such mappings are collineations since the mapping

$$\rho x' = x + 2y + z$$
$$\rho y' = 2x + y + z$$
$$\rho z' = x + y + z$$

is not a collineation. This can easily be verified.

The number ρ is always chosen so that the last non zero co-ordinate of a point is one.

The problem then is to determine the whole group of analytic collineations. In considering the transformation $\rho x_i' = \sum_{j=1}^{3} x_j a_{ij}$ $(i = 1,2,3)$, various cases must be considered depending on the scalar or non scalar characteristics of the a_{ij}. It is not necessary to consider any cases in which the matrix (a_{ij}) has a row or a column of zeros or in which a two by two submatrix consists entirely of zeros since such transformations do not have inverses. The problem will be broken down into various subcases as given in the following scheme:-



Case 1.

Let $a_{11} = s_1, a_{21} = s_2,$ and $a_{31} = s_3$ where s_1, s_2, s_3 are scalars.

Consider [3]

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s_1 & a_{12} & a_{13} \\ s_2 & a_{22} & a_{23} \\ s_3 & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$x + y\alpha + z\beta = 0$ maps into

$x' s_1 + y' a_{12} + z' a_{13} + (x' s_2 + y' a_{22} + z' a_{23})\alpha + (x' s_3 + y' a_{32} + z' a_{33})\beta = 0.$ *

For all α and β, necessary conditions for equation * to be the equation of a line are that two of s_2, a_{22}, a_{23} must be zero and two of s_3, a_{32}, a_{33} must be zero.

Suppose $a_{22} = a_{23} = 0$, then $s_2 \neq 0$. The following possibilities occur -

$$\text{(a)} \quad s_3 = a_{32} = 0$$

$$\text{or} \quad \text{(b)} \quad s_3 = a_{33} = 0.$$

(a) If $s_3 = a_{32} = 0$, then $a_{33} \neq 0$, and $a_{12} \neq 0$.

Equation * becomes

---

3. When collineations are considered in this form it is to be understood that the transformation equations are to be written in the form

$$\rho x_i = x'_1 \, a_{i1} + x'_2 \, a_{i2} + x'_3 \, a_{i3} \quad \text{for} \quad i = 1,2,3$$

and not in the form

$$\rho x_i = a_{i1} x' + a_{i2} x'_2 + a_{i3} x'_3 \quad \text{for} \quad i = 1,2,3.$$

$x'(s_1 + s_2\alpha) + y'a_{12} + z'(a_{13} + a_{33}\beta) = 0.$

This is not an admissible line unless (i) $s_1 + s_2\alpha$ is a scalar, or, (ii) $a_{13} + a_{33}\beta = 0.$

(i) If $s_1 + s_2\alpha$ is a scalar for all α, then $s_2 = 0$, a contradiction.

(ii) If $a_{13} + a_{33}\beta = 0$, then either ($\bar{I}$) $a_{13} = a_{33} = 0$, or ($\bar{II}$) a_{13} and a_{33} are not zero.

($\bar{I}$) If $a_{13} = a_{33} = 0$, there is no collineation.

($\bar{II}$) If a_{13} and a_{33} are not zero, then $\beta = 2a_{33}^{-1}a_{13}$. But it is possible to find a β for which this is not true.

Therefore (a) does not lead to a collineation.

(b) If $s_3 = a_{33} = 0$, then $a_{13} \neq 0$, and $a_{32} \neq 0$.

Equation * becomes

$x'(s_1 + s_2\alpha) + y'(a_{12} + a_{32}\beta) + z'a_{13} = 0.$

This is not an admissible line unless (i) $s_1 + s_2\alpha$ is a scalar, or (ii) $a_{12} + a_{32}\beta = 0.$

(i) If $s_1 + s_2\alpha$ is a scalar, then $s_2 = 0$, a contradiction.

(ii) If $a_{12} + a_{32}\beta = 0$, then either ($\bar{I}$) $a_{12} = a_{32} = 0$, or, ($\bar{II}$) a_{12} and a_{32} are not zero.

($\bar{I}$) If $a_{12} = a_{32} = 0$, no collineation exists.

($\bar{II}$) If a_{12} and a_{32} are not zero, then

$\beta = 2a_{32}^{-1}a_{12}$. But there exists a β for which this does not hold.

Therefore (b) does not lead to a collineation.

Hence the supposition $a_{22} = a_{23} = 0$ is false. Therefore, since a_{22} and a_{23} cannot both be zero, s_2 is zero at least.

Now equation * has the form

$x's_1 + y'a_{12} + z'a_{13} + (y'a_{22} + z'a_{23})\alpha + (x's_3 + y'a_{32} + z'a_{33})\beta = 0$. **

For all α and β, necessary conditions for ** to be a line are that one of a_{22}, a_{23} must be zero and that two of s_3, a_{32}, a_{33} must be zero.

A. Suppose $a_{22} = 0$, then $a_{23} \neq 0$. The following possibilities occur -

(a) $a_{32} = a_{33} = 0$

or (b) $s_3 = a_{33} = 0$.

(a) If $a_{32} = a_{33} = 0$, then $s_3 \neq 0, a_{12} \neq 0$.

Equation ** becomes

$x'(s_1 + s_2\beta) + y'a_{12} + z'(a_{23}\alpha + a_{13}) = 0$.

For an admissible line $a_{23}\alpha + a_{13} = 0$ for all α.

Since $a_{23} \neq 0$, a_{13} cannot be zero. Hence, since a_{23} and a_{13} cannot be zero for all α, $\alpha = 2a_{23}^{-1}a_{13}$. But there exists an α for which this will not hold.

Therefore (a) does not lead to a collineation.

(b) If $a_{33} = s_3 = 0$, equation ** becomes

$x's_1 + y'(a_{12} + a_{32}\beta) + z'(a_{13} + a_{23}\alpha) = 0.$

Multiplying through by s_1^{-1}

$x' + y's_1^{-1}(a_{12} + a_{32}\beta) + z's_1^{-1}(a_{13} + a_{23}\alpha) = 0.$

Hence the equations

$$\rho x = x' + y'a_{12}^* + z'a_{13}^*$$
$$\rho y = z'a_{23}^*$$
$$\rho z = y'a_{32}^* \qquad \text{where } a_{23}^* \neq 0, \text{and } a_{32}^* \neq 0$$

determine a collineation.

B. Suppose $a_{23} = 0$ in equation **. Using a similar argument, the following collineation is obtained -

$$\rho x = x' + y'a_{12}^* + z'a_{13}^+$$
$$\rho y = y'a_{22}^*$$
$$\rho z = z'a_{33}^* \quad \text{where } a_{22}^* \neq 0, \text{ and } a_{33}^* \neq 0.$$

Case 11.

Let $a_{11} = s_1$ and $a_{21} = s_2$ where s_1 and s_2 are scalars.

Suppose a_{31} is non scalar.

Consider

$$\rho\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s_1 & a_{12} & a_{13} \\ s_2 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$z = 0$ maps into $x'a_{31} + y'a_{32} + z'a_{33} = 0.$

For an admissible line

(a) $a_{32} = 0$

or (b) $a_{33} = 0.$

(a) If $a_{32} = 0$, then

$$\rho\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s_1 & a_{12} & a_{13} \\ s_2 & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$y + z = 0$ maps into

$x'(s_2 + a_{31}) + y'a_{22} + z'(a_{23} + a_{33}) = 0.$

For an admissible line, since $s_2 + a_{31}$ is non scalar either (i) $a_{22} = 0$, or (ii) $a_{23} + a_{33} = 0.$

(i) If $a_{22} = 0$, then $a_{12} \neq 0$, and

$$\rho\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s_1 & a_{12} & a_{13} \\ s_2 & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$x+z=0$ maps into

$x'(s_1+a_{31})+y'a_{12}+z'(a_{13}+a_{33})=0.$

Since s_1+a_{31} is non scalar and $a_{12}\neq 0$,

$a_{13}+a_{33}=0$, if the line is to be admissible.

Now, two cases will arise - (I) $a_{13}=a_{33}=0$,

or (II) a_{13} and a_{33} are not zero.

(I) If $a_{13}=a_{33}=0$ then $a_{23}\neq 0$.

$x+y+z=0$ maps into

$x'(a_{31}+s_1+s_2)+y'a_{12}+z'a_{23}=0.$

$s_1+s_2+a_{31}$ is non scalar, $a_{12}\neq 0$ and $a_{23}\neq 0$.

The line is therefore not admissible.

(II) If $a_{13}\neq 0$ and $a_{33}\neq 0$, then

$x+z2=0$ maps into

$x'(s_1+2a_{31})+y'a_{12}+z'(a_{13}+2a_{33})=0$

where s_1+2a_{31} is non scalar and $a_{12}\neq 0$.

For an admissible line therefore $a_{13}+2a_{33}=0$.

But, since $a_{13}+a_{33}=0$, then $a_{13}+2a_{33}=0$.

Hence

$x'(s_1+2a_{31})+y'a_{12}+z'(a_{13}+2a_{33})=0$

is not an admissible line.

(ii) If $a_{23}+a_{33}=0$, then either (I) $a_{23}=a_{33}=0$,

or (II) a_{23} and a_{33} are not zero.

(I) If $a_{23}=a_{33}=0$ then $a_{13}\neq 0$ and

$$\rho\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s_1 & a_{12} & a_{13} \\ s_2 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Now $x+y+z=0$ maps into

$x'(s_1+s_2+a_{31})+y'(a_{12}+a_{22})+z'a_{13}=0.$

Since $s_1+s_2+a_{31}$ is non scalar and $a_{13}\neq 0$, then $a_{12}+a_{22}$ must be zero if the line is to be admissible. The only case to consider is the one in which a_{12} and a_{22} are non zero. Now, if a_{12} and a_{22} are not zero then

$x+y2+z=0$ maps into

$x'(s_1+2s_2+a_{31})+y'(a_{12}+2a_{22})+z'a_{13}=0.$

Since $s_1+2s_2+a_{31}$ is non scalar and $a_{13}\neq 0$, then $a_{12}+2a_{22}$ must be zero if the line is to be admissible. But, since $a_{12}+a_{22}=0$, then $a_{12}+2a_{22}\neq 0$. Hence

$x'(s_1+2s_2+a_{31})+y'(a_{12}+2a_{22})+z'a_{13}=0$

is not an admissible line.

(II) If $a_{23}\neq 0$ and $a_{33}\neq 0$,

$y+z2=0$ maps into

$x'(s_2+2a_{31})+y'a_{22}+z'(a_{23}+2a_{33})=0$

where s_2+2a_{31} is non scalar and $a_{23}+2a_{33}\neq 0$. [$a_{23}+2a_{33}\neq 0$ because the assumption is that $a_{23}+a_{33}=0$.] Hence for an admissible line, $a_{22}=0$ and the argument is similar to part (i) section (ii) of (a).

Hence (a) does not lead to a collineation. By taking $a_{33}=0$ instead of $a_{32}=0$, it is found, by a similar argument, that (b) does not lead to a collineation.

Therefore Case 11 is not one which brings about a collineation.

Case 111.

Let $a_{11}=s_1$ and $a_{31}=s_3$ where s_1 and s_3 are scalar. Suppose a_{21} is non scalar.

Consider

$$\rho\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} s_1 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ s_3 & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

The argument is similar to that of Case 11. Hence Case 111 does not lead to a collineation.

Case 1V.

Let $a_{11}=s_1$ where s_1 is a scalar. Suppose a_{21} and a_{31} are non scalars.

Consider

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s_1 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Then $y = 0$ maps into $x'a_{21} + y'a_{22} + z'a_{23} = 0$.

For an admissible line either $a_{22} = 0$ or $a_{23} = 0$.

Also, $z = 0$ maps into $x'a_{31} + y'a_{32} + z'a_{33} = 0$.

For an admissible line either $a_{32} = 0$ or $a_{33} = 0$.

Four cases can occur -

(i) $a_{22} = a_{32} = 0$

(ii) $a_{23} = a_{33} = 0$

(iii) $a_{22} = a_{33} = 0$

or (iv) $a_{23} = a_{32} = 0$.

(i) If $a_{22} = a_{32} = 0$, then $a_{12} \neq 0$, and

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s_1 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Now $x + y = 0$ maps into

$x'(s_1 + a_{21}) + y'a_{12} + z'(a_{13} + a_{23}) = 0$.

Since $s_1 + a_{21}$ is non scalar and $a_{12} \neq 0$, for an admissible line $a_{13} + a_{23} = 0$. Now the only case to be considered is the one in which a_{13} and a_{33} are both non zero. If a_{13} and a_{33} are non zero then $x + y2 = 0$ maps into $x'(s_1 + 2a_{21}) + y'a_{12} + z'(a_{13} + 2a_{23}) = 0$ where $s_1 + 2a_{21}$ is non scalar and $a_{12} \neq 0$. Hence for an admissible line $a_{13} + 2a_{23} = 0$. But $a_{13} + a_{23} = 0$, and so $a_{13} + 2a_{23} \neq 0$.

Therefore $x'(s_1 + 2a_{21}) + y'a_{12} + z'(a_{13} + 2a_{23}) = 0$ is not an admissible line.

Therefore (i) does not lead to a collineation.

Similarly by considering (ii) instead of (i) the same result is obtained, namely that (ii) does not lead to a collineation if a_{21} and a_{31} are non scalars.

(iii) If $a_{22} = a_{33} = 0$, then

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s_1 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$y + z = 0$ maps into $x'(a_{21} + a_{31}) + y'a_{32} + z'a_{23} = 0$.

For an admissible line (a) $a_{32} = 0$, (b) $a_{23} = 0$, or (c) $a_{21} + a_{31}$ is a scalar.

(a) If $a_{32} = 0$, then $a_{12} \neq 0$ and

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s_1 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{21} & 0 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$x+y=0$ maps into

$x'(s_1+a_{21})+y'a_{12}+z'(a_{13}+a_{23})=0$ where

$a_{12}\neq 0$ and s_1+a_{21} are non scalar. Hence, for an admissible line, $a_{13}+a_{23}=0$ and the only possibility to be considered is the one in which a_{13} and a_{23} are non zero. Now, if a_{13} and a_{23} are both non zero, $x+y2=0$ maps into

$x'(s_1+2a_{21})+y'a_{12}+z'(a_{13}+2a_{23})=0$ where

$a_{12}\neq 0$ and s_1+2a_{21} are non scalar. Hence for an admissible line $a_{13}+2a_{23}=0$. But since $a_{13}+a_{23}=0$, then $a_{13}+2a_{23}\neq 0$.

Therefore $x'(s_1+2a_{21})+y'a_{12}+z'(a_{13}+2a_{23})=0$ is not an admissible line.

(b) If $a_{23}=0$, then $a_{13}\neq 0$ and

$$\rho\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} s_1 & a_{12} & a_{13} \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$x+y=0$ maps into $x'(s_1+a_{21})+y'a_{12}+z'a_{13}=0$ where $a_{13}\neq 0$ and s_1+a_{21} is non scalar. Hence for an admissible line $a_{12}=0$ and then $a_{32}\neq 0$.

Now $x+y+z=0$ maps into

$x'(s_1+a_{21}+a_{31})+y'a_{32}+z'a_{13}=0$ where $a_{13}\neq 0$

and $a_{32} \neq 0$. For an admissible line $s_1 + a_{21} + a_{31}$ must be scalar. i.e. $a_{21} + a_{31}$ is scalar.

Now $x + y2 + z = 0$ maps into

$x'(s_1 + 2a_{21} + a_{31}) + y'a_{32} + z'a_{13} = 0$ where $a_{32} \neq 0$ and $a_{13} \neq 0$. For an admissible line $s_1 + 2a_{21} + a_{31}$ is scalar. i.e. $2a_{21} + a_{31}$ is scalar. Since $2a_{21} + a_{31}$ is scalar and $a_{21} + a_{31}$ is scalar, a_{21} is scalar contrary to supposition.

(c) If $a_{21} + a_{31}$ is a scalar, then $y2 + z = 0$ maps into $x'(2a_{21} + a_{31}) + y'a_{32} + 2z'a_{23} = 0$. For an admissible line (1) $a_{32} = 0$, (2) $a_{23} = 0$, or (3) $2a_{21} + a_{31}$ is scalar.

(1) If $a_{32} = 0$ the argument will be the same as part (a) of (iii).

(2) If $a_{23} = 0$ the argument will be the same as part (b) of (iii).

(3) If $2a_{21} + a_{31}$ is scalar then since $a_{21} + a_{31}$ is scalar, a_{21} is scalar contrary to the supposition.

Hence case (iii) does not lead to a collineation. Similarly by considering case (iv) instead of case (iii) the same result is arrived at namely that if a_{31} and a_{21} are non scalar (iv) does not lead to a collineation.

Hence case (iv) cannot be considered as a case which brings about a collineation.

Therefore Case 1V does not lead to a collineation.

Case V.

Let $a_{21} = s_2$ and $a_{31} = s_3$ where s_2 and s_3 are scalars.

Suppose a_{11} is non scalar.

Consider

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ s_2 & a_{22} & a_{23} \\ s_3 & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$x = 0$ maps into $x'a_{11} + y'a_{12} + z'a_{13} = 0$. For an admissible line

(a) $a_{12} = 0$

or (b) $a_{13} = 0$.

(a) If $a_{12} = 0$ then

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{13} \\ s_2 & a_{22} & a_{23} \\ s_3 & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$x + y = 0$ maps into $x'(a_{11} + s_2) + y'a_{22} + z'(a_{13} + a_{23}) = 0$

where $a_{11} + s_2$ is non scalar. For an admissible line

(i) $a_{22} = 0$, or (ii) $a_{13} + a_{23} = 0$.

(i) If $a_{22} = 0$ then $a_{32} \neq 0$ and then

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{13} \\ s_2 & 0 & a_{23} \\ s_3 & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$x + z = 0$ maps into

$x'(a_{11} + s_3) + y'a_{32} + z'(a_{13} + a_{33}) = 0$ where

$a_{32} \neq 0$ and $a_{11} + s_3$ is non scalar. For an

admissible line $a_{13} + a_{33} = 0$. Two cases arise

(I) $a_{13} = a_{33} = 0$, or (II) a_{13} and a_{33} are non zero.

(I) If $a_{13} = a_{33} = 0$ then $a_{23} \neq 0$.

$x + y + z = 0$ maps into

$x'(s_2 + s_3 + a_{11}) + y'a_{32} + z'a_{23} = 0$ where

$a_{32} \neq 0$, $a_{23} \neq 0$, and $s_2 + s_3 + a_{11}$ is non scalar.

Hence $x'(s_2 + s_3 + a_{11}) + y'a_{32} + z'a_{23} = 0$ is not an admissible line.

(II) If $a_{13} \neq 0$ and $a_{23} \neq 0$,

$x + z2 = 0$ maps into

$x'(a_{11} + 2s_3) + 2y'a_{32} + z'(a_{13} + 2a_{33}) = 0$

where $a_{32} \neq 0$ and $a_{11} + 2s_3$ is non scalar.

Hence for an admissible line $a_{13} + 2a_{33} = 0$, but $a_{13} + a_{33} = 0$, and so $a_{13} + 2a_{33}$ cannot be zero.

Therefore $x'(a_{11} + 2s_3) + 2y'a_{32} + z'(a_{13} + 2a_{33}) = 0$ is not an admissible line.

(ii) If $a_{13} + a_{23} = 0$, then (I) $a_{13} = a_{23} = 0$, or (II) a_{13} and a_{23} are not zero.

(I) If $a_{13} = a_{23} = 0$, then $a_{33} \neq 0$ and

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ s_2 & a_{22} & 0 \\ s_3 & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$x + y + z = 0$ maps into

$x'(s_2 + s_3 + a_{11}) + y'(a_{22} + a_{32}) + z'a_{33} = 0$

where $a_{33} \neq 0$ and $s_2 + s_3 + a_{11}$ is non scalar. For an admissible line $a_{22} + a_{32} = 0$. The only possibility occuring is the one in which a_{22} and a_{32} are not zero. Hence, if $a_{22} \neq 0$ and $a_{32} \neq 0$, then $x + y + z2 = 0$ maps into

$x'(s_2 + 2s_3 + a_{11}) + y'(a_{22} + 2a_{32}) + z'a_{33} = 0$

where $s_2 + 2s_3 + a_{11}$ is non scalar and $a_{33} \neq 0$. For an admissible line $a_{22} + 2a_{32}$ is zero but $a_{22} + a_{32} = 0$, and so $a_{22} + 2a_{32}$ cannot be zero.

Hence

$x'(s_2 + 2s_3 + a_{11}) + y'(a_{22} + 2a_{32}) + 2z'a_{33} = 0$

is not an admissible line.

(ii) If $a_{13} \neq 0$ and $a_{23} \neq 0$, then

$x + y2 = 0$ maps into

$x'(a_{11} + 2s_2) + 2y'a_{22} + z'(a_{13} + 2a_{23}) = 0$

where $a_{11} + 2s_2$ is non scalar and $a_{13} + 2a_{23} \neq 0$.

[$a_{13} + 2a_{23} \neq 0$ because $a_{13} + a_{23} = 0$.]

Hence for an admissible line $a_{22} = 0$, but then the argument of (i) is repeated.

Hence case (a) does not lead to a collineation.

Similarly by taking $a_{13} = 0$ instead of $a_{12} = 0$ the result is that (b) does not lead to a collineation.

Therefore Case V cannot bring about a collineation.

Case VI.

Let $a_{21} = s_2$ where s_2 is scalar. Suppose a_{11} and a_{31} are non scalar.

Consider

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ s_2 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$x = 0$ maps into $x' a_{11} + y' a_{12} + z' a_{13} = 0$.

For an admissible line $a_{12} = 0$ or $a_{13} = 0$.

$z = 0$ maps into $x' a_{31} + y' a_{32} + z' a_{33} = 0$.

For an admissible line $a_{32} = 0$, or $a_{33} = 0$.

Four cases occur:

(i) $a_{12} = a_{32} = 0$

(ii) $a_{13} = a_{33} = 0$

(iii) $a_{12} = a_{33} = 0$

or (iv) $a_{13} = a_{32} = 0$.

(i) If $a_{12} = a_{32} = 0$, then $a_{22} \neq 0$ and

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{13} \\ s_2 & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$x + y = 0$ maps into

$x'(a_{11} + s_2) + y' a_{22} + z'(a_{13} + a_{23}) = 0$ where $a_{22} \neq 0$ and $a_{11} + s_2$ is scalar. For an admissible line $a_{13} + a_{23} = 0$. Two cases can occur - (1) $a_{13} = a_{23} = 0$, or (2) a_{13} and a_{23} are not zero.

(1) If $a_{13} = a_{23} = 0$, then $a_{33} \neq 0$.

$x + y + z = 0$ maps into

$x'(a_{11} + s_2 + a_{31}) + y' a_{22} + z' a_{33} = 0$ where $a_{22} \neq 0$ and $a_{33} \neq 0$. For an admissible line $a_{11} + s_2 + a_{31}$

is scalar. i.e. $a_{11} + a_{31}$ is scalar.

Now, $x + y + z2 = 0$ maps into

$x'(a_{11} + s_2 + 2a_{31}) + y'a_{22} + z'a_{33} = 0$ where $a_{22} \neq 0$ and $a_{33} \neq 0$. For an admissible line $a_{11} + s_2 + 2a_{31}$ is scalar. i.e. $a_{11} + 2a_{31}$ is scalar.

Since $a_{11} + 2a_{31}$ is scalar and $a_{11} + a_{31}$ is scalar a_{31} is scalar which is contrary to the supposition.

(2) If $a_{13} \neq 0$ and $a_{23} \neq 0$, then $x + y2 = 0$ maps into $x'(a_{11} + 2s_2) + 2y'a_{22} + z'(a_{13} + 2a_{23}) = 0$ where $a_{22} \neq 0$, and $a_{11} + 2s_2$ is non scalar. Since $a_{13} + a_{23} = 0$, then $a_{13} + 2a_{23} \neq 0$. Hence the line $x + y + z = 0$ does not have a line as its image.

Hence (i) does not lead to a collineation.

Similarly by using (ii) instead of (i) the same result is obtained.

(iii) If $a_{12} = a_{33} = 0$, then

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{13} \\ s_2 & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$x + z = 0$ maps into $x'(a_{11} + a_{31}) + y'a_{32} + z'a_{23} = 0$.

For an admissible line - (1) $a_{32} = 0$, (2) $a_{13} = 0$, or (3) $a_{11} + a_{31}$ is scalar.

(1) If $a_{32} = 0$, then $a_{22} \neq 0$, and

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{13} \\ s_2 & a_{22} & a_{23} \\ a_{31} & 0 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$x + y = 0$ maps into $x'(a_{11} + s_2) + y'a_{22} + z'(a_{13} + a_{23}) = 0$ where $a_{22} \neq 0$ and $a_{11} + s_2$ is non scalar. For an admissible line $a_{13} + a_{23} = 0$. Two cases can occur- ($\overline{\text{I}}$) $a_{13} = a_{23} = 0$, or ($\overline{\text{II}}$) a_{13} and a_{23} are non scalars.

($\overline{\text{I}}$) If $a_{13} = a_{23} = 0$, no collineation exists.

($\overline{\text{II}}$) If $a_{13} \neq 0$, and $a_{23} \neq 0$, then $x + y2 = 0$ maps into $x'(a_{11} + 2s_2) + y'a_{22} + z'(a_{13} + 2a_{23}) = 0$ where $a_{22} \neq 0$, and $a_{11} + 2s_2$ is non scalar. Since $a_{13} + a_{23} = 0$, then $a_{13} + 2a_{23} \neq 0$. Hence the image is not a line.

(2) If $a_{13} = 0$, then $a_{23} \neq 0$, and

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ s_2 & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$x+y=0$ maps into $x'(a_{11}+s_2)+y'a_{22}+z'a_{23}=0$ where $a_{11}+s_2$ is non scalar and $a_{23}\neq 0$. For an admissible line $a_{22}=0$ and hence $a_{32}\neq 0$.

$x+y+z=0$ maps into

$x'(a_{11}+s_2+a_{31})+y'a_{32}+z'a_{23}=0$ where $a_{32}\neq 0$ and $a_{23}\neq 0$. For an admissible line $a_{11}+s_2+a_{31}$ is scalar. i.e. $a_{11}+a_{31}$ is scalar.

Now, $x+y+z2=0$ maps into

$x'(a_{11}+s_2+2a_{31})+2y'a_{32}+z'a_{23}=0$ where $a_{32}\neq 0$ and $a_{23}\neq 0$. For an admissible line $a_{11}+s_2+2a_{31}$ is a scalar. i.e. $a_{11}+2a_{31}$ is scalar. Since $a_{11}+a_{31}$ is scalar and $a_{11}+2a_{31}$ is scalar, then a_{31} is scalar contrary to supposition.

(3) If $a_{11}+a_{31}$ is scalar, then $x+z2=0$ maps into $x'(a_{11}+2a_{31})+2y'a_{32}+za_{13}=0$. For an admissible line there are three possibilities - ($\overline{\text{i}}$) $a_{32}=0$, ($\overline{\text{ii}}$) $a_{13}=0$, or ($\overline{\text{iii}}$) $a_{11}+2a_{31}$ is scalar.

($\overline{\text{i}}$) If $a_{32}=0$, the argument is the same as that of part (1) section (ii).

($\overline{\text{ii}}$) If $a_{13}=0$, the argument is the same as that of part (2) section (iii).

($\overline{\text{iii}}$) If $a_{11}+2a_{31}$ is scalar then since $a_{11}+a_{31}$ is scalar, then a_{31} is scalar contrary to the assumption.

Therefore (iii) does not lead to a collineation. Similarly, by using (iv) instead of (iii) the same result is obtained.

Therefore Case V1 does not lead to a collineation.

<u>Case V11.</u>

Let $a_{31} = s_3$ where s_3 is a scalar. Suppose a_{11} and a_{21} are non scalars.

Consider

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ s_3 & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$x = 0$ maps into $x'a_{11} + y'a_{12} + z'a_{13} = 0$. For an admissible line $a_{12} = 0$, or $a_{13} = 0$.

$y = 0$ maps into $x'a_{21} + y'a_{22} + z'a_{23} = 0$. For an admissible line $a_{22} = 0$, or $a_{23} = 0$. Four cases occur -

(i) $a_{12} = a_{22} = 0$

(ii) $a_{13} = a_{23} = 0$

(iii) $a_{12} = a_{23} = 0$

or (iv) $a_{13} = a_{22} = 0$.

Similar arguments hold for these four cases as held for those of Case Vl. Likewise, Case Vll does not lead to a collineation.

Case Vlll.

Suppose a_{11}, a_{21}, a_{31} are non scalars. Consider

$$\rho\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$x = 0$ maps into $x'a_{11} + y'a_{12} + z'a_{13} = 0$.

For an admissible line $a_{12} = 0$ or $a_{13} = 0$.

$y = 0$ maps into $x'a_{21} + y'a_{22} + z'a_{23} = 0$.

For an admissible line $a_{22} = 0$ or $a_{23} = 0$.

$z = 0$ maps into $x'a_{31} + y'a_{32} + z'a_{33} = 0$.

For an admissible line $a_{32} = 0$ or $a_{33} = 0$.

The following two cases can arise:

Case 1. a) $a_{12} = a_{22} = a_{33} = 0$

b) $a_{12} = a_{23} = a_{33} = 0$

or c) $a_{13} = a_{22} = a_{33} = 0$.

Case 2. a) $a_{13} = a_{23} = a_{32} = 0$

b) $a_{13} = a_{22} = a_{32} = 0$

or c) $a_{12} = a_{23} = a_{32} = 0$.

Case 1.

a) If $a_{12} = a_{22} = a_{33} = 0$, then $a_{32} \neq 0$ and

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$y + z = 0$ maps into $x'(a_{21} + a_{31}) + y'a_{32} + z'a_{23} = 0$ where $a_{32} \neq 0$. For an admissible line (I) $a_{23} = 0$, or (II) $a_{21} + a_{31}$ is scalar.

(I) If $a_{23} = 0$, then $a_{13} \neq 0$ and

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$x + y + z = 0$ maps into

$x'(a_{11} + a_{21} + a_{31}) + y'a_{32} + z'a_{12} = 0$ where $a_{32} \neq 0$, $a_{13} \neq 0$. For an admissible line $a_{11} + a_{21} + a_{31}$ is a scalar. Now $x + y2 + z = 0$ maps into

$x'(a_{11} + 2a_{21} + a_{31}) + y'a_{32} + z'a_{13} = 0$ where $a_{32} \neq 0$, $a_{13} \neq 0$. For an admissible line $a_{11} + 2a_{21} + a_{31}$ is a scalar. Since $a_{11} + a_{21} + a_{31}$ is scalar and

$a_{11} + 2a_{21} + a_{31}$ is scalar, then a_{21} is scalar contrary to assumption.

(ii) If $a_{21} + a_{31}$ is scalar, $y + z2 = 0$ maps into $x'(a_{21} + 2a_{31}) + 2y'a_{32} + z'a_{23} = 0$. For an admissible line (1) $a_{23} = 0$, or (2) $a_{21} + 2a_{31}$ is scalar.

(1) If $a_{23} = 0$, the discussion of part a) section (I) is repeated.

(2) If $a_{21} + 2a_{31}$ is scalar, then, since $a_{21} + a_{31}$ is scalar, a_{31} is scalar contrary to supposition.

b) If $a_{12} = a_{23} = a_{33} = 0$, then $a_{13} \neq 0$, and

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$x + y = 0$ maps into $x'(a_{11} + a_{21}) + y'a_{22} + z'a_{13} = 0$ where $a_{13} \neq 0$. For an admissible line either (I) $a_{22} = 0$, or (II) $a_{11} + a_{21}$ is scalar.

(I) If $a_{22} = 0$, then $a_{32} \neq 0$.

$x + z = 0$ maps into $x'(a_{11} + a_{31}) + y'a_{32} + z'a_{13} = 0$ where $a_{32} \neq 0$, and $a_{13} \neq 0$.

For an admissible line $a_{11} + a_{31}$ is scalar.

$x2 + z = 0$ maps into

$x'(2a_{11} + a_{31}) + y'a_{32} + 2z'a_{13} = 0$ where $a_{32} \neq 0$, and $a_{13} \neq 0$. For an admissible line $2a_{11} + a_{31}$ is a scalar. Since $a_{11} + a_{32}$ is scalar and $2a_{11} + a_{31}$ is scalar, then a_{11} is scalar contrary to supposition.

(ii) If $a_{11} + a_{21}$ is scalar, $x2 + y = 0$ maps into $x'(a_{11} + 2a_{21}) + y'a_{22} + z'a_{13} = 0$ where $a_{13} \neq 0$. For an admissible line either (1) $a_{22} = 0$, or (2) $2a_{11} + a_{21}$ is scalar.

(1) If $a_{22} = 0$, the discussion of part b) section (I) is repeated.

(2) If $2a_{11} + a_{21}$ is scalar, then, since $a_{11} + a_{21}$ is scalar, a_{11} is scalar contrary to the supposition.

c) If $a_{13} = a_{22} = a_{33} = 0$, then $a_{23} \neq 0$ and

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$x + y = 0$ maps into $x'(a_{11} + a_{21}) + y'a_{12} + z'a_{23} = 0$ where $a_{23} \neq 0$. For an admissible line either (I) $a_{12} = 0$, or

(II) $a_{11} + a_{21}$ is scalar.

(I) If $a_{12} = 0$, then $a_{32} \neq 0$.

$y + z = 0$ maps into $x'(a_{21} + a_{31}) + y'a_{32} + z'a_{23} = 0$, where $a_{32} \neq 0$, and $a_{23} \neq 0$. For an admissible line $a_{21} + a_{31}$ is scalar. Now $y2 + z = 0$ maps into $x'(2a_{21} + a_{31}) + y'a_{32} + 2z'a_{23} = 0$ where $a_{32} \neq 0$ and $a_{23} \neq 0$. For an admissible line $2a_{21} + a_{31}$ is a scalar. Since $a_{21} + a_{31}$ is scalar also, then a_{21} is scalar contrary to supposition.

(II) If $a_{11} + a_{21}$ is scalar, then $x + y2 = 0$ maps into $x'(a_{11} + 2a_{21}) + y'a_{12} + 2z'a_{23} = 0$ where $a_{23} \neq 0$. For an admissible line either (1) $a_{12} = 0$, or (2) $a_{11} + 2a_{21}$ is a scalar.

(1) If $a_{12} = 0$, the discussion of part c) section (I) is repeated.

(2) If $a_{11} + 2a_{21}$ is scalar, then, since $a_{11} + a_{21}$ is scalar, a_{21} is scalar contrary to the supposition.

Hence Case 1. does not lead to a collineation.
Using a similar type of argument, it is found that Case 2. yields the same result.

Hence Case V111 does not lead to a collineation.

Therefore the analytic collineations have the following forms:

$$\rho x = x' + y'a_{12} + z'a_{13}$$
$$\rho y = y'a_{22}$$
$$\rho z = z'a_{33}$$

where $a_{22} \neq 0$, and $a_{33} \neq 0$.

$$\rho x = x' + y'a_{12} + z'a_{13}$$
$$\rho y = z'a_{23}$$
$$\rho z = y'a_{32}$$

where $a_{23} \neq 0$, and $a_{32} \neq 0$.

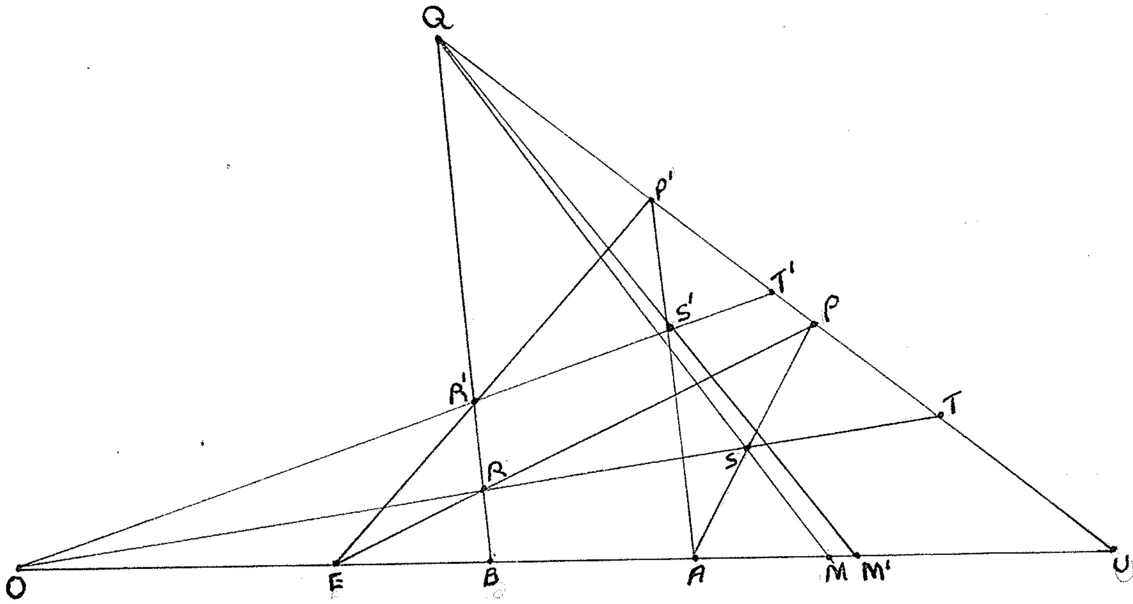
There are thus 9 X 9 X 8 X 8 X 2 analytic collineations, and combining these with the collineation $\rho x_i' = \varphi(x_i)$ there are 9 X 9 X 8 X 8 X 2 X 6 collineations.

More general transformations of the type

$$x' = xa + yb + zc + \sum_i (xa_i + yb_i + zc_i)\alpha_i$$
$$y' = xa' + yb' + zc' + \sum_i (xa_i' + yb_i' + zc_i')\alpha_i$$
$$z' = xa'' + yb'' + zc'' + \sum (xa_i'' + yb_i'' + zc_i'')\alpha_i$$

are not considered since they do not lead to any new collineations in view of the remark made at the end of PART II. There may be other collineations of the geometry but any such other collineations cannot be expressed in the given analytic form.

That in general a projective transformation on a line cannot be extended to a collineation in a plane is shown in the following example.



There exists a projective transformation which sends O into O, A into A, U into U, but which sends M into M'.(loc. cit. PART I. page 8.) Suppose co-ordinates are assigned to these points. Let O,P,R,S,P',R',S',Q,E,A,B,U,M,M' be the points $(2+j,0,1)$, $(0,1,1)$, $(0,j,1)$, $(2+2j,1,0)$, $(1,2,1)$, $(2,1,1)$, $(j,j,1)$, $(1+j,2+j,1)$, $(0,0,1)$, $(j+1,0,1)$, $(1+2j,0,1)$, $(2,0,1)$, $(1,0,1)$, $(2j,0,1)$ respectively. Hence under the existing projective transformation

$$
A \quad \begin{cases}
(x', y', z') \longrightarrow (x, y, z) \\
(2+j, 0, 1) \longrightarrow (2+j, 0, 1) \\
(1+j, 0, 1) \longrightarrow (1+j, 0, 1) \\
(2, 0, 1) \longrightarrow (2, 0, 1) \\
(1, 0, 1) \longrightarrow (2j, 0, 1)
\end{cases}
$$

Consider the analytic collineations

$$
I \quad \begin{cases}
\rho x' = x + yA + zB \\
\rho y' = yC \\
\rho z' = zD
\end{cases}
$$

$$
II \quad \begin{cases}
\rho x' = x + yA + zB \\
\rho y' = zC \\
\rho z' = yD
\end{cases}
$$

Upon substituting the co-ordinates of the points in (A) into (I), the following equations are obtained

$$2 + j = D^{-1}(2 + j + B) \qquad (1)$$

$$1 + j = D^{-1}(1 + j + B) \qquad (2)$$

$$2 = D^{-1}(2 + B) \qquad (3)$$

$$1 = D^{-1}(2j + B) \qquad (4)$$

Therefore from (3) and (4)

$$D = 2 + j, \quad B = 2 + 2j.$$

For these values of D and B, is $1 + j + B = D(1 + j)$?

Substituting for D and B in the latter equation, it is found that

$$1 + j + 2 + 2j = (2 + j)(1 + j) .$$

Hence there exists no transformation of type I, and by a similar argument no transformation of type II which will extend a projective transformation on a line to a collineation in the plane.

It is conjectured here that there are no collineations not of the forms

$$\begin{aligned} x' &= \varphi(x) + \varphi(y)A + \varphi(z)B \\ y' &= \varphi(y)C \\ z' &= \varphi(z)D \end{aligned}$$

or

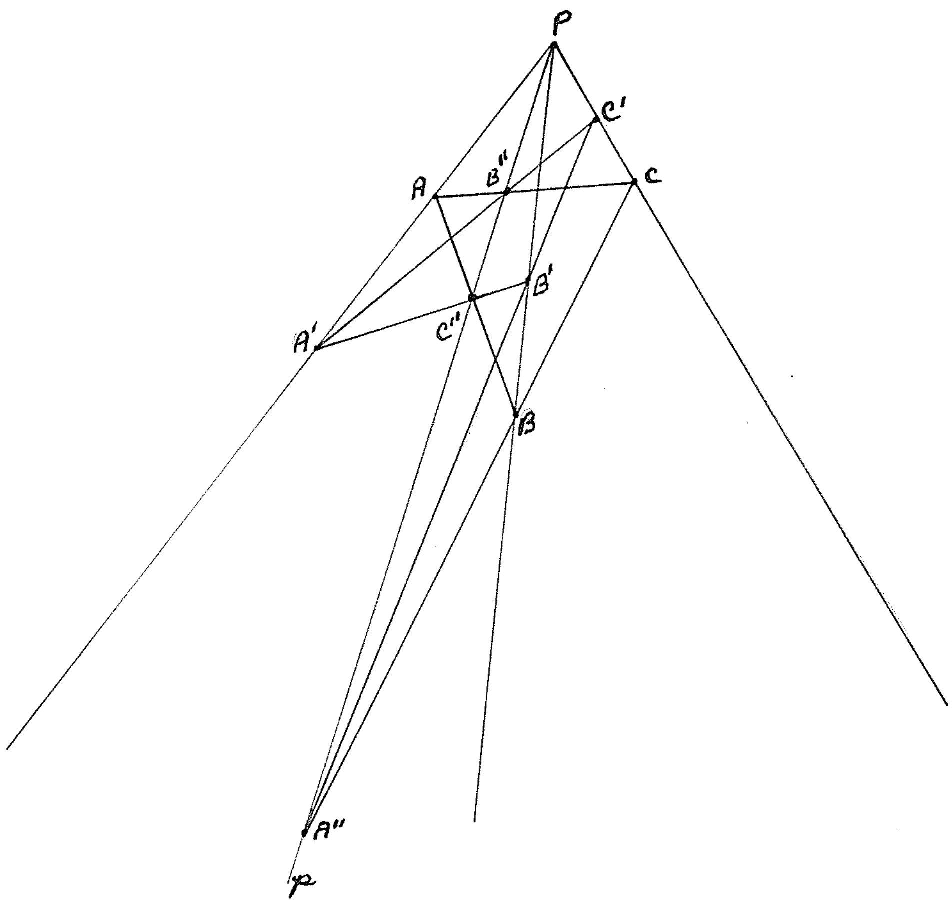
$$\begin{aligned} x' &= \varphi(x) + \varphi(y)A + \varphi(z)B \\ y' &= \varphi(z)C \\ z' &= \varphi(y)D \end{aligned}$$

but a proof does not seem to be easy to obtain.

The relationship between the little Desargues's theorem and collineations is given by the following theorem:

Theorem 1. Let P be any point on a line p and let A and A' be two points collinear with P but not on p. A necessary and sufficient condition that there is a collineation which keeps every line through P fixed and which keeps every point on p fixed, and which maps A into A' is that if ABC and A'B'C' are any two triangles centrally perspective from P, and if two pairs of corresponding sides of these triangles meet on p then so does the third pair of corresponding sides.

Proof: Let AC meet A'C' at B" on p, and AB meet A'B' at C" on p.



Necessity -

Suppose that BC meets B'C' at A" on p. It is now shown that the collineation is well defined. Let the image of

B in the collineation be B^*. Since A maps into A', and B maps into B^*, the line AB maps into $A'B^*$. Since C" is on AB and C" is fixed, C" is on $A'B^*$. Furthermore, B^*B passes through P. Therefore B^* is on BP and on A'C". Hence $B^* = B'$. In the same way, using the mapping A into A', it follows that C maps into C', and using the mapping B into B' it also follows that C maps into C'. Hence the image of C is uniquely determined by any pair of points A A'. That collinear points have collinear images is obvious.

Sufficiency -

Assume that the collineation exists. Let BC meet p at A". From the mapping A into A', by the same argument as in the case of necessity, B maps into B'. From the mapping A into A', it again follows that C maps into C'. But from the mapping B into B' and the mapping C into C', it follows that the line BC maps into the line B'C'. But A" is on BC and is fixed. Hence A" is on B'C'.

An immediate corollary is

<u>Theorem 1.1</u>. Let P be any point in the plane. A necessary and sufficient condition for the little Desargues's theorem to hold for all pairs of triangles centrally perspective from

P and for which two pairs of corresponding sides meet in points collinear with P, is that for any line p through P and any points B and B' collinear with P but not on p the collineation which maps B into B' and which keeps every point on p fixed and every line through P fixed, exists.

Theorem 2. The transformations

$$\rho x' = x + y\alpha + z\beta$$
$$\rho y' = y$$
$$\rho z' = z$$

are all possible collineations which satisfy the conditions of the above theorem for the point P with co-ordinates (1,0,0).

Proof:

Let P be the centre of the elation with axis p and such that B maps into B'. Consider the transformation

$$\rho x' = x + yC + zD$$
$$\rho y' = y$$
$$\rho z' = z \, .$$

The most general line through P has the equation

$$y + z\beta = 0.$$

Hence under the transformation

$$0 = y + z\beta \longrightarrow \rho(y' + z'\beta) = 0$$

and this transformation keeps every line through P fixed.

The axis is $y + zDC^{-1} = 0$ if $C \neq 0$, and if $C = 0$ it is the line $z = 0$. Let B have co-ordinates (1,1,1) and B' have co-ordinates $(\alpha, 1, 1)$ where $\alpha \neq 1$. Hence B,B', and P are on the line which has the equation $y + z2 = 0$. Since (1,1,1) maps into $(\alpha, 1, 1)$, then $1 + C + D = \alpha$. Let $DC^{-1} = \gamma$ where $\gamma \neq 2$. Then $D = \gamma C$, and $1 + C + D = 1 + C + \gamma C = \alpha$. Hence $C + \gamma C = \alpha - 1$ provided $\gamma \neq 2$. Therefore any line with the equation $y + z\gamma = 0$ for $\gamma \neq 2$ can be the axis and B can be made to map into any point B' simply by choosing $D = \gamma C$ and by taking $C + \gamma C = \alpha - 1$.

Now, if $y + z2 = 0$ is the axis of the elation then

$$DC^{-1} = 2.$$

$$\text{i.e.} \quad D = 2C.$$

Hence the transformation becomes

$$\text{(i)} \quad \begin{cases} \rho x' = x + yC + z2C \\ \rho y' = y \\ \rho z' = z \end{cases}$$

Let N be any point not on $y + z2 = 0$. Take the co-ordinates of N to be (0,1,0) and the co-ordinates of N' to be (k,1,0). It is now necessary to choose k in order that N map into N'. Substituting the co-ordinates of N and N' in (i)

$$\rho k = C$$

$$\text{and} \quad \rho = 1.$$

Hence k = C and (i) becomes

$$\rho x' = x + yk + z(2k)$$
$$\rho y' = y$$
$$\rho z' = z$$

This completes the proof of the theorem.

As a corollary we have

Theorem 2.1. The little Desargues's theorem is true for all pairs of triangles centrally in perspective from P with co-ordinates (1,0,0) and for which two pairs of corresponding sides meet in points collinear with P.

Theorem 3. If the little Desargues's theorem fails for a pair of triangles, then it also fails for the images of these triangles in any collineation.

Proof:

The proof is obvious from the definition of a collineation.

DEFINITION: Points A and B are said to be conjugate if there is a collineation which maps A into B.

Theorem 4. Under the group of analytic collineations the following are the sets of conjugate points:

$S_1 = (1,0,0)$

$S_2 = (\alpha,1,0)$ and $(\beta,0,1)$ where α and β are arbitrary

and $S_3 = (\gamma,\alpha,1)$ where γ is arbitrary and α is not zero.

Proof:

Consider the transformations

$$\text{I} \quad \begin{cases} \rho x' = x + yA + zB \\ \rho y' = yC \\ \rho z' = zD \end{cases}$$

and

$$\text{II} \quad \begin{cases} \rho x' = x + yA + zB \\ \rho y' = zC \\ \rho z' = yD \end{cases} .$$

Transformations I and II map $(1,0,0)$ into $(1,0,0)$.

Hence $(1,0,0)$ is self conjugate.

Transformation I maps $(0,1,0)$ into $(C^{-1}A,1,0)$.

Hence by choosing $A = \alpha$ and $C = 1$, $(0,1,0)$ maps into $(\alpha,1,0)$.

Transformation II maps $(0,1,0)$ into $(D^{-1}A,0,1)$.

Also, by the choice $A = \beta$, $D = 1$, $(0,1,0)$ maps into $(\beta,0,1)$.

Hence points with co-ordinates $(\alpha,1,0)$ and $(\beta,0,1)$ are conjugate points. There are eighteen such points.

Transformation I maps $(1,1,1)$ into $(D^{-1}\{1 + A + B\}, D^{-1}C, 1)$

and transformation II maps $(1,1,1)$ into the same point.

By the choice $D = 1, C = \alpha, B = 0, A = \gamma - 1$, $(1,1,1)$ maps into $(\gamma, \alpha, 1)$. Hence points with co-ordinates $(\gamma, \alpha, 1)$ where $\alpha \neq 0$ are conjugate. There are seventy-two such points. The elements of S_2 and S_3 do not form a single conjugate set since there are ninety of them, and ninety does not divide the order of the group. Hence S_2 and S_3 are distinct conjugate sets. A more direct proof could be obtained by showing that neither of the transformations I or II map $(0,1,0)$ into $(1,1,1)$.

<u>Theorem 5</u>. Theorem 2.1. is false for any other point.

Proof:

That the theorem is false for a point of S_3 was shown in a previous example.(loc. cit. PART II. page 20.)

For a point of S_2, the transformation

$$\begin{aligned} \rho x' &= x \\ \rho y' &= y + z \\ \rho z' &= z \end{aligned}$$

is not a collineation. The transformation keeps every line through $(0,1,0)$ fixed, and every point on $z = 0$ fixed. Also it maps $(1,1,1)$ into $(1,2,1)$ which are collinear with $(0,1,0)$. Hence there is no collineation which keeps every line through $(0,1,0)$ fixed and every point on $z = 0$ fixed, and which maps $(1,1,1)$ into $(1,2,1)$.

PART IV.

SOME PROPERTIES OF THE ANALYTIC COLLINEATION GROUP

Let G be the set of all transformations of the form

$$\begin{aligned} \rho x' &= x + yA + zB \\ \rho y' &= yC \\ \rho z' &= zD \end{aligned}$$

and

$$\begin{aligned} \rho x' &= x + yA + zB \\ \rho y' &= zD \\ \rho z' &= yC \end{aligned}$$

where $CD \neq 0$, and, A and B are arbitrary.

Now, this group contains two subgroups H and K where H is the set of all transformations of the form

$$\begin{aligned} \rho x' &= x + yA + zB \\ \rho y' &= y \\ \rho z' &= z \end{aligned}$$

and K is the set of all transformations of either of the forms

$$\begin{aligned} \rho x' &= x \\ \rho y' &= yE \\ \rho z' &= zF \; ; \end{aligned}$$

$$\rho x' = x$$
$$\rho y' = zF$$
$$\rho z' = yE$$

where $EF \neq 0$.

Clearly H and K are subgroups of order 81 and 128 respectively.

The group G has the following properties:

1. a) $G = KH$, and, b) H and K are normal subgroups of G.

2. a) H is an abelian subgroup of G and b) H is the direct product of four groups of order three.

3. a) A group K is generated by four elements a,b,c,d subject to these conditions

$a^8 = b^8 = d^2 = I$;

$a^2 = b^6$, $aba = b^3$;

$c^4 = b^4$, $bcb = aca$, $cac = b^7$, $c^2b = ac^2$;

$dcd = c^3$, $dad = a^3$, $dbd = b^3$;

and b) these relationships uniquely determine K apart from isomorphisms.

<u>Proof of 1.</u>

a) Take g_1 to be $\begin{pmatrix} 1 & \alpha & \beta \\ 0 & \gamma & 0 \\ 0 & 0 & \delta \end{pmatrix}$ in G.

Then $\begin{pmatrix} 1 & \alpha & \beta \\ 0 & \gamma & 0 \\ 0 & 0 & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Hence g_1 can be expressed as an element of K[4] and an element of H.

Let g_2 be $\begin{pmatrix} 1 & \alpha_1 & \beta_1 \\ 0 & 0 & \delta_1 \\ 0 & \gamma_1 & 0 \end{pmatrix}$.

Then $\begin{pmatrix} 1 & \alpha_1 & \beta_1 \\ 0 & 0 & \delta_1 \\ 0 & \gamma_1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \delta_1 \\ 0 & \gamma_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \alpha_1 & \beta_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Hence g_2 can be expressed as an element of K and an element of H.

Therefore it follows that $G = KH$.

---

4. Analytically, collineations may be regarded as aspects of the theory of matrices. The collineation

$$x'_i = \sum_{j=1}^{3} a_{ij} x_j \qquad (i = 1,2,3)$$

may be conveniently represented by the matrix A of the coefficients a_{ij}. The product of two collineations $A = (a_{ij})$ and $B = (b_{ij})$ is then given by the product of their matrices.

b) Let g, h, and k be elements of G, H, and K respectively.

Take g_1 to be $\begin{pmatrix} 1 & \alpha & \beta \\ 0 & \gamma & 0 \\ 0 & 0 & \delta \end{pmatrix}$.

Hence g_1^{-1} is $\begin{pmatrix} 1 & 2\alpha\gamma^{-1} & 2\beta\delta^{-1} \\ 0 & \gamma^{-1} & 0 \\ 0 & 0 & \delta^{-1} \end{pmatrix}$.

Take g_2 to be $\begin{pmatrix} 1 & \alpha & \beta \\ 0 & 0 & \gamma^* \\ 0 & \delta^* & 0 \end{pmatrix}$.

Hence g_2^{-1} is $\begin{pmatrix} 1 & 2\beta\gamma^{*-1} & 2\alpha\delta^{*-1} \\ 0 & 0 & \delta^{*-1} \\ 0 & \gamma^{*-1} & 0 \end{pmatrix}$.

Let h be $\begin{pmatrix} 1 & A & B \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Let k_1 be $\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{pmatrix}$

and k_2 be $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & c_1 \\ 0 & d_1 & 0 \end{pmatrix}$.

Hence $g_1^{-1} h g_1$ and $g_2^{-1} h g_2$ are elements of H, and $g_1^{-1} k_1 g_1$, $g_1^{-1} k_2 g_1$, $g_2^{-1} k_1 g_2$, $g_2^{-1} k_2 g_2$ are elements of K. These follow from a direct calculation. Therefore H and K are normal subgroups of G.

<u>Proof of 2.</u>

a) Let h_1 and h_2 be elements of H.

Take h_1 to be $\begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

and h_2 to be $\begin{pmatrix} 1 & \alpha^* & \beta^* \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Then $h_1 h_2$ is $\begin{pmatrix} 1 & \alpha^* + \alpha & \beta^* + \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $h_2 h_1$ is $\begin{pmatrix} 1 & \alpha + \alpha^* & \beta + \beta^* \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Hence H is an abelian subgroup.

b) Let h_1, h_2, h_3, h_4 be elements of H.

Take h_1 to be $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

then h_1^2 is $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$;

h_2 to be $\begin{pmatrix} 1 & j & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ then h_2^2 is $\begin{pmatrix} 1 & 2j & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$;

h_3 to be $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ then h_3^2 is $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$;

h_4 to be $\begin{pmatrix} 1 & 0 & j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ then $h_4^2 = \begin{pmatrix} 1 & 0 & 2j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Hence $h_i^3 = I$ where $i = 1,2,3,4$.

Take H_1 to be a subgroup of H consisting of the elements I, h_1, h_1^2 ; take H_2 to be a subgroup consisting of the elements I, h_2, h_2^2 ; take H_3 to be a subgroup consisting of the elements I, h_3, h_3^2; take H_4 to be a subgroup consisting of the elements I, h_4, h_4^2.

Then if $h = \begin{pmatrix} 1 & a+bj & c+dj \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

where a,b,c,d are all either 0,1,or 2, $h = h_1^a h_2^b h_3^c h_4^d$ and this expression for h is uniquely determined.

Proof of 3.

a) To show that K is so generated it is easily verified that the elements

$a = \begin{pmatrix} 0 & 1+j \\ 2+j & 0 \end{pmatrix}$ $b = \begin{pmatrix} 0 & 1+j \\ 1+2j & 0 \end{pmatrix}$

$c = \begin{pmatrix} 0 & j \\ 2+2j & 0 \end{pmatrix}$ $d = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

satisfy these equations and that any element in K can be expressed as a product using only the elements a,b,c,d.

b) Let H_1 be a group generated by b, then $H_1 + H_1 a$ is the group generated by a and b. This follows by a direct computation from the relationships

$$a^8 = b^8 = I, \quad a^2 = b^6, \quad aba = b^3 .$$

Furthermore $H_1 + H_1 a + H_1 c + H_1 ac + H_1 c^2 + H_1 ac^2 + H_1 ca + H_1 aca$ is the group generated by a,b, and c. This follows by a direct computation from the relationships

$$a^8 = b^8 = I, \; a^2 = b^6, \; aba = b^3, \; c^4 = b^4,$$
$$bcb = aca, \; cac = b^7, c^2 b = ac^2.$$

Finally, the group generated by a,b,c, and d is $H_1 + H_1 a + H_1 c + H_1 ac + H_1 c^2 + H_1 ac^2 + H_1 ca + H_1 aca + H_1 d + H_1 ad + H_1 cd + H_1 acd + H_1 c^2 d + H_1 ac^2 d + H_1 cad + H_1 acad$. This follows by a direct computation from the relationships

$$a^8 = b^8 = d^2 = I, \; a^2 = b^6, \; aba = b^3, \; c^4 = b^4,$$
$$bcb = aca, \; cac = b^7, \; c^2 b = ac^2, \; dcd = c^3, \; dad = a^3,$$
$$dbd = b^3.$$

Hence the given relationships determine K.

Besides the analytic collineations it is clear that the transformations $x' = \varphi(x)$, $y' = \varphi(y)$, $z' = \varphi(z)$ are also collineations since the line $x' + y'A + z'B = 0$ has as its image $\varphi(x) + \varphi(y)A + \varphi(z)B = 0$. If $A = \varphi(A')$ and $B = \varphi(B')$ this latter equation may be written as $\varphi(x + yA' + zB') = 0$ and since φ is an automorphism this implies $x + yA' + zB' = 0$. Hence lines of type $x' + y'A + z'B = 0$ map into lines. Similarly lines of types $y' + z'B = 0$, and $z' = 0$ map into lines.

The group of all automorphisms of the near-field is now studied. Since the near-field is generated by j the automorphism is completely determined when the image of j is given. Let φ map $j \longrightarrow \varphi(j) = \alpha + \beta j$ where α and β are scalars. In the first place $\beta \neq 0$ since in that case the mapping is not one to one. If $\alpha = 0$, either of the mappings $j \longrightarrow j$, and $j \longrightarrow 2j$ yield automorphisms. If $\alpha \neq 0$, then, since $\varphi(j) = \alpha + \beta j$, $-1 = \varphi(-1) = \varphi(j^2) = \varphi(j)\varphi(j) = (\alpha + \beta j)(\alpha + \beta j) = \alpha^2 + \beta^2$. Now this equation is always satisfied whenever α and β are scalars, both distinct from zero. Hence the following mappings are automorphisms

$I : j \longrightarrow j$

$\psi : j \longrightarrow 2j$

$\tau : j \longrightarrow 1 + j$

$\lambda : j \longrightarrow 2 + j$

$\mu : j \longrightarrow 1 + 2j$

$\nu : j \longrightarrow 2 + 2j$

These form a group of order six which is isomorphic to the symmetric group on three elements under the mapping

$$\begin{aligned} I &\longrightarrow I \\ \psi &\longrightarrow (12) \\ \mu &\longrightarrow (13) \\ \gamma &\longrightarrow (23) \\ \lambda &\longrightarrow (123) \\ \tau &\longrightarrow (132) \end{aligned}$$

These combined with the analytic collineations yield a group of order (81 X 8 X 8 X 2) X 6. It is verifiable by a direct computation that the analytic collineations are a normal subgroup of this larger collineation group.

BIBLIOGRAPHY

1 O. Veblen and J.H.M. Wedderburn, Non-Desarguesian and non-Pascalian geometries, Trans. Amer. Math. Soc., vol. 8, 1907, pp. 379-383.

2 O. Veblen and J.H.M. Wedderburn, Finite projective geometries, Trans. Amer. Math. Soc., vol. 7, 1906, pp. 241-259.

3 R.D. Carmichael, Groups of Finite Order, Ginn and Co., 1937.

4 Marshall Hall, Projective planes, Trans. Amer. Math. Soc., vol. 54, 1943, pp. 229-277. Correction, vol. 65, 1949, pp. 473-474.

5 Marshall Hall, Uniqueness of the projective plane with 57 points, Proc. Amer. Math. Soc., vol. 4, 1953, pp. 912-916.

6 R. Moufang, Alternativkorper und der Satz von vollstamdegen Viersert (D_q), Abh. Math. Sem. Hamburg University, vol. 9, 1933, pp. 207-222.

7 H.S.M. Coxeter, Non-Euclidean Geometry, The University of Toronto Press, 1942.

8 O. Veblen and J.W. Young, Projective Geometry, vol. 1, Ginn and Co., 1910.

9 N.S. Mendelsohn, A group theoretic characterization of the general projective collineation group, Trans. Royal Soc. of Can., vol. XL, 1946, pp. 37-58.

10 G. De B. Robinson, The Foundations of Geometry, The University of Toronto Press, 1940.

11 R. Baer, Homogeneity of projective planes, Bull. Amer. Math. Soc., vol. 51, 1945, pp. 903-906.

12 R. Bruck and H. Ryser, The nonexistence of certain finite projective planes, Can. J. Math., vol. 1, 1949, pp. 88-93.