CONCERNING BOUNDARY VALUE PROBLEMS OF CLASS M_2 AND M_3

A Thesis

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ABSTRACT

In this thesis, boundary value problems involving the solution of the differential equation $y'' = f(x,y)$ are discussed by setting up finite difference schemes. An account of the existence and uniqueness of the solutions as well as an error analysis are also given. The basic methods are improved by varying the meshsize and modifying the finite difference equations at the boundary. Some examples are considered for numerical evaluation to illustrate the methods. The numerical solutions are also compared with those done by earlier workers using similar techniques.

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INTRODUCTION

Several authors have considered the numerical solution of two point boundary value problems of the form

$$y'' = f(x,y), y(a) = A, \ y(b) = B, \ -\infty < a < b < \infty,$$

where A and B are arbitrary constants. The existence and uniqueness of solutions along with finite difference methods of finding numerical solutions are discussed in detail by Henrici [1] An extension of the above problem with infinity as a boundary point is considered by Froese [2]. Varga [3] has obtained error bounds when $f(x,y)$ is continuous and linear in y and $f_y(x,y)$ being non-negative. Lees [4] continues the work of Varga by weakening the conditions on $f(x,y)$ and presents some results for the non-linear case. Bramble and Hubbard [5] discuss high order difference equations to obtain accurate error estimation. Usmani [6] uses similar techniques with more accurate results. Keller [7] discusses a more general case of the two point second order boundary value problem where the function f contains y' also explicitly.

The object of the present thesis is to consider the general problem

$$y'' = f(x,y), \alpha y(a) + \beta y'(a) = A, \ \gamma y(b) + \delta y'(b) = B,$$

where α, β, γ, δ are constants with certain restrictions. Two basic existence theorems are proved. An analysis of error and numerical results are given for linear and nonlinear cases. Also certain modifications are suggested for the difference equations at the boundary points. All calculations were made on an IBM 360/75 computer with double-precision arithmetic. The programs were written in FORTRAN using WATFOR.

CHAPTER I

PRELIMINARIES

1. Boundary Value Problems

The nth order differential equation in $y(x)$ is given by

$$F(x,y(x),y'(x),y''(x),\ldots\ldots,y^n(x)) = 0. \tag{1.1}$$

The equation has a real function $\phi(x)$ as a general solution when $y(x)$ and its derivatives are replaced by $\phi(x)$ and its derivatives respectively make (1.1) an identity. $\phi(x)$ normally depends on n parameters which are determined by prescribing combinations of $\phi(x)$ and its first n-1 derivatives. The boundary conditions can have the form

$$\psi_\nu(\phi_{x_1},\phi'_{x_1}, \ldots, \phi^{n-1}_{x_1}, \phi_{x_2},\phi'_{x_2}, \ldots, \phi^{n-1}_{x_2},\ldots,\phi_{x_k},\phi'_{x_k},\ldots, \phi^{n-1}_{x_k}) = 0,$$

$$\nu = 0,\ 1,\ \ldots,\ n-1, \tag{1.2}$$

where x_ν are prescribed points which may include $\pm\infty$ and

$$\phi^m_{x_\nu} = \left(\frac{\partial^m\phi}{\partial x^m}\right)_{x\,=\,x_\nu} .$$

In what follows $y(x)$ will be used in the sense of $\phi(x)$. An example of a second order boundary value problem will be

$$y'' = f(x,y), \quad \alpha y(a) + \beta y'(a) = A, \quad \gamma y(b) + \delta y'(b) = B, \tag{1.3}$$

where α, β, γ, δ are constants.

2. Some Results for Initial Value Problems

For the initial value problem of first order

$$y' = f(x,y), \; y(a) = A, \tag{1.4}$$

where A is given, we have the following existence and uniqueness theorems [7]. The existence, uniqueness and continuity properties of the solution depend on continuity or smoothness properties of $f(x,y)$ in the neighbourhood of (a,A).

Theorem 1.1

Let $f(x,y)$ be defined and continuous in the strip S where

$$S = \{(x,y) : x\varepsilon[a,b], \; y\varepsilon(-\infty,\infty)\} \tag{1.5}$$

and satisfy a Lipschitz condition. Then there exists exactly one function $y(x)$ which is continuous and differentiable for $x\varepsilon[a,b]$, and satisfying (1.4).

Theorem 1.2

Let $y^*(x,\alpha)$ be the solution of the initial value problem

$$y' = f(x,y), \; y(a) = \alpha \tag{1.6}$$

where f satisfies the conditions stated in Theorem 1. Then $y^*(x,\alpha)$ and its partial derivative with respect to α, denoted by $y^*_{\alpha}(x,\alpha)$, are both continuous functions of x and α. The above theorems can be easily extended for a system of ordinary differential equations of first order.

3. M Classification of Boundary Value Problems

We will mainly be concerned with boundary value problem (b.v.p.) of the form

$$y'' = f(x,y), \tag{1.7}$$

with the boundary conditions

$$\begin{aligned} &\alpha y(a) + \beta y'(a) = A, \\ &\gamma y(b) + \delta y'(b) = B, \\ &-\infty < a < b < \infty, \quad |\alpha|+|\beta| \neq 0, \ |\gamma|+|\delta| \neq 0, \end{aligned} \tag{1.8}$$

where A and B are arbitrary constants and the function f(x,y) is defined and is continuous in the strip S, where S is defined in (1.5). The partial derivative of f(x,y) with respect to y is assumed to be continuous and bounded in S, thus ensuring that f(x,y) satisfies a Lipschitz condition, and also to satisfy

$$f_y(x,y) \geq K \tag{1.9}$$

in the strip S, where K will be defined later.

If, in the above problem, we set

$$\alpha = \gamma = 1, \quad \beta = \delta = 0, \tag{1.10}$$

4. A Finite Difference Scheme for Class M

For the numerical approximation of

$$y'' = f(x,y), \quad a \le x \le b, \tag{1.14}$$

we set up a finite set of grid points $x_0, x_1, \ldots, x_{N+1}$, where

$$x_n = a + nh, \; n = 0, 1, 2, \ldots, N+1,$$

$$x_0 = a, \; x_{N+1} = b \tag{1.15}$$

and

$$h = (b-a)/(N+1),$$

N being an appropriate integer.

Denoting the true solution of the b.v.p. at x_n by $y(x_n)$, a method is designed to find the numbers y_n which approximate closely to $y(x_n)$. A convenient way to obtain such a scheme is to demand that the numbers y_n satisfy a difference equation of the form

$$\sum_{0}^{k} \alpha_i y_{n+i} = h^2 \sum_{i=0}^{k} \beta_i y''_{n+i}, \; n = 0,1,2,\ldots,N-k+1, \tag{1.16}$$

where $y''_i = f(x_i, y_i)$,

$\alpha_k \neq 0$,

$|\beta_0| + |\alpha_0| \neq 0$.

k is called the order of the difference equation and the equation may be normalized by choosing $\alpha_k = 1$.

Equation (1.16) leads to N-k+2 equations involving

$y_1, y_2, \ldots, y_N$, possibly in a nonlinear form. y_0 and y_{N+1} are determined by the boundary conditions. For $k > 2$, we need additional equations. We may assume that the true solution of the b.v.p. to satisfy a difference equation of the form

$$\sum_{i=0}^{k} \alpha_i y(x_{n+i}) = h^2 \sum_{i=0}^{k} \beta_i y''(x_{n+i}) + T_{n+k} \tag{1.17}$$

where T_{n+k} is the truncation error and will be denoted by T.E. Defining the discretisation error e_n by

$$e_n = y(x_n) - y_n \tag{1.18}$$

we have from (1.16) and (1.17)

$$\sum_{i=0}^{k} \alpha_i e_{n+i} = h^2 \sum_{i=0}^{k} \beta_i [y''(x_{n+i}) - y''_{n+i}] + T_{n+k} \tag{1.19}$$

$$= h^2 \sum_{i=0}^{k} \beta_i \, e_{n+i} \, \sigma_{n+i} + T_{n+k}$$

where $\sigma_{n+i} = f_y(x_{n+i}, \xi) \quad \xi \in (y(x_{n+i}), y_{n+i})$

Assuming that $y(x)$ has continuous derivatives of sufficiently higher orders, we associate with (1.16) the operator

$$L[y(x);h] = \sum_{i=0}^{k} \alpha_i y(x+ih) - h^2 \sum_{i=0}^{k} \beta_i y''(x+ih) . \tag{1.20}$$

Expanding (1.20) using Taylor's theorem, we get

$$L[y(x);h] = \sum_{n=0}^{\infty} C_n h^n y^{(n)}(x) , \tag{1.21}$$

where