CENTRAL COLLINEATIONS OF FINITE PROJECTIVE PLANES

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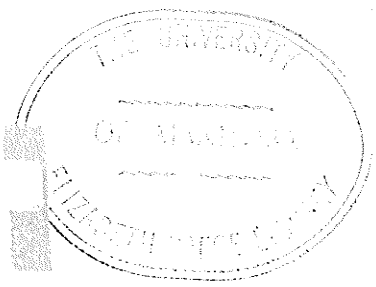

ABSTRACT

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It is well-known that the structure of the finite projective plane is determined to a great extent by the structure of the collineation group of the plane. In this thesis certain assumptions are made concerning the nature of the action of the collineation group considered as a permutation group on the points and lines of the plane. Assumptions are also made concerning the number and nature of the central collineations that occur in the collineation group, and the way in which these assumptions determine the structure of the plane is investigated. The approach used is that employed in recent papers of Piper and Wagner. In order to carry out this investigation, a development of the elementary theory of the finite projective plane and of aspects of the theory of permutation groups is also given.

The writer wishes to express his thanks to Dr. W. J. Jonsson of the Department of Mathematics for his advice and helpfulness, to Mrs. Elaine Halliday for her patience during the typing of this thesis, and to the late Tom Lehrer, one of whose songs provided the basic idea of the thesis.

TABLE OF CONTENTS

INTRODUCTION

The purpose of this thesis is to develop several recent results concerning the structure of the projective plane. It is shown that the structure of the plane can be deduced to a great extent from a knowledge of the structure of the collineation group of the plane, and more particularly from a knowledge of the properties of the central collineations of the plane.

The thesis is divided into five chapters. In the first chapter a number of elementary properties of central collineations are obtained. In the second chapter the method of co-ordinatizing the projective plane by means of a ternary ring is developed, and the important theorem that a projective plane is alternative if and only if its ternary ring is an alternative field is proved. The third chapter is devoted to a study of finite alternative fields, and it is shown that all finite alternative fields are commutative fields. This, coupled with the result of Chapter 2, yields the important result that all finite alternative planes are Desarguesian.

In the fourth chapter a number of combinatorial theorems, many relying heavily on the theory of permutation groups, are proved. In addition, a purely group-

theoretic result (theorem 4.1) is obtained. The results of Chapter 4 are used repeatedly in Chapter 5.

Chapter 5 is essentially a synthesis of the results of several recent papers of Wagner and Piper (10), (11), and (13). Certain conditions imposed on the collineation group of the projective plane are shown to be sufficient to ensure that the plane is Desarguesian; conditions under which the plane is a translation plane, or the dual of a translation plane, are also found. Thus the structure of the projective plane is shown to be determined to a great extent by the properties of the collineation group of the plane considered as a permutation group on the points and lines of the plane.

FOREWORD

In the body of this thesis it is assumed that the reader is familiar with the terminology and notation of projective geometry, and with the basic properties of the projective plane. The purpose of this foreword is to summarize these basic properties and to define notation not defined elsewhere. The results quoted below can be found in Pickert (9) and in Hall (5).

A projective plane π is a triple $(\mathcal{P},\mathcal{L},\varepsilon)$ consisting of a set $\mathcal{P}$ whose elements are called points, a collection $\mathcal{L}$ of distinguished subsets of $\mathcal{P}$, and the set-theoretic membership relation ε relating elements of $\mathcal{P}$ and elements of $\mathcal{L}$. The elements of $\mathcal{L}$ are called lines. If ℓ is a line and P is a point, then "$P\varepsilon\ell$" is defined to mean that P is a member of the distinguished subset ℓ of $\mathcal{P}$. Geometrical language is used throughout; hence "P is on ℓ", "P belongs to ℓ", "ℓ is a line through P", "ℓ contains P", "P is incident with ℓ", and "P is a point of ℓ", are all phrases meaning "$P\varepsilon\ell$". If $P_1,\cdots,P_n$ are all on the same line ℓ, then the points $P_1,\cdots,P_n$ are said to be collinear, and this is symbolized by writing $\equiv P_1,\cdots,P_n$. Similarly, if lines $\ell_1,\cdots,\ell_n$

all pass through the same point P, then the lines $\ell_1, \ldots, \ell_n$ are said to be concurrent. On occasion the symbol ε will be used in its more general sense of denoting set membership. The use of the symbol will always be clear from the context.

A projective plane π obeys the following axioms of incidence :

(1) If $P_1 \varepsilon \mathcal{P}$, $P_2 \varepsilon \mathcal{P}$, $P_1 \neq P_2$, then there exists exactly one line $\ell \varepsilon \mathcal{L}$ such that $P_1 \varepsilon \ell$, $P_2 \varepsilon \ell$.

(2) If $\ell_1 \varepsilon \mathcal{L}$, $\ell_2 \varepsilon \mathcal{L}$, $\ell_1 \neq \ell_2$, then there exists exactly one $P \varepsilon \mathcal{P}$ such that $P \varepsilon \ell_1$, $P \varepsilon \ell_2$.

(3) There exist four distinct points of $\mathcal{P}$, no three of which are collinear.

It immediately follows that there exist four distinct lines of $\mathcal{L}$, no three concurrent. Since a knowledge of two distinct points P_1 and P_2 on a line uniquely determines the line, we shall often denote by P_1P_2 the (unique) line containing both P_1 and P_2. Similarly, if ℓ_1 and ℓ_2 are distinct lines, $\ell_1 \cap \ell_2$ will denote the unique point incident with each.

Suppose that the number of points of a projective plane π is finite (such a plane is called a finite projective plane). Then the following statements are shown to be equivalent:

(1) One line contains exactly (n+1) points.

(2) One point is on exactly (n+1) lines.

(3) Every line contains exactly (n+1) points.

(4) Every point is on exactly (n+1) lines.

(5) There are exactly (n^2+n+1) points in $\mathcal{P}$.

(6) There are exactly (n^2+n+1) lines in $\mathcal{L}$.

These equivalences will be used repeatedly. The order of a finite projective plane π will be said to be n if some line of π contains exactly (n+1) points.

Although the lines of π were defined to be distinguished subsets of the points of π, it is evident from the axioms of incidence that an equivalent characterization of the plane can be obtained by considering the lines to be the primitive elements and defining the points of π to be distinguished subsets of the lines of π; thus a point could be considered to be the set of all lines passing through it. Consequently if the triple $\pi = (\mathcal{P}, \mathcal{L}, \varepsilon)$ is a projective plane, the triple $\pi^* = (\mathcal{L}, \mathcal{P}, \varepsilon^*)$ is also a projective plane where the binary relation ε is defined by

$$\ell \, \varepsilon^* P \iff P \varepsilon \ell \quad \text{for all } P \varepsilon \mathcal{P}, \ \ell \varepsilon \mathcal{L}.$$

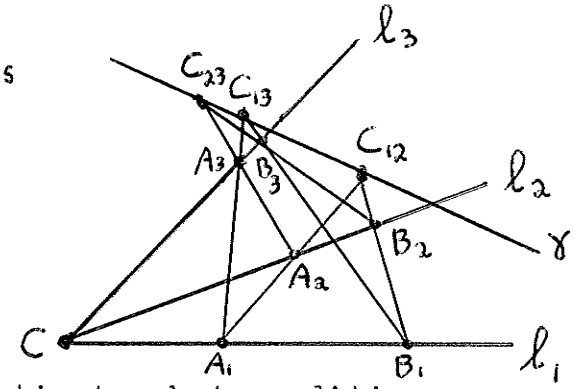
That π^* is indeed a projective plane can be verified by noting that π^* satisfies the axioms of incidence for a projective plane. π^* is called the projective plane dual to π. It is evident that $(\pi^*)^* = \pi$. Thus

every statement about a projective plane π can be "dualized" to a statement about π^* by interchanging the roles of points and lines and replacing ε by ε^*. It follows that if an assertion A is true of a projective plane π, the "dual" assertion A^* will be true of π^*. More generally, if all projective planes satisfying hypotheses H have property K, then all projective planes satisfying the dual hypotheses H^* will have the dual property K^*. This "principle of duality" will be used repeatedly throughout the thesis.

If π is a projective plane and ℓ is a line of π, then by the affine plane π_ℓ we shall mean the projective plane π with the line ℓ and the points thereof deleted. The line ℓ will be called the "line at infinity". Points not on ℓ and lines distinct from ℓ will be called affine points and lines. The concept of the affine plane will be used chiefly to facilitate notation and to aid in the co-ordinatization of the projective plane (see chapter II).

Let C be a point of π and γ a line of π. Let ℓ_1, ℓ_2, ℓ_3 be three arbitrary distinct lines through C (and $\neq \gamma$) and let A_i and B_i be two distinct points of $\ell_i - \{C\}$ $(i = 1,2,3)$. If, for all such A_i, B_i, and ℓ_i, $(A_1A_3 \cap B_1B_3)\varepsilon\gamma$ and $(A_1A_2 \cap B_1B_2)\varepsilon\gamma$ together imply that $(A_2A_3 \cap B_2B_3)\varepsilon\gamma$, then we shall say that Desargues'

(C,γ) theorem holds. If Desargues (C,γ) theorem holds for all points C and lines γ of π, then π will be said to be Desarguesian. The fundamental problem of this thesis will be to investigate what conditions determine the number of point-line pairs (C,γ) for which Desargues' (C,γ) theorem holds in a given projective plane.



The theory of groups, and in particular the theory of permutation groups, is used extensively throughout the thesis. A self-contained development of the theory of permutation groups appears in chapter IV, and several abstract group theoretical results are proved there as well. However, it is assumed that the reader is familiar with elementary abstract group theory, and with the standard notation employed in that subject. The results used can be found, for instance, in Hall (5).

Lemmas and theorems are numbered independently. Thus for example there is both a lemma 4.4 and a theorem 4.4, and these are distinct.

CHAPTER I

ELEMENTARY PROPERTIES OF COLLINEATIONS

In this chapter several elementary lemmas and theorems about collineations will be proved. Continual reference to these will be made throughout the rest of the paper.

Lemma 1.1 (i) The product of two collineations is a collineation.

(ii) The inverse of a collineation is a collineation.

Proof: (i) Let π be a projective plane with a point set $\mathcal{P}$ and a line set $\mathcal{L}$. Let σ and τ be two collineations of π.

Define a mapping $\sigma\tau$ as follows:

$$P \in \mathcal{P} \Longrightarrow P^{(\sigma\tau)} = (P^{\sigma})^{\tau}$$
$$l \in \mathcal{L} \Longrightarrow l^{(\sigma\tau)} = (l^{\sigma})^{\tau}.$$

As $\mathcal{P} \xrightarrow{\sigma} \mathcal{P}$, $\mathcal{L} \xrightarrow{\sigma} \mathcal{L}$ are one-to-one onto mappings and as τ is similarly one-to-one onto, $\sigma\tau$ is a one-to-one onto mapping of $\mathcal{P} \longrightarrow \mathcal{P}$ and $\mathcal{L} \longrightarrow \mathcal{L}$.

To demonstrate that the mapping $\sigma\tau$ preserves incidence, suppose that for $P \in \mathcal{P}$ and $l \in \mathcal{L}$, $P \in l$. Then $P^{\sigma} \in l^{\sigma}$ (as σ is a collineation), and similarly $(P^{\sigma})^{\tau} \in (l^{\sigma})^{\tau}$ (as τ is a collineation). By definition of $\sigma\tau$, this implies that $P^{(\sigma\tau)} \in l^{(\sigma\tau)}$. Hence

$$P \in l \Longrightarrow P^{(\sigma\tau)} \in l^{(\sigma\tau)};$$

thus $\sigma\tau$ preserves incidence and by definition is a collineation.

(ii) Let σ be a collineation of π projective plane π.

Define a mapping $\mathcal{P} \xrightarrow{\sigma^{-1}} \mathcal{P}$ and $\mathcal{L} \xrightarrow{\sigma^{-1}} \mathcal{L}$ by

$$P^{\sigma^{-1}} = Q \iff Q^{\sigma} = P \quad (P, Q \in \mathcal{P})$$

$$\ell^{\sigma^{-1}} = m \iff m^{\sigma} = \ell \quad (\ell, m \in \mathcal{L}).$$

Then σ^{-1} is a one-to-one onto mapping, since σ is. In addition, σ^{-1} preserves incidence; for suppose that it did not. Then there exist $P \in \mathcal{P}$ and $\ell \in \mathcal{L}$ such that

$$P \in \ell \text{ but } P^{\sigma^{-1}} \notin \ell^{\sigma^{-1}}.$$

But as σ is a collineation, it preserves non-incidence; hence

$$(P^{\sigma^{-1}})^{\sigma} \notin (\ell^{\sigma^{-1}})^{\sigma};$$

$$P \notin \ell \quad \text{(from the definition of } \sigma^{-1}).$$

This contradicts the assumption that $P \in \ell$, and thus σ^{-1} is an incidence-preserving mapping and hence a collineation. It is the inverse of σ since by definition of σ^{-1}, the mappings $\sigma\sigma^{-1}$ and $\sigma^{-1}\sigma$ fix π elementwise.

Corollary: The set of all collineations of a projective plane π forms a group.

Proof: This follows from the theorem and from the associativity of mappings.

Definition: The trivial collineation (also called the identity collineation) is the collineation that fixes every point and line of the plane.

Lemma 1.2 Let π be a projective plane and σ a non-trivial collineation of π. Let there exist a line $\ell \in \pi$ such that σ fixes every point on ℓ. Then there exists a point $A \in \pi$ such that σ fixes every line through A.

Proof: Pick an arbitrary point $P \in \pi$ such that $P \notin \ell$, and consider the point P^σ. Then $P^\sigma \notin \ell$; for otherwise $(P^\sigma)^\sigma = P^\sigma$, and application of σ^{-1} gives $P^\sigma = P$, which implies that $P \in \ell$, contrary to hypothesis. There are now two cases:

(i) $P^\sigma \neq P$. Then $PP^\sigma \cap \ell$ is a well-defined point which we will denote as Q.

Now $\equiv P, P^\sigma, Q$;

thus $$(PQ)^\sigma = P^\sigma Q^\sigma = P^\sigma Q \quad (\text{as } Q \in \ell)$$
$$= PQ$$

and thus the line PQ is fixed by σ.

Pick an arbitrary point R, $R \notin \ell$, $R \notin PQ$, and consider the line RR^σ (assuming that $R \neq R^\sigma$). It too is fixed by σ, by the above argument; thus the point

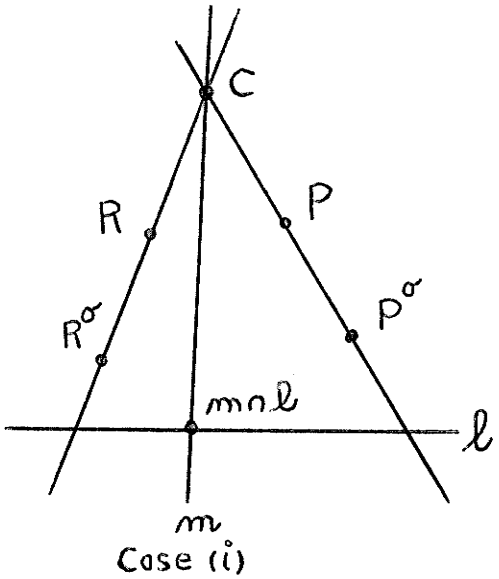
$RR^\sigma \cap PP^\sigma$ is also fixed by σ.

Let $RR^\sigma \cap PP^\sigma = C$.

(ii) $P^\sigma = P$ (or $R^\sigma = R$). In this case sub-

stitute P (or R) in place of C in the following argument.

Again there are two cases; either $C \notin \ell$ or $C \in \ell$.



Case (i)

(i) Suppose $C \notin \ell$. Let m be an arbitrary line through C. Then $m \cap \ell \neq C$, and we have

$$m = (m \cap \ell)C.$$

Thus

$$m^\sigma = \left[(m \cap \ell)C\right]^\sigma = (m \cap \ell)^\sigma C^\sigma = (m \cap \ell)C = m.$$

Thus all lines through C are fixed by σ, and C is the desired point A.

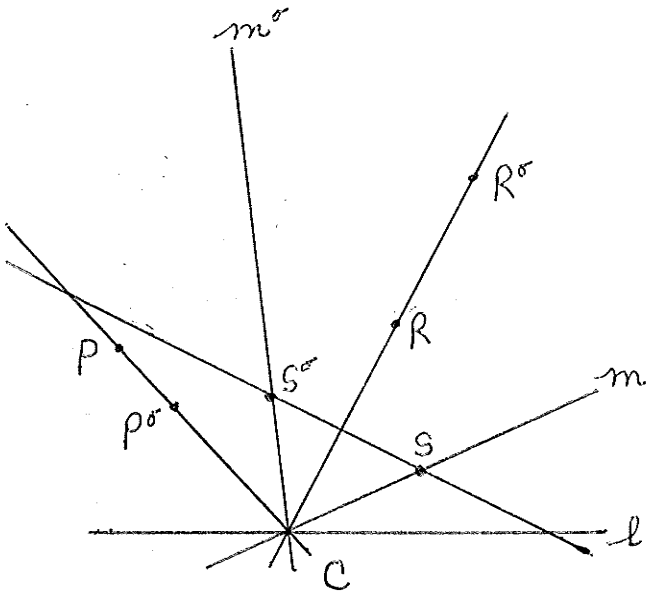
(ii) Suppose $C \in \ell$. Let m be an arbitrary line through C and let S be a point on m ($S \neq C$). Then $S^\sigma \in m^\sigma$, and by the argument used above, SS^σ is fixed by σ. Hence if $S^\sigma \in m$, we have

$$m = CS$$

$$m^\sigma = C^\sigma S^\sigma = CS^\sigma = m$$ (as $\ell^\sigma = \ell$, and if $m \neq \ell$, $S \notin \ell$ implies $S^\sigma \notin \ell$)

and m is fixed by σ. If $S^\sigma \notin m$, then $SS^\sigma \neq m$ and $SS^\sigma \notin m^\sigma$. Thus

$$SS^\sigma \cap PP^\sigma \neq SS^\sigma \cap RR^\sigma$$ (see diagram)



and so $SS^\sigma \cap PP^\sigma$ and $SS^\sigma \cap RR^\sigma$ are distinct points $\notin \ell$ and fixed by σ. By the argument of case (i), it follows that all lines through each point are fixed, and thus σ fixes all points and lines of π. This contradicts the assumption that σ is non-trivial; hence $S^\sigma \notin m$ is impossible and all lines through C are fixed.

The dual of this theorem is also true:

Corollary: If σ is a collineation of π and P a point of π such that σ fixes all lines through P, then there exists a line ℓ of π such that σ fixes all points on ℓ.

Definition: A collineation σ that fixes all points on the line ℓ and all lines through the point C is called a (C,ℓ)-collineation, or a central collineation. ℓ is called the axis of the collineation, and C is called its centre.

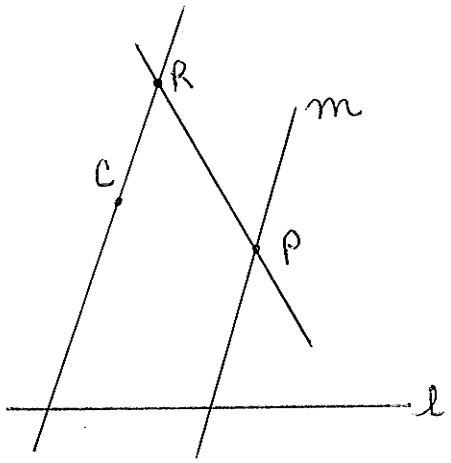
If $C \varepsilon \ell$, then σ is called a (C,ℓ)-elation; if $C \notin \ell$, then σ is called a (C,ℓ)-homology.

Lemma 1.3 A (C,ℓ)-collineation σ that fixes a point P, $P \neq C$ and $P \notin \ell$, is the identity collineation.

Proof: Let m be an arbitrary line through P. Then m is of the form PQ, where $Q = m \cap \ell$ and hence $Q \neq P$. Thus

$$m^\sigma = (PQ)^\sigma = P^\sigma Q^\sigma = PQ$$

since P and Q are both fixed points of σ. The two distinct points P and C then have the property that a line through either of them is fixed. Let R be a point not on CP. Then $R = PR \cap CR$ and $R^\sigma = (PR \cap CR)^\sigma = PR \cap CR = R$. Thus all points of the plane not on CP are fixed by σ. A similar argument in which P is replaced by $\bar{P}$, where $\bar{P} \notin CP$, $\bar{P} \in \ell$, shows that all points on CP are fixed by σ. Hence σ fixes all points of the plane, and as it preserves incidence, it fixes all lines of the plane. Hence $\sigma = 1$.



By the principle of duality, we have the

<u>Corollary</u> A (C, ℓ)-collineation σ that fixes a line m, $m \neq \ell$ and $C \notin m$, is the identity collineation.

<u>Lemma 1.4</u> A (C, ℓ)-collineation σ is determined by the image under σ of a point P (or, by the principle of duality, a line m) if $P \neq C$ and $P \notin \ell$ (dually, $m \neq \ell$ and $C \notin m$).

<u>Proof:</u> Let σ_1 and σ_2 be two (C, ℓ)-collineations. By lemma 1, $\sigma_1 \sigma_2^{-1}$ is also a (C, ℓ)-collineation.

Let P be a point such that $P^{\sigma_1} = P^{\sigma_2}$ ($P \neq C$, $P \notin \ell$). Then

$$P^{\sigma_1\sigma_2^{-1}} = P,$$

and by the previous lemma, $\sigma_1\sigma_2^{-1} = 1$. As inverses are unique, this means that $\sigma_1 = \sigma_2$, which proves the lemma.

Lemma 1.5 Let σ_1 be a (C_1, ℓ)-elation and σ_2 a (C_2, ℓ)-elation. Then either one of the following occurs:

(i) $C_1 = C_2$ and $\sigma_1\sigma_2$ is a (C_1, ℓ)-elation.

(ii) $C_1 \neq C_2$ and $\sigma_1\sigma_2$ is a (C_3, ℓ)-elation

for some point C_3 with $C_1 \neq C_3 \neq C_2$.

Proof: (i) If P is a point and m a line such that $P \varepsilon \ell$ and $C \varepsilon m$, then

$$P^{\sigma_1} = P = P^{\sigma_2},\ m^{\sigma_1} = m = m^{\sigma_2};$$

hence $P^{\sigma_1\sigma_2} = P$ and $m^{\sigma_1\sigma_2} = m$. Thus $\sigma_1\sigma_2$ is a (C_1, ℓ)-elation.

(ii) As both σ_1 and σ_2 fix each point on ℓ, $\sigma_1\sigma_2$ does. Hence by lemma 2, there exists a point C_3 such that $\sigma_1\sigma_2$ fixes all lines through C_3. In order to prove that $C_3 \varepsilon \ell$, it suffices to show that $\sigma_1\sigma_2$ fixes no point of $\pi - \{\ell\}$. If $P \notin \ell$ and $P^{\sigma_1\sigma_2} = P$, then $P^{\sigma_1} = P^{\sigma_2^{-1}}$. But σ_2^{-1} is evidently a (C_2, ℓ)-collineation, so $P^{\sigma_1} \varepsilon C_1P$ and $P^{\sigma_1} \varepsilon C_2P$. Thus as $\neq C_1, C_2, P$, it follows that $P^{\sigma_1} = P^{\sigma_2^{-1}} = P$; thus both σ_1 and σ_2 are trivial (i.e. are the identity collineation) by lemma 1.3, contrary to hypothesis. Hence $\sigma_1\sigma_2$ fixes only points on ,

and is thus an elation.

If $C_3 = C_1$, then $\sigma_2 = \sigma_1^{-1}(\sigma_1\sigma_2)$ is a (C_1, ℓ)-elation (by case (i)), contradicting the hypothesis that $C_1 \neq C_2$. Thus $C_1 \neq C_3 \neq C_2$.

Corollary I: The set of all (C_1, ℓ)-elations forms a group, provided that the identity collineation is counted as a (C, ℓ)-elation for all point-line pairs (C, ℓ).

Corollary II: The set of all elations with a given axis ℓ forms a group, provided that the identity collineation is considered to be such an elation.

Definition: A projective plane π is said to be (C, ℓ)-transitive if, for arbitrary points $P, Q \notin \ell$ such that $\equiv P, Q, C$, and $P \neq C \neq Q$ there is a (C, ℓ)-collineation σ such that $P^\sigma = Q$.

Lemma 1.6 Let π be a projective plane that is (C, γ)-transitive, and let σ be a (C, γ)-collineation and φ an arbitrary collineation. Then $\varphi^{-1}\sigma\varphi$ is a $(C^\varphi, \gamma^\varphi)$-collineation and π is $(C^\varphi, \gamma^\varphi)$-transitive.

Proof: Let $P \varepsilon \gamma^\varphi$; then $P^{\varphi^{-1}} \varepsilon \gamma$, and $P^{\varphi^{-1}\sigma} \varepsilon \gamma^\sigma = \gamma$ (as σ is a (C, γ)-collineation). Hence $P^{\varphi^{-1}\sigma\varphi} \varepsilon \gamma^\varphi$. But as $P^{\varphi^{-1}} \varepsilon \gamma$, $P^{\varphi^{-1}\sigma} = P^{\varphi^{-1}}$ as σ fixes points on γ. Hence $P^{\varphi^{-1}\sigma\varphi} = P^{\varphi^{-1}\varphi} = P$, i.e. $\varphi^{-1}\sigma\varphi$ fixes points on γ^φ. The dual argument gives that $\varphi^{-1}\sigma\varphi$ fixes all lines through C^φ, and hence $\varphi^{-1}\sigma\varphi$ is a $(C^\varphi, \gamma^\varphi)$-collineation.

Pick distinct points A and B in the plane,

arbitrary except that $\equiv A,B,C^{\varphi}$, $A \neq C^{\varphi}$, $B \neq C^{\varphi}$, and $A,B \not\in \varphi$. As φ^{-1} is a collineation, this means that $\equiv A^{\varphi^{-1}},B^{\varphi^{-1}},C$. As the plane is (C,γ)-transitive, there exists a (C,γ)-collineation σ such that $(A^{\varphi^{-1}})^{\sigma} = B^{\varphi^{-1}} \therefore A^{\varphi^{-1}\sigma\varphi} = B^{\varphi^{-1}\varphi} = B$. But $\varphi^{-1}\sigma\varphi$ is a $(C^{\varphi},\gamma^{\varphi})$-collineation; hence, as A and B were arbitrary, the plane is $(C^{\varphi},\gamma^{\varphi})$-transitive.

Definition: (a) A projective plane π is said to be a translation plane with respect to the line ℓ if π is (C,ℓ)-transitive for all points C on ℓ.

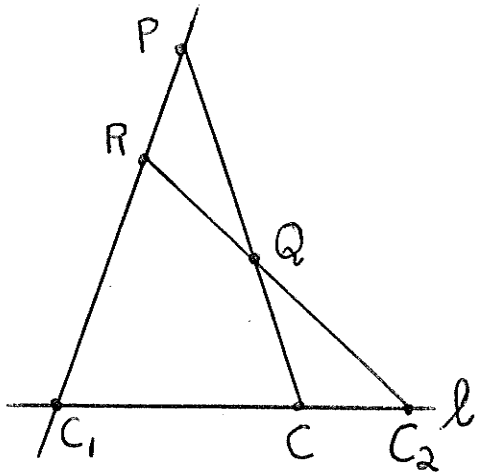
(b) The projective plane π is said to be the dual of a translation plane with respect to the point P if π is (P,ℓ)-transitive for all lines through P.

(c) If π is a translation plane with respect to a line ℓ, then the group of all elations with axis ℓ is called the translation group of π.

Lemma 1.7 Let π be a projective plane containing a line ℓ and distinct points C_1 and C_2 on ℓ. If π is (C_1,ℓ)- and (C_2,ℓ)-transitive, then it is a translation plane with respect to ℓ.

Proof: Let C be an arbitrary point on ℓ, $C_1 \neq C \neq C_2$. Let P and Q be arbitrary points of π such that $\equiv P,Q,C$ and $P,Q \not\in \ell$. As $C_1 \neq C_2$, the point $C_1P \cap C_2Q$ is well-defined; denote it by R.

As π is (C_1, ℓ)-transitive and as $\equiv P,R,C_1$, there exists a (C_1, ℓ)-elation σ_1 such that $P^{\sigma_1} = R$. Similarly there exists a (C_2, ℓ)-elation σ_2 such that $R^{\sigma_2} = Q$. Hence $P^{\sigma_1\sigma_2} = Q$. But by lemma 1.5, $\sigma_1\sigma_2$ is a (C, ℓ)-elation, and evidently it maps P onto R. Hence as C was arbitrary in π, and as P and Q were arbitrary points satisfying $\equiv P,Q,C$, it follows that π is (C, ℓ)-transitive for all $C \varepsilon \ell$. Hence π is a translation plane with respect to ℓ.



Lemma 1.8 Let π be a translation plane with respect to the line ℓ. If α is a collineation then π is a translation plane with respect to ℓ^α.

Proof: Let C be an arbitrary point of ℓ^α. Then there exists a point $\bar{C} \varepsilon \ell$ such that $\bar{C}^\alpha = C$. As π is a translation plane with respect to ℓ, π is $(\bar{C}, \ell)$-transitive. Then by lemma 1.6, π is $(\bar{C}^\alpha, \ell^\alpha)$-transitive, i.e. (C, ℓ^α)-transitive. As C was arbitrary on ℓ^α, π is a translation plane with respect to ℓ^α.

Corollary: If π is a translation plane with respect to two lines ℓ_1 and ℓ_2 intersecting at a point P, then it

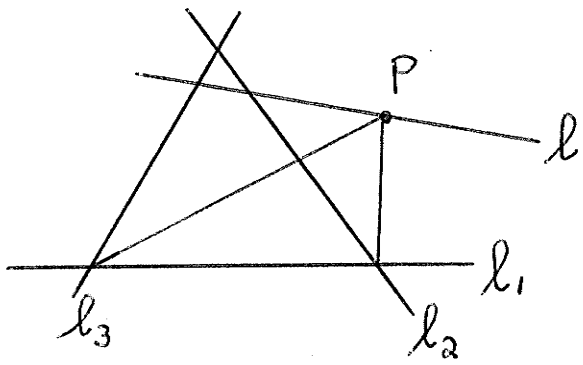
is a translation plane with respect to every line of the plane that passes through P.

Proof: Let m be an arbitrary line through P ($\ell_1 \neq m$). As π is a translation plane with respect to ℓ_1, there is a (C, ℓ_1)-elation σ (C $\neq$ P) such that $\ell_2{}^{\sigma} = m$. By lemma 1.8, since π is a translation plane with respect to ℓ_2, it is also a translation plane with respect to m. As m was arbitrary, π is a translation plane with respect to all lines through P.

Definition: An alternative plane π is a projective plane that is a translation plane with respect to every line of the plane.

Lemma 1.9 If π is a translation plane with respect to three non-concurrent lines ℓ_1, ℓ_2, ℓ_3, then it is an alternative plane.

Proof: Let ℓ be an arbitrary line of π. If ℓ passes through any of $\ell_1 \cap \ell_2$, $\ell_1 \cap \ell_3$, $\ell_2 \cap \ell_3$, then by the corollary of lemma 1.8, π is a translation plane with respect to ℓ . If ℓ passes through none of these, choose an arbitrary point P$\varepsilon\,\ell$, and without loss of generality, assume P$\not\varepsilon\,\ell_1$. Then as above, π is a translation plane with respect to the distinct lines $(\ell_1 \cap \ell_2)$P and



$(\ell_1 \cap \ell_3)P$. Thus it is a translation plane with respect to all lines through P, and in particular ℓ. As ℓ was arbitrary, π must be an alternative plane.

<u>Theorem 1.1</u> Let π be a projective plane and let ℓ be a line of π. Let there exist non-trivial elations σ_1 and σ_2 with axis ℓ and with centres C_1 and C_2, $C_1 \neq C_2$. Then $G(\ell)$, the group of all elations with axis ℓ, is either infinite abelian or elementary abelian.

<u>Proof</u>: We first prove that $G(\ell)$ is abelian. By lemma 1.4, it suffices to show that for an arbitrary point $A \notin \ell$, $A^{\sigma_1\sigma_2} = A^{\sigma_2\sigma_1}$.

Since σ_2 fixes all lines through C_2,

$$(A^{\sigma_1}C_2)^{\sigma_2} = A^{\sigma_1}C_2$$

But $(A^{\sigma_1}C_2)^{\sigma_2} = A^{\sigma_1\sigma_2}C_2$

and so $\equiv A^{\sigma_1}, A^{\sigma_1\sigma_2}, C_2$.

However, $\equiv A^{\sigma_1}, A, C_1$

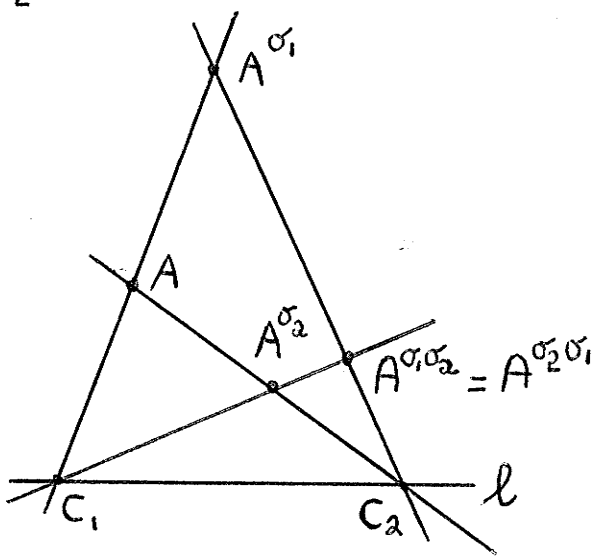
and hence $\equiv A^{\sigma_1\sigma_2}, A^{\sigma_2}, C_1$.

Thus as $A^{\sigma_2}C_1 \neq A^{\sigma_1}C_2$ (as $C_1 \neq C_2$), it follows that

$$A^{\sigma_1\sigma_2} = C_1A^{\sigma_2} \cap C_2A^{\sigma_1}$$

Analogously $A^{\sigma_2\sigma_1} = C_2A^{\sigma_1} \cap C_1A^{\sigma_2}$

$$= A^{\sigma_1\sigma_2}.$$



Thus $\sigma_1\sigma_2 = \sigma_2\sigma_1$. Evidently by the same argument, any two elations with distinct centres and the same axis ℓ will commute. Further, two distinct elations of

$G(\ell)$ with the same centre will commute; for let σ_1 and σ_1^* be (C_1, ℓ)-elations. Then by lemma 5, $\sigma_1^*\sigma_2$ and $\sigma_1\sigma_2$ have centres $\neq C_1, C_2$; thus by the above

$$\begin{aligned}\sigma_2(\sigma_1\sigma_1^*) &= (\sigma_1\sigma_1^*)\sigma_2 = \sigma_1(\sigma_1^*\sigma_2)\\ &= (\sigma_1^*\sigma_2)\sigma_1 = (\sigma_2\sigma_1^*)\sigma_1\\ &= \sigma_2(\sigma_1^*\sigma_1).\end{aligned}$$

Multiplying on the left by σ_2^{-1},

$$\sigma_1\sigma_1^* = \sigma_1^*\sigma_1.$$

Thus $G(\ell)$ is abelian as claimed.

If $G(\ell)$ has an element of finite order, then it has an element σ_1 with centre C_1 of prime order p. If σ_2 is an arbitrary non-trivial element of $G(\ell)$ with centre $C_2 \neq C_1$, then

$$\begin{aligned}(\sigma_1\sigma_2)^p &= \sigma_1^p\sigma_2^p \qquad \text{(as } G(\ell) \text{ is abelian)}\\ &= \sigma_2^p.\end{aligned}$$

As any power of an elation has the same centre as the elation, $(\sigma_1\sigma_2)^p$ has centre C_2, as σ_2^p has. This contradicts lemma 5 unless $(\sigma_1\sigma_2)^p = 1$, i.e. unless

$$\sigma_2^p = 1.$$

Thus all elations of $G(\ell)$ with centre $\neq C_1$, are of order p. By extension of the above argument with σ_2 playing the role of σ_1, all elations with centre C_1 are of order p as well. Hence $G(\ell)$ is an elementary abelian group.

By the principle of duality we obtain the following

Corollary: Let π be a projective plane and P a point of π. Let there exist non-trivial elations σ_1 and σ_2 with centre P and axes ℓ_1 and ℓ_2, $\ell_1 \neq \ell_2$. Then $G(P)$, the group of all elations with centre P, is either infinite abelian or elementary abelian.

Definition: An involution is a (C, ℓ)-collineation σ such that $\sigma \neq 1$ but $\sigma^2 = 1$. If σ is an elation (homology) of order 2, it is said to be an involutory elation (homology).

Theorem 1.2 Let σ be an involution of a projective plane π of order n. Then if n is even, σ is an elation; if n is odd, σ is a homology.

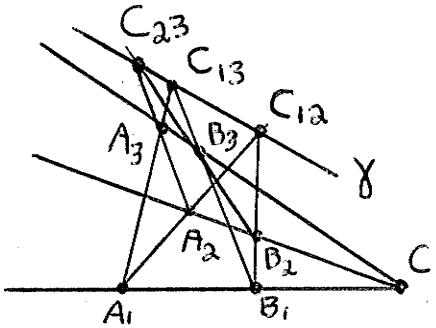
Proof: Let σ have axis ℓ and centre C, and let m be any line $\neq \ell$ such that $C \varepsilon m$. Then σ interchanges points of $m - \{C \cup (m \cap \ell)\}$ in pairs; thus $m - \{C \cup (m \cap \ell)\}$, considered as a point set, has an even number of points. If σ is an elation, then $C = m \cap \ell$ and $m - \{C \cup (m \cap \ell)\}$ contains n points; hence n is even. If σ is a homology, then $C \neq m \cap \ell$ and $m - \{C \cup (m \cap \ell)\}$ contains n-1 points. Thus n-1 is even, i.e. n is odd.

Theorem 1.3 Desargues' (C, γ) theorem holds in a projective plane if and only if the plane is (C, γ)-transitive.

Proof: First suppose that for a particular line γ and point C the plane is (C, γ)-transitive. Let

C, A_1, A_2, A_3, B_1, B_2, B_3 be seven distinct points such that $\equiv C,A_1,B_1$, $\equiv C,A_2,B_2$, and $\equiv C,A_3,B_3$. Suppose that $C_{12} = A_1A_2 \cap B_1B_2 \varepsilon \gamma$ and that $C_{13} = A_1A_3 \cap B_1B_3 \varepsilon \gamma$. It must be shown that $C_{23} = A_2A_3 \cap B_2B_3$ also lies on γ.

From the above conditions it follows that γ does not pass through A_i or B_i $(i = 1,2,3)$, so as there exists (C,γ)-transitivity, there exists a (C,γ)-collineation σ such that $A_1^{\sigma} = B_1$.



Thus $A_2^{\sigma} = (C_{12}A_1 \cap CA_2)^{\sigma}$

$= C_{12}^{\sigma}A_1^{\sigma} \cap CA_2$

$= C_{12}B_1 \cap CA_2 = B_2$

A similar argument shows that $A_3^{\sigma} = B_3$.

Thus

$$A_2A_3 \cap \gamma = (A_2A_3 \cap \gamma)^{\sigma} = A_2^{\sigma}A_3^{\sigma} \cap \gamma = B_2B_3 \cap \gamma .$$

Hence $A_2A_3 \cap B_2B_3 \varepsilon \gamma$, and Desargues' (C,γ) theorem holds.

Conversely, assume that Desargues' (C,γ) theorem holds for a particular point-line pair C and γ. Let A_1 and B_1 be any pair of distinct points $\neq C$ and not on γ such that $\equiv A_1,B_1,C$. Construct a mapping σ, defined on the affine plane (obtained by considering CA_1B_1 to be the line at infinity) as follows:

$$A_1^{\sigma} = B_1$$

$$C^{\sigma} = C$$

$$P \in \gamma \Rightarrow P^{\sigma} = P$$

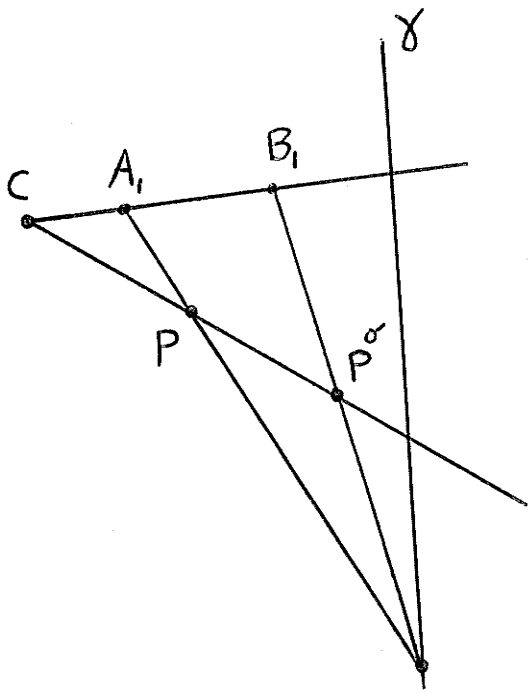
$$P \notin \gamma,\ P \notin A_1C, \Longrightarrow P^{\sigma} = CP \cap ((A_1P \cap \gamma)B_1).$$

It will now be shown that the mapping σ preserves incidence in the affine plane, i.e. that $P \not\in CA_1$, $P \in \ell, \Longrightarrow P^{\sigma} \in \ell^{\sigma}$. This will imply that σ maps parallel classes of lines into parallel classes of lines, and hence that a point on CA_1 is mapped into another point on CA_1. Hence σ will be shown to be an incidence-preserving mapping, and in fact to be a (C,γ)-collineation sending $A_1 \longrightarrow B_1$. Thus as A_1 and B_1 were arbitrary as described above, the plane will be shown to be (C,γ)-transitive.

Let P_1 and P_2 be two distinct points not in $\{\gamma \cup C\}$. Let $R = P_1P_2 \cap \gamma$.

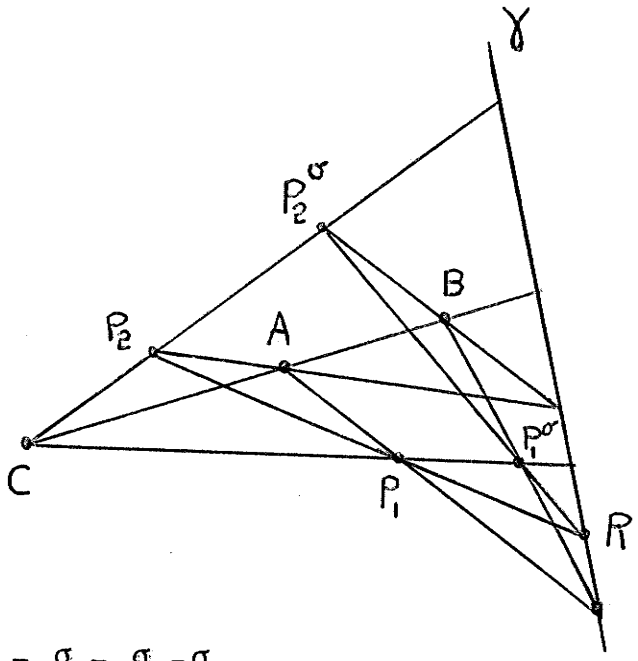
If $\equiv C,P_1,P_2$ then by definition of σ both P_1^{σ} and P_2^{σ} are on CR, i.e. $\equiv C,P_1^{\sigma},P_2^{\sigma},R,$ and thus

$$\equiv P_1,P_2,R \Rightarrow \equiv P_1^{\sigma},P_2^{\sigma},R.$$



If $\not\equiv C,P_1,P_2$, construct P_1^σ and P_2^σ and apply Desargues (C,γ) theorem; as $AP_1 \cap BP_1^\sigma \varepsilon \gamma$ and $AP_2 \cap BP_1^\sigma \varepsilon \gamma$, it follows that $P_1^\sigma P_2^\sigma \cap P_1P_2 \varepsilon \gamma$, i.e. that

$$\equiv P_1^\sigma, P_2^\sigma, R.$$



Hence in general

$$\equiv P_1,P_2,R \Longrightarrow \equiv P_1^\sigma,P_2^\sigma,R^\sigma.$$

Thus suppose that A,B,D are arbitrary distinct collinear points not = C nor on γ.

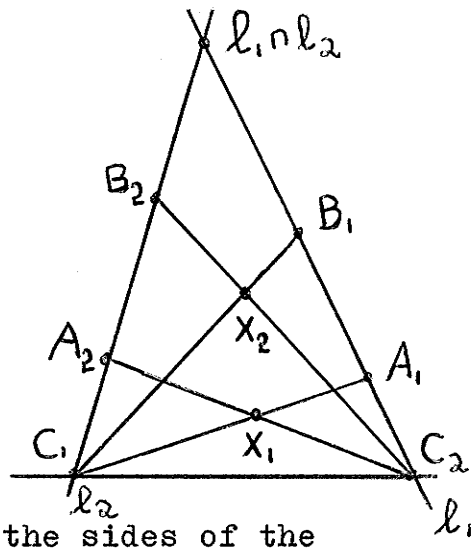
Let $ABD \cap \gamma = R$.

Thus $\equiv A,B,R$ and $\equiv B,D,R$. Hence by the above work, $\equiv A^\sigma,B^\sigma,R$ and $\equiv B^\sigma,D^\sigma,R$, so $\equiv A^\sigma,B^\sigma,D^\sigma$. Hence σ preserves incidence and hence is a collineation; it follows, as remarked earlier, that the plane is (C,γ)-transitive.

Lemma 1.10 (See Ostrom (6), lemma 6.) Let π be a projective plane and let σ_1 and σ_2 be two involutory homologies of π with centres C_1 and C_2 respectively and axes ℓ_1 and ℓ_2 respectively. If $C_1 \varepsilon \ell_2$ and $C_2 \varepsilon \ell_1$, then $\sigma_1\sigma_2$ is an involutory homology with centre $\ell_1 \cap \ell_2$ and axis C_1C_2.

Proof: As the points C_1,C_2, and $\ell_1 \cap \ell_2$ are fixed by

both σ_1 and σ_2, they are fixed by $\sigma_1\sigma_2$. Similarly the lines ℓ_1, ℓ_2, and $C_1 \cap C_2$ are fixed by σ_1, σ_2, and $\sigma_1\sigma_2$. Let A_1 and B_1 be two points of ℓ_1 interchanged by σ_2 (as $\sigma_2^2 = 1$, points of π are either fixed by σ_2 or are interchanged in pairs). Since σ_1 fixes all points on ℓ_1, $\sigma_1\sigma_2$ also interchanges A_1 and B_1. Similarly, if σ_1 interchanges A_2 and B_2 on ℓ_2, then $\sigma_1\sigma_2$ interchanges these also. Thus points on ℓ_1 and ℓ_2 (with the exception of C_1, C_2, and $\ell_1 \cap \ell_2$) are interchanged in pairs by $\sigma_1\sigma_2$.



Let X_1 be any point not on the sides of the triangle $C_1C_2(\ell_1 \cap \ell_2)$. Let $C_1X_1 \cap \ell_1 = A_1$ and $C_2X_1 \cap \ell_2 = A_2$. Let $A_1 \overleftrightarrows B_1$ and $A_2 \overleftrightarrows B_2$ under $\sigma_1\sigma_2$. Then if $C_1B_1 \cap C_2B_2 = X_2$, evidently $X_1 \overleftrightarrows X_2$ under $\sigma_1\sigma_2$. Thus all points of π are either fixed or interchanged in pairs by $\sigma_1\sigma_2$, so $(\sigma_1\sigma_2)^2 = 1$.

Let $X_1X_2 \cap C_1C_2 = R$.

Then $(X_1X_2 \cap C_1C_2)^{\sigma_1\sigma_2} = (X_1X_2)^{\sigma_1\sigma_2} \cap (C_1C_2)^{\sigma_1\sigma_2}$

$= X_1X_2 \cap C_1C_2 = R$

as $\sigma_1\sigma_2$ fixes the line X_1X_2 (as it interchanges X_1 and X_2) and the line C_1C_2 (as both σ_1 and σ_2 do). Thus all points on C_1C_2, and dually all lines through $\ell_1 \cap \ell_2$,

are fixed by $\sigma_1\sigma_2$. Hence $\sigma_1\sigma_2$ is an involutary homology with centre $\ell_1 \cap \ell_2$ and axis C_1C_2.

Lemma 1.11 (See Piper (10), result 5.) Let π be a finite projective plane and let ℓ be a line of π. Let ℓ possess a point Q with the property that $P \varepsilon \ell (P \neq Q)$ implies that there exists a non-trivial (P, ℓ)-elation. Then there exists a non-trivial (Q, ℓ)-elation.

Proof: Let π have order n and let the points of ℓ be labelled $P_1, \cdots, P_n, Q$. Let m be an arbitrary line through Q ($m \neq \ell$). Let α_i be a non-trivial (P_i, ℓ)-elation, i = 1 to n.

Then

$$m^{\alpha_i} \neq m, \quad i = 1 \text{ to } n \text{ (as } m \cap \ell \neq P_i, \quad i = 1 \text{ to } n).$$

Hence m has n images under the set $\{\alpha_i\}$, but no more than (n-1) of these can be distinct, as m and ℓ are not possible images and there are (n+1) lines through Q. Hence there exist j and k ($1 \leq j \leq n$, $1 \leq k \leq n$) such that

$$m^{\alpha_j} = m^{\alpha_k}, \qquad j \neq k.$$

Thus $m^{\alpha_j \alpha_k^{-1}} = m$ and $\alpha_j \alpha_k^{-1} \neq 1$ as $j \neq k$.

But by lemma 1.5, $\alpha_j \alpha_k^{-1}$ is an elation with axis ℓ, and as it fixes m, its centre must be Q. Hence $\alpha_j \alpha_k^{-1}$ is a non-trivial (Q, ℓ)-elation.

CHAPTER II

THE CO-ORDINATIZATION OF THE PROJECTIVE PLANE

The following treatment is patterned after that of Pickert (8). However, the ternary ring introduced by Hall (5) is also discussed, and its relation to that of Pickert is treated in some detail.

THE ASSIGNING OF CO-ORDINATES

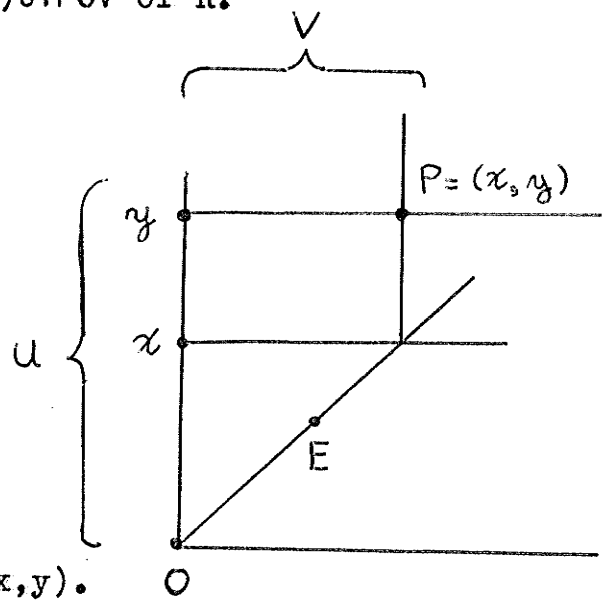
Let π be an arbitrary projective plane, and let UVOE be an arbitrarily chosen non-degenerate quadrangle of π. The line UV, henceforth called the line at infinity, together with its points, is now deleted from π, and the resulting affine plane is denoted as $\bar{\pi}$. The points of $\bar{\pi}$ are called affine points. Let

$$R = \{P \mid P \varepsilon OV,\ P \varepsilon \bar{\pi}\}.$$

If π is a finite plane of order n, then R possesses n distinct elements, which henceforth shall be denoted by $\{o, 1, a, b, \cdots\}$. The special elements o and 1 are identified with the points O and $UE \cap OV$ respectively. Because $\bar{\pi}$ can be extended in one and only one way (apart from isomorphism) to a projective plane, diagrams can without confusion be drawn in the affine plane. Braces will indicate pencils of lines passing through the same point on the line at infinity.

The points of π are now co-ordinatized as follows. To each point P of $\bar{\pi}$ is assigned an ordered pair of

elements of R. The members of this pair are called the "co-ordinates" of P, and are assigned as follows: P has co-ordinates (x,y), where y is the element $UP \cap OV$ of R and x is the element $(PV \cap OE)U \cap OV$ of R.



It is customary to identify a point of $\bar{\pi}$ with its co-ordinate pair, as there is a one-to-one correspondence between the points of the affine plane and the elements of RxR. Thus if $P \varepsilon \bar{\pi}$ has co-ordinates (x,y), one writes P = (x,y).

The "points at infinity", i.e. those on the line UV, are assigned singleton co-ordinates as follows:

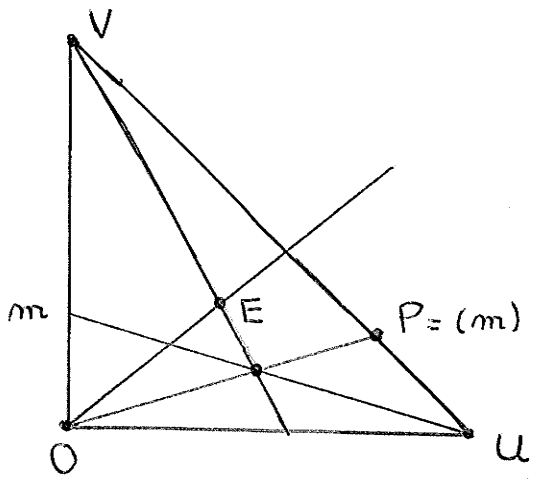
(1) V is given the co-ordinate (∞)

(2) $P \neq V$ is given the co-ordinate (m)

where $OP \cap VE = (1,m)$.

It is easily verified that the singleton co-ordinate (m) of (2) is well-defined and that $P_1 = (m)$, $P_2 = (m)$ if and only if $P_1 = P_2$.

In co-ordinatizing lines of the affine plane, a distinction is made between the pencil of lines through V and the remaining lines of the affine plane. If $\ell \varepsilon \bar{\pi}$ and $V \varepsilon \ell$, then ℓ



is assigned the singleton co-ordinate $[x]$, where $\ell \cap OE = (x,x)$ defines the element $x \varepsilon R$. As $\ell \cap OE$ is well-defined, x also is.

If $\ell \varepsilon \bar{\pi}$ and $V \not\varepsilon \ell$, ℓ is assigned the ordered pair $[m,b]$ as its co-ordinates, where m is defined by $\ell \cap UV = (m)$ and b is defined by $\ell \cap OV = (o,b)$. The co-ordinate m of a line $[m,b]$ is called the "slope" of the line; affine lines passing through V are said to have "infinite slope".

The line UV is assigned the singleton co-ordinate $[\infty]$. This completes the co-ordinatization of the points and lines of π.

The following special cases are of interest, and are easily verified:

(1) $O = (o,o)$, $E = (1,1)$

If $P \varepsilon \bar{\pi}$:

(2) $P \varepsilon OE \Leftrightarrow P = (x,x)$ for some $x \varepsilon R$

(3) $P \varepsilon OU \Leftrightarrow P = (x,o)$ for some $x \varepsilon R$

(4) $P \varepsilon OV \Leftrightarrow P = (o,y)$ where y is the element of R identified with P.

(5) $P \varepsilon EV \Leftrightarrow P = (1,y)$ for some $y \varepsilon R$

(6) $P \varepsilon EU \Leftrightarrow P = (x,1)$ for some $x \varepsilon R$

(7) $OV = [o]$, $EV = [1]$

(8) $V \not\varepsilon \ell$, $O \varepsilon \ell \Leftrightarrow \ell = [m,o]$ for some $m \varepsilon R$

(9) $U \varepsilon \ell$, $\ell \neq (\infty) \Leftrightarrow \ell = [o,b]$ for some $b \varepsilon R$

(10) $OU = [o,o]$.

THE TERNARY RING OF π

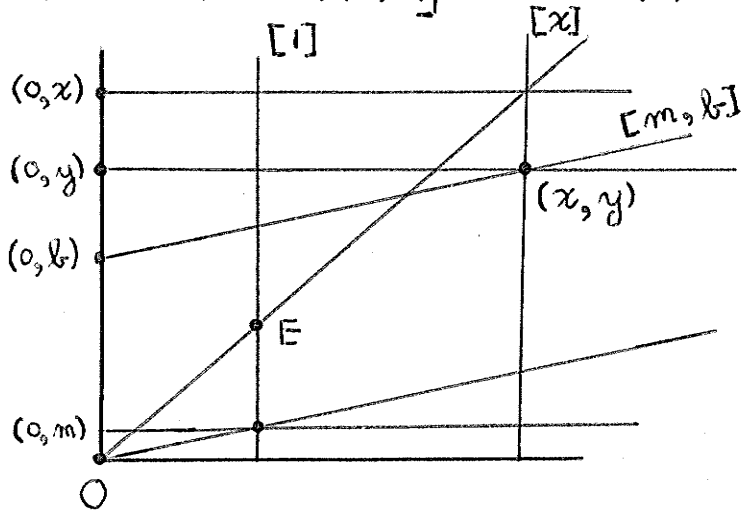
Closely associated with the co-ordinate set R of a projective plane π is the ternary ring of π. The ternary ring is defined to be the ordered pair (R,T) where R is the set defined previously and T is a ternary function mapping RxRxR onto R and defined as follows: if (m,x,b) is an arbitrary ordered triple of RxRxR, then $y = T(m,x,b) \iff (x,y) \varepsilon\ m,b$ where we have identified points and lines with their co-ordinates.

Lemma 2.1 T is a well-defined mapping from RxRxR onto R.

Proof: It must be shown that given $(m,x,b) \varepsilon RxRxR$, T(m,x,b) is uniquely determined. Let P be an affine point on $[m,b]$ with first co-ordinate x. Then $PV \cap OE = (x,x)$, and so $P \varepsilon [x]$. Hence $P = [x] \cap [m,b]$ and as two lines intersect in a unique point, P is uniquely determined. Thus the point (x,y) that lies on $[m,b]$ is uniquely determined, and hence y is. In fact it can easily be verified that

$$(o,y) = \left[((o,x)U \cap OE)V \cap (((o,m)U \cap EV)O \cap UV)(o,b)\right] U \cap OV \quad \ldots\ldots (1)$$

As y is uniquely determined, the ternary operation T is well-defined. It is onto because for any $y \varepsilon R$, any affine line $[m,b]$ has incident on it an affine



point with second co-ordinate y (this follows from the axioms of incidence).

The function T will be called the "Pickert ternary function" and the algebra (R, T) will be called the "Pickert ternary ring", as this is Pickert's version of Hall's ternary ring (see Pickert (9)).

Two binary operations mapping RxR onto R are now defined in terms of the ternary function T. These are addition (symbolized +) and multiplication (symbolized $\cdot$), and they are defined as follows:

(1) For any $x,b \in R$,

$x+b = T(1,x,b)$

(2) For any $m,x \in R$,

$m \cdot x = T(m,x,o)$.

Lemma 2.2

(1) Addition amongst elements of R is a loop.

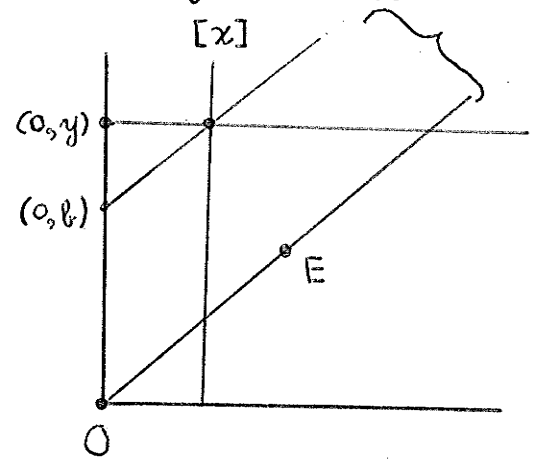
(2) Multiplication amongst elements of $R-\{o\}$ is a loop.

Proof: (1) By the definition of addition and the expression (in the proof of lemma 2.1) for (o,y) in terms of m,x, and b, one obtains that

$$y = x+b \iff (o,y) = ([x] \cap [1,b]) \cup \cap OV.$$

By lemma 2.1, f,x, and b are specified, (x+b) is uniquely determined. Further, if x and b are specified, $[x]$ is seen to be the line joining the points V and

$(o,y)U \cap (o,b)(1)$; by the incidence axioms, $[x]$, and therefore x, is uniquely defined. If y and x are specified, (o,b) is seen to be the point of intersection of $(x,y)(1)$ and OV; thus (o,b), and hence b, are uniquely determined. Hence addition is a loop.



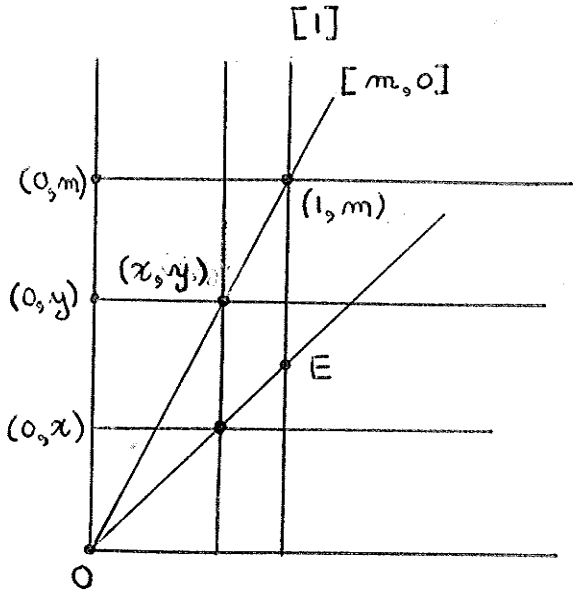
(2) By the definition of multiplication, it is easily seen that

$y = mx \Leftrightarrow (o,y) = [((o,x)U \cap OE)V \cap ((o,m)U \cap EV)O]U \cap OV.$

By lemma 2.1, y is uniquely determined if m and x are specified. If y and x are specified in $R-\{o\}$, it is seen that

$(o,m) = OV \cap (EV \cap O(x,y))U$

and hence by the incidence axioms, (o,m) is uniquely determined (note that the line O(x,y) is well-defined as $x \neq 0 \neq y$).



If y and m are specified, it is seen that

$$[x] = ((o,y)U \cap (1,m)O)V$$

and so by the incidence axioms, $[x]$ and thus x are uniquely determined.

Hence multiplication over $R-\{o\}$ is a loop.

Lemma 2.3 For all $a,b \in R$

(i) $ab = o \iff a = o$ or $b = o$

(ii) $o+a = a+o = a$

(iii) $1 \cdot a = a \cdot 1 = a$.

Proof (i) First suppose that $ab = o$ and that $a \neq o$. Now $ab = T(a,b,o)$, so

$$o = T(a,b,o).$$

Hence $(b,o) \in [a,o]$. Now $(b,o) \in OU$ while $[a,o]$ is the line joining (a) and O. As $a \neq o$, $[a,o] \neq OU$. Hence

$$(b,o) = [a,o] \cap OU = (o,o).$$

Hence $b = o$.

Conversely, consider the product $o \cdot a$. Let $y = o \cdot a$; then $y = T(o,a,o)$.

Consequently by equation (1) in the proof of lemma 2.1,

$$\begin{aligned}(o,y) &= \big[((o,a)U \cap OE)V \cap (((o,o)U \cap EV)O \cap UV)(o,o)\big]U \cap OV \\ &= [(a,a)V \cap OU]U \cap OV \\ &= OU \cap OV\end{aligned}$$

i.e. $(o,y) = (o,o)$.

Thus $y = o$ and $o \cdot a = o$.

Similarly, consider the product $a \cdot o$ and let $z = a \cdot o$; then $z = T(a,o,o)$.

Thus as above,

$$\begin{aligned}(o,z) &= \big[((o,o)U \cap OE)V \cap (((o,a)U \cap EV)O \cap UV)(o,o)\big]U \cap OV \\ &= [OV \cap (a)O]U \cap OV \\ &= OU \cap OV\end{aligned}$$

i.e. $(o,z) = (o,o)$.

Thus $z = o$ and $a \cdot o = o$.

(ii) Let $y = o+a$. Then $y = T(1,o,a)$.

Hence by equation (1) in the proof of lemma 2.1,

$$(o,y) = \left[((o,o)U \cap OE)V \cap (((o,1)U \cap EV)O \cap UV)(o,a)\right] U \cap OV$$
$$= \left[OV \cap (1)(o,a)\right] U \cap OV$$
$$= (o,a)U \cap OV$$

i.e. $(o,y) = (o,a)$.

Thus $y = a$ and $o+a = a$.

Similarly, set $z = a+o$; then $z = T(1,a,o)$.

Thus

$$(o,z) = \left[((o,a)U \cap OE)V \cap (((o,1)U \cap EV)O \cap UV)(o,o)\right] U \cap OV$$
$$= \left[(a,a)V \cap OE\right] U \cap OV$$
$$= (a,a)U \cap OV$$

i.e. $(o,z) = (o,a)$.

Thus $z = a$ and so $a+o = a$.

(iii) Let $y = 1 \cdot a$. Then $y = T(1,a,o)$, and by the last argument in (ii), it immediately follows that $y = a$, i.e. that $a = 1 \cdot a$.

Similarly, let $z = a \cdot 1$. Then $z = T(a,1,o)$.

Thus as above

$$(o,z) = \left[((o,1)U \cap OE)V \cap (((o,a)U \cap EV)O \cap UV)(o,o)\right] U \cap OV$$
$$= \left[EV \cap ((1,a)O \cap UV)(o,o)\right] U \cap OV$$
$$= (a,a)U \cap OV$$

i.e. $(o,z) = (o,a)$.

Thus $z = a$ and $a \cdot 1 = a$.

THE HALL TERNARY RING

Marshall Hall (Hall (5), chapter 20) co-ordinatizes the projective plane in essentially the same way as Pickert does, but defines a ternary function H mapping $RxRxR \longrightarrow R$ in a somewhat different manner. If $(m,x,b) \varepsilon RxRxR$, then the element $H(m,x,b)$ of R is defined as follows:

$$y = H(m,x,b) \iff (m,y) \varepsilon [x,b] .$$

Thus for any three elements $m,x,b \varepsilon R$, it follows that $T(m,x,b) = H(x,m,b)$.

Hall defines the binary operation of addition, mapping $RxR \longrightarrow R$, as follows:

$$y = x+b \iff (x,y) \varepsilon [1,b] ;$$

$$\text{i.e.} \quad \iff \quad y = H(x,1,b).$$

Thus addition as defined by Hall is the same function as addition as defined by Pickert.

Hall defines multiplication, a binary function mapping $RxR \longrightarrow R$ which we shall denote by $\dot{}_H$, as follows:

$$y = x \dot{}_H m \iff (x,y) \varepsilon [m,o] .$$

$$\text{i.e. } y = H(x,m,o) = T(m,x,o) = m \dot{}_T x,$$

where $\dot{}_T$ denotes the Pickert operation of multiplication. Thus for arbitrary $m,x \varepsilon R$,

$$m \dot{}_T x = x \dot{}_H m ,$$

and as multiplication is in general non-commutative, the

Pickert and Hall multiplications are in general distinct functions.

ALGEBRAIC CONSEQUENCES OF (C, γ)-TRANSITIVITY IN π

If UVOE is a non-degenerate quadrangle as above, it follows from lemma 1.9 that π is an alternative plane if and only if it is a translation plane with respect to the non-concurrent lines UV, OV, and OU. By lemma 1.7 , this will occur if and only if the plane is (U,UV)-transitive, (V,UV)-transitive, (V,OV)-transitive, (O,OV)-transitive, (O,OU)-transitive and (U,OU)-transitive. The algebraic properties of the ternary ring that are associated with the various types of transitivity mentioned above are now developed.

First two splitting laws (linearity conditions) that ternary rings obey under certain circumstances are mentioned. They are as follows:

L_1 : If, for all $m,x,b \in R$, the identity

$$T(m,x,b) = m \cdot x + b$$

i.e. $$T(m,x,b) = T(1,T(m,x,o),b)$$

holds, then R is said to obey the "first splitting law", denoted by L_1.

L_2 : If, for all $m,x,b \in R$ the identity

$$T(m,x,mb) = m \cdot (x+b),$$

i.e. $$T(m,x,mb) = T(m,T(1,x,b),o)$$

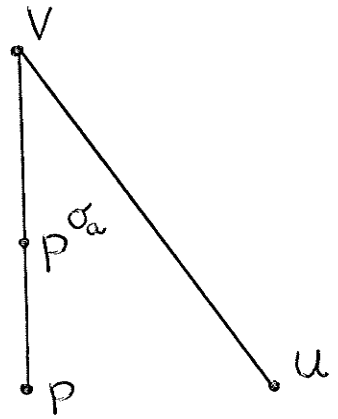
holds, then R is said to obey the "second splitting law", denoted by L_2.

Theorem 2.1 A projective plane π is (V,UV)-transitive if and only if L_1 holds and if addition is associative in the ternary ring of π.

Proof: First assume that π is (V,UV)-transitive. Then for arbitrary $a \varepsilon R$ there exists a collineation σ_a with axis UV and centre V such that $(o,o)^{\sigma_a} = (o,a)$. As points on UV and lines through V are fixed by σ_a, it follows that for arbitrary $m \varepsilon R$,

$$[m]^{\sigma_a} = [m]$$
$$(m)^{\sigma_a} = (m)$$
$$(UV)^{\sigma_a} = UV$$

and $(\infty)^{\sigma_a} = (\infty)$.



If $P = (x,y)$ is an arbitrary affine point of π, then $P \varepsilon [x]$. Consequently as $\equiv P, P^{\sigma_a}, V$, it follows that $P^{\sigma_a} \varepsilon [x]$. In particular if $x = o$, then $(o,y)^{\sigma_a} \varepsilon [o]$. Hence the collineation σ_a induces a permutation on the affine points of $[o]$, and thus on the elements of R. This permutation will be symbolized by writing

$$(o,y)^{\sigma_a} = (o,y^{\sigma_a}) \quad ;$$

thus $y \longrightarrow y^{\sigma_a}$ is a permutation of R. Note in particular that from the definition of σ_a, $o^{\sigma_a} = a$. It follows that

$$(x,y)^{\sigma_a} = (x,y^{\sigma_a}) \ ,$$

for $\equiv (o,y),(x,y),U \Rightarrow \equiv (o,y^{\sigma_a}),(x,y)^{\sigma_a},U$
and all affine points of a given line through U have the same y-co-ordinate.

The mapping τ of elements of R is now defined by

$$[m,b]^{\sigma_a} = [m,b^{\tau}] \quad \text{for all } m,b \in R.$$

(Note that as $(m) \varepsilon [m,b]$ and as $(m)^{\sigma_a} = (m)$, the image of $[m,b]$ under σ_a will also have slope m). However, $(x,y) \varepsilon [m,b]$ if and only if $(x,y)^{\sigma_a} \varepsilon [m,b]^{\sigma_a}$; i.e.

$$y = T(m,x,b) \iff y^{\sigma_a} = T(m,x,b^{\tau}) \ ;$$

thus $$(T(m,x,b))^{\sigma_a} = T(m,x,b^{\tau}) \qquad \ldots\ldots (1)$$

for all $m,x,b \in R$. In particular, since $T(m,o,b) = b$ for all $m,b \in R$, setting $x = o$ in equation (1) gives $b^{\sigma_a} = b^{\tau}$ for all $b \in R$; thus $\sigma_a = \tau$ and equation (1) becomes

$$(T(m,x,b))^{\sigma_a} = T(m,x,b^{\sigma_a}) \text{ for all } m,x,b,a \in R.$$

Setting $m = 1$, and using the fact that $T(1,x,b) = x+b$, one obtains that

$$(x+b)^{\sigma_a} = x+b^{\sigma_a} \qquad \ldots\ldots (2)$$

Setting $b = o$

$$(x+o)^{\sigma_a} = x+o^{\sigma_a}$$

i.e. $$x^{\sigma_a} = x+a \qquad \text{for all } x \in R.$$

Hence from equation (2) it follows that

$$(x+b)+a = x+(b+a).$$

But as a,x, and b are arbitrary in R, it follows that addition in R is associative.

Also, to each $x \varepsilon R$ there exists a unique additive inverse; for suppose that $x+q = o$. Then $T(1,x,q) = o$, i.e. $(x,o)\varepsilon(1,q)$. It follows that

$$(o,q) = OV \cap (OE \cap UV)((UX \cap OE)V \cap OU)$$

which by the axioms of incidence is a well-defined affine point of π. As $x+o = x$ for all $x \varepsilon R$ (lemma 2.3, (ii)) and as $(R,+)$ is a loop (lemma 2.2, (1)), it follows that $(R,+)$ is a group.

To derive the first splitting law, note that as $(T(m,x,b))^{\sigma_a} = T(m,x,b^{\sigma_a})$, it follows that

$$T(m,x,b)+a = T(m,x,b+a) \quad \text{for all } m,x,b,a \varepsilon R.$$

Setting $b = o$,

$$T(m,x,o)+a = T(m,x,a)$$

i.e. $$mx+a = T(m,x,a)$$

and thus L_1 holds.

Conversely, suppose that $(R,+)$ is a group and that L_1 holds, and consider the mapping σ_a of points and lines of π defined, for any given $a \varepsilon R$, as follows:

$$(x,y)^{\sigma_a} = (x,y+a) \qquad [m,b]^{\sigma_a} = [m,b+a]$$
$$(m)^{\sigma_a} = (m) \qquad [c]^{\sigma_a} = [c]$$
$$(\infty)^{\sigma_a} = (\infty) \qquad [\infty]^{\sigma_a} = [\infty]$$

Obviously σ_a fixes all points on $[\infty]$ and all lines through V; thus to show that it is a (V,UV)-collineation it suffices to show that σ_a preserves incidence. The verification that this occurs is trivial except for the case in which it is shown that

$(x,y)\varepsilon[m,b] \Longleftrightarrow (x,y)^{\sigma_a}\varepsilon[m,b]^{\sigma_a}$.

But by L_1

$$(x,y)\varepsilon[m,b] \Longleftrightarrow y = mx+b$$
$$\Longleftrightarrow y+a = (mx+b)+a$$
$$= (mx+(b+a));$$

hence $(x,y)\varepsilon[m,b] \Longleftrightarrow y+a = T(m,x,b+a)$

i.e. $(x,y)\varepsilon[m,b] \Longleftrightarrow (x,y)^{\sigma_a}\varepsilon[m,b]^{\sigma_a}$.

Hence σ_a preserves incidence and is thus a (V,UV)-collineation. As "a" was arbitrary in R, it follows that π is (V,UV)-transitive.

Theorem 2.2 A projective plane π is (U,UV)-transitive if and only if (R,+) is a group and L_2 holds.

Proof: First assume that π is (U,UV)-transitive. Then for arbitrary $b\varepsilon R$, there is a (U,UV)-elation σ_b such that $(o,o)^{\sigma_b} = (b,o)$. Because σ_b fixes the line $[o,o]$, it induces a permutation on the affine points of $[o,o]$, and thus on the elements of R, which will be denoted by $(x,o)^{\sigma_b} = (x^{\sigma_b},o)$; thus $x \longrightarrow x^{\sigma_b}$, where x^{σ_b} is defined as above, is the permutation induced on R by σ_b. In particular $o^{\sigma_b} = b$.

From incidence considerations it immediately follows that

$$[o,a]^{\sigma_b} = [o,a] \quad \text{for all } a\varepsilon R;$$
$$(\infty)^{\sigma_b} = (\infty)$$

$$(m)^{\sigma_b} = (m)$$

and $$[\infty]^{\sigma_b} = [\infty] .$$

Further, as for arbitrary $x \varepsilon R$ $(x,o) \varepsilon [x]$, it follows that $(x,o)^{\sigma_b} \varepsilon [x]^{\sigma_b}$; i.e.

$$(x^{\sigma_b}, o) \varepsilon [x]^{\sigma_b}.$$

But $(x^{\sigma_b}, o) \varepsilon [x^{\sigma_b}]$, and so

$$[x]^{\sigma_b} = [x^{\sigma_b}].$$

Hence for an arbitrary affine point (x,y), since $\equiv (x,y),(x,y)^{\sigma_b},U$ and since $(x,y)^{\sigma_b} \varepsilon [x]^{\sigma_b}$, it follows that $(x,y)^{\sigma_b} = (x^{\sigma_b},y)$.

Similarly, as $(m)^{\sigma_b} = (m)$, an affine line $[m,c]$ will be mapped as follows:

$$[m,c]^{\sigma_b} = [m,c^{\varphi_b}]$$

where this defines the permutation φ_b of elements of R.

As σ_b is a collineation,

$$(x,y) \varepsilon [m,c] \Longleftrightarrow (x,y)^{\sigma_b} \varepsilon [m,c]^{\sigma_b} ;$$

i.e. $y = T(m,x,c) \Longleftrightarrow y = T(m,x^{\sigma_b},c^{\varphi_b})$.

Thus for all $m,x,b,c \varepsilon R$

$$T(m,x,c) = T(m,x^{\sigma_b},c^{\varphi_b}) \qquad \ldots\ldots (1).$$

Setting $x = o$,

$$c = T(m,b,c^{\varphi_b}).$$

Hence equation (1) becomes

$$T(m,x, T(m,b,c^{\varphi_b})) = T(m,x^{\sigma_b},c^{\varphi_b}) \qquad \ldots\ldots (2).$$

Setting $m = 1$ and recalling the definition of addition,

$$x+(b+c^{\varphi_b}) = x^{\sigma_b}{}_{+c^{\varphi_b}} \qquad \ldots\ldots (3)$$

Choosing c such that $c^{\varphi_b} = o$ (this is possible as φ_b is a permutation on R), equation (3) becomes

$$x+b = x^{\sigma_b} \quad \text{for all } x,b \varepsilon R.$$

Using this, equation (3) becomes, in general,

$$x+(b+c^{\varphi_b}) = (x+b)+c^{\varphi_b} \quad \text{for all } x,bc \varepsilon R.$$

Hence addition is associative and (by the reasoning used in theorem 2.1), (R,+) is a group.

Choosing c so that $c^{\varphi_b} = o$, equation (2) becomes

$$T(m,x,mb) = T(m,x+b,o)$$
$$= m(x+b)$$

and hence L_2 holds.

Conversely, assume that L_2 holds and that (R,+) is a group. For arbitrary $c \varepsilon R$, define a mapping σ_c as follows:

$$(\infty)^{\sigma_c} = (\infty) \qquad [\infty]^{\sigma_c} = [\infty]$$
$$(m)^{\sigma_c} = (m) \qquad [x]^{\sigma_c} = [x+c]$$
$$(x,y)^{\sigma_c} = (x+c,y) \qquad [m,b] = [m,T(m,\alpha,b)]$$

where α is defined by $c+\alpha = o$.

The verification that σ_c preserves incidence, and hence is a collineation, is trivial except for the verification that

$$(x,y) \varepsilon [m,b] \Longleftrightarrow (x,y)^{\sigma_c} \varepsilon [m,b]^{\sigma_c} \qquad \ldots\ldots (4).$$

But $(x,y) \varepsilon [m,b] \Longleftrightarrow y = T(m,x,b)$

while $(x,y)^{\sigma_c} \varepsilon [m,b]^{\sigma_c} \Longleftrightarrow y = T(m,x+c,T(m,\alpha,b))$.

If $\bar{b}$ is defined by $m\bar{b} = b$ (and $\bar{b}$ is well-defined as

$(R,\cdot)$ is a loop), it follows that

$$\begin{aligned} T(m,x+c,\ T(m,\alpha,b)) &= T(m,x+c,\ T(m,\alpha,m\bar{b})) \\ &= T(m,x+c,m\ T(1,\alpha,\bar{b})) \\ &= mT(1,x+c,\alpha+\bar{b}) \text{ (by successive applications of } L_2) \\ &= m((x+c)+(\alpha+\bar{b})) \\ &= m((x+(c+\alpha))+\bar{b}) \\ &= m(x+b) \text{ (associativity of addition)} \\ &= T(m,x,m\bar{b}) \end{aligned}$$

i.e. $T(m,x+c,T(m,\alpha,b))= T(m,x,b)$.

Hence condition (4) holds and σ_c is a (U,UV)-collineation (as it fixes all lines through U and all points on UV). As c was arbitrary in R, it follows that π is (U,UV)-transitive.

Theorem 2.3 A projective plane π is a translation plane with respect to the line UV if and only if:

(i) $(R,+)$ is an abelian group,

(ii) L_1 and L_2 are valid.

Proof: By lemma 1.7, π is a translation plane with respect to UV if and only if π is (V,UV)- and (U,UV)-transitive.

If π is (V,UV)- and (U,UV)-transitive, by the last two theorems, L_1 and L_2 both hold and $(R,+)$ is a group. Also, on the assumption that π is a trans-

lation plane with respect to UV, by theorem 1.1 , the group of all elations with axis UV is abelian. For arbitrary $a,b \in R$, there exist (V,UV) elations σ_a and σ_b such that $(o,o)^{\sigma_a} = (o,a)$ and $(o,o)^{\sigma_b} = (o,b)$. Then, as in the proof of theorem 2.1,

$$(o,o)^{\sigma_a\sigma_b} = (o,a)^{\sigma_b} = (o,a+b).$$

Similarly $(o,o)^{\sigma_b\sigma_a} = (o,b+a)$.

But $\sigma_a\sigma_b = \sigma_b\sigma_a$; hence for all $a,b \in R$

$$a+b = b+a$$

and (R,+) is abelian.

Conversely, if L_1 and L_2 are valid and (R,+) is a group, by the two previous theorems π is (U,UV)-transitive and (V,UV)-transitive; hence it is a translation plane with respect to UV.

Corollary: If π is a translation plane with respect to UV, then the left distributive law is valid in C; i.e. for all $a,b,c \in R$, $a(b+c) = ab+ac$.

Proof: If π is a translation plane with respect to UV, then both L_1 and L_2 hold. Hence

$$T(a,b,ac) = ab+ac \qquad \text{by } L_1 ;$$

but

$$T(a,b,ac) = aT(1,b,c)$$

$$= a(b+c) \qquad \text{by } L_2 ;$$

hence $ab+ac = a(b+c)$ as claimed.

—

Note that under the Hall definition of multiplication, a right distributive law is obtained when π is a translation plane with respect to UV; i.e. for all $a,b,c \varepsilon R$, $ba+ca = (b+c)a$.

—

Theorem 2.4 If a projective plane π is (U,UV)- and (V,OV)-transitive, then the right distributive law holds; i.e. for all $a,b,c \varepsilon R$, $ba+ca = (b+c)a$.

Proof: As π is (V,OV)-transitive, for arbitrary $m \varepsilon R$ there exists a (V,OV)-elation σ_m such that $(o)^{\sigma_m} = (m)$. Then as $[\infty]$ is fixed by σ_m, σ_m permutes the points of $[\infty] - \{(\infty)\}$ amongst themselves, and hence induces a permutation $a \longrightarrow a^{\sigma_m}$ of the elements of R defined by $(a)^{\sigma_m} = (a^{\sigma_m})$. Hence in particular $o^{\sigma_m} = m$. Incidence considerations immediately give that

$$[\infty]^{\sigma_m} = [\infty] \qquad (o,b)^{\sigma_m} = (o,b)$$

$$[x]^{\sigma_m} = [x] \qquad (\infty)^{\sigma_m} = (\infty).$$

As $(a)^{\sigma_m} \varepsilon [a,b]^{\sigma_m}$ and $(o,b)^{\sigma_m} \varepsilon [a,b]^{\sigma_m}$, it follows that $[a,b]^{\sigma_m} = [a^{\sigma_m},b]$ and in particular $[o,b]^{\sigma_m} = [m,b]$.

For an arbitrary affine point (x,y), since $(x,y) \varepsilon [x]$, it follows that

$$(x,y)^{\sigma_m} \varepsilon [x]^{\sigma_m} = [x].$$

As π is (U,UV)-transitive, by theorem 2.1 L_1 holds and from this it follows that $(x,y) \varepsilon [o,y]$; hence

$$(x,y)^{\sigma_m} \varepsilon [o,y]^{\sigma_m} = [m,y].$$

Thus $(x,y)^{\sigma_m} = [x] \cap [m,y]$, and from L_1 it immediately

follows that

$$(x,y)^{\sigma_m} = (x,mx+y). \qquad \ldots\ldots (1)$$

As $(x,y)\varepsilon[c,b] \Longleftrightarrow (x,y)^{\sigma_m}\varepsilon[c,b]^{\sigma_m}$, it follows from L_1 that

$$y = cx+b \Longleftrightarrow mx+y = c^{\sigma_m}x+b,$$

i.e. that $mx+(cx+b) = c^{\sigma_m}x+b$

for all m,c,x, and $b\varepsilon R$. As $(R,+)$ is a group (since π is (U,UV)-transitive), this can be rewritten as

$$(mx+cx)+b = c^{\sigma_m}x+b,$$

i.e. $$mx+cx = c^{\sigma_m}x \qquad \ldots\ldots (2)$$

for all $m,x,c\varepsilon R$. Setting $x = 1$ in equation (1) gives $(1,y)^{\sigma_m} = (1,m+y)$ for all $m,y\varepsilon R$.

But by L_1, $(1,c)\varepsilon[c,o]$; hence

$$(1,c)^{\sigma_m}\varepsilon[c,o]^{\sigma_m},$$

i.e. $$(1,m+c)\varepsilon[c^{\sigma_m},o].$$

Hence by L_1, $m+c = c^{\sigma_m}$. Substitution in equation (1) gives

$$mx+cx = (m+c)x \qquad \text{for all } m,c,x\varepsilon R,$$

and hence the right distributive law holds.

<u>Theorem 2.5</u> If π is a translation plane with respect to the lines UV and OV, then for arbitrary $c\varepsilon R-\{o\}$ there exists $c^{-1}\varepsilon R$ such that for arbitrary $b\varepsilon R$, $(bc)c^{-1} = b$.

<u>Proof:</u> As π is (O,OV)-transitive, for each $c\varepsilon R-\{o,-1\}$

there exists an (0,0V)-elation σ_c such that $(o)^{\sigma_c} = (-1-c,o)$. Since, for arbitrary $b\varepsilon R$, σ_c permutes the lines through $(o,b+bc)$ amongst themselves, it follows that

$$[o,b+bc]^{\sigma_c} = [\bar{b},b+bc]$$

for some element $\bar{b}$ which will be in general a function of b. Since $(o)\varepsilon[o,b+bc]$ it follows that

$$(o)^{\sigma_c}\varepsilon[o,b+bc]^{\sigma_c},$$

i.e. $$(-1-c,o)\varepsilon[\bar{b},b+bc].$$

As π is a translation plane with respect to UV, by theorem 2.3 L_1 is valid and hence

$$o = \bar{b}(-1-c)+(b+bc).$$

Using the easily verified fact that $(-a)b = -(ab) = a(-b)$, and the fact that by the hypotheses and theorems 2.3 and 2.4 both distributive laws are valid, it follows that

$$o = (-\bar{b}+b)(1+c).$$

As $c \neq -1$, it follows from lemma 2.3 that $-\bar{b}+b = o$, i.e. that $\bar{b} = b$. Consequently

$$[o,b+bc]^{\sigma_c} = [b,b+bc] \qquad \text{for arbitrary } b\varepsilon R.$$

Also, as $(o,o)\varepsilon[\bar{b}+bc,o]$ for arbitrary $b\varepsilon R$, it follows that $[b+bc,o]^{\sigma_c} = [b+bc,o]$ for all $b\varepsilon R$. But by L_1, for all $b\varepsilon R$, $(1,b+bc)\varepsilon[b+bc,o]$ and $(1,b+bc)\varepsilon[o,b+bc]$. Hence

$$(1,b+bc)^{\sigma_c} = [b+bc,o]^{\sigma_c} \cap [o,b+bc]^{\sigma_c}$$

$$= [b+bc, o] \cap [b, b+bc].$$

If $c \neq o$ this is an affine point. On this assumption, if $(1, b+bc)^{\sigma c} = (h, k)$, by L_1 one obtains

$$k = (b+bc)h$$

and

$$k = bh+(b+bc).$$

Using the distributive laws, these equations give

$$bh+(bc)h = bh+b+bc$$

i.e. $$(bc)h = b+bc \quad \cdots\cdots (1)$$ for all $b \varepsilon R$.

However, for any $b \varepsilon R$, $(1, b+bc) \varepsilon [1]$, so $(1, b+bc)^{\sigma c} \varepsilon [1]^{\sigma c}$, and thus $[1]^{\sigma c} = h$. Hence h is a function of c alone; by setting $b = 1$ in equation (1) one obtains

$$ch = 1+c$$

which specifies h uniquely as a function of c. Set $h = u+1$ (this uniquely determines u). Substitution in equation (1) gives

$$(bc)(u+1) = b+bc$$

i.e. $$(bc)u = b$$ (using the left distributive law)

for all $b \varepsilon R$. As u is a function of c alone, it follows that u is the c^{-1} of the hypotheses, if $c \neq -1$. If $c = -1$, it is evident that $(b(-1))(-1)=b$ for all $b \varepsilon R$. Hence for all $c \varepsilon R - \{o\}$, there exists $c^{-1} \varepsilon R - \{o\}$ such that $(bc)c^{-1} = b$ for arbitrary $b \varepsilon R$.

Corollary I For all $c \varepsilon R$, $cc^{-1} = 1$.

Proof: Set $b = 1$ in $(bc)c^{-1} = b$.

Corollary II For all $c \varepsilon R$, $c^{-1}c = 1$.

Proof: Setting $b = c^{-1}$ in the above, one obtains $(c^{-1}c)c^{-1} = c^{-1}$. But $1 \cdot c^{-1} = c^{-1}$ so by the loop property of $(R - \{o\}, \cdot)$, $c^{-1}c = 1$.

Theorem 2.6 If π is a translation plane with respect to UV and OV, then the following algebraic laws hold:

(i) $(R, +)$ is an abelian group.

(ii) L_1 and L_2 are valid.

(iii) $(R - \{o\}, \cdot)$ is a loop with a right inverse that obeys the inverse condition $(bc)c^{-1} = b$ for all $c \varepsilon R - \{o\}$ where $c^{-1} \varepsilon R - \{o\}$ is so chosen that $cc^{-1} = 1$.

(iv) Both distributive laws hold.

Proof: This follows immediately from theorems 2.3, 214, and 2.5.

Theorem 2.7 If a projective plane π obeys the algebraic laws enunciated in theorem 2.6, it is a translation plane with respect to all lines through V.

Proof: By theorem 2.3, π will be a translation plane with respect to UV. Hence to show that π is a translation plane with respect to OV, it suffices by lemma 1.8 to show that there is a collineation of π mapping UV into

another line through V.

Consider the mapping φ of π defined by

$(\infty)^{\varphi} = (\infty)$

$(m)^{\varphi} = (1,m)$

$(c,d)^{\varphi} = ((1+c^{-1})^{-1}, d(1+c)^{-1})$ for $c \neq 0,-1$

$(0,d)^{\varphi} = (0,d)$

$(-1,d)^{\varphi} = (-d)$

$(UV)^{\varphi} = [1]$ $\qquad$ $[1]^{\varphi} = UV$

$[c]^{\varphi} = [(1+c^{-1})^{-1}]$, $c \neq 0, -1$ $\qquad$ $[m,b]^{\varphi} = [m-b,b]$

$[0]^{\varphi} = [0]$

Evidently φ fixes V. To show that φ is a collineation of π, it must be verified that φ preserves incidence. This is trivial for all cases except for showing that

$$(c,d)\varepsilon[m,b] \Longleftrightarrow (c,d)^{\varphi}\varepsilon[m,b]^{\varphi} \quad (c \neq 0,-1).$$

By L_1, $(c,d)\varepsilon[m,b]$ if and only if $d = mc+b$.

Similarly $(c,d)^{\varphi}\varepsilon[m,b]^{\varphi}$ if and only if

$$((1+c^{-1})^{-1}, d(1+c)^{-1})\varepsilon[m-b,b],$$

i.e. if and only if

$$d(1+c)^{-1} = (m-b)(1+c^{-1})^{-1}+b.$$

Hence $(c,d)\varepsilon[m,b] \Longleftrightarrow (c,d)^{\varphi}\varepsilon[m,b]^{\varphi} (c \neq 0,-1)$

if and only if

$$(mc+b)(1+c)^{-1} = (m-b)(1+c^{-1})^{-1}+b \qquad \ldots\ldots (1)$$

is an identity for all $m,b,c\varepsilon R$ $(c \neq 0,-1)$.

However, for all $m,b,c\varepsilon R$ $(c \neq 0,-1)$

$$m = m$$

and thus $m = [m(1+c^{-1})^{-1}](1+c^{-1})$

$= m(1+c^{-1})^{-1}+[m(1+c^{-1})^{-1}]c^{-1}.$

Thus $mc = [m(1+c^{-1})^{-1}]c+m(1+c^{-1})^{-1}$

$= [m(1+c^{-1})^{-1}](c+1)$

$= [m(1+c^{-1})^{-1}](1+c).$

Hence $mc(1+c)^{-1} = ([m(1+c^{-1})^{-1}](1+c))(1+c)^{-1}$

i.e. $mc(1+c)^{-1} = m(1+c^{-1})^{-1}$ ..... (2)

and equation (2) is an identity for all $m,c\varepsilon R$ $(c \neq o,-1)$.

Similarly $o = -b+b$

$= (-b)[(1+c^{-1})^{-1}(1+c^{-1})]+b$

$= [(-b)(1+c^{-1})^{-1}](1+c^{-1})+b$

$= (-b)(1+c^{-1})^{-1}+[(-b)(1+c^{-1})^{-1}]c^{-1}+b.$

Multiplying on the right by c,

$o = [(-b)(1+c^{-1})^{-1}]c+(-b)(1+c^{-1})^{-1}+bc.$

Thus $b = [(-b)(1+c^{-1})^{-1}]c+(-b)(1+c^{-1})^{-1}+bc+b$

$= [(-b)(1+c^{-1})^{-1}](c+1)+b(c+1).$

Hence $b(1+c)^{-1} = (-b)(1+c^{-1})^{-1}+b$ ..... (3)

is an identity for all $b,c\varepsilon R$ $(c \neq o,-1)$.

But upon expanding (1) one obtains

$(mc)(1+c^{-1})+b(1+c)^{-1} = m(1+c^{-1})^{-1}+(-b)(1+c^{-1})^{-1}+b$

and from (2) and (3) it is seen that this is an identity for all $b,c,m\varepsilon R$ $(c \neq o,-1)$.

Hence φ is indeed a collineation and π is a translation plane with respect to all lines through V.

—

Note that if Hall multiplication is used in the

co-ordinatization of π, then π is a translation plane with respect to all lines through V if and only if the algebraic laws cited in theorem 2.6 hold, with the exception that the existence of a right inverse is replaced by the existence of a left inverse; i.e. for any $c \neq o$, $c \varepsilon R$, there exists c_L^{-1} such that $c_L^{-1}(cb) = b$ for any $b \varepsilon R$.

Theorem 2.8 A projective plane π is an alternative plane if and only if it is co-ordinatized by an alternative field in which L_1 is valid.

Proof: The plane π will be an alternative plane if and only if π is a translation plane with respect to UV, OV, and OU, by lemma 1.9 . Hence, using the results of theorems 2.6 and 2.7, it suffices to show:

(i) If π is a translation plane with respect to all lines through V, and if there exists (O,OU)-transitivity, then to each $c \varepsilon R - \{o\}$ there exists a unique $c_L^{-1} \varepsilon R - \{o\}$ such that $c_L^{-1}(cb) = b$ for all $b \varepsilon R$, and in fact c_L^{-1} is the c^{-1} of theorem 2.5. This will prove that an alternative plane is co-ordinatized by an alternative field.

(ii) A plane co-ordinatized by an alternative field in which L_1 is valid possesses a collineation moving V. Then by the fact that V is a translation plane with respect to all lines through V, it follows

by lemma 1.6 that π is a translation plane with respect to three non-concurrent lines, and hence by lemma 1.9, π is an alternative plane.

To prove (i), suppose π is alternative. Then π is (0,0U)-transitive and there exists an (0,0U)-elation σ such that $(\infty)^\sigma = (o,-1)$. Evidently $(a,o)^\sigma = (a,o)$ for all $a\varepsilon R$, and $[m,o]^\sigma = [m,o]$ for all $m\varepsilon R$. Now $[a] = (\infty)(a,o)$, and so

$$[a]^\sigma = (\infty)^\sigma(a,o)^\sigma = (o,-1)(a,o).$$

If $a \neq o$, $[a]^\sigma$ is evidently of the form $[m,b]$ and by L_1 it is found that

$$[a]^\sigma = [a^{-1},-1] \qquad (a \neq o)$$

where a^{-1} is as defined in theorem 2.5.

Since for all $r\varepsilon R$ $(o,o)\varepsilon[r,o]$, it follows that $[r,o]^\sigma = [r,o]$ for all $r\varepsilon R$. Thus for $a \neq o \neq b$, consider how the point $(a,1-ba)$ maps under σ. By L_1 and the distributive laws it is found that $(a,1-ba)\varepsilon[a^{-1}-b,o]$. Also $(a,1-ba)\varepsilon[a]$; hence

$$\begin{aligned}(a,1-ba)^\sigma &= [a^{-1}-b,o]^\sigma \cap [a]^\sigma \\ &= [a^{-1}-b,o] \cap [a^{-1},-1]. \qquad \ldots\ldots (1)\end{aligned}$$

As $b \neq o$, $(a,1-ba)^\sigma$ is an affine point of π of the form (h,k). Applying L_1 and equation (1), one obtains

$$h = b^{-1},\ k = a^{-1}b^{-1}-1.$$

Consequently $(a,1-ba)^\sigma = (b^{-1},a^{-1}b^{-1}-1)$ ..... (2)

for $a \neq o \neq b$.

Next consider how the point $(1,1-ab)$ maps under σ for $a \neq o \neq b$. By L_1, $(1,1-ba) \varepsilon [1-ba,o]$. Consequently

$$(1,1-ba) = [1] \cap [1-ba,o]$$

Hence
$$(1,1-ba)^\sigma = [1]^\sigma \cap [1-ba,o]^\sigma$$
$$= [1,-1] \cap [1-ba,o].$$

As $a \neq o \neq b$, $(1,1-ba)^\sigma$ is an affine point of the form (h,k). By L_1, the distributive laws, and the existence of the right alternative law, it is found that

$$h = (ba)^{-1}, \quad k = (ba)^{-1}-1.$$

Hence
$$(1,1-ba)^\sigma = ((ba)^{-1},(ba)^{-1}-1).$$

As σ is an (O,OU)-elation, lines through U are permuted amongst themselves by σ. Hence for $a \neq o \neq b$,

$$[o,1-ba]^\sigma = [o,k] \quad \text{(assuming } [o,1-ba] \neq UV\text{)}.$$

But $(1,1-ba) \varepsilon [o,1-ba]$; hence

$$((ba)^{-1},(ba)^{-1}-1) \varepsilon [o,1-ba]^\sigma$$

and evidently $[o,1-ba]^\sigma$ is an affine line. Hence by L_1,

$$(ba)^{-1}-1 = k.$$

Consequently $[o,1-ba]^\sigma = [o, (ba)^{-1}-1]$. ..... (3)

Now $(a,1-ba) \varepsilon [o,1-ba]$ and hence $(a,1-ba)^\sigma \varepsilon [o,1-ba]^\sigma$; consequently from equations (2) and (3)

$$(b^{-1},a^{-1}b^{-1}-1) \varepsilon [o,(ba)^{-1}-1] \quad (a \neq o \neq b).$$

Thus
$$a^{-1}b^{-1}-1 = (ba)^{-1}-1,$$

i.e.
$$a^{-1}b^{-1} = (ba)^{-1}, \quad a \neq o \neq b.$$

However, from theorem 2.5 and the hypotheses, $b^{-1} = (b^{-1}a^{-1})(a^{-1})^{-1}$, and as $(a^{-1})(a^{-1})^{-1} = 1$ and $a^{-1}a = 1$, by the loop properties of $(R-\{o\},\cdot)$, $a = (a^{-1})^{-1}$; hence $b^{-1} = (b^{-1}a^{-1})a$.

Thus
$$\begin{aligned} b = (b^{-1})^{-1} &= [(b^{-1}a^{-1})a]^{-1} \\ &= a^{-1}(b^{-1}a^{-1})^{-1} \\ &= a^{-1}((a^{-1})^{-1}(b^{-1})^{-1}) \\ &= a^{-1}(ab) \end{aligned}$$
(by several applications of $(ba)^{-1} = a^{-1}b^{-1}$)

for $a \neq o \neq b$. If $b = o$ then $b = a^{-1}(ab)$ is trivially true. Hence for any $a \varepsilon R-\{o\}$, a^{-1} has the property that $b = a^{-1}(ab)$ for any $b \varepsilon R$. Hence R is an alternative field.

To prove (ii), consider the mapping φ of π defined as follows:

$(a,b)^{\varphi} = (b,a)$	$[c]^{\varphi} = [o,c]$
$(m)^{\varphi} = (m^{-1}), m \neq o$	$[m,b]^{\varphi} = [m^{-1},-m^{-1}b]$, $m \neq o$
$(o)^{\varphi} = (\infty)$	$[o,b]^{\varphi} = [b]$
$(\infty)^{\varphi} = (o)$	$[\infty]^{\varphi} = [\infty]$.

Evidently this mapping moves V; to show that φ is a collineation it must be verified that φ is one-to-one onto and preserves incidence. This is trivial, the most complicated case being the following:

$(a,b) \varepsilon [m,c]$ $(m \neq o)$ holds if and only if $b = ma+c$; $(b,a) \varepsilon [m^{-1},-m^{-1}c]$ holds if and only if

$a = m^{-1}b-m^{-1}c$. But $b = ma+c$ if and only if $a = m^{-1}b-m^{-1}c$, since $a = m^{-1}(ma+c)-m^{-1}c$ is an identity as $m^{-1}(ma) = a$. Thus

$$(a,b)\varepsilon\left[m,c\right] \Longleftrightarrow (a,b)^{\varphi}\varepsilon\left[m,b\right]^{\varphi}.$$

Hence φ is a collineation and it follows that π is an alternative plane.

CHAPTER III

THE THEORY OF FINITE ALTERNATIVE FIELDS

In Chapter II it was shown that an alternative plane can be co-ordinatized by an alternative field. In this chapter it will be proved that any finite alternative field is a commutative field. It is well-known (Pickert (9), page 136) that the ternary ring of a projective plane is a field if and only if the plane is Pappian; it is also well-known (Ibid. page 144) that all Pappian planes are Desarguesian. Hence it follows that all finite alternative planes are Desarguesian.

DEFINITION OF AN ALTERNATIVE FIELD

Recall that an alternative field is a triple $(A,+,\cdot)$ (where A is a set with special elements 0 and I, and + and $\cdot$ are binary operations defined on A) obeying the following axioms:

1. Addition is an abelian group with neutral element 0.
2. The left and right distributive laws hold; i.e. for all $a,b,c \in A$,
$$a(b+c) = ab+ac$$
$$(b+c)a = ba+ca$$
3. If $x,y,z \in A-\{0\}$, and if the values of any two of x,y,z are known, then the equation $xy=z$ uniquely specifies the value of the

third. (I.e. multiplication, excluding 0, is a loop.)

4. To each $a \in A - \{0\}$ there corresponds a unique element $a^{-1} \in A - \{0\}$ such that $a^{-1}a = aa^{-1} = 1$, and for all $b \in A$,

$$(ba)a^{-1} = b$$

$$\text{and } a^{-1}(ab) = b$$

5. $1 \cdot a = a \cdot 1 = a$ for all $a \in A$.

Note that multiplication need not be either associative or commutative.

Lemma 3.1 For all $a \in A$ and $b \in A$:

(i) $a \cdot 0 = 0 \cdot a = 0$

(ii) $(-1)(a) = (a)(-1) = -a$

(iii) $-(-a) = a$

(iv) $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$

where "$-a$" denotes the additive inverse of "a".

Proof: (i) By axiom 2 and the fact that 0 is the neutral element for addition

$$a(1) = a(1+0) = a(1)+a(0).$$

Hence as addition is a group,

$$a \cdot 0 = 0 .$$

Similarly $0 \cdot a = 0$

(ii) From axiom 2,

$$(1+-1) \cdot a = 1 \cdot a + (-1) \cdot a .$$

Thus $$0 \cdot a = a + (-1) \cdot a$$

i.e. $$0 = a + (-1)a$$

Thus by definition of $-a$,

$$(-1)a = -a$$

A similar argument yields $a(-1) = -a$

(iii) By definition of $-(-a)$,

$$-a+-(-a) = 0$$

But by definition of $-a$,

$$-a+a = 0$$

Hence $-(-a) = a$ as A is a group under addition.

(iv) Assume $ab = 0$ and that $a \neq 0$. Then by axiom 4 $a^{-1} \in A-\{0\}$ exists such that

$$b = a^{-1}(ab) = a^{-1}\cdot 0 = 0 \text{ (by (i))}$$

Thus $b = 0$.

Theorem 3.1 For all $a,b \in A$,

(i) $(ba)a^{-1} = b \Rightarrow (ba)a = ba^2$

(ii) $a^{-1}(ab) = b \Rightarrow a(ab) = a^2b$

Proof: (i) If $a = 0$, then

$$(ba)a = 0 = ba^2$$

and if $a = -1$, then

$$(ba)a = -(-b) = b \qquad \text{(lemma 3.1)}$$

while $$ba^2 = b\left[(-1)(-1)\right] = b \qquad \text{(lemma 3.1)}$$

so $$ba^2 = (ba)a \qquad \text{for } a = 0, -1$$

Now assume that $a \neq 0, -1$ and for arbitrary $c \in A$, consider

$$(ca)(a^{-1}-(a+1)^{-1})$$

(both a^{-1} and $(a+1)^{-1}$ are defined as $a \neq 0, -1$)

By axiom 2,

$$(ca)(a^{-1}-(a+1)^{-1}) = (ca)a^{-1}-(ca)(a+1)^{-1}$$
$$= c-(ca)(a+1)^{-1}.$$

Hence

$$\left[(ca)(a^{-1}-(a+1)^{-1})\right] (a+1) = \left[c-(ca)(a+1)^{-1}\right](a+1)$$
$$= c(a+1)-\left[(ca)(a+1)^{-1}\right](a+1)$$
$$= c(a+1)-ca$$
$$= ca+c-ca$$
$$= c.$$

Solving for ca, we obtain

$$ca = \left[c(a+1)^{-1}\right]\left[a^{-1}-(a+1)^{-1}\right]^{-1} \qquad \ldots\ldots (1)$$

This holds for arbitrary $c \in A$. Hence in particular it is true if $c = a+1$. Substituting this in equation (1) we obtain

$$(a+1)a = \left[a^{-1}-(a+1)^{-1}\right]^{-1};$$

substituting this in equation (1) gives

$$ca = \left[c(a+1)^{-1}\right]\left[(a+1)a\right] \qquad \ldots\ldots (2)$$

As c is arbitrary, set $c = b(a+1)$ and substitute into equation (2). This gives

$$\left[b(a+1)\right]a = \left[\{b(a+1)\}\ (a+1)^{-1}\right]\left[\ (a+1)a\right]$$

or $\quad (ba+b)a = ba^2+ba$

i.e. $\quad (ba)a+ba = ba^2+ba.$

As addition is a group, this implies

$$(ba)a = ba^2. \qquad \ldots\ldots (3)$$

As c was arbitrary in A, so is b, and thus equation (3) holds for all $b, a \in A$.

A completely analogous argument verifies (ii).

ASSOCIATORS AND COMMUTATORS

Let a,b,c be three arbitrary elements of A. The associator $[a,b,c]$ of the ordered triple (a,b,c) is defined to be the element $(ab)c-a(bc)$. The commutator $[a,b]$ of the ordered pair (a,b) is defined to be the element $ab-ba$. Thus the associator is a function mapping $AxAxA \to A$, and the commutator is a function mapping $AxA \to A$.

Lemma 3.2 The associator function is linear in each argument.

Proof: For arbitrary $a,\bar{a},b,c \in A$, consider the associator $[a+\bar{a},b,c]$. By definition

$$\begin{aligned}[a+\bar{a},b,c] &= ((a+\bar{a})b)c-(a+\bar{a})(bc) \\ &= (ab+\bar{a}b)c-a(bc)-\bar{a}(bc) \\ &= (ab)c-a(bc)+(\bar{a}b)c-\bar{a}(bc) \\ &= [a,b,c]+[\bar{a},b,c].\end{aligned}$$

Thus the associator is linear in its first argument. Similar reasoning yields that it is linear in its second and third arguments as well.

Lemma 3.3 If two arguments of an associator are the same, the associator is zero.

Proof: Let a and b be arbitrary elements of A.

Then $[a,a,b] = (a^2)b-a(ab) = 0$ (theorem 3.1)

Similarly $[b,a,a] = 0$

Thus $0 = [a,b+a,b+a]$

$= [a,b,b]+[a,b,a]+[a,a,b]+[a,a,a]$

(by lemma 3.2);

Thus $0 = [a,b,a]$

and the lemma holds.

Corollary: For all $a,b \in A$, $a(ba) = (ab)a$; for

$0 = [a,b,a] = (ab)a-a(ba)$.

Lemma 3.4: Interchanging two arguments of an associator changes its sign.

Proof: Let a,b,c be arbitrary in A. By lemmas 3.2 and 3.3,

$$\begin{aligned} 0 &= [a+b,a+b,c] \\ &= [a+b,b+a,c] \\ &= [a,b,c]+[a,a,c]+[b,b,c]+[b,a,c] \\ &= [a,b,c]+[b,a,c]. \end{aligned}$$

Hence

$$[a,b,c] = -[b,a,c].$$

Similarly, by expanding $[a,b+c, b+c]$ one obtains

$$[a,c,b] = -[a,b,c]$$

and by expanding $[a+c,b,a+c]$ one obtains

$$[c,b,a] = -[a,b,c]$$

Lemma 3.5: For all $a,b,c \in A$,

$$[a,b,c] - [a,c,b] + [c,a,b] = [ab,c]-a[b,c]+[a,c]b$$

Proof: Expanding the right-hand side of the above, we obtain

$$\begin{aligned} &[ab,c]-a[b,c]+[a,c]b \\ &= (ab)c-c(ab)-a(bc)+a(cb)+(ac)b-(ca)b \end{aligned}$$

$= [a,b,c]-[a,c,b]+[c,a,b]$ = left-hand side.

Lemma 3.6 For all $a,b,c,d \in A$,

$$[ab,c,d]-[a,bc,d]+[a,b,cd] = a[b,c,d]+[a,b,c]d$$

Proof: Upon expanding,

$[ab,c,d]-[a,bc,d]+[a,b,cd]$

$= ((ab)c)d-(ab)(cd)-(a(bc))d+a((bc)d)+(ab)(cd)-a(b(cd))$

$= a((bc)d-b(cd))+((ab)c-a(bc))d$

$= a[b,c,d]+[a,b,c]d.$

Theorem 3.2 Let A be an alternative field with commutative multiplication. Then if $1+1+1 \neq 0$, multiplication in A is associative.

Proof: Let a,b,c be arbitrary in A. By lemma 3.5,

$$[ab,c] = a[b,c]+[a,c]b+[a,b,c]-[a,c,b]+[c,a,b]$$

Applying lemma 3.4, this becomes

$$[ab,c] = a[b,c]+[a,c]b+3[a,b,c]$$

(where $3 = 1+1+1$)

As A is commutative, all commutators are zero; hence

$$3[a,b,c] = 0$$

This implies, as $3 \neq 0$, that

$$[a,b,c] = 0 ;$$

that is, $(ab)c = a(bc)$

As a,b,c are arbitrary, A is associative.

The "f" Function: Define a function $f: AxAxAxA \longrightarrow A$ as follows: for arbitrary $w,x,y,z \in A$,

$$f(w,x,y,z) = [wx,y,z]-[x,y,z]w-x[y,z,w]$$

Several properties of the "f" function follow.

Lemma 3.7 The function f is linear in each argument.

Proof: Let $w,\bar{w},x,y$, and z be arbitrary in A, and consider

$$
\begin{aligned}
& f(w+\bar{w},x,y,z) \\
= \; & [(w+\bar{w})x,y,z]-[x,y,z](w+\bar{w})-x[y,z,w+\bar{w}] \\
= \; & [wx+\bar{w}x,y,z]-[x,y,z]w+[x,y,z]\bar{w}-x[y,z,w]-x[y,z,\bar{w}] \\
& \quad \text{(by lemma 3.2)} \\
= \; & [wx,y,z]-[x,y,z]w-x[y,z,w]+[\bar{w}x,y,z]-[x,y,z]\bar{w}-x[y,z,\bar{w}] \\
= \; & f(w,x,y,z)+f(\bar{w},x,y,z).
\end{aligned}
$$

Thus f is linear in its first argument. The verification that it is linear in the other three arguments is similar.

Theorem 3.3 The f function is alternative; that is, interchanging any two of the arguments of $f(w,x,y,z)$ changes the sign of $f(w,x,y,z)$.

Proof: Define a function $S:AxAxAxA \longrightarrow A$ as follows: for arbitrary $w,x,y,z \varepsilon A$,

$S(w,x,y,z) = f(w,x,y,z)-f(x,y,z,w)+f(y,z,w,x)$.

Upon expansion we obtain

$$
\begin{aligned}
S(w,x,y,z) = \; & [wx,y,z]-[x,y,z]w-x[y,z,w] \\
& -[xy,z,w]+[y,z,w]x+y[z,w,x] \\
& +[yz,w,x]-[z,w,x]y-z[w,x,y] \qquad \ldots\ldots (1)
\end{aligned}
$$

However, by application of lemma 3.4 and lemma 3.5, one obtains

$$
\begin{aligned}
& [wx,y,z]-[xy,z,w]+[yz,w,x] \\
= \; & w[x,y,z]+[w,x,y]z
\end{aligned}
$$

Hence equation (1) becomes

$$S(w,x,y,z) = w[x,y,z]+[w,x,y]z - [x,y,z]w-x[y,z,w]+[y,z,w]x + y[z,w,x]-[z,w,x]y-z[w,x,y].$$

Recalling the definition of the commutator, this becomes

$$S(w,x,y,z) = [y,[z,w,x]] -[x,[y,z,w]] - [z,[w,x,y]]+[w,[x,y,z]].$$

Cyclic permutation of the arguments of S gives

$$S(w,x,y,z) = -S(x,y,z,w).$$

Hence by definition of S,

$$f(x,y,z,w)-f(y,z,w,x)+f(z,w,x,y) = -f(w,x,y,z)+f(x,y,z,w)-f(y,z,w,x).$$

Thus $$f(z,w,x,y) = -f(w,x,y,z). \quad \ldots\ldots (2)$$

Also,

$$f(w,x,y,z) = [wx,y,z]-[x,y,z]w-x[y,z,w]$$
$$= -[wx,z,y]+[x,z,y]w+x[z,y,w]$$

(using lemma 3.4);

$$f(w,x,y,z) = -f(w,x,z,y). \quad \ldots\ldots (3)$$

It is easily seen that equations (2) and (3) together imply that interchanging any two of the arguments of f changes the sign of f; for example,

$$f(w,x,y,z) = -f(x,y,z,w) = f(y,z,w,x) = -f(y,z,x,w) = f(w,y,z,x) ;$$

i.e. $f(w,x,y,z) = -f(w,y,x,z)$ by repeated use of equations (2) and (3).

Corollary I If two arguments of f are equal, then $f = 0$.

Corollary II (i) Since $f(x,x,y,z) = 0$, it follows that $[x^2,y,z] = [x,y,z]x + x[y,z,x]$.

(ii) Since $f(z,x,y,z) = 0$, it follows that $[zx,y,z] = [x,y,z]z + x[y,z,z]$ i.e. $[zx,y,z] = [x,y,z]z$.

(iii) Since $f(w,x,y,x) = 0$, it follows that $[wx,y,x] = x[y,x,w] = x[w,y,x]$ (by lemma 3.4).

Theorem 3.4 For arbitrary $a,b,c \in A$:

(i) $(ab)(ca) = a((bc)a)$

(ii) $((ab)a)c = a(b(ac))$

(iii) $(a(bc))a = (ab)(ca)$

(iv) $c(a(ba)) = ((ca)b)a$

Proof: (i) $(ab)(ca) = a(b(ca)) + [a,b,ca]$

$= a(b(ca)) - [ca,b,a]$

$= a(b(ca)) - a[c,b,a]$ (from theorem 3.3, Corollary II, (iii)).

$= a(b(ca) + [b,c,a])$

$= a(b(ca) + (bc)a - b(ca))$

i.e. $(ab)(ca) = a((bc)a)$

(ii) $((ab)a)c = [ab,a,c] + (ab)(ac)$

$= [b,a,c]\,a + (ab)(ac)$ (from Corollary II, (iii) above)

$$= [b,a,ac]+(ab)(ac)$$

$$= (ab)(ac)-[a,b,ac]$$

$$= (ab)(ac)-(ab)(ac)+a(b(ac))$$

i.e. $((ab)a)c = a(b(ac))$

(iii) $(a(bc))a = ((ab)c-[a,b,c])a$

$$= ((ab)c)a-[a,b,c]a$$

$$= ((ab)c)a-[ab,c,a]$$ (corollary II (ii) above)

$$= ((ab)c)a-((ab)c)a+(ab)(ca)$$

i.e. $(a(bc))a = (ab)(ca)$

(iv) $c(a(ba))= (ca)(ba)-[c,a,ba]$

$$= (ca)(ba)-[ba,c,a]$$

$$= (ca)(ba)-a[b,c,a]$$ (corollary II (iii)

$$= (ca)(ba)+[ca,b,a]$$

$$= (ca)(ba)+((ca)b)a-(ca)(ba)$$

i.e. $c(a(ba)) = ((ca)b)a$.

These four identities are sometimes called the Moufang identities , after Ruth Moufang (Hall (5), page 424) who first enunciated them.

SOME ELEMENTARY PROPERTIES OF FIELDS

Some basic properties of finite fields are now stated. Proofs of these results can be found in van der Waerden.[1]

(1) A finite field F can be regarded as a vector space (of finite dimension, say n) over the additive

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[1]Van der Waerden, Modern Algebra, Vol. I, Ungar Publishing Company, 1953.

group generated by the element 1. As all finite fields have prime characteristic p, the order of a finite field F will be p^n.

(2) If F is a finite field of order p^n, then $F-\{0\}$, which will be denoted by F^*, is a multiplicative group of order p^n-1.

(3) If K is a subfield of a finite field F, and if F has order p^n, then K has order p^m where m divides n.

(4) The intersection of two subfields of a field F is itself a subfield of F.

DEFINITION OF THE CENTRALIZER

For an arbitrary element $x\varepsilon F$, define the centralizer $C_F(x)$ of x in F as follows:

$$C_F(x) = \{f\varepsilon F \mid xf = fx\} .$$

Lemma 3.8 Let F be a finite field. For arbitrary $x\varepsilon F$, $C_F(x)$ is a subfield of F.

Proof: Trivially $0\varepsilon C_F(x)$ and $1\varepsilon C_F(x)$, for any $x\varepsilon F$.

To show that $C_F(x)$ is a subfield of F it suffices to verify closure of addition and multiplication; the associativity of addition and multiplication and the commutativity of addition in $C_F(x)$ then follow from the corresponding facts about F.

Let y and z be arbitrary elements of $C_F(x)$.

Then $$xy = yx , \quad yz = zx.$$

Thus $$(y+z)x = yx+zx$$
$$= xy+xz$$

i.e. $$(y+z)x = x(y+z)$$

and thus $y+z\varepsilon C_F(x)$.

Hence $C_F(x)$ is closed under addition.

Similarly,

$$(yz)x = y(zx) = y(xz)$$
$$= (yx)z = (xy)z$$

i.e. $(yz)x = x(yz)$

and thus $yz\varepsilon C_F(x)$.

Hence $C_F(x)$ is closed under multiplication.

For a given $z\varepsilon C_F^*(x)$, the mapping

$$\varphi_z : y \longrightarrow yz \qquad (z, y\varepsilon C_F(x))$$

is a one-to-one mapping of $C_F(x)$ onto $C_F(x)$.

Since $1\varepsilon C_F(x)$, there exists $\bar{z}\varepsilon C_F(x)$ such that $\bar{z}z = 1$;

hence $z^{-1}\varepsilon C_F^*(x)$ if $z\varepsilon C_F^*(x)$.

Similarly $(-z)\varepsilon C_F(x)$ if $z\varepsilon C_F(x)$.

This completes the verification that $C_F(x)$ is a field.

Theorem 3.5

Wedderburn's Theorem. Every finite field is a commutative field.

Proof: Let F be a finite field of order p^n, and let x be arbitrary in F. Let

$$|C_F(x)| = p^{i_x}$$

and let $|Z(F)| = p^m$.

Then for all $x\varepsilon F$, i_x divides n and m divides i_x.

For arbitrary $x\varepsilon F^*$, define C_x, the conjugate class of the multiplicative group F^*, as

$$C_x = \{g^{-1}xg \,|\, g\varepsilon F^*\}.$$

Then F^* can be partitioned into disjoint conjugate classes, as it is easily verified that being in a given conjugate

class is an equivalence relation. Hence the order of F^* can be written as the sum of the orders of the conjugate classes of F^*; this equality is called the class equation of F^*.

Now

$$|C_z| = 1 \iff z\varepsilon Z^*(F).$$

Hence there are (p^m-1) singleton conjugate classes of F^*.

Also, $$|C_x| = \left[F^*: C_F^*(x)\right];$$

for let $C_F(x)g_1$ and $C_F(x)g_2$ be two distinct cosets of $C_F^*(x)$ in F^*, and let

$$\lambda_1 \varepsilon C_F^*(x)g_1, \quad \lambda_2 \varepsilon C_F^*(x)g_2.$$

Then

$$\lambda_1^{-1}x\lambda_1 = (fg_1)^{-1}x(fg_1), \quad f\varepsilon C_F^*(x)$$

$$= g_1^{-1}f^{-1}xfg_1$$

i.e. $$\lambda_1^{-1}x\lambda_1 = g_1^{-1}xg_1 .$$

Thus conjugation of x by any element from a given coset $C_F^*(x)g_1$ yields the same conjugate $g_1^{-1}xg_1$.

However, $\lambda_1^{-1}x\lambda_1 \neq \lambda_2^{-1}x\lambda_2$; for if this were not the case, then

$$g_1^{-1}xg_1 = g_2^{-1}xg_1$$

i.e. $$(g_1g_2^{-1})^{-1}x(g_1g_2^{-1}) = x$$

and thus $$g_1g_2^{-1}\varepsilon C_F(x),$$

contradicting the hypothesis that $C_F^*(x)g_1$ and $C_F^*(x)g_2$ were distinct cosets.

Hence $$|C_x| = \left[F^*: C_F^*(x)\right] = \frac{p^n-1}{p^{l_x}-1}$$

Thus the class equation for F is

$$p^n-1 = p^m-1+\sum \frac{p^n-1}{p^{i_x}-1} \qquad \ldots\ldots (1)$$

where the summation is over distinct conjugate classes of order > 1.

As m divides n and m divides i_x for all x, we can write

$$p^m = q$$

$$n = mr$$

$$i_x = mj_x, \quad \text{for all } x\varepsilon F^*.$$

Then equation (1) becomes

$$q^r-1 = q-1+\sum \frac{q^r-1}{q^{j_x}-1} \qquad \ldots\ldots (2)$$

Define $\Phi_r(q)$ by

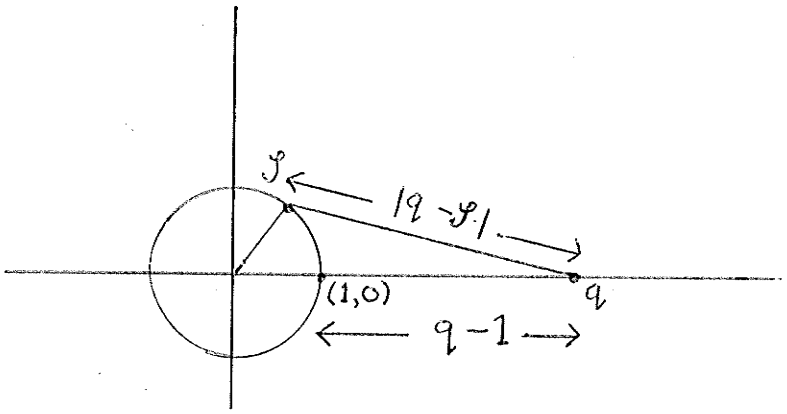
$$\Phi_r(q) = \pi_i(q-\zeta_i),$$

where the product is taken over all the rth primitive roots ζ_i of 1. Then $\Phi_r(q)$ divides q^r-1. Also, since $j_x < r$ for all $x\varepsilon F^*$(if we assume that $Z^*(F) \neq F^*$), $(q^{j_x}-1)$ contains no factors of the form $q-\zeta$, where ζ is an rth root of unity. Consequently $\Phi_r(q)$ divides $\sum \frac{q^r-1}{q^{j_x}-1}$. Hence from equation (2), it follows that

$$\Phi_r(q) \text{ divides } (q-1) \qquad \ldots\ldots (3)$$

Set up an Argand diagram of the complex plane, and consider the distribution of the rth primitive roots of unity about the unit circle. As q is a positive integer, it is seen that

$$|q-\zeta| > q-1 \quad \text{for each root } \zeta .$$



However, result (3) implies that

$$\pi(q-\mathcal{S}) \text{ divides } (q-1),$$

from which it follows that $r = 1$. Thus $n = m$, i.e. $Z(F) = F$, and as $Z(F)$ is commutative by definition, F is commutative.

Theorem 3.6 Every finite alternative field K is a commutative field.

Proof: The proof can be broken into four main assertions:

(1) If a and b are arbitrary in K, then $D = \{a,b,1\}$ generates a finite subfield F of K.

(2) By theorem 3.5, F is a commutative field, and so $ab = ba$.

(3) As a and b were arbitrary in K, K is commutative.

(4) A commutative alternative field is associative, and hence K is a commutative field.

Assertion (2) follows from (1), and (3) follows from (2), immediately. Hence it suffices to prove assertions (1) and (4).

To prove (1), some notation is first introduced. If A,B, and C are subsets of K, then $[A,B,C]$ is defined by $[A,B,C] = \{[a,b,c] \mid a\varepsilon A,\ b\varepsilon B,\ c\varepsilon C.\}$ By the statement "$[A,B,C] = 0$" we shall mean that

$$[a,b,c] = 0 \text{ , for all } a\varepsilon A,\ b\varepsilon B,\ c\varepsilon C.$$

Define a subset X of K by

$$X = \{x \mid [D,D,x] = 0,\ x\varepsilon K\}$$ (where D is $\{a,b,1\}$ as above).

Then $D\subseteq X$; for let $[d_1,d_2,d_3]\varepsilon[D,D,D]$. If $d_i = d_j$, $i \neq j$ $(i,j = 1,2,3)$ then by lemma 3.3, $[d_1,d_2,d_3] = 0$. If d_1,d_2,d_3 are distinct, then as $D = \{a,b,1\}$, one of d_1,d_2,d_3 is 1, and hence $[d_1,d_2,d_3] = 0$ as any associator with 1 as one of its elements is zero. Hence $[D,D,D] = 0$ and thus $D\subseteq X$.

Secondly, $DX\subseteq X$; for let $[d_1,d_2,d_3x]$ be arbitrary in $[D,D,DX]$. If $d_1 = d_2$, then $[d_1,d_2,d_3x] = 0$ by lemma 3.3. If d_1 or $d_2 = 1$, then $[d_1,d_2,d_3x] = 0$ as above. Hence without loss of generality assume that $d_1 = a$, $d_2 = b$.

If $d_3 = 1$, then $[d_1,d_2,d_3x] = 0$ by definition of X. If $d_3 = a$, then

$$\begin{aligned}[d_1,d_2,d_3x] &= [a,b,ax] \\ &= -[ax,b,a] \\ &= -[x,b,a]a \quad \text{(theorem 3.3, Corollary II, part (ii)),} \\ &= [a,b,x]a\end{aligned}$$

i.e. $[d_1,d_2,d_3x] = 0$

as $$[a,b,x]\varepsilon[D,D,X] = 0$$

If $d_3 = b$ the proof is similar. Hence in general $[D,D,DX] = 0$ and so $DX \subseteq X$.

Define a subset Y of K by

$$Y = \{y \mid y\varepsilon X,\ yX \subseteq X,\ [D,X,y] = 0.\}$$

Then $D \subseteq Y$. To verify this, suppose that $[d_1,x,d_2]$ is arbitrary in $[D,X,D]$. Then

$$[d_1,x,d_2] = -[d_1,d_2,x] = 0$$

and thus $[D,X,D] = 0$. As $D \subseteq X$ and $DX \subseteq X$, it follows that $D \subseteq Y$.

Define a subset R of K by

$$R = \{r \mid r\varepsilon Y,\ [X,Y,r] = 0.\}$$

Then $D \subseteq R$, as can be verified by the methods employed above. It follows that

$$D \subseteq R \subseteq Y \subseteq X.$$

In order to verify (1), it suffices to show that R is a field. By definition of X,Y, and R, it follows that $0\varepsilon R$ and $1\varepsilon R$.

Addition is closed in R; for if $r_1\varepsilon R$, $r_2\varepsilon R$, then $r_1+r_2\varepsilon R$ by the linear property of associators and the definitions of X, Y, and R.

Multiplication is closed in R, i.e. $RR = R$; for if r_1 and r_2 are arbitrary in R, then as $R \subseteq Y$ and $R \subseteq X$, $r_1\varepsilon Y$ and $r_2\varepsilon X$; thus

$$r_1r_2\varepsilon yX \subseteq X\ ,$$

and so $r_1r_2\varepsilon X$; in general $RR \subseteq X$. Further, as $[X,Y,R] = 0$, we have

$$(r_1r_2)X = r_1(r_2X).$$

As $r_2 \varepsilon Y$, $r_2X \subseteq X$ and so

$$(r_1r_2)X \subseteq r_1X \subseteq X \qquad \text{(as } r_1 \varepsilon Y\text{)}.$$

Also, $r_1r_2 \varepsilon X$, so in order to show that $r_1r_2 \varepsilon Y$, it remains to verify that $[D,X,r_1r_2] = 0$. However,

$$[D,X,r_1r_2] = 0 \quad \text{if and only if}$$

$$[X,D,r_1r_2] = 0 \quad \text{(by lemma 3.3)}$$

and this will certainly be true if

$$[X,Y,r_1r_2] = 0 \quad \text{as } D \subseteq Y.$$

For arbitrary $x \varepsilon X$, $y \varepsilon Y$, consider

$$f(y,x,r_1,r_2) = [yx,r_1,r_2] - [x,r_1,r_2]y - x[y,r_1,r_2].$$

As $[X,Y,R] = 0$ and as $R \subseteq Y \subseteq X$, the last two associators are zero; also $yx \varepsilon X$, so by similar reasoning, $[yx,r_1,r_2] = 0$. Hence

$$f(y,x,r_1,r_2) = 0.$$

By theorem 3.3, this implies that

$$0 = f(r_1,r_2,y,x)$$

$$= [r_1r_2,y,x] - [r_2,y,x]r_1 - r_2[y,x,r_1]$$

and by the alternative property of associators and the argument used above, one obtains

$$0 = [r_1r_2,y,x] = -[x,y,r_1r_2].$$

Hence

$$[X,Y,RR] = 0 \quad \text{so } RR \subseteq Y.$$

By the definition of R it immediately follows that $RR = R$, and so R is closed under multiplication.

Addition is associative and commutative in R, as it is in K. Also,

$$[R,R,R] \subseteq [X,Y,R] = 0$$

so multiplication in R is associative.

Finally, $r \in R^* \Longrightarrow r^{-1} \in R^*$

and $-r \in R \Longrightarrow -r \in R$

by the methods used to establish this in lemma 3.8. Hence R is a field, and as $D \subseteq R$, D generates a subfield of K.

This completes the proof of (1).

To prove (4), first note that by theorem 3.2, if $1+1+1 \neq 0$ in K, then K is associative. Hence all that need be considered now is the case in which $1+1+1 = 0$ (i.e. $3 = 0$).

Let u, v, and a be arbitrary in K. Consider the element $(uv)a^3$. As $\langle 1,uv,a \rangle$ is a field, and hence associative, it follows that

$$\begin{aligned}
(uv)a^3 &= (uv)((a^2)a) \\
&= ((uv)(a^2))a \\
&= (a^2(uv))a \\
&= a^2((uv)a) \\
&= a(a((uv)a)) \\
&= a((au)(va)) \qquad \text{(by theorem 3.4 (i))} \\
&= ((ua)(av))a \\
&= u(a((av)a)) \qquad \text{(by theorem 3.4 (iv))}
\end{aligned}$$

i.e. $(uv)a^3 = u(va^3)$ ..... (1)

(as $\langle 1,v,a \rangle$ is a field).

Choose a,b,c arbitrary in K^*. Define d by

$$(ab)c = (a(bc))d.$$

As any two elements of K, plus the identity, generate a field, for arbitrary $x,y \in K$:

$$x^3y^3 = (xy)^3.$$

Hence

$$\begin{aligned}(a^3b^3)c^3 &= (ab)^3c^3 = ((ab)c)^3 \\ &= ((a(bc))d)^3 \\ &= (a(bc))^3d^3 \\ &= (a^3(bc)^3)d^3\end{aligned}$$

i.e. $(a^3b^3)c^3 = (a^3(b^3c^3))d^3.$

By equation (1), expressions involving cubes as the third term are associative; hence

$$(a^3b^3)c^3 = ((a^3b^3)c^3)d^3$$

and thus $d^3 = 1,$

i.e. $d^3-1 = 0.$

As $1+1+1 = 0$, $d^3-1 = (d-1)^3$, so

$$(d-1)^3 = 0,$$

i.e. $d-1 = 0$ or $d = 1.$

Hence by the definition of d, for arbitrary $a,b,c \in K$,

$a(bc) = (ab)c$; i.e. K is associative.

Hence K is a commutative field, and therefore every finite alternative field is a commutative field.

CHAPTER IV

GROUP THEORETICAL AND COMBINATORIAL THEOREMS

The Elementary Theory of Permutation Groups

In this section a brief outline of the basic ideas of the theory of permutation groups will be given, with particular reference to how the theory relates to collineation groups of projective planes. In addition, several special results that will be needed later are proved. For a detailed treatment of this subject reference should be made to Wielandt (14).

Let S be a finite set. The number of elements in a subset T of S will be denoted by $|T|$ and will be called the order of T. A permutation of S is a one-to-one mapping of S onto itself. If g is a permutation of S and if $s \in S$, then the image of s under the permutation g will be written as s^g. If g and h are two permutations of S, their product gh is defined as the mapping sending $s \longrightarrow (s^g)^h$. Then gh is a permutation if g and h are, and multiplication of permutations is associative.

A permutation group G on a set S is a group each of whose elements is a permutation of S. The identity permutation 1 fixes every element of S while the inverse g^{-1} of a permutation g is defined by writing

$$s^{g^{-1}} = t \Longleftrightarrow t^g = s \quad \text{for all } s,t \varepsilon S.$$

Obviously g^{-1} is a permutation.

One subgroup of G deserves special mention. If $T \subseteq S$, then G_T is defined by

$$G_T = \left\{ g \varepsilon G \; t^g = t \quad \text{for all } t \varepsilon T \right\}.$$

Obviously G_T is a subgroup of G and thus also a permutation group of S. If in particular T is the singleton set $\{t\}$, then G_t is called the stabilizer of t.

The order of G will be denoted by $|G|$ and will always be assumed to be finite.

For arbitrary $s \varepsilon S$, the "orbit" of s, denoted Orb s, is defined as follows:

$$\text{Orb } s = \left\{ s^g \mid g \varepsilon G \right\}.$$

Orbits are sometimes called "transitive classes" of S under G. The set of orbits of S under G forms a partition of S into disjoint sets. In general Orb $(s) \subseteq S$; however, if for any $s \varepsilon S$ Orb $(s) = S$, then G is said to be "transitive" on S. Evidently G will be transitive on S if and only if for arbitrary $\alpha, \beta \varepsilon S$, there exists $g \varepsilon G$ such that $\alpha^g = \beta$.

More generally, an <u>action</u> of an abstract group G on a given set S is defined as a function mapping SxG onto S such that $s^1 = s$ for all $s \varepsilon S$ and such that $(s^{g_1})^{g_1} = s^{g_1 g_2}$ for all $s \varepsilon S$ and $g_1, g_2 \varepsilon G$. A given group G may have an action defined on more than one set, and the structure of G considered as a permutation group on one set may differ from the structure of G

considered as a permutation group on another. As an example, a collineation of a projective plane has an action on two distinct sets, the set of points of the plane and the set of lines of the plane. If a group G has an action on a set S, it can be regarded as a permutation group of S.

Of basic importance is the "fundamental" theorem of permutation groups:

Lemma 4.1 Let a group G have an action on a set S and let α be an arbitrary point of S. Then $|G_\alpha|\,|\text{Orb }(\alpha)| = |G|$.

Proof: For arbitrary $g_1, g_2 \in G$, it is seen that

$$\alpha^{g_1} = \alpha^{g_2} \Leftrightarrow \alpha^{g_1 g_2^{-1}} = \alpha$$

i.e. $$\Leftrightarrow g_1 g_2^{-1} \in G_\alpha$$

i.e. $\alpha^{g_1} = \alpha^{g_2} \Leftrightarrow g_1$ and g_2 are in the same right coset of G_α in G. Hence the number of images of α under G equals the number of right cosets of G_α in G, i.e.

$$|\text{Orb }(\alpha)| = [G:G_\alpha] = \frac{|G|}{|G_\alpha|}.$$

Hence $$|\text{Orb }(\alpha)|\,|G_\alpha| = |G|.$$

Lemma 4.2 Let a group G have an action on a set S and let T_1 and T_2 be two orbits of S under G. If $|T_1|$ and $|T_2|$ are relatively prime and if $\delta_1 \in T_1$, then G_{δ_1} is transitive on T_2.

Proof: Let $\delta_1 \varepsilon T_1$ and $\delta_2 \varepsilon T_2$. Then Orb $(\delta_1) = T_1$ and Orb $(\delta_2) = T_2$. Hence by lemma 4.1,

$$|T_1||G_{\delta_1}| = |G|$$

and

$$|T_2||G_{\delta_2}| = |G|.$$

Thus

$$|T_1||G_{\delta_1}| = |T_2||G_{\delta_2}|. \qquad \ldots\ldots (1)$$

As $|T_1|$ and $|T_2|$ are relatively prime, this implies that integers k_1 and k_2 exist such that

$$|G_{\delta_1}| = k_2|T_2|$$

and

$$|G_{\delta_2}| = k_1|T_1|.$$

Substitution into equation (1) gives

$$|T_1|k_2|T_2| = |T_2|k_1|T_1|$$

i.e.

$$k_1 = k_2 = k \text{ (say)}.$$

But considering G_{δ_1} as a permutation group on T_2 and using lemma 4.1, if the orbit of δ_2 under G_{δ_1} is denoted by Orb $G_{\delta_1}(\delta_2)$, it follows that

$$|\text{Orb } G_{\delta_1}(\delta_2)|\,|(G_{\delta_1})_{\delta_2}| = |G_{\delta_1}|. \qquad \ldots\ldots (2)$$

Similarly $|\text{Orb } G_{\delta_2}(\delta_1)|\,|(G_{\delta_2})_{\delta_1}| = |G_{\delta_2}|. \qquad \ldots\ldots (3)$

Noting that $(G_{\delta_1})_{\delta_2} = (G_{\delta_2})_{\delta_1}$ and dividing equation (2) by equation (3), one obtains

$$\frac{|\text{Orb } G_{\delta_1}(\delta_2)|}{|\text{Orb } G_{\delta_2}(\delta_1)|} = \frac{|G_{\delta_1}|}{|G_{\delta_2}|} = \frac{k|T_2|}{k|T_1|} = \frac{|T_2|}{|T_1|}$$

i.e.

$$|T_1|\,|\text{Orb } G_{\delta_1}(\delta_2)| = |T_2|\,|\text{Orb } G_{\delta_2}(\delta_1)|.$$

Thus as $|T_1|$ and $|T_2|$ are relatively prime, there exists an integer n such that

$$n|T_2| = |\text{Orb } G_{\delta_1}(\delta_2)|$$

i.e. $$|T_2| \; |\text{Orb } G_{\delta_1}(\delta_2)|.$$

As Orb $G_{\delta_1}(\delta_2) \subseteq T_2$, $n = 1$ and thus Orb $G_{\delta_1}(\delta_2) = T_2$.

Hence by definition G_{δ_1} is transitive on T_2.

Lemma 4.3 Let a group G have an action on a set S and let H be a subgroup of G. Let

$$T = \{ s \in S \mid s^h = s \text{ for all } h \in H \}.$$

Then $N_G(H)$, the normalizer of H in G, will permute the members of S amongst themselves.

Proof: Let n be arbitrary in $N_G(H)$, let t be arbitrary in T, and let h_1 be arbitrary in H.

Then $$nh_1n^{-1} = h_2 \in H.$$

Thus $${}_t{}^{nh_1n^{-1}} = {}_t h_2 = t \quad \text{by definition of T.}$$

Hence $$({}_t n)h_1 = {}_t n \; .$$

As h_1 was arbitrary in H, ${}_t{}^n \in T$; as n was arbitrary in $N_G(H)$, $N_G(H)$ permutes members of T amongst themselves.

Lemma 4.4 (see Gleason (4)) Let a group G have an action on a set S, and let $T \subseteq S$ be such that for any $t \in T$, there exists $g \in G$ of given prime order p such that g fixes t but no other element of S. Then T is contained in an orbit of S under G.

Proof: Choose $t \in T$ and $g \in G$ as specified in the hypotheses, and let D be the orbit of S under G to which t belongs. Then $\langle g \rangle$, the group generated by g, is a permutation group of order p on D and partitions it into disjoint

orbits. The orbit of t has order one and the orbits of all other members of D, since none is fixed by g, will have order p. Hence

$$|D| = 1+np \qquad \ldots\ldots (1)$$

for some integer n.

If $T \nsubseteq D$, then there exists $\bar{t} \in T \cap (S-D)$ and $\bar{g} \in G$ such that $\bar{t}^{\bar{g}} = \bar{t}$ but all other elements of S are moved by $\bar{g}$. Thus in particular all elements of D are moved by $\bar{g}$, so the permutation group $\langle \bar{g} \rangle$ partitions D into disjoint orbits each of order p. Hence $|D|$ is divisible by p, which contradicts equation (1). Hence the assumption that such a $\bar{t}$ exists is false and $T \subseteq D$.

Definitions

(1) A permutation group G transitive on a set S is said to be regular on S if $|G| = |S|$.

(2) A permutation group G is said to be a Frobenius group on a set S if:

a) G is transitive but not regular on S,

b) only the identity of G fixes two distinct elements of S.

(3) The kernel K of a Frobenius group F on a set S is defined as follows:

$$K = \{k \in F \mid s^k \neq s \text{ for all } s \in S\} \cup \{1\}.$$

(4) Let G be a permutation group transitive on a set S. Then G is said to be imprimitive if S can be partitioned into a collection $\{T_i\}$ of disjoint sets,

each of the same order t where $1 < t < |S|$, such that each element of G maps any T_i either onto itself or onto another of the sets. G is said to be primitive on S if it is not imprimitive on S.

(5) A permutation group G on a set S is said to be doubly transitive on S if for arbitrary $\alpha \varepsilon S$, G is transitive on S and G_α is transitive on $S-\{\alpha\}$.

Definition A fixed-point-free automorphism α of a finite group G is an automorphism of G such that $g^\alpha = g \Longleftrightarrow g = 1$ for all $g \varepsilon G$.

Lemma 4.5 If a finite group G has a fixed-point-free automorphism α of order 2, then G is abelian.

Proof: First note that the mapping $x \longrightarrow x^{-1}x^\alpha$, $x \varepsilon G$, is one-to-one; for if $x^{-1}x^\alpha = y^{-1}y^\alpha$ for $x,y \varepsilon G$ then $yx^{-1} = y^\alpha(x^\alpha)^{-1} = (yx^{-1})^\alpha$ and α fixes yx^{-1}. Thus $yx^{-1} = 1$, i.e. $y = x$ and the mapping is one-to-one as claimed. Hence as G is finite it is onto, so $y_1, y_2 \varepsilon G \Longrightarrow$ there exist $x_1, x_2 \varepsilon G$ such that $y_1 = x_1^{-1}x_1^\alpha$ and $y_2 = x_2^{-1}x_2^\alpha$, and x_1 and x_2 are unique. But $y_1 = x_1^{-1}x_1^\alpha \Longrightarrow y_1^\alpha = (x_1^{-1})^\alpha x_1^{\alpha^2} = (x_1^\alpha)^{-1}x_1 = y_1^{-1}$; consequently

$$(y_1y_2)^\alpha = y_1^\alpha y_2^\alpha = y_1^{-1}y_2^{-1}.$$

But $$(y_1y_2)^\alpha = (y_1y_2)^{-1} = y_2^{-1}y_1^{-1};$$

hence $y_1^{-1}y_2^{-1} = y_2^{-1}y_1^{-1}$ and taking inverses gives

$$y_1 y_2 = y_2 y_1.$$

As y_1 and y_2 were arbitrary in G, G is abelian.

Lemma 4.6 (see Wagner (12)). Let F be a Frobenius group of order 2n on a set S of order n, where n is odd. Then:

(1) If α and β are distinct elements of S, there exists $f \in F$ such that $f^2 = 1$ and $\alpha^f = \beta$.

(2) The kernel K of F is a characteristic subgroup of F and is regular on S.

(3) If H is a subgroup of F possessing r distinct elements of order 2, where $r > 1$, then $|H| = 2r$.

Proof: Any element f of order 2 in F interchanges elements of S in pairs, except for those elements that it fixes. As $|S|$ is odd, this implies that f fixes at least one element of S. As F is a Frobenius group, f fixes no more than one element of S; hence f fixes exactly one element of S. Thus to each element of order 2 in F there corresponds exactly one point of S fixed by it.

Conversely, for arbitrary $\alpha \in S$, by lemma 4.1

$$|\mathrm{Orb}\ (\alpha)|\ |F_\alpha| = |F|.$$

As F is transitive on S, $|\mathrm{Orb}(\alpha)| = n$; also $|F| = 2n$; hence $|F_\alpha| = 2$. Thus there is, for each element $\alpha \in S$, exactly one element of F of order 2 fixing α. Hence there is a one-to-one correspondence between points of

S and elements of F of order 2. The element of order 2 in F that fixes $\alpha \varepsilon S$ will be denoted f_α.

Assertion (1) is now proved. Let α and β be distinct points of S and let $\gamma \varepsilon S - \{\alpha\}$. Then $\alpha^{f_\gamma} \neq \alpha$ (as otherwise f_γ would fix both α and γ, which contradicts F being Frobenius). Now suppose that for some $\delta \varepsilon S$, $\alpha^{f_\gamma} = \alpha^{f_\delta}$. Then $\alpha^{f_\gamma f_\delta} = \alpha$, and

$$(\alpha^{f_\gamma})^{f_\gamma f_\delta} = \alpha^{f_\gamma^2 f_\delta} = \alpha^{f_\delta} = \alpha^{f_\gamma},$$

so $f_\gamma f_\delta$ fixes the distinct points α and α^{f_γ}. Thus $f_\gamma f_\delta = 1$, i.e. $f_\gamma = f_\delta$, and by the one-to-one correspondence established above, $\gamma = \delta$. Hence $\alpha^{f_\gamma} = \alpha^{f_\delta} \Longrightarrow \gamma = \delta$, and so as γ ranges over the (n-1) elements of $S - \{\alpha\}$, α^{f_γ} also ranges over these (n-1) elements; hence there must exist $\gamma \varepsilon S - \{\alpha\}$ such that $\alpha^{f_\gamma} = \beta$, and as $f_\gamma^2 = 1$, assertion (1) holds.

To prove assertion (2), note that from the proof of (1) K consists of all elements of F that are not of order 2; hence $|K| = 2n-n = n$. No two elements of order 2 in F interchange the same two elements of S; for suppose that $x,y \varepsilon F$ (and x and y are of order 2) and that α and β are distinct elements of S such that $\alpha^x = \beta$, $\alpha^y = \beta$. Then $\beta^x = \alpha$ and $\beta^y = \alpha$, and so $\beta^{xy} = \alpha^y = \beta$. Similarly $\alpha^{xy} = \alpha$, so xy fixes β and α: hence as F is Frobenius, $xy = 1$, i.e. $x = y$.

Let $x_1, \cdots, x_n$ be the n elements of order 2 in F and consider the set $x_1x_1, x_1x_2, \cdots, x_1x_n$. These are

n distinct elements of F and none but $x_1x_1 = 1$ fixes an element of S; for if x_1x_p fixed α, then both x_1 and x_p would interchange α and α^{x1}, in contradiction to the previous paragraph. Hence $K = \{x_1x_1, \cdots, x_1x_n\}$. But similarly $K = \{x_1x_1, x_2x_1, \cdots, x_nx_1\}$; thus if $k_1, k_2 \varepsilon K$ then $k_1 = x_px_1$ and $k_2 = x_1x_q$ for $1 \leq p \leq n$, $1 \leq q \leq n$ and $k_1k_2 = x_px_q \varepsilon K$ by the characterization of K. Hence K is closed under multiplication, and is thus a group. Evidently K is normal in F, and as it consists of all elements not of order 2 (plus the identity), it is characteristic in F. As no element of K (other than 1) fixes an element of S and $|K| = |S|$, it follows that K is regular on S.

To prove assertion (3), let f_γ be an element of order 2 in H (as H is a subgroup of F, all elements of order 2 in H will be of this form). Let T be the orbit of S under H containing γ, and let $|T| = c$. As f_γ fixes $\gamma \varepsilon T$ and interchanges other elements of T in pairs, c is odd. Now $|H_\gamma| \geq 2$ as H_γ contains both 1 and γ; however $|H_\gamma| \leq |F_\gamma| = 2$. Hence $|H_\gamma| = 2$. But by lemma 4.1,

$$|\text{Orb } \gamma||H_\gamma| = |H|$$

and thus

$$|H| = 2c.$$

Now consider the action of H on T. Only the identity of H fixes more than one point of T, as H is a subgroup of a Frobenius group; also, H is transitive but

not regular on T, and $|H| = 2c$ while $|T| = c$, where c is odd. Hence the same hypotheses hold on H and T as hold on F and S. Consequently there is a one-to-one correspondence between elements of T and elements of order 2 of H. Thus there are c elements of order 2 in H, i.e. $r = c$, and thus $|H| = 2r$.

Lemma 4.7 (see Wagner (12)) Let G be a doubly transitive permutation group on a set S consisting of (n+1) elements, where n is odd. Let $\alpha \in S$ be such that G_α contains a subgroup F_α of even order which is a Frobenius group on $S-\{\alpha\}$. Then G_α is primitive on $S-\{\alpha\}$.

Proof: For an arbitrary element $\beta \in S$, since G is transitive on S there exists $g \in G$ such that $\alpha^g = \beta$. Then $g^{-1}F_\alpha g$, which will be denoted as F_β, is a subgroup of $g^{-1}G_\alpha g = G_\beta$, and is a Frobenius group on $S-\{\beta\}$ (since F_β is isomorphic to F_α and thus plays the same role vis-a-vis β as F_α plays towards α).

Let K_β be the kernel of F_β. Then by lemma 4.6 part (2) K_β is a characteristic subgroup of F_β and is regular on $S-\{\beta\}$. Hence $|K_\beta| = |S-\{\beta\}| = n$. As $|F_\beta|$ is even, F_β contains an element b of order 2; as $|K_\beta|$ is odd, $b \notin K_\beta$ and hence by definition of the kernel, b fixes an element of $S-\{\beta\}$.

Let $\bar{F}_\beta$ be the group generated by K_β and b. As

K_β is normal in F_β (being characteristic), a typical element f of $\bar{F}_\beta$ is of the form $f = b^r k$ where $k \varepsilon K_\beta$ and r = o or 1. Hence $|\bar{F}_\beta| = 2|K_\beta| = 2n$. Also, $\bar{F}_\beta$ is transitive on $S-\{\beta\}$, as it contains K_β, but is not regular, as it contains $b \neq 1$ fixing an element of $S-\{\beta\}$. Only the identity of $\bar{F}_\beta$ fixes two distinct elements of $S-\{\beta\}$ as $\bar{F}_\beta$ is a subgroup of the Frobenius group F_β; hence $\bar{F}_\beta$ is a Frobenius group on $S-\{\beta\}$.

Now assume that G_α is not primitive on $S-\{\alpha\}$. Then $S-\{\alpha\}$ can be partitioned into sets of imprimitivity $T_1, T_2, \cdots, T_s$, each of order t where $1 < t < n$. Then st = n, and as n is odd, so are s and t. Let $T_1 = \{\delta_i \mid i = 1 \text{ to } t\}$ and let $\gamma \varepsilon T_s$. As $\bar{F}_\gamma$ is a Frobenius group on a set $S-\{\gamma\}$ of odd order n, as $\alpha \neq \gamma$ and $\beta_i \neq \gamma$, i = 1 to t, and as $\bar{F}_\gamma = 2n$, the hypotheses of part (1) of lemma 4.5 hold and so there exists a set of elements of $\bar{F}_\gamma$, namely $\{c_i \mid i = 1 \text{ to } t\}$, such that $c_i^2 = 1$ and $\alpha^{c_i} = \beta_i$, i = 1 to t.

Let $T_1 \cup \{\alpha\} = V$. Then $V^{c_i} = V$ for i = 1 to t. To prove this, note first that $\alpha^{c_i} = \beta_i \varepsilon V$ for i = 1 to t. Now suppose that there exist j and k, $1 \leq j \leq t$ and $1 \leq k \leq t$, such that $\beta_k^{c_j} = \delta$ and $\delta \notin V$. But as $\bar{F}_\delta$ is a Frobenius group of order 2n on the set $S-\{\delta\}$ of order n (n odd), and as $\alpha, \beta_j \neq \delta$ (since $\delta \notin V$), by lemma 4.6, part (1), there exists $d \varepsilon \bar{F}_\delta$ such

that $d^2 = 1$ and $\alpha^d = \beta_j$. Thus $\alpha^{c_j d} = \beta_j{}^d = \alpha$, and hence $c_j d \in G_\alpha$. But $\beta_j{}^{c_j d} = \alpha^d = \beta_j$, so as G_α is imprimitive on $S - \{\alpha\}$ by assumption and $c_j d$ fixes $\beta_j \in T_1$, $c_j d$ should map T_1 onto T_1. However, $\beta_k \in T_1$ and $\beta_k{}^{c_j d} = \delta^d = \delta$; hence as $\delta \notin T_1$, $c_j d$ does not map T_1 onto T_1. This contradiction implies that the assumed j and k do not exist and hence $V^{c_i} = V$, $i = 1$ to t, as claimed.

Let H be the subgroup of $\bar{F}_\gamma$ generated by the $\{c_i\}$. Also, H maps V onto itself by the above remarks, and from the way in which the $\{c_i\}$ were defined, H is transitive on V. Also, as $\bar{F}_\gamma$ is a Frobenius group on $S - \{\gamma\}$, each non-identity element of H either fixes no point of $S - \{\gamma\}$ or has order 2 and fixes precisely one point of $S - \{\gamma\}$ (as recall $|S - \{\gamma\}| = n$ and $|\bar{F}_\gamma| = 2n$). But $|V| = t+1$ and as t is odd, t+1 is even. Thus no element of order 2 in H can fix any point of V. Hence $|H_{\beta_i}| = 1$ and $|H_\alpha| = 1$. Thus by the fundamental theorem, $|H| = t+1$. But by lemma 4.6, part (3), as H possesses at least t elements of order 2, namely the $\{c_i\}$, it follows that $|H| \geq 2t$. Thus

$$t+1 \geq 2t, \quad \text{i.e.} \quad 1 \geq t$$

which contradicts the assumption that $t > 1$. Hence the assumption that G_α is imprimitive on $S - \{\alpha\}$ is false;

i.e. G_α is primitive on $S-\{\alpha\}$.

Lemma 4.8 Let G be a permutation group primitive on a set S and let N be a non-trivial normal subgroup of G. Then N is transitive on S.

Proof: As N is a permutation group on S it partitions S into disjoint orbits. Let α be arbitrary in S and let g be arbitrary in G. Let $\alpha^g = \beta$. If the orbit of α under N is denoted $Orb_N\alpha$, then $(Orb_N\alpha)^g = Orb_N\beta$; for let $\gamma \varepsilon Orb_N\alpha$. Then there exists $n \varepsilon N$ such that $\alpha = \gamma^n$. Thus $\alpha^g = \gamma^{ng}$. But as N is normal, there exists $\bar{n} \varepsilon N$ such that $ng = g\bar{n}$. Hence

$$\beta = \alpha^g = \gamma^{ng} = (\gamma^g)^{\bar{n}}$$

and thus $\gamma^g \varepsilon Orb_N\beta$. Hence $(Orb_N\alpha)^g \subseteq Orb_N\beta$, and by reversing the argument, $(Orb_N\beta) \subseteq (Orb_N\alpha)^g$. Thus $|Orb_N\beta| = |(Orb_N\alpha)^g| = |Orb_N\alpha|$. Evidently each $g \varepsilon G$ maps each orbit either onto itself or onto another orbit. As orbits are disjoint, they form sets of imprimitivity of G of order >1 (as $|N|>1$). Hence as G is imprimitive, $Orb_N\alpha = S$ and hence N is transitive on S.

Theorem 4.1 Let G_α, F_α, and K_α be as in lemma 4.7, and in addition assume that F_α is normal in G_α. Then K_α is an elementary abelian group and thus n is a power of a prime.

Proof: As K_α is characteristic in F_α, it is normal in

G_α. As $|F_\alpha|$ is even and $|K_\alpha| = n$ and thus is odd, F_α contains an element f of order 2 not in K_α. By definition of K_α there exists $\beta \in S-\{\alpha\}$ such that $\beta^f = \beta$. Then $k \longrightarrow f^{-1}kf$ for all $k \in K_\alpha$ is a fixed-point-free automorphism of K_α; for if not then there exists $k \in K_\alpha$, $k \neq 1$, such that $fk = kf$ (as $f^{-1} = f$). Then $\beta^{fk} = \beta^{kf}$ and so $\beta^k = (\beta^k)^f$. Thus f fixes the distinct points β and β^k of $S-\{\alpha\}$ in contradiction to the fact that F_α is a Frobenius group on $S-\{\alpha\}$. Hence K_α has a fixed-point-free automorphism of order 2, so by lemma 4.5 K_α is abelian.

Now K_α has no proper non-trivial characteristic subgroups; for let N be such a group. Then $|N| < |K_\alpha| = n$. Then as K_α is normal in G_α, N is normal in G_α. But as G_α is primitive on $S-\{\alpha\}$, by lemma 4.8 N is transitive on $S-\{\alpha\}$. This contradicts the fact that $|N| < n$, and so K_α has no proper characteristic subgroups.

Assume that p is a prime > 1 dividing $|K_\alpha|$. Then $|K_\alpha| = p^r$; for if not, K_α would have a p-Sylow subgroup P properly contained in it. As K_α is abelian, P would be characteristic in K_α, contradicting the result of the previous paragraph.

Lastly, all non-identity elements of K_α have order p; for as K_α is abelian, the set $K_\alpha{}^P = \{k^P \mid k \in K_\alpha\}$ is a proper characteristic subgroup of K_α, and hence

is the identity. Hence $k^p = 1$ for all $k \in K_\alpha$, and thus K_α is elementary abelian of order $n = p^r$ for some prime p.

The following lemmas and theorems are due to Gleason (4). Proofs are also given in Pickert (8), pages 26-28.

Lemma 4.9 Let ℓ be a line and C be a point (incident with ℓ) of a finite projective plane π of order n. Denote the group of all elations with centre C and axis ℓ as $G_{C,\ell}$ and the group of all elations with axis ℓ as G_ℓ. Then:

(1) $|G_{C,\ell}|$ divides n

(2) $|G_\ell|$ divides n^2

(3) $|G_\ell| = n^2 \iff \pi$ is a translation plane with respect to ℓ.

Proof: (1) Let m be an arbitrary line through C ($m \neq \ell$). Then as each element of $G_{C,\ell}$ fixes m, $G_{C,\ell}$ acts as a permutation group on the n points of $S = m - \{C\}$ (where m is thought of as a point set). By lemma 1.4, if $Q \in S$ and $g \in G_{C,\ell}$, then Q and Qg uniquely determine the element g. Hence $|\text{Orb } Q| = G_{C,\ell}$. By lemma 4.1, $|\text{Orb } Q|$ divides $|S|$ and hence $|G_{C,\ell}|$ divides n.

(2) The n^2 points of $\mathcal{P} - \{\ell\} = T$ (where ℓ is taken as a point set) are permuted amongst themselves

by elements of G_ℓ , and hence G_ℓ acts as a permutation group on T. By the argument used in (1), if $Q \varepsilon T$ then $|\text{Orb } Q| = |G_\ell|$. By lemma 4.1, $|\text{Orb } Q|$ divides $|T|$ and hence $|G_\ell|$ divides n^2.

(3) Suppose that $|G_\ell| = n^2$. Then $|\text{Orb } Q| = n^2$ for an arbitrary point $Q \varepsilon T$. Hence as $|T| = n^2$, if Q and R are any two points of T there exists $g \varepsilon G$ such that $Q^g = R$. But this clearly implies that π is (C, ℓ)-transitive for all $C \varepsilon \ell$; i.e. that π is a translation plane with respect to ℓ .

Conversely, if π is a translation plane with respect to ℓ and if Q and R are arbitrary in T, then there exists an elation g in G_ℓ such that $Q^g = R$. Thus $|\text{Orb } Q| = n^2$, but as $|\text{Orb } Q| = |G_\ell|$ it follows that $|G_\ell| = n^2$.

Lemma 4.10 Let ℓ be a line of a projective plane π of order n. If for each point $C \varepsilon \ell$ the elation group $G_{C,\ell}$ has order $m > 1$, then π is a translation plane with respect to ℓ .

Proof: Let $|G_\ell| = r$. By part (2) of the preceding lemma, $rk = n^2$ for some integer $k > 0$. By lemma 1.5 Corollary II, $G_\ell = \bigcup_{C \in \ell} G_{C,\ell}$, and evidently $|G_{C_1,\ell} \cap G_{C_2,\ell}| = 1$ if $C_1 \neq C_2$, so

$$|G_\ell| = (n+1)(|G_{C,\ell}| - 1) + 1$$

i.e.

$$r = (n+1)(m-1)+1.$$

As $m > 1$, $r > n$ and hence $k < n$.

Since $kr = n^2$, it follows that

$$n^2 - k = (r-1)k = (n+1)(m-1)k$$

i.e. $$k = n^2 - (m-1)(n+1)k$$

or $$k-1 = n^2 - 1 - (m-1)(n+1)k$$

$$= (n+1)((n-1)-(m-1)k).$$

Thus $(n+1)$ divides $(k-1)$. As $k < n$, this implies that $k-1 = 0$, i.e. $k = 1$. Hence $r = n^2$ and by part (3) of the preceding lemma, π is a translation plane with respect to ℓ.

Lemma 4.11 Let π be a projective plane possessing a line ℓ with the following properties: for each $C \varepsilon \ell$; (1) there exists a non-trivial (C, ℓ)-elation,

(2) there exists a line m, $C \varepsilon m$ and $m \neq \ell$, such that there exists a non-trivial (C,m)-elation.

Then π is a translation plane with respect to ℓ.

Proof: By theorem 1.1 and hypothesis (1), every non-trivial elation with axis ℓ has the same prime order p. For a fixed point $C \varepsilon \ell$ denote $\bigcup_{C \varepsilon m} G_{C,m}$, the set of all elations with centre C, by G_C. By the dual of Corollary II of lemma 1.5, G_C is a group. By the hypotheses and the dual of theorem 1.1, G_C will be an elementary abelian p-group for each $C \varepsilon \ell$.

Let H be the group generated by all non-trivial (C,m)-elations, of the type described in hypothesis (2), for each $C \varepsilon \ell$. Then each element of H fixes the line ℓ, and so H acts as a permutation group on the points of ℓ. Also, for each $C \varepsilon \ell$ there exists an element of H of order p that fixes C but no other point of ℓ (namely the non-trivial (C,m)-elation for that C); hence by lemma 4.4 H is transitive on the points of ℓ. Thus if C_1 and C_2 are arbitrary distinct points of ℓ there exists $h \varepsilon H$ such that $C_1^h = C_2$. By lemma 1.6, $G_{C_2,\ell} = h^{-1}(G_{C_1,\ell})h$. Thus $|G_{C_2,\ell}| = |G_{C_1,\ell}| > 1$ and as C_1 and C_2 were arbitrary on ℓ, it follows by lemma 4.10 that π is a translation plane with respect to ℓ.

Definition Let π be a projective plane. Then the collineation group generated by all elations of π is called the little projective group of π.

Theorem 4.2 Let π be a finite projective plane and let G be a collineation group of π such that for every line ℓ and point C such that $C \varepsilon \ell$, G possesses a non-trivial (C, ℓ)-elation. Then π is Desarguesian and G contains the little projective group of π as a subgroup.

Proof: From the hypotheses, every line of π satisfies the conditions imposed on the line ℓ in lemma 4.11. Hence from lemma 4.11 π is a translation plane with respect to every line of π, i.e. π is an alternative plane. As π is finite, from the conclusions of Chapter III it follows that π is Desarguesian. Evidently the elations in G generate all possible elations of π so G contains the little projective group of π.

The following result is due to Andre (1).

Theorem 4.3 Let γ be a line of a projective plane π and let γ be the axis of two non-trivial homologies σ_1 and σ_2 with distinct centres C_1 and C_2 respectively. Let G be the group generated by σ_1 and σ_2. Then:

(1) All elements of G are central collineations with axis γ.

(2) G has an action on the points of $\pi - \{\gamma\}$ and the set of centres of the homologies in G is contained in one orbit of G.

(3) There is an elation in G mapping C_1 into C_2.

Proof: Result (1) follows immediately from the fact that every element of G fixes γ pointwise.

Let $C_1C_2 = \ell$ and let F be the group of elations of G with centre $\ell \cap \gamma$. It is easily verified that F is a normal subgroup of G. Let $|F| = t$.

Regarding G as a permutation group on the points of $\pi-\{\gamma\}$, let $\{T_i \mid i = 1 \text{ to } N\}$ be the orbits of $\pi-\{\gamma\}$ under G. Regarding F as a permutation group on $\pi-\{\gamma\}$, it is seen that F fixes each T_i and hence can be regarded as a permutation group on T_i. F thus splits each T_i into orbits, each of which has length t since an element $f \varepsilon F$ is uniquely determined by P and P^f for any point $P \varepsilon \pi-\{\gamma\}$.

Let t_i be the number of orbits of T_i under F. Then $tt_i = |T_i|$ and as there are N orbits of the n^2 points of $\pi-\{\gamma\}$ under G, it follows that

$$n^2 = \sum_{i=1}^{N} tt_i \qquad \ldots\ldots (1)$$

Let P_1 and P_2 be two points in the same orbit of $\pi-\{\gamma\}$ under G. Then Orb P_1 = Orb P_2 and by the fundamental theorem it immediately follows that $|G_{P_1}| = |G_{P_2}|$. But G_{P_1}, the set of elements that fix P_1, is just the set of homologies with centre P_1 (plus the identity). Thus the number of homologies with centre P_1 equals the number of homologies with centre P_2, and hence without ambiguity the number of homologies (including the identity) whose centre is a given point in T_i can be denoted s_i.

Let $|G| = k$. As G can be partitioned into $F \cup (G-F)$, where G-F is the set of all homologies in G (excepting the identity), we can write

$$|G| = |F| + |G-F|$$

i.e. $$k = t + \sum_{i=1}^{N} tt_i(s_i-1) \qquad \ldots\ldots (2)$$

where $tt_i(s_i-1)$ is evidently the number of homologies in G with centres in T_i.

Let $C \varepsilon T_i$. Then applying the fundamental theorem,

$$|\text{Orb } C|\,|G_C| = |G|$$

i.e. $$tt_i s_i = k \qquad \ldots\ldots (3)$$

Substituting this in equation (2) gives

$$k = t + \sum_{i=1}^{N} k - \sum_{i=1}^{N} tt_i.$$

Combining this with equation (1) gives

$$k = t + Nk - n^2$$

i.e. $$k(N-1) = n^2 - t$$

so by equation (3),

$$tt_i s_i (N-1) = n^2 - t$$

i.e. $$(N-1)(t_i s_i) = n^2 t^{-1} - 1.$$

Summing over the N orbits, this becomes

$$\frac{N-1}{n^2 t^{-1}-1} \sum_{i=1}^{N} t_i = \sum_{i=1}^{N} \frac{1}{s_i}.$$

Using equation (1) this becomes

$$\left(\frac{N-1}{n^2 t^{-1}-1}\right)\left(\frac{n^2}{t}\right) = \sum_{i=1}^{N} \frac{1}{s_i}$$

i.e. $$\frac{N-1}{1-\frac{t}{n^2}} = \sum_{i=1}^{N} \frac{1}{s_i} \qquad \ldots\ldots (4)$$

Now the left-hand side, and hence the right-hand side, of equation (4) is greater than (N-1),

i.e. $$\sum_{i=1}^{N} \frac{1}{s_i} > N-1. \qquad \ldots\ldots (5)$$

Suppose that $s_i \geq 2$ for i = 1 to m and that $s_i = 1$ for i = (m+1) to N. Then $m \geq 1$ as the number of homologies with centre C_1 is ≥ 2 as σ_1 and the identity are two such. Thus $\frac{1}{s_i} \leq \frac{1}{2}$ for i = 1 to m and so

$$\sum_{i=1}^{m} \frac{1}{s_i} \leq \sum_{i=1}^{m} \frac{1}{2} = \frac{m}{2} \qquad \ldots\ldots (6)$$

As $s_i = 1$ for i = (m+1) to N, it follows that

$$\sum_{i=m+1}^{N} \frac{1}{s_i} = N-m.$$

Consequently

$$\sum_{i=1}^{m} \frac{1}{s_i} = \sum_{i=1}^{N} \frac{1}{s_i} - \sum_{i=m+1}^{N} \frac{1}{s_i}$$

$$> (N-1) - (N-m) \qquad \text{(using result (5))}$$

i.e. $$\frac{m}{2} > m-1 \qquad \text{(using result (6)).}$$

Thus $m > 2m-2$, and so $2 > m$. But as $m \geq 1$, it follows that $m = 1$. Hence only one orbit of $\pi - \{\gamma\}$ under G, namely T_1, is such that a point P of that orbit is the centre of more than one homology. Hence the centres of the non-trivial homologies are all contained in one orbit T_1 of $\pi - \{\gamma\}$ under G and the second claim is

true.

Setting i = 1 in equation (3) and substituting in equation (2), we obtain

$$tt_1s_1 = t+t \sum_{i=1}^{n} t_i(si^{-1}).$$

As $s_i = 1$ for $i > 1$, this becomes

$$t_1s_1 = 1+t_1s_1-t_1$$

i.e. $$t_1 = 1.$$

Hence recalling the definition of t_1, the orbit T_1 has only one suborbit under F; i.e. F is transitive on T_1. Hence as C_1 and $C_2 \varepsilon T_1$, there is an elation in F sending C_1 into C_2. Hence result (3) holds.

Theorem 4.4 Let π be a finite projective plane and let G be a collineation group of π.

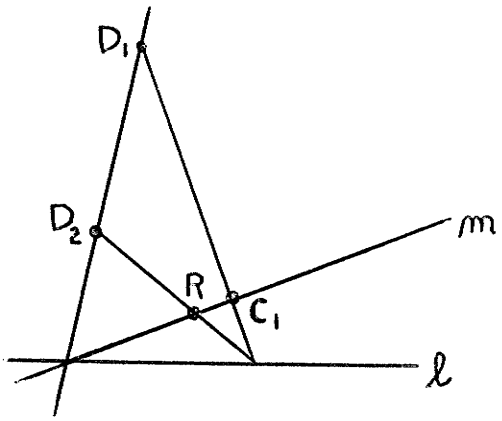
Then:

(1) If ℓ and m are distinct lines of π such that for every point $C \varepsilon m$ ($C \neq \ell \wedge m$) there exists a non-trivial (C, ℓ)-homology in G, then π is $(\ell \wedge m, \ell)$-transitive.

(2) If ℓ is a line such that $C \not\varepsilon \ell \Longrightarrow$ there exists a non-trivial (C, ℓ)-homology, then π is a translation plane with respect to ℓ.

(3) If for every point C and line ℓ of π, $C \not\varepsilon \ell$, there exists a non-trivial (C, ℓ)-homology in G then π is Desarguesian and G contains the little projective group of π.

Proof: (1) Let C_1 and C_2 be arbitrary distinct points of m- $\{\ell \cap m\}$ and let them be centres of non-trivial homologies σ_1 and σ_2(respectively) of G. By theorem 4.3 G contains an elation φ such that $C_1^{\varphi} = C_2$. Let D_1 and D_2 be arbitrary points of π- $\{\ell\}$ such that $\equiv D_1, D_2, \ell \cap m$.



Then let

$$(D_1C_1 \cap \ell)D_2 \cap m = R.$$

There is an elation mapping $C_1 \longrightarrow R$ and under this $D_1 \longrightarrow D_2$. Hence π is $(\ell, \ell \cap m)$-transitive.

(2) From (1) it immediately follows that π is (P, ℓ)-transitive for every $P \varepsilon \ell$, and hence π is a translation plane with respect to ℓ.

(3) From (2) it immediately follows that π is a translation plane with respect to all lines of π, and hence is a finite alternative plane. Hence by the result of chapter 3, π is Desarguesian and G contains all possible elations of π and hence the little projective group.

Lemma 4.12 Let φ be a collineation of a finite projective plane π of order n. Let N_{φ} be the number of points fixed by φ and $\bar{N}_{\varphi}$ the number of lines fixed by φ. Then $N_{\varphi} = \bar{N}_{\varphi}$.

Proof: Define

$$K = \{(P, \ell) \mid P \text{ a point}, \ell \text{ a line}, P \varepsilon \ell, P \varepsilon \ell^{\varphi}\}.$$

Two expressions for the order of K are now calculated. Each point P fixed by φ has (n+1) lines through it, and for each such line ℓ, $P \varepsilon \ell$ and $P \varepsilon \ell^{\varphi}$. Each point Q not fixed by φ (there are $(n^2+n+1)-N_{\varphi}$ of these) has exactly one line ℓ through it, namely $QQ^{\varphi^{-1}}$, such that $Q \varepsilon \ell$ and $Q \varepsilon \ell^{\varphi}$. Hence

$$|K| = N_{\varphi}(n+1)+(n^2+n+1-N_{\varphi}).$$

The dual argument gives

$$|K| = \bar{N}_{\varphi}(n+1)+(n^2+n+1-\bar{N}_{\varphi})$$

and it immediately follows that $N_{\varphi} = \bar{N}_{\varphi}$.

Lemma 4.13 Let G be a permutation group on a set S, and let k be the number of orbits of S under G. Then $k|G| = \sum r f_r$ where f_r is the number of elements of G that fix r elements of S, and summation is over r.

Proof: Let $|S| = n$. Set up a matrix $M = (m_{ij})$ with rows indexed by elements of G and columns indexed by elements of S. Then M has $|G|$ rows and n columns. Assign elements of M as follows:

$$m_{ij} = 1 \iff \alpha_j^{g_i} = \alpha_j$$

$$m_{ij} = 0 \iff \alpha_j^{g_i} \neq \alpha_j.$$

Let N be the total number of "1"s appearing in M. For $r = 0,1,2,\cdots$ the number of rows of M containing r "1"s is f_r; hence as N is the sum over all rows of M of the number of "1"s in each row, it follows that

$$N = \sum_r r f_r \qquad \ldots\ldots (1)$$

However, the number of "1"s appearing in a column of M indexed by $\alpha \varepsilon S$ is the number of elements of G fixing α, i.e. is $|G_\alpha|$. Thus

$\sum_{\alpha \varepsilon S} |G_\alpha| = N$ (as N is the sum over all columns of the number of "1"s in each column).

Let the k orbits of S under G be $\{T_i \mid i = 1 \text{ to } k\}$

Then

$$\sum_{\alpha \varepsilon S} |G_\alpha| = \sum_{i=1}^{k} \left(\sum_{\alpha \varepsilon T_i} |G_\alpha| \right) .$$

But by the fundamental theorem

$$\alpha_1 \varepsilon T_i, \alpha_2 \varepsilon T_i \Longrightarrow |G_{\alpha_1}| = |G_{\alpha_2}|$$

and hence $\sum_{\alpha \varepsilon T_i} |G_\alpha| = |\text{Orb } \alpha| \, |G_\alpha| = |G| .$

Thus $N = \sum_{\alpha \varepsilon S} |G_\alpha| = \sum_{i=1}^{k} |G| = k|G|$

so by equation (1),

$$\sum_r r f_r = k \, |G| .$$

<u>Theorem 4.5</u> Let G be a collineation group of a finite projective plane π. Then the number k of orbits of points of π under G equals the number $\bar{k}$

of orbits of lines of π under G.

Proof: By lemma 4.13,

$$k|G| = \sum_r r f_r$$

where f_r is the number of elements in G that fix r points of π. Similarly

$$\bar{k}\,|G| = \sum_r r \bar{f}_r$$

where $\bar{f}_r$ is the number of elements in G that fix r lines of π. But by lemma 4.12, if $g \in G$ then g fixes r points of π if and only if g fixes r lines of π. Thus for all r, $f_r = \bar{f}_r$. Consequently $\sum_r r f_r = \sum_r r \bar{f}_r$ and so $k|G| = \bar{k}|G|$; hence $k = \bar{k}$.

Definition: Let $\pi = \{\mathcal{P}, \mathcal{L}, \varepsilon\}$ be a projective plane with point set $\mathcal{P}$ and line set $\mathcal{L}$. Then a function φ mapping $\mathcal{P}$ onto $\mathcal{L}$ and $\mathcal{L}$ onto $\mathcal{P}$ such that $P \varepsilon \ell \iff \ell^{\varphi} \varepsilon P^{\varphi}$ is called a correlation of π.

As φ is one-to-one onto, its inverse exists and is a correlation. Evidently φ^2 is a collineation of π.

If φ is correlation such that $\varphi^2 = 1$, then φ is called a polarity of π. A point P is an absolute point of φ if $P \varepsilon P^{\varphi}$.

Theorem 4.6 (see Baer (2)) Every polarity φ of a finite projective plane π of order n possesses absolute points.

Proof: Define an m-cycle as an ordered m-tuplet of points $(P_1,\cdots,P_m)$, not all necessarily distinct, such that $P_i \varepsilon P_{i+1}^{\varphi}$ for i = 1 to (m-1) and $P_m \varepsilon P_1^{\varphi}$. Let Z_m be the number of ordered m-cycles of φ. Evidently the one point of a 1-cycle is an absolute point.

Let p be a prime; then $(Z_p - Z_1)$ is divisible by p. To prove this note that there are two types of p-cycles. The first is of the form $(P,\cdots,P)$ where the same point P is repeated p times. Evidently such a P is an absolute point and thus there are Z_1 such p-cycles. The second type is the set of p-cycles in which two or more distinct points occur in the cycle; there are $(Z_p - Z_1)$ p-cycles of this type.

Let $(P_1,\cdots,P_p)$ be a p-cycle of the second type and suppose that there exists an integer r, $0 < r < p$, such that $P_i = P_{i+r}$, i = 1 to p, (subscripts are taken modulo p). As the set of integers modulo a prime p forms a field, there exists an integer t such that $rt \equiv 1 \pmod{p}$. But for an arbitrary integer $k \geq 0$ it is evident by applying $P_i = P_{i+r}$ k times that $P_i = P_{i+kr}$, i = 1 to p; thus in particular $P_i = P_{i+rt}$, i = 1 to p. As indices are modulo p, this says that $P_i = P_{i+1}$ for i = 1 to p, i.e. that the given p-cycle is a p-cycle of the first type. This contradicts the assumption so no integer r as described exists. Hence p-cycles of the second type come in classes of p,

where a typical class is

$$\left\{(P_{1+r},\cdots,P_{p+r}) \mid r = 0 \text{ to } p-1\right\}$$

and each member of the class is distinct by the above argument. Hence $Z_p - Z_1$ is divisible by p.

There are now two cases, namely n even and n odd.

Case (1). If n is even, let (P_1,P_2) be a 2-cycle. Then $P_1 \varepsilon P_2^{\varphi}$ and $P_2 \varepsilon P_1^{\varphi}$; but as $\varphi^2 = 1$, $P_1 \varepsilon P_2^{\varphi} \Rightarrow P_1^{\varphi} \varepsilon P_2^{\varphi^2} = P_2$, so (P_1,P_2) is a 2-cycle if and only if $P_1 \varepsilon P_2^{\varphi}$, i.e. if and only if P_1 is any of the (n+1) points on P_2^{φ}. Hence as there are (n^2+n+1) points of π, there are $(n^2+n+1)(n+1)$ distinct ordered 2-cycles. As n is even, $(n^2+n+1)(n+1) = Z_2$ is odd so $Z_2 \equiv Z_1 \bmod 2$. But as 2 is a prime from the above $Z_2 \equiv Z_1 \bmod 2$, so $Z_1 \equiv 1 \bmod 2$. Hence $Z_1 \neq 0$ so there exist one-cycles and hence absolute points.

Case (2) Suppose that n is odd. Then for $m > 3$ consider an ordered set of (m-1) points, namely $\{P_1,\cdots,P_{m-1}\}$. If $P_i \varepsilon P_{i+1}^{\varphi}$ for i = 1 to m-2, such an ordered set is called an (m-1)-chain. There are $(n^2+n+1)(n+1)^{m-2}$ distinct ordered (m-1)-chains, as P_{m-1} can be chosen in (n^2+n+1) ways, P_{m-2} can be chosen to be any of the (n+1) points on P_{m-1}^{φ}, P_{m-3} can be any of the (n+1) points on P_{m-2}^{φ}, and so on.

A distinction is made between two types of (m-1)-chains. The first type is that in which $P_{m-1} = P_1$

and the second type is all other (m-1)-chains. To each (m-1)-chain of the first type there corresponds (n+1) distinct m-cycles, as the m-cycle can be completed by choosing P_m to be any of the (n+1) points of P_1^{φ}. Then $P_m \varepsilon P_1^{\varphi}$, so $P_1\varphi^2 \varepsilon P_m^{\varphi}$, i.e. $P_{m-1}\varepsilon P_m^{\varphi}$ and we indeed have an m-cycle. To each (m-1)-chain of the second type there corresponds one m-cycle, namely that for which $P_m = P_1^{\varphi} \cap P_{m-1}^{\varphi}$. This exhausts the m-cycles of φ.

However, the number of (m-1)-chains of the first type is Z_{m-2}, for the (m-1)-chains of the form $\{P_1,\cdots,P_{m-2},P_1\}$ can be put in one-to-one correspondence with the set of (m-2)-cycles $(P_1,\cdots,P_{m-2})$ (since $P_{m-2}^{\varphi}\varepsilon P_1$). Hence the number of m-cycles can be written by noting that there are (n+1) m-cycles for each (m-1)-chain of the first type and $((n^2+n+1)(n+1)^{m-2}-Z_{m-2})$ other m-cycles, i.e. one for each (m-1)-chain of the second type. Thus

$$Z_m = (n+1)Z_{m-2}+(n^2+n+1)(n+1)^{m-2}-Z_{m-2}$$

i.e. $$Z_m = nZ_{m-2}+(n^2+n+1)(n+1)^{m-2} \qquad \ldots\ldots (1)$$

Now for any integer $k \geq o$,

$$Z_{2k+1} = (n+1)^{2k+1}+n^k(Z_1-n-1) \qquad \ldots\ldots (2)$$

To prove this induct on k. If k = o it is evidently true. Suppose that it holds for (2k-1).

Then by equation (1),

$$Z_{2k+1} = nZ_{2k-1}+(n^2+n+1)(n+1)^{2k-1}$$

$$= n\left[(n+1)^{2k-1}+n^{k-1}(Z_1-n-1)\right]+(n^2+n+1)(n+1)^{2k-1}$$

(by the induction hypothesis)

$$= n(n+1)^{2k-1}+n^k(Z_1-n-1)+(n^2+n+1)(n+1)^{2k-1}$$

$$= (n+1)(n+1)^{2k-1}+n^k(Z_1-n-1)+n(n+1)(n+1)^{2k-1}$$

i.e. $$Z_{2k+1} = (n+1)^{2k+1}+n^k(Z_1-n-1)$$

and equation (2) evidently holds in general for all k.

As n is odd there exists an odd prime p, expressible as 2k+1 for some k, such that p divides n. In this case equation (2) becomes

$$Z_p = (n+1)^p+ps \qquad \text{for some integer } s$$

$$= n^p+pr+1+p \qquad \text{for integers } r \text{ and } s.$$

Hence $Z_p \equiv 1 \bmod p$; but as $Z_p \equiv Z_1 \bmod p$, $Z_1 \equiv 1 \bmod p$, so $Z_1 \neq 0$ and hence there exist absolute points.

Definition: An involutory collineation of a projective plane π is a collineation of order 2.

Lemma 4.14 Let π be a projective plane of order n containing a projective subplane $\mathcal{M}$ of order m. Let incidence of points and lines in $\mathcal{M}$ be the same as that in π, and let all points of π be on extended lines of $\mathcal{M}$. Then $n = m^2$.

Proof: Each line of $\mathcal{M}$ has (m+1) points and each line of π has (n+1) points. Hence in extending a line

of $\mathcal{M}$ to a line of π it is necessary to add $(n+1)-(m+1) = (n-m)$ points to the line. This must be done for each of the (m^2+m+1) lines of $\mathcal{M}$; hence $(m^2+m+1)(n-m)$ new points are added to $\mathcal{M}$ in this way. As all points of π are on extensions of lines of $\mathcal{M}$, these $(m^2+m+1)(n-m)$ points comprise all the points of $\pi-\mathcal{M}$.

These new points are all distinct; for if m_1 and m_2 are distinct lines of $\mathcal{M}$ and P is a point of $\pi-\mathcal{M}$ lying on both the extension of m_1 and the extension of m_2, then m_1 and m_2 (thought of now as lines of π) intersect both at a point of $\mathcal{M}$ and at $P\varepsilon\pi-\mathcal{M}$, contradicting the axioms of incidence. Thus as π has (n^2+n+1) points and $\mathcal{M}$ has (m^2+m+1) points, it follows that

$$n^2+n+1 = (m^2+m+1)+(m^2+m+1)(n-m)$$

i.e. $$(n^2-m^2)+(n-m) = (m^2+m+1)(n-m)$$

$$(n-m)(n+m+1) = (m^2+m+1)(n-m).$$

As $n > m$, this implies $n = m^2$.

Theorem 4.7 (see Baer (3)) Let σ be an involutory collineation of a finite projective plane of order n. Then either σ is a central collineation or σ fixes pointwise a subplane of π of order $\sqrt{n}$.

Proof: Every point of π lies on a line fixed by σ; for choose an arbitrary point $P\varepsilon\pi$. If P is not

fixed by σ then PP^{σ} is a well-defined line and $(PP^{\sigma})^{\sigma} = P^{\sigma}P^{\sigma^2} = P^{\sigma}P$. Hence PP^{σ} is fixed by σ and $P \varepsilon (PP^{\sigma})$. If P is fixed by σ, choose any line ℓ such that $P \notin \ell$ and $\ell^{\sigma} \neq \ell$ (such a line exists as $\sigma \neq 1$). Then $\ell \cap \ell^{\sigma} \neq P$ and so $P(\ell \cap \ell^{\sigma})$ is a well-defined line through P fixed by σ (as both P and $\ell \cap \ell^{\sigma}$ are). Thus P lies on a fixed line.

Let K be the set of all points and lines of π fixed by σ. Then two lines of K intersect at a unique point of K and two points of K are joined by a unique line of K. Hence if there are four points of K of which no three are collinear, the points and lines of K comprise a projective plane. As every point of π is on a line of π fixed by σ, i.e. on an extension of a line of K, it follows by lemma 4.13 that K has order $\sqrt{n}$.

If there are not four points of K of which no three are collinear, choose P such that $P^{\sigma} \neq P$ and let $\{\ell_i \mid i = 1 \text{ to } n+1\}$ be the (n+1) lines through P. Then there exists, for i = 1 to n+1, a point Q_i such that $Q_i \varepsilon \ell_i$ and $Q_i^{\sigma} = Q_i$ (and thus $Q_i \neq P$). As $Q_i \varepsilon K$, i = 1 to n+1, either all (n+1) points $\{Q_i\}$ are collinear or else n of them (say Q_1 to Q_n) are collinear. In the first case σ fixes a line point-wise and hence is a central collineation; in the

second case let S be the remaining point on the line $m = Q_1Q_n$. Then $m^\sigma = m$ so $S^\sigma \varepsilon m$; but $Q_i{}^\sigma = Q_i, i = 1$ to n, so $S^\sigma = S$ and σ fixes m point-wise and is thus a central collineation.

CHAPTER V

LOCALLY DESARGUESIAN PLANES

Let G be the collineation group of a finite projective plane π. In this chapter various assumptions are made concerning the action of G on the points and lines of π. Assumptions are also made concerning the number of elations and homologies in G. These assumptions are shown to imply in some cases that π is a translation plane or its dual, and in other cases that π is Desarguesian.

Two approaches are used; in the first, due to Wagner (13), the collineation group of the projective plane is regarded as a permutation group on the points and lines of the plane. Certain general assumptions about the orbits of the group are shown to imply that the projective plane in question is a translation plane, and a well-known theorem of Ostrom and Wagner (7), namely that if G is a collineation group doubly transitive on the points of π then π is Desarguesian, is shown to follow immediately as a corollary. The second approach, due to Wagner (11) and Piper (10), consists of postulating the existence of a certain number of central collineations and from this deducing information about the structure of the plane. The method is essentially an extension of the work of Gleason (4),

and Andre ( ).

Flags in Projective Planes

A flag in a projective plane π is an incident point-line pair. If the point is C and the line is ℓ, then the flag is symbolized (C, ℓ); C is called the centre of the flag and ℓ the axis of the flag. A collineation φ is said to map the flag (C_1, ℓ_1) onto the flag (C_2, ℓ_2) if and only if $C_1^{\varphi} = C_2$ and $\ell_1^{\varphi} = \ell_2$. This is symbolized by writing $(C_1, \ell_1)^{\varphi} = (C_2, \ell_2)$. Thus a collineation group G of a projective plane π has an action on the set F of flags of π. Hence the usual concepts of permutation group theory will be applicable to the collineation group G considered as a permutation group on the set F.

In the following work a very general condition on the collineation group of a finite projective plane π is shown to be sufficient to ensure that π is a translation plane. A number of preliminary lemmas must first be proved.

2-Subplanes of Projective Planes

Let π be a finite projective plane and G a collineation group of π. A 2-subgroup of G is defined to be a subgroup of G of order 2^{α} for some non-negative integer α. A non-degenerate projective subplane μ of π will be called a 2-subplane of π with respect to G

if there is a 2-subgroup H of G fixing every element of μ and no other element of π.

More generally, let G be a collineation group of a finite projective plane π containing a projective subplane μ, and suppose that G permutes the elements of μ amongst themselves. Then G has an action on μ, and induces a permutation group $\bar{G}$ on the elements of μ. It is easily seen that $\bar{G}$ will be a homomorphic image of G, and that the kernel K of the homomorphism will be the set of all elements of G that fix μ elementwise.

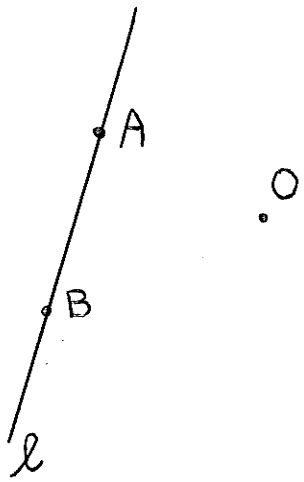
Lemma 5.1 Let π be a finite projective plane of order n and let μ be a 2-subplane of π with respect to the collineation group G of π. If μ has order m, then $n = m^{2^g}$ for some non-negative integer g.

Proof: Let H be the 2-subgroup of G that fixes μ, and only μ, elementwise. Then $H = 2^\alpha$ for some integer α. If $\alpha = 0$ then $\pi = \mu$, $g = 0$, and the theorem holds. If $\alpha > 0$ then $|Z(H)| > 1$ (where $Z(H) = \{h \in H \mid hk = kh$ for all $k \in H\}$). Consequently there exists an element $\varphi \in Z(H)$ of order 2. By theorem 4.7, φ is either a central collineation or fixes a subplane π_1 of π of order $\sqrt{n}$. As H fixes only elements of μ, the first alternative is impossible. Hence φ fixes a subplane π_1 elementwise.

As $\varphi \in Z(H)$, then $N_H(\langle\varphi\rangle) = H$ if $\langle\varphi\rangle$ is the cyclic group generated by φ; hence by lemma 4.3 H permutes the elements of π_1 amongst themselves. Thus

H has an action on π_1. Denote by H_1 the permutation group of π_1 induced by H and let K be the subgroup of H fixing π_1 elementwise. Then H_1 is isomorphic to $\frac{H}{K}$, as noted earlier. It follows that H_1 is a 2-group and a collineation group of π_1 that fixes only the elements of μ. If $H_1 = 1$, then $\pi_1 = \mu$ and $n = m^2$. If $H_1 \neq 1$, then the above argument is iterated with π_1 playing the role of π and H_1 playing the role of H (note that all the requisite hypotheses hold). As n is finite, after iterating the argument a finite number of times (say g), the group corresponding to H_1 (call it H_g) is the identity. Then the plane corresponding to π_1 will be μ, and it follows that $m^{2g} = n$.

Definition A projective plane π will be said to have the homology property if π contains a fixed line and a fixed point O, $O \notin \ell$, such that for any two distinct points A and B on ℓ, there exists an involutory homology of π with axis OA and centre B.



Lemma 5.2 Let π be a finite projective plane of order n possessing the homology property. Then $n = p^m$ for some prime p.

Proof: Let ℓ and O be the fixed line and point of π respectively, and let G be the group of collineations generated by all homologies with centre on ℓ and axis through O. Let A be an arbitrary point of ℓ, and define K_A to be the subgroup of G generated by all the involutory homologies of π with centre on ℓ and axis OA. Note that K_A is a proper subgroup of G_A, as G_A includes homologies with centre A.

The group G fixes ℓ and hence has an action on the points of ℓ; thus K_A also has such an action. Also, as π has the homology property, for any point B of $\ell - \{A\}$ there is a (B,OA)-homology of order 2 in K_A. Thus by theorem 4.4, K_A is (A,OA)-transitive, i.e. is transitive on the points of $\ell - \{A\}$. Further, all elements of K_A are, from its definition, central collineations with axis OA; hence by lemma 1.3 only the identity of K_A fixes two points of $\ell - \{A\}$ and K_A is thus a Frobenius group on the points of $\ell - \{A\}$.

By the above reasoning, for any distinct points A_1 and B_1 on ℓ, G_{A_1} is transitive on $\ell - \{A_1\}$ and G_{B_1} is transitive on $\ell - \{B_1\}$; hence G is transitive on the points of ℓ. Since, for an arbitrary $C \varepsilon \ell$, G_C is transitive on $\ell - \{C\}$, it follows that G is doubly transitive on the points of ℓ.

Finally, K_A is normal in G_A; for let α be a generator of K_A and σ a generator of G_A. Then α is a (C,OA)-

homology where $C \in \ell - \{A\}$; thus by lemma 1.6, $\sigma^{-1}\alpha\sigma$ is a $(C^\sigma, (OA)^\sigma)$-collineation. As G fixes O and G_A fixes A, $(OA)^\sigma = OA$, so $\sigma^{-1}\alpha\sigma$ is an involutory homology with axis OA and hence is in K_A. It follows that K_A is normal in G_A.

As K_A is generated by involutory homologies it has even order; by the definition of G, any generator of G interchanges all but two of the points of ℓ in pairs, so (n-1) is even, i.e. n is odd (i.e. any finite plane with the homology property has odd order). Hence all the hypotheses of theorem 4.1 are satisfied, with S being the set of points on ℓ, G_α being G_A, and F_α being K_A. Thus by theorem 4.1 n is a power of some prime. Consequently a finite plane with the homology property has order p^m for some prime p.

Lemma 5.3 Let π be a finite projective plane and let G be a collineation group of π fixing a line ℓ of π. Then the following three conditions are equivalent:

(1) G is transitive on the affine lines of π (where ℓ is considered to be the "line at infinity").

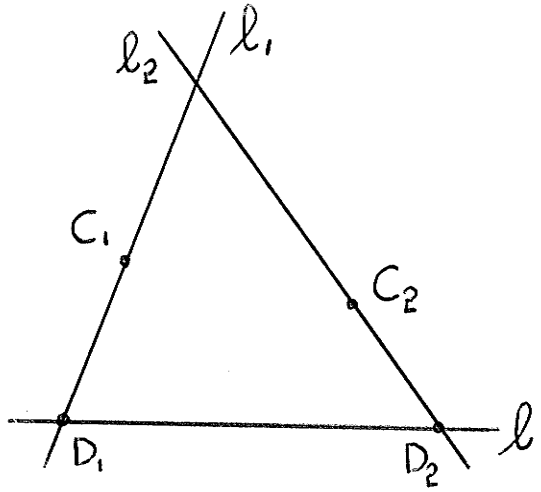
(2) G is transitive on the points of ℓ and on the affine points of π_ℓ.

(3) G is transitive on affine flags of π_ℓ (i.e. on flags (C,γ) with $C \notin \ell$ and $\gamma \neq \ell$).

Proof:

(1) $\Longrightarrow$ (2): Considered as a collineation group on the set $\mathcal{L}$ of lines of π, G partitions $\mathcal{L}$ into two disjoint orbits. By theorem 4.5, G will partition the points of π into two disjoint orbits. As G cannot map points of ℓ onto points not on ℓ (since G fixes ℓ), these two orbits must be the points of ℓ and the points of $\pi - \{\ell\}$. Hence (2) follows immediately.

(2) $\Longrightarrow$ (3): Let (C_1, ℓ_1) and (C_2, ℓ_2) be arbitrary affine flags of π. Let $\ell_1 \cap \ell = D_1$ and $\ell_2 \cap \ell = D_2$. By hypothesis the points of $\pi - \{\ell\}$ and the points of are orbits, under the action of G, of orders (n+1) and n^2 respectively. But (n+1) and n^2 are relatively prime, so by two applications of lemma 4.2 there exist



elements σ_1 and σ_2 in G such that $C_1^{\sigma_1} = C_1$, $D_1^{\sigma_1} = D_2$, and $D_2^{\sigma_2} = D_2$, $C_1^{\sigma_2} = C_2$. Thus $(C_1, \ell_1)^{\sigma_1\sigma_2} = (C_2, \ell_2)$; as (C_1, ℓ_1) and (C_2, ℓ_2) were arbitrary affine flags of π, G is transitive on such flags.

(3) $\Longrightarrow$ (1): This is trivially true.

Corollary: Let π be a finite projective plane of order n and let G be a collineation group of π fixing ℓ and transitive on the lines of $\pi - \{\ell\}$. Let $G_{C,\gamma}$ denote the

stabilizer in G of the affine flag (C,γ), let G_C denote the stabilizer in G of C, and let G_γ denote the stabilizer in G of γ. Then:

(1) $|G_{C,\gamma}|n^2(n+1) = |G|$

(2) $|G_{C,\gamma}|n^2 = |G_A|$ where G_A is the stabilizer in G of an arbitrary point $A \in \ell$.

(3) $|G_\gamma| = (n+1)|G_{C,\gamma}|$

(4) $|G_C| = n|G_{C,\gamma}|$

(5) $|G_{A,C}| = |G_{C,\gamma}|$ where $G_{A,C}$ is the subgroup of G fixing both C and a point $A \in \ell$.

Proof:

(1) There are n^2 affine points in π_ℓ, and $(n+1)$ affine lines through each of these. As each incident point-line pair with elements drawn from these points and lines comprises an affine flag of π, there are $n^2(n+1)$ affine flags in π. As G is transitive on affine flags of π_ℓ by lemma 5.3, $|\text{Orb}\ (C,\gamma)| = n^2(n+1)$. Thus $|G_{C,\gamma}|n^2(n+1) = |G|$ by lemma 4.1.

(2) As by lemma 5.3 G is transitive on the points of ℓ, it follows from lemma 4.1 that $|G_A|(n+1) = |G|$. Combining this with (1) immediately gives $|G_{C,\gamma}|n^2 = |G_A|$.

(3) By hypothesis G is transitive on the (n^2+n) affine lines of π, so by lemma 4.1 $|G_\gamma|(n^2+n) = |G|$. Combining this with (1) immediately gives $|G_\gamma| = n|G_{C,\gamma}|$.

(4) By lemma 5.3 G is transitive on the n^2 affine

points of π_ℓ. Thus by lemma 4.1, $|G_C|n^2 = |G|$. Combining this with (1) immediately gives $|G_C| = (n+1)|G_{C,\gamma}|$.

(5) It is evident that $G_{A,C} = G_{C,AC}$. As G is transitive on affine lines it follows from lemma 4.1 that $|G_{C,AC}| = |G_{C,\gamma}|$. Consequently $|G_{A,C}| = |G_{C,\gamma}|$.

Lemma 5.4 Let π be a finite projective plane and let G be a collineation group of π fixing a line ℓ of π. Then G is transitive on points of $\pi - \{\ell\}$ if and only if, for an arbitrary point $A \varepsilon \ell$, G_A is transitive on affine lines through A.

Proof: Let the action of G on the points of ℓ split these points into s orbits.

First assume that G is transitive on the points of $\pi - \{\ell\}$. Then there are (s+1) distinct orbits of points of π under the action of G. By theorem 4.5 there are (s+1) orbits of lines of π under the action of G. As ℓ is a fixed line, there are s orbits of affine lines of π under G. Let T_1 and T_2 be distinct orbits of points of ℓ under G, and let ℓ_1 and ℓ_2 be affine lines such that $\ell_1 \cap \ell \varepsilon T_1$ and $\ell_2 \cap \ell \varepsilon T_2$. Then if there exists $g \varepsilon G$ such that $\ell_1^g = \ell_2$, it would follow that $(\ell_1 \cap \ell)^g = \ell_2 \cap \ell$, contradicting the assumption that T_1 and T_2 are distinct orbits. Consequently the set of affine lines of π intersecting ℓ in the points of a given orbit of points of must comprise an orbit of lines of ℓ. Thus G is trans-

itive on affine lines through $A \varepsilon \ell$, and consequently G_A is also transitive on these lines (for arbitrary $A \varepsilon \ell$).

Conversely, suppose that for arbitrary $A \varepsilon \ell$, G_A is transitive on affine lines through A. Let T be an orbit of points of ℓ and let $P_1, P_2 \varepsilon T$. Then there exists $g \varepsilon G$ such that $P_1^g = P_2$. Hence if m_1 is an arbitrary affine line through P_1, m_1^g is an affine line through P_2. But G_{P_i} is transitive on affine lines through $P_i (i = 1,2)$, and P_1 and P_2 were arbitrary in T, so it follows that the set of affine lines intersecting ℓ in points of the same orbit T comprise an orbit of the lines of π under G. There are s such orbits, plus a singleton orbit $\{\ell\}$, and hence a total of (s+1) orbits of lines under G. By theorem 4.5, there are (s+1) orbits of points of π under G, of which s are comprised entirely of points of ℓ; it follows that G is transitive on the affine points of π.

Corollary Let π be a finite projective plane and G a collineation group of π fixing the line ℓ and transitive on the lines of $\pi - \{\ell\}$. If A and B are arbitrary points of ℓ and m_1 and m_2 are two affine lines through B, then $|G_{A,B,m_1}| = |G_{A,B,m_2}|$.

Proof: By part (2) of the corollary to lemma 5.3, G_A is transitive on the affine points of π. Thus by lemma 5.4, $G_{A,B}$ is transitive on affine lines through

B. Thus as $|\text{Orb}_{G_{A,B}}{}^{m_1}| = |\text{Orb}_{G_{A,B}}{}^{m_2}|$, it follows from lemma 4.1 that $|G_{A,B,m_1}| = |G_{A,B,m_2}|$.

Theorem 5.1 Let π be a finite projective plane of order n and let G be a collineation group of π fixing a line ℓ and transitive on the affine lines of π (where ℓ is regarded as the "line at infinity"). Then if either

(1) n is even

or (2) n is a power of an odd prime,

π is a translation plane with respect to ℓ and G contains the group of elations with axis ℓ.

Proof: By lemma 5.3, G is transitive on the points of ℓ. Hence if C_1 and C_2 are distinct points of ℓ, the elation groups $G_{C_1,\ell}$ and $G_{C_2,\ell}$ will be conjugate in G by lemma 1.6. Hence, by lemma 4.10, in order to show that π is a translation plane with respect to ℓ it suffices to show that G contains a non-trivial elation with axis ℓ.

Let (C,m) be an affine flag of π, and suppose $|G_{C,m}| = k$. By the notation $p^u \| x$ (for a prime p and integers u and x) we shall mean that p^u divides x but p^{u+1} does not divide x. There are two cases:

Case (1): n is even. Suppose $2^u \| n$ and $2^v \| k$. Then $u > 0$. Let M be a Sylow 2-subgroup of G. By the corollary of lemma 5.3, $|G_{C,m}|(n+1)n^2 = |G|$ and hence M has order 2^{2u+v}. As M has a non-trivial centre, choose

$\alpha \in Z(M)$ of order 2. Let S denote the set of all points of $\pi - \{\ell\}$ fixed by α, and let $|S| = s$. By theorem 4.7, either α is a central collineation or α fixes elementwise a projective subplane μ of π of order $\sqrt{n}$. If α is a central collineation, since n is even α is an elation by theorem 1.2. If α has axis ℓ, we are finished; if α has axis m ($m \neq \ell$), then S consists of the points of m, excepting $m \cap \ell$, and so $s = n$. If α fixes the points of a subplane μ, then α fixes $(n+\sqrt{n}+1)$ points of π, of which $\sqrt{n}+1$ are on ℓ restricted to μ; hence $s = n$ in this case as well.

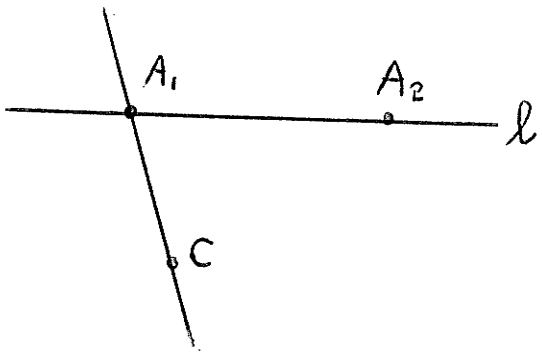
By lemma 4.3, the normalizer in M of the group generated by α permutes the members of S amongst themselves; as $\alpha \in Z(M)$, M is this normalizer, so M has an action on the points of S. Let T be an orbit of S under M and let P be an arbitrary element of T. Suppose $|T| = t$. Now M_P is a subgroup of G_P, and by the corollary of lemma 5.3, $|G_P| = (n+1)k$. Evidently $2^v \| (n+1)k$ (as (n+1) is odd), i.e. $2^v \| |G_P|$, and hence $|M_P|$ divides 2^v. But by lemma 4.1, $|M_P| t = |M| = 2^{2u+v}$, so it follows that 2^{2u} divides t. But as T was an arbitrary orbit in S, it follows that

$$2^{2u} \text{ divides } \sum_{\text{all orbits } T} t,$$

i.e. 2^{2u} divides n. As $u > 0$, this contradicts the hypothesis that $2^u \| n$. Consequently $s = n$ is impossible, α is an elation with axis ℓ, and π is a translation plane with respect to ℓ.

Case (2) $n = p^s$ for an odd prime p and a positive integer s.

Let $|G_{C,m}| = k$ as before, let $p^t \| k$, and let A_1 be an arbitrary point of ℓ. By the corollary to lemma 5.3, $|G_{A_1}| = n^2 |G_{C,m}|$. Thus $p^{t+2s} \| |G_{A_1}|$, so if M is a Sylow p-subgroup of G_{A_1}, $|M| = p^{2s+t}$. Let C be an arbitrary point of $\pi - \{\ell\}$. Then by lemma 4.1, the orbit of C under M will be of order p^u for some integer u; hence $|M_C| p^u = p^{2s+t}$, i.e. $|M_C| = p^{2s+t-u}$. But M_C is a subgroup of $G_{A_1,C}$, and by the corollary to lemma 5.3,



$|G_{A_1,C}| = |G_{C,m}| = k$.

Thus p^{2s+t-u} divides k. As $p^t \| k$, it follows that $p^{2s+t-u} \leq p^t$, and as $u \leq 2s$ from the definition of s and u, it follows that $u = 2s$. Hence $|\text{Orb } C| = p^u = p^{2s} = n^2$ and M is transitive on the affine points of π.

Now M has an action on the points of $\ell - \{A_1\}$, and hence partitions these points into orbits. Let $A_2 \varepsilon \ell$ and let A_2 be a member of an orbit of minimal length under M. Thus $B \varepsilon \ell - \{A_1\} \Longrightarrow |M_{A_2}| \geq |M_B|$. As M is transitive on affine points of π, by lemma 5.4 M_{A_2} is transitive on affine lines through A_2.

Evidently M_{A_2} is a p-subgroup of G_{A_2}; consequently