THE UNIVERSITY OF MANITOBA

ON THE THERMAL BUCKLING OF THIN WALL TUBES

By

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ABSTRACT

The investigation in this thesis concerns the thermal buckling of cylinders heated uniformly around the circumference and symmetrically with respect to the axial half-length.

Aluminum cylinders, machined to a thin wall, were threaded into a rigid frame and heated by means of a radiation type internal heater until buckling occurred. The temperature profile of the tube was recorded by thermocouples and this profile was used to simulate the test by using a successive approximation technique on an IBM 370/168 digital computer.

Axial load versus the centerline temperature plots were obtained for all specimens. The centerline radial displacement was also plotted as a function of the centerline temperature for several specimens. These results were compared to the successive approximation solutions and to the results of other investigators. Agreement with other works is noted, and any discrepancies are explained. Suggestions are also made as to areas with a need for further study to clarify the problem.

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Symbols

r	-	radius of curvature
L	-	length of cylinder
R	-	radius of cylinder
d	-	thickness of cylinder
D	-	flexural modulus of rigidity $D=Ed^3/12(1-\mu^2)$
E	-	modulus of elasticity
μ	-	Poisson's ratio
T	-	temperature rise above the unstrained state
u,v,w	-	displacement components in the median surface in the x,y,z, directions
P	-	normal primary surface force present prior to buckling
S	-	shear force present prior to buckling
ψ	-	Airy Stress function
N	-	total middle surface force
M	-	shell bending moment
Q	-	transverse shear force
ε	-	normal strain
σ	-	normal stress
γ	-	shear strain
τ	-	shear stress
U	-	elastic strain energy
V	-	potential energy of the external loads
ϕ	-	displacement function
α	-	coefficient of thermal expansion
λ	-	eigenvalue parameter
κ	-	curvature of shell
β	-	rotation of shell
q	-	general displacement term
z	-	dimensionless coefficient for geometry
k	-	dimensionless coefficient for loading
n	-	number of circumferential waves in buckling pattern
$\gamma_1 \cdots \gamma_8$	-	interpolation coefficients

Subscripts

o	-	in-plane component
x,y,z	-	denotes component in the axial, circumferential,radial direction
b	-	refers to bending component
B	-	refers to value at buckling
t	-	theoretical value
i	-	summation index
n	-	number of terms in the summation
m	-	maximum value
∇	-	indicates the change of the variable
ms	-	middle surface variable
E	-	experimental value
ψ	-	meridional direction
θ	-	circumferential direction

Superscripts

n	-	element node number
e	-	experimental value
t	-	theoretical value
1,2	-	refers to the type of stress coefficient
T	-	transpose of a matrix

CHAPTER I

INTRODUCTION

1.1 Statement of the Problem

Recent advances in the aerospace industry have imposed a great need for structures which have a very high strength to weight ratio. Lighter structures are also in demand for some of the key core components in nuclear reactors for low neutron absorption and better heat transfer capability. Shell type structures continue to be a main structural component in these industries.

In addition to an accurate stress analysis, the stability of shells as a function of the radius to thickness ratio is also an important design consideration. This stability problem may arise from a mechanical loading, or from a thermal loading from the harsh environment many of these shells must endure.

The use of long thin-walled tubes or pipes is common in engineering practise. Many of these tubes (or pipes) have to be held rigidly at the ends. These tubes are vulnerable to buckling due to the excess compressive longitudinal and circumferential stresses caused by the rising environmental temperature.

1.2 Scope of Thesis

This thesis deals with the analysis and subsequent experimental verification of the thermal buckling of thin-walled tubes rigidly held at both ends to a bulky attachment which acts as a heat sink, which in turn causes a non-uniform temperature profile along the tube length with the peak at the half-length.

The primary objectives of this thesis can be outlined as follows:

(1) To investigate the stability of thin-walled tubes that have higher length-to-radius and lower radius-to-thickness ratios than those tested by previous researchers.

(2) To compare the results derived from previous works to determine if the conclusions reached by other authors can be extended to the present work.

(3) To investigate if any conclusions drawn from the present work can be used to establish certain design criteria for thermally loaded shells.

(4) To recommend further studies necessary for a more complete understanding of the problem.

Chapter II reviews the related literature published on the thermal buckling of shells. Chapter III of this thesis reviews the basic equations of cylindrical shell stability problems and discusses some of the techiques available to solve these equations. Chapter IV describes the testing setup used in this work, and Chapter V discusses the results of that testing. The results are compared to other researcher's data and to a numerical solution of the problem. Chapter VI ends the thesis with the conclusions drawn from this work.

CHAPTER II

Literature Review

The equations for the stability of cylindrical shells have been made available for many years. An infinite series solution involving trigonometric functions was assumed by Lorenz in 1911 [1]* for solving the problem of a cylinder under uniform axial compression. Similar methods of solution were used by Southwell in 1913 [2] and von Mises in 1914 [3] for cylinders under uniform lateral pressure, and by Flügge in 1932 [4] for combined loading and bending.

In 1933 Donnell [5] proposed the use of a simpler form of stability equations in his solution for the buckling of cylinders subject to torsion. For simply supported cylinders,a solution was obtained by the use of an infinite trigonometric series.However,the problem of a cylinder with clamped ends could not be solved in this manner because of the divergence of the series solution. Singer [6] later showed that this result was due to the fact that Donnell's equation was an equilibrium equation and could not be used with the Galerkin method. Batdorf [7] proposed a modified equilibrium equation which could be used with the Galerkin method and proceeded to solve the problem of clamped shells under axial [8], shear [9], and combined axial and shear [10] loadings.

The first treatment of the thermal stability problem of

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*Number in brackets denote the reference number cited in this thesis.

shells was undertaken by Hoff [11] in his analysis of cylindrical shells subjected to hoop stresses varying in the axial direction. The three main types of thermal conditions that could cause a shell to buckle are: 1) a temperature gradient through the shell thickness, 2) a circumferential temperature gradient, and 3) an axial temperature gradient. The first condition was shown to be very unlikely to cause buckling [12]. The second condition has been investigated by several authors. Hoff, Chao & Madsen [13] and Hill [14] investigated the problem of buckling due to heating along a thin axial strip, while Ross, Mayers & Jaworski [15] extended their methods to include wide axial bonds, and Frum and Baruch [16] examined the buckling effect of heating along two opposite axial generators. It was found that buckling can easily be induced as a result of circumferential temperature gradients, even if the latter are fairly small.

The third temperature condition was first examined by Hoff [11] who concluded that simply supported cylinders were not likely to fail under uniform heating conditions. This work was extended by Anderson [17] to include both simply supported and clamped cylinders under combinations of axial pressure and uniform heating, using the Galerkin method along with Batdorf's modified equilibrium equation. Zuk [18] also presented a solution for the uniformly heated clamped shell using Donnell's equation with the Galerkin method; however, this method was found to be in error, as discussed previously. An experimental investigation of the clamped cylinder subject to uniform heating was presented by Ross, Hoff & Horton [19]. This problem has also been extended to investigate the non-linear aspects of the stability suggested by Hoff [20] and Ross [21] using a column-spring analogy. These papers were an extension of the work done by Tsein [22], who used this analogy to obtain a better understanding of some of the parameters involved in shell buckling.

In recent years much of the work in instability problems has been concerned with numerical techniques in order that solutions may be obtained for more complicated structures and loadings. The two principal methods that have been used widely are the finite difference method and the finite element method.

The finite element method was first introduced to analyse shell buckling by assuming the shell to be made of a series of truncated cones [23]. This method was later abandoned due to computational difficulties, and the principal approach recently has been to use curved shell elements and to approximate the displacement components by polynomials. This method is used in references [24], [25].

The finite difference technique has been used by many investigators to approximate shell buckling problems. The principal effort in recent years has been the development of computer programs based on the finite difference approximation to the variational problem. Some examples are given in references [26] [27]. An excellent comparison of the finite element and finite difference method is given by Bushnell [28].

CHAPTER III

THEORETICAL BACKGROUND

3.1 The Differential Equations of Buckling

The differential equations of a continuous system may be obtained either by considering the equilibrium of a deformed element, or by utilizing the principle of stationary potential energy and the calculus of variations.

For fairly simple systems the former method is usually the easiest and the most direct. For more complicated systems the latter method may be a more direct procedure for obtaining the solutions.

The consideration of equilibrium of a deformed element is explained in detail in the next section, for both small and large deflection theories. The stationary potential energy method is then explained as presented in detail in Appendix A.

3.1.1 Equilibrium Method

The differential equation which is most widely used in cylindrical buckling problems is the Donnell equation for small deflections. This can be derived as follows:

Using the notation given in figure 1; the equilibrium equations of in-plane forces in the x, and y directions are:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \qquad (1)$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{yx}}{\partial x} = 0 \qquad (2)$$

where N_x, N_y = the in-plane forces in the x and y directions

N_{yx}, N_{xy} = the in-plane shear forces

For the z direction, taking the equilibrium:

$$- N_x \frac{\partial w}{\partial x} dy + \left(N_x + \frac{\partial N_x}{\partial x} dx\right)\left(\frac{\partial w}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) d_x\right) d_y - N_y \frac{\partial w}{\partial y} dx$$

$$+\left(N_y + \frac{\partial N_y}{\partial y} dy\right)\left(\frac{\partial w}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) dy + \frac{dy}{R}\right) dx - N_{xy} \frac{\partial w}{\partial y} dy$$

$$+\left(N_{xy} + \frac{\partial N_{xy}}{\partial x} dx\right)\left(\frac{\partial w}{\partial y} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) dx\right) dy - N_{yx} \frac{\partial w}{\partial x} dy$$

$$+\left(N_{yx} + \frac{\partial N_{yx}}{\partial y} dy\right)\left(\frac{\partial w}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) dy\right) dx$$

where R = the radius of the cylinder

w = the radial displacement

After simplifying, neglecting terms of higher order, and using equations (1) and (2) the z components of the in-plane forces are:

$$N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y\left(\frac{\partial^2 w}{\partial y^2} + \frac{1}{R}\right) dx\, dy \qquad (3)$$

The shear forces must be added to this for the equilibrium of the z direction. These forces are:

$$\left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}\right)dx\ dy \tag{4}$$

where Q_x and Q_y = the normal shear forces

taking moments about the x axis yields

$$\left(\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} - \frac{1}{2}\frac{\partial Q_x}{\partial x}\ dx - Q_y - \frac{\partial Q_y}{\partial y}\ dy\right)dxdy = 0$$

where M_x, M_y, M_{xy} = the shell bending moments

After simplifying one gets

$$\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} - Q_y = 0 \tag{5}$$

Similarily for the x direction

$$\frac{\partial M_x}{\partial x} - \frac{\partial M_{yx}}{\partial y} - Q_x = 0 \tag{6}$$

Inserting Q_x and Q_y from equations (5) and (6) into equation (4) the total equilibrium in the z direction becomes:

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}$$

$$+ N_y\left(\frac{1}{R} + \frac{\partial^2 w}{\partial y^2}\right) = 0 \tag{7}$$

The moment curvature relationships for the cylinder will be derived from:

$$M_x = \int_{d/2}^{d/2} \sigma_x z dz \tag{8}$$

$$M_y = \int_{d/2}^{d/2} \sigma_y z dz \tag{9}$$

$$M_{xy} = \int_{d/2}^{d/2} \tau_{xy} z dz \tag{10}$$

The shell displacements u, v are separated into middle surface strains (resulting only from the in-plane forces N) and bending strains (resulting only from the moments M)

$$\left.\begin{aligned} u &= u_o + u_b \\ v &= v_o + v_b \\ \varepsilon_x &= \varepsilon_{xo} + \varepsilon_{xb} \\ \varepsilon_y &= \varepsilon_{yo} + \varepsilon_{yb} \\ \gamma_{xy} &= \gamma_{xyo} + \gamma_{xyb} \end{aligned}\right\} \quad (11)$$

The bending strains in the above expressions can be expressed in terms of displacements as following:

$$\left.\begin{aligned} \varepsilon_{xb} &= \frac{\partial u_b}{\partial x} \\ \varepsilon_{yb} &= \frac{\partial v_b}{\partial y} \\ \gamma_{xyb} &= \frac{\partial u_b}{\partial y} + \frac{\partial v_b}{\partial x} \end{aligned}\right\} \quad (12)$$

Since during bending plane sections are assumed to remain plane we have:

$$\left.\begin{aligned} u_b &= - z \frac{\partial w}{\partial x} \\ v_b &= - z \frac{\partial w}{\partial y} \end{aligned}\right\} \quad (13)$$

so the equations for total strain become

$$\varepsilon_x = - z \frac{\partial^2 w}{\partial x^2}$$

$$\left.\begin{aligned}\varepsilon_y &= -\ z\ \frac{\partial^2 w}{\partial y^2}\\[2ex] \gamma_{xy} &= -\ 2z\ \frac{\partial^2 w}{\partial x \partial y}\end{aligned}\right\} \quad (14)$$

Now, using the well-known stress-strain equations for plane stress this becomes

$$\left.\begin{aligned}\sigma_{xb} &= -\ \frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right)\\[2ex] \sigma_{yb} &= -\ \frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial y^2} + \mu\ \frac{\partial^2 w}{\partial x^2}\right)\\[2ex] \tau_{xyb} &= -\ \frac{Ez}{1+\mu}\left(\frac{\partial^2 w}{\partial x \partial y}\right)\end{aligned}\right\} \quad (15)$$

Substituting this into equations (8), (9), and (10) gives:

$$M_x = -\ D\left(\frac{\partial^2 w}{\partial x^2} + \mu\ \frac{\partial^2 w}{\partial y^2}\right) \quad (16)$$

$$M_y = -\ D\left(\frac{\partial^2 w}{\partial y^2} + \mu\ \frac{\partial^2 w}{\partial x^2}\right) \quad (17)$$

$$M_{xy} = D(1-\mu)\left(\frac{\partial^2 w}{\partial x \partial y}\right) \quad (18)$$

where $D = Ed^3/12(1-\mu^2)$

The middle surface strains for the element are:

$$\varepsilon_{xo} = \frac{\partial u_o}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{19}$$

$$\varepsilon_{yo} = \frac{\partial v_o}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - \frac{w}{R} \tag{20}$$

$$\gamma_{xyo} = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{21}$$

For small deflections these can be simplified to:

$$\varepsilon_{xo} = \frac{\partial u_o}{\partial x} \tag{22}$$

$$\varepsilon_{yo} = \frac{\partial v_o}{\partial y} - \frac{w}{R} \tag{23}$$

$$\gamma_{xyo} = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \tag{24}$$

Using the stress-strain relationships equations (15) the following middle surface force-deflection equations are obtained:

$$N_x^1 = \sigma x_o d = \frac{Ed}{1-\mu^2}\left[\frac{\partial u_o}{\partial x} + \mu \frac{\partial v_o}{\partial y} - \mu \frac{w}{R}\right] \tag{25}$$

$$N_y^1 = \sigma y_o d = \frac{Ed}{1-\mu^2}\left[\frac{\partial v_o}{\partial y} + \mu \frac{\partial u_o}{\partial x} - \frac{w}{R}\right] \tag{26}$$

$$N_{xy}^1 = \tau xy_o d = \frac{Ed(1-\mu)}{2(1-\mu^2)}\left[\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right] \tag{27}$$

These forces are due to loads present due to buckling. Setting the pre-buckling forces equal to:

$$N_x = P_x$$

$$N_y = P_y$$

$$N_{xy} = S_{xy}$$

and now introducing the secondary buckling forces, equations (25) to (27), the total forces are:

$$N_x = \frac{Ed}{1-\mu^2}\left(\frac{\partial u_o}{\partial x} + \mu\frac{\partial v_o}{\partial y} - \mu\frac{w}{R}\right) + P_x \qquad (28)$$

$$N_y = \frac{Ed}{1-\mu^2}\left(\frac{\partial v_o}{\partial y} + \mu\frac{\partial u_o}{\partial x} - \frac{w}{R}\right) + P_y \qquad (29)$$

$$N_{xy} = \frac{Ed(1-\mu)}{2(1-\mu^2)}\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) + S_{xy} \qquad (30)$$

Inserting the appropriate moment-deflection relationships in equations (16) to (18) and middle surface force-deflection equations (28) to (30) into the three equilibrium equations in (1), (2) and (7) the equations of equilibrium for a cylindrical shell using small deflection theory become:

$$\frac{\partial^2 u_o}{\partial x^2} + \frac{(1-\mu)}{2}\frac{\partial^2 u_o}{\partial y^2} + \frac{(1+\mu)}{2}\frac{\partial^2 v_o}{\partial x \partial y} - \frac{\mu}{R}\frac{\partial w}{\partial x} = 0 \qquad (31)$$

$$\frac{\partial^2 v_o}{\partial y^2} + \frac{1-\mu}{2}\frac{\partial^2 v_o}{\partial x^2} + \frac{(1+\mu)}{2}\frac{\partial^2 u_o}{\partial x \partial y} - \frac{1}{R}\frac{\partial w}{\partial y} = 0 \tag{32}$$

$$- D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + \left(N^1_x + P_x\right)\frac{\partial^2 w}{\partial x^2} + \left(N^1_y + P_y\right)\left(\frac{1}{R} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$+ 2(N^1{}_{xy} + S_{xy})\frac{\partial^2 w}{\partial x \partial y} = 0 \tag{33}$$

The initial curvature and primary middle surface forces in equation (33) are much larger than the curvatures due to bending, and the secondary middle surface forces. This makes it possible to re-arrange equation (33) to the form:

$$- D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + P_x\frac{\partial^2 w}{\partial x^2} + P_y\frac{\partial^2 w}{\partial y^2} + \frac{P_y}{R}$$

$$+ \frac{1}{R}\frac{Ed}{1-\mu^2}\left(\frac{\partial v_o}{\partial y} - \frac{w}{R} + \mu\frac{\partial u_o}{\partial x}\right) + 2 S_{xy}\frac{\partial^2 w}{\partial x \partial y} = 0 \tag{34}$$

Because all the secondary middle surface forces are not negligible in linear shell theory the three equilibrium equations (31), (32) and (34) are coupled and must be solved simultaneously. It is often more convenient to combine the three equations to obtain a single equation in w. If equation (32) is operated on by $\partial^2/\partial x \partial y$, and equation (31) by $\partial^2/\partial x^2$, and $\partial^2/\partial y^2$, one obtains three equations which may be reduced to:

$$\nabla^4 u = \frac{\mu}{R}\frac{\partial^3 w}{\partial x^3} - \frac{1}{R}\frac{\partial^3 w}{\partial y^2 \partial x} \tag{35}$$

where $\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$

Similarily if equation (31) is operated on by $\partial^2/\partial x\, \partial y$, and equation (32) by $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$ one obtains three equations which may be reduced to the form:

$$\nabla^4 v = \frac{\mu+2}{R}\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{1}{R}\frac{\partial^3 w}{\partial y^3} \tag{36}$$

Equation (34) is now operated on by ∇^4, yielding:

$$- D\nabla^8 w + \nabla^4\left(P_x \frac{\partial^2 w}{\partial x^2} + P_y \frac{\partial^2 w}{\partial y^2} + 2\, S_{xy} \frac{\partial^2 w}{\partial x \partial y}\right) +$$

$$\frac{1}{R}\frac{Ed}{(1-\mu^2)}\left(\nabla^4 \frac{\partial v}{\partial y} + \mu\, \nabla^4 \frac{\partial u}{\partial x} - \frac{1}{R}\nabla^4 w\right) = 0 \tag{37}$$

Operating on equation (35) by $\partial/\partial x$, and equation (36) by $\partial/\partial y$, and substituting the results in equation (37) one obtains:

$$D\nabla^8 w - \nabla^4\left(P_x \frac{\partial^2 w}{\partial x^2} + P_y \frac{\partial^2 w}{\partial y^2} + 2\, S_{xy} \frac{\partial^2 w}{\partial x \partial y}\right) + \frac{Ed}{R^2}\frac{\partial^4 w}{\partial x^4} = 0 \tag{38}$$

This equation is known as the Donnell small deflection equation for shell buckling. As was shown by Batdorf [7] the use of equation (38) implies certain boundary conditions on the solution. These are, for simply supported edges: