ON THREE THEOREMS CONCERNING THE RADICAL AND SEMI-SIMPLICITY

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An Abstract of a Thesis

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The radical N of a finite dimensional algebra, or more generally of a ring R which satisfies the descending chain condition (d.c.c.), is usually defined as the union of all nilpotent left (right) ideals in R. Other definitions of the radical, namely those for arbitrary rings, narings, clusters, and alternative rings, are generalizations of N; and they coincide with this ideal when considered in a ring with d.c.c.

Fundamental to the structure theory of rings which satisfy the d.c.c., (i.e., the so-called classical theory), is that the radical N is a two-ideal; that R/N is semi-simple; and that a ring is semi-simple if and only if it is a finite direct sum of simple rings. For the more general definitions of the radical, a number of interesting analogues of these theorems have been proven. Taking them collectively reveals the following general information:

1. The fact that the radical is a two-ideal, is proved differently for different definitions of the radical. Each proof, however, depends mainly on the <u>manner</u> in which the radical is defined. On classifying the different definitions, it is found that four general methods suffice.

2. The analogues of the second theorem, that a ring, naring, cluster, etc., modulo its radical is semi-simple, are proved by one of two different methods. The most common of the two, used for all but two definitions of the radical, is based on the fact that the radical can always be expressed in terms of a specified set of elements. The other slighty more general method makes use of a suitable application of the second theorem on homomorphic mappings.

3. Lastly, analogues of the third theorem, (as well as the theorem itself), are proved by one of three general methods. First, there is the original or classical proof based on a number of results of the theory of idempotents. Secondly, there is a method based on an intersection characterization of the radical, and a fundamental theorem on subdirect sums; and thirdly, there is a method which results from a special and rather unusual definition of the radical.

A brief description of each particular method, illustrated by one or more specific examples, comprises the present dissertation.

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CHAPTER I

INTRODUCTION

The main object of this thesis is to become acquainted with the following general theorems taken from the theory of the radical and semi-simplicity:

Theorem 1. The radical N is a two-ideal.

Theorem 2. The difference ring (naring, cluster, alternative ring, non-associative algebra, alternative algebra, Jordan algebra) S/N is semi-simple, meaning that it is either a ring with zero radical or else in a particular case a finite direct sum of simple rings.*

Theorem 3. A ring (naring, cluster, etc.) S is semi-simple if and only if S is isomorphic to a subdirect sum of rings S_i, the S_i's suitably generalizing simple rings.

For the most part, these theorems were developed since 1942; and all of them have resulted from various extensions of the classical notion of the radical for algebras. The third is of particular interest as it expresses the existence of a number of semi-simple systems, whose structures suitably generalize the finite direct sum decomposition of classically semi-simple algebras.

For our particular purpose, we will study the different methods of proof which have been found expedient for these theorems. We have determined that there are

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* Throughout, unless otherwise stated, semi-simplicity will be regarded in the zero radical sense.

four general methods for theorem 1, two for theorem 2, and three for theorem 3. Each method is discussed briefly, and illustrated by one or more specific examples.

To familiarize the reader with the general theory of the radical and semi-simplicity, we begin with a brief synopsis of the various definitions of the radical, their relationships, and the explicit forms of the theorems above.

It is assumed that the reader is familiar with such concepts as group, ring, vector space, algebra, field, ideal, nil and nilpotent ideals, descending chain condition, direct sum, union, intersection, homomorphism, etc., all as one might find in B. L. Van der Waerden's "Modern Algebra."

With regard to notation, rings have been designated by R; algebras (which are assumed to have finite dimension unless otherwise stated) by A; rings or algebras by S; and residue class rings, i.e., difference rings by $S/I \equiv \{[s];\ \text{modulo } I\}$.

CHAPTER II

THE RADICAL AND STRUCTURE THEORY

Rings

For rings which satisfy the descending chain condition (d.d.c.), the three theorems of the introduction are a direct outgrowth of Wedderburn's contribution to the structure of algebras over a non-modular field [27].*

Following Albert's exposition of Wedderburn's results [1], it is well known that every nilpotent left, right, or two-ideal of an algebra A over an arbitrary field is contained in a maximal nilpotent two-ideal N_1 called its radical. Furthermore the difference algebra A/N_1 is semi-simple; and an algebra A is semi-simple if and only if it is isomorphic to a direct sum of simple algebras each of which is isomorphic to a total matrix algebra over a division algebra. Similarly, by defining the radical N_2 of a ring R with d.c.c. to be the union of all nilpotent left (right) ideals in R, it can be shown that N_2 is a two-ideal; R/N_2 is semi-simple; and a ring R is semi-simple if and only if R is isomorphic to a direct sum of simple rings each of which is isomorphic to a total matrix ring over a division ring.

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* The numbers in the square brackets refer to the bibliography given at the end.

The radical N_2 coincides with N_1; both are referred to as the classical radicals; and all other definitions of the radical are either generalizations or characterizations of one, or the other.

In the early 1940's, after the above so-called Wedderburn-Artin structure theorems (of R) were fully realized, Baer and Levitzki independently proposed definitions of the radical for arbitrary rings. It was hoped that these definitions would lead to further generalizations of Wedderburn's original results.

Baer [7], called an ideal N of a ring R a radical ideal if N is a two-ideal, a nil-ideal, and if R/N contains no non-zero nilpotent left (right) ideals. He defined the lower radical N_3 of R to be the intersection of all radical ideals, and the upper radical N_4 to be the union of all radical ideals.

Levitzki [18], called a left, right, or two-ideal I of R, a semi-nilpotent ideal if every subring generated by a finite number of its elements is nilpotent. He defined the radical N_5 to be the union of all semi-nilpotent two-ideals in R.

Although N_3, N_4, and N_5, coincide with N_2 when R satisfies the d.c.c., none of these definitions led to a structure for (arbitrary) semi-simple rings.

In 1945, Jacobson [15] finally overcame the difficulties presented by rings which are not restricted to the d.c.c. On defining a completely new kind of radical he was able to provide an entirely satisfactory generalization of the Wedderburn-Artin theory. He abandoned the concepts of nil and semi-nilpotence in defining the radical, and used the notion of quasi-regularity introduced by Perlis [23]. The latter called an element $x \in R$ right quasi-regular if there exists a y such that $x + y + xy = 0$, and a right ideal quasi-regular if all of its elements are right quasi-regular. Jacobson defined the right (left) radical N_6 of R to be the union of all quasi-regular right (left) ideals of R.

He proved that N_6 the right radical coincides with the left radical; and that it is a two-ideal. He showed that the quotient R/N_6 is semi-simple, and that a ring R is semi-simple if and only if R is isomorphic to a subdirect sum of primitive rings* R_i each of which is isomorphic to a dense ring of linear transformations on a vector space over a division ring. This theorem is an immediate generalization of the classical result; for if R satisfies the d.c.c., the subdirect sum of primitive rings becomes a finite direct sum of simple rings.

This contribution created a great deal of interest, and in 1947 Brown and McCoy [10] contributed the G-radical.

They considered the two-ideal

$$G(a) \equiv \{ax - x + \sum(y_i a z_i - y_i z_i)\}$$

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* A ring R is called right primitive if it contains a maximal right ideal J such that the quotient ideal $(J:R) \equiv \{\alpha \in R;\ R\alpha \subseteq J\}$ is zero. A left primitive ring is defined similarly.

generated by the right ideal $\{ax - x\} \leq R$; and defined the G-radical N_7 to be the set of all $b \in R$ such that $a \in (b)$, the two-ideal generated by b, implies $a \in G(a)$. The element a is called G-regular.

A more general approach to this definition, of interest itself, is given as follows: Assume there is defined an arbitrary mapping $a \longrightarrow F(a)$ of R into the set of two-ideals F(R) of R such that if $a \longrightarrow \bar{a}$ is any homomorphism of R onto a ring $\bar{R}$, then $F(\bar{a}) = \overline{F(a)}$. Call an element a F-regular if it is in F(a), and an ideal F-regular if every element in the ideal is F-regular. Finally, define the F-radical N_8 of R to be the set of all $b \in R$ such that (b), the two-ideal generated by b, is F-regular.

Since $G(\bar{a}) = \overline{G(a)}$, the G-radical <u>is</u> clearly a special case of the F-radical. In addition $N_7 \geq N_6$, since an element $b \in R$ is in the Jacobson radical if and only if $a \in \{ax - x\}$ for every $a \in \{bi + bz\}$. If R satisfies the d.c.c., Brown and McCoy have shown that $N_7 = N_6$ - which then coincide with N_2.

It should be noted, however, that the F-radical (itself) will not in general reduce to the classical radical. Obviously, there is no assurance that an arbitrary F-mapping will define a generalization of N_2. The latter occurs only if the F-mapping is specially chosen.

In spite of this fact, Brown and McCoy discovered that R/N_8 has zero F-radical; and also that a ring has zero

F-radical if and only if it is isomorphic to a subdirect sum of subdirectly irreducible rings with zero F-radical. In the case of the G-radical, it was proved that a ring R is semi-simple with respect to N_7 if and only if R is a subdirect sum of simple rings with unity element.

An equivalent definition of the G-radical was later proposed by Andrunakievitch [5]. Andrunakievitch considered the operation

$$a \circ b \equiv a + b - ab$$

which we call quasi-multiplication. He defined a co-united ideal C of R to be a set satisfying:

1) that $a \in C$ and $r \in R$ implies $a \circ r \in C$;

2) for any a_1, a_2, $a_3 \in C$, $a_1 - a_2 + a_3 \in C$.

We note that this set is not an ideal in the ordinary sense, in fact it is not even closed under addition.

For the definition of the radical, Andrunakievitch considered the co-united ideal $R \circ a \circ R$ defined by

$$\left\{ \sum \lambda_i (x_i \circ a \circ y_i) \right\},$$

where $\sum \lambda_i = 1$. He called an element $b \in R$ a common radical element if $R \circ b \circ R = R$; and a two-ideal a common radical ideal if each of its elements is common radical. He defined the radical N_9 (which we call the Andrunakievitch radical) to be the union of all common radical ideals in R.

The notion of common radicality coincides with G-regularity, because an element $a \in R$ is common radical if and only if a is G-regular. Hence the Andrunakievitch radical, being $\{ b \in R;$ every $a \in (b)$ is a common radical element$\}$, coincides with the G-radical. Evidently this is another approach to the theory of Brown and McCoy.

Among these definitions of the radical, there is one more due to McCoy [21].

McCoy calls a set M of a ring R an m-system if a, $b \in M$ implies that $axb \in M$ for some $x \in R$. He defined the radical N(I) of an ideal $I \leq R$ to be the set of all elements $x \in R$ with the property that every m-system which contains x contains an element of I. The radical N_{10} of R is defined to be the radical of the zero ideal. The set N_{10} is referred to as the p-radical.

It is shown that N_{10} is a nil ideal which contains every nilpotent right ideal of R; and in the presence of d.c.c. it coincides with the classical radical.

McCoy also obtained a form of the third theorem. He proved that a ring R is semi-simple with respect to N_{10} if and only if R is isomorphic to a subdirect sum of prime rings. It is interesting to note that a prime ring has zero p-radical, and that a prime ring with minimal left (right) ideals is a primitive ring.

Non-Associative Systems

Albert was the first to propose a definition of the radical for a non-associative system. In 1942, see [2], he showed that if a non-associative algebra A is homomorphic to a direct sum of non-zero simple algebras, then there exists a minimal two-ideal $N_{11} \subseteq A$, such that N_{11} is the intersection of all two-ideals $B \subseteq A$ with the property that A/B is a direct sum of non-zero simple algebras. Since the proof of the existence of N_{11} established that A/N_{11} is a finite direct sum of non-zero simple algebras, Albert defined a semi-simple algebra to be one which is a finite direct sum of simple algebras. He called N_{11} the radical of A.

Much later, Jenner [16] proposed a similar theory for non-associative rings. He considered the transformation ring T(R),* (as did Albert for A), and defined R to be semi-simple if T(R) satisfies the d.c.c.; R contains no absolute divisors of zero; and if R is a finite direct sum of simple rings.

For his definition of the radical, Jenner assumed that R is isomorphic to a semi-simple algebra R/H (where $H \subseteq R$) and that T(R) satisfies the d.c.c. He defined the radical N_{12} of R to be the complete reciprocal image of a

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* T(R) is formed by taking all finite sums and products of the right and left multiplications of R.

set M_j under the natural homomorphism

$$R \longrightarrow R/R^N,$$

where: (1) the set N is the classical radical of T(R); (2) R^N is the set of all images of R under N; and (3) M_j, a two-ideal contained in R/R^N, is the set of all $[w] \in R/R^N$ such that both $[w][d]$ and $[d][w]$ are in M_{j-1} for all $[d] \in R/R^N$. The set $M_1 \equiv \{[x] \in R/R^N;\ [x][a] = [a][x] = [0]$ for all $[a] \in R/R^N\}$.

It is shown that R/N_{12} is isomorphic to a direct sum of simple narings, where N_{12} is minimal in R having this property. As such, N_{12} is justifiably called the radical of R.

The set N_{11} coincide with N_1 if R is associative; and similarly N_{12} coincides with N_2 if R is associative and satisfies the d.c.c.

Between the Albert and Jenner contributions, Smiley [26] furthered the development of the structure of non-associative systems when he discovered that the notion of the F-radical of Brown and McCoy applies directly to narings.

For his definition of the radical Smiley considered the F-mapping $a \longrightarrow F(a)$ where

$$F(a) = F_1(a) \equiv \{ax - x + ya - y\}.$$

He defined the F_1-radical N_{13} to be the set of all $b \in R$, such that if $a \in (b)$ then $a \in F_1(a)$.

With this particular choice of $F_1(a)$ - regularity, there is no difficulty regarding the lack of associativity. It is easy to prove, as did Smiley, that N_{13} is a two-ideal; that R/N_{13} is semi-simple; and finally that a naring R is semi-simple with respect to N_{13} if and only if R is isomorphic to a subdirect sum of simple narings with unity element.

By examining the relationships between the $F_1(a)$-radical and those defined by Albert and Zorn, (the latter which we discuss below), Smiley found that if R is a non-associative algebra with unity element, then N_{13} coincides with N_{11}; and if R is hypercomplex alternative the radical N_{13} coincides with the Zorn radical N_{14}.

<u>Alternative Systems</u>

M. Zorn [28] was able to define a radical for an alternative ring, when he imposed the restriction that it be hypercomplex.

A hypercomplex alternative ring is one in which there are chain conditions on modules paralleling the d.c.c. on ideals in ordinary rings.

To define the radical, Zorn considered elements $a_{ik} \in R_{ik}$ such that $\{a_{ik} \cdot r_{ik}\}$ is nilpotent for each $r_{ik} \in R_{ik}$. The set R_{ik} is a component of R decomposed as a direct sum of special "Peirce" decomposition, and the elements a_{ik} are called singular.

Zorn thus assumed that there exists at least one finite set of idempotents $e_1, e_2, \cdots e_n$, with $e_i \cdot e_k = 0$ for $i \neq k$, such that

$$R = \sum_{i,k=1}^{n} R_{ik},$$

where $R_{ik} \equiv \{ x_{ik} \in R;\ e_j x_{ik} = \delta_{ji} x_{ik}, \text{ and } x_{ik} e_j = \delta_{kj} x_{ik} \}$. He defined the radical N_{14} to be the set of all $\sum_{i,k=1}^{n} n_{ik}$ such that $n_{ik} \cdot R_{ik}$ is singular.

He showed that the radical N_{14} is the set of all nilpotent elements of R, and also that a hypercomplex alternative ring which is semi-simple with respect to N_{14} is a direct sum of simple rings.

For the definition of the radical of ordinary alternative rings, Smiley [25] considered an extension of Jacobson's radical. Having shown that a naring is semi-simple (re $F_1(a)$) if and only if it is isomorphic to a sub-direct sum of simple narings with unity element, the $F_1(a)$-radical naturally applies to alternative rings. He was, however, interested in defining something smaller. Thus, instead of considering $F_1(a) \equiv \{ax - x + ya - y\}$ he considered $Q(a) \equiv \{ax - x\}$.

In other words, Smiley considered the operation $x \circ y = x + y + xy$ which he called the quasi product of x with y. He defined the right radical N_r of an alternative ring R to be the set of all $b \in R$ such that $(b)_r$, the right ideal generated by b, has the property that each $a \in (b)_r$ yields $a \circ z = 0$ for some $z \in R$. Equivalently this means that each $a \in (b)_r$ is contained in $\{az - z\} = Q(a)$. A left radical N_l was defined similarly and N_{15} the radical of R is taken either as N_r or N_l.

Smiley was able to show that $N_r = N_l$, and that if R is hypercomplex alternative then N_{15} coincides with Zorn's radical. This is to be expected since N_{15} is contained in the $F_1(a)$-radical which, as we stated, also coincides with Zorn's radical if the naring R is hypercomplex alternative. He was, however, unable to show that N_{15} led to a generalization of the Wedderburn-Artin structure theorem for semi-simple rings.

For alternative algebras with unity element there are three definitions of the radical proposed by Dubish and Perlis [11]. The first of these radicals, which is called the N-radical N_{16} of A is defined to be the totality of elements $n \in A$ such that $g + n$ is regular for every regular $g \in A$. The Q-radical N_{17} is defined to be the sum of all quasi-regular right ideals, and the S-radical (similar to Albert's N_{11}) is taken to be the intersection of all two-ideals B such that A/B is a direct sum of simple alternative algebras.

It is shown that the N-radical coincides with the S and Q-radical; and the proof of the existence of the S-radical provides a structure for semi-simple alternative algebras in a manner similar to Albert's result for non-associative algebras.

In the case that A is associative the above radicals coincide with the classical radical N_1. Actually it was shown by Perlis [23] that N_1 the classical radical coincides with the set of all elements $r \in A$ such that $x + \alpha r$ is quasi-regular for every quasi-regular $x \in A$ and α in the base field. Furthermore, if A contains a unity element then N_1 coincides with the set of all $n \in A$ such that $g + n$ is regular for every regular $g \in A$.

Jordan Algebras

The structure of Jordan algebras is due almost entirely to Albert. For the definition of the radical [3], he considered the derived series of a Jordan algebra A which is the sequence $A = A^{(o)} \geqslant A^{(1)} \geqslant A^{(2)} \geqslant \cdots \geqslant A^{(n)}$. By definition $A^{(k+1)}$ is the square of $A^{(k)}$, and it is either zero or a subalgebra of A. The elements of $A^{(k)}$ are finite sums of quantities of the form $aS_1 \cdots S_k$, where each S_i is a right multiplication of the transformation algebra T(A), and $a \in A$. Albert called an algebra A solvable if $A^{(k)} = 0$ for some interger k. He established that there exists a maximal solvable two-ideal, the radical N_{17}, and proved that a Jordan algebra modulo this two-ideal is a direct sum of simple Jordan algebras each of which is a Jordan algebra of linear transformations with the exception of algebras of degree 3 and 27 over their centers.