Posteriori Error Analysis for the P-version of the Finite Element Method

by

Xiaofeng Yang

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University of Manitoba

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Contents

Abstract

In the framework of the Jacobi-weighted Sobolev space, we design the a-posterior error estimators and error indicators associated with residuals and jumps of normal derivatives on internal edges with appropriate Jacobi weights for the *hp*-version of the finite element method. With the help of quasi Jacobi projection operators, the upper bounds and the lower bounds of indicators and estimators are analyzed, which shows that such a-posteriori error estimation is quasi optimal. The indicators and estimators are computed for some model problems and programmed in C++. The numerical results show the reliability of our indicators and estimators.

CHAPTER 1

Introduction

Numerical Simulation plays an extremely important role for solving practical problems arising from various fields of engineering, science, finance, and health science as computer technology has rapidly developed in the recent decades. Numerical simulation involves modeling and computing. Validation of models and verification of computational results are two fundamental issues for numerical simulation. The verification includes numerical methods, error analysis, corresponding adopted numerical method, assessment of error computed solutions, adaptive algorithms. The finite element method (FEM) is the today's mostly adopted numerical methods for numerical simulation. The high-order FEM such as p and h-p versions provides reliable computational solutions with high accuracy and lower computational cost than lower order FEM referred as the h-version.

There are two kinds of error assessments: a-priori error estimation and a-posteriori error estimation. The former gives the convergence rate of the finite element solution for a given problem in certain functional spaces, which is derived by theoretical analysis prior to the computation. the latter provides quantitative information of error in the finite element solutions measured in norms such as energy norm and infinity norm based on the computed data accumulated in the process of computation. Since the solutions are unknown for practical problem, except those manufacture problems, the errors in finite element solutions are not known and not computable. Therefore, a-priori error estimation provides only theoretical guidance for computation, but it can not be directly applied to the verification of the accuracy of the finite element solutions in decision making progress for important industrial, engineering, science and problems related to various national interests. A-posteriori error estimation can be applied to quantitative error assessment, and play a decisive role in verification and decision making process. Although various progress on a-posteriori estimation of the finite element method have made, the ongoing research on a-posteriori error estimation on the p-version of FEM lacks solidly theory and decisive impact on practical computation. Therefore the theme of the thesis is of great significance for the adaptive algorithm of p-version of FEM and its applications to engineering and scientific computing.

Since Babuška and Rheinboldt proposed [**14, 15**] in the later 1970's the revolutionary concept: a-posteriori error estimation and the adaptivity, the adaptive finite element algorithms have become powerful and reliable computational tools for large-scale scientific and engineering computations. The adaptive method has rapidly developed in the past decade theoretically and practically, and it has integrated in many commercial and research codes such as Pro/MECHANICA(Parametric Technology Corporation), PolyFEM(IBM), ProPHLEX(Altair Engineering, Inc., Texas), STRESSCHECK(Engineering Software Research and Development, Missouri), STRIPE(Aeronautical Research Institute of Sweden), 2Dhp90 and 3Dhp90(ICES, University of Texas at Austin, [**33, 34**]).

There are two basic approaches for a-posteriori error estimation of FEM. The first approach is based on the solutions of local auxiliary problems. The second approach is based on the residuals, including the jump of normal derivative of finite element solutions on internal edges of elements. For the lower-order FEM, a-posteriori error estimation and the adaptive mesh generation have been well developed in the past two decades, we refer to the books [**4**] for an overview. The first approach for the h-version of FEM has been developed in various papers such as [**3, 5**]. The second approach is also widely used for the h-version, see. e.g., [**37**]. Various a posteriori error estimates for numerical solutions with finite elements of low order are known to be reliable and efficient [**31**]. They are used to create locally refined meshes in an adaptive way that fit to the elliptic problem under consideration.

In contrast to the lower-order FEM, a-posteriori error estimation for the high-order FEM such as the p-version of FEM, the hp version of FEM and the spectral method is much less developed and lacks of substantial progresses in the past two decades. There are a few papers available in the current literature [**15, 17, 18, 41**]. Therefore the research in this direction is now at a quite primary stage because of the obvious difficulties of high-order methods except in one dimension. Many useful and effective methods and techniques developed for the h-version of FEM can not be or have not been applied to the high-order FEM, such as the super-convergence and patch recovery [**48**]. The comprehensive analysis for the error estimators and indicators based on the residuals for the p- and hp of FEM in one dimension was well known in the mid of the 1980's [**41**], but the results for one dimension can not be exactly established in high dimensions, and techniques used for one-dimensional analysis can not be applied to two and three dimensions.

The first paper about residual based a-posteriori estimation for the p-version is [**22**], which shows the viability of this procedure computationally, meanwhile, theoretical analysis for the hN spectral method was provided in [**19**]. However, the analysis is restricted to meshes consisting of axi-parallel rectangles. The residual based error estimator and error indicator for p-version of FEM was implemented for mechanical problems in [**22, 33, 34**] without theoretical support. For theoretical analysis of such error estimators we find a few papers in the early 2000's [**16, 35, 36**]. The approach of solving local auxiliary problems was utilized in adaptive hp codes in the 1980s and 1990s [**37, 44**] . Unfortunately, these error indicators and estimators used by engineers lack of theoretical support. Under shadow of the success of a-posteriori error estimation for the h-version of FEM, it is not clear yet what mathematical framework should be adopted for a-posteriori error estimation of the high-order FEM in two and three dimensions. While lacking of theoretical progresses in past two decades, the engineers have implemented various types of error estimators and indicators based on their own experiences in commercial and research codes, we refer to [**37**]. Recently a-posteriori error estimation based on a solution of local auxiliary problems on a patch centered at the nodes of mesh was proposed in [**21**], where many theocratical and practical aspects were considered.

Since the later 1990's, the approximation theory for the p- and hp version of FEM has been well developed in the mathematical framework of the Jacobi-weighted Besov and Sobolev spaces [**9, 10, 11, 12**]. With help of this framework, a-priori error analysis leads to a rigorous proof of the optimal convergence for the p- and hp version (with quasi-uniform meshes) of FEM. The recent breakthrough of the a-priori error estimation for the p and $h-p$ version (with quasi-uniform meshes) of FEM raise a question, namely, whether the new mathematical framework can benefit the a-posteriori error estimation, and how to develop further this framework for the need of the a-posteriori error analysis.

The focus of the thesis is a-posteriori error estimation for the p-version on triangular elements, which is much more complicated and difficult than on square and quadrilateral elements, and has never been addressed in literature. After studying of the literature on a-priori error estimation on triangle and a-posteriori error analysis on square and quadrilateral, the following approaches are adopted:

- Design error indicators and estimators of residual-type on triangles;
- Utilize theoretical setting of the Jacobi-weighted space on triangles;
- Generalize the error estimators and indicators on square to triangle by a singular mapping which maps a square onto a triangle.

In this paper we will analyze the residual-based a-posteriori error estimation for the p-version of FEM in the newly-developed mathematical framework of Jacobi-weighted Sobolev spaces. The rest of the paper is organized as follows. The error indicators and estimators on triangle with Jacobi weights are defined in Chapter 2, and two major theorems are stated, for which the proof are given in Chapter 4. The Jacobi-weighted spaces on triangles are introduced in Chapter 3, which provides a mathematical framework for a-priori and a-posteriori error analysis of the high-order FEM on triangles. In Chapter 4 we construct a quasi-Jacobi projection operator and analyze its properties, which play an essential role in proof of quasi-optimality of the error indicators and error estimators. The computational results are presented in Chapter 5, and followed by various comments. In the last section, we make some concluding remarks.

CHAPTER 2

Residual-based Error Indicators and Estimators

Consider a boundary value problem on a polygon Ω:

$$\begin{cases} -\triangle u + u = f, & \text{in } \Omega \\ u\big|_{\Gamma_D} = 0, & \text{on } \Gamma = \partial\Omega. \end{cases} \tag{2.1}$$

The corresponding variational problem is to find $u \in H_0^1(\Omega)$ such that

$$B(u,v) = F(v) \qquad \forall v \in H_0^1(\Omega) \tag{2.2}$$

where B is a bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$:

$$B(u,v) = \int_\Omega (\nabla u \cdot \nabla v + uv)dx,$$

and F is a linear functional on $H^1(\Omega)$:

$$F(v) = \int_\Omega fv dx \qquad \forall v \in H^1(\Omega).$$

Let $\mathcal{T} = \{K_i, 1 \le i \le M\}$ be a partition of Ω with shape-regular quadrilateral elements S_i or triangle elements T_i, and let $\partial\mathcal{T} = \{\gamma_l, 1 \le l \le L\}$ with inter-element edges γ_l. By F_i we denote a mapping of the standard element $S = (-1,1)^2$ onto elements S_i or

$T = \{(\xi,\eta)|0 \le \xi,\eta,\xi+\eta \le 1\}$ onto elements T_i. Then the subspaces of continuous and piecewise polynomials over $\mathcal{T}$ for the p-version of FEM is defined as usual, i.e.

$$S_0^{p,1}(\Omega,\mathcal{T}) = S^{p,1}(\mathcal{T})\bigcap H_0^1(\Omega) = \{\varphi|_{T_i} = \psi_p \circ F_i, \psi_p \in \mathcal{P}_p(K)\}\bigcap H_0^1(\Omega), K = T$$

or

$$S_0^{p,1}(\Omega,\mathcal{T}) = S^{p,1}(\mathcal{T})\bigcap H_0^1(\Omega) = \{\varphi|_{S_i} = \psi_p \circ F_i, \psi_p \in \mathcal{P}_p(K)\}\bigcap H_0^1(\Omega), K = S$$

where $\mathcal{P}_p(K)$ is a set of polynomials of total degree $\le p$ on K if $K = T$ or a set of polynomials of separate degree $\le p$ on K if $K = S$.

The p-version FEM solution $u_S \in S_0^{p,1}(\mathcal{T})$ satisfies

$$B(u_S, v) = F(v) \qquad \forall v \in S_0^{p,1}(\Omega,\mathcal{T}). \tag{2.3}$$

By e,e_i,r,r_i and R,R_{γ_l} we denote the error, residue and jump of normal derivative along the internal edges γ_l for the finite element solution u_S:

$$\begin{aligned} e &= u - u_S, \qquad e_i = e|_{K_i}; \\ r &= f + \Delta u_S - u_S, r_i = r|_{K_i}; \\ R &= \left[\frac{\partial u_S}{\partial n}\right], \quad R_{\gamma_l} = \left[\frac{\partial u_S}{\partial n}\right]|_{\gamma_l}. \end{aligned}$$

A local error indicator η_{T_i} associated with the residual r_i is defined as

$$\eta_{T_i} = (p+1)^{-1}\|r_i\|_{L^2_\beta(K_i)} \tag{2.4}$$

and the indicators $\eta_{\gamma_l^i}$ associated with the jump of $R = \left[\frac{\partial u_S}{\partial n}\right]$ on the internal edge γ_l^i

$$\eta_{\gamma_l^i} = (p+1)^{-\beta}\|R\|_{L^2_\beta(\gamma_l^i)}. \tag{2.5}$$

The estimator η is defined as

$$\eta = \Big(\sum_{K_i\in\mathcal{T}} \eta^2_{K_i}\Big)^{\frac{1}{2}} + \Big(\sum_{\gamma_l^i\in\partial\mathcal{T}} \eta^2_{\gamma_l^i}\Big)^{\frac{1}{2}}. \tag{2.6}$$

In general the residual $r_i \notin \mathcal{P}_{p_i}(K_i)$, and the jump $R\,|_{\gamma_l^i} \notin \mathcal{P}_{p_i}(\gamma_l^i)$ if the basis functions are the images of shape functions defined on the standard square element and the corresponding mapping is not linear. For the analyzing the indicators as the lower bound of the error, we need to modify the indicators η_{K_i} and $\eta_{\gamma_l^i}$.

Let $\Pi^\beta_{K_i}$ denote the $L^2_\beta(K_i)$-projection on $\mathcal{P}_{p_i}(K_i)$, and let $\Pi^\beta_{\gamma_l^i}$ denote the $L^2_\beta(\gamma_l^i)$-projection on $\mathcal{P}_{p_i}(\gamma_l^i)$.

$$\tilde\eta_{K_i} = (p+1)^{-1}\|r_{i,p}\|_{L^2_\beta(K_i)} = (p+1)^{-1}\|\Pi^\beta_{K_i} r_i\|_{L^2_\beta(K_i)} = (p+1)^{-1}\|\Pi^\beta_{K_i} f + \Delta u_S - u_S\|_{L^2_\beta(K_i)} \tag{2.7}$$

and

$$\tilde{\eta}_{\gamma_l^i} = (p+1)^{-\beta}\|R_p\|_{L^2_\beta(\gamma_l^i)} = (p+1)^{-\beta}\|\Pi^\beta_{\gamma_l^i} R\|_{L^2_\beta(\gamma_l^i)}. \tag{2.8}$$

The modified error estimator η is defined as

$$\tilde{\eta} = \Big(\sum_{K_i\in\mathcal{T}} \tilde{\eta}^2_{K_i}\Big)^{\frac{1}{2}} + \Big(\sum_{\gamma_l^i\in\partial\mathcal{T}} \tilde{\eta}^2_{\gamma_l^i}\Big)^{\frac{1}{2}}. \tag{2.9}$$

If there is no confusion, we drop the indices i and l in K_i and γ_l^i for the simplicity. Then K denotes one of elements K_i, and γ denote one of internal edges γ_l^i.

CHAPTER 3

A Mathematical Framework of Jacobi-weighted Spaces

3.1. Jacobi polynomials and associated properties

The Jacobi polynomial of degree $n = 0, 1, 2, \dots$ is defined as

$$J_n^{\alpha,\beta}(x) = \frac{(-1)^n (1-x)^{-\alpha}(1+x)^{-\beta}}{2^n\, n!}\, \frac{d^n\, (1-x)^{\alpha+n}(1+x)^{\beta+n}}{dx^n} \tag{3.1}$$

with $\alpha, \beta > -1$. These polynomial posses important properties, which are essential to the approximation of the high-order finite element method and the special method.

$J_n^{\alpha,\beta}(x)$ are orthogonal with Jacobi weight

$$\int_I J_m^{\alpha,\beta}(x)\, J_n^{\alpha,\beta}(x)\, (1-x)^{\alpha}(1+x)^{\beta}\, dx = \begin{cases} \gamma_n^{\alpha,\beta} & m = n \\ 0 & m \neq n \end{cases}$$

with

$$\gamma_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}. \tag{3.2}$$

By the Stirling formula

$$\Gamma(s+1) = \sqrt{2\pi s} s^s e^{-s}\big(1 + O(s^{-1/5})\big) \tag{3.3}$$

we have asymptotic estimation for $n \geq 1$

$$\gamma_n^{\alpha,\beta} \simeq \frac{2^{\alpha+\beta+1}}{(2n+\alpha+\beta+1)}. \tag{3.4}$$

For $x \in [-1,1]$, there holds

$$|J_n^{\alpha,\beta}(x)| \leq C(n+1)^{\max\{\alpha,\beta,-1/2\}} \tag{3.5}$$

with C independent of α and β, and

$$|J_n^{\alpha,\beta}(1)| \leq C(\alpha)(n+1)^{\alpha},\ |J_n^{\alpha,\beta}(-1)| \leq C(\beta)(n+1)^{\beta} \tag{3.6}$$

with $C(\alpha) = \frac{C_0}{\Gamma(1+\alpha)}$ and $C(\beta) = \frac{C_0}{\Gamma(1+\beta)}$.

3.2. Jacobi-weighted Sobolev and Besov spaces on Triangle

Let $T = \{(x,y) | 0 \leq x, y \leq 1, 0 \leq x+y \leq 1\}$. For $\beta = (\beta_1, \beta_2, \beta_3)$ with $\beta_i > -1$, we define a weight function on T

$$W_\beta(x,y) = W_{\beta_1,\beta_2,\beta_3}(x,y) = x^{\beta_1} y^{\beta_2} (1-x-y)^{\beta_3}.$$

The weighted space $L^2_\beta(T)$ is the closure of C^∞ function under the norm:

$$\|u\|_{L^2_\beta(T)} = (\int_T u^2 W_\beta(x,y)dxdy)^{1/2} \tag{3.7}$$

$L^2_\beta(T)$ is an inner product space with:

$$(u,v)_{L^2_\beta(T)} = \int_T uvW_\beta(x,y)dxdy. \tag{3.8}$$

For $\beta = (\beta_1, \beta_2, \beta_3)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $\beta_i > -1$ and integers $\alpha_i \geq 0$, we define a weight function on T

$$W_{\beta\pm\alpha}(x,y) = W_{\beta_1\pm\alpha_1,\beta_2\pm\alpha_2,\beta_3\pm\alpha_3}(x,y) = x^{\beta_1\pm\alpha_1}y^{\beta_2\pm\alpha_2}(1-x-y)^{\beta_3\pm\alpha_3}. \tag{3.9}$$

Define the Jacobi-weighted Sobolev spaces $H^{k,\beta}(T)$, $k \geq 0$ are furnished with the norm and semi-norm [**32**]:

$$\|u\|^2_{H^{k,\beta}(T)} = \sum_{l=0}^{k} |u|^2_{H^{l,\beta}(T)}, \qquad |u|^2_{H^{l,\beta}(T)} = \sum_{n_1+n_2+n_3=l} \frac{l!}{n_1!n_2!n_3!} \|D^n u\|^2_{L^2_{n^*+\beta}(T)}$$

for $n = (n_1, n_2, n_3)$, $n^* = (n_1+n_3, n_2+n_3, n_1+n_2)$ and $D^n = \partial_x^{n_1}\partial_y^{n_2}(\partial_x - \partial_y)^{n_3}$. $H^{k,\beta}(T)$ is an inner product space with

$$(u,v)_{H^{k,\beta}(T)} = \sum_{l=0}^{k} \sum_{n_1+n_2+n_3=l} \frac{l!}{n_1!n_2!n_3!} \int_T |D^n u||D^n v| W_{\beta+n^*}(x,y)dxdy. \tag{3.10}$$

Let $\mathcal{B}^{s,\beta}_{2,q}(T)$ be the interpolation spaces defined by the K-method

$$\Big(H^{\ell,\beta}(T), H^{k,\beta}(T)\Big)_{\theta,q}$$

where $0<\theta<1, 1\le q\le\infty$, $s=(1-\theta)\ell+\theta k$, ℓ and k are integers, $\ell<k$,

$$\|u\|_{\mathcal{B}^{s,\beta}_{2,q}(T)} = \Big(\int_0^\infty t^{-q\theta}\,|K(t,u)|^q \frac{dt}{t}\Big)^{1/q},\ 1\le q<\infty; \tag{3.11}$$

and

$$\|u\|_{\mathcal{B}^{s,\beta}_{2,\infty}(T)} = \sup_{t>0} t^{-\theta}\,K(t,u) \tag{3.12}$$

where

$$K(t,u) = \inf_{u=v+w}\Big(\|v\|_{H^{\ell,\beta}(T)} + t\|w\|_{H^{k,\beta}(T)}\Big).$$

In particular, we are interested in the cases $q=2$ and $q=\infty$. We shall write for $s\ge 0$ and $q=2$

$$H^{s,\beta}(T) = \mathcal{B}^{s,\beta}_{2,2}(T) = \Big(H^{\ell,\beta}(T), H^{k,\beta}(T)\Big)_{\theta,2}$$

with $0<\theta<1$ and $s=(1-\theta)\ell+\theta k$. This space is called the Jacobi-weighted Sobolev space with fractional order if s is not an integer. It has been proved that $\mathcal{B}^{s,\beta}_{2,2}(Q) = H^{m,\beta}(Q)$ if s is an integer m in two dimensions[**9**].

The $H^{1,\beta}$-norm on triangle is defined as:

$$\|u\|^2_{H^{1,\beta}(T)} = \left\|\frac{\partial u}{\partial x}\right\|^2_{L^2_{\beta+l_1}(T)} + \left\|\frac{\partial u}{\partial y}\right\|^2_{L^2_{\beta+l_2}(T)} + \left\|\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\right\|^2_{L^2_{\beta+l_3}(T)} + \|u\|^2_{L^2_\beta(T)},$$

where

$$\begin{aligned}
\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2_{\beta+l_1}(T)} &= \int_T \left|\frac{\partial u}{\partial x}\right|^2 x^{\beta_1+1} y^{\beta_2} (1-x-y)^{\beta_3+1} dxdy,\\
\left\|\frac{\partial u}{\partial y}\right\|^2_{L^2_{\beta+l_2}(T)} &= \int_T \left|\frac{\partial u}{\partial y}\right|^2 x^{\beta_1} y^{\beta_2+1} (1-x-y)^{\beta_3+1} dxdy,\\
\left\|\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\right\|^2_{L^2_{\beta+l_3}(T)} &= \int_T \left|\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right|^2 x^{\beta_1+1} y^{\beta_2+1} (1-x-y)^{\beta_3} dxdy,\\
\|u\|^2_{L^2_\beta(T)} &= \int_T |u|^2 x^{\beta_1} y^{\beta_2} (1-x-y)^{\beta_3} dxdy.
\end{aligned}$$

with $l_1 = (1,0,1)$, $l_2 = (0,1,1)$, $l_3 = (1,1,0)$.

The $\tilde{H}^{1,\beta}(T)$ is defined as a dual to $H^{1,\beta}(T)$ with the norm

$$\|u\|^2_{\tilde{H}^{1,\beta}(T)} = \left\|\frac{\partial u}{\partial x}\right\|^2_{L^2_{\beta-l_1}(T)} + \left\|\frac{\partial u}{\partial y}\right\|^2_{L^2_{\beta-l_2}(T)} + \left\|\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\right\|^2_{L^2_{\beta-l_3}(T)} + \|u\|^2_{L^2_\beta(T)},$$

where

$$\begin{aligned}
\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2_{\beta-l_1}(T)} &= \int_T \left|\frac{\partial u}{\partial x}\right|^2 x^{\beta_1-1} y^{\beta_2} (1-x-y)^{\beta_3-1} dxdy,\\
\left\|\frac{\partial u}{\partial y}\right\|^2_{L^2_{\beta-l_2}(T)} &= \int_T \left|\frac{\partial u}{\partial y}\right|^2 x^{\beta_1} y^{\beta_2-1} (1-x-y)^{\beta_3-1} dxdy,\\
\left\|\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\right\|^2_{L^2_{\beta-l_3}(T)} &= \int_T \left|\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right|^2 x^{\beta_1-1} y^{\beta_2-1} (1-x-y)^{\beta_3} dxdy,
\end{aligned}$$

with $l_1 = (1, 0, 1)$, $l_2 = (0, 1, 1)$, $l_3 = (1, 1, 0)$.

We have a very good results for pure quadrilateral elements in [**25**]. In this thesis, we discuss the pure triangle cases. In Chapter 3 and Chapter 4, we only consider the pure triangle condition, which means $K_i = T_i$ and $\mathcal{T} = \{K_i, 1 \leq i \leq M\} = \{T_i, 1 \leq i \leq M\}$.

We next to define Jacobi-weighted spaces $H^{k,\beta}(T_i)$ and $\tilde{H}^{k,\beta}(T_i)$ over the elements T_i. Let F_i be the mapping of T onto element T_i, and $\tilde{u}_i = u \circ F_i$. We define

$$\|u\|_{\tilde{H}^{k,\beta}(T_i)} = \|\tilde{u}\|_{\tilde{H}^{k,\beta}(T)}, \qquad \|u\|_{H^{k,\beta}(T_i)} = \|\tilde{u}\|_{H^{k,\beta}(T)}.$$

The Jacobi-weighted spaces $H^{k,\beta}(\mathcal{T})$ and $\tilde{H}^{k,\beta}(\mathcal{T})$ are defined as

$$H^{k,\beta}(\mathcal{T}) = \bigcap_{K} H^{k,\beta}(T_i), \qquad \tilde{H}^{k,\beta}(\mathcal{T}) = \bigcap_{K} \tilde{H}^{k,\beta}(T_i)$$

furnished with "broken" norms

$$\|u\|_{H^{k,\beta}(\mathcal{T})} = \sum_{i=1}^{M} \|u\|_{H^{k,\beta}(T_i)}, \qquad \|u\|_{\tilde{H}^{k,\beta}(\mathcal{T})} = \sum_{i=1}^{M} \|u\|_{\tilde{H}^{k,\beta}(T_i)}.$$

In the framework of Jacobi-weighted spaces $H^{1,-\beta}(\mathcal{T})$ and $\tilde{H}^{1,\beta}(\mathcal{T})$ with $0 \leq \beta_1 = \beta_2 = \beta_3 < 1$ we re-formulate the elliptic problem (2.1) and its variational equation. Suppose that $f \in L^2_\beta(\mathcal{T})$. Then we seek $u \in \tilde{H}^{1,\beta}_0(\mathcal{T})$ such that

$$B(u, v) = F(v) \quad \forall v \in H^{1,-\beta}_0(\mathcal{T})$$

Here $B(u,v)$ is a bilinear forms on $\tilde{H}_0^{1,\beta}(\mathcal{T})\times H_0^{1,-\beta}(\mathcal{T})$, $F(v)$ is a linear functional on $L^2_{-\beta}(\mathcal{T})$, and $\tilde{H}_0^{1,\beta}(\mathcal{T})$ and $H_0^{1,\beta}(\mathcal{T})$ are subspaces of $\tilde{H}^{1,\beta}(\mathcal{T})$ and $H^{1,-\beta}(\mathcal{T})$ containing functions vanishing on Γ_D, respectively. We introduce a new norm for the error e denoted by $|||e|||_K$ and $|||e|||$

$$|||e|||_{T_i} = \sup_{\|v\|_{H^{1,-\beta}(T_i)}=1} |B(e,v)_{T_i}| \leq \|e\|_{\tilde{H}^{1,\beta}(T_i)} \tag{3.13}$$

and

$$|||e||| = \sup_{\|v\|_{H^{1,-\beta}(\mathcal{T})}=1} |B(e,v)| \leq \|e\|_{\tilde{H}^{1,\beta}(\mathcal{T})}. \tag{3.14}$$

where

$$B(e,v)_{T_i} = \int_{T_i} \Big(\nabla e\cdot\nabla v + ev\Big)dxdy.$$

3.3. Orthogonal polynomials in Jacobi-weighted spaces on Triangle

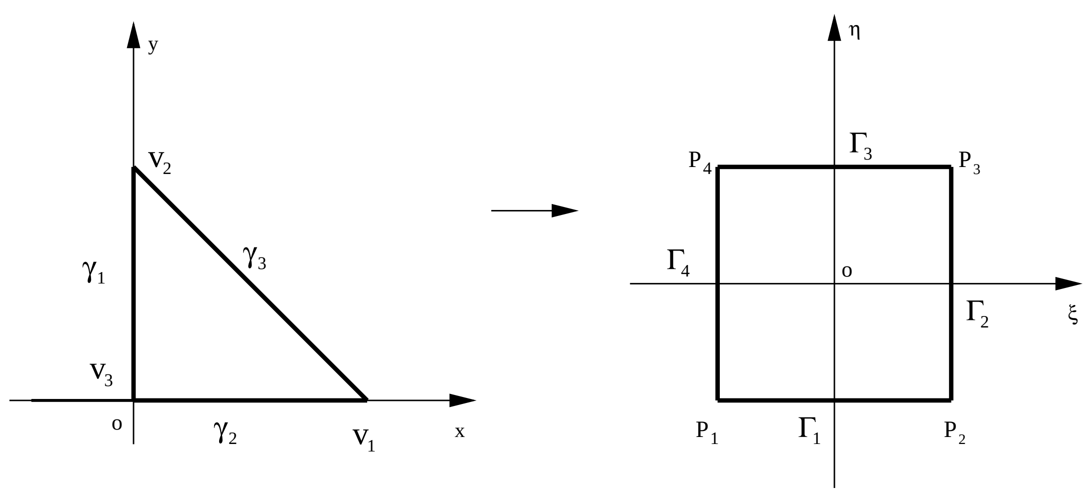

Fig. 3.1 Transform from standard Triangle to Square

Let F_1 be the collapse mapping of T onto standard square $S = (-1,1)^2$:

$$F_1 : \begin{cases} \xi = \frac{x+2y-1}{1-x} \\ \eta = 1-2x \end{cases}, \qquad F_1^{-1} : \begin{cases} x = \frac{1}{2}(1-\eta) \\ y = \frac{1}{4}(1+\xi)(1+\eta) \end{cases}$$

which maps γ_1 onto Γ_3, γ_2 onto Γ_4, γ_3 onto Γ_2, and V_1 to Γ_1.

We define the orthogonal polynomial $\tilde{K}^{\beta}_{ij}(x,y)$ by the mapping F_1

$$\begin{aligned} \widetilde{K}^{\beta}_{i,j}(x,y) &= J_i^{\beta_3,\beta_2}(\xi)\Big(\frac{1+\eta}{2}\Big)^i J_j^{\beta_1,2i+\beta_2+\beta_3+1}(\eta) \\ &= (1-x)^i J_i^{\beta_3,\beta_2}\Big(\frac{x+2y-1}{1-x}\Big) J_j^{\beta_1,2i+\beta_2+\beta_3+1}(1-2x). \end{aligned}$$

Then we have orthogonality of $\tilde{K}^{\beta}_{ij}(x,y)$ on T,

$$\int_T \widetilde{K}^{\beta}_{i,j}(x,y)\widetilde{K}^{\beta}_{i',j'}(x,y)W_\beta(x,y)dxdy = \frac{1}{4}\int_Q \Big(\frac{1-\xi}{2}\Big)^{\beta_3}\Big(\frac{1+\xi}{2}\Big)^{\beta_2}\Big(\frac{1-\eta}{2}\Big)^{\beta_1}$$

$$\times\Big(\frac{1+\eta}{2}\Big)^{i+i'+\beta_2+\beta_3+1} J_i^{\beta_3,\beta_2}(\xi)J_{i'}^{\beta_3,\beta_2}(\xi)J_j^{\beta_1,2i+\beta_2+\beta_3+1}(\eta)J_{j'}^{\beta_1,2i'+\beta_2+\beta_3+1}(\eta)d\xi d\eta$$

$$= \begin{cases} 2^{-(2i+\lceil\beta\rfloor+\beta_2+\beta_3+3)}\gamma_i^{\beta_3,\beta_2}\gamma_j^{\beta_1,2i+\beta_2+\beta_3+1} & (\text{ if } i=i' \text{ and } j=j') \\ 0 \quad \text{otherwise} \end{cases}.$$

where $\lceil\beta\rfloor = \beta_1+\beta_2+\beta_3$.

Similarly let F_2 be the collapse mapping of T onto standard square $S = (-1,1)^2$:

$$F_2 : \begin{cases} \xi = \frac{1-2x-y}{1-y} \\ \eta = 1-2y \end{cases} \qquad F_2^{-1} : \begin{cases} x = \frac{1}{4}(1-\xi)(1+\eta) \\ y = \frac{1}{2}(1-\eta). \end{cases}$$

which maps γ_1 onto Γ_2, γ_2 onto Γ_3, γ_3 onto Γ_4 and V_2 to Γ_1. We define another orthogonal polynomial $\bar{K}^{\beta}_{ij}(x,y)$ by the mapping F_2

$$\begin{aligned}\overline{K}^{\beta}_{i,j}(x,y) &= J_i^{\beta_1,\beta_3}(\xi)\Big(\frac{1+\eta}{2}\Big)^i J_j^{\beta_2,2i+\beta_1+\beta_3+1}(\eta)\\ &= (1-y)^i J_i^{\beta_1,\beta_3}\Big(\frac{1-2x-y}{1-y}\Big)J_j^{\beta_2,2i+\beta_1+\beta_3+1}(1-2y),\end{aligned}$$

Then we have orthogonality of $\bar{K}^{\beta}_{ij}(x,y)$ on T,

$$\int_T \overline{K}^{\beta}_{i,j}(x,y)\overline{K}^{\beta}_{i',j'}(x,y)W_\beta(x,y)dxdy = \frac{1}{4}\int_Q \Big(\frac{1-\xi}{2}\Big)^{\beta_1}\Big(\frac{1+\xi}{2}\Big)^{\beta_3}\Big(\frac{1-\eta}{2}\Big)^{\beta_2}$$
$$\times\Big(\frac{1+\eta}{2}\Big)^{i+i'+\beta_1+\beta_3+1} J_i^{\beta_1,\beta_3}(\xi)J_{i'}^{\beta_1,\beta_3}(\xi)J_j^{\beta_2,2i+\beta_1+\beta_3+1}(\eta)J_{j'}^{\beta_2,2i'+\beta_1+\beta_3+1}(\eta)d\xi d\eta$$
$$= \begin{cases} 2^{-(2i+\lceil\beta\rfloor+\beta_1+\beta_3+3)}\gamma_i^{\beta_1,\beta_3}\gamma_j^{\beta_2,2i+\beta_1+\beta_3+1} & (\text{ if } i=i' \text{ and } j=j')\\ 0 \qquad \text{otherwise} & \end{cases}.$$

F_3 be the collapse mapping of T onto standard square $S=(-1,1)^2$:

$$F_3:\begin{cases}\xi=\frac{x-y}{x+y}\\ \eta=2x+2y-1\end{cases}\qquad F_3^{-1}:\begin{cases}x=\frac{1}{4}(1+\xi)(1+\eta)\\ y=\frac{1}{4}(1-\xi)(1+\eta)\end{cases}$$

which maps γ_1 onto Γ_4, γ_2 onto Γ_2, γ_3 onto Γ_3 and V_3 to Γ_1.

We define the third orthogonal polynomial $K^{\beta}_{i,j}(x,y)$ by the mapping F_3

$$\begin{aligned} K^{\beta}_{i,j}(x,y) &= J^{\beta_2,\beta_1}_i(\xi)\Big(\frac{1+\eta}{2}\Big)^i J^{\beta_3,2i+\beta_1+\beta_2+1}_j(\eta) \\ &= (y+x)^i J^{\beta_2,\beta_1}_i\Big(\frac{x-y}{y+x}\Big) J^{\beta_3,2i+\beta_1+\beta_2+1}_j(2x+2y-1) \end{aligned}$$

and

$$\int_T K^{\beta}_{i,j}(x,y)K^{\beta}_{i',j'}(x,y)W_\beta(x,y)dxdy = \frac{1}{4}\int_Q \Big(\frac{1-\xi}{2}\Big)^{\beta_2}\Big(\frac{1+\xi}{2}\Big)^{\beta_1}\Big(\frac{1-\eta}{2}\Big)^{\beta_3}$$
$$\times\Big(\frac{1+\eta}{2}\Big)^{i+i'+\beta_1+\beta_2+1} J^{\beta_2,\beta_1}_i(\xi)J^{\beta_2,\beta_1}_{i'}(\xi)J^{\beta_3,2i+\beta_1+\beta_2+1}_j(\eta)J^{2i'+\beta_1+\beta_2+1,\beta_3}_{j'}(\eta)d\xi d\eta$$
$$= \begin{cases} 2^{-(2i+\lceil\beta\rfloor+\beta_1+\beta_2+3)}\gamma^{\beta_2,\beta_1}_i\gamma^{\beta_3,2i+\beta_1+\beta_2+1}_j & (\text{ if } i=i' \text{ and } j=j') \\ 0 \quad \text{otherwise} & \end{cases}.$$

For $u\in L^2_\beta(T)$ we have three Jacobi expansions of u on T, and

$$u = \sum_{i,j\geq 0} a_{ij}K^{\beta}_{i,j}(x,y) = \sum_{i,j\geq 0} c_{ij}\overline{K}^{\beta}_{i,j}(x,y) = \sum_{i,j\geq 0} d_{ij}\widetilde{K}^{\beta}_{i,j}(x,y),$$

where

$$\begin{cases} a_{ij} = \frac{2^{(2i+\lceil\beta\rfloor+\beta_1+\beta_2+3)}}{\gamma^{\beta_2,\beta_1}_i\gamma^{\beta_3,2i+\beta_1+\beta_2+1}_j}\int_T u(x,y)K_{i,j}(x,y)W_\beta(x,y)dxdy, \\ c_{ij} = \frac{2^{(2i+\lceil\beta\rfloor+\beta_1+\beta_3+3)}}{\gamma^{\beta_1,\beta_3}_i\gamma^{\beta_2,2i+\beta_1+\beta_3+1}_j}\int_T u(x,y)\overline{K}_{i,j}(x,y)W_\beta(x,y)dxdy, \\ d_{ij} = \frac{2^{(2i+\lceil\beta\rfloor+\beta_2+\beta_3+3)}}{\gamma^{\beta_3,\beta_2}_i\gamma^{\beta_1,2i+\beta_2+\beta_3+1}_j}\int_T u(x,y)\widetilde{K}_{i,j}(x,y)W_\beta(x,y)dxdy. \end{cases} \tag{3.15}$$

Lemma 3.1. *Let* $u \in L^2_\beta(T)$, *then* $\sum_{i,j\geq 0, i+j\leq p} a_{ij}K^\beta_{i,j}(x,y)$, $\sum_{i,j\geq 0, i+j\leq p} c_{ij}\bar{K}^\beta_{i,j}(x,y)$, $\sum_{i,j\geq 0, i+j\leq p} d_{ij}\tilde{K}^\beta_{i,j}(x,y)$ *with* a_{ij}, c_{ij}, d_{ij} *given in (3.15) are projection of* u *in* $\mathcal{P}_p(T)$ *denoted by* $\Pi^\beta_{p,T}u$ *and*

$$\|u - \Pi^\beta_{p,T}u\|_{L^2_\beta(T)} = \min_{\varphi - \mathcal{P}_p(T)} \|u - \varphi\|_{L^2_\beta(T)} \tag{3.16}$$

Proof. First we prove $\sum_{i,j\geq 0, i+j\leq p} a_{ij}K^\beta_{i,j}(x,y)$, consider polynomial $K^\beta_{k,l}(x,y)$ with $k+l \leq p$, then

$$\begin{aligned}\langle u - \sum_{i,j\geq 0, i+j\leq p} a_{ij}K^\beta_{i,j}, K^\beta_{k,l}\rangle_{L^2_\beta(T)} &= \langle u, K^\beta_{k,l}\rangle_{L^2_\beta(T)} - \langle \sum_{i,j\geq 0, i+j\leq p} a_{ij}K^\beta_{i,j}, K^\beta_{k,l}\rangle_{L^2_\beta(T)} \\ &= 0. \end{aligned} \tag{3.17}$$

$\sum_{i,j\geq 0, i+j\leq p} a_{ij}K^\beta_{i,j}$ is the projection of u on $\mathcal{P}_p(T)$. Similarly, we can prove that $\sum_{i,j\geq 0, i+j\leq p} c_{ij}\bar{K}^\beta_{i,j}(x,y)$, $\sum_{i,j\geq 0, i+j\leq p} d_{ij}\tilde{K}^\beta_{i,j}(x,y)$ are the projection of u on $\mathcal{P}_p(T)$, which can be written as

$$\Pi^\beta_{p,T}u = \sum_{i,j\geq 0, i+j\leq p} a_{ij}K^\beta_{i,j}(x,y) = \sum_{i,j\geq 0, i+j\leq p} c_{ij}\bar{K}^\beta_{i,j}(x,y) = \sum_{i,j\geq 0, i+j\leq p} d_{ij}\tilde{K}^\beta_{i,j}(x,y).$$

From (3.17), we can get $\Pi^{\beta}_{p,T}u - u$ is orthogonal to any $\varphi_p \in \mathcal{P}_p(T)$. Then

$$\begin{aligned}
\|\varphi_p - u\|^2_{L^2_\beta(T)} &= \|(\varphi_p - \Pi^{-\beta}_{p,T}u) + (\Pi^{-\beta}_{p,T}u - u)\|^2_{L^2_\beta(T)} \\
&= \|\varphi_p - \Pi^{-\beta}_{p,T}u\|^2_{L^2_\beta(T)} + 2\langle \varphi_p - \Pi^{-\beta}_{p,T}u, \Pi^{-\beta}_{p,T}u - u\rangle_{L^2_\beta(T)} + \|\Pi^{-\beta}_{p,T}u - u\|^2_{L^2_\beta(T)} \\
&\geq \|\Pi^{-\beta}_{p,T}u - u\|^2_{L^2_\beta(T)},
\end{aligned}$$

which implies (3.16). □

Theorem 3.2. *For $u \in L^2_\beta(T)$, then holds in $L^2_\beta(T)$,*

$$\lim_{p\to\infty} \|u - \Pi^{-\beta}_{p,T}u\|_{L^2_\beta(T)} = 0. \tag{3.18}$$

Proof. We first assume that $u \in C^0(\bar{T})$, then for any $\varepsilon > 0$, there exist N such that a polynomial $\varphi_p \in \mathcal{P}_p(T)$, $p > N$ satisfy

$$\|u - \varphi_p\|_{C^0(\bar{T})} \leq \varepsilon.$$

Then we have

$$\begin{aligned}
\|u - \Pi^{-\beta}_{p,T}u\|^2_{L^2_\beta(T)} &\leq \|u - \varphi_p\|^2_{L^2_\beta(T)} \\
&= \int_T (u(x,y) - \varphi_p)^2 W_\beta(x,y)dxdy. \\
&\leq \|u - \varphi_p\|^2_{C^0(\bar{T})} \int_T W_\beta(x,y)dxdy \leq M\varepsilon,
\end{aligned}$$

which implies

$$\lim_{p\to\infty} \|u - \Pi_{p,T}^{-\beta} u\|_{L^2_\beta(T)} = 0. \tag{3.19}$$

Since $C^0(T)$ is dense in $L^2_\beta(T)$, then for $u \in L^2_\beta(T)$ there exists a $v \in C^0(T)$ such that

$$\|u - v\|_{L^2\beta(T)} \leq \varepsilon.$$

Then for $p \geq N$

$$\begin{aligned} \|u - \Pi_{p,T}^{\beta} u\| &\leq \|u - v\|_{L^2_\beta(T)} + \|v - \Pi_{p,T}^{\beta} v\|_{L^2_\beta(T)} + \|\Pi_{p,T}^{\beta} u - \Pi_{p,T}^{\beta} v\|_{L^2_\beta(T)} \\ &\leq 2\|u - v\|_{L^2_\beta(T)} + \|v - \Pi_{p,T}^{\beta} v\|_{L^2_\beta(T)} \leq 3\varepsilon, \end{aligned}$$

which implies (3.18). □

Lemma 3.3. *(Lemma 3.5 of* [**32**] *) Let $\sigma \in N_0$ and $\beta \in Z^3$. If $u \in H^{\sigma,\beta}(T)$, then*

$$\begin{aligned} |D^n u|^2_{H^{s,\beta+n^*}(T)} &= \sum_{m\geq 0} \mu^s_{m-(n_1+n_2+n_3),\beta+n} \|D^n u_m\|^2_{L^2_{\beta+n^*}(T)}, \\ \textit{for } 0 \leq s &\leq \sigma - (n_1+n_2+n_3), n \in Z^3. \end{aligned}$$

In particular,

$$|u|^2_{H^{s,\beta}(T)} = \sum_{m\geq 0} \mu^s_{m,\beta} \|u_m\|^2_{L^2_\beta(T)}, \qquad 0 \leq s \leq \sigma \tag{3.20}$$

where

$$\mu^s_{m,\beta} = \frac{\Gamma(m+1)}{\Gamma(m-s+1)} \frac{\Gamma(m+s+(\beta_1+\beta_2+\beta_3)+2)}{\Gamma(m+(\beta_1+\beta_2+\beta_3)+2)}.$$

We can easily get

$$\|u\|^2_{H^{k,\beta}(T)} = \sum_{l=0}^{k} |u|^2_{H^{l,\beta}(T)} = \sum_{l=0}^{k} \sum_{m\geq 0} \mu^l_{m,\beta} \|u_m\|^2_{L^2_\beta(T)} \tag{3.21}$$

where

$$\|u_m\|^2_{L^2_\beta(T)} = \sum_{i+j=m} |a_{ij}|^2 \|K_{i,j}\|^2_{L^2_\beta(T)}.$$

Therefore

$$\|u\|^2_{H^{k,\beta}(T)} \quad = \quad \sum_{l=0}^{k} \sum_{i,j\geq 0} \mu^l_{i+j,\beta} |a_{ij}|^2 \|K^\beta_{i,j}(x,y)\|^2_{L^2_\beta(T)}.$$

It will lead to

$$\begin{aligned} &\|u\|^2_{H^{k,\beta}(T)} \\ &= \sum_{l=0}^{k} \sum_{i,j\geq 0}^{\infty} 2^{-(2i+\lceil\beta\rfloor+\beta_1+\beta_2+3)} \frac{\Gamma(i+j+1)\Gamma(i+j+l+\lceil\beta\rfloor+2)}{\Gamma(i+j-l+1)\Gamma(i+j+\lceil\beta\rfloor+2)} |a_{ij}|^2 \gamma_i^{\beta_2,\beta_1} \gamma_j^{\beta_3,2i+\beta_1+\beta_2+1}. \end{aligned} \tag{3.22}$$

Similarly we can get

$$\|u\|^2_{H^{k,\beta}(T)} = \sum_{l=0}^{k}\sum_{i,j\geq 0}^{\infty} 2^{-(2i+\lceil\beta\rfloor+\beta_1+\beta_3+3)}\frac{\Gamma(i+j+1)\Gamma(i+j+l+\lceil\beta\rfloor+2)}{\Gamma(i+j-l+1)\Gamma(i+j+\lceil\beta\rfloor+2)}|c_{ij}|^2\gamma_i^{\beta_1,\beta_3}\gamma_j^{\beta_2,2i+\beta_1+\beta_3+1}, \tag{3.23}$$

and

$$\|u\|^2_{H^{k,\beta}(T)} = \sum_{l=0}^{k}\sum_{i,j\geq 0}^{\infty} 2^{-(2i+\lceil\beta\rfloor+\beta_2+\beta_3+3)}\frac{\Gamma(i+j+1)\Gamma(i+j+l+\lceil\beta\rfloor+2)}{\Gamma(i+j-l+1)\Gamma(i+j+\lceil\beta\rfloor+2)}|d_{ij}|^2\gamma_i^{\beta_3,\beta_2}\gamma_j^{\beta_1,2i+\beta_2+\beta_3+1}. \tag{3.24}$$

By Stirling formula, we have

$$\begin{aligned}\frac{\Gamma(i+j+1)\Gamma(i+j+l+\lceil\beta\rfloor+2)}{\Gamma(i+j-l+1)\Gamma(i+j+\lceil\beta\rfloor+2)} &\simeq e^{-2l}\frac{(i+j)^{i+j+1/2}(i+j+l+\lceil\beta\rfloor+1)^{i+j+l+\lceil\beta\rfloor+3/2}}{(i+j+\lceil\beta\rfloor+1)^{i+j+\lceil\beta\rfloor+3/2}(i+j-l)^{i+j-l+1/2}} \\ &\simeq Ce^{-2l}(i+j+1)^{2l}.\end{aligned}$$

Using the asymptotic of $\gamma_n^{\alpha,\beta}$ given in (3.4), we introduced an equivalent norm for $H^{k,\beta}(T)$,

$$\|u\|^2_{H^{k,\beta}(T)} \simeq C\sum_{i,j\geq 0}^{\infty} e^{-2k}\frac{(i+j+1)^{2k-1}}{i}|a_{ij}|^2 = |\|u\||^2_{H^{k,\beta}(T)}.$$

Similarly we can get

$$\|u\|^2_{H^{k,\beta}(T)} \simeq C\sum_{i,j\geq 0}^{\infty} e^{-2k}\frac{(i+j+1)^{2k-1}}{i}|c_{ij}|^2 = |\|u\||^2_{H^{k,\beta}(T)},$$

and

$$\|u\|^2_{H^{k,\beta}(T)} \simeq C\sum_{i,j\geq 0}^{\infty} e^{-2k}\frac{(i+j+1)^{2k-1}}{i}|d_{ij}|^2 = |\|u\||^2_{H^{k,\beta}(T)}.$$

Theorem 3.4. *(Theorem 3.3 of* **[32]** *) Let $k>0$ and $\beta=(\beta_1,\beta_2,\beta_3)$. If $u\in H^{k,\beta}(T)$, and let $u_p=\Pi^{\beta}_{p,T}u$. Then there hold for $0\leq l\leq k$, $p\geq 0$*

$$|u-u_p|_{H^{l,\beta}(T)} \leq (p+1)^{-(k-l)}|u|_{H^{k,\beta}(T)}. \tag{3.25}$$

Theorem 3.5. *(Theorem 3.5 of* **[17]** *) Let β be a real number such that $-1<\beta$. Then the following inverse inequality holds for any polynomial $\phi_p(x)$ in $\mathcal{P}_p(I)$, $I=(-1,1)$:*

$$\int_{-1}^{1}\Big(\frac{d}{dx}\phi_p\Big)^2(x)(1-x^2)^{\beta+1}dx \leq C(p+1)^2\int_{-1}^{1}\phi_p^2(x)(1-x^2)^{\beta}dx. \tag{3.26}$$

We next recall the quadrature rule of Gauss-Jacobi-Lobbato type over the interval $I=(-1,1)$, which will be used to determine the $L^2_\beta(I)$-norm of the Lagrange-Jacobi interpolation polynomials.

Let $\xi^{\beta}_{m,j}$ be zeros of $(J^{\beta}_m(\xi))'=\frac{m+1+2\beta}{2}J^{\beta+1}_{m-1}(\xi)$, $j=1,2,\cdots,m-1$, and let $\xi^{\beta}_{m,0}=-1$, $\xi^{\beta}_{m,m}=1$. These $(m+1)$ points are called Gauss-Jacobi-Lobbato (GJL) points. Let $l^{\beta}_{m,j}(\xi)$

be the Lagrange-Jacobi interpolation polynomials of degree m, $0 \le j \le m$ such that

$$l^{\beta}_{m,j}(\xi^{\beta}_{m,i}) = \delta_{i,j} \qquad 0 \le i, j \le m.$$

We have the Jacobi-weighted L^2-norm of $l^{\beta}_{m,0}(\xi)$ and $l^{\beta}_{m,m}(\xi)$.

Theorem 3.6. *(Theorem 3.2 of* [**25**] *) For $j = 0$, m, $\beta > -1$,*

$$\|l^{\beta}_{m,j}\|_{L^2_{\beta}(I)} \le C\Gamma^{1/2}(1+\beta)(m+1)^{-(1+\beta)} \tag{3.27}$$

with the constant C independent of m and β.

CHAPTER 4

Quasi Jacobi Projection and Its Approximation Properties

4.1. Quasi Jacobi Projection on Triangle and Its Approximation Properties

Let $\Pi_{p,T}^{-\beta} v$ be the Jacobi projection of $v \in H^{1,-\beta}(T)$ on $P_p(T)$ with $\beta_i < 1$. Then $H^{1,-\beta}(\mathcal{T})$ is a Jacobi-weighted space furnished with norm

$$\|v\|_{H^{1,-\beta}(\mathcal{T})} = \sum_{i=1}^{M} \|v\|_{H^{1,-\beta}(T_i)}, \qquad \forall v \in H^{1,-\beta}(\mathcal{T}). \tag{4.1}$$

Then $\Pi_{p,\mathcal{T}}^{-\beta} v|_{T_i} = \Pi_{p,T_i}^{-\beta} v$ is a piecewise polynomial in $\mathcal{T}$ for any v in $H^{1,-\beta}(\mathcal{T})$.

Lemma 4.1. *Let $\tilde{l}_{p,j}^{\beta}(\xi) = l_{p,j}^{\beta}(\frac{x+1}{2})$ for $\xi \in I' = (0,1)$, where $l_{p,j}^{\beta}(x)$ be the Lagrange-Jacobi interpolation polynomials of degree p, $j = 0$ or p on $[-1,1]$ with $\beta > -1$,*

$$\|\tilde{l}_{p,j}^{\beta}\|_{L^2_{\beta}(I')} \leq C\Gamma^{1/2}(1+\beta)(p+1)^{-(1+\beta)} \tag{4.2}$$

and

$$\|\tilde{l}'^{\beta}_{p,j}\|_{L^2_{\beta}(I')} \leq C\Gamma^{1/2}(1+\beta)(p+1)^{-\beta} \tag{4.3}$$

with the constant C independent of p and β.

PROOF. Due to Theorem 3.6, we have by the mapping $\xi = \frac{x+1}{2}$

$$\|\tilde{l}^{\beta}_{p,j}\|_{L^2_\beta(I')} = \frac{1}{\sqrt{2}}\|l^{\beta}_{p,j}\|_{L^2_\beta(I)} \leq C\Gamma^{1/2}(1+\beta)(p+1)^{-(1+\beta)}. \tag{4.4}$$

By Theorem 3.5, we have

$$\|\tilde{l}'^{\beta}_{p,j}\|_{L^2_\beta(I')} \leq Cp\|\tilde{l}^{\beta}_{p,j}\|_{L^2_\beta(I')} \leq C\Gamma^{1/2}(1+\beta)(p+1)^{-\beta}. \tag{4.5}$$

□

For any vertex V of $\mathcal{T}$, we define an average as

$$\overline{v}_p(V) = \frac{\sum_{T\in Q_V} v_T(V)}{n_V} \tag{4.6}$$

where Q_V is a patch center at the vertex V, and n_V denotes the number of elements in the patch Q_V. We modify v_T as

$$v_T^{(1)} = v_T + w_T^{(1)}$$

with

$$\begin{aligned} w_T^{(1)} &= (\overline{v}_p(V_1) - v_T(V_1))(x+y)^{[\frac{p}{3}]}\tilde{l}^{-\beta}_{[\frac{p}{3}],[\frac{p}{3}]}\Big(\frac{x}{x+y}\Big)\tilde{l}^{-\beta}_{[\frac{p}{3}],[\frac{p}{3}]}(x+y) \\ &+ (\overline{v}_p(V_2) - v_T(V_2))(1-x)^{[\frac{p}{3}]}\tilde{l}^{-\beta}_{[\frac{p}{3}],[\frac{p}{3}]}\Big(\frac{y}{1-x}\Big)\tilde{l}^{-\beta}_{[\frac{p}{3}],0}(x) \\ &+ (\overline{v}_p(V_3) - v_T(V_3))(1-y)^{[\frac{p}{3}]}\tilde{l}^{-\beta}_{[\frac{p}{3}],0}\Big(\frac{x}{1-y}\Big)\tilde{l}^{-\beta}_{[\frac{p}{3}],0}(y) \end{aligned}$$

where $\tilde{l}_{p,j}^{-\beta}(\xi)$, $j=0,\ p$ is a Bouzitat-type polynomial (Lagrange-Jacobi interpolation polynomials) of degree p such that

$$\tilde{l}_{p,j}^{-\beta}(0)=1 \quad (\text{or } 0), \quad \tilde{l}_{p,j}^{-\beta}(1)=0 \quad (\text{or } 1), \quad j=0 \quad (\text{or } p).$$

Then $v_T^{(1)}(V_i)=\overline{v}_{[\frac{p}{2}]}(V_i)$, $1\le i\le 3$.

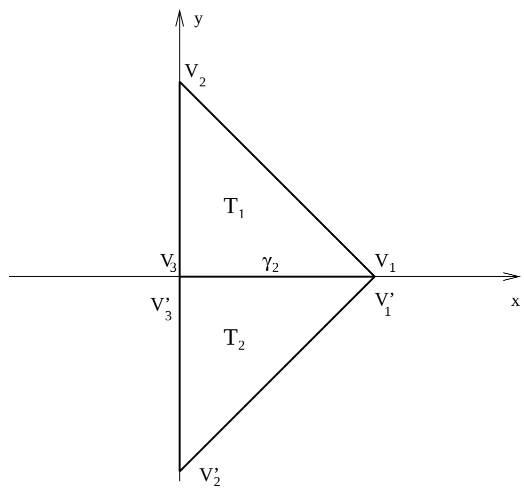

Fig. 4.1 Two neighborhood triangle mesh

We further modify $v_p^{(1)}$ on each internal edge γ which is shared by a pair of elements T_1 and T_2. We may assume $\gamma_2=\{(x,0)|0\le x\le 1\}$, $T_1=\{(x,y)|0\le x,y\le 1, 0\le x+y\le 1\}$, $T_2=\{(x,y)|0\le x\le 1,-1\le y\le 0,-1\le y-x\le 0\}$. Let

$$\phi_\gamma(x)=v_{T_2}^{(1)}(x,0)-v_{T_1}^{(1)}(x,0),$$

note that $\phi_\gamma(x)|_{\gamma_3}=v_{T_2}^{(1)}(0,0)-v_{T_1}^{(1)}(0,0)=0$. And $\phi_\gamma(1)=v_{T_2}^{(1)}(1,0)-v_{T_1}^{(1)}(1,0)=0$, let $\widetilde{\phi}_\gamma(x)=\frac{\phi_\gamma(x)}{1-x}$, $\widetilde{\phi}_\gamma(x)$ is a polynomial.

We extend this function in the patch of $\gamma_2 : \overline{T}_1 \cup \overline{T}_2$ by

$$w_\gamma = \begin{cases} \frac{1}{2}\widetilde{\phi}_\gamma(x)(1-x)\tilde{l}_{1,1}^{-\beta}(\frac{1-x-y}{1-x})\tilde{l}_{[\frac{p}{3}]-1,[\frac{p}{3}]-1}^{-\beta}(1-y) & = \frac{1}{2}\phi_\gamma(x)\tilde{l}_{1,1}^{-\beta}(\frac{1-x-y}{1-x})\tilde{l}_{[\frac{p}{3}]-1,[\frac{p}{3}]-1}^{-\beta}(1-y) \\ & = w_{\gamma,1} \\ -\frac{1}{2}\widetilde{\phi}_\gamma(x)(1-x)\tilde{l}_{1,1}^{-\beta}(\frac{1-x+y}{1-x})\tilde{l}_{[\frac{p}{3}]-1,[\frac{p}{3}]-1}^{-\beta}(y+1) & = -\frac{1}{2}\phi_\gamma(x)\tilde{l}_{1,1}^{-\beta}(\frac{1-x+y}{1-x})\tilde{l}_{[\frac{p}{3}]-1,[\frac{p}{3}]-1}^{-\beta}(y+1) \\ & = w_{\gamma,2}. \end{cases}$$

Note that w_γ vanishes on the boundary of Q_γ, and can be extended to whole domain Ω by a zero extension outside of Q_γ. Let

$$w_T^{(2)} = \sum_{\gamma \in \partial T \bigcap \partial \mathcal{T}} w_\gamma, \quad w^{(2)} = \sum_{\gamma \in \partial \mathcal{T}} w_\gamma,$$

and let $v_p^{(2)}$ be a piecewise polynomial on Ω

$$v_p^{(2)} = v_p^{(1)} + w^{(2)}.$$

Then $\Pi_{\mathcal{T}}^{-\beta} v = v_p^{(2)} \in S^{p,1}(\Omega, \mathcal{T})$, and

$$\Pi_{\mathcal{T}}^{-\beta} v = v_p + w^{(1)} + w^{(2)}. \tag{4.7}$$

We have completed the construction of the quasi Jacobi projection which has desired properties which are essential in the prof of the upper bound of the error. To prove the theorems, we need several lemmas.

Lemma 4.2. *(Lemma 4.1 of* [**25**] *) Let $\varphi_p(\xi)$ be a polynomial of degree $p > 0$ on $I = (-1, 1)$. Then for $s > 1 + \beta$ and $\beta \geq -\frac{1}{2}$,*

$$\|\varphi_p\|_{C^0(\overline{I})} \leq \frac{C}{(s - \beta - 1)^{1/2}} \|\varphi_p\|_{H^{s,\beta}(I)} \tag{4.8}$$

and

$$\|\varphi_p\|_{C^0(\overline{I})} \leq C \log^{\frac{1}{2}}(p + 1) \|\varphi_p\|_{H^{1+\beta,\beta}(I)} \tag{4.9}$$

with C independent of p.

For $s > \frac{1}{2}$ and $-1 < \beta \leq -\frac{1}{2}$

$$\|\varphi_p\|_{C^0(\overline{I})} \leq \frac{C}{(s - \frac{1}{2})^{1/2}} \|\varphi_p\|_{H^{s,\beta}(I)} \tag{4.10}$$

with C independent of p and β.

Furthermore, there hold for $\beta > -1$ and $s > 1 + \beta$

$$|\varphi_p(\pm 1)| \leq \frac{C}{\Gamma(1 + \beta)(s - \beta - 1)^{\frac{1}{2}}} \|\varphi_p\|_{H^{s,\beta}(I)} \tag{4.11}$$

and for $s = 1 + \beta$

$$|\varphi_p(\pm 1)| \leq \frac{C \log^{\frac{1}{2}}(p + 1)}{\Gamma(1 + \beta)} \|\varphi_p\|_{H^{1+\beta,\beta}(I)} \tag{4.12}$$

with C independent of p and β.

Lemma 4.3. *For $\beta > -1$, there holds for $s > 1+\beta$*

$$\|w_T^{(1)}\|_{L^2_\beta(T)} \leq \frac{C3^{2(\beta+1)}\log^{\frac{1}{2}}(p+1)}{(p+1)^{2(\beta+1)}} \sum_{T',T''\in Q_T} \sum_{\gamma=T'\bigcap T''} \|v_{T'} - v_{T''}\|_{H^{s,\beta}(\gamma)}. \tag{4.13}$$

PROOF. By Lemma 4.1, let $p^* = [\frac{p}{3}]$ we can easily get for $\beta_1 = \beta_2 = \beta_3 = \beta > -1$

$$\begin{aligned} & \|(1-y)^{[\frac{p}{3}]}\tilde{l}^\beta_{[\frac{p}{3}],j}(\frac{x}{1-y})\tilde{l}^\beta_{[\frac{p}{3}],j}(y)\|^2_{L^2_\beta(T)} \leq \|\tilde{l}^\beta_{p^*,j}(\frac{x}{1-y})\tilde{l}^\beta_{p^*,j}(y)\|^2_{L^2_\beta(T)} \\ = \; & \int_T \Big(\tilde{l}^\beta_{p^*,j}(\frac{x}{1-y})\tilde{l}^\beta_{p^*,j}(y)\Big)^2 x^\beta y^\beta (1-x-y)^\beta dy dx \\ = \; & C\int_S (\tilde{l}^\beta_{p^*,j}(\xi)\tilde{l}^\beta_{p^*,j}(\eta))^2 \xi^\beta(1-\xi)^\beta \eta^\beta (1-\eta)^{2\beta+1} d\xi d\eta \\ \leq \; & C\int_0^1 (\tilde{l}^\beta_{p^*,j}(\xi))^2 \xi^\beta (1-\xi)^\beta d\xi \cdot \int_0^1 (\tilde{l}^\beta_{p^*,j}(\eta))^2 \eta^\beta (1-\eta)^\beta d\eta \\ \leq \; & C\Gamma^2(1+\beta)(p^*)^{-4(\beta+1)} \leq C\Gamma^2(1+\beta)\Big(\frac{p+1}{3}\Big)^{-4(\beta+1)}. \end{aligned} \tag{4.14}$$

The above argument can leads to an analogies estimate

$$\|(1-x)^{[\frac{p}{3}]}\tilde{l}^\beta_{[\frac{p}{3}],j}(\frac{y}{1-x})\tilde{l}^\beta_{[\frac{p}{3}],j}(x)\|^2_{L^2_\beta(T)} \leq C\Gamma^2(1+\beta)\Big(\frac{p+1}{3}\Big)^{-4(\beta+1)}, \tag{4.15}$$

Similarly we can get

$$
\begin{aligned}
(4.16)\qquad & \|(x+y)^{[\frac{p}{3}]}\tilde{l}^{\beta}_{[\frac{p}{3}],j}(\frac{x}{x+y})\tilde{l}^{\beta}_{[\frac{p}{3}],j}(x+y)\|^2_{L^2_\beta(T)} \leq \|\tilde{l}^{\beta}_{p^*,j}(\frac{x}{x+y})\tilde{l}^{\beta}_{p^*,j}(x+y)\|^2_{L^2_\beta(T)} \\
= & \int_T \Big(\tilde{l}^{\beta}_{[p^*],j}(\frac{x}{x+y})\tilde{l}^{\beta}_{[p^*],j}(x+y)\Big)^2 x^\beta y^\beta (1-x-y)^\beta dydx \\
= & \; C\int_S (\tilde{l}^{\beta}_{p^*,j}(\xi)\tilde{l}^{\beta}_{p^*,j}(\eta))^2 \xi^\beta(1-\xi)^\beta \eta^\beta (1-\eta)^{2\beta+1} d\xi d\eta \\
\leq & \; C\int_0^1 (\tilde{l}^{\beta}_{p^*,j}(\xi))^2 \xi^\beta (1-\xi)^\beta d\xi \cdot \int_0^1 (\tilde{l}^{\beta}_{p^*,j}(\eta))^2 \eta^\beta(1-\eta)^\beta d\eta \\
\leq & \; C\Gamma^2(1+\beta)(p^*)^{-4(\beta+1)} \leq C\Gamma^2(1+\beta)\Big(\frac{p+1}{3}\Big)^{-4(\beta+1)}.
\end{aligned}
$$

Let V be a vertex of T and let $Q_V = \sum_{i=1}^m T_i$ with $T_1 = T$ and $\gamma_i = T_i \bigcap T_{i+1}$, $1 \leq i \leq m$, where the edges sharing the common vertex V which are not on $\partial\Omega$.

Note that

$$\overline{v}_p(V) - v_{T_1}(V) = \frac{1}{n_V}\sum_{1\leq i\leq m} v_{T_i}(V) - v_{T_1}(V) = \frac{1}{n_V}\sum_{1\leq i\leq m-1}(v_{T_{i+1}}(V) - v_{T_1}(V)).$$

Then for $i = 1$, by (4.12)

$$|v_{T_2}(V) - v_{T_1}(V)| \leq \frac{C}{\Gamma(1+\beta)}\log^{\frac{1}{2}}(p+1)\|v_{T_2} - v_{T_i}\|_{H^{1+\beta,\beta}(\gamma_i)},$$

and for $i \neq 1$

$$\begin{aligned} |v_{T_{i+1}}(V) - v_{T_1}(V)| &\leq \sum_{j=1}^{i} |v_{T_{j+1}}(V) - v_{T_j}(V)| \\ &\leq \frac{C}{\Gamma(1+\beta)} \log^{\frac{1}{2}}(p+1) \sum_{j=1}^{i} \|v_{T_{j+1}} - v_{T_j}\|_{H^{1+\beta,\beta}(\gamma_j)}. \end{aligned} \tag{4.17}$$

Therefore combining (4.14)-(4.16) we can obtain (4.13). □

Lemma 4.4. *If $u \in H^{t,\beta}(T)$ where $\beta = (\beta_1, \beta_2, \beta_3)$, with $t > 1 + s + \beta_i$, $1 \leq i \leq 3$ for some $s \geq 0$ and $\beta_i > -1$, there holds on the side γ_i of T*

$$\|u\|_{H^{s,\beta_{\gamma_i}}(\gamma_i)} \leq \Phi(t,s,\beta)\|u\|_{H^{t,\beta}(T)}. \tag{4.18}$$

where $\Phi(t,s,\beta) = \frac{C_0 2^{\lceil \beta \rfloor}}{2^{\beta_i}(t-s-\beta_i-1)\Gamma^2(1+\beta_i)}$ and $\beta_{\gamma_1} = (\beta_3, \beta_2)$, $\beta_{\gamma_2} = (\beta_1, \beta_3)$, $\beta_{\gamma_3} = (\beta_2, \beta_1)$.

Proof. By Lemma 3.3, we may write

$$u = \sum_{i,j \geq 0}^{\infty} a_{ij} (x+y)^i J_i^{\beta_2,\beta_1}\Big(\frac{x-y}{x+y}\Big) J_j^{\beta_3, 2i+\beta_1+\beta_2+1}(2x+2y-1).$$

On $\gamma_3 = \{(x,y) | x+y = 1, 0 \leq x, y \leq 1\}$ we have

$$\begin{aligned} u|_{\gamma_3} &= u(x, 1-x) = \sum_{i,j \geq 0}^{\infty} a_{ij} J_i^{\beta_2,\beta_1}(2x-1) J_j^{\beta_3, 2i+\beta_1+\beta_2+1}(1) \\ &= \sum_{i \geq 0}^{\infty} \big(\sum_{j \geq 0}^{\infty} a_{ij} J_j^{\beta_3, 2i+\beta_1+\beta_2+1}(1) \big) J_i^{\beta_2,\beta_1}(2x-1) = \sum_{i \geq 0}^{\infty} b_i J_i^{\beta_2,\beta_1}(\xi), \end{aligned}$$

with

$$b_i = \sum_{j\geq 0}^{\infty} a_{ij} J_j^{\beta_3, 2i+\beta_1+\beta_2+1}(1),$$

and $\xi = 2x-1, -1 \leq \xi \leq 1$.

Therefore there holds for $s \geq 0$, $\beta_{\gamma_3} = (\beta_2, \beta_1)$:

$$\|u\|^2_{H^{s,\beta_{\gamma_3}}(\gamma_1)} \leq C \sum_{i\geq 0}^{\infty} |b_i|^2 \gamma_i^{\beta_2,\beta_1}(1+i^2)^s.$$

Note that

$$\begin{aligned} |b_i|^2 &= |\sum_{j\geq 0}^{\infty} a_{ij} J_j^{\beta_3,2i+\beta_1+\beta_2+1}(1)|^2 \\ &\leq \Big(\sum_{j\geq 0}^{\infty} |a_{ij}|^2 \gamma_j^{\beta_3,2i+\beta_1+\beta_2+1}(1+i^2+j^2)^{t-s}\Big)\Big(\sum_{j\geq 0}^{\infty} \frac{|J_j^{\beta_3,2i+\beta_1+\beta_2+1}(1)|^2}{\gamma_j^{\beta_3,2i+\beta_1+\beta_2+1}(1+i^2+j^2)^{t-s}}\Big). \end{aligned}$$

Due to the property (3.6) of the Jacobi polynomials we have for $t > s+1+\beta_3$,

$$\begin{aligned} \sum_{j\geq 0}^{\infty} \frac{|J_j^{\beta_3,2i+\beta_1+\beta_2+1}(1)|^2}{\gamma_j^{\beta_3,2i+\beta_1+\beta_2+1}(1+i^2+j^2)^{t-s}} &\leq \frac{C_0}{\Gamma^2(1+\beta_3)}\Big(\sum_{j\geq 0}^{\infty} \frac{(1+j)^{2\beta_3}}{\gamma_j^{\beta_3,2i+\beta_1+\beta_2+1}(1+i^2+j^2)^{t-s}}\Big) \\ &\leq \frac{C_0 2^{-(2i+\lceil\beta\rfloor+2)}}{\Gamma^2(1+\beta_3)}\Big(\sum_{j\geq 0}^{\infty} \frac{1}{(j+1)^{2(t-s)-2\beta_3-1}}\Big) \\ &\leq \frac{C_0 2^{-(2i+\lceil\beta\rfloor+2)}}{(t-s-\beta_3-1)\Gamma^2(1+\beta_3)}, \end{aligned}$$

which implies

$$
\begin{aligned}
&\|u\|^2_{H^{s,\beta}(\gamma_3)} \\
\leq\ & \frac{C_0}{(t-s-\beta_3-1)\Gamma^2(1+\beta_3)} \sum_{i+j\geq 0, i,j\geq 0}^{\infty} 2^{-(2i+\lceil\beta\rfloor+2)} |a_{ij}|^2 \gamma_i^{\beta_2,\beta_1} \gamma_j^{\beta_3,2i+\beta_1+\beta_2+1} (1+i^2+j^2)^t.
\end{aligned}
$$

As we know by (3.24)

$$
\begin{aligned}
&\|u\|^2_{H^{t,\beta}(T)} \\
=\ & \sum_{l=0}^{t} \sum_{i,j\geq 0}^{\infty} 2^{-(2i+\lceil\beta\rfloor+\beta_1+\beta_2+3)} \frac{\Gamma(i+j+1)\Gamma(i+j+l+\lceil\beta\rfloor+2)}{\Gamma(i+j-l+1)\Gamma(i+j+\lceil\beta\rfloor+2)} |a_{ij}|^2 \gamma_i^{\beta_2,\beta_1} \gamma_j^{\beta_3,2i+\beta_1+\beta_2+1}.
\end{aligned}
$$

By Stirling formula, we have

$$
\begin{aligned}
\frac{\Gamma(i+j+1)\Gamma(i+j+l+\lceil\beta\rfloor+2)}{\Gamma(i+j-l+1)\Gamma(i+j+\lceil\beta\rfloor+2)} &\simeq e^{-2l} \frac{(i+j)^{i+j+1/2}(i+j+l+\lceil\beta\rfloor+1)^{i+j+l+\lceil\beta\rfloor+3/2}}{(i+j+\lceil\beta\rfloor+1)^{i+j+\lceil\beta\rfloor+3/2}(i+j-l)^{i+j-l+1/2}} \\
&\simeq Ce^{-2l}(i+j+1)^{2l}.
\end{aligned}
$$

We can get

$$
\begin{aligned}
\|u\|^2_{H^{t,\beta}(T)} &\simeq C \sum_{i+j\geq 0, i,j\geq 0}^{\infty} 2^{-(2i+\lceil\beta\rfloor+\beta_1+\beta_2+3)} |a_{ij}|^2 \gamma_i^{\beta_2,\beta_1} \gamma_j^{\beta_3,2i+\beta_1+\beta_2+1} \sum_{l=0}^{t} e^{-2l}(i+j+1)^{2l} \\
&\simeq C \sum_{i+j\geq 0, i,j\geq 0}^{\infty} 2^{-(2i+\lceil\beta\rfloor+\beta_1+\beta_2+3)} |a_{ij}|^2 \gamma_i^{\beta_2,\beta_1} \gamma_j^{\beta_3,2i+\beta_1+\beta_2+1} (1+i^2+j^2)^t,
\end{aligned}
$$

which implies

$$\|u\|^2_{H^{s,\beta_{\gamma_3}}(\gamma_3)} \le \frac{C_0 2^{\beta_1+\beta_2}}{(t-s-\beta_3-1)\Gamma^2(1+\beta_3)}\|u\|^2_{H^{t,\beta}(T)}.$$

In particular, when $s=0$ and $t>1+\beta_3$, we have

$$\|u\|_{H^{0,\beta_{\gamma_3}}(\gamma_3)} \le \Phi(t,\beta)\|u\|_{H^{t,\beta}(T)},$$

with

$$\Phi(t,\beta) = \frac{C_0 2^{\beta_1+\beta_2}}{\Gamma(1+\beta_3)(t-1-\beta_3)^{1/2}}.$$

Due to Lemma 3.3,

$$u = \sum_{i,j\ge 0} c_{ij}\widetilde{K}^{\beta}_{ij}(x,y),$$

and by (3.24), we have

$$\begin{aligned}&\|u\|^2_{H^{k,\beta}(T)}\\ = &\sum_{l=0}^{k}\sum_{i,j\ge 0}^{\infty} 2^{-(2i+\lceil\beta\rfloor+\beta_2+\beta_3+3)}\frac{\Gamma(i+j+1)\Gamma(i+j+l+\lceil\beta\rfloor+2)}{\Gamma(i+j-l+1)\Gamma(i+j+\lceil\beta\rfloor+2)}|c_{ij}|^2\gamma_i^{\beta_3,\beta_2}\gamma_j^{\beta_1,2i+\beta_2+\beta_3+1}.\end{aligned}$$

Then we have

$$\begin{aligned} u|_{\gamma_1} &= u(0,y) = \sum_{i,j\geq 0}^{\infty} c_{ij} J_i^{\beta_3,\beta_2}(2y-1) J_j^{\beta_1,2i+\beta_2+\beta_3+1}(1) \\ &= \sum_{i\geq 0}^{\infty} \big(\sum_{j\geq 0}^{\infty} c_{ij} J_j^{\beta_1,2i+\beta_2+\beta_3+1}(1) \big) J_i^{\beta_3,\beta_2}(2y-1) = \sum_{i\geq 0}^{\infty} b_i J_i^{\beta_3,\beta_2}(\xi), \end{aligned}$$

with

$$b_i = \sum_{j\geq 0}^{\infty} c_{ij} J_j^{\beta_1,2i+\beta_2+\beta_3+1}(1),$$

and $\xi = 2y-1, -1 \leq \xi \leq 1$. Therefore there holds for $s \geq 0$, $\beta_{\gamma_1} = (\beta_3, \beta_2)$:

$$\|u\|^2_{H^{s,\beta_{\gamma_1}}(\gamma_1)} \leq C \sum_{i\geq 0}^{\infty} |b_i|^2 \gamma_i^{\beta_3,\beta_2} (1+i^2)^s.$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned} |b_i|^2 &= |\sum_{j\geq 0}^{\infty} c_{ij} J_j^{\beta_1,2i+\beta_2+\beta_3+1}(1)| \\ &\leq \Big(\sum_{j\geq 0}^{\infty} |c_{ij}|^2 \gamma_j^{\beta_1,2i+\beta_2+\beta_3+1} (1+i^2+j^2)^{t-s} \Big) \Big(\sum_{j\geq 0}^{\infty} \frac{|J_j^{\beta_1,2i+\beta_2+\beta_3+1}(1)|^2}{\gamma_j^{\beta_1,2i+\beta_2+\beta_3+1}(1+i^2+j^2)^{t-s}} \Big). \end{aligned}$$

Due to the property (3.6) of the Jacobi polynomials we have for $t > s+1+\beta_1$,

$$\begin{aligned}\sum_{j\geq 0}^{\infty}\frac{|J_j^{\beta_1,2i+\beta_2+\beta_3+1}(1)|^2}{\gamma_j^{\beta_1,2i+\beta_2+\beta_3+1}(1+i^2+j^2)^{t-s}} &\leq \frac{C_0}{\Gamma^2(1+\beta_1)}\Big(\sum_{j\geq 0}^{\infty}\frac{(1+j)^{2\beta_1}}{\gamma_j^{\beta_1,2i+\beta_2+\beta_3+1}(1+i^2+j^2)^{t-s}}\Big)\\ &\leq \frac{C_0 2^{-(2i+\lceil\beta\rfloor+2)}}{\Gamma^2(1+\beta_1)}\Big(\sum_{j\geq 0}^{\infty}\frac{1}{(j+1)^{2(t-s)-2\beta_1-1}}\Big)\\ &\leq \frac{C_0 2^{-(2i+\lceil\beta\rfloor+2)}}{(t-s-\beta_1-1)\Gamma^2(1+\beta_1)}\end{aligned}$$

which implies

$$\begin{aligned}&\|u\|^2_{H^{s,\beta_{\gamma_1}}(\gamma_1)}\\ \leq\ & \frac{C_0}{(t-s-\beta_1-1)\Gamma^2(1+\beta_1)}\sum_{i,j\geq 0}^{\infty}2^{-(2i+\lceil\beta\rfloor+2)}|c_{ij}|^2\gamma_i^{\beta_3,\beta_2}\gamma_j^{\beta_1,2i+\beta_2+\beta_3+1}(1+i^2+j^2)^t.\end{aligned}$$

We can get

$$\begin{aligned}\|u\|^2_{H^{t,\beta}(T)} &\simeq C\sum_{i,j\geq 0}^{\infty}2^{-(2i+\lceil\beta\rfloor+\beta_2+\beta_3+3)}|c_{ij}|^2\gamma_i^{\beta_3,\beta_2}\gamma_j^{\beta_1,2i+\beta_2+\beta_3+1}\sum_{l=0}^{t}e^{-2l}(i+j+1)^{2l}\\ &\simeq C\sum_{i+j\geq 0, i,j\geq 0}^{\infty}2^{-(2i+\lceil\beta\rfloor+\beta_2+\beta_3+3)}|c_{ij}|^2\gamma_i^{\beta_3,\beta_2}\gamma_j^{\beta_1,2i+\beta_2+\beta_3+1}(1+i^2+j^2)^t\end{aligned}$$

which implies

$$\|u\|^2_{H^{s,\beta_{\gamma_1}}(\gamma_1)} \leq \frac{C_0 2^{\beta_2+\beta_3}}{(t-s-\beta_1-1)\Gamma^2(1+\beta_1)}\|u\|^2_{H^{t,\beta}(T)}.$$

In particular, when $s = 0$ and $t > 1 + \beta_1$, we have

$$\|u\|_{H^{0,\beta_{\gamma_1}}(\gamma_1)} \leq \Phi(t,\beta)\|u\|_{H^{t,\beta}(T)},$$

with

$$\Phi(t,\beta) = \frac{C_0 2^{\beta_2+\beta_3}}{\Gamma(1+\beta_1)(t-1-\beta_1)^{1/2}}.$$

Similarly, we have the result on $\gamma_2 = \{(x,0)|0 \leq x \leq 1\}$ with $\beta_{\gamma_2} = (\beta_1, \beta_3)$:

$$\|u\|^2_{H^{s,\beta_{\gamma_2}}(\gamma_2)} \quad \leq \quad \frac{C_0 2^{\beta_1+\beta_3}}{(t-s-\beta_2-1)\Gamma^2(1+\beta_2)}\|u\|^2_{H^{t,\beta}(T)},$$

when $t > 1 + s + \beta_2$.

$\square$

Lemma 4.5. *Let $\gamma = \overline{T}_1 \bigcap \overline{T}_2$ shown in Fig 4.1. For $\beta = \beta_1 = \beta_2 = \beta_3 \in (-1, -1/2)$ and $t \in (2+2\beta, 1]$ there holds*

$$\|v_{T_1} - v_{T_2}\|_{H^{1+\beta,\beta}(\gamma)} \leq \frac{C_2(t)}{\Gamma(1+\beta)}(p+1)^{t-1}\sum_{m=1,2}\|v\|_{H^{1,\beta}(T_m)} \tag{4.19}$$

and for $\beta \in (-1, 0)$ and $t \in (1+\beta, 1]$ there holds

$$\|v_{T_1} - v_{T_2}\|_{L^2_\beta(\gamma)} \leq \frac{C_1(t)}{\Gamma(1+\beta)}(p+1)^{t-1}\sum_{m=1,2}\|v\|_{H^{1,\beta}(T_m)} \tag{4.20}$$

with $C_i(t) = C_0 2^{2\beta}(t - i(1+\beta))^{-1/2}$, $i = 1, 2$.

PROOF. Note that

$$\|v_{T_1} - v_{T_2}\|_{H^{1+\beta,\beta}(\gamma)} \leq \sum_{m=1,2} \|v - v_{T_m}\|_{H^{1+\beta,\beta}(\gamma)}.$$

Applying Lemma 4.4 with $s = 1+\beta$ and Theorem 3.4, we have for $t \in (2(1+\beta), 1]$,

$$\begin{aligned}\|v - v_{T_m}\|_{H^{1+\beta,\beta}(\gamma)} &\leq \frac{C_0 2^{2\beta}(t-2-2\beta)^{-1/2}}{\Gamma(1+\beta)}\|v - v_{T_m}\|_{H^{t,\beta}(T_m)} \\ &\leq \frac{C_0 2^{2\beta}(t-2-2\beta)^{-1/2}}{\Gamma(1+\beta)}(p+1)^{t-1}\|v\|_{H^{1,\beta}(T_m)},\end{aligned}$$

which implies (4.19).

Similarly, we have for $s = 0$ and $t \in (1+\beta, 1]$

$$\begin{aligned}\|v_{T_1} - v_{T_2}\|_{L^2_\beta(\gamma)} &\leq \sum_{m=1,2} \|v - v_{T_m}\|_{L^2_\beta(\gamma)} \\ &\leq \frac{C_0 2^{2\beta}(t-1-\beta)^{-1/2}}{\Gamma(1+\beta)} \sum_{m=1,2} \|v - v_{T_m}\|_{H^{t,\beta}(T_m)} \\ &\leq \frac{C_0 2^{2\beta}(t-1-\beta)^{-1/2}}{\Gamma(1+\beta)}(p+1)^{t-1} \sum_{m=1,2} \|v\|_{H^{1,\beta}(T_m)}.\end{aligned}$$

□

Lemma 4.6. *Let $\gamma = T_1 \bigcap T_2$ shown in Fig 4.1 and w_γ be defined in. Then there holds for $\beta \in (-1, -1/2)$*

$$\sum_{m=1,2} \|w_\gamma\|_{L^2_\beta(T_m)} \leq C(\epsilon,\beta)(p+1)^{\epsilon-1} log^{1/2}(p+1) \sum_{m=1,2} \|v\|_{H^{1,\beta}(T_m)}. \tag{4.21}$$

PROOF. In T_1, for $\gamma = \{(x,0)|0 \le x \le 1\}$

$$w_\gamma = w_{\gamma,1} = \frac{1}{2}\phi_\gamma(x)\tilde{l}^\beta_{1,1}(\frac{1-x-y}{1-x})\tilde{l}^\beta_{[\frac{p}{3}]-1,[\frac{p}{3}]-1}(1-y).$$

In $T_1 = \{(x,y)|0 \le x, y, x+y \le 1\}$, we can easily get:

$$-1 \le \frac{1-x-y}{1-x} \le 1,$$

then we can get

$$\tilde{l}^\beta_{1,1}(\frac{1-x-y}{1-x}) \le C.$$

By Theorem 3.6

$$\begin{aligned} \|w_\gamma\|^2_{L^2_\beta(T_1)} &\le C\frac{1}{4}\|\tilde{l}^\beta_{[\frac{p}{3}]-1,[\frac{p}{3}]-1}(1-y)\|^2_{L^2_\beta(\gamma)}\|v^{(1)}_{T_1} - v^{(1)}_{T_2}\|^2_{L^2_\beta(\gamma)} \\ &\le C\Gamma(1+\beta)(p+1)^{-2(\beta+1)}\|v^{(1)}_{T_1} - v^{(1)}_{T_2}\|^2_{L^2_\beta(\gamma)}. \end{aligned}$$

Note that

$$(v^{(1)}_{T_1} - v^{(1)}_{T_2})|_\gamma = (v_{T_1} - v_{T_2})|_\gamma + (w^{(1)}_{T_1} - w^{(1)}_{T_2})|_\gamma.$$

By Lemma 4.5 there holds for $t \in (1+\beta, 1)$.

$$\|v_{T_1} - v_{T_2}\|_{L^2_\beta(\gamma)} \le \frac{C_0 2^{2\beta}(t-1-\beta)^{-1/2}}{\Gamma(1+\beta)}(p+1)^{t-1}\sum_{m=1,2}\|v\|_{H^{1,\beta}(T_m)}.$$

Select $t = 1+\beta+\epsilon < 1$ with $\epsilon \in (0, 1+\beta)$ we have

$$\|v_{T_1} - v_{T_2}\|_{L^2_\beta(\gamma)} \leq \frac{C_0 2^{2\beta}\epsilon^{-1/2}}{\Gamma(1+\beta)}(p+1)^{\epsilon+\beta} \sum_{m=1,2} \|v\|_{H^{1,\beta}(T_m)}. \tag{4.22}$$

And by the definition we have:

$$(w^{(1)}_{T_1} - w^{(1)}_{T_2})|_\gamma = (v_{T_2}(V_1) - v_{T_1}(V_1))x^{[\frac{p}{3}]}\tilde{l}^\beta_{[\frac{p}{3}],[\frac{p}{3}]}(x) + (v_{T_2}(V_3) - v_{T_1}(V_3))\tilde{l}^\beta_{[\frac{p}{3}],0}(x).$$

Due to Lemma 4.1, there holds

$$\begin{aligned}
\|w^{(1)}_{T_1} - w^{(1)}_{T_2}\|^2_{L^2_\beta(\gamma)} &\leq |v_{T_1}(V_1) - v_{T_2}(V_1)|^2 \|x^{[\frac{p}{3}]}\tilde{l}^\beta_{[\frac{p}{3}],[\frac{p}{3}]}(x)\|^2_{L^2_\beta(\gamma)} \\
&+ |v_{T_1}(V_3) - v_{T_2}(V_3)|^2 \|\tilde{l}^\beta_{[\frac{p}{3}],0}(x)\|^2_{L^2_\beta(\gamma)} \\
&\leq C_0\Gamma(1+\beta)2^{2(\beta+1)} \sum_{l=1,3}(p+1)^{-2(1+\beta)}|v_{T_1}(V_l) - v_{T_2}(V_l)|^2.
\end{aligned}$$

Applying Lemma 4.2 and Lemma 4.5 with $t = 2+2\beta+\epsilon$, $\epsilon \in (0, 2+2\beta)$ and $\beta \in (-1, -1/2)$, we obtain

$$\begin{aligned}
|v_{T_1}(V_l) - v_{T_2}(V_l)| &\leq \frac{C_0 \log^{1/2}(p+1)}{\Gamma(1+\beta)} \|v_{T_1} - v_{T_2}\|_{H^{1+\beta,\beta}(\gamma)} \\
&\leq \frac{C_0 \log^{1/2}(p+1)}{\Gamma^2(1+\beta)}(p+1)^{1+\epsilon+2\beta}\epsilon^{-1/2} \sum_{m=1,2}\|v\|_{H^{1,\beta}(T_m)},
\end{aligned}$$

which implies

$$\|w^{(1)}_{T_1} - w^{(1)}_{T_2}\|^2_{L^2_\beta(\gamma)} \leq \frac{C_0}{\Gamma^{3/2}(1+\beta)}\epsilon^{-1/2}(p+1)^{\epsilon+\beta}\log^{1/2}(p+1)\sum_{m=1,2}\|v\|_{H^{1,\beta}(T_m)}. \tag{4.23}$$

Note that for $\beta \in (-1,-1/2)$

$$0 < 2+2\beta < 1+\beta < 1.$$

Then a combination leads to

$$\sum_{m=1,2}\|w_\gamma\|_{L^2_\beta(T_m)} \leq C(\epsilon,\beta)(p+1)^{\epsilon-1} log^{1/2}(p+1)\sum_{m=1,2}\|v\|_{H^{1,\beta}(T_m)}, \tag{4.24}$$

with $C(\epsilon,\beta) = C_0\epsilon^{-1/2}\Gamma^{-1/2}(1+\beta)$ and $\epsilon \in (0, 2+2\beta)$. □

Lemma 4.7. *Let $\gamma = \overline{T_1}\bigcap\overline{T_2}$ be a common side of elements T_1 and T_2 with the ending points V_1 and V_2. Then there holds for $\beta \in (-1,-1/2)$*

$$\|w^{(1)}\|_{L^2_\beta(\gamma)} \leq C(\epsilon,\beta)(p+1)^{\beta+\epsilon}\log^{1/2}(p+1)\sum_{l=1,2}\sum_{T'\subset Q_{V_l}}\|v\|_{H^{1,\beta}(T')} \tag{4.25}$$

with $C(\epsilon,\beta) = C_0\epsilon^{-1/2}\Gamma^{-3/2}(1+\beta)$ and $\epsilon \in (0, 2+2\beta)$.

PROOF. We consider $\gamma = \gamma_2 = \{(x,0)|0 \le x \le 1\}$, due to the definition of $w^{(1)}$,

$$w^{(1)}|_{\gamma_2} = \begin{cases} (\overline{v}_{[\frac{p}{2}]}(V_1) - v_{T_1}(V_1))x^{[\frac{p}{3}]}\tilde{l}^{\beta}_{[\frac{p}{3}],[\frac{p}{3}]}(x) + (\overline{v}_{[\frac{p}{2}]}(V_3) - v_{T_1}(V_3))\tilde{l}^{\beta}_{[\frac{p}{3}],0}(x) = w^{(1)}_{\gamma,1} \text{ on } \gamma_2 \bigcap \overline{T}_1 \\ (\overline{v}_{[\frac{p}{2}]}(V_1) - v_{T_1}(V_1))x^{[\frac{p}{3}]}\tilde{l}^{\beta}_{[\frac{p}{3}],[\frac{p}{3}]}(x) + (\overline{v}_{[\frac{p}{2}]}(V_3) - v_{T_1}(V_3))\tilde{l}^{\beta}_{[\frac{p}{3}],0}(x) = w^{(1)}_{\gamma,2} \text{ on } \gamma_2 \bigcap \overline{T}_2 \end{cases}.$$

It is sufficient to show

$$\|(\overline{v}_{[\frac{p}{2}]}(V_3) - v_{T_j}(V_3))\tilde{l}^{\beta}_{[\frac{p}{3}],0}(x)\|_{L^2_\beta(\gamma_2)} \le C(\epsilon,\beta)(p+1)^{\beta+\epsilon}\log^{1/2}(p+1)\sum_{T'\subset Q_{V_3}}\|v\|_{H^{1,\beta}(T')},$$

and

$$\|(\overline{v}_{[\frac{p}{2}]}(V_1) - v_{T_j}(V_1))x^{[\frac{p}{3}]}\tilde{l}^{\beta}_{[\frac{p}{3}],[\frac{p}{3}]}(x)\|_{L^2_\beta(\gamma_2)} \le C(\epsilon,\beta)(p+1)^{\beta+\epsilon}\log^{1/2}(p+1)\sum_{T'\subset Q_{V_1}}\|v\|_{H^{1,\beta}(T')}.$$

By Theorem 3.6, we get

$$\|\tilde{l}^{\beta}_{[\frac{p}{3}],0}(x)\|_{L^2_\beta(\gamma_2)} \le C_0\Gamma^{1/2}(1+\beta)(p+1)^{-(1+\beta)}, \tag{4.26}$$

and

$$\|x^{[\frac{p}{3}]}\tilde{l}^{\beta}_{[\frac{p}{3}],[\frac{p}{3}]}(x)\|_{L^2_\beta(\gamma_2)} \le C_0\Gamma^{1/2}(1+\beta)(p+1)^{-(1+\beta)}. \tag{4.27}$$

Due to (4.17)

$$(4.28)\ |\overline{v}_{[\frac{p}{2}]}(V_l)-v_{T_1}(V_l)|\leq\frac{C\log^{1/2}(p+1)}{\Gamma(1+\beta)}\sum_{T',T''\subset Q_{V_l}}\sum_{\gamma_3=\overline{T}'\bigcap\overline{T}''}\|v_{T'}-v_{T''}\|_{H^{1+\beta,\beta}(\gamma_2)}.$$

Applying Lemma 4.5 with $t=2+2\beta+\epsilon$, $\epsilon\in(0,2+2\beta)$ and $\beta\in(-1,-1/2)$ we have

$$(4.29)\quad |\overline{v}_{[\frac{p}{2}]}(V_l)-v_{T_1}(V_l)|\leq\frac{C\epsilon^{-1/2}\log^{1/2}(p+1)}{\Gamma^2(1+\beta)}(p+1)^{1+2\beta+\epsilon}\sum_{T'\subset Q_{V_l}}\|v\|_{H^{1,\beta}(T')}.$$

Combing (4.28) and (4.29) we obtain proof for $w^{(1)}_{\gamma,1}$, which can be proved similarly for $w^{(1)}_{\gamma,2}$. □

Lemma 4.8. *Let $\gamma=\overline{T_1}\bigcap\overline{T_2}$ be a common side of elements T_1 and T_2 with the ending points V_1 and V_2. Then there holds for $\beta\in(-1,-1/2)$*

$$(4.30)\qquad \|w^{(2)}\|_{L^2_\beta(\gamma)}\leq\frac{C\epsilon^{-1/2}}{\Gamma(1+\beta)}(p+1)^{\beta+\epsilon}\log^{1/2}(p+1)\sum_{m=1,2}\|v\|_{H^{1,\beta}(T_m)}.$$

Proof. Consider $\gamma=\gamma_2=\{(x,0)|0\leq x\leq 1\}$, due to the definition of $w^{(2)}$

$$w^{(2)}|_{\gamma_2}=\begin{cases}\frac{1}{2}\phi_\gamma(x)=w^{(2)}_{\gamma,1}=w_{\gamma,1} & \text{on } \gamma_2\bigcap\overline{T}_1\\ -\frac{1}{2}\phi_\gamma(x)=w^{(2)}_{\gamma,2}=w_{\gamma,2} & \text{on } \gamma_2\bigcap\overline{T}_2\end{cases}$$

with $\phi_\gamma(x)=(v^{(1)}_{T_2}-v^{(1)}_{T_1})|_{\gamma_2}=(v_{T_2}-v_{T_1})|_{\gamma_2}+(w^{(1)}_{T_2}-w^{(1)}_{T_1})|_{\gamma_2}$.

Due to (4.22)

$$\|v_{T_2} - v_{T_1}\|_{L^2_\beta(\gamma_2)} \le \frac{C\epsilon^{-1/2}}{\Gamma(1+\beta)}(p+1)^{\epsilon+\beta} \sum_{m=1,2} \|v\|_{H^{1,\beta}(T_m)},$$

with $\epsilon \in (0, 1+\beta)$, and by (4.23)

$$\|w^{(1)}_{T_1} - w^{(1)}_{T_2}\|_{L^2_\beta(\gamma_2)} \le \frac{C\epsilon^{-1/2}}{\Gamma^{3/2}(1+\beta)}(p+1)^{\epsilon+\beta} \log^{1/2}(p+1) \sum_{m=1,2} \|v\|_{H^{1,\beta}(T_m)},$$

with $\epsilon \in (0, 2+2\beta)$, which lead to

$$\begin{aligned} \|w^{(2)}\|_{L^2_\beta(\gamma_2)} &\le \|(v_{T_2} - v_{T_1})\|_{L^2_\beta(\gamma_2)} + \|(w^{(1)}_{T_2} - w^{(1)}_{T_1})\|_{L^2_\beta(\gamma_2)} \\ &\le \frac{C\epsilon^{-1/2}}{\Gamma(1+\beta)}(p+1)^{\beta+\epsilon} \log^{1/2}(p+1) \sum_{m=1,2} \|v\|_{H^{1,\beta}(T_m)}. \end{aligned}$$

□

Theorem 4.9. *For $v \in H^{1,-\beta}(\tau)$, there holds for $\beta \in (1/2, 1)$*

$$\|v - \Pi^{-\beta}_{\mathcal{T}} v\|_{L^2_{-\beta}(T)} \le C(p+1)^{-1} \log(1+p) \sum_{T' \in Q_T} \|v\|_{H^{1,-\beta}(T')} \tag{4.31}$$

with C independent of p and β.

Proof. Due to the definition of the quasi Jacobi projection

$$\|v - \Pi^{-\beta}_{\mathcal{T}} v\|_{L^2_{-\beta}(T)} \le \|v - v_T\|_{L^2_{-\beta}(T)} + \|w^{(1)}_T\|_{L^2_{-\beta}(T)} + \sum_{\gamma \subset \partial T \bigcap \partial\tau} \|w_\gamma\|_{L^2_{-\beta}(T)}. \tag{4.32}$$

By property of Jacobi projection, there holds

$$\|v - v_T\|_{L^2_{-\beta}(T)} \le C(p+1)^{-1}\|v\|_{H^{1,-\beta}(T)}. \tag{4.33}$$

Applying Lemma 4.3 and Lemma 4.5 with $t = 2 - 2\beta + \epsilon$ and $\epsilon \in (0, 2-2\beta)$ we have

$$\begin{aligned} \|w^{(1)}\|_{L^2_{-\beta}(T)} &\le C(p+1)^{2(\beta-1)}\log^{\frac{1}{2}}(p+1)\sum_{T',T''\in Q_T}\sum_{\gamma\in T''\bigcap T'}\|v_{T''} - v_{T'}\|_{H^{1-\beta,-\beta}(\gamma)} \\ &\le \frac{C\epsilon^{-1/2}}{\Gamma(1-\beta)}(p+1)^{-1+\epsilon}\log^{\frac{1}{2}}(p+1)\sum_{T'\in Q_T}\|v\|_{H^{1,-\beta}(T')}. \end{aligned} \tag{4.34}$$

By Lemma 4.6 with $\epsilon \in (0, 2-2\beta)$

$$\|w_\gamma\|_{L^2_{-\beta}(T)} \le \frac{C_0\epsilon^{-1/2}}{\Gamma^{1/2}(1-\beta)}(p+1)^{-1+\epsilon}\log^{\frac{1}{2}}(1+p)\sum_{T'\in Q_\gamma}\|v\|_{H^{1,-\beta}(T')} \tag{4.35}$$

A combination of (4.32)-(4.35) leads to

$$\begin{aligned} &\|v - \Pi_{\mathcal{T}}^{-\beta} v\|_{L^2_{-\beta}(T)} \le \\ &(\widetilde{C}(\epsilon,-\beta)(p+1)^{\epsilon-1}\log^{\frac{1}{2}}(1+p) + C_0(p+1)^{-1})\sum_{T'\in Q_T}\|v\|_{H^{1,-\beta}(T')} \end{aligned} \tag{4.36}$$

Selecting $\epsilon = \frac{1}{2\log(p+m)} < 1-\beta$ for some integer $m \ge 0$, we have

$$\epsilon^{-1/2}(p+1)^{\epsilon} = \sqrt{2e}\log^{\frac{1}{2}}(p+m). \tag{4.37}$$

Substituting (4.37) into (4.36) we obtain (4.31). □

Theorem 4.10. *Let $\gamma = \bar{T}_1 \bigcap \bar{T}_2$. If $v \in H^{1,-\beta}(\tau)$, there holds for $\beta \in (1/2, 1)$*

$$\|v - \Pi_{\mathcal{T}}^{-\beta} v\|_{L^2_{-\beta}(\gamma)} \leq C\Gamma^{-1}(1-\beta)(p+1)^{-\beta} \log(p+1) \sum_{l=1,2} \sum_{T' \subset Q_{V_l}} \|v\|_{H^{1,-\beta}(T')} \tag{4.38}$$

with C independent of p and β, where V_l, $l = 1, 2$ are two end points of γ.

Proof.

$$(v - \Pi_{\mathcal{T}}^{-\beta} v)|_\gamma = (v - v_{T_1})|_\gamma + w_{\gamma,1}^{(1)} + w_{\gamma,1}^{(2)}$$

By theorem 3.3 of [**32**] and Lemma 4.4 with $= 1 - \beta + \epsilon$ and $\epsilon \in (0, 1-\beta)$ we have

$$\begin{aligned} \|v - v_{T_1}\|_{L^2_{-\beta}(\gamma)} &\leq \Phi(t, -\beta)\|v - v_{T_1}\|_{H^{t,-\beta}(T_1)} \\ &\leq \frac{C\epsilon^{-1/2}}{\Gamma(1-\beta)}(p+1)^{\epsilon-\beta}\|v\|_{H^{1,-\beta}(T_1)} \end{aligned} \tag{4.39}$$

Applying Lemma 4.7 and Lemma 4.8 we have

$$\|w_{\gamma,1}^{(1)}\|_{L^2_{-\beta}(\gamma)} \leq \frac{C\epsilon^{-1/2}}{\Gamma^{3/2}(1-\beta)}(p+1)^{\epsilon-\beta} \sum_{l=1,2} \sum_{T' \subset Q_{V_l}} \|v\|_{H^{1,-\beta}(T')}$$

and

$$\|w_{\gamma,1}^{(2)}\|_{L^2_{-\beta}(\gamma)} \leq \frac{C\epsilon^{-1/2}}{\Gamma(1-\beta)}(p+1)^{\epsilon-\beta} \log^{\frac{1}{2}}(p+1) \sum_{m=1,2} \|v\|_{H^{1,-\beta}(T_m)},$$

which together with (4.39) leads to

$$\|v - \Pi_{\mathcal{T}}^{-\beta} v\|_{L^2_{-\beta}(\gamma)} \leq C\Gamma^{-1}(1-\beta)\epsilon^{-1/2}(p+1)^{\epsilon-\beta} \log^{1/2}(p+1) \sum_{l=1,2} \sum_{T' \subset Q_{V_l}} \|v\|_{H^{1,-\beta}(T')}$$

Selecting ϵ as in (4.37) we obtain (4.38). □

4.2. Upper and Lower bound of Error in term of Estimator and Indicator

Theorem 4.11. *Let $e = u - u_s$ be the error of the finite element solution in $S^{p,1}(\Omega, \tau)$. Then there holds for $\beta \in (\frac{1}{2}, 1)$,*

$$|||e||| \leq C_1(p)\eta \tag{4.40}$$

where $C_1(p) = C_0 \log(p+1)$ with C_0 independent of p and β.

Proof. First, let $v \in C_0^\infty(\Omega)$, and let $\Pi_{\mathcal{T}}^{-\beta} v$ be its quasi Jacobi projection on $S^{p,1}(\Omega, \mathcal{T})$. Then

$$B(e, v) = B(e, v - \Pi_{\mathcal{T}}^{-\beta} v) = \sum_{T_i \in \mathcal{T}} \int_{T_i} r(v - \Pi_{\mathcal{T}}^{-\beta} v)dx + \sum_{\gamma \in \partial\mathcal{T}} \int_\gamma R(v - \Pi_{\mathcal{T}}^{-\beta} v)ds. \tag{4.41}$$

By Theorem 4.9 and Theorem 4.10, we have for $\epsilon \in (0, 1-\beta)$

$$\begin{aligned} & |\sum_{T_i \in \mathcal{T}} \int_{T_i} r(v - \Pi_{\mathcal{T}}^{-\beta} v) dx| \leq \sum_{T_i \in \mathcal{T}} \|r\|_{L^2_\beta(T_i)} \|v - \Pi_{\mathcal{T}}^{-\beta} v\|_{L^2_{-\beta}(T_i)} \\ \leq & (\sum_{T_i \in \mathcal{T}} \frac{1}{(p+1)^2} \|r\|^2_{L^2_\beta(T_i)})^{\frac{1}{2}} (\sum_{T_i \in \mathcal{T}} (p+1)^2 \|v - \Pi_{\mathcal{T}}^{-\beta} v\|^2_{L^2_{-\beta}(T_i)})^{\frac{1}{2}} \\ \leq & C \log(p+1) \|v\|_{H^{1,-\beta}(\mathcal{T})} (\sum_{T \in \mathcal{T}} \eta^2_{T_i})^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} & |\sum_{\gamma \in \partial\mathcal{T}} \int_{\gamma} R(v - \Pi_{\mathcal{T}}^{-\beta} v) ds| \\ \leq & (\sum_{\gamma \in \partial\mathcal{T}} \frac{1}{(p+1)^{2\beta}} \|R\|^2_{L^2_\beta(\gamma)})^{\frac{1}{2}} (\sum_{\gamma \in \partial\mathcal{T}} (p+1)^{2\beta} \|v - \Pi_{\mathcal{T}}^{-\beta} v\|^2_{L^2_{-\beta}(\gamma)})^{\frac{1}{2}} \\ \leq & C \log(p+1) \|v\|_{H^{1,-\beta}(\mathcal{T})} (\sum_{\gamma \in \partial\mathcal{T}} \eta^2_{\gamma})^{\frac{1}{2}} \end{aligned} \tag{4.42}$$

A combination of (4.42) and (4.42) gives

$$|B(e,v)| \leq C \log(p+1) \|v\|_{H^{1,-\beta}(\mathcal{T})} \cdot \eta \tag{4.43}$$

The above estimation is valid for $v \in C_0^\infty(\Omega)$. By a density argument it can be proved for all $v \in H_0^{1,-\beta}(\mathcal{T})$. Therefore

$$\sup_{v \in H^{1,-\beta}(\mathcal{T})} \frac{|B(e,v)|}{\|v\|_{H^{1,-\beta}(\mathcal{T})}} \leq C \log(p+1) \cdot \eta$$

which proves (4.40). □

Lemma 4.12. *The following inverse inequality holds for any polynomial $\phi_p(x,y)$ in $\mathcal{P}_p(T)$, $T=\{(x,y)|0\le x,y,x+y\le 1\}$:*

$$\begin{aligned}\int_T |\frac{d}{dx}\phi_p(x,y)|^2 x^{\beta+1}y^{\beta}(1-x-y)^{\beta+1}dxdy \\ \le\ C(p+1)^2\int_T |\phi_p(x,y)|^2 x^{\beta}y^{\beta}(1-x-y)^{\beta}dxdy,\end{aligned} \tag{4.44}$$

$$\begin{aligned}\int_T |\frac{d}{dy}\phi_p(x,y)|^2 x^{\beta}y^{\beta+1}(1-x-y)^{\beta+1}dxdy \\ \le\ C(p+1)^2\int_T |\phi_p(x,y)|^2 x^{\beta}y^{\beta}(1-x-y)^{\beta}dxdy,\end{aligned} \tag{4.45}$$

and

$$\begin{aligned}\int_T |\frac{d}{dx}\phi_p(x,y)-\frac{d}{dy}\phi_p(x,y)|^2 x^{\beta+1}y^{\beta+1}(1-x-y)^{\beta}dxdy \\ \le\ C(p+1)^2\int_T |\phi_p(x,y)|^2 x^{\beta}y^{\beta}(1-x-y)^{\beta}dxdy.\end{aligned} \tag{4.46}$$

Proof. For any fixed $y\in[0,1]$, by inverse inequality and mapping $x=\frac{(1-y)(t+1)}{2}$, we have

$$\begin{aligned}&\int_0^{1-y} |\frac{d}{dx}\phi_p(x,y)|^2 x^{\beta+1}(1-x-y)^{\beta+1}dx \\ =\ &\int_{-1}^{1} |\frac{d\phi_p(\frac{(1-y)(t+1),y}{2})}{dt}\frac{dt}{dx}|^2(1-y)^{\beta+1}\Big(\frac{t+1}{2}\Big)^{\beta+1}\Big(1-\frac{(1-y)t}{2}-y\Big)^{\beta+1}\frac{1-y}{2}dt \\ =\ &\frac{1}{2}(1-y)^{2\beta+1}\int_{-1}^{1}|\frac{d}{dt}\phi_p(\frac{(1-y)(t+1),y}{2})|^2\Big(\frac{t+1}{2}\Big)^{\beta+1}\Big(\frac{1-t}{2}\Big)^{\beta+1}dt.\end{aligned}$$

By the inverse inequality [**17**], (3.3), for $\beta > -1$, $\varphi_p(x) \in \mathcal{P}_p(I)$ we have

$$\begin{aligned} &\int_{-1}^{1} (\frac{d}{dt}\varphi_p(\frac{(1-y)(t+1)}{2}, y))^2 \Big(\frac{1-t}{2}\Big)^{\beta+1} \Big(\frac{1+t}{2}\Big)^{\beta+1} dt \\ \leq\ & C(p+1)^2 \int_{-1}^{1} (\varphi_p(\frac{(1-y)(t+1)}{2}, y))^2 \Big(\frac{1-t}{2}\Big)^{\beta} \Big(\frac{1+t}{2}\Big)^{\beta} dt. \end{aligned} \tag{4.47}$$

Then we can get

$$\begin{aligned} &\int_0^{1-y} |\frac{d}{dx}\phi_p(x,y)|^2 x^{\beta+1}(1-x-y)^{\beta+1} dx \\ \leq\ & C(1-y)^{2\beta+1}(p+1)^2 \int_{-1}^{1} (\phi_p(\frac{(1-y)(t+1)}{2}, y))^2 \Big(\frac{t+1}{2}\Big)^{\beta} \Big(\frac{1-t}{2}\Big)^{\beta} dt \\ =\ & C(p+1)^2 \int_0^{1-y} (\phi_p(x,y))^2 \frac{x^\beta}{(1-y)^\beta} \Big(1 - \frac{x}{1-y}\Big)^{\beta} \frac{1}{1-y} (1-y)^{2\beta+1} dx \\ =\ & C(p+1)^2 \int_0^{1-y} |\phi_p(x,y)|^2 x^\beta (1-x-y)^\beta dx. \end{aligned}$$

Integrating y which implies (4.44).

Similarly we can fixed $x \in [0,1]$ and get

$$\begin{aligned} &\int_T |\frac{d}{dy}\phi_p(x,y)|^2 x^\beta y^{\beta+1}(1-x-y)^{\beta+1} dxdy \\ \leq\ & C(p+1)^2 \int_T |\phi_p(x,y)|^2 x^\beta y^\beta (1-x-y)^\beta dxdy, \end{aligned} \tag{4.48}$$

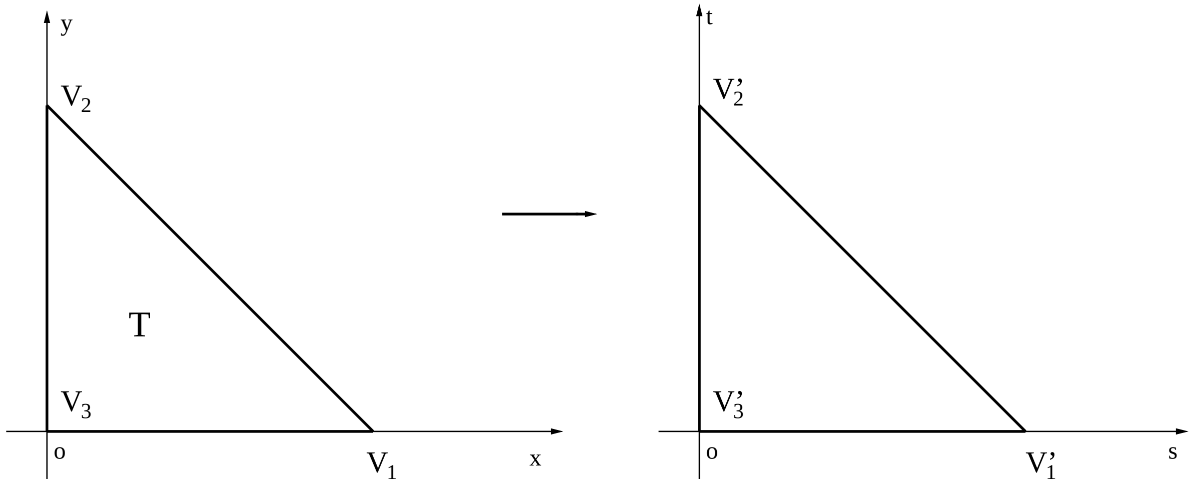

Fig. 4.2 Transform from triangle to triangle

Let M be the mapping of T onto T':

$$M : \begin{cases} s = y \\ t = 1 - x - y \end{cases} \qquad M^{-1} : \begin{cases} x = 1 - s - t \\ y = s. \end{cases}$$

which maps V_1 to V_3', V_2 to V_1', V_3 to V_2'. Let $\tilde{\phi}_p(s,t) = \phi_p(1-s-t, s)$, by (4.44), we have

$$\begin{aligned}
& \int_T |\frac{d}{dx}\phi_p(x,y) - \frac{d}{dy}\phi_p(x,y)|^2 x^{\beta+1} y^{\beta+1} (1-x-y)^\beta dxdy \\
= & \int_{T'} |\frac{d}{ds}\tilde{\phi}_p(s,t)\frac{ds}{dx} + \frac{d}{dt}\tilde{\phi}_p(s,t)\frac{dt}{dx} - \frac{d}{ds}\tilde{\phi}_p(s,t)\frac{ds}{dy} - \frac{d}{dt}\tilde{\phi}_p(s,t)\frac{dt}{dy}|^2 \\
& s^{\beta+1} t^\beta (1-s-t)^{\beta+1} dsdt \\
= & \int_{T'} |\frac{d}{ds}\tilde{\phi}_p(s,t)|^2 s^{\beta+1} t^\beta (1-s-t)^{\beta+1} dsdt \\
\leq & \; C(p+1)^2 \int_{T'} |\tilde{\phi}_p(s,t)|^2 s^\beta t^\beta (1-s-t)^\beta dsdt \\
= & \; C(p+1)^2 \int_T |\phi_p(x,y)|^2 x^\beta y^\beta (1-x-y)^\beta dxdy.
\end{aligned}$$

□

Lemma 4.13. *The following inverse inequality holds for any polynomial $\phi_p(x,y)$ in $\mathcal{P}_p(T)$, $T = \{(x,y)|0 \le x, y, x+y \le 1\}$:*

(4.49)
$$\begin{aligned}&\int_T |\phi_p(x,y)|^2(1-y)^2x^{\beta-1}y^{\beta}(1-x-y)^{\beta-1}dxdy\\ \le\ & C(p+1)^2\int_T |\phi_p(x,y)|^2x^{\beta}y^{\beta}(1-x-y)^{\beta}dxdy.\end{aligned}$$

PROOF. For any fixed $y \in [0,1]$, by mapping $x = \frac{(1-y)(t+1)}{2}$, we have

$$\begin{aligned}&\int_0^{1-y}|\phi_p(x,y)|^2x^{\beta-1}(1-x-y)^{\beta-1}dx\\ =\ & \int_{-1}^{1}|\phi_p(\frac{(1-y)(t+1)}{2},y)|^2(1-y)^{\beta-1}\Big(\frac{t+1}{2}\Big)^{\beta-1}\Big(1-\frac{(1-y)(t+1)}{2}-y\Big)^{\beta-1}\frac{1-y}{2}dt\\ =\ & (1-y)^{2\beta-1}\int_{-1}^{1}|\phi_p(\frac{(1-y)(t+1)}{2},y)|^2\Big(\frac{t+1}{2}\Big)^{\beta-1}\Big(\frac{1-t}{2}\Big)^{\beta-1}dt.\end{aligned}$$

By the inverse inequality (7.9), we have

$$\begin{aligned}&\int_0^{1-y}|\phi_p(x,y)|^2x^{\beta-1}(1-x-y)^{\beta-1}dx\\ \le\ & C(1-y)^{2\beta-1}(p+1)^2\int_{-1}^{1}|\phi_p(\frac{(1-y)(t+1),y}{2})|^2\Big(\frac{t+1}{2}\Big)^{\beta}\Big(\frac{1-t}{2}\Big)^{\beta}dt\\ =\ & C(p+1)^2(1-y)^{-2}\int_0^{1-y}|\phi_p(x,y)|^2\frac{x^\beta}{(1-y)^\beta}\Big(1-\frac{x}{1-y}\Big)^{\beta}\frac{1}{1-y}(1-y)^{2\beta+1}dx\\ =\ & C(p+1)^2(1-y)^{-2}\int_0^{1-y}|\phi_p(x,y)|^2x^{\beta}(1-x-y)^{\beta}dx.\end{aligned}$$

Integrating y which implies (4.50). □

Lemma 4.14. *The following inverse inequality holds for* $-1 < \alpha < \beta$*, any polynomial* $\phi_p(x,y)$ *in* $\mathcal{P}_p(T)$, $T = \{(x,y)|0 \leq x, y, x+y \leq 1\}$*:*

$$\begin{aligned} &\int_T |\phi_p(x,y)|^2(1-y)^{2(\beta-\alpha)}x^\alpha y^\beta(1-x-y)^\alpha dxdy \\ \leq\ & C(p+1)^{2(\beta-\alpha)}\int_T |\phi_p(x,y)|^2 x^\beta y^\beta(1-x-y)^\beta dxdy, \end{aligned} \tag{4.50}$$

$$\begin{aligned} &\int_T |\phi_p(x,y)|^2(1-x)^{2(\beta-\alpha)}x^\beta y^\alpha(1-x-y)^\alpha dxdy \\ \leq\ & C(p+1)^{2(\beta-\alpha)}\int_T |\phi_p(x,y)|^2 x^\beta y^\beta(1-x-y)^\beta dxdy \end{aligned} \tag{4.51}$$

and

$$\begin{aligned} &\int_T |\phi_p(x,y)|^2(x+y)^{2(\beta-\alpha)}x^\alpha y^\alpha(1-x-y)^\beta dxdy \\ \leq\ & C(p+1)^{2(\beta-\alpha)}\int_T |\phi_p(x,y)|^2 x^\beta y^\beta(1-x-y)^\beta dxdy \end{aligned} \tag{4.52}$$

with C *depending on* α *and* β*, but not on* p *and* ϕ_p*.*

Proof. For any fixed $y \in [0,1]$, by mapping $x = \frac{(1-y)(t+1)}{2}$, we have

$$\begin{aligned} &\int_0^{1-y} |\phi_p(x,y)|^2 x^\alpha(1-x-y)^\alpha dx \\ =\ & \int_{-1}^1 |\phi_p(\frac{(1-y)(t+1)}{2},y)|^2(1-y)^\alpha\Big(\frac{t+1}{2}\Big)^\alpha\Big(1-\frac{(1-y)(t+1)}{2}-y\Big)^\alpha\frac{1-y}{2}dt \\ =\ & 2^{-2\alpha-1}(1-y)^{2\alpha+1}\int_{-1}^1 |\phi_p(\frac{(1-y)(t+1)}{2},y)|^2(t+1)^\alpha(1-t)^\alpha dt. \end{aligned}$$

By the inverse inequality (7.9), we have

$$
\begin{aligned}
&\int_0^{1-y} |\phi_p(x,y)|^2 x^\alpha (1-x-y)^\alpha dx \\
\leq\ & C2^{-2\alpha-1}(1-y)^{2\alpha+1}(p+1)^{2(\beta-\alpha)} \int_{-1}^{1} |\phi_p(\frac{(1-y)(t+1), y}{2})|^2 (t+1)^\beta (1-t)^\beta dt \\
=\ & C2^{2(\beta-\alpha)}(p+1)^{2(\beta-\alpha)}(1-y)^{2\alpha+1} \int_0^{1-y} |\phi_p(x,y)|^2 \frac{x^\beta}{(1-y)^\beta}\Big(1-\frac{x}{1-y}\Big)^\beta \frac{1}{1-y} dx \\
=\ & C2^{2(\beta-\alpha)}(p+1)^{2(\beta-\alpha)}(1-y)^{-2(\beta-\alpha)} \int_0^{1-y} |\phi_p(x,y)|^2 x^\beta (1-x-y)^\beta dx.
\end{aligned}
$$

Integrating y which implies (4.50).

Similarly, we fixed any $x \in [0,1]$, by mapping $y = \frac{(1-x)(t+1)}{2}$, it can be shown by same argument that

$$
\begin{aligned}
&\int_T |\phi_p(x,y)|^2 (1-x)^{2(\beta-\alpha)} x^\beta y^\alpha (1-x-y)^\alpha dxdy \\
\leq\ & C(p+1)^{2(\beta-\alpha)} \int_T |\phi_p(x,y)|^2 x^\beta y^\beta (1-x-y)^\beta dxdy.
\end{aligned}
$$

Let M be the mapping of T onto T' shown as Fig 4.2:

$$
M : \begin{cases} s = y \\ t = 1-x-y \end{cases} \qquad M^{-1} : \begin{cases} x = 1-s-t \\ y = s. \end{cases}
$$

which maps V_1 to V_3', V_2 to V_1', V_3 to V_2'. Due to (4.50), let $\tilde{\phi}_p(s,t) = \phi_p(1-s-t,s)$, we have

$$\begin{aligned} & \int_T |\phi_p(x,y)|^2 (x+y)^{2(\beta-\alpha)} x^\alpha y^\alpha (1-x-y)^\beta dxdy \\ = & \int_{T'} |\tilde{\phi}_p(s,t)|^2 (1-t)^{2(\beta-\alpha)} s^\alpha t^\beta (1-s-t)^\alpha dsdt \\ \leq & \; C(p+1)^{2(\beta-\alpha)} \int_{T'} |\tilde{\phi}_p(s,t)|^2 s^\beta t^\beta (1-s-t)^\beta dsdt \\ = & \; C(p+1)^{2(\beta-\alpha)} \int_T |\phi_p(x,y)|^2 x^\beta y^\beta (1-x-y)^\beta dxdy. \end{aligned}$$

□

Theorem 4.15. *If $f \in \mathcal{P}_p(T)$, then for $\beta \in (0,1)$*

$$|||e|||_T \geq C\eta_T. \tag{4.53}$$

with C independent of p.

Proof. Let $v = rW_{\beta,T} = x^\beta y^\beta (1-x-y)^\beta r$. Then v vanishes on ∂T, and it can be extended by zero extension outside of T. Substituting v in to (4.41), we have

$$\|r\|^2_{L^2_\beta(T)} = (r,v) = B(e,v) \leq \|v\|_{H^{1,-\beta}(T)} |||e|||_T. \tag{4.54}$$

Note that

$$\|v\|_{L^2_{-\beta}(T)} = \|r\|_{L^2_\beta(T)} \tag{4.55}$$

and

$$\begin{aligned}
|v|^2_{H^{1,-\beta}(T)} &= \|(x(1-x-y))^{1/2}v_x\|^2_{L^2_{-\beta}(T)} \\
&+ \|(y(1-x-y))^{1/2}v_y\|^2_{L^2_{-\beta}(T)} + \|(xy)^{1/2}(v_x-v_y)\|^2_{L^2_{-\beta}(T)}.
\end{aligned}$$

By a simple calculation we have

$$\begin{aligned}
&\|(x(1-x-y))^{1/2}v_x\|^2_{L^2_{-\beta}(T)} \\
= &\|((\beta(1-x-y)-\beta x)x^{\beta-1}y^{\beta}(1-x-y)^{\beta-1}r + r_x x^{\beta}y^{\beta}(1-x-y)^{\beta})x^{\frac{1}{2}}(1-x-y)^{\frac{1}{2}}\|^2_{L^2_{-\beta}(T)} \\
\leq &C(\|x^{\beta-1/2}y^{\beta}(1-x-y)^{\beta-1/2}(1-2x-y)r\|^2_{L^2_{-\beta}(T)} + \|r_x x^{\beta+1/2}y^{\beta}(1-x-y)^{\beta+1/2}\|^2_{L^2_{-\beta}(T)}) \\
= &C\Big(\int_T |r|^2((1-y)^2-4x(1-x-y))x^{\beta-1}y^{\beta}(1-x-y)^{\beta-1}dxdy \\
+ &\int_T |r_x|^2 x^{\beta+1}y^{\beta}(1-x-y)^{\beta+1}dxdy\Big) \\
\leq &C\Big(\int_T |r|^2(1-y)^2x^{\beta-1}y^{\beta}(1-x-y)^{\beta-1}dxdy + \int_T |r|^2x^{\beta}y^{\beta}(1-x-y)^{\beta}dxdy \\
&+ \int_T |r_x|^2x^{\beta+1}y^{\beta}(1-x-y)^{\beta+1}dxdy\Big).
\end{aligned}$$

Since $f \in \mathcal{P}_p(T)$, $r = f + \Delta u_S - u_S \in \mathcal{P}_p(T)$, and by Lemma 4.12-4.13, we have

$$\int_T |r|^2(1-y)^2x^{\beta-1}y^{\beta}(1-x-y)^{\beta-1}dxdy \leq C(p+1)^2\int_T |r|^2x^{\beta}y^{\beta}(1-x-y)^{\beta}dxdy$$

and

$$\int_T |r_x|^2 x^{\beta+1} y^\beta (1-x-y)^{\beta+1} dxdy \leq C(p+1)^2 \int_T |r|^2 x^\beta y^\beta (1-x-y)^\beta dxdy,$$

which implies

$$\|(x(1-x-y))^{1/2} v_x\|_{L^2_{-\beta}(T)} \leq C(p+1)\|r\|_{L^2_\beta(T)}. \tag{4.56}$$

Similarly, it holds that

$$\|(y(1-x-y))^{1/2} v_y\|_{L^2_{-\beta}(T)} \leq C(p+1)\|r\|_{L^2_\beta(T)} \tag{4.57}$$

and because

$$v_x - v_y = (\beta(y-x)x^{\beta-1}y^{\beta-1}(1-x-y)^\beta)r + (r_x - r_y)x^\beta y^\beta (1-x-y)^\beta,$$

we can get

$$\begin{aligned} & \|(xy)^{1/2}(v_x - v_y)\|^2_{L^2_{-\beta}(T)} \leq C(\|x^{\beta-1/2}y^{\beta-1/2}(1-x-y)^\beta r\|^2_{L^2_{-\beta}(T)} \\ + \quad & \|(r_x - r_y)x^{\beta+1/2}y^{\beta+1/2}(1-x-y)^\beta\|^2_{L^2_{-\beta}(T)} \\ \leq \quad & C\Big(\int_T |r|^2 x^{\beta-1}y^{\beta-1}(1-x-y)^\beta dxdy + \int_T |r_x - r_y|^2 x^{\beta+1}y^{\beta+1}(1-x-y)^\beta dxdy\Big). \end{aligned}$$

So by the Lemma 4.12-4.13 it can easily get

$$\|(xy)^{1/2}(v_x - v_y)\|_{L^2_{-\beta}(T)} \leq C(p+1)\|r\|_{L^2_\beta(T)} \tag{4.58}$$

which yields

$$\|v\|_{H^{1,-\beta}(T)} \leq C(p+1)\|r\|_{L^2_\beta(T)}. \tag{4.59}$$

Therefore

$$|||e|||_T \geq \frac{\|r\|^2_{L^2_\beta(T)}}{\|v\|_{H^{1,-\beta}(T)}} \geq C(p+1)^{-1}\|r\|_{L^2_{-\beta}(T)} = C\eta_T. \tag{4.60}$$

□

Theorem 4.16. *If $R_\gamma = R|_\gamma \in \mathcal{P}_p(\gamma)$,$\gamma \in \partial\mathcal{T}$, there holds for $\beta \in (0,1)$*

$$\sum_{T \in Q_\gamma} |||e|||_T \geq C\Gamma^{-1/2}(1-\beta)(p+1)^{-\beta}\eta_\gamma, \qquad \forall\gamma \in \partial\tau \tag{4.61}$$

where Q_γ is a pair of elements sharing γ.

Proof. Suppose $\gamma = \{(0,y)|0 \leq y \leq 1\} = \bar{T}_1 \bigcap \bar{T}_2$ as Fig 4.3 shown. Let $\psi_\gamma = y^\beta(1-y)^{\frac{3\beta}{2}}R_\gamma$ with $\beta \in (0,1)$ and let $v_\gamma = \psi_\gamma(y)\tilde{l}^{-\beta}_{p,0}(\frac{|x|}{1-y})$. Then v_γ vanishes on ∂Q_γ, and it can be extended by a zero extension outside of Q_γ.

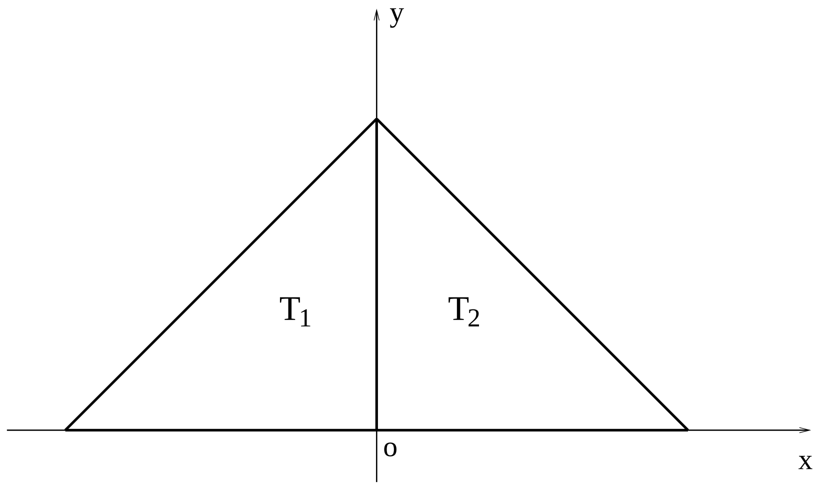

Fig. 4.3

Note that

$$\|\psi_\gamma\|_{L^2_{-\beta}(\gamma)} = \|R_\gamma\|_{L^2_\beta(\gamma)}. \tag{4.62}$$

Introducing the mapping

$$\begin{cases} x = \xi(1-\eta) \\ y = \eta \end{cases},$$

which maps T_2 onto square S: $(0,1)\times(0,1)$, we have by Lemma 4.1

$$\begin{aligned} \|v_\gamma\|^2_{L^2_{-\beta}(T_2)} &= \int_{T_2} \psi^2_\gamma(y)(\tilde{l}^{-\beta}_{p,0}(\frac{x}{1-y}))^2 x^{-\beta}y^{-\beta}(1-x-y)^{-\beta}dxdy \\ &= \int_S \psi^2_\gamma(\eta)(\tilde{l}^{-\beta}_{p,0}(\xi))^2\xi^{-\beta}(1-\xi)^{-\beta}\eta^{-\beta}(1-\eta)^{1-2\beta}d\xi d\eta \\ &= \int_S R^2_\gamma(\eta)(\tilde{l}^{-\beta}_{p,0}(\xi))^2\xi^{-\beta}(1-\xi)^{-\beta}\eta^{\beta}(1-\eta)^{1+\beta}d\xi d\eta \\ &\leq C\Gamma(1-\beta)(p+1)^{2(\beta-1)}\|R_\gamma\|^2_{L^2_\beta(\gamma)}. \end{aligned} \tag{4.63}$$

Similarly, introducing the mapping

$$\begin{cases} x = -\xi(1-\eta) \\ y = \eta \end{cases},$$

which maps T_1 onto square S: $(0,1)\times(0,1)$. We have the similarly results by the same argument:

$$\|v_\gamma\|^2_{L^2_{-\beta}(T_1)} \leq C\Gamma(1-\beta)(p+1)^{2(\beta-1)}\|R_\gamma\|_{L^2_\beta(\gamma)}. \tag{4.64}$$

Note that

$$\frac{\partial}{\partial x}(v_\gamma) = \frac{\psi_\gamma(y)}{1-y}\tilde{l}'^{-\beta}_{p,0}(\frac{|x|}{1-y}).$$

By Lemma 4.1, we have for $m = 1, 2$

$$
\begin{aligned}
(4.65)\qquad & \|(|x|(1-|x|-y))^{\frac{1}{2}}\frac{\partial}{\partial x}(v_\gamma)\|^2_{L^2_{-\beta}(T_m)} \\
= & \int_{T_m}|\frac{\psi_\gamma(y)}{1-y}|^2|\tilde{l}'^{-\beta}_{p,0}(\frac{|x|}{1-y})|^2|x|(1-|x|-y)|x|^{-\beta}y^{-\beta}(1-|x|-y)^{-\beta}dxdy \\
= & \int_S |R_\gamma(\eta)|^2|\tilde{l}'^{-\beta}_{p,0}(\xi)|^2\xi^{-\beta+1}(1-\xi)^{-\beta+1}\eta^\beta(1-\eta)^{1+\beta}d\xi d\eta \\
\leq & C\Gamma(1-\beta)(p+1)^{2\beta}\|R_\gamma\|^2_{L^2_\beta(\gamma)}.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
\frac{\partial}{\partial y}(v_\gamma) = & (\beta y^{\beta-1}(1-y)^{\frac{3}{2}\beta-1}(1-\frac{5}{2}y)R_\gamma(y)+y^\beta(1-y)^{\frac{3}{2}\beta}R_{\gamma,y})\tilde{l}^{-\beta}_{p,0}(\frac{|x|}{1-y}) \\
& +y^\beta(1-y)^{\frac{3}{2}\beta}R_\gamma(y)\tilde{l}'^{-\beta}_{p,0}(\frac{|x|}{1-y})\cdot\frac{|x|}{(1-y)^2}
\end{aligned}
$$

and

$$
\begin{aligned}
& \|(y(1-|x|-y))^{\frac{1}{2}}\frac{\partial}{\partial y}(v_\gamma)\|_{L^2_{-\beta}(T_m)} \\
\leq & C(\|y^{\beta-1}(1-y)^{\frac{3}{2}\beta-1}(1-\frac{5}{2}y)R_\gamma(y)\tilde{l}^{-\beta}_{p,0}(\frac{|x|}{1-y})(y(1-|x|-y))^{\frac{1}{2}}\|_{L^2_{-\beta}(T_m)} \\
+ & \|y^\beta(1-y)^{\frac{3}{2}\beta}R_{\gamma,y}\tilde{l}^{-\beta}_{p,0}(\frac{|x|}{1-y})(y(1-|x|-y))^{\frac{1}{2}}\|_{L^2_{-\beta}(T_m)} \\
+ & \|y^\beta(1-y)^{\frac{3}{2}\beta}R_\gamma(y)(\tilde{l}'^{-\beta}_{p,0}(\frac{|x|}{1-y}))\frac{|x|}{(1-y)^2}(y(1-|x|-y))^{\frac{1}{2}}\|_{L^2_{-\beta}(T_m)}).
\end{aligned}
$$

By Lemma 4.1 and Theorem 7.9, we have

$$
\begin{aligned}
(4.66)\quad & \|y^{\beta-1}(1-y)^{\frac{3}{2}\beta-1}(1-\frac{5}{2}y)R_\gamma(y)\tilde{l}_{p,0}^{-\beta}(\frac{|x|}{1-y})(y(1-|x|-y))^{\frac{1}{2}}\|^2_{L^2_{-\beta}(T_m)} \\
= & \int_{T_m}\Big(y^{\beta-1}(1-y)^{\frac{3}{2}\beta-1}(1-\frac{5}{2}y)R_\gamma(y)\tilde{l}_{p,0}^{-\beta}(\frac{|x|}{1-y})\Big)^2 y(1-|x|-y)x^{-\beta}y^{-\beta}(1-|x|-y)^{-\beta}dxdy \\
= & \int_S |\tilde{l}_{p,0}^{-\beta}(\xi)|^2\xi^{-\beta}(1-\xi)^{-\beta+1}|R_\gamma(\eta)|^2\eta^{\beta-1}(1-\eta)^{\beta}(1-\frac{5}{2}\eta)^2 d\xi d\eta \\
\leq & \ C\int_0^1 |\tilde{l}_{p,0}^{-\beta}(\xi)|^2\xi^{-\beta}(1-\xi)^{-\beta}d\xi\int_0^1 |R_\gamma(\eta)|^2\eta^{\beta-1}(1-\eta)^{\beta-1}d\eta \\
\leq & \ C\Gamma(1-\beta)(p+1)^{2\beta}\|R_\gamma\|^2_{L^2_\beta(\gamma)},
\end{aligned}
$$

by Lemma 4.1 and Theorem 3.5, we have

$$
\begin{aligned}
(4.67)\quad & \|y^{\beta}(1-y)^{\frac{3}{2}\beta}R_{\gamma,y}\tilde{l}_{p,0}^{-\beta}(\frac{|x|}{1-y})(y(1-|x|-y))^{\frac{1}{2}}\|^2_{L^2_{-\beta}(T_m)} \\
= & \int_{T_m}\Big(y^{\beta}(1-y)^{\frac{3}{2}\beta}R_{\gamma,y}\tilde{l}_{p,0}^{-\beta}(\frac{|x|}{1-y})\Big)^2 y(1-|x|-y)x^{-\beta}y^{-\beta}(1-|x|-y)^{-\beta}dxdy \\
= & \int_S |\tilde{l}_{p,0}^{-\beta}(\xi)|^2\xi^{-\beta}(1-\xi)^{-\beta+1}|R_{\gamma,\eta}(\eta)|^2\eta^{\beta+1}(1-\eta)^{3+\beta}d\xi d\eta \\
\leq & \ C\int_0^1 |\tilde{l}_{p,0}^{-\beta}(\xi)|^2\xi^{-\beta}(1-\xi)^{-\beta}d\xi\int_0^1 |R_{\gamma,\eta}(\eta)|^2\eta^{\beta+1}(1-\eta)^{\beta+1}d\eta \\
\leq & \ C\Gamma(1-\beta)(p+1)^{2\beta}\|R_\gamma\|^2_{L^2_\beta(\gamma)}
\end{aligned}
$$

and

$$\begin{aligned}
(4.68) \qquad & \|y^{\beta}(1-y)^{\frac{3}{2}\beta}R_{\gamma}(y)(\tilde{l}'^{-\beta}_{p,0}(\frac{|x|}{1-y}))\cdot\frac{|x|}{(1-y)^2}(y(1-|x|-y))^{\frac{1}{2}}\|^2_{L^2_{-\beta}(T_m)} \\
= & \int_{T_m} y^{2\beta}(1-y)^{3\beta}R^2_{\gamma}(y)((\tilde{l}'^{-\beta}_{p,0}(\frac{|x|}{1-y})))^2\frac{x^2}{(1-y)^4}y(1-|x|-y)x^{-\beta}y^{-\beta}(1-|x|-y)^{-\beta}dxdy \\
= & \int_S ((\tilde{l}^{-\beta}_{p,0}(\xi))')^2\xi^{2-\beta}(1-\xi)^{1-\beta}R^2_{\gamma}(\eta)\eta^{\beta+1}(1-\eta)^{\beta}d\xi d\eta \\
\leq & C\Gamma(1-\beta)(p+1)^{4\beta}\int_0^1 R^2_{\gamma}(\eta)\eta^{\beta}(1-\eta)^{\beta}d\eta \\
\leq & C\Gamma(1-\beta)(p+1)^{4\beta}\|R_{\gamma}\|^2_{L^2_{\beta}(\gamma)}.
\end{aligned}$$

Combining (4.66)-(4.68) we have

$$(4.69) \qquad \|(y(1-|x|-y))^{\frac{1}{2}}\frac{\partial}{\partial y}(v_{\gamma})\|_{L^2_{-\beta}(T_m)} \leq C\Gamma^{1/2}(1-\beta)(p+1)^{2\beta}\|R_{\gamma}\|_{L^2_{\beta}(\gamma)}.$$

Similarly we have

$$\begin{aligned}
\|(|x|y)^{\frac{1}{2}}(\frac{\partial}{\partial x}(v_{\gamma})-\frac{\partial}{\partial y}(v_{\gamma}))\|_{L^2_{-\beta}(T_m)} \leq & \|(|x|y)^{\frac{1}{2}}\frac{\partial}{\partial x}(v_{\gamma})\|_{L^2_{-\beta}(T_m)}+\|(|x|y)^{\frac{1}{2}}\frac{\partial}{\partial y}(v_{\gamma})\|_{L^2_{-\beta}(T_m)} \\
\leq & C\Gamma^{1/2}(1-\beta)(p+1)^{2\beta}\|R_{\gamma}\|_{L^2_{\beta}(\gamma)},
\end{aligned}$$

which yields

$$(4.70) \qquad \|v_{\gamma}\|_{H^{1,-\beta}(T_m)} \leq C\Gamma^{1/2}(1-\beta)(p+1)^{2\beta}\|R_{\gamma}\|_{L^2_{\beta}(\gamma)}.$$

From the definition of residual

$$B(e, v_\gamma) = \sum_{T \in Q_\gamma} \int_T r v_\gamma dxdy + \int_\gamma R_\gamma v_\gamma ds \tag{4.71}$$

there holds

$$\begin{aligned}
\|R_\gamma\|^2_{L^2_\beta(\gamma)} &= \sum_{T \in Q_\gamma} (B(e, v_\gamma)_T - \int_T r v_\gamma dxdy) \\
&\leq \sum_{T \in Q_\gamma} (|||e|||_T \|v_\gamma\|_{H^{1,-\beta}(T)} + \|r\|_{L^2_\beta(T)} \|v_\gamma\|_{L^2_{-\beta}(T)}) \\
&\leq \sum_{T \in Q_\gamma} (|||e|||_T \|v_\gamma\|_{H^{1,-\beta}(T)} + (p+1)|||e|||_T \|v_\gamma\|_{L^2_{-\beta}(T)}) \\
&\leq C\Gamma^{1/2}(1-\beta)(p+1)^{2\beta} \|R_\gamma\|_{L^2_\beta(\gamma)} \sum_{T \in Q_\gamma} |||e|||_T,
\end{aligned}$$

which implies (4.61). □

4.3. The modified indicators $\widetilde{\eta}_T$, $\widetilde{\eta}_\gamma$ and modified estimator $\widetilde{\eta}$

In general the residual $r_i \notin \mathcal{P}_p(T_i)$, and the jump $R\,|_{\gamma_l^i} \notin \mathcal{P}_p(\gamma_l^i)$ if the basis functions are the images of shape functions defined on the standard triangle element and the corresponding mapping is not linear. Therefore, Theorem 4.15 and 4.16 do not hold in general. We need to investigate the modified indicators $\widetilde{\eta}_T$ and $\widetilde{\eta}_\gamma$ as the lower bound of the error.

Theorem 4.17. *Let $\widetilde{\eta}, \widetilde{\eta}_T$ and $\widetilde{\eta}_\gamma$ be the modified error estimator and indicators defined in (2.7)-(2.9), and let Π_T^β and Π_γ^β be the Jacobi projection operators on $\mathcal{P}_p(T)$ and $\mathcal{P}_p(\gamma)$ with*

$\beta \in (1/2, 1)$ *respectively. Then*

$$|||e||| \le C_1(p)\left(\tilde{\eta} + (p+1)^{-1}\sum_{T\in\mathcal{T}}\|f - \Pi_T^\beta f\|_{L^2_\beta(T)} + (p+1)^{-\beta}\sum_{\gamma\in\partial\mathcal{T}}\|R - \Pi_\gamma^\beta R\|_{L^2_\beta(\gamma)}\right) \tag{4.72}$$

with $C_1(p) = C_0 \log(p+1)$ *and* C_0 *independent of* p, β.

PROOF. Let $r_p = \Pi_T^\beta r$, $f_p = \Pi_T^\beta f$, and let $R_p = \Pi_\gamma^\beta R$. Then

$$\left(r, v - \Pi_{\mathcal{T}}^{-\beta} v\right)_T = \left(r_p, v - \Pi_{\mathcal{T}}^{-\beta} v\right)_T + \left(f - f_p, v - \Pi_{\mathcal{T}}^{-\beta} v\right)_T$$

and

$$\left(R, v - \Pi_{\mathcal{T}}^{-\beta} v\right)_\gamma = \left(R_p, v - \Pi_{\mathcal{T}}^{-\beta} v\right)_\gamma + \left(R - R_p, v - \Pi_{\mathcal{T}}^{-\beta} v\right)_\gamma.$$

Then we have from (4.40)

$$
\begin{aligned}
\Big|B(e,v)\Big| \le & \sum_{T\in\mathcal{T}} \Big(\|r_p\|_{L^2_\beta(T)} \|v-\Pi^{-\beta}_{\mathcal{T}} v\|_{L^2_{-\beta}(T)} + \|f-f_p\|_{L^2_\beta(T)} \|v-\Pi^{-\beta}_{\mathcal{T}} v\|_{L^2_{-\beta}(T)} \Big) \\
& + \sum_{\gamma\in\partial\mathcal{T}} \Big(\|R_p\|_{L^2_\beta(\gamma)} \|v-\Pi^{-\beta}_{\mathcal{T}} v\|_{L^2_{-\beta}(\gamma)} + \|R-R_p\|_{L^2_\beta(\gamma)} \|v-\Pi^{-\beta}_{\mathcal{T}} v\|_{L^2_{-\beta}(\gamma)} \Big) \\
\le & \Big((\sum_{T\in\mathcal{T}} (p+1)^{-2} \|r_p\|^2_{L^2_\beta(T)})^{\frac12} + (\sum_{T\in\mathcal{T}} (p+1)^{-2} \|f-f_p\|^2_{L^2_\beta(T)})^{\frac12} \Big) \\
& \Big(\sum_{T\in\mathcal{T}} (p+1)^2 \|v-\Pi^{-\beta}_{\mathcal{T}} v\|^2_{L^2_{-\beta}(T)} \Big)^{\frac12} \\
+ & \Big((\sum_{\gamma\in\partial\mathcal{T}} (p+1)^{-2\beta} \|R_p\|^2_{L^2_\beta(\gamma)})^{\frac12} + (\sum_{\gamma\in\partial\mathcal{T}} (p+1)^{-2\beta} \|R-R_p\|^2_{L^2_\beta(\gamma)})^{\frac12} \Big) \\
& \Big(\sum_{\gamma\in\partial\mathcal{T}} (p+1)^{2\beta} \|v-\Pi^{-\beta}_{\mathcal{T}} v\|^2_{L^2_{-\beta}(\gamma)} \Big)^{\frac12} \\
\le & \Big((\sum_{T\in\mathcal{T}} \widetilde{\eta}_T^2)^{1/2} + (\sum_{T\in\mathcal{T}} (p+1)^{-2} \|f-f_p\|^2_{L^2_\beta(T)})^{\frac12} \Big) \Big(\sum_{T\in\mathcal{T}} (p+1)^2 \|v-\Pi^{-\beta}_{\mathcal{T}} v\|^2_{L^2_{-\beta}(T)} \Big)^{\frac12} \\
+ & \Big((\sum_{\gamma\in\partial\mathcal{T}} \widetilde{\eta}_\gamma^2)^{1/2} + (\sum_{\gamma\in\partial\mathcal{T}} (p+1)^{-2\beta} \|R-R_p\|^2_{L^2_\beta(\gamma)})^{\frac12} \Big) \Big(\sum_{\gamma\in\partial\mathcal{T}} (p+1)^{2\beta} \|v-\Pi^{-\beta}_{\mathcal{T}} v\|^2_{L^2_{-\beta}(\gamma)} \Big)^{\frac12}.
\end{aligned}
$$

Due to Theorem 4.9 and Theorem 4.10 we have

$$
\Big|B(e,v)\Big| \le C\log(p+1)\Big(\tilde{\eta} + (p+1)^{-1} \sum_{T\in\mathcal{T}} \|f-\Pi^\beta_T f\|_{L^2_\beta(T)} + (p+1)^{-\beta} \sum_{\gamma\in\partial\mathcal{T}} \|R-\Pi^\beta_\gamma R\|_{L^2_\beta(\gamma)}\Big),
$$

which implies (4.72).

$\square$

Theorem 4.18. *Let $\tilde{\eta}_T$ be the modified error indicator defined in (2.8). Then*

$$\tilde{\eta}_T \leq C\,(p+1)^{1-\beta}\,|||e|||_T + (p+1)^{-\beta}\,\|f - \Pi_T^\beta f\|_{L^2_\beta(T)} \tag{4.73}$$

or

$$|||e|||_T \geq C\Big(\tilde{\eta}_T - (p+1)^{-1}\,\|f - \Pi_T^\beta f\|_{L^2_\beta(T)}\Big). \tag{4.74}$$

Proof. Let $v = W_{\beta,T} r_p$, where r_p is $L^2_\beta(T)$ projection of r on $\mathcal{P}_p(T)$. Then v vanishes on ∂T and is extend by zero extension outside of T. Substituting v into (4.41), we have

$$B(e,v)_T = (r,v) = \|r_p\|^2_{L^2_\beta(T)} + (f - f_p, v) \leq \|v\|_{H^{1,-\beta}(T)}\,|||e|||_T \tag{4.75}$$

Since $r_p \in \mathcal{P}_p(T)$, (4.55) and (4.59) hold, i.e.

$$\|v\|_{L^2_{-\beta}(T)} = \|r_p\|_{L^2_\beta(T)} \tag{4.76}$$

and

$$\|v\|_{H^{1,-\beta}(T)} \leq C\,(p+1)\,\|r_p\|_{L^2_\beta(T)}, \tag{4.77}$$

which lead to immediately

$$\|r_p\|^2_{L^2_\beta(T)} \leq C\,(p+1)\,\|r_p\|_{L^2_\beta(T)}\,|||e|||_T + \|f - \Pi_T^\beta f\|_{L^2_\beta(T)}\,\|r_p\|_{L^2_\beta(T)}.$$

Then (4.73) and (4.74) follow immediately. □

Theorem 4.19. *Let $\tilde{\eta}_\gamma$ be the modified error indicator on internal edge $\gamma = \bar{T}_1 \cap \bar{T}_2$, then*

(4.78)

$$\tilde{\eta}_\gamma \leq C\,\Gamma^{\frac{1}{2}}(1-\beta) \sum_{T\in Q_\gamma} \Big((p+1)^\beta |||e|||_T + (p+1)^{-1}\|f-f_p\|_{L^2_\beta(T)}\Big) + (p+1)^{-\beta}\,\|R-R_p\|_{L^2_\beta(\gamma)}$$

or

(4.79)

$$\sum_{T\in Q_\gamma} |||e|||_T \geq C\,\Gamma^{-\frac{1}{2}}(1-\beta)(p+1)^{-\beta}\Big(\eta_\gamma - p^{-\beta}\,\|R-R_p\|_{L^2_\beta(\gamma)}\Big) - (p+1)^{-\beta-1} \sum_{T\in Q_\gamma} \|f-f_p\|_{L^2_\beta(T)}.$$

Proof. Suppose $\gamma = \{(0,y) | 0 \leq y \leq 1\} = \bar{T}_1 \bigcap \bar{T}_2$ as shown in Fig 4.3. Let R_p be the $L^2_\beta(\gamma)$ projection of R on $\mathcal{P}(\gamma)$, and let $\psi_\gamma = W_{\beta,\gamma} R_p = y^\beta (1-y)^\beta R_p$ with $\beta \in (0,1)$ and let $v_\gamma = \psi_\gamma(y) l^{-\beta}_{p,0}(2|\frac{x}{1-y}| - 1)$. Then v_γ vanishes on ∂Q_γ, and it can be extended by a zero extension outside of Q_γ.

Since $R_p \in \mathcal{P}_p(\gamma)$, (4.63)-(4.70) are valid, i.e. for $m = 1, 2$

$$\|v_\gamma\|_{L^2_{-\beta}(T_m)} \leq C\,\Gamma^{1/2}(1-\beta)\,(p+1)^{\beta-1}\,\|R_p\|_{L^2_\beta(\gamma)} \tag{4.80}$$

and

$$|v_\gamma|_{H^{1,-\beta}(T_m)} \leq C\,\Gamma^{1/2}(1-\beta)\,(p+1)^{2\beta}\,\|R_p\|_{L^2_\beta(\gamma)}. \tag{4.81}$$

For v_γ there holds

$$\begin{aligned} B(e, v_\gamma) &= \sum_{m=1,2} \int_{T_m} r\, v_\gamma dx + \int_\gamma R\, v_\gamma ds \\ &= \sum_{m=1,2} \int_{T_m} r_p\, v_\gamma dx + \int_{T_m} (r - r_p)\, v_\gamma dx + \int_\gamma R_p\, v_\gamma ds + \int_\gamma (R - R_p)\, v_\gamma ds \end{aligned}$$

which together with (4.80)-(4.81) gives

$$\begin{aligned} &\|R_p\|_{L^2_\beta(\gamma)} = \int_\gamma R_p\, v_\gamma ds \\ &= \sum_{m=1,2} \Big(B(e, v_\gamma)_{T_m} - \int_{T_m} r_p\, v_\gamma dx + \int_{T_m} (f_p - f)\, v_\gamma dx \Big) + \int_\gamma (R_p - R)\, v_\gamma ds \\ &\leq \sum_{m=1,2} \Big(|||e|||_{T_m} \|v_\gamma\|_{H^{1,-\beta}(T_m)} + \|r_p\|_{L^2_\beta(T_m)} \|v_\gamma\|_{L^2_{-\beta}(T_m)} \Big) \\ &\quad + \sum_{m=1,2} \|f - f_p\|_{L^2_\beta(T_m)} \|v_\gamma\|_{L^2_{-\beta}(T_m)} + \|R_p - R\|_{L^2_\beta(\gamma)} \|R_p\|_{L^2_\beta(\gamma)}. \end{aligned}$$

Due to (4.80) and (4.81) we further have

$$\begin{aligned} \|R_p\|_{L^2_\beta(\gamma)} &\leq C\, \Gamma^{\frac{1}{2}}(1-\beta)\,(p+1)^\beta \|R_p\|_{L^2_\beta(\gamma)} \sum_{m=1,2} \Big((p+1)^\beta |||e|||_{T_m} + (p+1)^{-1} \|f - f_p\|_{L^2_\beta(T_m)} \Big) \\ &\quad + \|R_p - R\|_{L^2_\beta(\gamma)} \|R_p\|_{L^2_\beta(\gamma)}, \end{aligned}$$

which implies to (4.78) and (4.79).

□

CHAPTER 5

The computation of Finite Element Method of p-version

The finite element solution $u_N \in S_N \subset E(\Omega)$ satisfies

$$B(u_N, v) = F(v) \qquad \forall v \in S_N. \tag{5.1}$$

Selecting basis functions $\varphi_i(x,y), 1 \le i \le N$ and substituting

$$u_N = \sum_{j=1}^{N} c_j \varphi_j,$$

and

$$v = \varphi_i, \quad i = 1, 2, \cdots, N.$$

into (5.1), we have

$$B(\sum_{j=1}^{N} c_j \varphi_j, \varphi_i) = F(\varphi_i), \quad i = 1, 2, \cdots, N.$$

Since B is bilinear,we get

$$\sum_{j=1}^{N} c_j B(\varphi_j, \varphi_i) = F(\varphi_i), \quad i = 1, 2, \cdots, N.$$

Letting $A_{ij}^{N\times N} = (a_{ij})_{i,j=1}^{N}$ with $a_{ij} = B(\varphi_j, \varphi_i) = B(\varphi_i, \varphi_j)$, and $b^{N\times 1} = (b_1, b_2, \cdots, b_N)^T$ with $b_i = F(\varphi_i)$, we have finite element formulation in matrix form

$$A\vec{C} = b \tag{5.2}$$

with $\vec{C} = (c_1, c_2, \cdots, c_N)^T$.

In this chapter we concentrate on computational aspects of the finite element solutions, including the computation of the matrix A and the vector b, numerical integration, the structure of indicator of finite element algorithm.

We consider the base of the finite element space $S_N \subset H^1(\Omega)$ (or $H_0^1(\Omega)$). Usually piecewise polynomials are used for the base of S_N, we need to divide the domain Ω into small subdomains $\Omega_i, 1 \leq i \leq M$. The union of subdomains $\bigcup_{i=1}^{M} \Omega_i$ is called a mesh on Ω or a partition of Ω. Each subdomain Ω_i is called an element. Ω_i is a triangle or a quadrilateral (curved or straight line), and there holds

$$\bar{\Omega} = \bigcup_{i=1}^{M} \bar{\Omega}_i$$

We now construct basis functions of S_N. The global basis functions for S_N, which are piecewise and continuous, are defined on each element Ω_m, then are pieced together on Ω. For efficiency of the program, modern finite element method use the following strategy in construction of basis functions:

(1) Select a standard element(Master element),and define a series of standard function: $N_j(\xi,\eta)$, which are polynomial in ξ and η of degree p_i. These polynomials are called shape functions. There are two type of standard elements : a square $S = (-1,1)\times(-1,1)$ and an equilateral triangle T.

(2) Define a series mappings M_m of S (or T) onto $\Omega_m, 1 \le m \le M$,

$$M_m : x = X_m(\xi,\eta), y = Y_m(\xi,\eta).$$

The inverse M_m^{-1} is written as

$$M_m^{-1} : \xi = \Phi_m(x,y), \eta = \Psi_m(x,y).$$

(3) Define local basis functions $\phi_{m,j}(x,y)$ are "pull back polynomial" defined on S or T.

$$\phi_{m,j} = N_j(M_m^{-1}(x,y)) = N_j(\Phi_m(x,y),\Psi_m(x,y)).$$

(4) Global basis function $\varphi_i(x,y)$ combining local basis functions $\phi_{m,l}$ such that $\varphi_i(x,y)$ is continuous and $supp.\varphi_i(x,y)$ is as small as possible, i.e., the global basis functions are zero almost everywhere except a few elements, where $supp.\varphi_i(x,y)$ is a measurement of the set $\{(x,y) : \varphi_i(x,y) \neq 0\}$.

The shape functions $N_i(\xi,\eta)$ are polynomials defined on the standard square $S = (-1,1)\times(-1,1)$ or triangle $T = \{(\xi,\eta) \mid 0 \le \xi,\eta,\xi+\eta \le 1\}$ with vertices P_i and sides $\Gamma_i, 1 \le i \le 4$ or 3 as shown.

1. Local basis functions

Let M_m be the mapping: $S($ or $T) \longrightarrow \Omega_m$,

$$\mathbf{M}_m : \begin{cases} x = \mathbf{X}_m(\xi, \eta) \\ y = \mathbf{Y}_m(\xi, \eta) \end{cases},$$

and let M_m^{-1} be its inverse: $\Omega_m \longrightarrow S($ or $T)$,

$$\mathbf{M}_m^{-1} : \begin{cases} \xi = \mathbf{\Phi}_m(x, y) \\ \eta = \mathbf{\Psi}_m(x, y) \end{cases}.$$

The local basis functions on the element Ω_m are defined as images of shape functions under the inverse mapping M_m^{-1}, i.e.

$$\phi_{m,i} = N_i(\mathbf{\Phi}_m(x, y), \mathbf{\Psi}_m(x, y)) \tag{5.3}$$

where $N_i(\xi, \eta)$ are the shape functions on $S($ or $T)$, $1 \leq m \leq M$ and $1 \leq i \leq I(p)$. $I(p)$ is the number of local basis functions for the polynomial degree p on the element Ω_m with $I(p) = 4p$ for $p \leq 3$ and $I(p) = 4p + \frac{(p-2)(p-3)}{2}$ for $p \geq 4$ on a quadrilateral element, and $I(p) = 3p$ for $p \leq 2$ and $I(p) = 3p + \frac{(p-1)(p-2)}{2}$ for $p \geq 3$ on a triangular element.

2. Global basis functions

The N-dimensional finite element subspace S^N has N global basis functions: $\varphi_1(x,y), \varphi_2(x,y), \cdots,$ $\varphi_N(x,y)$. Those global basis functions are selected such that

$$\varphi_i(x,y) = \begin{cases} \phi_{m,j}(x,y) & \text{on a few elements } \Omega_m \\ 0 & \text{on most elements} \end{cases}$$

and $\varphi_i(x,y)$ is a continuous piecewise polynomial with the smallest possible support.

1. Stiffness matrix

The finite element formulation results in a linear system

$$AC = b$$

where $A = (a_{ij})_{i,j=1}^N$ and $b = (b_1, b_2, \cdots b_N)^T$ with

$$a_{ij} = B(\varphi_i, \varphi_j) = \int_\Omega (\nabla\varphi_i \cdot \nabla\varphi_j + c\varphi_i\varphi_j)dxdy$$

and

$$b_i = F(\varphi_i) = \int_\Omega f\varphi_i dxdy + \int_{\Gamma_N} g\varphi_i ds.$$

We decompose A into K and M such that

$$A = K + M = (k_{ij})_{i,j=1}^N + (m_{ij})_{i,j=1}^N$$

with

$$k_{ij} = \int_{\Omega} \nabla\varphi_i \cdot \nabla\varphi_j dxdy, \quad m_{ij} = \int_{\Omega} c\varphi_i\varphi_j dxdy.$$

K, M and b are called stiffness matrix,mass matrix and load vector,respectively.

$$k_{ij} = \int_{\Omega} \nabla\varphi_i \cdot \nabla\varphi_j dxdy = \sum_{m=1}^{M_e} \int_{\Omega_m} (\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x} + \frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial y})dxdy$$

where M_e is the total number of elements. If $Supp.\varphi_i \bigcap Supp.\varphi_j = \emptyset$, or $Supp.\varphi_i \bigcap Supp.\varphi_j \neq \emptyset$ but $\Omega_m \not\subset Supp.\varphi_i \bigcap Supp.\varphi_j$, the value of integral over Ω_m is zero. If $\Omega_m \subset Supp.\varphi_i \bigcap Supp.\varphi_j$, then

$$\begin{aligned} \int_{\Omega_m} (\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x} &+ \frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial y})dxdy = \int_S \Big\{ (\frac{\partial N_l}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial N_l}{\partial\eta}\frac{\partial\eta}{\partial x})(\frac{\partial N_n}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial N_n}{\partial\eta}\frac{\partial\eta}{\partial x}) \\ &+ (\frac{\partial N_l}{\partial\xi}\frac{\partial\xi}{\partial y} + \frac{\partial N_l}{\partial\eta}\frac{\partial\eta}{\partial y})(\frac{\partial N_n}{\partial\xi}\frac{\partial\xi}{\partial y} + \frac{\partial N_n}{\partial\eta}\frac{\partial\eta}{\partial y}) \Big\} |J_m| d\xi d\eta = k_{l,n}^{(m)} \end{aligned}$$

where $|J_m| = |det(J_m)|$,

$$J_m = \begin{pmatrix} \frac{\partial X_m}{\partial\xi} & \frac{\partial X_m}{\partial\eta} \\ \frac{\partial Y_m}{\partial\xi} & \frac{\partial Y_m}{\partial\eta} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}.$$

$x = X_m(\xi, \eta)$ and $y = Y_m(\xi, \eta)$ are the mapping functions for M_m.

In practical computation we compute the local quantity for $k_{l,n}^{(m)}$ in each element Ω_m, which form local stiffness matrices $K^{(m)}$, then contribute these local quantities to k_{ij} of the global stiffness matrix.

Computation of local stiffness matrix

$$K^{(m)} = \begin{pmatrix} k_{11}^{(m)} & \cdots & k_{1,n_m}^{(m)} \\ \cdots & \cdots & \cdots \\ k_{n_m,1}^{(m)} & \cdots & k_{n_m,n_m}^{(m)} \end{pmatrix}$$

where n_m is the number of shape functions for degree p_m, which is called element degree of freedom (EDOF).

The each entry of $K^{(m)}$ is given

$$k_{l,n}^{(m)} = \int_S \Big\{ (\frac{\partial N_l}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_l}{\partial \eta}\frac{\partial \eta}{\partial x})(\frac{\partial N_n}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_n}{\partial \eta}\frac{\partial \eta}{\partial x}) + (\frac{\partial N_l}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_l}{\partial \eta}\frac{\partial \eta}{\partial y})(\frac{\partial N_n}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_n}{\partial \eta}\frac{\partial \eta}{\partial y})\Big\} |J_m| d\xi d\eta$$

where N_l and N_n are shape functions,$1 \leq l, n \leq n_m$.

2. Mass matrix

Similarly,we compute $M = (m_{ij})_{i,j=1}^{N}$ with

$$m_{ij} = \sum_{\Omega_m \subset Supp.\varphi_i \cap Supp.\varphi_j} \int_{\Omega_m} c\varphi_i\varphi_j dxdy$$

If $Supp.\varphi_i \cap Supp.\varphi_j = \emptyset$, or $\Omega_m \not\subset Supp.\varphi_i \cap Supp.\varphi_j$, then $m_{ij} = 0$. For $\Omega_m \subset Supp.\varphi_i \cap Supp.\varphi_j$

$$\int_{\Omega_m} c\varphi_i\varphi_j dxdy = \int_S cN_l N_n |J_m| d\xi d\eta = m_{ln}^{(m)}$$

where $|J_m|$ is the same as in computation of stiffness matrix.

These local quantities $m_{l,n}^{(m)}$ form a local mass matrix

$$M^{(m)} = (m_{l,n}^{(m)})_{l,n=1}^{N_m}.$$

3. Load vector

According to finite element formulation, the component of the load vector

$$\begin{aligned} b_i &= \int_\Omega f\varphi_i dxdy + \int_{\Gamma_N} g\varphi_i ds \\ &= \sum_{m=1}^{M_e} \int_{\Omega_m} f\varphi_i dxdy + \sum_{\partial\Omega_m\cap\Gamma_N\neq\phi} \int_{\Gamma_N\cap\partial\Omega_m} g\varphi_i ds \\ &= f_i + g_i \end{aligned}$$

We denote $(f_1, f_2, \cdots, f_N)^T$ and $(g_1, g_2, \cdots, g_N)^T$ by F and G.

Obviously, $f_i = 0$ if $\Omega_m \not\subset Supp.\varphi_i$. For $\Omega_m \subset Supp.\varphi_i$

$$\int_{\Omega_m} f\varphi_i dxdy = \int_S f(X_m(\xi,\eta), Y_m(\xi,\eta)) N_l(\xi,\eta)|J_m| d\xi d\eta = f_l^{(m)}$$

where $|J_m|$ is the same as above, The local quantities $f_l^{(m)}$ form a local vector $F^{(m)} = (f_1^{(m)}, \cdots, f_{N_m}^{(m)})^T$ with

$$f_l^{(m)} = \int_S f(X_m(\xi,\eta), Y_m(\xi,\eta)) N_l(\xi,\eta) |J_m| d\xi d\eta.$$

5.1. Computation of the Indicator and estimator with the FEM Solution

By e,e_i,r,r_i and R,R_{γ_l} we denote the error, residue and jump of normal derivative along the internal edges γ_l for the finite element solution u_{FE}:

$$\begin{aligned} e &= u - u_{FE}, \qquad e_i = e|_{\Omega_i}; \\ r &= f + \Delta u_{FE} - u_{FE}, r_i = r|_{\Omega_i}; \\ R &= \Big[\frac{\partial u_{FE}}{\partial n}\Big], \quad R_{\gamma_l} = \Big[\frac{\partial u_{FE}}{\partial n}\Big]|_{\gamma_l}. \end{aligned}$$

A local error indicator η_{Ω_i} associated with the residual r_i is defined as

$$\eta_{\Omega_i} = (p+1)^{-1} \|r_i\|_{L^2_\beta(\Omega_i)} \tag{5.4}$$

and the indicators $\eta_{\gamma_l^i}$ associated with the jump of $R = \left[\frac{\partial u_{FE}}{\partial n}\right]$ on the internal edge γ_l^i

$$\eta_{\gamma_l^i} = (p+1)^{-\beta} \|R\|_{L^2_\beta(\gamma_l^i)}. \tag{5.5}$$

The estimator η is defined as

$$\eta^2 = \sum_{\Omega_i \in \mathcal{T}} \eta^2_{\Omega_i} + \sum_{\gamma_l^i \in \partial\mathcal{T}} \eta^2_{\gamma_l^i}. \tag{5.6}$$

Suppose that $\varphi_i(x)$, $1 \le i \le N$ are global basis functions of S_N and u_{FE} is the finite element solution in S_N. Then $u_{FE} = \sum_{i=1}^N c_i\varphi_i$, and

$$\begin{aligned} \|r_i\|^2_{L^2_\beta(\Omega_i)} &= \|f + \Delta u_{FE} - u_{FE}\|^2_{L^2_\beta(\Omega_i)} \\ &= \int_{\Omega_i} (f + \Delta u_{FE} - u_{FE})^2 \widetilde{W}_\beta(x,y)dxdy \\ &= \int_K (\tilde{f} + \Delta\tilde{u}_{FE} - \tilde{u}_{FE})^2 W_\beta(\xi,\eta)|J_i|d\xi d\eta, \qquad K = S \text{ or } T, \end{aligned}$$

where

$$W_\beta(\xi,\eta) = (1-\xi^2)^\beta(1-\eta^2)^\beta, \widetilde{W}_\beta(x,y) = W_\beta \circ M_i^{-1}, \qquad K = S \tag{5.7}$$

and $W_\beta(\xi,\eta)$ is given in (3.9) for $\alpha = 0$ if $K = T$, and

$$\tilde{f} = f(X_m(\xi,\eta), Y_m(\xi,\eta)), \tilde{u}_{FE} = \sum_{i=1}^N c_i N_i(\xi,\eta).$$

And for $\Delta\tilde{u}_{FE}$, we first calculate

$$\frac{\partial u_{FE}}{\partial x} = \frac{\partial u_{FE}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_{FE}}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial u_{FE}}{\partial y} = \frac{\partial u_{FE}}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u_{FE}}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\begin{aligned}
\frac{\partial^2 u_{FE}}{\partial x^2} &= \frac{\partial}{\partial x}\Big(\frac{\partial u_{FE}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_{FE}}{\partial \eta}\frac{\partial \eta}{\partial x}\Big) \\
&= \frac{\partial}{\partial x}\Big(\frac{\partial u_{FE}}{\partial \xi}\Big)\frac{\partial \xi}{\partial x} + \frac{\partial u_{FE}}{\partial \xi}\Big(\frac{\partial^2 \xi}{\partial x^2}\Big) + \frac{\partial}{\partial x}\Big(\frac{\partial u_{FE}}{\partial \eta}\Big)\frac{\partial \eta}{\partial x} + \frac{\partial u_{FE}}{\partial \eta}\Big(\frac{\partial^2 \eta}{\partial x^2}\Big) \\
&= \frac{\partial^2 u_{FE}}{\partial \xi^2}\Big(\frac{\partial \xi}{\partial x}\Big)^2 + \frac{\partial^2 u_{FE}}{\partial \eta^2}\Big(\frac{\partial \eta}{\partial x}\Big)^2 + \frac{\partial u_{FE}}{\partial \xi}\Big(\frac{\partial^2 \xi}{\partial x^2}\Big) + \frac{\partial u_{FE}}{\partial \eta}\Big(\frac{\partial^2 \eta}{\partial x^2}\Big) + 2\frac{\partial^2 u_{FE}}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 u_{FE}}{\partial y^2} &= \frac{\partial}{\partial y}\Big(\frac{\partial u_{FE}}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u_{FE}}{\partial \eta}\frac{\partial \eta}{\partial y}\Big) \\
&= \frac{\partial}{\partial y}\Big(\frac{\partial u_{FE}}{\partial \xi}\Big)\frac{\partial \xi}{\partial y} + \frac{\partial u_{FE}}{\partial \xi}\Big(\frac{\partial^2 \xi}{\partial y^2}\Big) + \frac{\partial}{\partial y}\Big(\frac{\partial u_{FE}}{\partial \eta}\Big)\frac{\partial \eta}{\partial y} + \frac{\partial u_{FE}}{\partial \eta}\Big(\frac{\partial^2 \eta}{\partial y^2}\Big) \\
&= \frac{\partial^2 u_{FE}}{\partial \xi^2}\Big(\frac{\partial \xi}{\partial y}\Big)^2 + \frac{\partial^2 u_{FE}}{\partial \eta^2}\Big(\frac{\partial \eta}{\partial y}\Big)^2 + \frac{\partial u_{FE}}{\partial \xi}\Big(\frac{\partial^2 \xi}{\partial y^2}\Big) + \frac{\partial u_{FE}}{\partial \eta}\Big(\frac{\partial^2 \eta}{\partial y^2}\Big) + 2\frac{\partial^2 u_{FE}}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}
\end{aligned}$$

which leads to

$$\begin{aligned}
\frac{\partial^2 \tilde{u}_{FE}}{\partial x^2} &= \sum_{i=1}^{N} c_i \frac{\partial^2 N_i}{\partial \xi^2}\Big(\frac{\partial \xi}{\partial x}\Big)^2 + \sum_{i=1}^{N} c_i \frac{\partial^2 N_i}{\partial \eta^2}\Big(\frac{\partial \eta}{\partial x}\Big)^2 + \sum_{i=1}^{N} c_i \frac{\partial N_i}{\partial \xi}\Big(\frac{\partial^2 \xi}{\partial x^2}\Big) \\
&\quad + \sum_{i=1}^{N} c_i \frac{\partial N_i}{\partial \eta}\Big(\frac{\partial^2 \eta}{\partial x^2}\Big) + 2\sum_{i=1}^{N} c_i \frac{\partial^2 N_i}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \tilde{u}_{FE}}{\partial y^2} &= \sum_{i=1}^{N} c_i \frac{\partial^2 N_i}{\partial \xi^2}\Big(\frac{\partial \xi}{\partial y}\Big)^2 + \sum_{i=1}^{N} c_i \frac{\partial^2 N_i}{\partial \eta^2}\Big(\frac{\partial \eta}{\partial y}\Big)^2 + \sum_{i=1}^{N} c_i \frac{\partial N_i}{\partial \xi}\Big(\frac{\partial^2 \xi}{\partial y^2}\Big) \\
&\quad + \sum_{i=1}^{N} c_i \frac{\partial N_i}{\partial \eta}\Big(\frac{\partial^2 \eta}{\partial y^2}\Big) + 2\sum_{i=1}^{N} c_i \frac{\partial^2 N_i}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}.
\end{aligned}$$

Similarly, we have

$$\|R\|^2_{L^2_\beta(\gamma_l)} = \sum_{j=1}^{l} \|[\frac{\partial u_{FE}}{\partial n}]\|^2_{L^2_\beta(\gamma_l)},$$

in which

$$\frac{\partial u_{FE}}{\partial n}|_{\gamma_l} = \frac{\partial u_{FE}}{\partial x}\cos\widehat{nx} + \frac{\partial u_{FE}}{\partial y}\sin\widehat{ny},$$

where

$$\frac{\partial u_{FE}}{\partial x} = \sum_{i=1}^{N} c_i \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \sum_{i=1}^{N} c_i \frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial x}$$

and

$$\frac{\partial u_{FE}}{\partial y} = \sum_{i=1}^{N} c_i \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial y} + \sum_{i=1}^{N} c_i \frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial y}.$$

$\cos\widehat{nx}$ and $\cos\widehat{ny}$ are given by

$$\cos\widehat{nx} = \frac{y_2 - y_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}, \qquad \cos\widehat{ny} = \frac{x_1 - x_2}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

where γ_l is internal edge from point (x_1, y_1) to point (x_2, y_2).

In these formulas we need to calculate the $\frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y}$, $\frac{\partial^2 \xi}{\partial x^2}, \frac{\partial^2 \xi}{\partial y^2}$ and $\frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y}$, $\frac{\partial^2 \eta}{\partial x^2}, \frac{\partial^2 \eta}{\partial y^2}$. We can calculate as follows:

(i) Differentiating $x = X_m(\xi, \eta)$ and $y = Y_m(\xi, \eta)$ with respect to x,we get

$$\begin{cases} 1 = \frac{\partial X_m}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial X_m}{\partial \eta}\frac{\partial \eta}{\partial x} \\ 0 = \frac{\partial Y_m}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial Y_m}{\partial \eta}\frac{\partial \eta}{\partial x} \end{cases}, \tag{5.8}$$

which implies

$$\begin{cases} \frac{\partial \xi}{\partial x} &= \frac{\partial Y_m/\partial \eta}{det(J)} = \frac{J_{22}}{det(J)} \\ \frac{\partial \eta}{\partial x} &= \frac{-\partial Y_m/\partial \xi}{det(J)} = \frac{-J_{21}}{det(J)} \end{cases}. \tag{5.9}$$

Differentiating equation with respect to x for the second time, we get

$$\begin{cases} 0 = \frac{\partial X_m}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial X_m}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 X_m}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + \frac{\partial^2 X_m}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 \\ 0 = \frac{\partial Y_m}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial Y_m}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 Y_m}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + \frac{\partial^2 Y_m}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 \end{cases}. \tag{5.10}$$

Let

$$\begin{cases} A = -\frac{\partial^2 X_m}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 - \frac{\partial^2 X_m}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 \\ B = -\frac{\partial^2 Y_m}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 - \frac{\partial^2 Y_m}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 \end{cases}, \tag{5.11}$$

we can get

$$\begin{cases} \frac{\partial^2 \xi}{\partial x^2} = \frac{A\frac{\partial Y_m}{\partial \eta} - B\frac{\partial X_m}{\partial \eta}}{\det(J)} \\ \frac{\partial^2 \eta}{\partial x^2} = \frac{-A\frac{\partial Y_m}{\partial \xi} + B\frac{\partial X_m}{\partial \xi}}{\det(J)} \end{cases}. \tag{5.12}$$

(ii) Differentiating $x = X_m(\xi, \eta)$ and $y = Y_m(\xi, \eta)$ with respect to y,we have

$$\begin{cases} 0 = \frac{\partial X_m}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial X_m}{\partial \eta}\frac{\partial \eta}{\partial y} \\ 1 = \frac{\partial Y_m}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial Y_m}{\partial \eta}\frac{\partial \eta}{\partial y} \end{cases}, \tag{5.13}$$

which implies

$$\begin{cases} \frac{\partial \xi}{\partial y} &= \frac{-\partial X_m/\partial \eta}{det(J)} = \frac{-J_{12}}{det(J)} \\ \frac{\partial \eta}{\partial y} &= \frac{\partial X_m/\partial \xi}{det(J)} = \frac{J_{11}}{det(J)} \end{cases}. \tag{5.14}$$

Differentiating equation with respect to y for the second time, we get

$$\begin{cases} 0 = \frac{\partial X_m}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial X_m}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} + \frac{\partial^2 X_m}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + \frac{\partial^2 X_m}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 \\ 0 = \frac{\partial Y_m}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial Y_m}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} + \frac{\partial^2 Y_m}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + \frac{\partial^2 Y_m}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 \end{cases}. \tag{5.15}$$

Let

$$\begin{cases} C = -\frac{\partial^2 X_m}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 - \frac{\partial^2 X_m}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 \\ D = -\frac{\partial^2 Y_m}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 - \frac{\partial^2 Y_m}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 \end{cases}, \tag{5.16}$$

we can get

$$\begin{cases} \frac{\partial^2 \xi}{\partial y^2} = \frac{C\frac{\partial Y_m}{\partial \eta} - D\frac{\partial X_m}{\partial \eta}}{\det(J)} \\ \frac{\partial^2 \eta}{\partial y^2} = \frac{-C\frac{\partial Y_m}{\partial \xi} + D\frac{\partial X_m}{\partial \xi}}{\det(J)} \end{cases}, \tag{5.17}$$

which leads what we want.

5.2. Gauss-Jacobi-Quadrature

In computation of error indicator η_i we need to use Gauss-Jacobi quadrature on $S = (-1,1)^2$

$$\int_S F(\xi,\eta)(1-\xi^2)^\alpha(1-\eta^2)^\beta d\xi d\eta \approx \sum_{j=1}^{NG\eta}\sum_{i=1}^{NG\xi} w_i w_j F(\xi_i,\eta_j) \tag{5.18}$$

where ξ_i and η_j are Gauss-Jacobi point and w_i and w_j are Gauss-Jacobi weight. The number $NG\xi$ and $NG\eta$ of Gauss-Jaocbi points are determined in the following principle:

Suppose that $F(\xi,\eta)$ is a polynomial of separate degree less or equal to $2p$ the Gauss-Jaocbi quadrature is exact. Hence $2p \leq 2NG\xi - 1$, $2p \leq 2NG\eta - 1$ by the property of Gauss-Jaocbi quadrature [**1**] , i.e. $NG\xi \geq p + \frac{1}{2}$, $NG\eta \geq p + \frac{1}{2}$. Therefor we usually take $NG\xi = NG\eta = p + 1$. If the function $F(\xi,\eta)$ is not very smooth we may select $NG\xi = NG\eta = p + 2, p + 3, \cdots$.

For special values of α and β e.g. $\alpha = \beta = 0$ or $\alpha = \beta = -\frac{1}{2}$, the Gauss-Jacobi point and Gauss-Jacobi weight can be found in Mathematical Handbook, e.g [**1**]. For general α,β, we have to develop an effective algorithm to compute the Gauss-Jacobi point and weight.

The Gauss-Jacobi points ξ_i and η_j are the roots of Jacobi polynomial $J_n^\alpha(x)$, and the roots are distinct on $(-1,1)$. The roots x_i, $i = 1, 2, \cdots, NG$ satisfy the algebraic equation

$$J_n^\alpha(x_i) = 0, \qquad i = 1, 2, \cdots, NG. \tag{5.19}$$

We use Newton iteration method to solve for x_i, $i = 1, \cdots, n$

$$\begin{cases} x_i^{(k+1)} = x_i^{(k)} - \frac{J_n^\alpha(x_i^{(k)})}{\frac{d}{dx}J_n^\alpha(x_i^{(k)})} = x_i^{(k)} - \frac{J_n^\alpha(x_i^{(k)})}{\frac{n+2\alpha+1}{2}J_{n-1}^{\alpha+1}(x_i^{(k)})}, \\ x_i^{(0)} = z_i. \end{cases} \tag{5.20}$$

where z_i are initial guess for the root x_i. Since Newton method is very sensitive to the selection of the initial point, we have to select z_i very carefully such that the Newton method converge to each root x_i. We adopt the following algorithm to compute z_i, $i = 1, \cdots, n$, [**45, 2, 42**].

For $i = 1$

$$\begin{aligned} z_1 &= 1 - \frac{r_1}{r_2} \\ r_1 &= (1+\alpha)(\frac{2.78}{4+N^2} + 0.76\frac{\alpha}{N^2}) \\ r_2 &= 1 + 1.48\frac{\alpha}{N} + 0.96\frac{\beta}{N} + 0.452\frac{\alpha^2}{N^2} + 0.83\frac{\alpha\beta}{N^2}; \end{aligned} \tag{5.21}$$

for $i = 2$

$$\begin{aligned} z_2 &= z_1 - (1-z_1)r_1r_2r_3 \\ r_1 &= \frac{4.1+\alpha}{(1+\alpha)(1+0.156\alpha)} \\ r_2 &= 1 + 0.06(N-8)\frac{1+0.12\alpha}{N} \\ r_3 &= 1 + 0.012\beta\frac{1+0.25|\alpha|}{N}; \end{aligned} \tag{5.22}$$

for $i = 3$

$$z_3 = z_2 - (x_1 - z_2) r_1 r_2 r_3 \tag{5.23}$$
$$r_1 = \frac{1.67 + 0.28\alpha}{1 + 0.37\alpha}$$
$$r_2 = 1 + 0.22 \frac{N-8}{N}$$
$$r_3 = 1 + \frac{8\beta}{(6.28 + \beta) N^2};$$

for $4 \leq i \leq N-2$

$$z_i = 3x_{i-1} - 3x_{i-2} + x_{i-3}; \tag{5.24}$$

for $i = N-1$

$$z_{N-1} = z_{N-2} + (z_{N-2} - x_{N-3}) r_1 r_2 r_3 \tag{5.25}$$
$$r_1 = \frac{1 + 0.235\beta}{0.766 + 0.119\beta}$$
$$r_2 = \frac{1}{1 + \frac{0.639(N-4)}{1+0.71(N-4)}}$$
$$r_3 = \frac{1}{1 + \frac{20\alpha}{(7.5+\alpha)N^2}};$$

and for $i = N$

$$
\begin{aligned}
z_N &= z_{N-1} + (z_{N-1} - x_{N-2}) r_1 r_2 r_3 \\
r_1 &= \frac{1 + 0.37\beta}{1.67 + 0.28\beta} \\
r_2 &= \frac{1}{1 + \frac{0.22(N-8)}{N}} \\
r_3 &= \frac{1}{1 + \frac{8\alpha}{(6.28+\alpha)N^2}}.
\end{aligned}
\tag{5.26}
$$

Using this series of z_i, we solve the equation $J_N^{\alpha,\beta}(x_i) = 0$, $1 \le i \le N$ by Newton-method to be initial value of x_i. To calculate the weight w_i, corresponding Gauss-Jacobi point x_i, based on the Gauss-Jacobi quadrature for polynomial of degree $\le N-1$

$$\int_{-1}^{1} J_k^{\alpha,\beta}(x) W_{\alpha,\beta}(x) dx = \sum_{i=1}^{N} J_k^{\alpha,\beta}(x_i) w_i, \quad for \quad 0 \le k \le N-1$$

we need to solve the linear equation system

$$
\begin{pmatrix}
J_0^{\alpha,\beta}(x_1) & J_0^{\alpha,\beta}(x_2) & \cdots & J_0^{\alpha,\beta}(x_N) \\
J_1^{\alpha,\beta}(x_1) & J_1^{\alpha,\beta}(x_2) & \cdots & J_1^{\alpha,\beta}(x_N) \\
\vdots & \vdots & \vdots & \vdots \\
J_{N-1}^{\alpha,\beta}(x_1) & J_{N-1}^{\alpha,\beta}(x_2) & \cdots & J_{N-1}^{\alpha,\beta}(x_N)
\end{pmatrix}
\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix}
=
\begin{pmatrix}
\int_{-1}^{1} J_0^{\alpha,\beta}(x) W_{\alpha,\beta}(x) dx \\
\int_{-1}^{1} J_1^{\alpha,\beta}(x) W_{\alpha,\beta}(x) dx \\
\vdots \\
\int_{-1}^{1} J_{N-1}^{\alpha,\beta}(x) W_{\alpha,\beta}(x) dx
\end{pmatrix}.
\tag{5.27}
$$

Note that $J_0^{\alpha,\beta}(x) = 1$ and orthogonal to $J_k^{\alpha,\beta}(x)$, $1 \le k \le N-1$ with weight function $W_{\alpha,\beta}(x)$, it is easily to seen

$$\int_{-1}^{1} J_k^{\alpha,\beta}(x) W_{\alpha,\beta}(x) dx = 0 \quad \text{for} \quad 1 \le k \le N-1,$$

which leads:

$$\begin{pmatrix} J_0^{\alpha,\beta}(x_1) & J_0^{\alpha,\beta}(x_2) & \cdots & J_0^{\alpha,\beta}(x_N) \\ J_1^{\alpha,\beta}(x_1) & J_1^{\alpha,\beta}(x_2) & \cdots & J_1^{\alpha,\beta}(x_N) \\ \vdots & \vdots & \vdots & \vdots \\ J_{N-1}^{\alpha,\beta}(x_1) & J_{N-1}^{\alpha,\beta}(x_2) & \cdots & J_{N-1}^{\alpha,\beta}(x_N) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix} = \begin{pmatrix} \gamma_0^{\alpha,\beta} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{5.28}$$

Solve the linear system equation we will get the weight value w_i.

5.3. Structure of Computation Program

After computing the finite element method solution u_{FE}, the indicate can be computed.

First, the internal can be insured by the following algorithm:

For m=1 to MaxElementNum

initialize $Eta_{in}[m]$;

cal=0;

$p = p[i]$, $NG = p + 1$, calculate Jacobipoint[g], Jacobiweight[g] $g = 1, 2, \cdots NG$ $(*1)$

determine Ndof of Ω_m;

for g1=1 to NG

for g2=1 to NG

s=0;

$\xi = Jaocobipoint[g1]$, $w_1 = Jacobipoint[g1]$;

$\eta = Jacobipoint[g2]$, $w_2 = Jacobipoint[g2]$;

$functionvalue = f(X_m(\xi,\eta), Y_m(\xi,\eta))$;

$s = functionvalue$;

$DUD\xi = \sum_{n=1}^{NDOF} C[Nic(m,n)]DND\xi(n,\xi,\eta)$;

$DUD\eta = \sum_{n=1}^{NDOF} C[Nic(m,n)]DND\eta(n,\xi,\eta)$;

$DDUDD\xi = \sum_{n=1}^{NDOF} C[Nic(m,n)]DDNDD\xi(n,\xi,\eta)$;

$DDUDD\eta = \sum_{n=1}^{NDOF} C[Nic(m,n)]DDNDD\eta(n,\xi,\eta)$;

$DDUD\xi D\eta = \sum_{n=1}^{NDOF} C[Nic(m,n)]DDND\xi D\eta(n,\xi,\eta)$;

$J_{11} = \frac{\partial X_m}{\partial \xi}(\xi,\eta)$,$J_{12} = \frac{\partial X_m}{\partial \eta}(\xi,\eta)$ (*2)

$J_{21} = \frac{\partial Y_m}{\partial \xi}(\xi,\eta)$,$J_{22} = \frac{\partial Y_m}{\partial \eta}(\xi,\eta)$

$detJ = \sqrt{J_{11}^2 + J_{21}^2}$;

$B_{11} = \frac{J_{22}}{detJ}, \qquad B_{12} = -\frac{J_{12}}{detJ}$

$B_{21} = -\frac{J_{21}}{detJ}, \qquad B_{22} = \frac{J_{11}}{detJ}$

$A = -\frac{\partial^2 X_m}{\partial \xi^2}(\xi,\eta)B_{11}^2 - -\frac{\partial^2 X_m}{\partial \eta^2}(\xi,\eta)B_{21}^2$; (*3)

$B = -\frac{\partial^2 Y_m}{\partial \xi^2}(\xi,\eta)B_{11}^2 - -\frac{\partial^2 Y_m}{\partial \eta^2}(\xi,\eta)B_{21}^2$;

$C = -\frac{\partial^2 X_m}{\partial \xi^2}(\xi,\eta)B_{12}^2 - -\frac{\partial^2 X_m}{\partial \eta^2}(\xi,\eta)B_{22}^2$;

$D = -\frac{\partial^2 Y_m}{\partial \xi^2}(\xi,\eta)B_{12}^2 - -\frac{\partial^2 Y_m}{\partial \eta^2}(\xi,\eta)B_{22}^2$;

$C_{11} = \frac{AJ_{22} - BJ_{12}}{detJ}, \qquad C_{12} = \frac{-AJ_{21} + BJ_{11}}{detJ}$

$C_{21} = \frac{CJ_{22} - DJ_{12}}{detJ}, \qquad C_{22} = \frac{-CJ_{21} + DJ_{11}}{detJ}$

$DELTU = DDUDD\xi\Big(B_{11}^2 + B_{12}^2\Big) + DDUDD\eta\Big(B_{21}^2 + B_{22}^2\Big)$

$DELTU = DELTU + DUD\xi\Big(C_{11} + C_{21}\Big) + DUD\eta\Big(C_{12} + C_{22}\Big)$

$DELTU = DELTU + 2DDUD\xi D\eta\Big(B_{11}B_{21} + B_{12}B_{22}\Big)$

$s = s + DELTU;$

$s = s - \sum_{n=1}^{NDOF} C[Nic(m, n)] \times N(n, \xi, \eta);$

$cal = cal + s^2 \times w_1 \times w_2 \times detJ;$

end

end

$Eta_{in}[m] = cal;$

end

where array $Eta_{in}[m]$, $m = 1, \cdots, MaxElemntNum$ return the internal indicator for each element.

Remark

(∗1) If uniform degree p is deployed, then $NG = p+1$ and Jacobipoints[g] and Jacobiweight[g] $g = 1, 2, \cdots NG$ should be calculated outside of the side-loop to avoid repeating the computation of Jacobi points and weights, which is quite cpu consuming.

(∗2) $XX(t), YY(t)$: x,y coordinate for nodes of element Ω_m which are needed for mapping functions $X_m(\xi, \eta)$ and $Y_m(\xi, \eta)$.

(∗3) For quadrilateral element with straight-line sides

$$\frac{\partial X_m}{\partial \xi}(\xi,\eta) = \sum_{i=4}^{4} XX(i)\partial N\partial\xi(i,\xi,\eta), \qquad \frac{\partial X_m}{\partial \eta}(\xi,\eta) = \sum_{i=4}^{4} XX(i)\partial N\partial\eta(i,\xi,\eta),$$

$$\frac{\partial Y_m}{\partial \xi}(\xi,\eta) = \sum_{i=4}^{4} YY(i)\partial N\partial\xi(i,\xi,\eta), \qquad \frac{\partial Y_m}{\partial \eta}(\xi,\eta) = \sum_{i=4}^{4} YY(i)\partial N\partial\eta(i,\xi,\eta),$$

and

$$\begin{aligned}\frac{\partial^2 X_m}{\partial \xi^2}(\xi,\eta) &= \sum_{i=4}^{4} XX(i)\partial^2 N\partial\xi^2(i,\xi,\eta), \qquad \frac{\partial^2 X_m}{\partial \eta^2}(\xi,\eta) = \sum_{i=4}^{4} XX(i)\partial^2 N\partial\eta^2(i,\xi,\eta),\\ \frac{\partial^2 Y_m}{\partial \xi^2}(\xi,\eta) &= \sum_{i=4}^{4} YY(i)\partial^2 N\partial\xi^2(i,\xi,\eta), \qquad \frac{\partial^2 Y_m}{\partial \eta^2}(\xi,\eta) = \sum_{i=4}^{4} YY(i)\partial^2 N\partial\eta^2(i,\xi,\eta).\end{aligned}$$

For quadrilateral element with curved side

$$\gamma_l = \begin{cases} x = \psi_l(\xi) & -1 \le \xi \le 1, l = 1,3 \\ y = \chi_l(\xi) & -1 \le \xi \le 1, l = 1,3 \end{cases},$$

$$\gamma_l = \begin{cases} x = \psi_l(\eta) & -1 \le \eta \le 1, l = 2,4 \\ y = \chi_l(\eta) & -1 \le \eta \le 1, l = 2,4 \end{cases},$$

Then

$$\begin{aligned}\frac{\partial X_m}{\partial \xi} &= \frac{1}{2}(1-\eta)\psi_1'(\xi)+\frac{1}{2}\psi_2(\eta)+\frac{1}{2}(1+\eta)\psi_3'(\xi)-\frac{1}{2}\psi_4(\eta)\\ &+ \sum_{i=4}^{4} XX(i)\partial N\partial\xi(i,\xi,\eta);\end{aligned}$$

$$\begin{aligned}\frac{\partial X_m}{\partial \eta} &= -\frac{1}{2}\psi_1(\xi)+\frac{1}{2}(1+\xi)\psi_2'(\eta)+\frac{1}{2}\psi_3(\xi)+\frac{1}{2}(1-\xi)\psi_4'(\eta)\\ &+ \sum_{i=4}^{4} XX(i)\partial N\partial\eta(i,\xi,\eta);\end{aligned}$$

$$\begin{aligned}\frac{\partial Y_m}{\partial \xi} &= \frac{1}{2}(1-\eta)\chi_1'(\xi)+\frac{1}{2}\chi_2(\eta)+\frac{1}{2}(1+\eta)\chi_3'(\xi)-\frac{1}{2}\chi_4(\eta)\\ &+ \sum_{i=4}^{4} YY(i)\partial N\partial\xi(i,\xi,\eta);\end{aligned}$$

$$\begin{aligned}\frac{\partial X_m}{\partial \eta} &= -\frac{1}{2}\chi_1(\xi)+\frac{1}{2}(1+\xi)\chi_2'(\eta)+\frac{1}{2}\chi_3(\xi)+\frac{1}{2}(1-\xi)\chi_4'(\eta)\\ &+ \sum_{i=4}^{4} YY(i)\partial N\partial\eta(i,\xi,\eta);\end{aligned}$$

and

$$\frac{\partial^2 X_m}{\partial \xi^2} = \frac{1}{2}(1-\eta)\psi_1''(\xi)+\frac{1}{2}(1+\eta)\psi_3''(\xi)+\sum_{i=4}^{4} XX(i)\partial^2 N\partial\xi^2(i,\xi,\eta);$$

$$\frac{\partial^2 X_m}{\partial \eta^2} = \frac{1}{2}(1+\xi)\psi_2''(\eta) + \frac{1}{2}(1-\xi)\psi_4''(\eta) + \sum_{i=4}^{4} XX(i)\partial^2 N \partial \eta^2 (i,\xi,\eta);$$

$$\frac{\partial^2 Y_m}{\partial \xi^2} = \frac{1}{2}(1-\eta)\chi_1''(\xi) + \frac{1}{2}(1+\eta)\chi_3''(\xi) + \sum_{i=4}^{4} YY(i)\partial^2 N \partial \xi^2 (i,\xi,\eta);$$

$$\frac{\partial^2 X_m}{\partial \eta^2} = \frac{1}{2}(1+\xi)\chi_2''(\eta) + \frac{1}{2}(1-\xi)\chi_4''(\eta) + \sum_{i=4}^{4} YY(i)\partial^2 N \partial \eta^2 (i,\xi,\eta).$$

There is an informal algorithm for the boundary indicator:

For l=1 to MaxElesideNum

initialize $Eta_{bd}[l]$;

i=sideNb(l,1), j=sideNb(l,2)

$p = \max\{p(i), p(j)\}$, $NG = p + 1$, calculate Jacobipoint[g1], Jacobiweight[g1] $g1 = 1, 2, \cdots NG$

If j=0{ (∗4)

If $Eleside[l] \subset \Gamma_N${

find k such that globeside (i,k)=l;

determine $XX(t), YY(t), 1 \le t \le 4$ Ndof (∗5)

determine $n_1 = \cos\widehat{nx}$,$n_2 = \cos\widehat{ny}$

for g1=1 to NG

$\xi = Jacobipoint[g1]$, $w = Jacobiweight[g1]$;

if k=1 {

$J_{11} = \frac{\partial X_m}{\partial \xi}(\xi,-1)$,$J_{12} = \frac{\partial X_m}{\partial \eta}(\xi,-1)$ (∗6)

$J_{21} = \frac{\partial Y_m}{\partial \xi}(\xi,-1)$,$J_{22} = \frac{\partial Y_m}{\partial \eta}(\xi,-1)$

$E = \sqrt{J_{11}^2 + J_{21}^2}$

$DUD\xi = \sum_{n=1}^{NDOF} C[Nic(m,n)]DND\xi(n,\xi,-1);$

$DUD\eta = \sum_{n=1}^{NDOF} C[Nic(m,n)]DND\eta(n,\xi,-1);$

$gfungionvalue = g(X_m(\xi,-1), Y_m(\xi,-1));$

}

elseif k=2{

$J_{11} = \frac{\partial X_m}{\partial \xi}(1,\xi)$,$J_{12} = \frac{\partial X_m}{\partial \eta}(1,\xi)$

$J_{21} = \frac{\partial Y_m}{\partial \xi}(1,\xi)$,$J_{22} = \frac{\partial Y_m}{\partial \eta}(1,\xi)$

$E = \sqrt{J_{12}^2 + J_{22}^2}$

$DUD\xi = \sum_{n=1}^{NDOF} C[Nic(m,n)]DND\xi(n,1,\xi);$

$DUD\eta = \sum_{n=1}^{NDOF} C[Nic(m,n)]DND\eta(n,1,\xi);$

$gfungionvalue = g(X_m(1,\xi), Y_m(1,\xi));$

}

elseif k=3{

$J_{11} = \frac{\partial X_m}{\partial \xi}(\xi,1)$,$J_{12} = \frac{\partial X_m}{\partial \eta}(\xi,1)$

$J_{21} = \frac{\partial Y_m}{\partial \xi}(\xi,1)$,$J_{22} = \frac{\partial Y_m}{\partial \eta}(\xi,1)$

$E = \sqrt{J_{11}^2 + J_{21}^2}$

$DUD\xi = \sum_{n=1}^{NDOF} C[Nic(m,n)]DND\xi(n,\xi,1);$

$DUD\eta = \sum_{n=1}^{NDOF} C[Nic(m,n)]DND\eta(n,\xi,1);$

$gfungionvalue = g(X_m(\xi,1), Y_m(\xi,1));$

}

elseif k=4{

$J_{11} = \frac{\partial X_m}{\partial \xi}(-1,\xi)$,$J_{12} = \frac{\partial X_m}{\partial \eta}(-1,\xi)$

$J_{21} = \frac{\partial Y_m}{\partial \xi}(-1,\xi)$,$J_{22} = \frac{\partial Y_m}{\partial \eta}(-1,\xi)$

$E = \sqrt{J_{12}^2 + J_{22}^2}$

$DUD\xi = \sum_{n=1}^{NDOF} C[Nic(m,n)]DND\xi(n,-1,\xi);$

$DUD\eta = \sum_{n=1}^{NDOF} C[Nic(m,n)]DND\eta(n,-1,\xi);$

$gfungionvalue = g(X_m(-1,\xi), Y_m(-1,\xi));$

}

$detJ = J_{11}J_{22} - J_{12}J_{21}$

$\nu_1 = \frac{J_{22}}{detJ}n_1 - \frac{J_{12}}{detJ}n_2$ (∗8)

$\nu_2 = -\frac{J_{21}}{detJ}n_1 + \frac{J_{11}}{detJ}n_2$

$DUD\nu = \nu_1 DUD\xi + \nu_2 DUD\eta$

$s = s + (gfunctionvalue - DUD\nu)^2 \times w \times E$

end

}

}

else {

for m=1 to M

If m=i or m=j{

find k such that globeside(m,k)=l;

determine $n_1 = \cos\widehat{nx}$,$n_2 = \cos\widehat{ny}$

s=0;

for g1=1 to NG

$\xi = Jacobipoint[g1]$, $w = Jacobiweight[g1]$;

if k=1 {

$J_{11} = \frac{\partial X_m}{\partial \xi}(\xi,-1)$,$J_{12} = \frac{\partial X_m}{\partial \eta}(\xi,-1)$

$J_{21} = \frac{\partial Y_m}{\partial \xi}(\xi,-1)$,$J_{22} = \frac{\partial Y_m}{\partial \eta}(\xi,-1)$

$E = \sqrt{J_{11}^2 + J_{21}^2}$

$DUD\xi = \sum_{n=1}^{NDOF} C[Nic(m,n)]DND\xi(n,\xi,-1);$

$DUD\eta = \sum_{n=1}^{NDOF} C[Nic(m,n)]DND\eta(n,\xi,-1);$

}

elseif k=2{

$J_{11} = \frac{\partial X_m}{\partial \xi}(1,\xi)$,$J_{12} = \frac{\partial X_m}{\partial \eta}(1,\xi)$

$J_{21} = \frac{\partial Y_m}{\partial \xi}(1,\xi)$,$J_{22} = \frac{\partial Y_m}{\partial \eta}(1,\xi)$

$E = \sqrt{J_{12}^2 + J_{22}^2}$

$DUD\xi = \sum_{n=1}^{NDOF} C[Nic(m,n)]DND\xi(n,1,\xi);$

$DUD\eta = \sum_{n=1}^{NDOF} C[Nic(m,n)]DND\eta(n,1,\xi);$

}

elseif k=3{

$J_{11} = \frac{\partial X_m}{\partial \xi}(\xi, 1)$,$J_{12} = \frac{\partial X_m}{\partial \eta}(\xi, 1)$

$J_{21} = \frac{\partial Y_m}{\partial \xi}(\xi, 1)$,$J_{22} = \frac{\partial Y_m}{\partial \eta}(\xi, 1)$

$E = \sqrt{J_{11}^2 + J_{21}^2}$

$DUD\xi = \sum_{n=1}^{NDOF} C[Nic(m,n)]DND\xi(n, \xi, 1);$

$DUD\eta = \sum_{n=1}^{NDOF} C[Nic(m,n)]DND\eta(n, \xi, 1);$

}

elseif k=4{

$J_{11} = \frac{\partial X_m}{\partial \xi}(-1, \xi)$,$J_{12} = \frac{\partial X_m}{\partial \eta}(-1, \xi)$

$J_{21} = \frac{\partial Y_m}{\partial \xi}(-1, \xi)$,$J_{22} = \frac{\partial Y_m}{\partial \eta}(-1, \xi)$

$E = \sqrt{J_{12}^2 + J_{22}^2}$

$DUD\xi = \sum_{n=1}^{NDOF} C[Nic(m,n)]DND\xi(n, -1, \xi);$

$DUD\eta = \sum_{n=1}^{NDOF} C[Nic(m,n)]DND\eta(n, -1, \xi);$

}

$detJ = J_{11}J_{22} - J_{12}J_{21}$

$\nu_1 = \frac{J_{22}}{detJ}n_1 - \frac{J_{12}}{detJ}n_2$

$\nu_2 = -\frac{J_{21}}{detJ}n_1 + \frac{J_{11}}{detJ}n_2$

$DUD\nu = \nu_1 DUD\xi + \nu_2 DUD\eta$

$s = s + (DUD\nu)^2 \times w \times E$

end

}

$Eta_{bd}[l] = Eta_{bd}[l] + s;$

end

}

end

where array $Eta_{bd}[m]$, $m = 1, \cdots, MaxElemntNum$ return the boundary indicator for each element.

(∗4) this side is on the boundary

(∗5) $XX(t), YY(t)$: x,y coordinate for nodes of element Ω_m which are needed for mapping functions $X_m(\xi, \eta)$ and $Y_m(\xi, \eta)$.

(∗6) For quadrilateral element with straight-line sides

$$\frac{\partial X_m}{\partial \xi}(\xi, \eta) = \sum_{i=4}^{4} XX(i)\partial N\partial\xi(i, \xi, \eta), \qquad \frac{\partial X_m}{\partial \eta}(\xi, \eta) = \sum_{i=4}^{4} XX(i)\partial N\partial\eta(i, \xi, \eta),$$
$$\frac{\partial Y_m}{\partial \xi}(\xi, \eta) = \sum_{i=4}^{4} YY(i)\partial N\partial\xi(i, \xi, \eta), \qquad \frac{\partial Y_m}{\partial \eta}(\xi, \eta) = \sum_{i=4}^{4} YY(i)\partial N\partial\eta(i, \xi, \eta).$$

For quadrilateral element with curved side

$$\gamma_l = \begin{cases} x = \psi_l(\xi) & -1 \le \xi \le 1, l = 1, 3 \\ y = \chi_l(\xi) & -1 \le \xi \le 1, l = 1, 3 \end{cases},$$

$$\gamma_l = \begin{cases} x = \psi_l(\eta) & -1 \le \eta \le 1, l = 2, 4 \\ y = \chi_l(\eta) & -1 \le \eta \le 1, l = 2, 4 \end{cases},$$

Then

$$\frac{\partial X_m}{\partial \xi} = \frac{1}{2}(1-\eta)\psi_1'(\xi) + \frac{1}{2}\psi_2(\eta) + \frac{1}{2}(1+\eta)\psi_3'(\xi) - \frac{1}{2}\psi_4(\eta) + \sum_{i=4}^{4} XX(i)\partial N\partial\xi(i,\xi,\eta);$$

$$\frac{\partial X_m}{\partial \eta} = -\frac{1}{2}\psi_1(\xi) + \frac{1}{2}(1+\xi)\psi_2'(\eta) + \frac{1}{2}\psi_3(\xi) + \frac{1}{2}(1-\xi)\psi_4'(\eta) + \sum_{i=4}^{4} XX(i)\partial N\partial\eta(i,\xi,\eta);$$

$$\frac{\partial Y_m}{\partial \xi} = \frac{1}{2}(1-\eta)\chi_1'(\xi) + \frac{1}{2}\chi_2(\eta) + \frac{1}{2}(1+\eta)\chi_3'(\xi) - \frac{1}{2}\chi_4(\eta) + \sum_{i=4}^{4} YY(i)\partial N\partial\xi(i,\xi,\eta);$$

$$\frac{\partial X_m}{\partial \eta} = -\frac{1}{2}\chi_1(\xi) + \frac{1}{2}(1+\xi)\chi_2'(\eta) + \frac{1}{2}\chi_3(\xi) + \frac{1}{2}(1-\xi)\chi_4'(\eta) + \sum_{i=4}^{4} YY(i)\partial N\partial\eta(i,\xi,\eta).$$

For $l = 1$, $\hat{n} = (\chi_1'(\xi), -\psi_1'(\xi))$,

$$\cos\widehat{nx} = \frac{\chi_1'(\xi)}{\sqrt{(\psi_1'(\xi))^2 + (\chi_1'(\xi))^2}} \quad \cos\widehat{ny} = \frac{-\psi_1'(\xi)}{\sqrt{(\psi_1'(\xi))^2 + (\chi_1'(\xi))^2}};$$

$l = 2$, $\hat{n} = (\chi_2'(\eta), -\psi_2'(\eta))$,

$$\cos\widehat{nx} = \frac{\chi_2'(\eta)}{\sqrt{(\psi_2'(\eta))^2 + (\chi_2'(\eta))^2}} \quad \cos\widehat{ny} = \frac{-\psi_2'(\eta)}{\sqrt{(\psi_2'(\eta))^2 + (\chi_2'(\eta))^2}};$$

$l = 3$, $\hat{n} = (-\chi_3'(\xi), \psi_3'(\xi))$,

$$\cos\widehat{nx} = \frac{-\chi_3'(\xi)}{\sqrt{(\psi_3'(\xi))^2 + (\chi_3'(\xi))^2}} \quad \cos\widehat{ny} = \frac{\psi_3'(\xi)}{\sqrt{(\psi_3'(\xi))^2 + (\chi_3'(\xi))^2}};$$

$l = 4$, $\hat{n} = (-\chi_4'(\eta), \psi_4'(\eta))$,

$$\cos\widehat{nx} = \frac{-\chi_4'(\eta)}{\sqrt{(\psi_4'(\eta))^2 + (\chi_4'(\eta))^2}} \quad \cos\widehat{ny} = \frac{\psi_4'(\eta)}{\sqrt{(\psi_4'(\eta))^2 + (\chi_4'(\eta))^2}}.$$

(*7) If uniform degree p is deployed, then $NG = p + 1$ and Jacobipoints[g1] and Jacobiweight[g1] $g1 = 1, 2, \cdots NG$ should be calculated outside of the side-loop to avoid repeating the computation of Jacobi points and weights, which is quite cpu consuming.

(*8) If $\cos\widehat{nx}$ and $\cos\widehat{ny}$ on edges of Ω_m are not constants, then compute ν_1 and ν_2 with J_{11}, J_{12} and J_{21}, J_{22} for different k.

5.4. Results and Analysis of Computation on quadrilateral elements

The estimator η is defined on quadrilateral elements as

$$\eta = \Big(\sum_{K_i \in \mathcal{T}} \eta_{K_i}^2\Big)^{\frac{1}{2}} + \Big(\sum_{\gamma_l^i \in \partial\mathcal{T}} \eta_{\gamma_l^i}^2\Big)^{\frac{1}{2}} = \Big(\sum_{S_i \in \mathcal{T}} \eta_{S_i}^2\Big)^{\frac{1}{2}} + \Big(\sum_{\gamma_l^i \in \partial\mathcal{T}} \eta_{\gamma_l^i}^2\Big)^{\frac{1}{2}}. \tag{5.29}$$

The main results of [**25**] are contained the following theorems on quadrilateral mesh:

Theorem 5.1. *For* $\beta \in (1/2, 1)$*, there holds*

$$|||e||| \leq C_1(p)\,\eta \tag{5.30}$$

with $C_1(p) = C_0 \log(p+1)$ *and* C_0 *independent of* p *and* β*.*

Theorem 5.2. *If* $f \in S^p(\mathcal{T})$ *and* $\beta \in (0, 1)$*, there holds*

$$|||e|||_K \geq C\,\eta_K, \forall K \in \mathcal{T}, \tag{5.31}$$

with C *independent of* p *and* β*, and if* $R\,|_\gamma \in \mathcal{P}_p(\gamma)$*, there holds*

$$\sum_{K \in Q_\gamma} |||e|||_K \geq C_2(\beta)\,\eta_\gamma, \forall \gamma \in \mathcal{T} \tag{5.32}$$

with $C_2(\beta) = C_0 \Gamma^{-1/2}(1-\beta)$ *and* C_0 *independent of* p *and* β*, where* Q_γ *is a pair of elements sharing* γ*.*

Theorem 5.3. *For* $\beta \in (1/2, 1)$*, there holds*

$$|||e||| \leq C_3(p)\left(\tilde{\eta}_K + \sum_{K \in \mathcal{T}} p^{-1} \|f - f_p\|_{L^2_\beta(K)} + \sum_{\gamma \in \partial\mathcal{T}} p^{-\beta} \|R - R_p\|_{L^2_\beta(\gamma)}\right) \tag{5.33}$$

with $C_3(p) = \tilde{C}_0 \log(p+1)$ *and* $\tilde{C}_0$ *independent of* p *and* β*.*

Theorem 5.4. *For $\beta \in (0,1)$, there hold*

$$|||e|||_K \geq C\Big(\tilde{\eta}_\gamma - p^{-\beta}\|R - R_p\|_{L^2_\beta(\gamma)}\Big), \forall K \in \mathcal{T} \tag{5.34}$$

and

$$\sum_{K \in Q_\gamma} |||e|||_K \geq C_4(\beta)\Big(\tilde{\eta}_\gamma - p^{-\beta}\|R - R_p\|_{L^2_\beta(\gamma)}\Big), \forall \gamma \in \mathcal{T} \tag{5.35}$$

with $C_4(\beta) = \tilde{C}_0\Gamma^{-1/2}(1-\beta)$ and $\tilde{C}_0$ independent of p and β, where Q_γ is a pair of elements sharing γ.

To illustrate the convergence behaviors of the indicator, we report below some numerical results. An important purpose of these tests is to verify the correctness of relationship between indicator and error.

By (3.14), we have

$$|||e||| \leq \|u - u_{FE}\|_{\tilde{H}^{1,\beta}(\Omega)}$$

and

$$\eta = \Big(\sum_{S_i \in \mathcal{T}} \eta^2_{S_i}\Big)^{\frac{1}{2}} + \Big(\sum_{\gamma^i_l \in \partial\mathcal{T}} \eta^2_{\gamma^i_l}\Big)^{\frac{1}{2}},$$

we need the computation result to check

$$C_1 \log(p+1)\eta \geq |||e||| \geq C_2\eta.$$

We define C as the ratio of indicator and error:

$$C = \frac{|||e|||}{\eta}, \tag{5.36}$$

then

$$C_1 \log(p+1) \geq C \geq C_2$$

where C_1 and C_2 should independent of p.

In generally computation, $|||e|||$ is not computable for unknown u. We can use $\|e\|_{\tilde{H}^{1,\beta}(\Omega)}$ instead, noting that

$$|||e||| \leq \|e\|_{\tilde{H}^{1,\beta}(\Omega)}.$$

We define C' and C^* as

$$C' = \frac{\|e\|_{H^1(\Omega)}}{\eta}, \qquad C^* = \frac{\|e\|_{H^1(\Omega)}}{(p+1)^{1-\beta}\eta}.$$

Example 5.5. We consider the system problem

$$
\begin{cases} -\triangle u + u = f, & \text{in } \Omega = [-1,1] \times [0,1] \\ u\big|_{\Gamma_D} = 0, u\big|_{\Gamma_A} = g & \text{on } \Gamma = \partial\Omega \end{cases} \tag{5.37}
$$

with the exact solution

$$
u = r^{\frac{1}{2}} \sin \frac{\theta}{2}
$$

We use the 8 element FEM solution and $p = 1$ to 8. The element division is shown as Fig 5.1

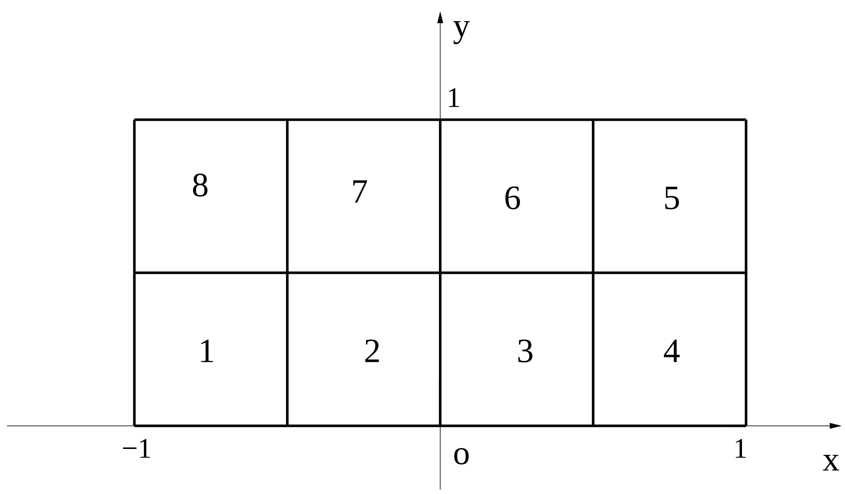

Fig. 5.1

Note that u is the singular function and $(0,0)$ is the singular point. The results of C, C' and C^* of $\beta = 0.5$, $\beta = 0.6$ and $\beta = 0.1$ can be shown on Table 5.1-5.9 respect. We choose Element 2 and 5 arbitrary, which mean the most singularity and least singularity to draw picture on Fig 5.2-5.8.

Then the computation result for

$$C \simeq \frac{\|e\|_{\tilde{H}^{1,\beta}(\Omega)}}{\eta} \tag{5.38}$$

is as follow:

Table 5.1. $\beta = 0.5$ the value of C of Example 5.5

P	1	2	3	4	5	6	7	8
*element*1	0.3546054	1.916975	1.489814	1.449855	1.243637	1.280566	1.263176	1.287194
*element*2	0.1841958	2.232435	1.737537	1.502246	1.497187	1.498202	1.498061	1.496997
*element*3	0.06705665	0.8474819	0.7297016	0.6346869	0.6442866	0.6668664	0.6878502	0.7040870
*element*4	0.1566148	0.6770542	0.7667121	1.038311	1.073772	1.025067	1.079446	1.060833
*element*5	0.6563276	0.4817893	0.3738726	0.3179768	0.4716319	0.4833468	0.5276855	0.5414586
*element*6	0.4910612	1.566565	1.328895	1.438147	1.336041	1.318696	1.335259	1.343957
*element*7	0.1831352	2.033474	1.458770	0.9263470	0.7282947	0.7139753	0.6883884	0.6545495
*element*8	0.2524968	0.8703775	0.2592347	0.2395992	0.3094178	0.3245788	0.3525714	0.3617923
ratio of estimator	0.1961852	1.474916	1.204633	1.069366	1.059119	1.075733	1.092886	1.103980

Table 5.2. $\beta = 0.5$ the value of C' of Example 5.5

P	1	2	3	4	5	6	7	8
$element1$	0.4167860	2.064899	1.696324	1.505012	1.323930	1.356382	1.369377	1.381077
$element2$	0.2277853	3.372163	2.707570	2.348627	2.391858	2.451698	2.502518	2.538148
$element3$	0.09379224	1.260400	1.064072	0.9394112	0.9626781	1.004564	1.043437	1.072852
$element4$	0.1847198	0.7655634	0.9475798	1.315330	1.238495	1.221017	1.255469	1.271251
$element5$	0.6191113	0.3937685	0.3015284	0.2630278	0.3927791	0.4074133	0.4448578	0.4587146
$element6$	0.4809652	1.721406	1.580194	1.682715	1.493352	1.504878	1.531488	1.558894
$element7$	0.2390759	2.164808	1.561172	0.8692449	0.7444363	0.7022076	0.6917037	0.6452357
$element8$	0.3191930	0.8756222	0.2389358	0.2301110	0.3051850	0.3204616	0.3482065	0.3585230
ratio of estimator	0.2453741	1.939252	1.618432	1.463589	1.488972	1.552799	1.612577	1.656191

Table 5.3. $\beta = 0.5$ the value of C^* of Example 5.5

P	1	2	3	4	5	6	7	8
$element1$	0.2947122	1.408401	0.9620285	0.8237812	0.6642239	0.6502068	0.6091993	0.5903846
$element2$	0.1610686	1.886269	1.328150	1.035700	0.9647178	0.9169437	0.8769915	0.8396065
$element3$	0.06632113	0.7841206	0.5717251	0.4411789	0.4074375	0.3904258	0.3772083	0.3641811
$element4$	0.1306166	0.6847208	0.6765711	0.8776342	0.7383721	0.6750033	0.6416147	0.6141583
$element5$	0.4377778	0.4552255	0.3235979	0.2054958	0.3159759	0.2751293	0.3023579	0.2767817
$element6$	0.3400937	1.192144	0.9105511	0.9099144	0.7382693	0.7049684	0.6630242	0.6450598
$element7$	0.1690522	1.683512	0.9829418	0.5619368	0.4461902	0.4160517	0.3848085	0.3455370
$element8$	0.2257036	0.9763175	0.2545456	1.912024	0.2496484	0.2277119	0.2438039	0.2267792
ratio of estimator	0.1735057	1.279452	0.9113799	0.7293048	0.6684544	0.6380374	0.6144863	0.5905753

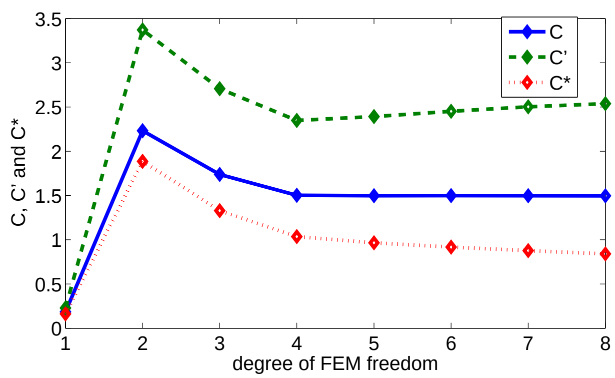

Fig. 5.2 Relationship of indicator and degree of Element 2, β=0.5

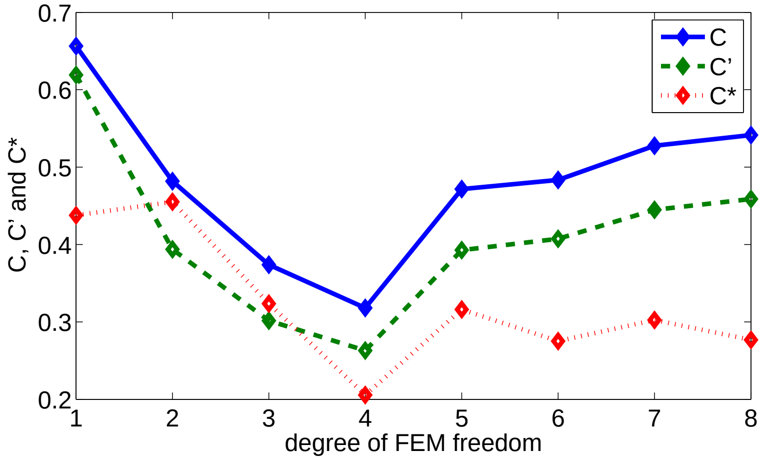

Fig. 5.3 Relationship of indicator and degree of Element 5, β=0.5

Table 5.4. $\beta = 0.6$ the value of C of Example 5.5

P	1	2	3	4	5	6	7	8
$element1$	0.3825977	2.208179	1.681430	1.721968	1.458148	1.514589	1.487386	1.534171
$element2$	0.1980606	2.408966	1.884558	1.648824	1.660845	1.672731	1.681090	1.686587
$element3$	0.07442364	1.034001	0.8865842	0.7677308	0.7783027	0.8058351	0.8320138	0.8527742
$element4$	0.1670287	1.041267	1.113901	1.582064	1.542497	1.493327	1.514827	1.504296
$element5$	0.6826354	0.9065989	0.7489889	0.5113781	0.8250236	0.7801722	0.8771632	0.8667067
$element6$	0.5122140	1.850744	1.559010	1.775514	1.572588	1.594242	1.579606	1.619180
$element7$	0.1997634	2.677699	1.778424	1.299165	1.033927	1.028522	1.000060	0.9619142
$element8$	0.2744794	1.651156	0.5606994	0.4438635	0.5867427	0.5842721	0.6438861	0.6392886
ratio of estimator	0.2129918	1.773351	1.432099	1.279385	1.260621	1.278838	1.296754	1.310764

Table 5.5. $\beta = 0.6$ the value of C' of Example 5.5

P	1	2	3	4	5	6	7	8
$element1$	0.4014437	2.314897	1.804917	1.707820	1.484003	1.536972	1.514314	1.532281
$element2$	0.2193242	3.116516	2.505303	2.147125	2.147710	2.165651	2.177003	2.176096
$element3$	0.09022211	1.294589	1.078307	0.9141870	0.9072382	0.9226157	0.9373679	0.9454606
$element4$	0.1773875	1.125688	1.260849	1.822735	1.655892	1.602854	1.598972	1.598291
$element5$	0.5964318	0.7498273	0.6168783	0.4263447	0.7002416	0.6481672	0.7485286	0.7159808
$element6$	0.4631458	1.961832	1.705478	1.887619	1.652442	1.668927	1.649430	1.675669
$element7$	0.2302334	2.761755	1.849470	1.162787	0.9896361	0.9798663	0.9552239	0.8950021
$element8$	0.3074989	1.592650	0.4852093	0.3986980	0.5532981	0.5365561	0.6035931	0.5866456
ratio of estimator	0.2362524	2.110005	1.716314	1.511928	1.489147	1.507477	1.526120	1.531633

Table 5.6. $\beta = 0.6$ the value of C^* of Example 5.5

P	1	2	3	4	5	6	7	8
$element1$	0.3042375	1.491706	1.036653	0.8971275	0.7247265	0.7057110	0.6591434	0.6362699
$element2$	0.1662166	2.008264	1.438919	1.127897	1.048854	0.9943730	0.9475957	0.9036100
$element3$	0.06837557	0.8342257	0.6193249	0.4802275	0.4430583	0.4236250	0.4080131	0.3925965
$element4$	0.1344346	0.7253865	0.7241676	0.9574929	0.8086704	0.7359608	0.6959931	0.6636803
$element5$	0.4520108	0.4831842	0.3543035	0.2239613	0.3419695	0.2976102	0.3258160	0.2973065
$element6$	0.3509989	1.264193	0.9795398	0.9915768	0.8069856	0.7662987	0.7179560	0.6958110
$element7$	0.1744843	1.779658	1.062241	0.6108187	0.4832980	0.4499120	0.4157854	0.3716440
$element8$	0.2330406	1.026294	0.2786796	0.2094383	0.2702083	0.2463633	0.2627291	0.2436008
ratio of estimator	0.1790458	1.359674	0.9857637	0.7942241	0.7272389	0.6921679	0.6642822	0.6360007

Table 5.7. $\beta = 0.1$ the value of C of Example 5.5

P	1	2	3	4	5	6	7	8
$element1$	0.1688832	0.9850620	0.7407881	0.7733417	0.6280775	0.6621527	0.6264020	0.6464943
$element2$	0.08292735	0.8697086	0.6991381	0.5877832	0.5741027	0.5653512	0.5601803	0.5560933
$element3$	0.02694504	0.3048894	0.2573468	0.2098070	0.2045374	0.2063870	0.2096092	0.2121237
$element4$	0.07916384	0.3883816	0.4202885	0.5299856	0.6151644	0.5626710	0.6399172	0.5899239
$element5$	0.3925952	0.4181391	0.3446354	0.2732776	0.4144720	0.4040645	0.4542573	0.4507377
$element6$	0.2832714	0.8191897	0.6656809	0.7063180	0.6739354	0.6493678	0.6520234	0.6437799
$element7$	0.08333065	1.084997	0.7676226	0.6599038	0.4306465	0.5244697	0.4728761	0.5111516
$element8$	0.1163421	0.5246502	0.1779615	0.1685094	0.2365048	0.2577928	0.2963505	0.3130865
ratio of estimator	0.08756595	0.6170683	0.5036195	0.4217010	0.4000780	0.3958444	0.3945836	0.3929812

Table 5.8. $\beta = 0.1$ the value of C' of Example 5.5

P	1	2	3	4	5	6	7	8
$element1$	0.5014869	3.151178	2.632870	2.652120	2.534706	2.978074	3.246886	3.621586
$element2$	0.2740021	4.123039	3.547957	3.342361	3.750773	4.214830	4.700884	5.175724
$element3$	0.1132874	1.719795	1.527548	1.426855	1.582663	1.789808	2.011142	2.226909
$element4$	0.2256717	1.530617	1.905359	2.811840	2.770968	3.026606	3.379369	3.718949
$element5$	0.7440917	1.009136	0.8196980	0.6598717	1.258378	1.280770	1.643442	1.723320
$element6$	0.5794231	2.652957	2.510568	2.923426	2.793776	3.206859	3.521098	3.940411
$element7$	0.2879727	3.798578	2.654791	1.822150	1.760697	1.935743	2.061663	2.139293
$element8$	0.3836324	2.255438	0.6450041	0.6001206	0.9938401	1.059089	1.324915	1.411858
ratio of estimator	0.2955349	2.820277	2.452438	2.352741	2.590244	2.927904	3.286328	3.630146

Table 5.9. $\beta = 0.1$ the value of C^* of Example 5.5

P	1	2	3	4	5	6	7	8
$element1$	0.2687402	1.172368	0.7560933	0.6230460	0.5053491	0.5168295	0.4996731	0.5012802
$element2$	0.1468341	1.533940	1.018883	0.7852001	0.7477986	0.7314621	0.7234334	0.7163955
$element3$	0.06070921	0.6398343	0.4386731	0.3352021	0.3155385	0.3106119	0.3095008	0.3082367
$element4$	0.1209345	0.5694524	0.5471706	0.6605680	0.5524530	0.5252520	0.5200613	0.5147567
$element5$	0.3987489	0.3754402	0.2353964	0.1550196	0.2508852	0.2222711	0.2529143	0.2385326
$element6$	0.3105051	0.9870091	0.7209715	0.6867822	0.5570004	0.5565339	0.5418725	0.5454103
$element7$	0.1543207	1.413227	0.7623885	0.4280665	0.3510334	0.3359382	0.3172757	0.2961093
$element8$	0.2055835	0.8391156	0.1852288	0.1409826	0.1981438	0.1837995	0.2038952	0.1954218
ratio of estimator	0.1583732	1.049259	0.7042778	0.5527148	0.5164217	0.5081227	0.5057430	0.5024650

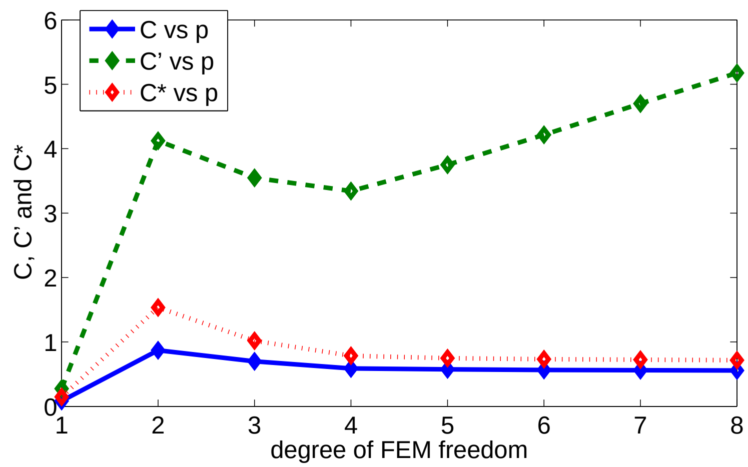

Fig. 5.4 Relationship of indicator and degree of Element 2, β=0.1

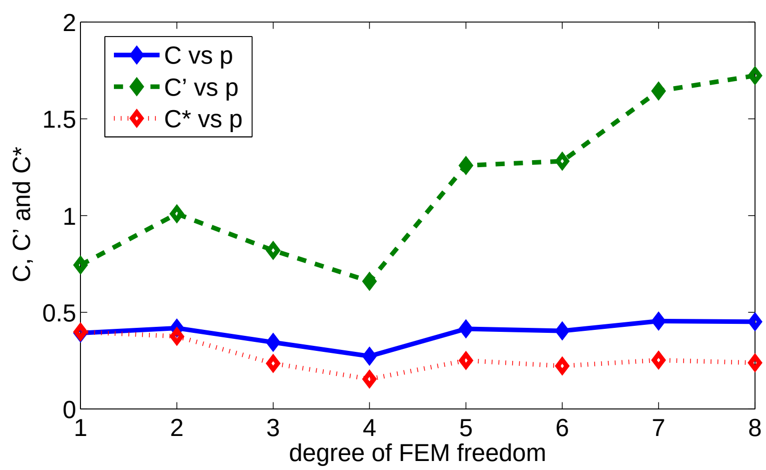

Fig. 5.5 Relationship of indicator and degree of Element 5, β=0.1

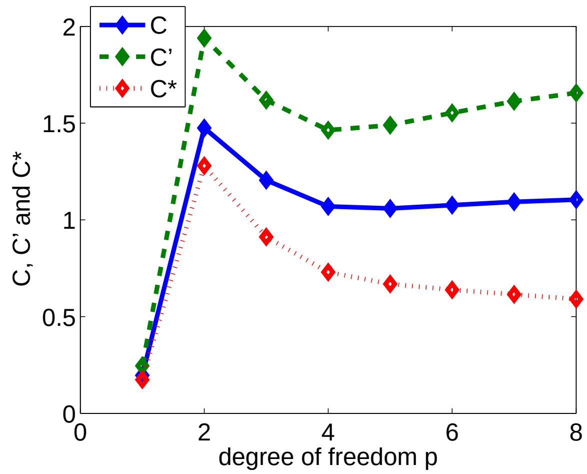

Fig.5.6 Relationship between estimator and p, β=0.5

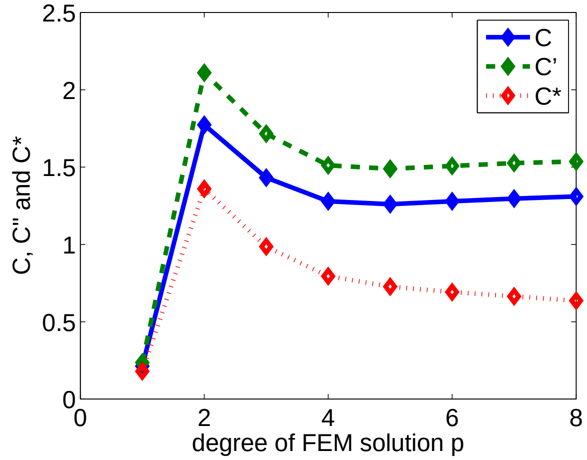

Fig.5.7 Relationship between estimator and p, β=0.6

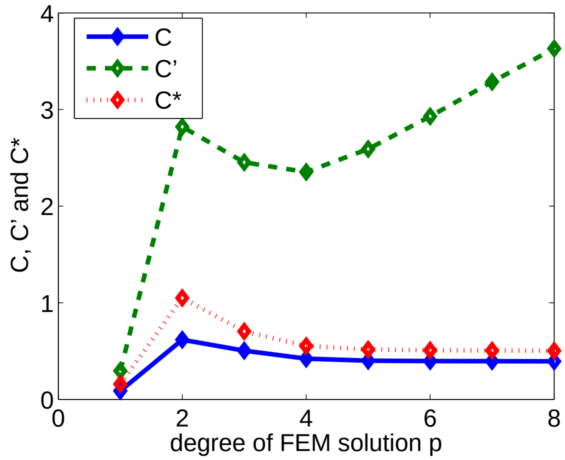

Fig.5.8 Relationship between estimator and p, β=0.1

From the Fig 5.2-5.8, it can be shown that C is almost stable and C^* is more stable than C' which means

$$\|e\|_{\tilde{H}^{1,\beta}(\Omega)} \approx C\eta$$

and

$$\|e\|_{H^1(\Omega)} \approx C^*(p+1)^{1-\beta}\eta$$

for $\beta \in (0,1)$.

Example 5.6. We consider the same system problem of Example 5.5 with the exact solution

$$u = \exp x \sin y$$

Note here the solution is smooth function. The value of C, C' and C^* is also shown in Table 5.10-5.15. We choose Element 2 to draw the picture Fig 5.9-5.10 for $\beta = 0.5$ and $\beta = 0.1$.

Table 5.10. $\beta = 0.5$ the value of C of Example 5.6

P	1	2	3	4	5	6	7	8
$element1$	0.1698364	3.562792	1.110317	1.490398	1.563095	1.304650	2.093791	1.126606
$element2$	0.08424815	3.563470	1.119973	1.526060	1.705096	1.394833	2.097348	1.172471
$element3$	0.04326418	3.589605	1.359761	1.558718	1.868114	1.616847	2.101287	1.397948
$element4$	0.02002801	3.599816	1.340093	1.641904	1.732577	1.646114	2.087522	1.279839
$element5$	0.1131966	3.560698	0.9453431	1.557408	1.418325	1.642523	2.076687	1.463068
$element6$	0.07179227	3.554727	1.028773	1.621974	1.540957	1.659847	2.079491	1.471384
$element7$	0.04787812	3.544471	0.9716293	1.574164	1.519539	1.633786	2.077227	1.456666
$element8$	0.02194089	3.565364	1.066554	1.618502	1.459680	1.441685	2.067143	1.375851
ratio of estimator	0.04819009	3.572263	1.101828	1.592857	1.622784	1.562088	2.083196	1.354518

Table 5.11. $\beta = 0.5$ the value of C' of Example 5.6

P	1	2	3	4	5	6	7	8
$element1$	0.2199330	5.173336	1.230879	1.924352	2.298552	2.198209	4.493788	1.192063
$element2$	0.1121375	5.176072	1.272630	2.051617	2.557616	2.390704	4.458901	1.267474
$element3$	0.05852590	5.199421	1.584984	2.050031	2.788755	2.783714	4.450888	1.470974
$element4$	0.02727774	5.203984	1.607065	2.257523	2.630762	2.852937	4.372758	1.339756
$element5$	0.1517595	5.177828	1.214920	2.147038	2.179004	2.664131	4.501430	1.590440
$element6$	0.09702606	5.164904	1.352347	2.233525	2.400556	2.791056	4.480614	1.569778
$element7$	0.06494055	5.150042	1.246941	2.144609	2.360323	2.761695	4.471374	1.531663
$element8$	0.02983248	5.182075	1.438469	2.210651	2.269978	2.491887	4.424963	1.438446
ratio of estimator	0.06520873	5.182479	1.400408	2.159956	2.477788	2.671371	4.437237	1.429014

Table 5.12. $\beta = 0.5$ the value of C^* of Example 5.6

P	1	2	3	4	5	6	7	8
$element1$	0.1555161	2.986827	0.6154396	0.8605965	0.9383800	0.8308451	1.588794	0.3973542
$element2$	0.07929319	2.988407	0.6363150	0.9175109	1.044142	0.9036012	1.576460	0.4224913
$element3$	0.04138406	3.001887	0.7924919	0.9168019	1.138505	1.052145	1.573626	0.4903248
$element4$	0.01928828	3.004522	0.8035326	1.009595	1.074004	1.078309	1.546003	0.4465855
$element5$	0.1073102	2.989421	0.6074601	0.9601846	0.8895748	1.006947	1.591496	0.5301466
$element6$	0.06860779	2.981959	0.6761735	0.9988628	0.9800230	1.054920	1.584136	0.5232592
$element7$	0.04591990	2.973378	0.6234707	0.9590984	0.9635978	1.043822	1.580869	0.5105544
$element8$	0.02109475	2.991872	0.7192344	0.9886334	0.9267145	0.9418446	1.564461	0.4794819
ratio of estimator	0.04610953	2.992106	0.7002039	0.9659617	1.011553	1.009683	1.568800	0.4763379

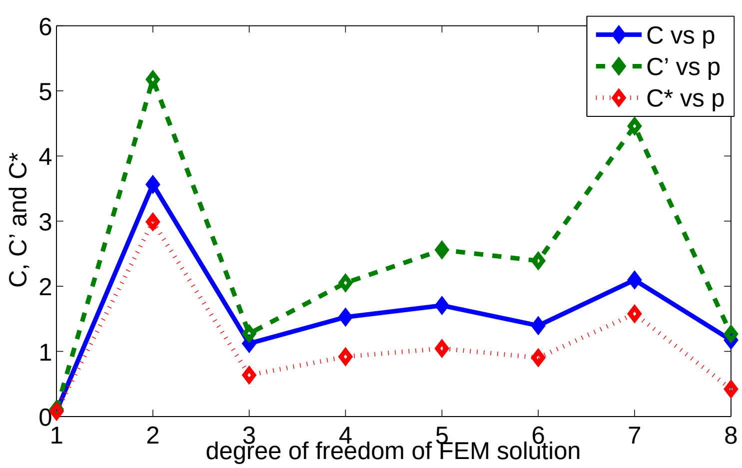

Fig. 5.9 Relationship of indicator and degree of Element 2, β=0.5

Table 5.13. $\beta = 0.1$ the value of C of Example 5.6

P	1	2	3	4	5	6	7	8
$element1$	0.07530275	1.538022	0.4981561	0.6604110	0.6280833	0.5273382	0.8749726	1.116812
$element2$	0.03701747	1.538101	0.5021711	0.6790485	0.6983787	0.5707063	0.8737064	1.067840
$element3$	0.01889098	1.550288	0.6098679	0.6983629	0.7819850	0.6905484	0.8765480	1.332419
$element4$	0.008688309	1.555783	0.6007415	0.7150834	0.7191943	0.6964237	0.8712504	1.274681
$element5$	0.05073051	1.542332	0.3999655	0.6773445	0.5578095	0.7019497	0.8665126	1.173274
$element6$	0.03210541	1.540891	0.4351499	0.7064246	0.6152184	0.7043796	0.8664560	1.183585
$element7$	0.02136229	1.536504	0.4124835	0.6845713	0.6023413	0.6933275	0.8655542	1.179383
$element8$	0.009731258	1.545327	0.4481214	0.6945356	0.5814584	0.5855698	0.8610392	1.119739
ratio of estimator	0.02137828	1.545680	0.4753796	0.6955707	0.6579191	0.6546223	0.8685910	1.183125

Table 5.14. $\beta = 0.1$ the value of C' of Example 5.6

P	1	2	3	4	5	6	7	8
$element1$	0.2646790	6.943280	1.655958	2.683048	2.964524	3.146851	7.999146	1.827856
$element2$	0.1352723	6.946922	1.709969	2.768779	3.288409	3.462240	7.968255	1.970905
$element3$	0.07037032	6.978128	2.101916	2.730223	3.645384	4.142898	7.965013	2.263885
$element4$	0.03244144	6.984172	2.103337	2.946269	3.385573	4.214201	7.940622	2.056933
$element5$	0.1823026	6.947945	1.498596	2.849773	2.856101	4.118338	7.977015	2.554835
$element6$	0.1165846	6.930172	1.661720	2.929556	3.141388	4.206246	7.957151	2.521870
$element7$	0.07799282	6.910636	1.543358	2.832500	3.085136	4.158434	7.950907	2.459223
$element8$	0.03567211	6.952786	1.751120	2.940344	2.995368	3.580738	7.933683	2.306570
ratio of estimator	0.07828281	6.954559	1.763255	2.863588	3.226377	3.947590	7.952104	2.253905

Table 5.15. $\beta = 0.1$ the value of C^* of Example 5.6

P	1	2	3	4	5	6	7	8
$element1$	0.1418380	2.583185	0.4755489	0.6303118	0.5910426	0.5461199	1.231013	0.2530018
$element2$	0.07249063	2.584540	0.4910596	0.6504520	0.6556162	0.6008540	1.226259	0.2728020
$element3$	0.03771052	2.596150	0.6036170	0.6413943	0.7267870	0.7189787	1.225760	0.3133547
$element4$	0.01738494	2.598399	0.6040248	0.6921487	0.6749880	0.7313530	1.222007	0.2847095
$element5$	0.09769353	2.584921	0.4303586	0.6694794	0.5694262	0.7147163	1.227607	0.3536264
$element6$	0.06247613	2.578309	0.4772037	0.6882224	0.6263043	0.7299723	1.224550	0.3490635
$element7$	0.04179532	2.571040	0.4432131	0.6654217	0.6150894	0.7216748	1.223589	0.3403922
$element8$	0.01911621	2.586722	0.5028771	0.6907567	0.5971921	0.6214186	1.220939	0.3192629
ratio of estimator	0.04195072	2.587382	0.5063621	0.6727249	0.6432488	0.6850840	1.223774	0.3119732

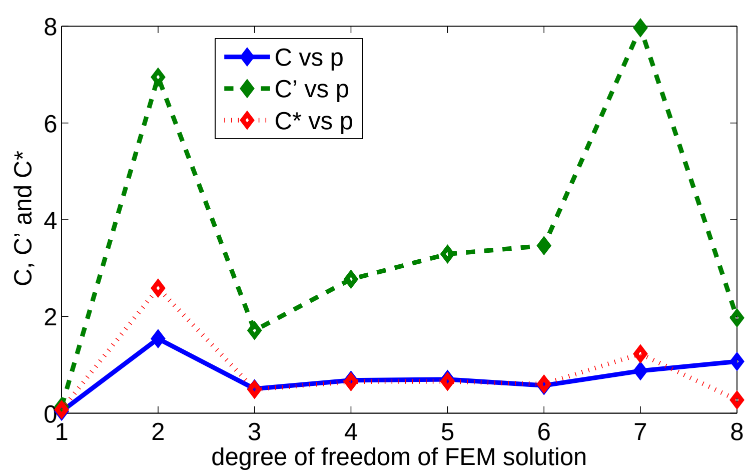

Fig. 5.10 Relationship of indicator and degree of Element 2, β=0.1

From the Fig 5.9 and 5.10, we have the same conclusion of Example 1: C is almost stable when p is large and C^* is more stable than C'.

5.5. Results and Analysis of Computation on triangle elements

We consider the Example 5.5 again on triangle elements

EXAMPLE 5.7.

$$\begin{cases} -\triangle u + u = f, & \text{in } \Omega = [-1,1] \times [0,1] \\ u\big|_{\Gamma_D} = 0, u\big|_{\Gamma_A} = g & \text{on } \Gamma = \partial\Omega \end{cases} \tag{5.39}$$

with the exact solution

$$u = r^{\frac{1}{2}} \sin \frac{\theta}{2}$$

We use the 16 element FEM solution and $p = 1$ to 8. The element division is shown as Fig 5.11

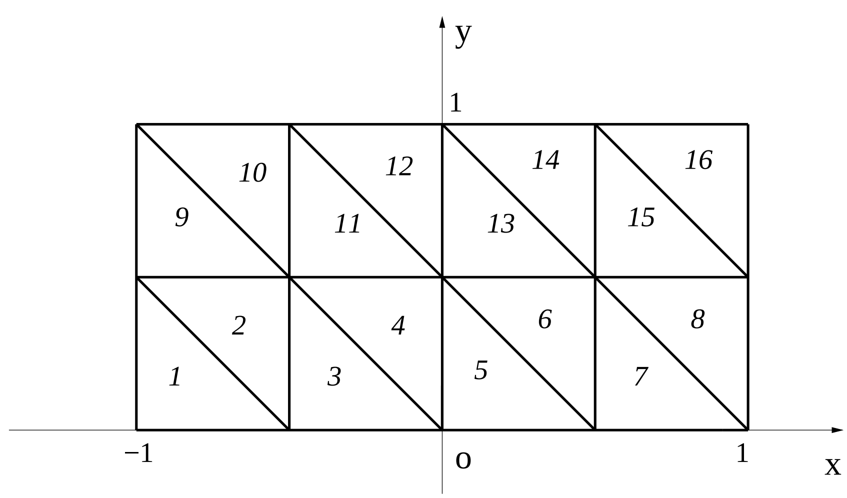

Fig. 5.11

Note that u is the singular function and $(0,0)$ is the singular point. The results of C, C' and C^* of $\beta = 0.5$ and $\beta = 0.1$ can be shown on Table 5.16-5.19 respect. We choose Element

3, 4, 5, and 9, 16 arbitrary, which mean the most singularity and least singularity to draw picture on Fig 5.12-5.19.

Table 5.16. $\beta = 0.1$ the value of η of Example 5.8

P	Element 3	Element 4	Element 5	Element 9	Element 16
1	1.510881×10^{-1}	1.641327×10^{-1}	1.886113×10^{-1}	3.359294×10^{-2}	1.563206×10^{-2}
2	8.718364×10^{-2}	9.069076×10^{-2}	1.192141×10^{-1}	9.401045×10^{-3}	4.043327×10^{-3}
3	6.146152×10^{-2}	6.348272×10^{-2}	8.218730×10^{-2}	4.011037×10^{-3}	1.844593×10^{-3}
4	4.903582×10^{-2}	4.946429×10^{-2}	6.062228×10^{-2}	2.443599×10^{-3}	1.106987×10^{-3}
5	4.414569×10^{-2}	4.010761×10^{-2}	4.709984×10^{-2}	1.629502×10^{-3}	7.387514×10^{-4}
6	3.234384×10^{-2}	3.384249×10^{-2}	4.383007×10^{-2}	1.165667×10^{-3}	5.2833973×10^{-4}
7	2.927761×10^{-2}	2.983220×10^{-2}	3.646246×10^{-2}	8.749789×10^{-4}	3.965952×10^{-4}
8	2.637618×10^{-2}	2.640278×10^{-2}	3.108375×10^{-2}	6.809232×10^{-4}	3.086326×10^{-4}

Table 5.17. $\beta = 0.2$ the value of η of Example 5.8

P	Element 3	Element 4	Element 5	Element 9	Element 16
1	3.667498×10^{-2}	3.215730×10^{-2}	1.414287×10^{-2}	1.437780×10^{-2}	5.051119×10^{-3}
2	2.456383×10^{-1}	1.984695×10^{-1}	2.179433×10^{-1}	1.393661×10^{-2}	2.647991×10^{-3}
3	1.949622×10^{-1}	1.630239×10^{-1}	1.771420×10^{-1}	1.097272×10^{-3}	1.041595×10^{-3}
4	1.601197×10^{-1}	1.344464×10^{-1}	1.472293×10^{-1}	1.856938×10^{-4}	9.739372×10^{-5}
5	1.376084×10^{-1}	1.156107×10^{-1}	1.271117×10^{-1}	3.314775×10^{-5}	2.046295×10^{-5}
6	1.220157×10^{-1}	1.024662×10^{-1}	1.129137×10^{-1}	6.121243×10^{-6}	2.500212×10^{-6}
7	1.106275×10^{-1}	9.283575×10^{-2}	1.024829×10^{-1}	1.034598×10^{-6}	4.571697×10^{-7}
8	1.019485×10^{-1}	8.548883×10^{-2}	9.451557×10^{-2}	1.538569×10^{-7}	5.733097×10^{-8}

Table 5.18. $\beta = 0.3$ the value of η of Example 5.8

P	Element 3	Element 4	Element 5	Element 9	Element 16
1	2.973109×10^{-2}	2.609624×10^{-2}	1.138482×10^{-2}	1.164745×10^{-2}	4.076704×10^{-3}
2	1.988653×10^{-1}	1.606767×10^{-1}	1.764130×10^{-1}	1.128364×10^{-2}	2.143249×10^{-3}
3	1.529198×10^{-1}	1.278687×10^{-1}	1.389252×10^{-1}	8.604916×10^{-4}	8.170658×10^{-4}
4	1.220902×10^{-1}	1.024910×10^{-1}	1.126036×10^{-1}	1.423702×10^{-4}	7.471479×10^{-5}
5	1.021501×10^{-1}	8.583140×10^{-2}	9.483401×10^{-2}	2.511681×10^{-5}	1.536013×10^{-5}
6	8.831517×10^{-2}	7.420600×10^{-2}	8.219907×10^{-2}	4.563876×10^{-6}	1.842419×10^{-6}
7	7.820650×10^{-2}	6.568946×10^{-2}	7.288944×10^{-2}	7.560779×10^{-7}	3.336786×10^{-7}
8	7.050848×10^{-2}	5.919753×10^{-2}	6.577225×10^{-2}	1.119680×10^{-7}	4.098094×10^{-8}

Table 5.19. $\beta = 0.4$ the value of η of Example 5.8

P	Element 3	Element 4	Element 5	Element 9	Element 16
1	2.418827×10^{-2}	2.125099×10^{-2}	9.200767×10^{-3}	9.471186×10^{-3}	3.304224×10^{-3}
2	1.616209×10^{-1}	1.305834×10^{-1}	1.433510×10^{-1}	9.170886×10^{-3}	1.741470×10^{-3}
3	1.206349×10^{-1}	1.008726×10^{-1}	1.095822×10^{-1}	6.787062×10^{-4}	6.446229×10^{-4}
4	9.384235×10^{-2}	7.876164×10^{-2}	8.678539×10^{-2}	1.099647×10^{-4}	5.773800×10^{-5}
5	7.662910×10^{-2}	6.439252×10^{-2}	7.147253×10^{-2}	1.918122×10^{-5}	1.163458×10^{-5}
6	6.475598×10^{-2}	5.443450×10^{-2}	6.060702×10^{-2}	3.434215×10^{-6}	1.373329×10^{-6}
7	5.613495×10^{-2}	4.718698×10^{-2}	5.263527×10^{-2}	5.590623×10^{-7}	2.462311×10^{-7}
8	4.961309×10^{-2}	4.169863×10^{-2}	4.657216×10^{-2}	8.237853×10^{-8}	2.972422×10^{-8}

The Table 5.16-5.19 describe the performance of the p version on the uniform mesh. We know $u = r^{\frac{1}{2}} \sin\frac{\theta}{2}$ is singular solution and u_{FE} will be the most singularity on element 3,4,5 and the least singularity on element 9,16. From the Table 5.16-5.19, for different β, the indicator is convergence with p. The relationship is plotted in Fig 5.12-5.19.

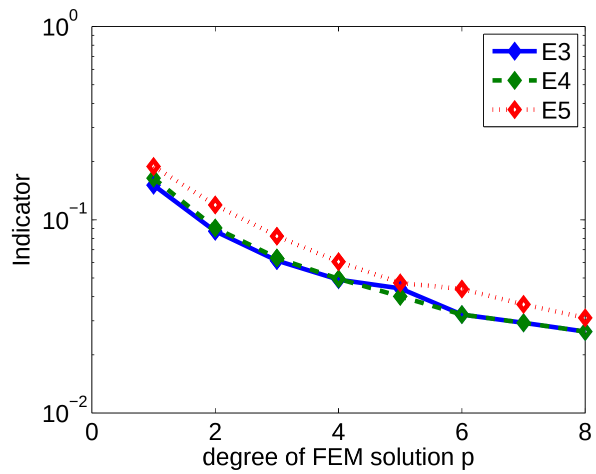

Fig. 5.12 $\beta = 0.1$ relationship between p and E3,E4,E5

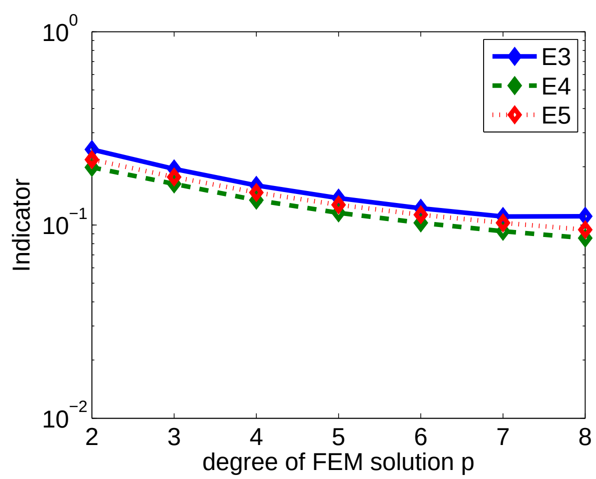

Fig. 5.13 $\beta = 0.2$ relationship between p and E3,E4,E5

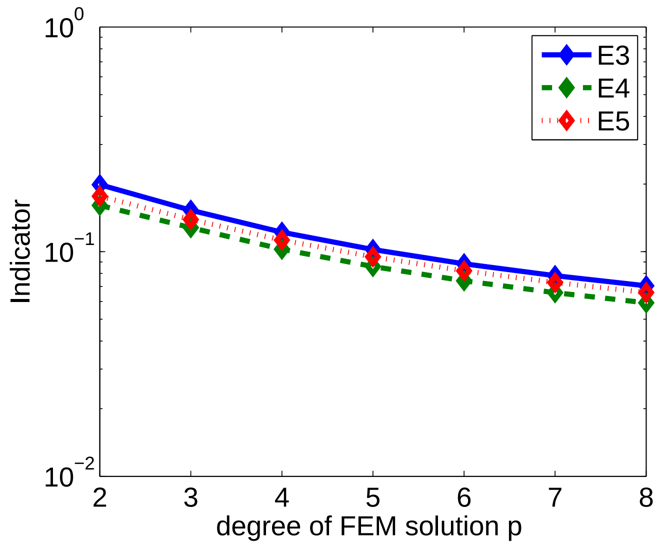

Fig. 5.14 $\beta = 0.3$ relationship between p and E3,E4,E5

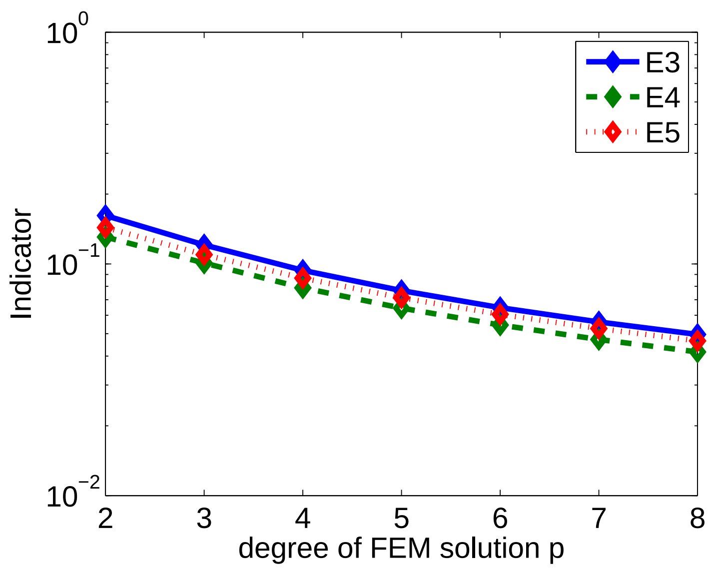

Fig. 5.15 $\beta = 0.4$ relationship between p and E3,E4,E5

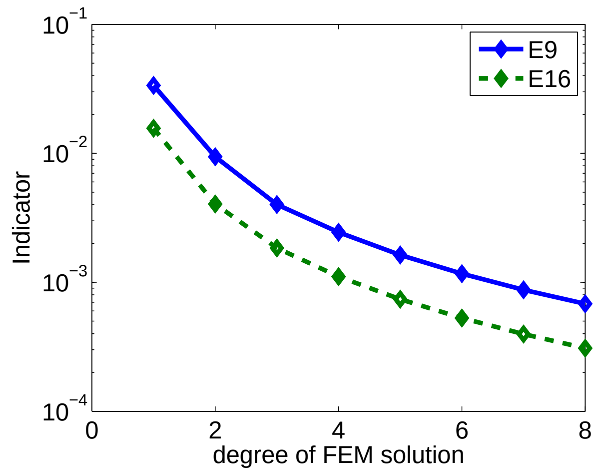

Fig. 5.16 $\beta = 0.1$ relationship between p and E9,E16

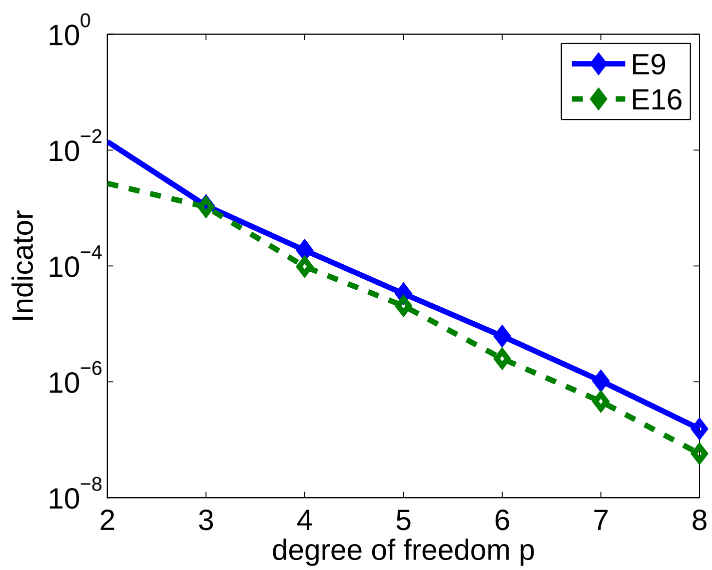

Fig. 5.17 $\beta = 0.2$ relationship between p and E9,E16

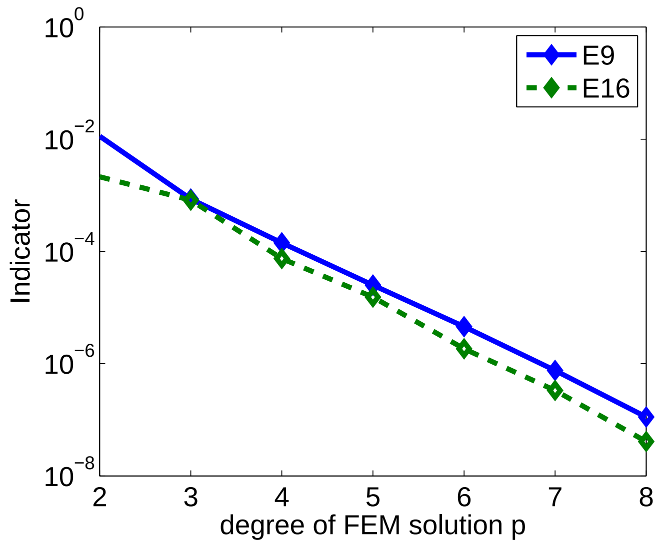

Fig. 5.18 $\beta = 0.3$ relationship between p and E9,E16

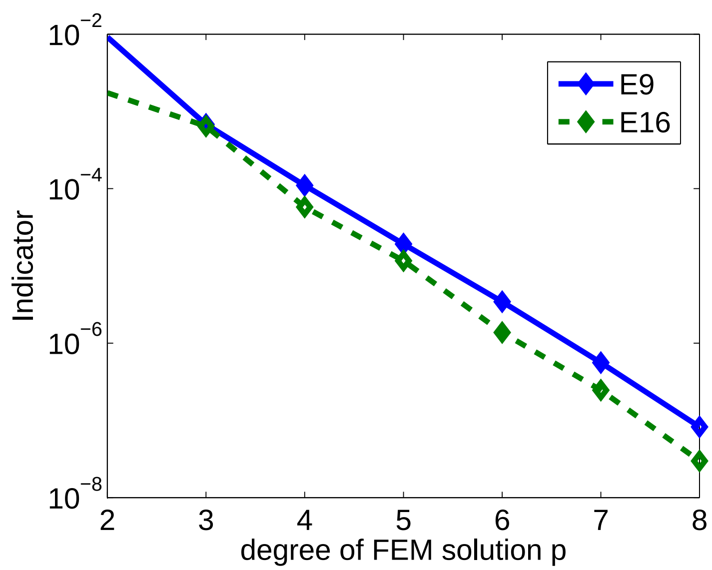

Fig. 5.19 $\beta = 0.4$ relationship between p and E9,E16

The computation results in Table 5.16-5.19 show that the theory of the residual-based a-posterior error estimation is very reliable and coincides with the computation.

CHAPTER 6

Conclusion

In this thesis, we have established the residual-based a-posteriori error estimation on triangles for the p-version of FEM with triangular mesh in the framework of Jacobi-weighted Sobolev spaces. The quasi optimality of the error indicators and estimators is proved, which is parallel with result on square without substantial comprising of the optimality, the results are novel and not seen in current literature. But for the collapse mapping, the lower bound of indicator on triangle element is $(p+1)^{-\beta}$ worse than on quadrilateral element, which needs to be further improved.

The theorems and analysis are verified by computations on square and triangular mesh. The computations also explore the numerical aspects of the error indicators and estimators, which are not easy to addressed theoretically. The reliability of the error indicator and estimator with respect to the error in weighted Sobolev norm $\tilde{H}^{1,\beta}(\mathrm{T})$ and $\tilde{H}^{1,\beta}(\Omega)$ is maintained for singular and smooth solutions. Furthermore, the observation of the numerical experiments guess the relationship between the error in real energy norm $H^1(T)$ or $H^1(\Omega)$ and the error indicator and estimator, for which further investigation is needed.

The quasi optimal error indicators and estimation can provide important information of the magnitude of the error and the distribution of the error on each element, which significantly improve the efficiency of adaptive algorithm for the p-version for elliptic problems such as elasticity problems on polyhedral domains, the heat problems and magnetic-electric problems.

The error indicators and estimators proposed and analyzed can be easily generalized to the h-p version on triangular mesh, the results and techniques developed can be used to a-priori error analysis for p and h-p versions on triangular mesh, e.g., the equivalence of the Jacobi-weighted spaces via three different mapping of a square onto a triangle provide a powerful tool for error estimation of the p and h-p version.

CHAPTER 7

Appendix: Inverse Inequality

Lemma 7.1.

$$\int_{-1}^{1} |\frac{d}{dx}J_n^{\alpha}(x)|^2(1-x^2)^{\alpha}dx \le Cn^2 \tag{7.1}$$

for $\alpha > -1$ *and* $n \geq 1$.

PROOF. From the Jacobi polynomial property [**29**] (J4), we have

$$(2n+2\alpha+1)J_n^{\alpha}(x) = \frac{n+2\alpha+1}{n+\alpha+1}(\frac{d}{dx}J_{n+1}^{\alpha}(x)) - \frac{n+\alpha}{n+2\alpha}(\frac{d}{dx}J_{n-1}^{\alpha}(x)).$$

It can easily get

$$(\frac{d}{dx}J_{n+1}^{\alpha}(x)) = \frac{(2n+2\alpha+1)(n+\alpha+1)}{n+2\alpha+1}J_n^{\alpha}(x) + \frac{(n+\alpha)(n+\alpha+1)}{(n+2\alpha)(n+2\alpha+1)}(\frac{d}{dx}J_{n-1}^{\alpha}(x)),$$

and

$$\begin{aligned}(\frac{d}{dx}J_{n+1}^{\alpha}(x))^2 &= \frac{(2n+2\alpha+1)^2(n+\alpha+1)^2}{(n+2\alpha+1)^2}(J_n^{\alpha}(x))^2 + \frac{(n+\alpha)^2(n+\alpha+1)^2}{(n+2\alpha)^2(n+2\alpha+1)^2}(\frac{d}{dx}J_{n-1}^{\alpha}(x))^2 \\ &+ \frac{2(2n+2\alpha+1)(n+\alpha)(n+\alpha+1)^2}{(n+2\alpha+1)^2(n+2\alpha)}J_n^{\alpha}(x)(\frac{d}{dx}J_{n-1}^{\alpha}(x)).\end{aligned}$$

Because of the orthogonality of the Jacobi polynomials, we have

$$\int_{-1}^{1} J_n^\alpha(x)(\frac{d}{dx}J_{n-1}^\alpha(x))(1-x^2)^\alpha dx = 0,$$

and

$$\int_{-1}^{1} |J_n^\alpha(x)|^2(1-x^2)^\alpha dx = \frac{2^{2\alpha+1}\Gamma^2(n+\alpha+1)}{(2n+2\alpha+1)\Gamma(n+1)\Gamma(n+2\alpha+1)}. \tag{7.2}$$

We can get

$$\int_{-1}^{1} |\frac{d}{dx}J_n^\alpha(x)|^2(x)(1-x^2)^\alpha dx = \frac{(2n+2\alpha-1)^2(n+\alpha)^2}{(n+2\alpha)^2}\int_{-1}^{1} |J_{n-1}^\alpha(x)|^2(1-x^2)^\alpha dx$$
$$+ \quad \frac{(n+\alpha-1)^2(n+\alpha)^2}{(n+2\alpha-1)^2(n+2\alpha)^2}\int_{-1}^{1} |\frac{d}{dx}J_{n-2}^\alpha(x)|^2(1-x^2)^\alpha dx.$$

Similarly there hold

$$\int_{-1}^{1} |\frac{d}{dx}J_{n-2}^\alpha(x)|^2(1-x^2)^\alpha dx = \frac{(2n+2\alpha-5)^2(n+\alpha-2)^2}{(n+2\alpha-2)^2}\int_{-1}^{1} (J_{n-3}^\alpha(x))^2(1-x^2)^\alpha dx$$
$$+ \quad \frac{(n+\alpha-3)^2(n+\alpha-2)^2}{(n+2\alpha-3)^2(n+2\alpha-2)^2}\int_{-1}^{1} |\frac{d}{dx}J_{n-4}^\alpha(x)|^2(1-x^2)^\alpha dx$$

and for any integer $s \leq \frac{n}{2}$, there holds

$$\begin{aligned}&\int_{-1}^{1}|\frac{d}{dx}J_{n-2(s-1)}^{\alpha}(x)|^2(1-x^2)^{\alpha}dx\\ =&\ \frac{(2n+2\alpha-4s+3)^2(n+\alpha-2s+2)^2}{(n+2\alpha-2s+2)^2}\int_{-1}^{1}(J_{n-2s+1}^{\alpha}(x))^2(1-x^2)^{\alpha}dx\\ +&\ \frac{(n+\alpha-2s+1)^2(n+\alpha-2s+2)^2}{(n+2\alpha-2s+1)^2(n+2\alpha-2s+2)^2}\int_{-1}^{1}|\frac{d}{dx}J_{n-2s}^{\alpha}(x)|^2(1-x^2)^{\alpha}dx,\end{aligned}$$

which imply

$$\begin{aligned}&\int_{-1}^{1}|\frac{d}{dx}J_{n}^{\alpha}(x)|^2(x)(1-x^2)^{\alpha}dx\\ =&\ (\sum_{i=1}^{s}(2(n-2i+1)+2\alpha+1)^2\int_{-1}^{1}|J_{n-2i+1}^{\alpha}(x)|^2(1-x^2)^{\alpha}dx\cdot\prod_{j=1}^{2i-1}\frac{(n+\alpha-j+1)^2}{(n+2\alpha-j+1)^2})\\ +&\ \prod_{i=1}^{2s}\frac{(n+\alpha-i+1)^2}{(n+2\alpha-i+1)^2}\int_{-1}^{1}|\frac{d}{dx}J_{n-2s}^{\alpha}(x)|^2(1-x^2)^{\alpha}dx, \qquad s<\frac{n}{2}.\end{aligned} \tag{7.3}$$

Note that by [[**1**], [**29**]]

$$\begin{aligned}&\int_{-1}^{1}|J_{n-2i+1}^{\alpha}(x)|^2(1-x^2)^{\alpha}dx\\ =&\ \frac{2^{2\alpha+1}\Gamma^2(n-2i+\alpha+2)}{(2n-4i+2\alpha+3)\Gamma(n-2i+2)\Gamma(n-2i+2\alpha+2)},\end{aligned} \tag{7.4}$$

and

$$\prod_{j=1}^{2i-1}\frac{(n+\alpha-j+1)^2}{(n+2\alpha-j+1)^2}=\frac{\Gamma^2(n+\alpha+1)}{\Gamma^2(n+\alpha-2i+2)}\frac{\Gamma(n+2\alpha-2i+2)}{\Gamma(n+2\alpha+1)}. \tag{7.5}$$

Furthermore,

$$\int_{-1}^{1} |\frac{d}{dx} J_{n-2s}^{\alpha}(x)|^2 (1-x^2)^{\alpha} dx = 0, \qquad s = [\frac{n}{2}] \tag{7.6}$$

where $[\frac{n}{2}]$ denote the largest integer $\leq \frac{n}{2}$ and by Strling formula, it can easily get:

$$\begin{aligned} & \sum_{i=1}^{s} (2(n-2i+1)+2\alpha+1)^2 \int_{-1}^{1} |J_{n-2i+1}^{\alpha}(x)|^2 (1-x^2)^{\alpha} dx \cdot \prod_{j=1}^{2i-1} \frac{(n+\alpha-j+1)^2}{(n+2\alpha-j+1)^2} \\ = & \sum_{i=1}^{s} 2^{2\alpha+1} (2n-4i+2\alpha+3) \frac{n+\alpha}{n+2\alpha} \\ & ((\frac{n+\alpha}{n+2\alpha})^{n+2\alpha})^2 (\frac{n+2\alpha-2i+1}{n-2i+1})^{n-2i+3/2} (\frac{n+2\alpha-2i+1}{n+\alpha})^{2\alpha}. \end{aligned}$$

It is easy to verify that

$$\frac{n+\alpha}{n+2\alpha}, \left(\frac{n+\alpha}{n+2\alpha}\right)^{n+2\alpha}, \frac{n+2\alpha-2i+1}{n-2i+1}, \left(\frac{n+2\alpha-2i+1}{n+\alpha}\right)^{2\alpha} \leq C, \text{for } \alpha > -1, \tag{7.7}$$

Combinatin of (7.3)-(7.7) leads to

$$\int_{-1}^{1} |\frac{d}{dx} J_{n}^{\alpha}(x)|^2 (1-x^2)^{\alpha} dx \leq C \sum_{i=1}^{k} (n-2i) \leq Cn^2. \tag{7.8}$$

□

Theorem 7.2. *Let α and β be two real numbers such that $-1 < \alpha < \beta$. Then the following inverse inequality holds for any polynomial $\phi_p(x)$ in $\mathcal{P}_p(I)$, $I = (-1, 1)$:*

$$\int_{-1}^{1} \phi_p^2(x)(1-x^2)^{\alpha} dx \leq C(p+1)^{2(\beta-\alpha)} \int_{-1}^{1} \phi_p^2(x)(1-x^2)^{\beta} dx. \tag{7.9}$$

Proof. Writing each $\phi_p(x)$ in $\mathcal{P}_p(I)$ as

$$\phi_p(x) = \sum_{n=1}^{p+1} a_n \Big(\frac{d}{dx} J_n^{\alpha}(x) \Big),$$

the Jacobi polynomials have property (J3) and (J4) from [**29**] :

$$\begin{aligned} \int_{-1}^{1} |\frac{d}{dx} J_n^{\alpha}(x)|^2 (1-x^2)^{\alpha+1} dx &= \frac{1}{4}(n+2\alpha+1)^2 \int_{-1}^{1} |J_{n-1}^{\alpha+1}(x)|^2 (1-x^2)^{\alpha} dx \\ &= \frac{1}{4}(n+2\alpha+1)^2 \gamma_{n-1}^{\alpha+1,\alpha+1} \\ &= \frac{2^{2\alpha+1} n(n+2\alpha+1)\Gamma^2(n+\alpha+1)}{(2n+2\alpha+1)\Gamma(n+1)\Gamma(n+2\alpha+1)}, \end{aligned}$$

where

$$\gamma_{n-1}^{\alpha+1,\alpha+1} = \frac{2^{2\alpha+3}\Gamma^2(n+\alpha+2)}{(2n+2\alpha+1)\Gamma(n)\Gamma(n+2\alpha+2)}.$$

We get

$$\int_{-1}^{1} \phi_p^2(x)(1-x^2)^{\alpha+1} dx = \sum_{n=1}^{p+1} a_n^2 \frac{2^{2\alpha+1} n(n+2\alpha+1)\Gamma^2(n+\alpha+1)}{(2n+2\alpha+1)\Gamma(n+1)\Gamma(n+2\alpha+1)} \simeq C \sum_{n=1}^{p+1} a_n^2 n.$$

By lemma (7.1) and Cauchy inequality, we have

$$\begin{aligned}
\int_{-1}^{1} \phi_p^2(x)(1-x^2)^\alpha dx &\leq \int_{-1}^{1} \Big((\sum_{n=1}^{p+1} a_n^2 n)(\sum_{n=1}^{p+1} \frac{1}{n} |\frac{d}{dx} J_n^\alpha(x)|^2) \Big) (1-x^2)^\alpha dx \\
&= \Big(\sum_{n=1}^{p+1} |a_n|^2 n \Big) \Big(\sum_{n=1}^{p+1} \frac{1}{n} \int_{-1}^{1} |\frac{d}{dx} J_n^\alpha(x)|^2 (1-x^2)^\alpha dx \Big) \\
&= C \Big(\sum_{n=1}^{p+1} a_n^2 n \Big) \Big(\sum_{n=1}^{p+1} n \Big) \\
&\leq C(p+1)^2 \int_{-1}^{1} \phi_p^2(x)(1-x^2)^{\alpha+1} dx.
\end{aligned}$$

The 7.9 is proved for $\beta = \alpha + 1$, and iterating the argument, it can be easily proved for any $\beta - \alpha$ be a positive integer. For the condition $\beta - 1 < \alpha < \beta$, let $\alpha = \frac{\beta-1}{p} + \frac{\beta}{q}$ in which $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{p} = \beta - \alpha < 1$. Then by Holder's inequality, we get

$$\begin{aligned}
\int_{-1}^{1} \phi_p^2(x)(1-x^2)^\alpha dx &= \int_{-1}^{1} \phi_p^{\frac{2}{p}}(x)(1-x^2)^{\frac{\beta-1}{p}} \phi_p^{\frac{2}{q}}(x)(1-x^2)^{\frac{\beta}{q}} dx \\
&\leq (\int_{-1}^{1} \phi_p^2(x)(1-x^2)^{\beta-1} dx)^{\frac{1}{p}} (\int_{-1}^{1} \phi_p^2(x)(1-x^2)^{\beta} dx)^{\frac{1}{q}} \\
&\leq (C(p+1)^2 \int_{-1}^{1} \phi_p^2(x)(1-x^2)^{\beta} dx)^{\frac{1}{p}} (\int_{-1}^{1} \phi_p^2(x)(1-x^2)^{\beta} dx)^{\frac{1}{q}} \\
&\leq C(p+1)^{2(\beta-\alpha)} \int_{-1}^{1} \phi_p^2(x)(1-x^2)^{\beta} dx.
\end{aligned}$$

For $\alpha < \beta - 1$, let $\beta = \beta' + m$ where m is a positive integer and $\beta' - 1 < \alpha < \beta'$, then

$$\begin{aligned}
\int_{-1}^{1} \phi_p^2 (1-x^2)^{\alpha} dx &\leq C(p+1)^{2m} \int_{-1}^{1} \phi_p^2 (1-x^2)^{\alpha+m} dx \\
&\leq C(p+1)^{2m}(p+1)^{2(\beta-\alpha-m)} \int_{-1}^{1} \phi_p^2 (1-x^2)^{\beta} dx \\
&\leq C(p+1)^{2(\beta-\alpha)} \int_{-1}^{1} \phi_p^2 (1-x^2)^{\beta} dx.
\end{aligned}$$

Thus we prove the theorem in general. □

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