CLASSICAL MESON THEORY

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ABSTRACT

The subject of the thesis is classical meson theory. The purpose in writing this thesis was to present meson theory from an elementary point of view, a point of view which could be understood by students in physics not familiar with quantum field theory. Thus the development of the meson field equations is carried through by analogy with familiar classical fields, such as the electromagnetic field. The various formulations of meson field theory are considered: scalar, pseudo-scalar, vector, and pseudo-vector. The charged and symmetric meson theories are introduced by regarding the isotopic spin of a nucleon as a classical vector in an abstract charge space.

A sufficient background is developed to enable the reader to follow the analysis of several interesting problems in meson physics. The questions of nuclear forces, the structure of the deuteron, meson production in nucleon-nucleon and photon-nucleon collisions, meson scattering by nucleons, are treated by applications of the classical theory.

Finally the classical results are compared with quantum mechanics and with experiment.

TABLE OF CONTENTS

1. INTRODUCTION:

The mesons we shall be mainly concerned with are the π -mesons, believed to be responsible for the specific nuclear interactions. We assume the reader is familiar with the early historical development of meson theory. Hideki Yukawa[1] was the first to suggest that nuclear forces might be explained by a field theory similar to electromagnetic theory. Because of the short range nature of nuclear forces, Yukawa found it necessary to associate with his field, particles of finite mass intermediate between that of the proton and electron. He was thus able to predict the existence and mass of a hitherto unobserved elementary particle, the meson. Several years after the publication of Yukawa's first paper on meson theory, a particle of approximately the correct mass was discovered as a constituent of cosmic rays[2]. It was subsequently demonstrated that this particle, later called the μ-meson, interacted only weakly

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1. Hideki Yukawa: 'On the Interaction of Elementary Particles, I' Proceedings of the Physico-Mathematical Society of Japan (3), 17, 48 (1935). Reprinted in 'Foundations of Nuclear Physics', Dover, 1949.

2. First observed by Neddermeyer, Anderson (1939) and independently by Blackett, Wilson (1939).

with nucleons. However, further research[3] uncovered the presence of yet another meson in cosmic rays. This latter meson, the π -meson, did in fact seem to possess the properties of Yukawa's field particles.

Present day meson theory is based on the formalism of quantum field theory. It is felt that many physicists possess insufficient mathematical background to permit them to follow the rather complicated structure of meson theory, but are otherwise quite capable of grasping the physics involved. Perhaps, then, there is a need for an introductory, classical theory of mesons - a classical theory which could provide a basis for the reading of current literature on experimental meson physics. It is true that there are present in the literature many classical or semi-classical treatments of problems in meson physics. However, nowhere does there exist a unified account of meson theory from an elementary point of view.

It is our purpose to discuss the classical field theory for the π -meson. We begin by establishing the defining equations for the various types of meson fields: scalar, pseudo-scalar, vector, pseudo-vector. This is accomplished by considering well known classical fields as analogies, the source terms of the meson field being patterned

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3. Powell et al, at Bristol, (1947).

after the source terms of the electrostatic, magnetostatic, and electromagnetic fields. Thus, just as one regards the electrostatic field as arising from a distribution of electric charge, so must one regard the meson field as arising from a distribution of nuclear matter.

By interpreting the meson as a field particle we find it possible to introduce electric charge into the meson field, and hence to set up the symmetric formulation of meson physics. It is in terms of this symmetric theory that we are able to discuss many interesting problems in meson physics. A solution of the static, symmetric, meson field equations makes possible a discussion of nuclear forces, and in addition yields a rather crude explanation of meson production in nucleon-nucleon collisions.

Within the classical framework, we also consider the scattering of mesons by nucleons and the production of mesons in photon-nucleon collisions.

Finally the results of the applications of our classical meson theory are compared with quantum theory and with experiment. The latter comparison is carried out to make the reader aware of the strengths and weaknesses of present day meson theory.

2. FORMALISM: FIELD EQUATIONS FOR MESON FIELD

2.1 Scalar Meson Field.

We have indicated that our purpose is to treat meson theory in a classical or semi-classical fashion. In doing so we will discuss two main formulations: first a scalar, and then a vector development of the meson field. We shall begin by considering the simpler of the two treatments, the scalar theory. One ~~would~~ says that a physical field ~~was~~ is scalar in nature, if it is possible to completely describe the field by specifying the value of a scalar $\phi(\vec{r},t)$ at an arbitrary space-time point $(\vec{r},t)$. Associated with this field ϕ are its sources, which may be given as definite functions $S(\vec{r},t)$ of time and position[1]. The manner in which the sources S give rise to the field ϕ is summed up in a field equation

$$D(\vec{r},t)\,\phi(\vec{r},t) = S(\vec{r},t) \tag{1}$$

where D is a multiplicative and differential operator.

Our problem with regard to the meson field ϕ is to choose appropriate functions characteristic of the nuclear sources for the field, and to relate these sources with ϕ through a set of field equations.

It will be instructive to consider a familiar scalar field, the classical electrostatic field arising from

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1. In such case one ignores the reaction of the field on its sources S.

a stationary distribution of electric charge. Let us suppose that in some region of space the field is described by the function $\psi(\vec{r})$ and the charge distribution by a density $\rho_Q(\vec{r})$. Then, it is well known that the behaviour of $\psi(\vec{r})$ will be governed by the Poisson equation

$$\nabla^2 \psi = -\rho_Q \tag{2}$$

(units chosen so dielectric constant = 4π)

At this point one often considers the force exerted on an elementary volume (dv) of charge, thereby introducing the notion of a field intensity. The force on dv will be

$$d\vec{F} = (\rho_Q \, dv)\, \vec{E} \tag{3}$$

where the field intensity is defined as

$$\vec{E} = -\nabla \psi \tag{4}$$

In terms of $\vec{E}$ it is possible to replace the single second order field equation (2) by a pair of first order equations. The first of these is the defining equation (4) for $\vec{E}$, and the second is obtained by substitution from (4) into (2).

The result is $\nabla \cdot \vec{E} = \rho_Q$ (5)

We are in particular interested in computing the field due to a single charged particle situated say at $\vec{r} = 0$. Such a charge distribution would be described by a density

$$\rho_Q = Q\,\delta(\vec{r}) \tag{6}$$

where Q is the charge on the particle and $\delta(\vec{r})$ is the Dirac delta function. Now a general solution of the Poisson equation (2) may be obtained by a Green's method. One uses the Green's function

$$\frac{1}{|\vec{r}-\vec{r}'|}$$

and obtains

$$\psi(\vec{r}) = \frac{1}{4\pi}\int \rho_Q(\vec{r}')\,\frac{dv'}{|\vec{r}-\vec{r}'|} \tag{7}$$

The field at $\vec{r}$ due to a single particle at the origin then is

$$\psi(\vec{r}) = \frac{Q}{4\pi r} \tag{8}$$

One notes that though the field $\psi(\vec{r})$ in (8) approaches zero as distance from the source increases, the fall off is rather gradual. For our present work, this is an important point. Experiments have shown that perhaps the main feature of specific nuclear forces is their very short range[±]. If then one is to develop a meson theory capable of explaining nuclear forces, one must incorporate in such a theory the short range nature of these forces.

This may be accomplished by selecting for the meson field a proper wave equation. One simple generalization of

~~1. One might say that nuclear forces determine the scale of the nucleus, coulomb electrostatic forces that of the atom. Since the nuclear radius $\sim 10^{-13}$ cm. and atomic radius $\sim 10^{-8}$ cm., one sees that the nuclear forces posses a much shorter range.~~

the Poisson equation (2) which will define a field, ϕ , of definite range is the following[1]:

$$(\nabla^2 - \kappa^2)\,\phi(\vec{r}) = \rho_n(\vec{r}) \tag{9}$$

(κ is some constant of dimensions $(\text{length})^{-1}$)

For suppose one seeks the solution of (9) corresponding to a point source, i.e.

$$\rho_n(\vec{r}) = g\,\delta(\vec{r}) \tag{10a}$$

g being some constant whose nature we leave unspecified at the moment.

A general solution of (9) is obtained through use of a Green's Function

$$\frac{e^{-\kappa|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}$$

The result is

$$\phi(\vec{r}) = -\frac{1}{4\pi}\int \rho_n(\vec{r}\,')\,\frac{e^{-\kappa|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\,dv' \tag{10b}$$

and the choice $\rho_n = g\,\delta(\vec{r})$ leads to a field

$$\phi(\vec{r}) = -\frac{g}{4\pi}\frac{e^{-\kappa r}}{r} \tag{11}$$

It is immediately evident that the field described by equation (11) will fall off far more rapidly for large r than will the coulomb field (8). One can in fact associate with the

1. The sign of ρ_n in the equation (9) is opposite to that of ρ_Q in the electrostatic field equation. The reason for this is that we wish to abide by standard convention for meson notation. The same question arises with reference to the meson field intensity.

field (11) a characteristic range $\frac{1}{K}$.

If one takes for $\frac{1}{K}$ a value in agreement with the experimentally determined range of nuclear forces, then perhaps equation (9) may be used as a basis for a scalar meson theory.

By analogy with the coulomb field, the source function $\rho_n(\vec{r})$ for the meson field will be taken as a density of nuclear source material. We shall in fact set $\rho_n(\vec{r}) = g\rho$, where $\rho(\vec{r})$ is to be the number of nuclear particles per unit volume. This would imply that we attribute to each nucleon, whether proton or neutron, a mesic charge g . In this respect the meson field would seem to parallel the gravitational field more closely than the electrostatic field - we have only one type of mesic charge.

Equation (11) will then give the field due to a single nucleon.

We may fix the units of g by considering the energy possessed by a nucleon placed in an external meson field. In electrostatics a charge Q placed in an external field ψ possesses energy $Q\psi$. Similarly, we shall take as the energy of a nuclear particle placed in a meson field ϕ , the quantity $g\phi$. Clearly then, $\frac{g^2 e^{-Kr}}{4\pi r}$ is to have the dimensions of energy, and g those of an electric charge.

It is also possible to introduce in meson field theory an intensity defined by

$$\vec{\chi} = \nabla \phi \tag{12}$$

Substitution into (9) then yields

$$\nabla \cdot \vec{\chi} = \kappa^2 \phi + g \rho(\vec{r}) \tag{13}$$

The equation (9) or the pair of equations (12) and (13) may be taken as defining equations for the scalar meson field. Actually such equations describe a rather limited field ϕ, in fact a static field. One would like to be able to discuss time varying fields arising from a moving distribution of nuclear matter. We shall demand that the structure of any time dependent meson theory be in conformity with the requirements of special relativity, i.e. we ask that the field equations be Lorentz invariant.[1]

We achieve a relativistic generalization of (9) or (12) and (13) by replacing all 3-dimensional quantities by corresponding 4-dimensional quantities. Such a transformation may be described by the following scheme:

$$\phi(\vec{r}) \longrightarrow \phi(\vec{r}, t)$$
$$\vec{\chi}(\vec{r}) \longrightarrow \chi_\nu \equiv (\vec{\chi}, \chi_4)$$
$$\nabla \cdot \vec{\chi} \longrightarrow \nabla \cdot \vec{\chi} + \frac{1}{ic} \frac{\partial \chi_4}{\partial t} \equiv \partial_\nu \chi_\nu$$

Finally we must obtain a replacement for the nuclear density $\rho(\vec{r})$. This is most readily done by introducing a velocity field $\vec{v}(\vec{r}, t)$ to describe the flow of nuclear matter. We

1. See Appendix A.

~~would~~ refer to $\vec{J} = \rho\vec{v}$ as a current density for this flow, and ~~would~~ then combine $\vec{J}$, ρ to form a 4-current $J_\nu \equiv (\vec{J}, ic\rho)$. The magnitude of this 4-current would be given by

$$J_\nu J_\nu = -c^2\rho^2(1 - v^2/c^2)$$

Hence it would follow

$$\rho\sqrt{1-v^2/c^2} = \sqrt{-\tfrac{1}{c^2} J_\nu J_\nu}$$

is a 4-scalar or invariant.

For the static field $v \to 0$, and $\sqrt{-\tfrac{1}{c^2} J_\nu J_\nu} \longrightarrow \rho$. Clearly then $\sqrt{-\tfrac{1}{c^2} J_\nu J_\nu}$ provides us with a suitable generalization of ρ.

The time dependent meson field equations, then, may be written down

$$\vec{\chi} = \nabla\phi \qquad \text{(a)}$$

$$\chi_4 = \frac{1}{ic}\frac{\partial\phi}{\partial t} \qquad \text{(b)} \ldots\ldots \text{(14)}$$

$$\nabla\cdot\vec{\chi} + \frac{1}{ic}\frac{\partial\chi_4}{\partial t} = K^2\phi + \rho g\sqrt{1-v^2/c^2} \qquad \text{(c)}$$

The Lorentz invariance of the field equations becomes apparent if one makes use of the 4-dimensional notation

$$\chi_\nu = \partial_\nu\phi \qquad \text{(a)}$$

$$\partial_\nu\chi_\nu = K^2\phi + g\sqrt{-\tfrac{1}{c^2} J_\nu J_\nu} \qquad \text{(b)} \ldots\ldots \text{(15)}$$

Either set, (14) or (15), is equivalent to the single wave equation

$$(\Box^2 - K^2)\phi = g\rho\sqrt{1-v^2/c^2} = g\rho\beta \qquad (16)$$

$$\left(\begin{array}{l} \beta = \sqrt{1-v^2/c^2} \\ \Box^2 \equiv \partial_\nu\partial_\nu \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \end{array} \right)$$

We see that the presence of nuclear matter makes itself felt only in the divergence equation (15b), through the scalar term $g\sqrt{-\frac{1}{c^2}J_\nu J_\nu}$.

We refer to such a situation as a scalar coupling of the meson field to its sources. A vector coupling may be introduced by adding to $\partial_\nu\phi$, in the definition of χ_ν, some 4-vector describing the nuclear sources. A rather obvious choice is the 4-current J_ν. We then might replace equation (15b) by

$$\chi_\nu = \partial_\nu\phi + g_z J_\nu \tag{15b'}$$

where g_z is a coupling constant. It will now be shown that the vector coupling contributes nothing to the field ϕ.

If we assume conservation of nuclear matter, then the flow of such matter must obey a continuity equation

$$\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0 \tag{16}$$

Consider the wave equation resulting from both scalar and vector coupling, i.e. from equation (15a) and (15b'). This wave equation will be

$$(\Box^2 - K^2)\phi = g\rho\sqrt{1-v^2/c^2} - g_z\partial_\nu J_\nu \tag{17}$$

The contribution from vector coupling vanishes by virtue of the continuity equation (16). Hence we will not lose any of the physical content of our theory if we ignore the vector coupling.

In later applications of meson theory we shall be interested in energy flow in the meson field. As a necessary part of our formalism, then, we must consider the possibility of energy retention in the field, and of energy contribution to the field by its source. We attempt to sum up such considerations in a continuity equation describing the transport of energy by the field.

To form this continuity equation we examine the vector

$$\vec{S} = \frac{c}{2} \chi_4 \vec{\chi} = -\dot{\phi} \nabla \phi , \quad \left(\frac{\partial \phi}{\partial t} \equiv \dot{\phi} \right) \tag{18}$$

We compute the divergence of this vector and substitute from the wave equation (17) to obtain

$$\nabla \cdot \vec{S} + \frac{\partial}{\partial t} (H_0) = -g \beta \dot{\phi} \rho \tag{19}$$

where $$H_0 = \frac{1}{2} \kappa^2 \phi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2c^2} \dot{\phi}^2$$

This latter equation (19) is a continuity equation expressing the conservation of some physical entity characteristic of the field ϕ . To deduce the nature of this entity we examine the term $-g \rho \beta \dot{\phi}$ which is clearly a source for this field entity. By analogy with electrostatics we might attribute an energy density $g\rho\beta\phi$ to a distribution of nuclear matter, ρ , immersed in a meson field ϕ . It is then reasonable to suppose $-g\beta\rho\dot{\phi}$ represents the rate at which the nuclear sources contribute energy to the meson field[1]. This leads us to designate $\dot{H}_0$ as the energy density for the meson field, and $\vec{S}$ as the energy current density.

In quantum mechanics one refers to $g\beta\rho\phi$ as an interaction energy density peculiar to the nuclear sources, and writes the field energy density in the form

$$H = H_0 + g\beta\rho\phi$$

1. Let us suppose that the meson field is set up by a single nucleon. We expect that there is a true energy conservation in the system nucleon plus meson field. Thus, by integrating the right side of (19) over the nuclear volume we obtain as the time rate of change of the nucleon total energy $\mathcal{H}$

$$\frac{d\mathcal{H}}{dt} = g\beta\,\dot{\phi}(\vec{r}_n, t)$$

($\dot{\phi}$, β being evaluated at the nucleon position.)
Now one has as a theorem in particle mechanics $\frac{d\mathcal{H}}{dt} = -\frac{\partial L}{\partial t}$,

where $\mathcal{H}$ is the total energy or Hamiltonian for the particle and L is its Lagrangian. It is a simple matter to compute the Lagrangian for the nucleon - one adds to the free particle Lagrangian a contribution from the meson field, an interaction Lagrangian.

The relativistic free particle Lagrangian is

$$L_0 = -Mc^2\beta$$

M being the nucleon mass,

while the interaction Lagrangian is

$$L' = -\int_{\substack{\text{nucleon}\\\text{volume}}} g\beta\rho\phi\, dv = g\beta\phi(\vec{r}_n,t)\ , \quad \rho = \delta(\vec{r}-\vec{r}_n)$$

Hence the total nucleon Lagrangian is

$$L = -(Mc^2 + g\phi)\beta$$

and one finds

$$\frac{d\mathcal{H}}{dt} = g\beta\dot{\phi}(\vec{r}_n,t)$$

Thus energy conservation in the system nucleon plus meson field is guaranteed.

The fact that one has labelled $g\beta\rho\phi(\vec{r},t)$ as an interaction energy density might suggest the potential energy of a nucleon in a meson field is $g\beta\phi(\vec{r}_n,t)$.

Actually the total energy for the nucleon is

$$\mathcal{H} = \frac{mc^2}{\beta} + \frac{g\phi(\vec{r}_n,t)}{\beta}$$

and hence the nucleon potential energy is $\frac{g\phi(\vec{r}_n,t)}{\beta}$.

In any case, the nucleon at rest in a scalar field ϕ possesses an energy $g\phi(\vec{r}_n)$.

2.2 Pseudo-Scalar Meson Field.

Suppose in place of electrostatics we had selected as a guiding analogy for meson theory the magnetic field set up by magnetized materials. One generally pictures magnetic material as consisting of elementary magnetic dipoles. The result is that on a macroscopic level the material possesses a net dipole moment per unit volume, $M(\vec{r})$ say.

The magnetic field intensity due to such a distribution of magnetic material is

$$\vec{B}(\vec{r}) = -\nabla\psi(\vec{r}) + \vec{M}(\vec{r}) \tag{1}$$

where ψ is to be considered as a potential for the magnetostatic field. As a consequence of the non-existence of magnetic monopoles one has

$$\nabla\cdot\vec{B} = 0 \tag{2}$$

From (1) and (2) one could deduce as the basic equation for the magnetic field ψ ,

$$\nabla^2\psi = \nabla\cdot\vec{M} \tag{3}$$

We note that $\vec{M}$ is not a true 3-vector. In fact, the magnetic moment $\vec{M}$ is very similar to an angular momentum, in that it possesses the properties of a cross product of two vectors. We suppose $\vec{a}$, $\vec{b}$ are a pair of 3-vectors, and we form the product $\vec{a} \times \vec{b}$. Then we consider a reflection of all three co-ordinate axes used in representation