The Impedance Matching and Reduction - of - Variation Capabilities of Cascaded Identical Twoports

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ABSTRACT

The input impedance to a cascade of identical twoports approaches, as the cascade lengthens, an impedance fixed by the parameters of the twoports no matter what termination is used except for a few special cases of no practical interest. For a variable load impedance, the input impedance at the n^{th} twoport lies within a given range of this fixed impedance. With the use of circles which enclose the variations in load and input impedances, equations are given which relate these circles to the ABCD parameters of the network. For certain values of the Thevenin equivalent impedance connected to the input of the network, the ABCD parameters are obtained for one twoport. The insertion loss is shown to be dependent upon the parameters of the network for the above fixed values of the Thevenin equivalent impedance.

A secondary result is the expressing of the Chebychev polynomial of the second kind as a ratio, rather than as a sum, of terms. A discussion on the use of the technique presented in this thesis as a method of designing a cascaded network for matching a variable load impedance to a fixed impedance is presented. Examples illustrating the design of a network of one section, for a given load condition, are also presented.

CONTENTS

LIST OF DIAGRAMS

CHAPTER I

INTRODUCTION

Purpose and Outline

The purpose of this thesis is to investigate the impedance matching capabilities at a fixed frequency of a network of cascaded identical twoports and to determine the parameters required for these twoports to match a variable load condition. To help in the evaluation of this method of impedance matching, the ensuing insertion loss is to be considered.

It is shown that variations in load impedance can be reduced by the use of the cascade of twoports. It is also shown that the insertion loss is dependent only upon the parameters of the twoports if certain conditions on one of the parameters are satisfied.

For the specific case of one twoport, the parameters are obtained for any variation in load impedance. Some examples illustrating the use of the equations obtained are presented.

A discussion of the results of this thesis is presented with some remarks on possible extensions and expansions of the work.

Background Material

Most of the following introductory material can be found in Johnson[1] and Eldring and Johnson[2]. Because this thesis follows a different line of reasoning than that presented in the above papers, the emphasis on the material presented in this section will be different.

Consider a cascade of identical twoports with a variable terminating impedance, Z_L (see Fig. 1). Each section is to be described by its ABCD, or chain, parameters, so that for the $(k + 1)^{th}$ section the following applies:

$$\begin{bmatrix} V_{k+1} \\ I_{k+1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} V_k \\ I_k \end{bmatrix} \tag{1}$$

$$= \underline{M} \begin{bmatrix} V_k \\ I_k \end{bmatrix} \tag{2}$$

and,

$$Z_{k+1} = \frac{A\ Z_k + B}{C\ Z_k + D} \tag{3}$$

where the symbols are those shown in Fig. 1.

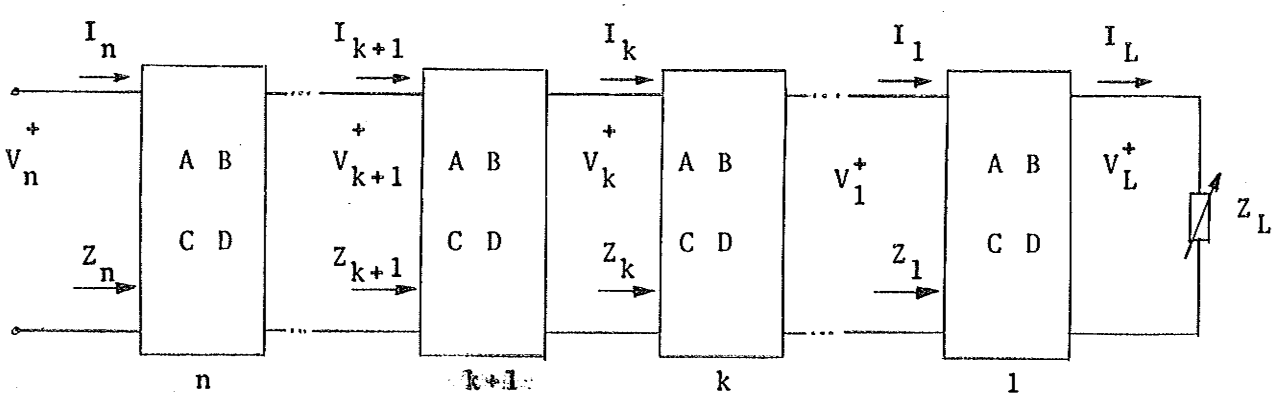

Fig. 1. Cascade of Identical Twoports.

The theorem presented by Johnson[1] makes use of the sequence of input impedances $\{Z_k\}$ and the fixed points of equation (3). It can be shown that the fixed points of (3) (i.e. for $Z_{k+1} = Z_k$) are given by:

$$Z_{s,u} = \frac{(A - D) \pm \sqrt{(A - D)^2 + 4BC}}{2C} \tag{4}$$

The value that is associated with Z_s is the value which, where possible, satisfies the condition:

$$\left| \frac{d\, Z_{k+1}}{d\, Z_k} \right|_{Z_k = Z_s} < 1 \tag{5}$$

The other fixed point, which will yield a magnitude of the above derivative greater than unity, is labelled Z_u.

The theorem states that the sequence of input impedances $\{Z_k\}$ will approach Z_s if the derivative condition can be satisfied. For this reason Z_s is referred to as the "stable" iterative impedance, while Z_u is referred to as the "unstable" iterative impedance. The case for which the derivative condition cannot be satisfied is for the magnitude of the derivative equal to unity. In this case the sequence $\{Z_k\}$ does not approach a definite limit but rather it oscillates about either Z_s or Z_u (see Eldring and Johnson[2]).

A parameter that is used in Ford[3] and Eldring and Johnson[2] to describe the locus of the sequence of input impedances $\{Z_k\}$ is given by equation (6) and is introduced here to simplify further calculations.

$$K = \frac{A - CZ_s}{A - CZ_u} \qquad (6)$$

The parameters A and C are two of the ABCD parameters of the twoport and Z_s and Z_u are as defined by (4) and the derivative conditions. The magnitude of K can be shown to be less than unity for all ABCD (see Appendix I).

With a little manipulation, the following useful relations can also be shown:

$$K + \frac{1}{K} = \frac{(A + D)^2}{AD - BC} - 2 \qquad (7)$$

and,

$$\frac{(A + D)^2}{AD - BC} = \frac{(K + 1)^2}{K} \qquad (8)$$

The parameter K will be used extensively in the following chapters.

It is desirable that the input impedance to the n^{th} section, Z_n, be related directly to the terminating (load) impedance, Z_L. This is accomplished by expressing, in closed form, the n^{th} power of the ABCD matrix, M. The following work has been presented in Johnson[1] and is presented here to define some symbols to be used later, and to give the full matrix equations that are derived.

It is convenient that the matrix to be taken to the n^{th} power have a unity determinant (the reciprocal case). However, (AD - BC) is not to be assumed equal to unity. Therefore, by normalizing the matrix $\underline{M}$ by dividing by $(AD - BC)^{1/2}$ the non-reciprocal case may be included. That is:

$$\underline{M}' = \begin{bmatrix} \frac{A}{(AD - BC)^{1/2}} & \frac{B}{(AD - BC)^{1/2}} \\ \frac{C}{(AD - BC)^{1/2}} & \frac{D}{(AD - BC)^{1/2}} \end{bmatrix}$$

$$= \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \tag{9}$$

for which the characteristic equation is:

$$\lambda^2 - (A' + D')\lambda + 1 = 0 \tag{10}$$

or,

$$\lambda^2 - 2T'\lambda + 1 = 0 \tag{11}$$

where,

$$T' = 1/2(A' + D') = \frac{1/2(A + D)}{(AD - BC)^{1/2}} \tag{12}$$

is the half trace of the matrix $\underline{M}$.

The normalizing procedure must be extended to the voltage-current matrix so that the matrix equation for the

$(k + 1)^{th}$ twoport becomes

$$\frac{1}{(AD - BC)^{1/2}} \begin{bmatrix} V_{k+1} \\ I_{k+1} \end{bmatrix} = \underline{M}' \begin{bmatrix} V_k \\ I_k \end{bmatrix} \tag{13}$$

The Cayley - Hamilton theorem states that, if the matrix $\underline{M}'$ be substituted for λ in its characteristic equation, a valid matrix equation is obtained. It is true, therefore, that

$$\underline{M}'^2 = 2T'\underline{M}' - \underline{U} \tag{14}$$

where $\underline{U}$ is the unit matrix of order 2. This leads directly to (see Pease[4])

$$\underline{M}'^n = U_n(T') \; \underline{M}' - U_{n-1}(T) \; \underline{U} \tag{15}$$

where $U_n(T')$ are the Chebyshev polynomials of the second kind given by

$$U_n(T') = \frac{\sinh (n \cosh^{-1} T')}{\sqrt{T'^2 - 1}} \tag{16}$$

(This form for $U_n(T')$ is used because T' will probably be complex. Equation (16) can be reduced to the equation given in Johnson[1] (note 11, page 36) for $|U_n(T')| < 1$.)

Now $\underline{M}'^n$ is the transfer matrix for the n - cascaded sections and so is given by

$$\underline{M}'^n = U_n(T') \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} - U_{n-1}(T') \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{M}'^{n} = \begin{bmatrix} U_n A' - U_{n-1} & U_n B' \\ U_n C' & U_n D' - U_{n-1} \end{bmatrix}$$

$$\equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (17)$$

where $U_n(T')$ has been replaced by U_n and $U_{n-1}(T')$ by U_{n-1}.

The normalization also affects the input voltage - current matrix as shown by (13). For the case of n sections instead of one section, (13) can be extended to:

$$\frac{1}{(AD - BC)^{n/2}} \begin{bmatrix} V_n \\ I_n \end{bmatrix} = \underline{M}'^{n} \begin{bmatrix} V_L \\ I_L \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} V_L \\ I_L \end{bmatrix} \quad (18)$$

where V_L and I_L are the load voltage and current and V_n and I_n are the input voltage and current to the n^{th} section, as shown in Fig. 1.

From (18) the input impedance to the n - cascaded sections terminated in Z_L is given by:

$$Z_n = \frac{aZ_L + b}{cZ_L + d} \quad (19)$$

where a, b, c, and d are defined by (17).

The cascade of n identical sections may now be replaced by one section with the input - output voltage - current relations given by (18) and (17). The input impedance has been found to depend upon the load impedance through the bilinear transformation of (19). While it seems that little has been gained by putting the equations describing the cascaded sections in this form, the next two chapters show how helpful equations (16) to (19) are.

CHAPTER II

INSERTION LOSS

The insertion loss is now to be considered because, in the evaluation of it, some assumptions are made which are used in later chapters and because the usefulness of the parameter K is illustrated.

The insertion loss is defined as "the decibel loss in power delivered to the load with the network inserted between the generator and load as compared with that when the generator and load are connected directly"[5].

The network to be considered is the n - cascade of identical sections used in Chapter I. By reference to Figures 2 and 3, the insertion loss is now to be calculated in terms of the ABCD parameters of each section. V_g and Z_g represent the Thevenin equivalent voltage source and impedance, respectively, of the network connected to the input terminals of the cascaded twoports.

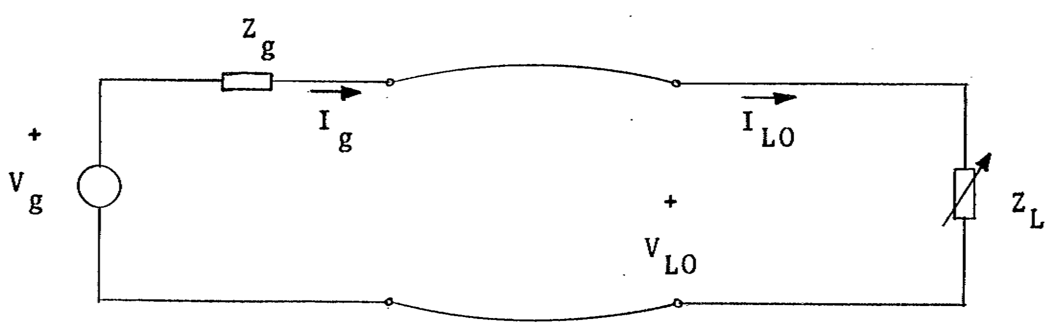

Fig. 2 - Load connected directly to generator

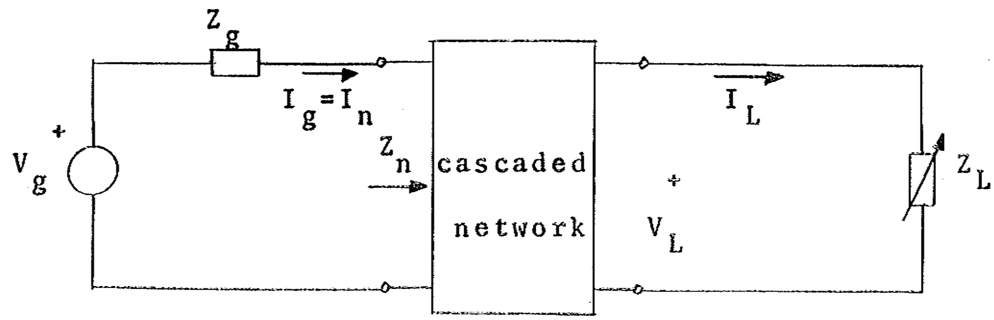

Fig. 3 - Network inserted between load and generator.

The insertion loss, I, can now be given as

$$I = 10 \log_{10} \frac{|V_{LO}|^2 / \; |Z_L|}{|V_L|^2 / \; |Z_L|}$$

$$= 10 \log \left| \frac{V_{LO}}{V_L} \right|^2$$

$$= 10 \log \; |I_v|^2 \qquad (20)$$

where I_v is defined as the insertion voltage ratio. I_v is now to be determined in terms of K, Z_s, and the ABCD parameters.

For the network in Fig. 3, equation (18) relates the terminal conditions and is rewritten in (21).

$$\begin{bmatrix} V_n \\ I_n \end{bmatrix} = (AD - BC)^{n/2} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} V_L \\ I_L \end{bmatrix} \qquad (21)$$

To solve for the output voltage, (21) can be written as

$$\begin{bmatrix} V_L \\ I_L \end{bmatrix} = \frac{1}{(ad - bc)(AD - BC)^{n/2}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot \begin{bmatrix} V_n \\ I_n \end{bmatrix} \tag{22}$$

or, particularly

$$V_L = \frac{dV_n - bI_n}{(ad - bc)(AD - BC)^{n/2}}$$

$$= \frac{(dZ_n - b)I_n}{(ad - bc)(AD - BC)^{n/2}} \tag{23}$$

where

$$Z_n \equiv \frac{V_n}{I_n} .$$

But, from Fig. 3, it is evident that

$$I_n = \frac{V_g}{Z_g + Z_n} \tag{24}$$

and so, there results:

$$V_L = \frac{(dZ_n - b)}{(ad - bc)(AD - BC)^{n/2}} \cdot \frac{V_g}{(Z_g + Z_n)} . \tag{25}$$

From Fig. 2, it is evident that the voltage across the load is given by (26).

$$V_{LO} = \frac{V_g Z_L}{Z_g + Z_L} \tag{26}$$

However, with the use of (19), Z_L may be put in terms of Z_n to yield:

$$V_{LO} = \frac{V_g(dZ_n - b)}{-cZ_nZ_g + dZ_n + aZ_g - b} \tag{27}$$

Division of (27) by (25) yields:

$$I_v = \frac{V_{LO}}{V_L} = \frac{(ad - bc)(AD - BC)^{n/2}(Z_g + Z_n)}{-cZ_nZ_g + dZ_n + aZ_g - b} \tag{28}$$

Division of both numerator and denominator of (28) by $(Z_n + Z_g)$ and the substitututyion of the values for a,b,c, and d as given in (17), yields (29).

$$I_v = \frac{(ad - bc)(AD - BC)^{n/2}}{\left(-C'Z_g + D' + \frac{C'Z_g^2 + (A' - D')Z_g - B'}{Z_n + Z_g}\right) U_n - U_{n-1}} \tag{29}$$

It can be shown that the roots of the expression

$$C'Z_g^2 + (A' - D')Z_g - B' = 0$$

are

$$Z_g = Z_s - \frac{A - D}{C} = - Z_u \tag{30-a}$$

and

$$Z_g = - Z_s \tag{30-b}$$

by making use of (4) and the facts that

$$Z_s = Z_s' ,$$

(where the prime indicates that the ABCDs are primed) and

$$\frac{A - D}{C} = \frac{A' - D'}{C'} \quad .$$

Therefore, if Z_g is equal to either $-Z_s$ or $-Z_u$, the insertion voltage ratio, and hence the insertion loss, becomes independent of the input impedance Z_n and has the value

$$I_v = \frac{(ad - bc)(AD - BC)^{n/2}}{(-C'Z_g + D')U_n - U_{n-1}} , \quad Z_g = -Z_s \text{ or } -Z_u \tag{31}$$

For the two values of Z_g given by (30), it is possible to obtain I_v in terms of K and n. Before proceeding, however, it is necessary that the Chebyshev polynomial U_n be evaluated in terms of K and n.

Recall that U_n is defined as

$$U_n = \frac{\sinh (n \cosh^{-1} T')}{\sqrt{T'^2 - 1}} \tag{16}$$

where

$$T' = 1/2(A' + D') = \frac{K + 1}{2K^{1/2}} \tag{32}$$

It is known that

$$\cosh^{-1}T' = \log_e(T' + \sqrt{T'^2 - 1})$$

and when T' is replaced by $\frac{(K + 1)}{2K^{1/2}}$ equation (33) results.

$$\cosh^{-1}T' = \ln K^{1/2} \tag{33}$$

By use of (33), the expression

$$\sinh(n \cosh^{-1}T') = 1/2(e^{n \cosh^{-1}T'} - e^{-n \cosh^{-1}T'})$$

becomes

$$\sinh(n \cosh^{-1}T') = \frac{K^n - 1}{2K^{n/2}} \quad . \tag{34}$$

Therefore,

$$U_n = \frac{K^n - 1}{2K^{n/2}\sqrt{\frac{(K + 1)^2}{4K} - 1}}$$

$$= \frac{K^n - 1}{(K - 1)K^{(n-1)/2}} \tag{35}$$

From equation (4) it is known that

$$Z_s + Z_u = \frac{A - D}{C}$$

and hence (6) becomes

$$K = \frac{A - CZ_s}{D + CZ_s} = \frac{A' - C'Z_s}{D' + C'Z_s} \tag{36}$$

Now, by substituting (32) into (36), it may be seen that

$$D' + C'Z_s = \frac{1}{K^{1/2}} \tag{37}$$

and

$$A' - C'Z_s = K^{1/2} \tag{38}$$

For (ad - bc) equal to unity (see Appendix II) and for Z_g equal to $-Z_s$, I_v is found to be given by (39) when (35) and (37) are substituted into (31) and some cancellations are carried out.

$$I_v = K^{n/2}(AD - BC)^{n/2} \tag{39}$$

Also, for Z_g equal to $-Z_u$, the use of (35) and (38) produces

$$I_v = \frac{(AD - BC)^{n/2}}{K^{n/2}} \tag{40}$$

It now is possible to write the insertion loss in terms of K, n and the determinant of the chain matrix for two specific values of the generator impedance. That is,

$$I = 10 \log |K(AD - BC)|^n , \quad Z_g = -Z_s \tag{41}$$

and

$$I = 10 \log \left| \frac{AD - BC}{K} \right|^n , \qquad Z_g = -Z_u = Z_s - \frac{A - D}{C} \tag{42}$$

As (AD - BC) can be adjusted to any desired value, the insertion loss is essentially dependent only upon K. For equations (41) and (42) to be equal, K must be equal to unity, thus yielding the situation where Z_s and Z_u are the same impedance. While this condition is theoretically possible (see Ford[3] and Eldring and Johnson[2]) it is unstable and any slight change in parameters would cause a difference between Z_s and Z_u and hence a reduction in the magnitude of K.

Now, since K is always less than unity (for this thesis, at any rate), then, for a given (AD - BC), equation (41) will yield a smaller insertion loss than (42), for all n. However, (41) requires that either Z_g or Z_s must have a negative real part (except where both are lossless). Since Z_s is approximately the input impedance to the cascaded network, the network must have active sources if Z_s is to have a negative real part. Equation (42) suggests that Z_g and Z_s may both be positive real functions and hence the network may be passive. The use of active sources would seem to be indicative of a smaller insertion loss than for a passive network with the same magnitude of (AD - BC). The condition presented by (42) does not rule out active sources in the network, but it does indicate that the network may be composed of passive elements.

The problem of reducing the variations in load impedance by means of the cascade of identical twoports still remains. The equations involved in showing that there is a reduction in load impedance variations are presented in the next chapter.

CHAPTER III

REDUCTION IN IMPEDANCE VARIATIONS

This chapter presents equations which show that the cascade of identical twoports reduces variations in load impedance and changes the impedance about which these variations can be considered to revolve. This is accomplished by employing circles which enclose the load impedance variations and the input impedance variations of the network.

It has been stated in the Introduction that for a cascade of n - identical twoports, the sequence of input impedances $\{Z_k\}$ will approach a fixed impedance Z_s, which is independent of the load impedance Z_L (see Fig. 1 for diagram). If the input impedance to the n - cascaded twoports, Z_n, is required to be within a certain percentage of Z_s, then it is possible to obtain the ABCD parameters of the sections for a given variation in Z_L. By restricting the value of Z_n in this way, the matching of the input impedance and the generator impedance is made possible.

Suppose that Z_n is to be within a value ε of Z_s. That is,

$$|Z_n - Z_s| \leq \varepsilon \tag{43}$$

where

$$\varepsilon = \varepsilon' |Z_s|$$

and ε' is less than unity. Thus, Z_n lies within a circle of radius ε and center Z_s.

Because the variation in load impedance is arbitrary and therefore probably difficult to describe mathematically in the impedance plane, a circle which encloses the complete locus of Z_L may be used to advantage. If the bilinear transformation given by (19) is applied to this circle as well as to the load impedance, it can be shown that Z_n will also remain within the transformed circle. Therefore, wherever this transformed circle falls within the circle drawn about Z_s, the locus of Z_n will fall within ε of Z_s.

The evaluation of the radii and centers of each of the circles in terms of the parameters of the other circle, the ABCD parameters, K, and n is now considered. To aid in this evaluation, however, it is convenient to consider the circle about Z_s of radius ε to be the load impedance circle after n transformations by (3) (or by one transformation (19)).

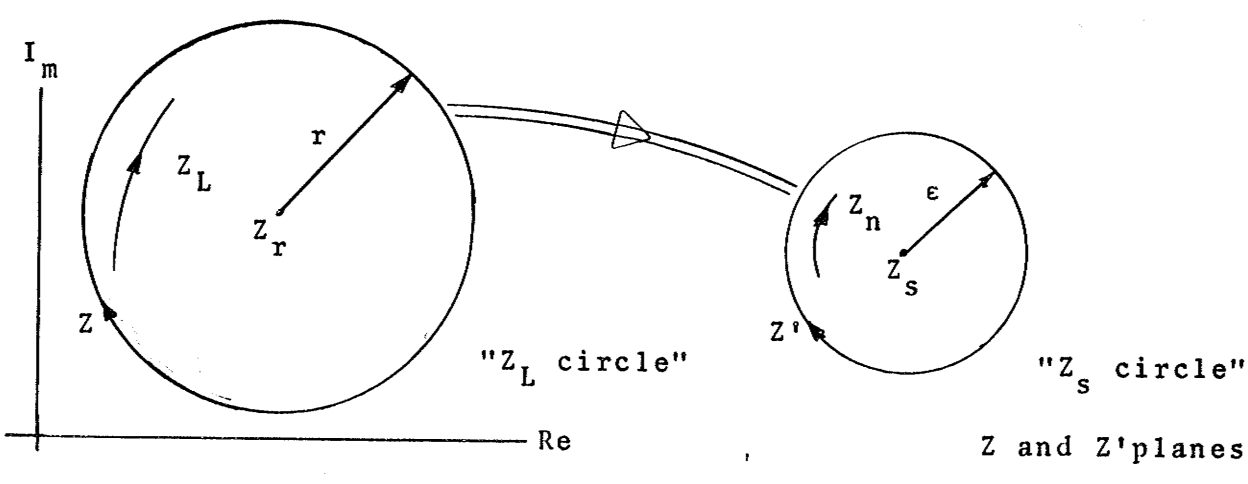

Fig. 4 - Load impedance and input impedance circles, (see below for definition of symbols).

Fig. 4 indicates the transformation to be carried out as well as the relation of the circles to each other. The Z_n and Z_L are arbitrarily drawn and are used only as an illustration of their relative positions in the impedance planes. The equation presented in Fig. 4 is (19) with Z_L replaced by Z and Z_n replaced by Z'. While each circle and the corresponding impedance variation should be in different planes, it is convenient to consider them on the same plane and denote their difference by a change in the plane variable (i.e. Z becomes Z' when Z_L has been transformed to Z_n). It is shown later in this chapter that Z_s must lie within the load impedance circle, but to simplify the diagram this restriction has been overlooked.

The problem now to be solved is that of expressing the center and radius of the load impedance circle (the "Z_L circle") in terms of the center and radius of the input impedance circle (the "Z_s circle"), K, and any of the ABCD parameters that cannot be eliminated. In the next chapter the equations presented here are used to solve for the ABCD parameters for various conditions on the network and the generator impedance.

Using standard notation (see Ford[3]) for the coefficients, the "Z_L circle" is given, in the Z plane, by

$$A_L Z\overline{Z} + B_L Z + \overline{B}_L \overline{Z} + C_L = 0 \qquad (44)$$

where the coefficients A_L and C_L are real and the bar over a symbol represents the complex conjugate of that symbol. Similarly, the "Z_s circle" is given, in the Z' plane, by

$$A_s Z'\overline{Z'} + B_s Z' + \overline{B}_s \overline{Z'} + C_s = 0 \tag{45}$$

where the coefficients A_s and C_s are real.

The transformation taking the "Z_L circle" to the "Z_s circle" is given by (46).

$$Z' = \frac{aZ + b}{cZ + d} \tag{46}$$

where a, b, c, and d are as defined by (17).

The coefficients of the "Z_L circle" are written in terms of those of the "Z_s circle" by substituting (46) into (45) and putting the resulting equations in the form of (44) so that by comparison with (44) the following results:

$$\left.\begin{aligned} A_L &= a\overline{a}A_s + a\overline{c}B_s + c\overline{a}\overline{B}_s + c\overline{c}C_s \\ B_L &= a\overline{b}A_s + a\overline{d}B_s + c\overline{b}\overline{B}_s + c\overline{d}C_s \\ C_L &= b\overline{b}A_s + b\overline{d}B_s + d\overline{b}\overline{B}_s + d\overline{d}C_s \end{aligned}\right\} \tag{47}$$

The center, Z_r, and radius, r, of the "Z_L circle" are given in terms of the coefficients of (44) by (48) and (49).

$$Z_r = \frac{-\overline{B}_L}{A_L} \tag{48}$$

$$r = \frac{(B_L\overline{B}_L - A_L C_L)^{1/2}}{A_L} \tag{49}$$

To obtain Z_r and r in terms of ε, Z_s, K and the A'B'C'D' parameters it is necessary that the coefficients of the "Z_s circle" be expressed in terms of its center and radius. The "Z_s circle" is given, in the Z' plane, by

$$|Z' - Z_s| = \varepsilon$$

or

$$|Z' - Z_s|^2 = (Z' - Z_s)(\overline{Z}' - \overline{Z}_s) = \varepsilon^2$$

and

$$Z'\overline{Z}' - \overline{Z}_s Z' - Z_s\overline{Z}' + (Z_s\overline{Z}_s - \varepsilon^2) = 0 \tag{50}$$

Comparison of (50) with (45) produces:

$$\left.\begin{aligned} A_s &= 1 \\ B_s &= -\overline{Z}_s \\ C_s &= Z_s\overline{Z}_s - \varepsilon^2 \end{aligned}\right\} \tag{51}$$

When the above values of A_s, B_s, and C_s are substituted into (47), A_L becomes

$$A_L = |a - cZ_s|^2 - |c|^2\varepsilon^2$$

With the use of the values of a and c as given by (17)

$$A_L = |U_n(A' - C'Z_s) - U_{n-1}|^2 - \varepsilon^2|C'|^2|U_n|^2$$

For U_n given by (35), $(A' - C'Z_s)$ equal to $K^{1/2}$ (equation (38)), and some manipulation and cancellation, equation (52) is obtained.

$$A_L = |K|^n - \frac{\varepsilon^2 |C'|^2 |K^n - 1|^2}{|K|^{n-1} \quad |K-1|^2} \qquad (52)$$

$\bar{B}_L$ and thence Z_r are obtained in a manner similar to that used for obtaining A_L. The work is presented in Appendix III with the final value of Z_L given by (53).

$$Z_r = \frac{1}{C'} \left[-D' + \frac{1}{K^{1/2}} + \frac{\varepsilon^2 |C'|^2 \, (\bar{K}^n - 1)(K-1)}{K^{1/2} (|K|^{2n-1} |K-1|^2 - \varepsilon^2 |C'|^2 |K^n - 1|^2)} \right] \qquad (53)$$

Since it is possible to prove equation (54),

$$(B_L \bar{B}_L - A_L C_L)^{1/2} = \varepsilon \qquad (54)$$

(See Appendix III - B for details) r is given by (55) when the value of A_L in (52) is used in (49).

$$r = \frac{\varepsilon |K|^{n-1} |K - 1|^2}{|K|^{2n-1} |K - 1|^2 - \varepsilon^2 |C'|^2 |K^n - 1|^2} \qquad (55)$$

The center Z_r and the radius r of the "Z_L circle" have now been found in terms of C', K, n, ε, and Z_s (D' is a function of C', K and Z_s, - see (37)). Therefore, for any given n, the load impedance circle is known if K, C' and the "Z_s circle" are given. The converse problem, that of finding K, Z_s, and C' when the load impedance circle is known, is solved in the next chapter, but for one section only.

By use of equations (53) and (55) it is possible to relate the difference between Z_r and Z_s to r. Substituting for D' as given in (37) into (53) and using (55) to simplify the last term of (53), Z_r is given by (56).

$$Z_r = Z_s + \frac{r\varepsilon\overline{C}' \, (\overline{K}^n - 1)}{K^{1/2}|K|^{n-1} \, (\overline{K} - 1)} \tag{56}$$

and

$$|Z_r - Z_s|^2 = \frac{r^2\varepsilon^2|C'|^2}{|K|^{2n-1}} \left|\frac{(\overline{K}^n - 1)}{(K - 1)}\right|^2 \tag{57}$$

But from (55), ε may be obtained in terms of r as

$$\varepsilon = \frac{|K|^{n-1}}{2r|C'|^2} \left|\frac{K - 1}{K^n - 1}\right|^2 \left[-1 \pm \sqrt{1 + 4r^2|C'|^2|K| \left|\frac{K^n - 1}{K - 1}\right|^2} \right] \tag{58}$$

For ε to be greater than zero, the plus sign before the radical of (58) must be used. By substitution of the value for ε given by (58) into (57) and reduction of the resulting equation to its simplest form, equation (59) results:

$$|Z_r - Z_s|^2 = r^2 - \frac{\varepsilon r}{|K|^n} \tag{59}$$

That is, the stable iterative impedance Z_s must lie within the load impedance circle for all n. This definitely is a restriction upon the circle that may be drawn about the load impedance, or, conversely, it is a restriction upon Z_s. Since Z_s may have to be set at a specific value determined either by the insertion loss or the matching problem, there may be some difficulty in describing a load impedance circle which encloses the load impedance as well as Z_s.

It is seen from (59) that the distance between Z_r and Z_s is dependent upon the magnitude of K and the number of sections n. For $|K|^n$ close to unity, which produces minimum insertion loss in the passive network case, Z_s and Z_r will have the maximum separation that is possible. As $|K|^n$ decreases, Z_s moves towards Z_r, and for $|K|^n$ equal to ε/r, Z_s equals Z_r.

While it has been assumed that ε is known, it is, in reality, unknown because it depends upon $|Z_s|$ which is usually unknown for a given problem. However, (59) indicates that the magnitude of Z_s is within r of the magnitude of Z_r and, in most practical cases, probably much less. Therefore, Z_s may be approximated by Z_r so that ε may be given as

$$\varepsilon \approx \varepsilon' |Z_r| . \tag{60}$$

This means that ε may be assumed known at all times and if ε, as calculated from Z_s, is much different from ε found by using Z_r, then the calculations can be repeated for the new ε.

Up to this point it has been assumed that ε is less than r. While this assumption might not be evident from the work presented, equation (59) shows that it must be true. That is, for (59) to hold,

$$|K|^n \geq \frac{\varepsilon}{r} . \tag{61}$$

Because $|K|$ is less than unity at all times, ε must be less than r at all times. Therefore, there definitely is a reduction in impedance variation from load to input. Equation (61) also indicates that for increasing n, with K and r constant, ε may be decreased thus reducing the load variation even more.

The problem of solving for some of the parameters in terms of the remaining ones for the specific case of n equal to one, is to be considered in the next chapter.

CHAPTER IV

EVALUATION OF PARAMETERS FOR ONE SECTION (n = 1), AND EXAMPLES

The equations needed to solve for some of the parameters in terms of other parameters for one section, or twoport, are now considered. There are many combinations of known and unknown parameters that may be used but only a few of these combinations are presented. In all cases, the final result is the determination of the ABCD parameters.

It is assumed, in all cases, that a load variation is known and must either be matched to a generator (or fixed) impedance, or must have its variation reduced. A network of one twoport is found that satisfies either or both of these conditions. The equations to be used viz: (53), (56), and (55) for n = 1 are, respectively:

$$Z_r = \frac{1}{C'}\left(-D' + \frac{\bar{K}^{1/2}}{|K| - \varepsilon^2|C'|^2}\right) \tag{62}$$

$$= Z_s + \frac{r\varepsilon\bar{C}'}{K^{1/2}} \tag{63}$$

and

$$r = \frac{\varepsilon}{|K| - \varepsilon^2|C'|^2} \tag{64}$$

The two values of the generator impedance, Z_g, obtained in Chapter II are used extensively in the remainder of this chapter.

Both values are functions of Z_s, K and C', although, at first glance

$$Z_g = -Z_u = Z_s - \frac{A' - D'}{C'} \qquad (30\text{-a})$$

is not. However, when (37) and (38) are used for A' and D', (30-a) becomes

$$Z_g = -Z_s + \frac{1-K}{C'K^{1/2}} \qquad (65)$$

<u>Case 1</u>

Consider a circle drawn about the load variation thereby fixing Z_r and r. Also assume that ε is known. The problem is to solve for C', K, and Z_s and thence A', B', and D'. To do this, however, it is not enough that Z_r, r and ε are known. Therefore, Z_g is introduced as a known parameter and must have the value of either $-Z_s$ or $-Z_u$, thus ensuring a calculable insertion loss.

<u>(a)</u> $Z_g = -Z_u = Z_s - \dfrac{A' - D'}{C'}$

Substituting for A' from (38) into the above equation, it can be shown that

$$D' = K^{1/2} + C'Z_g \qquad (66)$$

By the use of (66), equation (62) can be put in the following form:

$$C'(Z_r + Z_g) = -K^{1/2} + \frac{\overline{K}^{1/2}}{|K| - \varepsilon^2 \; |C'|^2} \qquad (67)$$

With

$$\frac{r}{\varepsilon} = \frac{1}{|K| - \varepsilon^2 |C'|^2}$$

from (64), (67) becomes

$$\varepsilon C'(Z_r + Z_g) = -\varepsilon K^{1/2} + r\bar{K}^{1/2} \tag{68}$$

K may be found from (68) by using:

$$C'(Z_r + Z_g) \equiv a_1 + ja_2 = |C'| \; |Z_r + Z_g| e^{j\phi} \tag{69}$$

and

$$K \equiv |K| e^{j\theta} = |K| \; (\cos\theta + j\sin\theta) \; . \tag{70}$$

Substituting (69) and (70) into (68) and equating real and imaginary terms yields:

$$|K|^{1/2} \cos\theta/2 = \frac{\varepsilon \; a_1}{r - \varepsilon}$$

and

$$|K|^{1/2} \sin\theta/2 = -\frac{\varepsilon \; a_2}{r + \varepsilon}$$

Therefore, K and $|K|$ are given by (71) and (72) respectively:

$$K = (K^{1/2})^2 = \left(\frac{\varepsilon \; a_1}{r - \varepsilon} - j \; \frac{\varepsilon \; a_2}{r + \varepsilon} \right)^2$$

$$= \varepsilon^2 |C'|^2 |Z_r + Z_g|^2 \left(\frac{\cos\phi}{r - \varepsilon} - j \frac{\sin\phi}{r + \varepsilon} \right)^2 \tag{71}$$

$$|K| = \varepsilon^2 |C'|^2 \; |Z_r + Z_g|^2 \; \left(\frac{\cos^2\phi}{(r-\varepsilon)^2} + \frac{\sin^2\phi}{(r+\varepsilon)^2} \right) \tag{72}$$

The dependence of K on the magnitude of C' may be removed by solving for $|C'|^2$ from (72) and (64) and substituting this value into (71) yielding:

$$|C'|^2 = \frac{1}{\epsilon r \left[|Z_r + Z_g|^2 \left(\frac{\cos^2\phi}{(r-\epsilon)^2} + \frac{\sin^2\phi}{(r+\epsilon)^2} \right) - 1 \right]} \tag{73}$$

and

$$K = \frac{\epsilon |Z_r + Z_g|^2 \left(\frac{\cos\phi}{r-\epsilon} - j \frac{\sin\phi}{r+\epsilon} \right)^2}{r \left[|Z_r + Z_g|^2 \left(\frac{\cos^2\phi}{(r-\epsilon)^2} + \frac{\sin^2\phi}{(r+\epsilon)^2} \right) - 1 \right]} \tag{74}$$

where

$$\phi = \arg (Z_r + Z_g) + \arg C' \tag{75}$$

While K and $|C'|^2$ still depend upon the argument of C', it is possible to choose this angle such that equations (73) and (74) are simplified. If this is done, C' is then completely determined.

For ϕ either 0° or 180°, $|C'|^2$ and K are given by (76) and (77):

$$|C'|^2 = \frac{1}{\epsilon r \left[\frac{|Z_r + Z_g|^2}{(r - \epsilon)^2} - 1 \right]} \tag{76}$$

$$K = \frac{\epsilon}{r \left[1 - \frac{(r-\epsilon)^2}{|Z_r + Z_g|^2} \right]} \tag{77}$$

For ϕ either 90° or 270°, $|C'|^2$ and K are given by (78) and (79);

$$|C'|^2 = \frac{1}{\varepsilon r\left[\frac{|Z_r + Z_g|^2}{(r+\varepsilon)^2} - 1\right]} \tag{78}$$

$$K = \frac{-\varepsilon}{r\left[1 - \frac{(r+\varepsilon)^2}{|Z_r + Z_g|^2}\right]} \tag{79}$$

The condition for the existence of $|C'|$ and therefore, K is

$$|Z_r + Z_g| > (r + \varepsilon)$$

It is noticed that K as given by (77) is real and positive while K as given by (79) has an angle of 180°. It is also seen that for $|Z_r + Z_g|$ much greater than $(r + \varepsilon)$:

$$|K| \approx \frac{\varepsilon}{r}$$

no matter what value ϕ has.

The stable iterative impedance Z_s is found by substituting the value for K given in (71) into (63) and solving for Z_s:

$$Z_s = Z_r - \frac{r \;\underline{/- \arg C'}}{|Z_r + Z_g|\left(\frac{\cos\phi}{r - \varepsilon} - j\,\frac{\sin\phi}{r + \varepsilon}\right)} \tag{80}$$

$$Z_s = Z_r - \frac{r \angle \tan^{-1}\left(\frac{r-\varepsilon}{r+\varepsilon}\tan\phi\right) - \arg C'}{|Z_r + Z_g|\left(\frac{\cos^2\phi}{(r-\varepsilon)^2} + \frac{\sin^2\phi}{(r+\varepsilon)^2}\right)^{1/2}} \tag{81}$$

<u>(b)</u> $Z_g = -Z_s$

K is obtained from (63) to yield:

$$K = \frac{(\varepsilon r)^2 \, \overline{C}'^2}{(Z_r - Z_s)^2} \tag{82}$$

or

$$K = \frac{(\varepsilon r)^2 \, |C'|^2}{|Z_r + Z_g|^2} \angle -2(\arg C' + \arg(Z_r + Z_g)) \tag{83}$$

To remove the dependence of K on C', take the magnitude of (83) and with (64) solve for $|C'|^2$. Then,

$$|C'|^2 = \frac{|Z_r + Z_g|^2}{\varepsilon r \, (r^2 - |Z_r + Z_g|^2)} \tag{84}$$

Substitution for $|C'|^2$ as given by (84) into (83) produces equation (85).

$$K = \frac{\varepsilon r \angle -2\phi}{r^2 - |Z_r + Z_g|^2} \tag{85}$$

where

$$\phi = \arg C' + \arg(Z_r + Z_g)$$

Therefore, K and C' have been obtained, although as with case 1(a), the argument of C' must be specified. In the case above, arg C' need not be made a multiple of 90° or 180°, although it may be desirable that C' be made such that K becomes real.