

## TWO NEW FINITE DIFFERENCE METHODS FOR COMPUTING EIGENVALUES OF A FOURTH ORDER LINEAR BOUNDARY VALUE PROBLEM

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**ABSTRACT.** This paper describes some new finite difference methods of order 2 and 4 for computing eigenvalues of a two-point boundary value problem associated with a fourth order differential equation of the form  $(py'''' + (q - \lambda r)y = 0$ . Numerical results for two typical eigenvalue problems are tabulated to demonstrate practical usefulness of our methods.

**KEY WORDS AND PHRASES.** Band-matrices, finite-difference methods, generalized eigenvalue problem, positive definite matrices, two-point boundary value problems.

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### 1. INTRODUCTION.

We shall consider the fourth order linear differential equation

$$\frac{d^2}{dx^2} \left[ p(x) \frac{d^2 y}{dx^2} \right] + [q(x) - \lambda r(x)]y = 0, \quad -\infty \leq a \leq x < b < \infty, \quad (1.1)$$

associated with the following pairs of homogeneous boundary conditions

$$y(a) = y(b) = y''(a) = y''(b) = 0. \quad (1.2)$$

Such boundary value problems occur in applied mathematics, engineering and modern physics, (see ref. [1-4]). In the differential equation (1.1) the functions  $p(x)$ ,  $q(x)$ ,  $r(x) \in C[a,b]$  and satisfy the conditions

$$p(x) > 0, q(x) \geq 0 \text{ and } r(x) > 0, x \in [a,b]. \quad (1.3)$$

We cannot compute the exact values of the eigenvalues  $\lambda$  for which the boundary value problem (1.1) - (1.2) has a nontrivial eigensolution  $y(x)$  for arbitrary choices of the functions  $p(x)$ ,  $q(x)$  and  $r(x)$ . We resort to numerical methods for computing approximate values of  $\lambda$ . The most commonly used technique for approximating  $\lambda$  for which the system (1.1) - (1.2) has a nontrivial eigenfunction  $y(x)$  is by finite difference methods.

Recently, the author [2] has analysed some new finite different methods of order 2 and 4 for computing eigenvalues of a two point boundary value problem involving the differential equation (1.1) with  $p(x) \equiv 1$  associated with one of the following pairs of homogeneous boundary conditions:

$$\begin{aligned} (a) \quad & y(a) = y(b) = y'(a) = y'(b) = 0 \\ (b) \quad & \text{the same boundary conditions as (1.2)} \\ (c) \quad & y(a) = y'(a) = y''(b) = y'''(b) = 0. \end{aligned} \tag{1.4}$$

Chawla and Katti [3] have developed a numerical finite difference method of order 2 for approximating the lowest eigenvalue  $\lambda$  of the system (1.1) - (1.4(a)) with  $p(x) \equiv 1$ . A fourth order method was later developed by Chawla [4] for the numerical treatment of the same problem. This latter method leads to a generalized seven-band symmetric matrix eigenvalue problem.

Let  $\lambda$  be any eigenvalue of the system (1.1) - (1.2) and let  $y(x) \neq 0$  be the corresponding eigenfunction. Then on multiplying (1.1) by  $y(x)$  and integrating the resulting equation from  $a$  to  $b$ , we find after integration by parts and on using (1.2), that

$$\lambda = \frac{\int_a^b p(y'')^2 dx + \int_a^b qy^2 dx}{\int_a^b ry^2 dx} > 0 \tag{1.5}$$

in view of (1.3).

The purpose of this brief report is to present two new finite difference methods for computing approximate values of  $\lambda$  for the system (1.1) - (1.2). These methods lead to generalized five-band and nine-band symmetric matrix eigenvalue problems and provide  $O(h^2)$  and  $O(h^4)$ -convergent approximations for the eigenvalues.

## 2. A SECOND ORDER METHOD

For a positive integer  $N \geq 5$ , let  $h = (b - a)/(N + 1)$  and  $x_i = a + ih$ ,  $i = 0(1)N + 1$ . We shall designate  $y_i = y(x_i)$ ,  $p_i = p(x_i)$ ,  $q_i = q(x_i)$  and  $r_i = r(x_i)$ . Note that the differential system (1.1) - (1.2) is equivalent to

$$\begin{aligned} (a) \quad & y''(x) = v(x)/p(x), \quad y(a) = y(b) = 0, \\ (b) \quad & v''(x) + [q(x) - \lambda r(x)] y(x) = 0, \\ & v(a) = v(b) = 0. \end{aligned} \tag{2.1}$$

Now the central difference approximation to 2.1(a) is

$$-y_{i-1} + 2y_i - y_{i+1} + h^2(v_i/p_i) + \frac{h^4}{12} y^{(4)}(\theta_i) = 0, \tag{2.2}$$

$$\theta_i \in (x_{i-1}, x_{i+1}), \quad i = 1(1)N.$$

The preceding system can be conveniently written in matrix form

$$JY + h^2 P^{-1}V + \frac{h^4}{12} T_1 = 0 \tag{2.3}$$

where  $Y = (y_i)$ ,  $V = (v_i)$ ,  $T_1 = (\rho_i)$  are  $N$ -dimensional column vectors with  $\rho_i = y^{(4)}(\theta_i)$ ,  $P = \text{diag}(p_i)$ , and  $J = (j_{mn})$  is a tridiagonal matrix so that

$$j_m^n = \begin{cases} 2, & m = n \\ -1, & |m - n| = 1 \\ 0, & |m - n| > 1. \end{cases} \quad (2.4)$$

In an analogous manner, on discretizing 2.1(b), we get

$$JV - h^2 QY + \lambda h^2 RY + \frac{h^4}{12} T_2 = 0 \quad (2.5)$$

where  $Q = \text{diag}(q_i)$ ,  $R = \text{diag}(r_i)$  and  $T_2 = (\sigma_i)$  with  $\sigma_i = v^{(4)}(\phi_i)$ ,  $\phi_i \in (x_{i-1}, x_{i+1})$ . Next, we eliminate  $v$  between (2.3) and (2.5) to obtain

$$AY \equiv (JPJ + h^4 Q)Y = \lambda h^4 RY + \Gamma, \quad (2.6)$$

where

$$\Gamma = \frac{1}{12} [h^6 T_2 - h^4 JPT_1]. \quad (2.7)$$

It can be verified that the matrix  $A = JPJ + h^4 Q$  is a five-band symmetric matrix. Now, in (2.6), neglect truncation error  $\Gamma$ , replace  $Y$  by  $\bar{Y}$ , then our method for computing approximations  $\Lambda$  for  $\lambda$  of the system (1.1) - (1.2) can be expressed as a generalized seven-band symmetric matrix eigenvalue problem

$$A\bar{Y} = \Lambda h^4 R\bar{Y}. \quad (2.8)$$

In fact the matrix  $JPJ$  is a positive definite matrix and hence for any step-size  $h > 0$ , the approximations  $\Lambda$  for  $\lambda$  by (2.8) are real and positive for all  $p(x) > 0$  and  $r(x) > 0$ . That our method provides  $O(h^2)$  convergent approximations  $\Lambda$  for  $\lambda$  can be established following Grigorieff [5]. We omit the proof of convergence for brevity.

### 3. A FOURTH ORDER METHOD

Following Shoosmith [6] the boundary value problems 2.1(a) and 2.1(b) are discretized by the finite difference scheme

$$\begin{aligned} \text{(a)} \quad & 14y_0 - 29y_1 + 16y_2 - y_3 = h^2[y_0'' + 12y_1''] , \\ \text{(b)} \quad & (1 - \frac{\delta^2}{12})\delta^2 y_i = h^2 y_i'' , \quad i = 2(1)N-1, \\ \text{(c)} \quad & -y_{n-2} + 16y_{N-1} - 29y_N + 14y_{N+1} = h^2[12y_N'' + y_{N+1}''] . \end{aligned} \quad (3.1)$$

It turns out the boundary value problem 3.1(a) gives rise to the linear equations

$$M\bar{Y} + 12h^2 p^{-1} \bar{V} = 0. \quad (3.2)$$

Similarly, for the system 2.1(b), we obtain the linear equations

$$M\bar{V} = 12h^2 Q\bar{Y} - 12\lambda h^2 R\bar{Y}, \quad (3.3)$$

where the five-band  $N \times N$  matrix  $M$  is given by

$$M = \begin{bmatrix} 29 & -16 & 1 & & & & & & \\ -16 & 30 & -16 & 1 & & & & & \\ 1 & -16 & 30 & -16 & 1 & & & & \\ \text{-----} & & & & & & & & \\ & & 1 & -16 & 30 & -16 & 1 & & \\ & & & 1 & -16 & 30 & -16 & & \\ & & & & 1 & -16 & 29 & & \end{bmatrix}. \quad (3.4)$$

The elimination of  $\tilde{V}$  from (3.2) and (3.3) gives our method for computing  $\Lambda$  for  $\lambda$  of (1.1) - (1.2) in the form

$$(MPM + 144h^4Q)\tilde{Y} = 144\Lambda h^4\tilde{R}\tilde{Y}, \quad (3.5)$$

where the matrix  $MPM$  is a nine-band positive definite matrix and hence for any step-size  $h > 0$ , the approximations  $\Lambda$  for  $\lambda$  by (3.5) are real and positive for all  $p(x), r(x) > 0$ . As before, it can be proved from the results of Grigorieff [5] that our present method provided  $O(h^4)$  convergent approximations  $\Lambda$  for  $\lambda$ .

#### 4. NUMERICAL RESULTS

In order to illustrate our methods of order 2 and 4 for the approximation of  $\lambda$  satisfying (1.1) - (1.2), we consider the eigenvalue problems:

$$[(1+x^2)y'']'' + \left[ \frac{1}{(1+x^2)} - \lambda(1+x)^4 \right] y = 0, \quad (4.1)$$

$$y(0) = y(1) = y''(0) = y''(1) = 0.$$

The smallest eigenvalue  $\lambda_1 = 22.754, 058, 480, \dots$

$$[e^x y'']'' + [\sin x - \lambda \cos x] y = 0, \quad (4.2)$$

$$y(0) = y(1) = y''(0) = y''(1) = 0.$$

The smallest eigenvalue of the system (4.2) is  $\lambda_1 = 181.345, 488, 233, \dots$ . We list the approximations  $\Lambda_1$  for  $\lambda_1$  and the relative errors  $\left| 1 - \frac{\lambda_1}{\Lambda_1} \right|$  for various values of the step-size  $h$ . It is readily verified that the relative errors (Table I) based on generalized eigenvalue problem (2.8) provide  $O(h^2)$ -convergent approximations for the smallest eigenvalue of the system (4.1) and (4.2). Similarly, the relative errors (Table II) based on the generalized eigenvalue problem (3.5) do indeed provide  $O(h^4)$ -convergent approximations for the smallest eigenvalue of the systems (4.1) and (4.2).

TABLE I

Results based on (2.8), second order approximations

Problem	N	$\Lambda_1$	$\left  1 - \frac{\lambda_1}{\Lambda_1} \right $
(4.1)	7	22.187	2.557-2*
	15	22.610	6.352-3
	31	22.718	1.586-3
	63	22.745	3.962-4
	127	22.752	9.907-5
	255	22.753	2.480-5
(4.2)	7	176.641	2.664-2
	15	180.159	6.588-3
	31	181.048	1.642-3
	63	181.271	4.103-4
	127	181.327	1.025-4
	255	181.341	2.560-5

\*We write 2.557-2 for  $2.557 \times 10^{-2}$ .

TABLE II

Results based on (3.5), 4th order approximations

Problem	N	$\Lambda_1$	$\left  1 - \frac{\lambda^1}{\Lambda_1} \right $
(4.1)	7	22.746, 419	3.358-4
	15	22.753, 574	2.129-5
	31	22.754, 027	1.358-6
	63	22.754, 056	1.078-7
(4.2)	7	181.244, 637	5.564-4
	15	181.339, 089	3.529-5
	31	181.345, 093	2.175-6
	63	181.345, 470	9.728-8

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