

ON CONTINUITY, DIFFERENTIABILITY
AND INTEGRABILITY

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CHAPTER 1

AIM, METHOD, AND JUSTIFICATION OF CONSIDERATIONS

PROPOSED

Herein it will be attempted to add to the understanding of the basic concepts of continuity, differentiability and integrability, by the consideration of functions which are not well-behaved. The more common functions in analysis are apt to be very misleading, as they do not show in all cases the requirements for full rigor in setting up necessary and sufficient conditions.

Historical development in analysis has shown this effect. It was at one time held that any continuous function has a derivative. Leibnitz and Newton defined $f(x) = F'(x)$, and the integral as the "anti-derivative"; ensuring that

$$\int_a^x f(x)dx = F(x) - F(a).$$

Thus the problem of having given an $f(x)$, and establishing the existence of $F(x)$ such that $F'(x) = f(x)$, did not arise. The constructive definition of the integral came much later, Riemann's integral being the first of this type. An assumption then made was that

$$\int_a^b F'(x)dx$$

always exists. Such conceptions were prevalent even up to the time of Fourier at the beginning of the nineteenth century. Examples to the contrary have proved them false.

Reversibility of operations and an intuitive behavior of elements with their combinations, introduce a simplification into any logical science. Thus with the discovery that the original simple ideas were

inadequate, an attempt was made to redefine integration. It was hoped that integration might form with differentiation a reversible process, and enable the beliefs of Newton to be true in this new sense. Of many generalized forms of integral, that of Lebesgue proved to be the most effective. It is neither so specialized as to exclude many functions from consideration, as does the Riemann integral, nor is it so generalized, as are some others, that it is lacking in power. The attempt for a new simplicity was only partially successful. While considering the failure of oversimplified concepts of functional behavior under Riemann integration, the extent to which they may hold using the integral of Lebesgue will be shown.

The method of counter-example, while at best an indirect form of proof, and having in general only the ability to deny, has been the most useful force in destroying these false ideas. This is essentially the method used here.

Functions which show the limitations on the concepts of analysis are in general characterized by discontinuities, either in the function itself or in its derivatives. Selected examples will be considered in an ordered classification. These range from the totally discontinuous, through increasing degrees of regularity, to a type continuous with a finite bounded derivative existing everywhere.

CHAPTER II

EXAMINATION OF FUNCTIONS DISCONTINUOUS EVERYWHERE

(1) The characteristic function of the rationals.

An example is first given of a function for which the Lebesgue integral exists while the Riemann integral does not. Consider the characteristic function of the set of rationals on an interval; ie on the interval (0,1) a function $f(x)$ such that $f(x)=1$ if x is rational and $f(x)=0$ if x is irrational, $f(x)$ is obviously discontinuous everywhere in the range of definition.

The Riemann integral is defined as the common limit as $\max(x_{p+1}-x_p) \rightarrow 0$ of the sums: $s_n = \sum_{p=1}^n m_p (x_{p+1}-x_p)$ and $S_n = \sum_{p=1}^n M_p (x_{p+1}-x_p)$; where the interval of variation (a,b) of the independent variable is subdivided by x such that $a=x_1 < x_2 < \dots < x_n = b$ and m_p M_p are the lower and upper bounds respectively of $f(x)$ in the interval

$$x_p < x < x_{p+1}$$

In the case of the characteristic function the m_p in each case is 0 and the M_p is 1. Thus the two sums are not equal in the limit, and the Riemann integral does not exist.

The Lebesgue integral is defined for a function bounded between the values a and b as the common limit of two similar sums. The interval of variation of the function is sub-divided by means of y such that

$$a=y_0 < y_1 < \dots < y_n = b, \text{ and}$$

the lower sum $s_n = \sum_{p=0}^{n-1} y_p m e_p$;
 where $m e_p$ the measure of the set e_p of values of the
 independent variable such that $y_p \leq y < y_{p+1}$. The upper sum

$$S_n = \sum_{p=0}^{n-1} y_{p+1} m e_p .$$

$\lim (S_n - s_n) = 0$ as $\max (y_{p+1} - y_p) \rightarrow 0$, due to the nature of a
 measurable function. It will now be shown that the Lebesgue
 integral of the characteristic function exists. The limit of
 the sum s_n is

$$\lim s_n = \lim_{n \rightarrow \infty} \sum_{p=0}^{n-1} y_p m e_p ,$$

taken over the set of all real x in $(0,1)$.

- $y_p = 0$ for $p=0$
- $m e_p = 0$ for all $p \neq 0$
- Thus $\lim s_n = 0$.

Hence the Lebesgue integral exists and is equal to zero.

(2) Example of Lebesgue

An evident requirement of a continuous function is that
 it assume within an interval all values intermediate to the
 functional values at the end-points; ie if $f(x)$ is continuous
 on the interval (a,b) and $f(a) \neq f(b)$, eg $f(a) < f(b)$,
 then given g such that

$$f(a) \leq g \leq f(b)$$

there exists a number c , $(a \leq c \leq b)$ such that $f(c) = g$.

It is shown below that this may also be a property of a
 function everywhere

discontinuous on the interval $(1)_*$

The function $f(x)$ is defined on the interval $0,1$. Let all real numbers x in the interval $0,1$ be expressed as infinite decimals, terminating decimals being represented as ending with an infinite succession of 9's. A number thus represented has the form

$$0.a_1 a_2 \dots a_n \dots . \text{ If the decimal expansion}$$

$0.a_1 a_3 a_5 \dots$ is not periodic the function $f(x)$ is given the value zero. If this expansion is periodic, the first period commencing with a_{2n-1} , $f(x)$ is given the value

$$0.a_{2n} a_{2n+2} a_{2n+4} \dots .$$

It will now be shown that in every sub-interval of $(0,1)$ $f(x)$ takes on all values between 0 and 1 and hence is everywhere discontinuous on $(0,1)$. Let A, B be any subinterval such that

$$0 \leq A < B \leq 1 \text{ and } \frac{1}{10^n} \leq B-A \leq \frac{1}{10^{n-1}} .$$

Their decimal expansions will be identical up to and including the $(n-2)^{\text{nd}}$ place but will differ in the $(n-1)^{\text{st}}$ place. It will be shown that elements T, U exist such that $A < T < U < B$ where T and U are of the form $T = 0.C_1 C_2 \dots C_n C_{n+1} 000 \dots$; $U = 0.C_1 C_2 \dots C_{n+1} 999 \dots$

Consider the set of intervals

$$0 \leq x \leq \frac{1}{10^{n+1}}, \frac{1}{10^{n+1}} \leq x \leq \frac{2}{10^{n+1}} \dots \frac{k}{10^{n+1}} \leq x \leq \frac{k+1}{10^{n+1}} \dots \frac{10^{n+1}-1}{10^{n+1}} \leq x \leq 1 .$$

Every point of the interval $0 \leq x \leq 1$ lies in one of these intervals.

(1)* Lebesgue, Lecons Sur L'Integration, 2^d Ed., p. 97.

Hence there is an integer K such that $\frac{k-1}{10^{n+1}} \leq A \leq \frac{k}{10^{n+1}}$. Take

$$T = \frac{k}{10^{n+1}}, \quad U = \frac{k+1}{10^{n+1}} \quad \text{Thus } A \leq T \leq U.$$

$$B-U = \frac{B-k+1}{10^{n+1}} = (B-A) + \frac{(A-k+1)}{10^{n+1}} \geq \frac{10}{10^{n+1}} - \frac{2}{10^{n+1}} > 0. \quad \text{Hence}$$

$A \leq T \leq U \leq B$. Let k in decimal notation be $0.C_1C_2 \dots C_{n+1}$.

$$T = 0.C_1 \dots C_{n+1} 000 \dots; \quad U = 0.C_1 \dots C_{n+1} 999 \dots$$

A point G will be shown to exist such that $T \leq G \leq U$ and such that for any number D , $0 \leq D \leq 1$, $f(G) = D$.

Let D in decimal notation be expressed as $D = 0.d_1d_2 \dots d_n \dots$.

If $n+1$ is even take $G = 0.C_1C_2 \dots C_{n+1} kd_1kd_2kd_3 \dots$ where k any integer except C_n . Then by definition $f(G) = D$. If $n+1$ is odd take

$$G = 0.C_1 \dots C_{n+1} 0 kd_1kd_2kd_3 \dots$$

where k any integer except C_{n+1} . Again $f(G) = D$. Hence $f(x)$ while everywhere discontinuous on $(0,1)$ assumes all intermediate values.

CHAPTER 111

CONTINUITY WITHOUT DIFFERENTIABILITY

The behavior of the above functions may not be surprising because of their discontinuous nature, but even continuous functions may show properties far removed from the regular behavior assigned to them by early mathematicians. In van der Waerden's example is seen a function continuous on an interval but having no differential coefficient at any point within it.(2)

This function $f(x)$ is defined on the interval $0,1$ as $\sum_{n=1}^{\infty} f_n(x)$ where $f_n(x)$ denotes the distance between x and the nearest number of the form $\frac{m}{10^n}$, where m is an integer.

$f(x)$ is easily shown to be continuous. Each $f_n(x)$ is continuous and $|f_n(x)| < 10^{-n}$. $f(x)$ is the sum of a uniformly convergent series of continuous functions and is therefore continuous.

It will now be shown that a differential coefficient $f'(x)$ does not exist. Let x , any number in the interval $(0,1)$, be expressed as an infinite decimal -- terminating decimals having a final infinite succession of 9's. If the q 'th digit be a 4 or 9 let $x_q = x - 10^{-q}$, otherwise $x_q = x + 10^{-q}$. Hence if $n < q$, the nearest number $\frac{m}{10^n}$ is the same for x and x_q and is either less than both of them or greater than both. It is seen that 4 and 9 must be considered specially from an example of x , x_q formed as above and $x_{q'}$ formed by treating a 4 or 9 as any other digit.

(2) Titchmarsh, The Theory of Functions, (Oxford: The Clarendon Press, 1932), p. 353.

Let $x = 0.3546 \dots$ $x_q = 0.3536 \dots$ $x_{q'} = 0.3556 \dots$ where the q 'th digit in x is 4. For x , x_q the nearest number of the form $\frac{m}{10^n}$ is 0.35, $n < q$ while for $x_{q'}$, the nearest such number is 0.36. Similarly if the q 'th figure be a 9. Let $x = 0.34396 \dots$ $x_q = 0.34386 \dots$ $x_{q'} = 0.34406 \dots$. The closest number $\frac{m}{10^n}$, $n < q$ is for x , $0.3440 > x$; for x_q , $0.3440 > x_q$; for $x_{q'}$, $0.3440 < x_{q'}$, hence $x_{q'}$ and x are not both of the same order of magnitude relative to the appropriate $\frac{m}{10^n}$.

For $n \geq q$ the numbers $\frac{m}{10^n}$, $\frac{m'}{10^n}$ corresponding to x , x_q respectively differ by $x - x_q$.

$$\begin{aligned} \text{Hence } f_n(x_q) - f_n(x) &= \pm (x_q - x), & (n < q) \\ &= 0, & (n \geq q) \end{aligned}$$

Thus $f(x_q) - f(x) = \sum_{n=1}^{q-1} \{ \pm (x_q - x) \} = p(x_q - x)$ where p is an integer, odd or even with $(q-1)$ since only an even multiple of $(x_q - x)$ may be removed by cancellation of positive and negative sign.

$$\text{As } q \rightarrow \infty, x_q \rightarrow x, \left\{ \frac{f(x_q) - f(x)}{x_q - x} - \frac{f(x_{q+1}) - f(x)}{x_{q+1} - x} \right\} \geq 1.$$

Therefore $\frac{f(x_q) - f(x)}{x_q - x}$ does not tend to a finite limit as $x_q \rightarrow x$. Hence the differential coefficient does not exist.

CHAPTER IV

CONTINUITY AND DIFFERENTIABILITY WITHOUT INTEGRABILITY OF DERIVATIVE

With an increasing degree of regularity in the function, deviation from supposedly normal properties seems even more strange. A function may however be continuous and possess a bounded differential coefficient; yet this differential coefficient may be non-integrable in the sense of Riemann. Volterra's example, the first such function constructed, is considered here. (1)

It will be shown that in the sense of Lebesgue the derivative does integrate back to the function.

(1) Volterra's example

The function will first be defined on an interval (a,b) . One requires a perfect non-dense set E of points on an interval (a,b) such that the measure of the set is greater than zero. Such a set is formed by taking the linear continuum $(0,1)$ as the interval (a,b) and deleting in the first stage an open interval of length $\frac{1}{4}$ centred at the point $\frac{1}{2}$. In the second stage are deleted open intervals with the properties:

- (1) they are of equal length
- (2) they are centred at the mid-points of the intervals remaining after the first stage
- (3) their total length is $1/8$

(1) Hobson, The Theory of Functions of a Real Variable, (Cambridge University Press, 1907), p.357.

In the n 'th stage open intervals are deleted such that:

- (1) they are of equal length $\frac{1}{2^{2n}}$,
- (2) they are centred at the mid-points of intervals remaining after the $(n-1)^{\text{st}}$ stage,
- (3) their total length is $\frac{1}{2^{n+1}}$.

This process is continued indefinitely, the points remaining forming the set E . E consists of the end points and the limiting points of the end points of the deleted intervals. It is seen that E is perfect since it is closed—being the complement of the open deleted set—and it is dense in itself. It is non-dense in the interval $(0,1)$, since any chosen interval will contain an interval of points of the complement, $C(E)$, of E . $C(E)$ is made up of the deleted open intervals. The measure of the complementary open set is the sum of the lengths of its intervals, ie $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 1/2 = m(C(E))$

Hence the measure of E is $1 - m(C(E)) = 1/2$.

Thus E is a perfect non-dense set of measure greater than zero.

Consider a general one of the complementary deleted intervals (A,B) . Define $\phi(x,A) = (x-A)^2 \sin \frac{1}{(x-A)}$. The derivative of $\phi(x,A)$ with respect to x is

$$\phi'(x,A) = 2(x-A) \sin \frac{1}{(x-A)} - \cos \frac{1}{(x-A)}; \text{ at } x = A, \phi(x,A) = 0 \text{ and } \phi'(x,A) = 0$$

$$\text{since } \phi'_A(x,A) = \lim_{h \rightarrow 0} \frac{\phi(A+h,A) - \phi(A)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} (A+h-A)^2 \sin \frac{1}{h} = 0.$$

It is also evident that $\phi'(x,A)$ vanishes at an infinite number of points in (A,B) . Of these zero points $(A+C)$ is chosen as the greatest value of x not exceeding $\frac{A+B}{2}$ for which $\phi'(x,A)$ vanishes. C is $< 1/8$ since the greatest

interval (A, B) is of magnitude $1/4$. For convenience let $(A+C)$ be the last such maximum of $\phi(x)$. The necessary requirement here for the construction of a continuous function is the choice of a zero point which is not an accumulation point of zero points. Thus it is not essential that one's choice should be the very last such zero point.

It will now be shown that such a last maximum in $(x-A)^2 \sin \frac{1}{x-A}$ in the range $A, \frac{A+B}{2}$ does exist. In the interval $A, \frac{A+B}{2}$; A is the only accumulation point of zero points of $\phi'(x, A)$. In any range $0 \leq x - A \leq x' - A$ where $A \leq x \leq x'$ the quantity $\frac{1}{x-A}$ varies from $\frac{1}{x'-A}$ to ∞ . In this range an infinite number of odd integral multiples of $\frac{\pi}{2}$ occur, at each of which is a zero of ϕ' . Thus any arbitrarily small interval about A contains infinitely many zero points. In the interval $(x', \frac{B+A}{2})$ the quantity $\frac{1}{x-A}$ varies throughout a finite range $(\frac{1}{x'-A}, \frac{1}{\frac{B+A}{2}-A})$ which contains only a finite number of odd integral multiples of $\frac{\pi}{2}$. Hence there are only a finite number of zero points in any range $(x', \frac{B+A}{2})$ where $x' > A$. If a maximum point t_1 of $\phi(x, A)$ is chosen such that $A < t_1 \leq \frac{A+B}{2}$ there are an infinite number of maxima in (A, t_1) but at most a finite number in $(t_1, \frac{A+B}{2})$. Hence the interval $t_1 \leq x \leq \frac{A+B}{2}$ contains a last maximum of $\phi(x, A)$ which may be t_1 . $A+C$ is the value of x at which this last maximum appears.

The function $F(x)$ is defined such that $F(x) = 0$ at every point of E . In each complementary interval (A, B) , $F(x)$ is defined as $\phi(x, A)$ for $A \leq x \leq A+C$; $F(x) = \phi(A+C, A)$ for $A+C \leq x \leq B-C$ and

$F(x) = \phi(B, x) = -\phi(x, B)$ for $B - C \leq x \leq B$. $F(x)$ so defined can be shown to be continuous with a finite bounded differential coefficient everywhere on the interval of definition $(0, 1)$.

$F(x)$ will be shown to be continuous by the manner of its definition. In E , $F(x) = \lim_{n \rightarrow \infty} F(X_n)$ where X_n an end point of a complementary interval to E and $X_n \rightarrow x$. On a complementary interval (A, B) , $F(x) = \phi(x, A)$ in $(A, A+C)$; $F(x) = \phi(A+C, A)$ in $(A+C, B-C)$ and $F(x) = \phi(B, x)$ in $(B-C, B)$. At the point $x = B-C$, $\phi(B, B-C) = (B-(B-C))^2 \sin \frac{1}{B-(B-C)} = C^2 \sin \frac{1}{C} = \phi(A+C, A)$. Hence it is seen that $F(x)$ is continuous on the interval (A, B) .

It will now be shown that the derivative of $F(x)$ with respect to x , $F'(x)$, exists everywhere, is bounded and measurable but has a discontinuity, of measure at least 2, at all points of E .

The derivative of $F(x)$ in $C(E)$ is zero for the ranges $(A_n + C, B_n - C)$ where $F(x)$ is constant and is $\phi'(x, A_n)$ on $(A_n, A_n + C)$; $\phi'(B_n, x)$ on $(B_n - C, B_n)$. It will now be shown that at a point x of E , $F'(x) = 0$; i.e. for any constant $\epsilon > 0$ there exists a constant $\delta > 0$ such that $|\frac{F(x+h) - F(x)}{h}| < \epsilon$ for all $|h| < \delta$. Given $\epsilon > 0$ take $\delta = \epsilon$ and $0 < |h| < \delta$. If $x+h$ in E , $\frac{F(x+h) - F(x)}{h} = \frac{0 - 0}{h} = 0$. If $x+h$ in $C(E)$ and $x - \epsilon \leq A - B \leq x + \epsilon$ there are three possible cases.

Case 1. If $x+h$ in the interval $A, A+C$ then

$$\left| \frac{F(x+h) - F(x)}{h} \right| = \frac{(x+h - A)^2 \sin \frac{1}{x+h-A}}{h}. \text{ The interval } A, B$$

must lie wholly to the left with $B \leq x$ or to the right with $x \leq A$ since a point of E could not be contained in a complementary interval. Therefore $|x+h-A| < |h|$ and $\left| \frac{F(x+h) - F(x)}{h} \right| < |h| < \epsilon$.

Case 2. If $x+h$ be in $A+C$, $B-C$ then $\left| \frac{F(x+h) - F(x)}{h} \right| = \frac{C^2}{h} \sin \frac{1}{h}$ and $C < /h/$. Therefore $\left| \frac{F(x+h) - F(x)}{h} \right| < /h/ < \epsilon$.

Case 3. If $x+h$ in $B-C$, B ; $\left| \frac{F(x+h) - F(x)}{h} \right| = \frac{(B-x-h)^2}{h} \sin \frac{1}{B-x-h}$ and $/B-x-h/ < /h/$. Hence $\left| \frac{F(x+h) - F(x)}{h} \right| < /h/ < \epsilon$.

If the complementary interval containing $x+h$ is such that

(1) $A < x - \epsilon < B \leq x$ or (2) $x \leq A < x + \epsilon < B$ the result follows as above. Here the location of $x \pm \epsilon$ in the interval A, B might exclude one or two of the above cases, eg. in (1) if $B - C < x - \epsilon \leq B$ then one would not find $x+h$ in the ranges $A, A+C$ or $A+C, B-C$.

Hence $F'(x) = 0$ at all points of E . In the ranges $(A, A+C)$; $(B-C, B)$ $F'(x)$ is respectively $2(x-A) \sin \frac{1}{x-A} - \cos \frac{1}{x-A}$ and $2(B-x) \sin \frac{1}{B-x} - \cos \frac{1}{B-x}$. These can never exceed $/2C+1/$ since $C < \frac{B-A}{2}$. Hence $F'(x)$ exists everywhere and is bounded.

$F'(x)$ will now be shown to have a discontinuity of measure at least 2 at every point of the set E . Consider any point x of first or second type in E . In every arbitrary interval about x is contained a complementary interval of E since E is nowhere dense on $0, 1$. In every complementary interval the function $F'(x)$ oscillates at least between $+1$ and -1 in the neighborhood of an end point. Thus for $|x' - x| < \text{arbitrary constant}$, $|F'(x') - F'(x)| \geq 2$ and a discontinuity of measure at least 2 exists at all points of E . Since E is of positive measure $1/2$, $F'(x)$ is not integrable in the Riemann sense. This is also readily shown by direct considerations. One may make a subdivision of the x -axis in the interval $0, 1$ by means of the points x_1, x_2, \dots, x_n and consider the upper and lower

Riemann sums $\sum_{s=1}^{n-1} M_s (x_{s+1} - x_s)$ and $\sum_{s=1}^{n-1} m_s (x_{s+1} - x_s)$. The requirement for the existence of the integral is

$$\lim_{n \rightarrow \infty} \sum_{s=1}^{n-1} (M_s - m_s) (x_{s+1} - x_s) = 0.$$

The summation over $(0,1)$ may be considered as the sum over intervals containing points of E and those not containing points of E .

since $F'(x)$ continuous on the set CE .

$$\lim_{n \rightarrow \infty} \sum_E (M_s - m_s) (x_{s+1} - x_s) \geq (2) \left(\frac{1}{2}\right) = 1$$

since $F'(x)$ has a discontinuity of measure at least 2 at all points of E and the measure of E is $\frac{1}{2}$. Thus the limit is not zero and a Riemann integral of $F'(x)$ over $(0,1)$ does not exist.

$F'(x)$ being bounded and measurable is integrable in the sense of Lebesgue. Moreover $F'(x)$ exists everywhere on the interval and

$$\left| \frac{F(x+h) - F(x)}{h} - F'(x - \theta h) \right| \leq M, \quad 0 < \theta < 1$$

Hence $\sum_{s=1}^n |F(x_s + h_s) - F(x_s)| \leq M \sum_{s=1}^n |h_s|$, i.e.

$F(x)$ is absolutely continuous and is thus a Lebesgue integral.

In the sense of Lebesgue therefore

$$F(x) = \int_a^x F'(t) dt - F(a),$$

the derivative is integrable and integrates back to the function.

A further example of a rather regular function (in fact continuous, monotone increasing, possessing a derivative bounded and measurable) which is not an integral, is constructed on the Cantor set. (2)

(2) Hille and Tamarkin, "Remarks on a Known Example Of a Monotone Continuous Function", American Mathematical Monthly, Vol.

36, May 1929, pp. 255-64.

(2) A function on the Cantor set.

The function will first be defined on the interval $(0,1)$. Consider a perfect set of points nowhere dense on the interval $(0,1)$, Cantor's set is such a set. To form the Cantor set $(0,1)$ is subdivided in three equal parts and the interior of the middle part removed. The remaining two parts are each subdivided in three and the interiors of the middle parts removed. This is repeated indefinitely--the number of intervals removed at the p^{th} stage being 2^{p-1} each of length 3^{-p} . The intervals of the p^{th} stage are denoted by D_{pk} , ($k = 1, 2, \dots, 2^{p-1}$). The total number removed in the first p stages is $2^p - 1$. Let E be the set of points not removed consisting of the end points of the intervals D_{pk} and their limiting points. The complementary set $C(E)$ is the sum of the open deleted intervals D_{pk} . Let the closed intervals remaining at the p^{th} stage be N_{pk} , ($k = 1, 2, \dots, 2^p$). E is always covered by the N_{pk} . For fixed p all N_{pk} have length 3^{-p} and the sum of their lengths is $(2/3)^p$ which goes to zero as p approaches infinity. Hence E is of measure zero. Denoting the points of E on the ternary scale as infinite decimal fractions, they are numbers of the type $0.a_1 a_2 \dots a_n \dots$ where only digits 0 and 2 are admitted. All numbers of this type may be approximated as closely as is desired by numbers of the same type. Thus E contains all its limiting points. Since each point of E is a limiting point, E is a perfect set. Every subinterval of $(0,1)$ contains intervals of $C(E)$, hence E is nowhere dense on $(0,1)$.

Using a to indicate digits 0 or 2 and b for $\frac{a}{2}$, then when $X = 0.a_1 a_2 \dots$
 (any point of E) the function $w(x)$ has the value $w(x) = 0.b_1 b_2 \dots$,

a number on the scale of 2. Considering values of $w(x)$ at end points of an interval D_{pk} , at the left-hand end point $X = 0.a_1 a_2 \dots a_{p-1} 0222\dots$ and $w(x) = 0.b_1 b_2 \dots b_{p-1} 0111\dots$. At the right-hand end point $x = 0.a_1 a_2 \dots a_{p-1} 2000\dots$ and $w(x) = 0.b_1 b_2 \dots b_{p-1} 1000\dots$. Therefore at the end points $w(x)$ has the same value $\frac{2^{k-1}}{2^p}$ which may be seen by considering that for fixed p and increasing k one has $w_{p1}(x) = \frac{1}{2^p}$, and for every increase of 1 in k there is an increase of $2/2^p$ in $w_{pk}(x)$. The value of $w(x)$ at all points of a D_{pk} is taken its value at the end points. Thus the function $w(x)$ is defined on the interval $(0,1)$.

It will now be shown $w(x)$ is monotone (non-decreasing) on $(0,1)$ and increases from 0 to 1 as x increases from 0 to 1. Intervals D_{pk} are intervals of constancy of $w(x)$, thus one needs only to consider the points of E . For $x' = 0.a_1^1 a_2^1 \dots$, $x'' = 0.a_1^{11} a_2^{11} \dots$ and $x'' \succ x'$ there exists a subscript n such that $a_i^1 = a_i^{11}$ for $i < n$, $a_n^{11} \succ a_n^1$. Thus

$$w(x'') = 0.b_1^{11} \dots b_n^{11} \dots \succ 0.b_1^1 \dots b_n^1 \dots = w(x')$$

It is here shown that $w(x)$ is continuous on $(0,1)$. Given an arbitrary positive constant ϵ there exists a positive constant d such that

$$|w(x') - w(x)| < \epsilon \text{ for } |x' - x| < d.$$

$w(x)$ as a constant is continuous on the D_{pk} and it is necessary to consider only x and x' of E . For the given $\epsilon > 0$ choose an n such that $2^{-n} < \epsilon$. It is now seen that $d = 3^{-n}$ satisfies the requirements, for $|x' - x| < d$ means x and x' are of the form:

$$x = 0.a_1 a_2 \dots a_{n+1} a_{n+2} \dots$$

$$x' = 0.a_1 a_2 \dots a_{n+1} a_{n+2}' \dots$$

on the scale of three. x and x' differ only after the $(n+1)^{\text{th}}$ place.

$$\text{then } |w(x) - w(x')| = |0.b_1 \dots b_{n+1} b_{n+2} \dots - 0.b_1 \dots b_{n+1} b'_{n+2} \dots| \\ < 2^{-n} < e.$$

Thus $w(x)$ is continuous on the interval $(0,1)$.

$w(x)$ is now proved to be not absolutely continuous and to have on $(0,1)$ a constant L variation of magnitude 1. L variation on an interval is defined as the upper limit of the sum

$$\sum_{k=1}^m |f(B_k) - f(A_k)|$$

over any finite set of non-overlapping sub-intervals (A_k, B_k) ; $k = 1, 2, \dots, m$,

of total length $\sum_{k=1}^m (B_k - A_k) \leq L$. Choosing $(A_k, B_k) = N_{pk}$ then

$$\sum |w(B_k) - w(A_k)| = \sum [w(B_k) - w(A_k)] = 1 \quad \text{but}$$

$$\sum N_{pk} = \left(\frac{2}{3}\right)^p \text{ and } \left(\frac{2}{3}\right)^p < e \text{ for } p > \frac{\log e}{\log 2/3}$$

Thus the L variation is constant at 1 and $w(x)$ does not satisfy the condition of absolute continuity which is:

For an arbitrary positive constant e given, there must exist a positive constant d such that

$$\sum_{s=1}^n |w(x_s + h_s) - w(x_s)| < e$$

for any set of non-overlapping intervals $(x_s, x_s + h_s)$ where

$$\sum |x_s + h_s - x_s| < d.$$

Since absolute continuity is the necessary and sufficient condition for a function to be an integral

$$w(x) \neq \int_a^x w'(x) dx + w(a).$$

The derivative $w'(x)$ is zero almost everywhere on $(0,1)$ since it is zero at all points of the set $C(E)$. Thus $w'(x)$ is bounded and measurable almost everywhere, hence integrable but not to $w(x)$. In fact

$$\int_a^x w'(x) dx = 0 \text{ for all } a, x \text{ in } (0,1).$$

It will now be shown that although $w(x)$ is monotone, hence of bounded variation and possessing a curve $y = w(x)$ of finite length between $(0,0)$ and $(1,1)$, the length is not given by the usual formula

$$\int_0^1 (1 + w'(x)^2)^{1/2} dx = 1$$

The actual length will first be found from consideration of the definition of curve length as the limit of the perimeter of an inscribed polygon. The perimeter of any such inscribed polygon, without double points, cannot exceed the sum of all the horizontal and vertical projections of its sides; which is 2 for a polygon running from $(0,0)$ to $(1,1)$. If the inscribed polygon is taken as the broken line with vertices at $(0,0)$, $(1,1)$ and the end points of the D_{pk}

$$(n \text{ fixed, } p = 1, 2, \dots, n, k = 1, 2, \dots, 2^{p-1})$$

the sum of the horizontal sides is

$$\sum_{pk} D_{pk} = \sum_{p=1}^n 2^{p-1} 3^{-p} = 1 - (2/3)^n.$$

All the inclined sides, 2^n in number are of equal length. Their common

length is $(2^{-2n} + 3^{-2n})^{1/2} = 2^{-n} \left(1 + \left(\frac{2}{3}\right)^{2n}\right)^{1/2}$. As $n \rightarrow \infty$ length of polygon,

$$1 - \left(\frac{2}{3}\right)^n + \left(1 + \left(\frac{2}{3}\right)^{2n}\right)^{1/2} \rightarrow 2.$$

Hence the length of the arc $y = w(x)$ between $(0,0)$ and $(1,1)$ is 2. The usual formula fails here because $w(x)$ is not absolutely continuous.

$w(x)$ will now be shown to satisfy a Lipschitz condition of order